MINIMUM ENERGY SPACE RENDEZVOUS

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ABSTRACT

MINIMUM ENERGY SPACE RENDEZVOUS

by Eugene Harrison

The space rendezvous maneuver is defined as one which is designed to move a space vehicle from one location to another in order to match the position and velocity of an object in orbit at the second location. It is equivalent to a space transfer in which terminal conditions and time of transfer are specified. An investigation to determine the minimum energy required to achieve a space rendezvous is reported.

Two methods were used to investigate the rendezvous energy problem—an analysis by the calculus of variations, and a trajectory perturbation procedure. For each method it is assumed that the only forces acting on the rendezvous vehicle are due to the inverse square gravity field and applied thrusts. It is also assumed that thrust levels are high so that velocity changes are essentially impulsive.

The analysis by the calculus of variations is based upon the use of linearized equations of relative motion which are reasonably accurate if the distance separating vehicle and target is in the order of 50 miles or less. Using the relative equations, a set of Euler-Lagrange equations were obtained; which, on examination, led to

the following conclusions:

(1) A minimum energy trajectory is comprised of segments flown with either zero or maximum thrust.

(2) A minimum energy rendezvous is accomplished using either two or three impulses.

The perturbation method of energy analysis determines the effect of perturbing a vehicle which would otherwise move along a nominal coasting trajectory joining space terminals. The perturbing velocity is assumed to be the residual from a partial nullification of the initial velocity relative to the coasting trajectory velocity. At an intermediate time an impulse is applied to bring the vehicle back onto the nominal trajectory at the destination point. The trajectory from the point of application of the intermediate impulse to the destination point is selected so that the total transit time on the perturbed path equals the transfer time along the nominal trajectory. The velocity at arrival via the two routes would, of course, be different, and the vector difference is termed the resultant velocity. A third impulse potential is defined in terms of the residual, intermediate, and resultant velocities.

The results of a parametric study in which the third impulse potential was used to examine the conditions under which a third impulse could be utilized are reported. A sample problem is presented in which the velocity required to rendezvous is shown to be greatly reduced by using three velocity impulses instead of two.

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By

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LIST OF SYMBOLS

PRIMARY SYMBOLS

A	Matrix or constant as noted at usage
Aq^1, Aq^2	Acceleration components in the generalized directions q^1 and q^2
A_R, A_T	Radial and tangential components of thrusting acceleration
A_x, A_y, A_z	Thrusting acceleration components
a	Apogee of transfer ellipse
a_1	Dimensionless group
a_{ij}	Matrix coefficients
B	A constant
b_1	Dimensionless group
C	Constant or Erdmann-Weierstrass corner condition as noted at usage
$\bar{c}$	Rocket exhaust velocity
D	A constant
E	Eccentric anomaly
ΔE	Difference between the eccentric anomalies of two points on an orbit

e	Eccentricity of transfer ellipse
$\bar{F}$	Gravity force per unit mass
g_i	Function of relative equation parameters
J	Angular momentum
m	Mass of rocket and fuel
n	Average angular velocity of an object in orbit
P	Period of orbit or space terminal as noted at usage
q	Generalized parameter defined at usage
R	Radius of orbiting object measured from the earth's center
T	Total transfer time
t	Time
u	Relative velocity in x direction
$\bar{V}$	Absolute or inertial velocity
ΔV	Characteristic velocity
v	Relative velocity (in y direction or total magnitude as noted at usage)
$\bar{v}$	Relative velocity vector

δv	Third impulse potential
x,y,x	Rectangular coordinates
y_i	Relative position and velocity parameters in first order equations of motion
δy_i	Variations from trial trajectory
x_i	Adjoint equation parameters
α	Angle between the thrust and velocity vectors
β	$-\dot{m}$
β_1	Direction of residual velocity at initial space terminal
β_2	Direction of resultant velocity at destination
γ	Flight path angle measured between velocity vector and horizontal direction
ζ	Phase angle
η	Ratio of the areas of the triangle and elliptical sector between two radii of an orbit
λ_q	Lagrange multipliers
μ	Earth gravity constant = $1.4077998 \times 10^{16} \frac{ft^3}{sec^2}$
ξ	Time dependent variable used to limit the thrust magnitude

