SPACES WITH THE UHOS PROPERTY

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P. H. Doyle

Major professor

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ABSTRACT

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By

Leon Brewster Hardy

A topological space has uniformly homeomorphic open sets (has the UHOS property) if all nonempty open sets are homeomorphic. We prove that the category of spaces with this property is large, and that the rational and irrational numbers in E^1 are in this category.

A topological space, X, is said to be invertible if for every open set U in X, $U \neq \emptyset$, there is a homeomorphism $h \circlearrowleft X$ satisfying $h(X - U) \subset U$. We prove, in Chapter II, that every topological space embeds in an invertible, UHOS-space.

Characterization theorems for the rational and irrational numbers with respect to the UHOS property are presented in Chapter III.

We prove, in Chapter IV, that compact, UHOS-spaces are connected.

In Chapter V, we prove that a topological space, X, has the UHOS property iff any nondense set $D \subset X$ may be taken into $X - D$ by a homemorphism $h : X \to X - \bar{D}$.

SPACES WITH THE UHOS PROPERTY

By

Leon Brewster Hardy

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To my wife Nellie, and my family

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INTRODUCTION

This thesis deals with topological spaces with the property that all non-void open sets in them are homeomorphic (UHOS-spaces). Questions about the nature and existence of nontrivial spaces of this type arose from discussions of general topology.

Chapter I explores the category C of such spaces, and establishes familiar spaces such as the rational numbers and irrational numbers as members of C. The morphisms of the category are quite unimportant as a rule and for convenience may be taken as maps (continuous functions). No separation axioms are assumed for topological groups in this chapter.

Chapter II establishes C as a universal embedding class for all topological spaces. The argument leads to a corollary showing that every topological space embeds in an invertible space [1]. A topological space X is invertible if each proper closed set W in X is carried to its complement in X by some homeomorphism h of X onto X; i.e. $h \circlearrowleft X$ and $h(W) \subset X - W$ [1]. Early examples suggested a strong relationship between invertible spaces and the objects in C. It is certain that in general invertible spaces are not UHOS spaces since the n-sphere is invertible, but not a UHOS-space. A simple example of a UHOS-space that is not invertible appears in Example 5.1. In addition, theorem 5.4 shows the weaker relation in a UHOS-space between a closed set and its complement.

Chapter III deals with metric UHOS-spaces. Our interest is largely confined to the separable case. The Menger-Urysohn definition of dimension is used. The most general result here is the existence of an infinite partition into open and closed sets of every infinite UHOS-space. Finally the rational and irrational numbers are characterized among UHOS-spaces.

Chapter IV studies properties of UHOS-spaces related to connectedness. Theorem 4.5.1 establishes the surprising result that compact UHOS-spaces are connected.

Finally Chapter V deals with the rather weak connections between UHOS-spaces and invertible ones.

CHAPTER I

NONTRIVIAL EXISTENCE

1. The Definition and Existence of Spaces with Uniformly Homeomorphic Open Sets.

In this section of Chapter 1, we prove that a large number of topological spaces have the UHOS-property. In particular, we prove that the familiar, but seemingly unlikely spaces R (the space of rational numbers in E^1 with the relative topology), and I (the space or irrational numbers in E^1 with the relative topology) have this property.

Definition 1.1.1. Let X be a non-empty topological space. X has uniformly homeomorphic open sets (UHOS), or is a UHOS-space or has the UHOS-property if all nonempty open sets in X are homeomorphic.

Observe that the class of UHOS-spaces along with the maps between them form a category C.

Lemma 1.1.2. C is a large category.

Proof: Any non-empty set with the indiscrete topology (at most two open sets) is in C.

Lemma 1.1.2. The only finite spaces in C have the indiscrete topology.

Proof: There exists no homeomorphism between two sets of different cardinality.

Theorem 1.1.3: The irrational numbers belong to C.

Proof: Let $I \subset E^1$ (E^n is euclidean n-space) be the irrationals. By a theorem of Hurewicz-Wallman, we can consider the irrationals on the real line [3], page 60. If $U \subset I$ is open, then $U = I \cap V$, where V is open in E^1. Now $V = \bigcup_1^n V_i$, where the V_i are disjoint open intervals, and $n = 1$, $1 < n < m$ for some positive integer m or $n = \infty$.

Case 1: $V = V_1$, a single open interval.

$V - (V - I)$ is the set of rational numbers in V; a countable dense set in V. There exists a homeomorphism $g:V \to E^1$, which is generally the composition of two homeomorphisms; the standard homeomorphism from $(-1,+1)$ to E^1 [2], and some linear map in E^1. This g takes the countable dense set $\{V - (V - I)\}$ to a countable dense set in E^1. By a theorem of Hurewicz-Wallman [3], page 44, there exists an onto homeomorphism $g':E^1 \to E^1$ which takes the countable dense set $g\{V - (V - I)\}$ to the rationals in E^1. Now the composition $g' \circ g:V \to E^1$ is a homeomorphism and takes the rationals in V to the rationals in E^1. Hence the irrational numbers $(V \cap I)$ in V are preserved under $g' \circ g$ and this completes the proof of case I.

<u>Case 2</u>: $V = \bigcup_1^n V_i$.

Again $\{V - (V \cap I)\}$ is the set of rational numbers in V, a countable dense set in V. There exists a homeomorphism $g_i : V_i \to E^1$, for each positive integer $i = 1, 2, \ldots, n$, as noted in case 1. Then $G : \bigcup_1^n V_i \to \bigcup_1^n E_i^1$ is a homeomorphism, where $G(V_i) = g_i(V_i)$, and each E_i^1 is a copy of E^1. G takes the countable dense set $\{V - (V \cap I)\}$ to a countable dense set in $\bigcup_1^n E_i^1$. Again using the theorem of Hurewicz-Wallman employed in case 1, we can construct a homeomorphism $G' : \bigcup_1^n E_i^1 \to \bigcup_1^n E_i^1$ taking the countable dense set $G\{V - (V \cap I)\}$ in $\bigcup_1^n E_i^1$ onto the rational numbers in $\bigcup_1^n E_i^1$ (G is defined on each $V_i \cap I$ after using the Hurewicz-Wallman result mentioned in case 1 on each $V_i \cap I$).

Now, as a result of case 1, a homeomorphism g_i' can be constructed (preserving rationals) from each E_i^1, $i = 1, \ldots, n$, to any open interval. However, we choose our intervals so that $g_1' : E_1^1 \to U_1 = (-\infty, 0)$, $g_2' : E_2^1 \to U_2 = (0, 1)$, $g_3' : E_3^1 \to (1, 2), \ldots, g_{n-1}' : E_{n-1}^1 \to (n-3, n-2)$; $g_n' : E_n^1 \to (n-2, +\infty)$. Observe that I is contained in the union of these n-intervals, and inherits the usual subspace topology from this union of open intervals. Now

$$G'' : \bigcup_1^n E_i^1 \to \bigcup_1^n U_i ,$$

where $G''(E_i^1) = g_i'(E_i^1) = U_i$ is a homeomorphism and preserves rational numbers.

The composition $(G'' \circ G' \circ G)$ on $\{V - (V \cap I)\}$ is a homeomorphism and preserves rational numbers. Hence $(G'' \circ G' \circ G)(V \cap I) = I$, and this completes the proof of case 2.

<u>Case 3</u>: $V = \bigcup_1^\infty V_i$.