ρ Radius from earth's center to vehicle

τ Time measured from point of intermediate thrust application

φ True anomaly of transfer ellipse

φ_q Constraint equations

$\Delta\varphi$ Difference between the true anomalies of two points on an ellipse

ω Angular velocity of circular orbit

$(\dot{\ })$ Denotes differentation with respect to time

$(\overset{*}{\ })$ Denotes evaluation along a trial trajectory

SUBSCRIPTS

c1,c2 Initial and final values on a nominal coasting trajectory

f Final value

i Intermediate value

i,k,q Identification subletters which take successive values as noted at usage

n Nominal

1,2 Denotes starting point and destination on a transfer trajectory

CHAPTER I

INTRODUCTION

The purpose of the investigation described in this thesis was to determine the minimum energy required to transfer a spacecraft between two points in an inverse square gravity field when the time to transfer and terminal velocities are specified. With these specifications, the transfer is equivalent to "space rendezvous"—a maneuver generally described as one designed to move a spacecraft from one location to another in order to match the position and velocity of some object in orbit at the second location. Hereafter, the designations "vehicle" and "target" will be used.

The need for information concerning the rendezvous problem is becoming increasingly important. A number of complex satellites have already been placed into orbit and many more are planned for the near future. Some of these will be manned and may possibly range from craft carrying a single operator to facilities that will serve as launching platforms for lunar or interplanetary missions. Rendezvous capability will be necessary in order to transfer personnel and supplies to and from these facilities and to provide emergency rescue capability.

Recent literature has been replete with articles concerning rendezvous. One author lists a bibliography containing fifty-nine references.[1] For the most part, these articles have dealt with the development of guidance

equations and various rendezvous techniques. The energies required to rendezvous by the various techniques differ, and in many cases by a significant amount. One of the reasons for attempting to determine the minimum energy required for rendezvous was to provide a comparison energy so that an efficiency number, insofar as energy or fuel expenditure is concerned, may be determined for each technique.

The total operation from earth launch to contact between vehicle and target may be described as consisting of the following phases:

(1) Launch from earth for a direct approach or into a parking orbit.

(2) Orbit transfer and/or injection into the target orbit to get an approximate position and velocity match.

(3) Rendezvous, which is usually described as beginning between 10 to 50 miles from the target and extending to close proximity.

(4) Final approach and docking which begins within a few feet of the target.

It was intended that the subject investigation would be restricted to the rendezvous phase, however, in many cases the results obtained could equally well apply to the orbit transfer phase.

CHAPTER II

PROBLEM DEFINITION AND BACKGROUND

PROBLEM STATEMENT

In terms of the generalized coordinates q^1 and q^2, the subject problem can be stated as that of transferring a spacecraft or rocket from condition $(q_1^1, q_1^2, \dot{q}_1^1, \dot{q}_1^2)$ to $(q_2^1, q_2^2, \dot{q}_2^1, \dot{q}_2^2)$ in time T with a minimum expenditure of energy. The motion of a rocket in space can be described by the vector equation

$$\frac{d\overline{V}}{dt} + \overline{F} = -(\overline{c}/m)\frac{dm}{dt} \tag{2.1}$$

where

$\overline{V}$ = rocket velocity

m = mass of rocket and fuel

$\overline{c}$ = rocket exhaust velocity

$\overline{F}$ = gravity force per unit mass.

Energy expended is usually measured in terms of characteristic velocity as determined by the integral of either side of Eq. (2.1). Hence, the energy of transfer (rendezvous) is determined either by the equation

$$\Delta V = \int_{m_o}^{m_T} -(\overline{c}/m)\frac{dm}{dt} = c \log (m_o/m_T) \tag{2.2}$$

or

$$\Delta V = \int_{V_o}^{V_T} d\overline{V} + \int_{o}^{T} \overline{F}\, dt\ . \tag{2.3}$$

The problem of minimizing energy can be considered as one of minimizing the change in mass, $(m_o - m_T)$, or the

characteristic velocity as defined by Eq. (2.3).

RELATED STUDIES

Previous studies pertaining to the transfer of a vehicle between space terminals more or less fell into two general catagories: (1) those in which time is specified and are rendezvous equivalent, and (2) those in which time is not specified. A number of the more pertinent investigations and their relationship to the subject investigation are briefly discussed in the following paragraphs.

Rendezvous Equivalent Transfer

In early investigations Lawden used the classical variational calculus approach to analyze the space transfer problem.[3,4,5] On the basis of a Taylor series expansion of the gravity potential in which terms greater than second order were neglected, results were derived which indicated that the minimum energy transfer trajectory would contain no segment flown with an intermediate thrust level, i.e. it would consist of arcs of maximum thrust followed by coasting arcs. However, both Lawden and Leitmann later showed that an intermediate thrust trajectory cannot be ruled out when the complete gravity potential is present.[6,7] In reference (7) Leitmann treats the problem of thrust mode selection and shows it to depend upon a switching function determined by the Weierstrass E-function. However, this function does not furnish apriori information for the above stated case (a non-linear gravity function). A running account must be kept of the switching function during the numerical integration of the Euler-Lagrange equations.

Thus, one of the primary problems in determining an optimal trajectory is the selection of the appropriate thrust mode. However, even going under the premise that a minimum energy trajectory in an inverse square gravity field would contain no intermediate thrust segment, there still remains a major problem. This problem is the lack of criteria for determining the number of powered arcs to achieve an optimum.

In many cases where the classical calculus of variations was used to analyze those space transfer problems which lend themselves to analysis, the procedure has been to formulate criteria for solution without presenting a solution. The difficulty lies in solving the resulting set of Euler-Lagrange equations. It is necessary to know all of the initial conditions before the equations can be numerically integrated. In most cases, terminal conditions are specified in part as initial, and in part as final conditions. Thus, an iterative scheme must be employed to obtain a solution.

To avoid the difficulty associated with the use of the methods of the calculus of variations, a number of investigators have proposed the use of direct methods of optimization. Among those that have been employed for trajectory optimization are the gradient theory methods, and a method analogous to the Rayleigh-Ritz methods.[9,10,11]

Reference (12) is a survey of the problem of optimizing aircraft and missile flight paths. Prior treatments are described and the problems of Bolza, Mayer, and Lagrange are developed. This reference contains an excellent bibliography pertaining to trajectory optimizing.

Orbit Transfer

The orbit transfer problems which specify terminal conditions but not the transfer time are not equivalent to the rendezvous problem, although some insight may be drawn from them. The difference may be seen in Fig. 2.1 which shows two coplanar orbits that are nearly identical in geometry but differ in orientation. Let it be assumed that an optimal orbit transfer trajectory would be similar to path P_1-P_2, where one impulse is applied at P_1 on orbit A to initiate the transfer trajectory and a second is applied at P_2 to enter orbit B. Now, it would be coincidental if the vehicle and target were phased in orbit such that this trajectory would result in an intercept. Further, since the orbital periods would be nearly equal, waiting until the proper phasing occurred might not be feasible. However, if time is of no consequence, the fuel required to rendezvous would, in the limit, be equal to that required to transfer orbits. Such a rendezvous would be carried out following the orbit transfer by applying a small retro thrust at the perigee (for minimum velocity impulse required/period change) of the entered orbit in order to go into a wait-