Again $\{V - (V \cap I)\}$ is a countable dense set in V. There exists a homeomorphism $g : V_i \to E_i^1$ for each positive integer. Then $G : \bigcup_1^\infty V_i \to \bigcup_1^\infty E_i^1$, where $G(V_i) = g_i(V_i)$ is a homeomorphism and $\bigcup_1^\infty E_i^1$ is a countable union of copies of E^1. G takes the countable dense set $\{V - (V \cap I)\}$ to a countable dense set in $\bigcup_1^\infty E_i^1$. The image of $\{V - (V \cap I)\}$ in each E_i^1 is countable and dense in E_i^1. Again using the theorem of Hurewicz-Wallman in case 1, there is a homeomorphism $G' : \bigcup_1^\infty E_i^1 \to \bigcup_1^\infty E_i^1$ taking countable dense $G\{V - (V \cap I\}$ in $\bigcup_1^\infty E_i^1$ to the rationals in $\bigcup_1^\infty E_i^1$. Now again by case 1, a homeomorphism can be constructed (preserving rationals) from each E_i^1 to some open interval in $\bigcup_k (k, k+1)$, $k = 0, \pm 1, \pm 2, \ldots,$ which contains I. The rational preserving $G'' : \bigcup E_i^1 \to E^1$ is then evident. The composition, $(G'' \circ G' \circ G)$ is then a homeomorphism from $(V \cap I)$ to the irrationals in E^1, and this completes the proof of case 3.

<u>Theorem 1.1.4</u>. The rational numbers belong to C.

Proof: Using the same theorems and similar procedures as in Theorem 1.1.3, this result follows.

The above results may lead to faulty intuition and we note: The Cantor ternary set C^* does not belong to C.

Proof: The Cantor ternary set C^* is compact and perfect. $C^* - \{0\}$ is open and not compact in C^* and hence not homeomorphic to C^*.

2. Examples of UHOS-Spaces.

1.2.1. Let $X \neq \emptyset$ be any set with indiscrete topology. X is a UHOS-space.

1.2.2. Let X be a one point space. X is a UHOS-space.

1.2.3. Let $X = E^1$ and the topology consists of $\emptyset$ and any open interval about zero (0). This space is UHOS. X is also T_0 but not T_1.

1.2.4. Let X_i be countably infinite, $i = 1,\ldots,\infty$; $X_i \cap X_j = \emptyset$ for $i \neq j$. Consider $Y = \bigcup_1^\infty X_i$.

U in Y is open if $\tilde{U} = (Y - U)$ is a union of finitely many X_i.

Y is then a UHOS-space. Y is not T_0.

1.2.5. Let X be a countably infinite set, and let $U \subset X$ be open if $\tilde{U}$ is finite. X is then a UHOS-space.

X is T_0, T_1 but not T_2. (X has the co-finite topology) [5].

1.2.6. Let X be uncountable, and let $U \subset X$ be open if $\tilde{U}$ is (i) finite or, (ii) countable. X is then a UHOS-space in both cases. X is T_0, T_1, but not T_2.

1.2.7. The rationals form a UHOS-space.

1.2.8. The irrationals form a UHOS-space.

1.2.9. Let $X = E^1$. $U \subset X$ is open if $U = \{x > a \mid a$ any real number$\}$. X is a UHOS-space. X is T_0 but not T_1!

1.2.10. Let (X,T) be a topological space which has the UHOS-property. Consider the collection, T, of open sets in X. We order the open sets in T by set inclusion, and the pair (T,"$\subset$") becomes a partial ordering. However, the pair (T,"$\subset$") need not be a total ordering.

Define a new topological space (X,T′) as follows:

1. While (T,"$\subset$") is not necessarily totally ordered, there are chains in T. Consider a maximal chain in (T,"$\subset$").
2. The open sets comprising this maximal chain will be the open sets in our new topology, T′.

Is (X,T') a topological space? A straightforward proof using the definition of chain with respect to set inclusion shows that our collection of open sets, T', is closed under arbitrary unions and finite intersections. The definition also shows that the sets $\emptyset$ and X are both in T'. (X,T') is therefore a topological space.

Is (X,T') a UHOS-space? Let U, V be open in T'. U, V are open in T, by definition of T', hence there is a homeomorphism $h: U \to V$. This same h applies in (X,T'). Therefore (X,T') is a UHOS-space.

This result yields a method of generating new UHOS-spaces from known UHOS-spaces.

1.2.11. A counterexample to the conjecture that the product of UHOS-spaces is a UHOS-space.

Let $X_i = Z^+$, with cofinite topology, for each $i = 1,2,\ldots$. Observe that X_i is compact and a UHOS-space for each i; hence

$$Y = \prod_1^\infty X_i \text{ is compact.}$$

But we check the UHOS property in Y.

Let $C_i = \{1_i, 2_i\}$, $i = 1,2,\ldots$, and consider $C = \prod_1^\infty \{1_i, 2_i\}$.

Claim: $Y - C$ is open.

<u>Proof</u>: Let $p = \{p_i\} \in Y - C$. Then there is an integer j with $p_j \geq 3_j$ in X_j. Let $U_j = \{3_j, 4_j, 5_j, \ldots\}$. Then $(p_1, p_2, \ldots, p_{j-1}, 3_j, p_{j+1}, \ldots)$ belongs to $(\prod_1^{j-1} X_i) \times U_j \times (\prod_{j+1}^{\infty} X_i)$, an open set in $Y - C$. Notice that U_j, $j = 1, 2, \ldots$ generates an infinite open covering of $Y - C$, namely,

$$A = \{(\prod_1^{j-1} X_i) \times U_j \times (\prod_{j+1}^{\infty} X_i)\}_{j=1}^{\infty} .$$

But points of the form,

$$(1_1, 1_2, \ldots, 1_{j-1}, 3_j, 1_{j+1}, \ldots),\ j = 1, 2, \ldots,\ \text{in } Y - C,$$

are found in only one such covering element. Hence a finite subcover of $Y - C$ from A is impossible. $Y - C$ is therefore not compact. We conclude then that Y is not a UHOS-space, and that the product of UHOS-spaces is not necessarily a UHOS-space.

1.2.12. Counterexample to the reasonable conjecture that a G_δ set in a UHOS-space is void or UHOS. (Recall that a G_δ set in a topological space is the intersection of countably many open sets [5]).

Consider the rationals R in E^1, which is as we showed earlier, a UHOS-space. Consider the open sets in R determined by $\{S_{\frac{1}{n}}(a) \cup S_{\frac{1}{n}}(b)\}$, $a < b$ rational, $n = 1, 2, \ldots$. Now,

$$\bigcap_{n=1}^{\infty} \{S_{\frac{1}{n}}(a) \cup S_{\frac{1}{n}}(b)\} = \{a,b\}\ , \text{ which}$$

has the discrete relative topology as a subspace of R. But since finite discrete topological spaces are not UHOS-spaces (see lemma 1.1.2), we conclude that G_δ sets in UHOS-spaces are not generally UHOS-spaces or void.

Note: Consider the result of requiring X to be a metric UHOS-space and each G_δ in X a UHOS-space (as a subspace). Using an argument like the one above, we establish that the two-point subspace, $\{p,q\}$, is a G_δ set, hence a UHOS-space. But again, $\{p,q\}$ has the discrete topology in X. It follows that X contains at most one point.

The same applies to Hausdorff first countable spaces.

It will appear in Chapter III that the G_δ property is useful in conjunction with the UHOS property to characterize the irrational numbers.

Another reasonable question is considered in the following example.

1.2.13. Do onto, open maps preserve the UHOS property?

Consider the rationals, R, which we showed earlier has the UHOS-property, and the subspace $\{0,1\}$ of the rationals. Here $\{0,1\}$ has the discrete subspace topology, and hence is not a

UHOS-space. But the function $F:R \rightarrow \{0,1\}$ defined so that $F\{x|x \text{ rational}; x < \Pi\} = 0$ and $F\{x|x \text{ rational}; x > \Pi\} = 1$, is open and continuous. Hence onto, open maps do not preserve the UHOS-property.

1.2.14. Examples of UHOS-spaces with special closed sets, i.e., UHOS-spaces in which all nonvoid closed sets are homeomorphic.

(i) One point spaces have this property.

(ii) Consider example 1.2.4 in which X_i is countably infinite, $i = 1,2,\ldots$; $X_i \cap X_j = \emptyset$, $i \neq j$. We defined

$$Y = \bigcup_1^\infty X_i ,$$

and defined $U \subset Y$ to be open iff $\tilde{U} = \bigcup_1^n X_i$. Y is a UHOS-space, and since nonvoid closed sets are all finite unions of X_i's, each X_i countable, we conclude that nonvoid closed sets are homeomorphic also.

Construction (ii) suggests a scheme for generating UHOS-spaces with the property that nonvoid closed sets are homeomorphic using the technique in example 1.2.4. We only need require that each X_i be "large enough", e.g. have cardinality $c, 2^c, \ldots,$ etc.

3. Spaces that are Groups.

Several of the examples introduced above carry a group structure. In this section we prove that topological groups ([4], [5], [6]), G, which have the UHOS-property retain this property in the quotient, G/A, for suitable A.

Definition 1.3.1. Let G be a topological group. Define a subset A of G as follows: a $\in$ A if every neighborhood of a contains e, and every neighborhood of e contains a, and A is maximal with respect to this property.

Lemma 1.3.2. A is a subgroup of G.

Proof: Let $a,b \in A$. Consider any neighborhood W of the product ab^{-1}. The continuity of the mapping $T: (x,y) \to xy^{-1}$ of $G \times G$ onto G guarantees the existence of a neighborhood V, of $(a,b) \ni T(V) \subset W$. But $(e,e) \in V$ (by definition of A and the fact that $V = M \times N$, M,N open sets containing e), hence $T(e,e) = ee^{-1} = e \in W$. Therefore, any neighborhood of ab^{-1} contains e.

Consider any neighborhood W' of e. Again the continuity of T at (e,e) guarantees the existence of a neighborhood V', of $(e,e) \ni T(V') \subset W'$. But $(a,b) \in V'$, hence $T(a,b) = ab^{-1} \in W'$. Therefore, any neighborhood of e contains ab^{-1}. We conclude then that A is a subgroup of G.

<u>Lemma 1.3.3</u>. A is an invariant subgroup of G.

<u>Proof</u>: The map $f \ni f(x) = g\,xg^{-1}$ is a homeomorphism and isomorphism of G onto G. Thus f must preserve A, the set of all points not meeting any separation conditions relative to e. Since $f(A) = gAg^{-1}$, and A is maximal, $gAg^{-1} = A$. We conclude then that A is an invariant subgroup of G.

<u>Theorem 1.3.4</u>. A is a closed subgroup of G.

<u>Proof</u>: Let x be a limit point of A.

1. Consider an arbitrary neighborhood U_x of x. x is a limit point of A, hence there is an $a \in A$ with $a \in U_x$. But $a \in U_x$ implies that $e \in U_x$.

2. Consider an arbitrary neighborhood, W_e, of e. Suppose $x \notin W_e$.

   Preliminaries: a) x^{-1} is a limit point of A.

   Proof: Clear (apply the homeomorphism $T: x \to x^{-1}$ of G onto G).

   b) Every neighborhood of x^{-1} contains e.

   Proof: Use a).

   Now, there is a neighborhood, W'_e, of e since otherwise $x^{-1} \notin A$ and we are done. Then $x, x^{-1} \notin W_e \cap W'_e$. However,

$$x \in x(W_e \cap W_e'),$$

an open set containing x which does not contain e.

$$(e \in x(W_e \cap W_e') \Rightarrow e = xy \Rightarrow x^{-1} = y \in W_e \cap W_e').$$

The preceding lemmas and theorem allow us to conclude that G/A satisfies the separation axioms T_i, $i = 0,1,2,3$, and is also a topological group (Husain [4], Chapter 3).

We are interested in whether or not the UHOS-property is retained in the quotient space G/A. The following example shows that the UHOS-property is not generally retained when passing to quotient spaces.

Example 1.3.5. Consider the rationals, Q, under multiplication, without zero. Let R be the equivalence relation that partitions the positive and negative rationals into two disjoint sets by means of the open sets,

$$S = (-\infty, \ldots, 0) \quad \text{and} \quad T = (0, \ldots, +\infty), \text{ i.e.,}$$

$x \sim y \Leftrightarrow x$ and y are both in S or in T. Now, Q/R is a discrete two point space, hence Q/R is not a UHOS-space.

Note that we might as well consider Q to be a topological group, in which case $Q/R \cong Z_2$, with the discrete topology. Consequently the result holds for topological groups as well.

The following result will be of great importance in dealing with the quotient space G/A in the upcoming theorem.

Lemma 1.3.6. A is the smallest closed set that contains e.

Proof: Suppose $e \in B$, a closed set and $B \subsetneqq A$. Then, there is an element $a \in A$ with $a \notin B$. Now, $\widetilde{B}$ is an open set and $a \in \widetilde{B}$, and $\widetilde{B} \cap \{e\} = \emptyset$. This however contradicts our definition of A, so we conclude that A is the smallest closed set containing e.

Observe that the equivalence classes, $\{gA : g \in G\}$, in G/A are closed and homeomorphic to A, since left translation determines a homeomorphism. As in the previous lemma, it is easy to show that an arbitrary equivalence class, say $\bar{g}A$, is the smallest closed set containing a representative $\bar{g}$. It is clear that under a homeomorphism, these closed sets are permuted, an observation that is used in the theorem that follows.

Theorem 1.3.7. If G is a UHOS-space, then G/A is a UHOS-space. Any set $\dot{S} = \{sA : s \in S\}$, and $\dot{m} = mA$.

Proof: We show that every non-empty open set $\dot{U} \subset G/A$ is homeomorphic to G/A. Consider the diagram,

$$\begin{tikzcd}
G/A & G & & \\
\cup & \cup & & \\
\dot{U} \arrow[r, "\emptyset^{-1}"] & \emptyset^{-1}(\dot{U}) \arrow[r, "h"] & G \arrow[r, "\emptyset"] & G/A, \\
\parallel & \parallel & & \\
\{uA : u \in U\} & \bigcup\limits_{u \in U} \{uA\} & &
\end{tikzcd}$$

where ϕ is the natural map, and h a homeomorphism from $\phi^{-1}(\dot{U})$ to G. (This h is guaranteed since G is a UHOS-space and $\phi^{-1}(\dot{U})$ is open).

(i) $\phi h\phi^{-1}$ is onto: Since h is onto of G, and ϕ is onto all of G/A, $\phi h\phi^{-1}$ is onto.

(ii) $\phi h\phi^{-1}$ is 1-1: Let $\dot{x} \neq \dot{y} \in \dot{U}$. Then $\phi^{-1}(\dot{x}) \neq \phi^{-1}(\dot{y})$ (because equivalent elements in G are identified in G/A under ϕ), and $h(\phi^{-1}(\dot{x}) \neq h(\phi^{-1}(\dot{y}))$ (h permutes the closed partition, $\{gA:g \in G\}$, of G). Finally, under ϕ, different equivalence classes in G are identified with different points in G/A, so $\phi[h(\phi^{-1}(\dot{x}))] \neq \phi[h(\phi^{-1}(\dot{y}))]$.

(iii) $\phi h\phi^{-1}$ is open: $\dot{V} \subset \dot{U} \subset G/A$ open $\Rightarrow \phi^{-1}(\dot{V}) \subset \phi^{-1}(\dot{U})$ open in $G \Rightarrow h[\phi^{-1}(\dot{V})]$ open in G. But $h[\phi^{-1}(\dot{V})]$ is the complete inverse image of an open set in G/A, hence $\phi\{h[\phi^{-1}(\dot{V})]\}$ is open by definition of open sets in G/A.

(iv) $\phi h\phi^{-1}$ is continuous: Let $\dot{V} \subset \dot{U} \subset G/A$ be open. Then consider $(\phi h\phi^{-1})^{-1}(\dot{V}) = \phi h^{-1}\phi^{-1}(\dot{V})$. Now $h^{-1}\phi^{-1}(\dot{V})$ is clearly open and is the complete inverse image of an open set in G/A. By definition of ϕ and open sets in G/A, $\phi[(h^{-1}\phi^{-1}(\dot{V})]$ is open. Hence $\phi h\phi^{-1}$ is continuous.