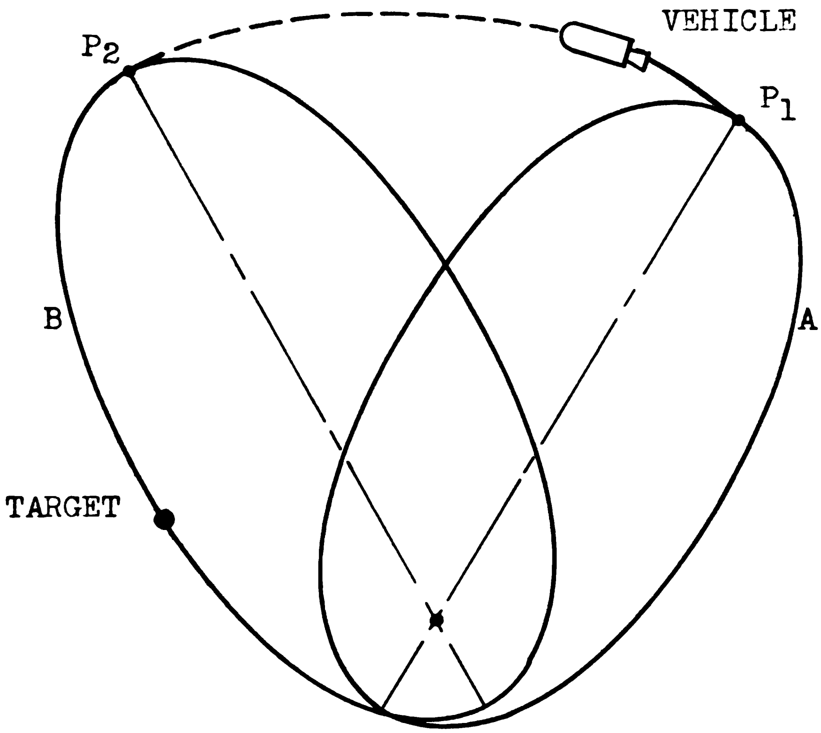


FIG. 2.1—ORBITAL TRANSFER WITH TIME UNSPECIFIED

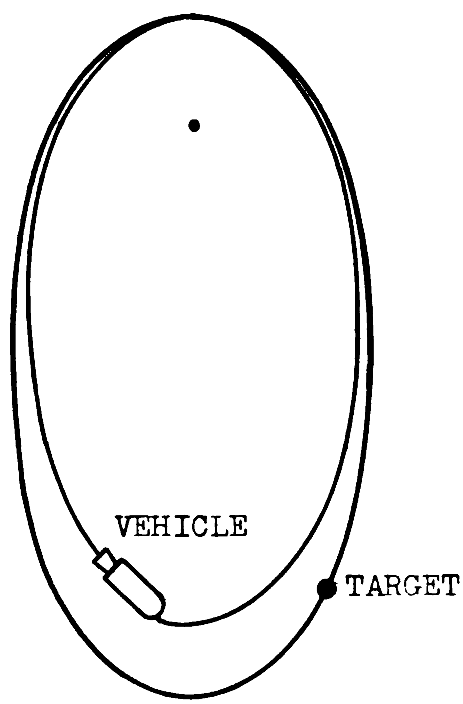


FIG. 2.2—WAITING ORBIT DESIGNED TO CORRECT PHASING PRIOR TO RENDEZVOUS

ing orbit with a slightly shorter period (Fig. 2.2). The waiting orbit should be tailored so that the vehicle and target would arrive at their coincidental perigee points simultaneously after a number of revolutions.

One of the first works concerned with the minimizing of orbit transfer energy was the derivation by Hohmann of the transfer ellipse named for him.[13] The method uses two impulses to transfer between circular orbits. The first is applied at a point on the initial orbit to establish an elliptical trajectory tangent to the two orbits. Transfer is accomplished during 180^{o} of orbit travel, and another impulse is applied to enter the second orbit at the point of trajectory tangency. Transfer can be to an orbit of either smaller or larger radius. The Hohmann transfer ellipse provides a minimum energy transfer if the radius ratio between the two orbits is not greater than 11.9.[14]

Other investigators have examined the more general problem of transfer between non-circular orbits. It has been shown that the optimum number of impulses to transfer from one coplanar orbit to another is either two or three, depending upon their orientation and element values.[15,16]

ANALYSIS METHODS USED IN SUBJECT STUDY

Early in the program three avenues of investigation were followed, more or less simultaneously. The three consisted of a study of the applicability of the calculus of variations and two direct minimization methods.