CHAPTER II

THE EMBEDDING THEOREM

This chapter considers the universal embedding character of UHOS-spaces, that is, the main result is that every topological space embeds in a UHOS-space. The final corollary proves that in fact, every topological space embeds as a closed subset in an invertible, UHOS-space.

<u>Theorem 2.1.1</u>. Each topological space embeds in a UHOS-space.

<u>Proof</u>: Let X be an arbitrary topological space, and let $A = \{U_\alpha\}_{\alpha\in\Omega}$ be the collection of open sets in X. Consider χ_0-copies of each of the U_α, $\alpha \in \Omega$, giving rise to the collection,

$$B = \{U_{\alpha n}\}_{\substack{\alpha\in\Omega \\ n\in Z^+}}$$

Now, we study the set,

$$Y = \bigvee_{\substack{\alpha\in\Omega \\ n\in Z^+}} U_{\alpha n} .$$

In our study, it would be helpful to think of each of the $U_{\alpha n}$'s in B as defining or representing a position in Y; for example we can speak of the γs position, $\gamma \in \Omega$, $s \in Z^+$.

Define a basis element in Y as Y, excluding at most a finite number of positions, those positions being filled or replaced with $\emptyset$, or some nonempty open subset of the representative of that position.

(a) The basis so defined is a basis for a topology in Y. For, let M and N be basis elements in Y, where

$$M = (Y - \{U_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}}) \vee \{U'_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}},$$

$$N = (Y - \{U_{\gamma k}\}_{\substack{\gamma\in\Theta \\ k\in S}}) \vee \{U'_{\gamma k}\}_{\substack{\gamma\in\Theta \\ k\in S}}, \quad \beta, m, \gamma, k \quad \text{indexed}$$

over finite subsets of Ω and Z^+, and $U'_{\sigma t}$ some open subset of $U_{\sigma t}$, where $\sigma \in \Delta \cup \Theta$, $t \in W \cup S$.

Now,

$$Y - \bigvee_{\substack{\Delta\cup\Theta \\ W\cup S}} (\{U_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}} \cup \{U_{\gamma k}\}_{\substack{\gamma\in\Theta \\ k\in S}}) \subset M \cap N,$$

and is by definition a basis element in Y. We conclude that we have defined a basis for a topology on Y. (Note that open sets in Y are of the same form as basis elements).

(b) X embeds in Y. Choose any one of the χ_0-copies of X in $\{U_{\alpha n}\}_{\substack{\alpha\in\Omega \\ n\in Z^+}}$, say $U_{\mu t}$. The embedding we want is of course defined by, $x \longrightarrow x_{\mu t}$, and we only need check that with the relative topology on $U_{\mu t}$, $X \cong U_{\mu t}$. But by definition of basic open set, open sets induced by the relative topology do not alter the structure of $U_{\mu t}$. (The relative topology in $U_{\mu t}$ is generated by copies of Y with the $\mu t^{\underline{th}}$ position occupied by open subsets of $U_{\mu t}$). We conclude that X embeds in Y.

(c) Y is a UHOS-space. Again, let M and N be two nonempty open sets in Y, where

$$M = (Y - \{U_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}}) \vee \{U'_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}},$$

$$N = (Y - \{U_{\gamma k}\}_{\substack{\gamma\in\Theta \\ k\in S}}) \vee \{U'_{\gamma k}\}_{\substack{\gamma\in\Theta \\ k\in S}},$$

Δ, Θ, W, S indexed over finite subsets of Ω and Z^+, and $U'_{\sigma t}$ some open subset of $U_{\sigma t}$, $\sigma \in \Delta \cup \Theta$, $t \in W \cup S$. In accordance with our preliminary discussion, we can consider the elements of the indexing set Ω as determining an "open set type". With this notion the desired homeomorphism $F{:}M \to N$ is easy to construct. Each U' in M and N is homeomorphic to each of the elements of some "open set type", while it may <u>not</u>

be homeomorphic to the element of $B = \{U_{\alpha n}\}_{\substack{\alpha\in\Omega \\ n\in Z^+}}$ whose position it occupies. Since the U' in M and N are finite in number, they can easily be associated with elements of the same "open set type", and since each "open set type" is represented by $\aleph_0$ members, a homeomorphism defined piecewise on open set types yields the homeomorphism F we desire.

For example, if we have

$$M = \cdots\cdot \vee U_{\eta b} \vee \cdots\cdot \vee U_{\delta 1} \vee U_{\delta 2} \vee \cdots\cdot \vee \cdots\cdot \text{, and}$$

$$N = \cdots\cdot \vee \cdots \vee \cdots\cdot \vee U_{\delta 1} \vee U_{\delta 2} \vee \cdots\cdots\cdots ,$$

where $U'_{\eta b} \cong U_{\delta n}$, $n \in Z^+$, F on the "δ open set types" might look like

$$\begin{array}{cccccccc} M = \cdots\cdots & \vee & U'_{\eta b} & \vee & U_{\delta 1} & \vee & U_{\delta 2} & \vee \cdots\cdot \\ & & \downarrow & & \downarrow & & \downarrow & \\ N = \cdots\cdots & \vee & U_{\delta 1} & \vee & U_{\delta 2} & \vee & \cdots\cdots & . \end{array}$$

We simply have $U'_{\eta b} \to U_{\delta 1}$, then $U_{\delta c} \to U_{\delta(c+1)}$, $c = 1,2,\ldots$. We conclude that Y is a UHOS-space.

<u>Corollary 2.1.2</u>. X is embedded as a closed set in Y.

<u>Proof</u>: Let $X \to X_{\alpha n}$ be the embedding. By definition, $\tilde{X}_{\alpha n} = Y - (X_{\alpha n})$ is a basic open set in Y, and X is embedded as a closed set in Y.

Definition 2.1.3. A topological space X is said to be invertible if for every non-empty open set $U \subset X$, there is a homeomorphism $h \circlearrowleft X$ (i.e. $h:X \to X$) satisfying $h(X-U) \subset U$ [1].

Corollary 2.1.3. Y is an invertible space.

Proof: Let $S \subseteq Y$ be an open set. We show that there is a homeomorphism $F:Y \to Y$ such that $F(Y-S) \subset S$. Represent S as

$$S = (Y - \{U_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}}) \vee \{U'_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}} \quad , \Delta , W$$

finite subsets of Ω and Z^+, and $U'_{\beta m} \subset U_{\beta m}$ some open subset for each β, m. For each $U_{\beta m}$, choose some $U_{\beta r} \not\in \{U_{\beta m}\}_{\substack{\beta\in\Delta \\ m\in W}}$, giving rise to pairs $\{(U_{\beta m}, U_{\beta r})\}$. Now, alter the identity map $I:Y \to Y$ as follows. Instead of $U_{\beta m} \to U_{\beta m}$ as required by I, map $U_{\beta m} \to U_{\beta r}$ and $U_{\beta r} \to U_{\beta m}$ for every pair in $\{(U_{\beta m}, U_{\beta r})\}$. Call this altered map $I':Y \to Y$. I' is clearly a homeomorphism and $I'(Y-S) \subset S$ by construction. We conclude that Y is invertible.

Corollary 2.1.4. Each topological space embeds in an invertible UHOS-space as a closed subset.

Proof: Apply theorem 2.1.1 and corollaries 2.1.2 and 2.1.3.

CHAPTER III

THE METRIC UHOS-SPACES

In this chapter we study the extent to which compact subsets of UHOS metric spaces, X, are moved by homeomorphisms $h \circlearrowleft X$, and the incompatibility of compactness and metrizability. We also characterize the familiar metric spaces R (rationals) and I (irrationals) in terms of the UHOS-property.

<u>Lemma 3.1.1</u>. Let X be a metric space and p a point in X. Then for any real numbers α, β, $\alpha < \beta$, the set $A = \{x | \alpha < d(x,p) < \beta\}$, is open in X.

<u>Proof</u>: $A = \{x|d(x,p) < \beta\} \cap \{x|d(x,p) > \alpha\}$, is the intersection of open sets; hence A is open.

<u>Lemma 3.1.2</u>. Let X be an infinite metric space or a metric space with a limit point. Then from X, a disconnected subset, which is the union of infinitely many disjoint open sets, can be constructed.

<u>Proof</u>: Let p be a limit point of X. Let $p_1, p_2, \ldots$ be a sequence of points in X converging to p, with $d(p_1,p) > d(p_2,p) > d(p_3,p) > \ldots$, $n = 1,2,3,\ldots$. Let $\alpha_n = \dfrac{d(p_n,p) + d(p_{n+1},p)}{2}$, $n = 1,2,3,\ldots$. Now,

$p_n \in A_n = \{x | \alpha_n < d(x,p) < \alpha_{n-1}\}$, $n = 2,3,\ldots,$ an open set by the lemma above. If we let $B_n = \{x | d(x,p) = \alpha_n\}$, $n = 1,2,\ldots$ then each B_n is closed, and $[\cup B_n \cup \{p\}]$ is closed in X. Hence $X - [\cup B_n \cup \{p\}]$ is open in X and is the disconnected open subset we wanted to construct. If X is discrete the result follows immediately.

Theorem 3.1.3. Let X be a UHOS, metric space with at least one limit point, and let $C \subset X$ be compact. Then there is a homeomorphism $h \subset X$ such that $h(C) \subset X - C$.

Proof: X metric and UHOS implies, as a result of the lemma above, that X is the union of disjoint open sets, $\{U_n\}$. Now $C \subset X$, and C compact implies that finitely many of these open sets suffice to cover C, say $C \subset U_1 \cup U_2 \cup \ldots \cup U_n$. Now then,

$$X = (U_1 \cup U_2 \cup \ldots \cup U_n) \cup (U_{n+1} \cup \ldots),$$

which is a separation of X, say $X = A \cup B$, where A, B are open in X. As in a previous theorem, there are homeomorphisms $h_1 : A \to B$, and $h_2 : B \to A$ which determine a homeomorphism $h \circlearrowleft X$ with $h(C) \subset X - C$. Simply define $h(A) = h_1(A)$ and $h(B) = h_2(B)$.

Theorem 3.1.4. Let X be a compact, nondegenerate, UHOS-space. Then X is not Hausdorff (T_2).

Proof: Let p be a point in X, and $\{U_n(p)\}_A$ the neighborhood filterbase of p. Then $\{U_\alpha(p)\}_A$ converges to p, and only to p, if X is T_2. (In fact $\{U_\alpha(p)\}$ has no accumulation points). Now, $\{U_\alpha(p)-p\}_A$ is a filterbase in $X-p$, but has no point of accumulation. Hence X cannot be compact [2].

Corollary 3.1.5. There are no infinite compact UHOS metric space.

Proof: Since metric spaces are T_2, they cannot be compact and UHOS!

We note that theorem 4.5.1. asserts that compact UHOS-spaces are connected. The above corollary is hardly surprising when this result is known

Definition 3.1.6. A topological space X has dimension O at a point p if p has arbitrarily small neighborhoods with empty boundaries. A non-empty space X has dimension O if X has dimension O at each of its points [3]. (This definition of dimension O in a space X assumes X to be a separable metric space.)

Lemma 3.1.7. If U is an open set of real numbers containing a non-countable, O-dimensional, UHOS-space N, and η is a positive number, then there exists an infinite sequence of non-overlapping open intervals $D_1, D_2, \ldots$;

each D_n has length $< \eta$, each $\bar{D}_n \subset U$, each $N \cap D_n$ is non-countable, and the set $N - (D_1 \cup D_2 \cup \ldots)$ is empty ($= \emptyset$).

Proof: N is 0-dimensional and separable (because N is a UHOS-space), hence can be embedded in the space of irrationals in E^1 since the space of irrationals is a universal 0-dimensional space [3], pg. 64. So without loss of generality, consider an open set, U, of real numbers, which contains N (as a subset of the irrationals). U is open in E^1, hence can be written as the at most countable union of disjoint open intervals, $\{U_i\}_1^\infty$.

Let x be any element of $N \subset U$. Then x is in some $U_i = (a,b)$. Two rational numbers α and β can be found so that $x \in (\alpha,\beta) \subset (a,b)$, $|\beta - \alpha| < \eta$, and $\overline{(\alpha,\beta)} \subset (a,b)$. Now a strictly increasing sequence of rational numbers, say $b_1, b_2, b_3, \ldots,$ can be chosen in the interval (β, b), converging to b, and a strictly decreasing sequence of rational numbers, say $a_1, a_2, a_3, \ldots,$ can be chosen in the interval (a,α), converging to a. In addition, sequences can be chosen so that $|\alpha - a_1| < \eta$, $|a_n - a_{n+1}| < \eta$ and $|b_n - b_{n+1}| < \eta$ and $|b_1 - \beta| < \eta$. Note that $\overline{(a_{n+1}, a_n)} \subset (a,b)$, $\overline{(b_n, b_{n+1})} \subset (a,b)$, and that the set $\{(a_{n+1}, a_n), (b_n, b_{n+1}), (\alpha,\beta), (a_1,\alpha), (\beta, b_1)\}_{n=1}^\infty$ is countable. Label these intervals $\{K_i\}_{i=1}^\infty$

We sum up the results of our construction as follows. We have a countable set, $\{K_n\}_1^\infty$, of disjoint open intervals;

each K_n has length $< \eta$, each $\overline{K_n} \subseteq U_i = (a,b)$; each $N \cap K_n$ is noncountable (because $N \cap K_n = \emptyset$ or is homeomorphic to N), and the set $(N \cap U_i) - (D_1 \cup D_2 \cup \cdots) = \emptyset$. This last claim is obvious (and essential in the following theorem) when we realize that every irrational in $(N \cap U_i)$ is contained in some K_i. (Remember $N \subseteq$ the irrationals).

But this construction can be duplicated for each one of the U_i's in $\{U_i\}_1^\infty$, and each U_i gives rise to a countable number of K_n's. The set of $\{K_n\text{'s}\}$ generated for all the U_i's is countable, and after relabelling them as $D_1, D_2, D_3, \ldots,$ our lemma will be proved.

<u>Definition 3.1.7</u>. A topological space is an absolute G_δ iff it is metrizable and is a G_δ in every metric space in which it is embedded

It can be shown [2], [5] that the irrationals is an absolute G_δ-space, so that the absolute G_δ, UHOS-spaces hypothesized in the following theorem do exist nontrivially.

<u>Theorem 3.1.9</u>. Let X be a 0-dimensional, noncountable, absolute G_δ set which satisfies the UHOS-property. Then X is homeomorphic to the irrational numbers.

<u>Proof</u>: By a theorem of Hurewicz-Wallman [3], X can be embedded in E^1 (dim $n \to$ dim $2n+1$), and since X is an absolute G_δ set, it is embedded as a G_δ set in E^1. Also call the image of this embedding in E^1, X.

Then there exists an infinite sequence of open sets G_n $(n = 1,2,\ldots)$ such that $X = G_1 \cap G_2 \cap \cdots$. Since X is noncountable etc, and $X \subseteq G_1$ and G_1 is open, we may apply our lemma after setting $\eta = 1$. Thus we obtain an infinite sequence $D_1, D_2, \ldots$ of non-overlapping open intervals, each D_n has length < 1, each $\overline{D_n} \subseteq G_1$, each $X \cap D_n$ is non-countable, and the set $X - (D_1 \cup D_2 \cup \cdots)$ is empty $(= \emptyset)$.