One of the direct methods, a trajectory perturbation technique, was the primary tool of subsequent investigation. The other direct method of analysis was based upon a variation of delta quantities to establish a gradient that would indicate the manner in which a non-minimum trajectory should be varied to reduce transfer energy. It was abandoned because of excessive computer time requirements and lack of proof that the determined trajectory was an extremum and not merely one with a stationary energy level.

Treatment of the rendezvous energy problem by the methods of the calculus of variations and the trajectory perturbation technique are presented in Chapters III and IV respectively.

GENERAL ASSUMPTIONS

The followint assumptions were made in order to facilitate the energy analysis and are not believed to detract from the results obtained:

(1) The earth is spherical. The perturbations of a satellite orbit caused by the earth's oblateness may be neglected insofar as rendezvous is concerned because the integrated effect of the perturbations would be small during the time interval of interest.[2] Also, since the distance between target and vehicle during rendezvous is small compared to the radius of the earth, the perturbation effects would be approximately the same on each and

would cancel insofar as relative motion is concerned.

(2) Gravity is the only outside disturbing force. The forces exerted by atmospheric drag, meteorite collision, and sun pressure are neglected.

(3) The target and vehicle are in coplanar orbits. Motion in a direction normal to the target orbit plane is, for all practical purposes, independent of in-plane motion for the relatively small separation distances being considered. (See Eq. A.24.) Thus, it is possible to treat the in-plane and out-of-plane motion independently and superimpose results.

(4) The vehicle is a point mass. The problem of attitude stablization and thrust vectoring was not investigated.

(5) Impulsive thrusts and instantaneous velocity charges are allowed. The time required for a vehicle to attain a desirable closing velocity in the usual situation would be small compared to the time to perform the terminal phase rendezvous maneuver.

CHAPTER III

ANALYSIS BY THE CALCULUS OF VARIATIONS

The analysis by the calculus of variations presented here relies heavily upon work described in references (5,7,12, and 17). The problem of energy minimization is analyzed using the linearized rendezvous equations (A.22 and A.23). Within the limitations of these equations, which are shown in Appendix A to be reasonably accurate, it will be shown that a minimum energy rendezvous trajectory contains no arc flown with intermediate thrust. In addition, it will be shown that if thrust is unbounded, a minimum energy trajectory is achieved with either two or three impulses.

If the angle between the thrust and velocity vectors is denoted by α, Eqs. (A.22 and A.23) can be written as the following set of first order equations:

$$\varphi_x = \dot{x} - u = 0 \tag{3.1}$$

$$\varphi_y = \dot{y} - v = 0 \tag{3.2}$$

$$\varphi_u = \dot{u} - 2\omega v - (c\beta/m) \operatorname{Cos} \alpha = 0 \tag{3.3}$$

$$\varphi_v = \dot{v} + 2\omega u - 3\omega^2 y - (c\beta/m) \operatorname{Sin} \alpha = 0 \tag{3.4}$$

$$\varphi_m = \dot{m} + \beta = 0 \; . \tag{3.5}$$

Assuming the rocket exhaust velocity, c, to be a constant, as is nearly so in the case of a chemical rocket, there are two control variables—α and β.

Following Lietmann,[7] a sixth equation is introduced in order to limit the mass flow rate such that

$$\beta_{min} \leq \beta \leq \beta_{max} .$$

The equation is

$$\varphi_\xi = (\beta - \beta_{min})(\beta_{max} - \beta) - \xi^2 = 0 \qquad (3.6)$$

where

$$\xi = \xi(t) .$$

Equations (3.1-3.6) are six restricting conditions to be imposed on the eight variables x, y, u, v, m, α, β, ξ, leaving two degrees of freedom.