Let n_1 denote a natural number. Since the sets G_2 and D_{n_1} are open the set $G_2 \cap D_1$ is open; since $X \subseteq G_2$ and $X \cap D_{n_1}$ is non-countable, the set $X \cap G_2 \cap D_{n_1}$ is no - countable. Applying the lemma to the sets $G_2 \cap D_{n_1}$, $X \cap D_{n_1}$, with $\eta = \frac{1}{2}$, we obtain an infinite sequence of non-overlapping open intervals $D_{n_{1,1}}, D_{n_{1,2}}, D_{n_{1,3}}, \ldots$; each $D_{n_1,n}$ has length $< \frac{1}{2}$, each $\bar{D}_{n_1,n} \subseteq G_2 \cap D_{n_1}$, each $X \cap D_{n_1,n}$ is non-countable, and the set $(X \cap D_{n_1}) - (D_{n_{1,1}} \cup D_{n_{1,2}} \cup \cdots)$ is empty $(= \emptyset)$.

Further, let n_1, n_2 be two rational numbers. Since the sets G_3 and D_{n_1,n_2} are open and the set $X \cap G_3 \cap D_{n_1,n_2}$ is non-countable, we may apply our lemma to $G_3 \cap D_{n_1,n_2}$, $E \cap D_{n_1,n_2}$, with $\eta = \frac{1}{3}$.

Continuing this argument, we obtain for every finite combination $n_1, n_2, \ldots, n_k$ (abbreviated to $n_{(k)}$) of natural numbers an open interval $D_{n(k)}$ such that

(i) Diameter $(D_{n(k)}) < \frac{1}{k}$,

(ii) $D_{n(k-1),p} \cap D_{n(k-1),q} = \emptyset, \quad p \neq q$

(iii) $\bar{D}_{n(k)} \subseteq G_k \cap D_{n(k-1)}$,

(iv) $X \cap D_{n(k)}$ is non-countable,

(v) $X \cap D_{n(k-1)} - D_{n(k-1),1} \cup D_{n(k-1),2} \cup \cdots) = \emptyset$.

Let N denote the set of all irrational numbers in the interval $(0,1)$, x a given number of N, and let

$$(1) \quad x = \frac{1}{m_1+} \frac{1}{m_2+} \frac{1}{m_3+} \cdots$$

be the development of x as a continued fraction. Put

$$(2) \quad F(x) = \bar{D}_{m_1} \cap \bar{D}_{m_1,m_2} \cap \bar{D}_{m_1,m_2,m_3} \cap \cdots \quad .$$

It follows from (iii) and (iv) that the set (2) is the intersection of a descending sequence of closed nonempty intervals and is therefore nonempty; moreover, by (1) $F(x)$ is contained in an interval of length $< \frac{1}{k}$, for $k = 1,2,3,\ldots$; hence $F(x)$ consists of a single element which we denoted by $f(x)$. From (2) and (iii) we have $f(x) \in G_k$ for $k = 1,2,3,\ldots$; so $f(x) \in X$. The set T of all the numbers $f(x)$ for $x \in N$ is therefore a subset of the set X. We next show that the set $X - T$ is empty $(=\emptyset)$.

To prove this, let

$$(3) \quad R = (X - S) \cup (\cup(X \cap D_{n(k)} - S_{n(k)}))$$

where the union extends over all finite combinations $n(k)$ of natural numbers, and where $S = D_1 \cup D_2 \cup \cdots$; while

$$(4) \quad S_{n(k)} = D_{n(k),1} \cup D_{n(k),2} \cup D_{n(k),3} \cup \cdots \quad .$$

It is evident from (4) and (v) that the terms of the sum (union) (3) are empty sets; consequently the set R is empty.

Let y denote a number of the set $X - R = X$. Then $y \in X$ and $y \notin R$; so, from (3), $y \notin X - S$. But $y \in X$; therefore $y \in S$ and since $S = D_1 \cup D_2 \cup \cdots$, there is an index m_1 such that $y \in D_{n_1}$. From $y \notin R$ and (3), we find that $y \notin (X \cap D_{m_1} - S_{m_1})$; but since $y \in X \cap D_{m_1}$, we have $y \in S_{m_1}$; hence from (4), there exists an index m_2 such that $y \in D_{m_1,m_2}$.

Continuing this argument, we obtain an infinite sequence $m_1, m_2, m_3, \ldots$ of indices such that

$$y \in D_{m(k)}, \quad k = 1,2,3,\ldots \quad .$$

From (2) we have $y \in F(x)$, where x is the number defined by (1); in virtue of the definition of the set T, this proves that $y \in T$. Hence $X - R = X \subseteq T$; this gives $X - T \subseteq R = \emptyset$ and, $X = T$. (This proof is due to Mazurkiewicz and was applied to our particular G_δ set X [7]).

It can be shown that $N \overset{f}{\cong} T$, ([7], pg. 239) and the proof is complete since $T = X$.

Lemma 3.1.10. A UHOS-space, X, containing more than one point contains no isolated points.

Proof: Let p be an isolated point in X. Then $\{p\}$ is open, hence is homeomorphic to X. But $\text{Card}\,\{p\} = 1$ and $\text{Card}\,X > 1$, and we are done.

Corollary 3.1.11. Each such UHOS-space is dense-in-itself.

Proof: By definition of "dense-in-itself".

Theorem 3.1.12: Let X be a countably infinite metric UHOS-space. Then X is homeomorphic to the rational numbers in E^1.

Proof: By the corollary above, X is dense-in-itself and countably infinite. The space of rationals R is dense-in-itself and countably infinite. By a theorem of Kuratowski [6], pg. 287, countably infinite dense-in-itself spaces are homeomorphic.

It should be noted that the classical characterization of the Cantor set C^* may be stated as follows: A compact, totally disconnected perfect metric space is homeomorphic to C^*. If perfect were used in the sense that each point is a limit point, the above theorem might be stated as follows: A 0-dimensional, absolute G_δ, perfect space is the irrationals.

We remark before proving the next theorem that it is trivially true for one point spaces, but after this consideration the cardinality of our sets is at least χ_0.

Theorem 3.1.13. A 0-dimensional, UHOS-space is homogeneous.

Proof: Let x,y be two distinct points in X. Because X is 0-dimensional and hence separable and metric [3], there exist disjoint neighborhoods U,V of x and y respectively, which contain clopen neighborhoods U_1 and V_1 of x and y.

Consider U_1 and $r_1 = d(x, X - U_1)$. We can find a clopen neighborhood U_2 of x with $U_2 \subset S_{r_1}(x)$, the spherical neighborhood of x of radius r_1. Choose $r_2 = d(x, X - U_2)$, and again we are able to find a clopen neighborhood, U_3, of x with $U_3 \subset S_{r_2}(x)$. Continuing in this manner we are able to construct a strictly decreasing sequence of clopen neighborhoods of x converging to x, say $U_1 \supset U_2 \supset \ldots$. (Note: 1. If $d(x, X - U_i) = 0$ for some i, then the existence of a basis of clopen sets at x is contradicted. 2. If $d(x, X - U_r) = r - 1$, i.e., U_r becomes a spherical clopen set possibly terminating the process above, we are able to resume the process by considering a clopen neighborhood of x contained in the spherical neighborhood $S_{\frac{1}{n}}(x)$, where $\frac{1}{n} < r - 1$, n some positive integer.). Likewise we can construct a strictly decreasing sequence of clopen neighborhoods of y converging to y, say $V_1 \supset V_2 \supset \ldots$.

Now, $U_i - U_{i+1}$ is open for each $i = 1,2,\ldots,$ and $V_i - V_{i+1}$ is open for each $i = 1,2,\ldots$. Because X is a UHOS-space there exist homeomorphisms $h_i : U_i - U_{i+1} \to V_i - V_{i+1}$ for each $i = 1,2,\ldots$. Define a function $H \circlearrowleft X$ as follows:

1. $H(X - (U_1 \cup V_1)) = id(X - (U_1 \cup V_1))$

2. $H(U_i - U_{i+1}) = h_i(U_i - U_{i+1})$

3. $H(V_i - V_{i+1}) = h_i^{-1}(V_i - V_{i+1})$

4. $H(x) = y$ and $H(y) = x$.

From our construction of the sets $U_i - U_{i+1}$ and $V_i - V_{i+1}$, it is clear that sequences converging to x and y respectively, converge to $H(x) = y$ and $H(y) = x$, after application of H. Hence, $H \circlearrowleft X$ is a homeomorphism and $H(x) = y$.

CHAPTER IV

CONNECTED UHOS-SPACES

In this chapter we study some properties of UHOS-spaces related to connectedness. The most surprising result is that compact UHOS-spaces are connected.

1. Closed and Open sets in UHOS-spaces; Components

Lemma 4.1.1. Let X be a connected UHOS-space, and let $W \subset X$, $W \neq \emptyset$, $W \neq X$ be closed. Then W contains no open sets.

Proof: Suppose $U \subset W$ is an open set and $U \neq \emptyset$. Then $\tilde{W}$ is open, $\tilde{W} \cup U$ is open and therefore $\tilde{W} \cup U \cong X$, contradicting the connectedness of X.

Corollary 4.1.2. Let X be a connected UHOS-space and let $U \subset X$, $U \neq \emptyset$, $U \neq X$ be open. Then $X - U$ contains no open sets.

Proof: $X - U$ is closed in X. Apply lemma 4.1.1.

Lemma 4.1.3. Let X be a connected UHOS-space, and let $U \subset X$, $U \neq \emptyset$, $U \neq X$ be open. Then, $\bar{U} = X$.

Proof: $\bar{U} \neq X$ implies $\tilde{\tilde{U}}$ is open and $\tilde{\tilde{U}} \neq \emptyset$. Then $U \cup \tilde{\tilde{U}}$ is open, disconnected and $U \cup \tilde{\tilde{U}} \cong X$, contradicting the connectedness of X.

As a result of lemma 4.1.3, we can say that non-empty open sets in a connected UHOS-space, X, are dense in X.

Lemma 4.1.4. All finite UHOS-spaces are connected.

Proof: A finite UHOS-space has the indiscrete topology.

Corollary 4.1.5. Disconnected UHOS-spaces have cardinality at least $\aleph_0$.

Proof: By lemma 4.1.4, if X is a UHOS-space and disconnected, it has to follow that X cannot have finite cardinality.

Lemma 4.1.6. Let X be a disconnected UHOS-space. Then the components of X are infinite in number.

Proof: Let $X = U_1 \cup U_2 \cup \cdots \cup U_n$ be the decomposition of X into components (i.e., maximal connected sets). U_i is closed for $1 = 1,2,\ldots,n$, hence $\tilde{U}_i = U_1 \cup U_2 \cup \cdots \cup U_{i-1} \cup U_{i+1} \cup \cdots \cup U_n$ is open and $U_1 \cup \cdots \cup U_{i-1} \cup U_{i+1} \cup \cdots \cup U_n \cong X$. But the number of components in a space is a topological invariant, hence we arrive at a contradiction.

2. Homeomorphic Closed Sets.

In this section, we study the result of requiring all nonvoid closed sets in a UHOS-space, X, to be homeomorphic.

Lemma 4.2.1. If X is indiscrete, the topology gives this property.

Proof: Clear.

Lemma 4.2.2. X and each of its closed sets will be connected, and the closure of a point is topologically X.

Proof: Let A be a connected subset of X. (There is at least one, a point for example!). Then $\bar{A}$ is closed and connected and homeomorphic to X and every other closed set in X.

Corollary 4.2.3. No proper closed set in X contains an open set.

Proof: Let S be a proper closed set in X, and assume that $A \subsetneqq S$ is open. Then $\tilde{S}$ is open and $A \cap \tilde{S} = \emptyset$. But then $X \cong A \cup \tilde{S}$, which contradicts Lemma 2.

3. Disconnected UHOS-spaces with Clopen Sets.

The following results assume that X is a disconnected (hence infinite), UHOS-space in which all open sets are also closed.

Lemma 4.3.1. No point in X is closed.

<u>Proof</u>: If $\{p\}$ is closed, then $X - \{p\} = \widetilde{\{p\}}$ is open, and by hypothesis also closed. Then $\widetilde{X - \{p\}} = \widetilde{\widetilde{\{p\}}} = \{p\}$ is open and homeomorphic to X. This is impossible.

<u>Remark 4.3.2</u>. This shows that no such topological space can be T_1.

<u>Lemma 4.3.3</u>. All closed sets in X are open.

<u>Proof</u>: Let $R \subset X$ be closed. Then $\tilde{R}$ is open and closed in X. Hence $\tilde{\tilde{R}} = R$ is open in X.

<u>Remark 4.3.4</u>. Note that every point, p, in a topological space X is connected, and hence $\overline{\{p\}}$ is connected in X.

<u>Lemma 4.3.5</u>. Let $p \in X$. Then $\overline{\{p\}}$ is the smallest open set containing p, (i.e., if $p \in U$, U open, then $\overline{\{p\}} \subset U$).

<u>Proof</u>: Clearly $\overline{\{p\}}$ is closed, open and connected, by lemma 1 and remark 2 above. Suppose there is an open set U with $p \in U$ and $\overline{\{p\}} \not\subset U$. Then $\overline{\{p\}} \cap U$ is open and closed, contains p, and is contained in $\overline{\{p\}}$. But $\overline{\{p\}}$ is connected and can contain <u>no</u> proper open and closed (clopen) sets.

<u>Theorem 4.3.6</u>. No infinite, UHOS-space, X, has all open sets closed unless it is $\emptyset$, a point, or indiscrete.

Proof: Such a space, X, is (i) disconnected, (ii) the empty set, (iii) a point, or (iv) has the indiscrete topology. But (i) is impossible since for every point, p, in X, $\overline{\{p\}}$ is open and connected and $\overline{\{p\}} \cong X$, which is impossible.

Lemma 4.3.7. A basis for X is a collection of point closures.

Proof: Let U be an open set in X. Then for each $p \in U$, $\overline{\{p\}}$ is open and $\overline{\{p\}} \subset U$. Then

$$U = \bigcup_{p \in U} \overline{\{p\}}.$$

4. Miscellaneous Results.

If a UHOS-space has a local or global property, then it may also have the corresponding global or local property. Three examples are:

a) X locally connected $\Rightarrow$ X connected

b) Each point in X lies in a compact open set $\Leftrightarrow$ X is compact

c) X arcwise connected $\Rightarrow$ X locally arcwise connected.

Lemma 4.4.1. Let X be a connected, T_2, UHOS-space. Let x,y be points in X. X is T_2 implies there are disjoint open sets U,V containing x,y respectively. But $U \cup V$ is open and by hypothesis $U \cup V \cong X$, which is impossible

since X is connected. Therefore connected, nondegenerate UHOS-spaces cannot be T_2. (Also not T_3, not T_4, etc.)

Note: This also makes it clear that connected UHOS-spaces contain no disjoint open sets.

Definition 4.4.2. A topological space X is said to be rigid if the only homeomorphism from X to itself is the identity map.

Lemma 4.4.3. If X is a rigid UHOS-space, then X is connected.

Proof: If X is not connected, then let $= A \cup B$ be a separation [5] of X. But A and B are open sets, hence there are homeomorphisms h and g with $h(A) = B$ and $g(B) = A$.

But $H: X \to X$ defined by $H(A) = h(A) = B$ and $H(B) = g(B) = A$ is a homeomorphism from X to itself and $h \neq Id_X$.

Remark: We note that subspaces of UHOS-spaces are not necessarily UHOS-spaces. For consider the subspace $\{1,2,3\}$ in example 5.1. . $\{1,2,3\}$ and $\{2,3\}$ are open in the subspace topology but $\{1,2,3\} \not\approx \{2,3\}$. However, open subsets of UHOS-spaces do inherit the UHOS-property in the relative topology.

5. The Main Theorem.

Theorem 4.5.1. Compact UHOS-spaces are connected.

Proof: Let X be a compact UHOS-space and assume that it is not connected. Then $X = U \cup V$, U, V open in X, $U \cap V = \emptyset$. Since $V \cong X$, V is not connected, hence $V = U_1 \cup V_1$, U_1, V_1 open in V (U_1, V_1 open in X also), $U_1 \cap V_1 = \emptyset$. Since $V_1 \cong X$, V_1 is not connected, hence $V_1 = U_2 \cup V_2$, U_2, V_2 open in V_1 (X also), $U_2 \cap V_2 = \emptyset$. Note that the U's and V's we generate with this process are compact, clopen sets in X. The V's generated by this process also give rise to a nested sequence,

$$V_1 \supset V_2 \supset V_3 \; \ldots\ldots \; .$$

Consider a maximal chain of clopen sets containing this sequence, say $\{V_\alpha\}_{\alpha \in \Gamma}$. We observe that,

$$\bigcap_{\Gamma} V_\alpha = D$$

is a compact, closed subset of X. D is not open (otherwise maximality is contradicted).

Lemma 4.5.2. For every open set U with $D = \bigcap_{\Gamma} V_\alpha \subset U$, there is some V_β with $D = \bigcap_{\Gamma} V_\alpha \subset V_\beta \subset U$.

Proof: Suppose no such V_β exists. Then $(V_\beta \cap \tilde{U}) \neq \emptyset$, for every $\beta \in \Gamma$. Clearly $(V_\beta \cap U) \neq \emptyset$, for every β. Note: $D \subset U$, $D \subset V_\alpha$, $\alpha \in \Gamma$, gives us $D \subset (U \cap V_\alpha)$, $\alpha \in \Gamma$, and $D \subset \bigcap_{\Gamma} (U \cap V_\alpha)$. Observe that

$$D = \bigcap_{\Gamma} V_\alpha = \bigcap_{\Gamma} [(V_\alpha \cap \tilde{U}) \cup (V_\alpha \cap U)]$$

$$= \bigcap_{\Gamma}(V_\alpha \cap \tilde{U}) \cup \bigcap_{\Gamma}(V_\alpha \cap U).$$

Now, if $\bigcap_{\Gamma}(V_\alpha \cap \tilde{U}) = \emptyset$, by the finite intersection property we have $\bigcap_{j=1}^{n} (V_j \cap \tilde{U}) = \emptyset$, n finite, and since the $(V_\alpha \cap \tilde{U})$ are nested, some $V_r \cap \tilde{U} = \emptyset$, and we are done.

Since $(V_\alpha \cap U) \neq \emptyset$, $\alpha \in \Gamma$, we have

$$D = A \cup B,\ A \cap B = \emptyset,\ D \subset B,\ A \neq \emptyset,\ B \neq \emptyset.$$

But this is impossible. We conclude that V_β exists with $D = \bigcap_{\Gamma} V_\alpha \subset V_\beta \subset U$.

We return to the proof of the theorem, and consider

$$X - D.$$

Since D is closed in X, $X - D$ is open and is compact. Construct a net, $\{v_\alpha\}_{\alpha \in \Gamma}$, from the properly nested sequence $\{V_\alpha\}_{\alpha \in \Gamma}$, such that $v_\gamma \in V_\gamma$, $v_\gamma \notin V_\sigma$; $v_\sigma \in V_\sigma$, $v_\sigma \notin V_\gamma, V_\delta$; etc. Now this sequence $\{v_\alpha\}_{\alpha \in \Gamma}$ in $X - D$ clusters, but not in $X - D$. This contradicts the compactness of $X - D$. We conclude that compact UHOS-spaces are connected.

CHAPTER V

THE INVERTIBLE CASE

In this chapter we begin to study the relationship between UHOS-spaces and invertible spaces (see Def. 2.1.3.). The following examples begin to illustrate the problem, but we are reminded of the result in Corollary 2.4. This result guarantees the existence of invertible UHOS-spaces.

Example 5.1. A UHOS-space is not necessarily invertible. Consider the integers $1,2,3,\ldots$ with the following topology: $\{\{n,n+1,n+2,\ldots,\} \mid n = 1,2,3,\ldots\}$ is the collection of open sets.

Claim: This is a UHOS-space! Clear!

Claim: This space is not invertible! In fact the only homeomorphism h of X onto X is the identity map (i.e., X is rigid). For suppose h is not the identity map, then there are integers $n,t, \ni n \neq t$, and $h(n) = t$. Suppose $n > t$. Under h the open set determined by n goes to an open set, which is determined by an integer of size t or smaller. But then our homeomorphism is required to take $(n-1)$ points to $(t-1)$ points. This is impossible. The case $n < t$ follows as above by considering h^{-1} as the homormorphism in question.

Example 5.2. An invertible space is not necessarily a UHOS-space.

Proof: The sphere, S^n, is invertible, but is not a UHOS-space since it is connected and Hausdorff, (See note in 4.4).

Example 5.3. Invertible, UHOS-spaces do exist.

Proof: See Chapter II, Corollary 2.4.

The next two theorems show that certain subsets of UHOS-spaces, X, are moved by homeomorphisms $h:X \to X$, h into; h onto respectively, in a manner reminiscent of the case with invertible spaces.

Theorem 5.4. A topological space X is a UHOS-space iff any nondense set $D \subset X$ may be taken into $X - D$ by a homeomorphism $h:X \to X - \bar{D}$.

Proof: If $D \subset X$ is nondense, then $\bar{D} \subsetneq X$. Then $X - \bar{D}$ is open, and the UHOS property gives a homeomorphism $h:X \to X - \bar{D}$. But $X - \bar{D} \subset X - D$, and we are done.

Let U be a proper open set in X. Then $X - U$ is closed in X and is nondense in X. Therefore, there is a homeomorphism $h:X \to X - (\overline{X - U}) = X - (X - U) = U$. We conclude that X has the UHOS property.

Note: This result indicates a priori that the UHOS property in a space falls short of invertibility. However,

Theorem 2.1.1 shows the remarkable compatibility of the two properties.

Theorem 5.5. If X is a metric UHOS-space and $C \subset X$ is connected, then there is a homeomorphism $h \circlearrowleft X \ni h(C) \subset X - C$.

Proof: 1) Assume X is not connected, so that $X = U \cup V$, where $U, V \neq \emptyset$ open in X and $U \cap V = \emptyset$. X is a UHOS-space, hence there is a homeomorphism $h: U \to V$, and a homeomorphism $g: V \to U$. Now $C \subset U$ or $C \subset V$ since C is connected; say $C \subset U$. Define $H: X \to X$ as follows: $h(u) = h(u)$, $u \in U$; $H(v) = g(v)$, $v \in V$. Clearly H is a homeomorphism, and $H(C) = h(C) \subset V \subset X - C$.

Note: X cannot be connected since X metric implies X is Hausdorff, and there exist no non-trivial connected UHOS-spaces.

Definition 5.6. If X is a topological space and U, V are open in X, then U and V have the same embedding type if there is an $h \circlearrowleft X$ and $h(U) = V$.

We remind the reader that the category of connected UHOS-spaces is large, and we prove the following theorem.

Theorem 5.7. A topological space X is in C (i.e., the category of connected, UHOS-spaces) iff each proper closed set in X lies in an open set of every embedding type under homeomorphisms $h \circlearrowleft X$.

Proof: Let $X \in C$, and suppose that each proper closed set W in X lies in an open set of every embedding type. Let $U \subset X$ be open. Then $X - U$ is closed. Since $X - U$ lies in an open set of the same embedding type as U, say V (i.e., there is a homeomorphism $h \circlearrowright X$ with $h(V) = U$) and $X - U \subset V = h^{-1}(U)$, then $h(X - U) \subset U$. Hence X is invertible.

Now let X be invertible, W a closed set in X, and $U \neq \emptyset$ an open set in X. By lemma 4.1.1., $U \not\subset W$. Then $(U - W)$ is open and there is a homeomorphism $h \circlearrowright X$ with $h\{X - (U - W)\} \subset U - W \subset U$. Since $W \subset \{X - (U - W)\}$, the proof is complete.

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