The problem at hand is to determine $\alpha(t)$ and $\beta(t)$ such that the rocket will traverse a trajeotory between designated space terminals in time T that minimizes $(m_o - m_T)$. As a first requisite for a solution, the Euler-Lagrange equations must be satisfied. These equations are

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}}\right) - \frac{\partial F}{\partial q} = 0; \quad q = x, y, u, v, m, \alpha, \beta, \xi$$

where

$$F = \sum \lambda_q \phi_q; \quad q = x, y, u, v, m, \xi$$

and the λ_q are the Lagrange multipliers. When applied to Eqs. (3.1-3.6), the resulting set of equations are as follows:

$$\dot{\lambda}x = 0 \tag{3.7}$$

$$\dot{\lambda}y + 3\omega^2\lambda v = 0 \tag{3.8}$$

$$\dot{\lambda}u - 2\omega\lambda v + \lambda x = 0 \tag{3.9}$$

$$\dot{\lambda}v + 2\omega\lambda u + \lambda y = 0 \tag{3.10}$$

$$\dot{\lambda}_m - (c\beta/m^2)(\lambda u \operatorname{Cos}\alpha + \lambda v \operatorname{Sin}\alpha) = 0 \tag{3.11}$$

$$(c/m)(\lambda u \operatorname{Cos}\alpha + \lambda v \operatorname{Sin}\alpha) - \lambda m + \lambda_\xi[(\beta - \beta min) - (\beta max - \beta)] = 0 \tag{3.12}$$

$$(c\beta/m)(\lambda u \operatorname{Sin}\alpha - \lambda v \operatorname{Cos}\alpha) = 0 \tag{3.13}$$

$$\lambda_\xi \xi = 0 \tag{3.14}$$

Solutions for λx, λy, λu, and λv become immediately available by first differentiating Eqs. (3.9 and 3.10) and substituting from Eqs. (3.7 and 3.8), whereupon, the following equations are obtained:

$$\ddot{\lambda}u - 2\omega\dot{\lambda}v = 0 \tag{3.15}$$

$$\ddot{\lambda}v + 2\omega\dot{\lambda}u - 3\omega^2\lambda v = 0 \ . \tag{3.16}$$

It is seen that with a substitution of λu for x and λv for y, these equations become identical to the linearized equations of motion (A.22 and A.23) with $Ax = Ay = 0$.

Leitmann proved in the case of an analogous problem, i.e. when the λu-λv equations can be uncoupled from the remaining set of Euler-Lagrange equations, that no arc flown with intermediate thrust can exist.[7] Thus, an optimum trajectory would be composed of arcs of maximum and minimum thrust. In the present investigation velocity impulses have been assumed, i.e. $\beta max \to \infty$.

Thrust direction is determined from Eq. (3.13) as

$$\operatorname{Tan} \alpha = \frac{\lambda v}{\lambda u} \, . \tag{3.17}$$

It is noted that the functions λu and λv remain defined between impulses since βmin is allowed to approach but not attain a zero value.

If an optimal trajectory is to be composed of more than one coasting arc, the Erdmann-Weierstrass corner conditions must be satisfied at each corner (interior junction). These conditions are expressed as

$$\left(\frac{\partial F}{\partial \dot{q}}\right)_- = \left(\frac{\partial F}{\partial \dot{q}}\right)_+$$

$$\left(F - \dot{q}\,\frac{\partial F}{\partial \dot{q}}\right)_- = \left(F - \dot{q}\,\frac{\partial F}{\partial \dot{q}}\right)_+$$

which define the equality of quantities immediately prior to and after a corner. Applied to the present problem they result in

$$\begin{aligned} \lambda q_- &= \lambda q_+ \; ; \quad q = x, y, u, v, m \\ C_- &= C_+ \end{aligned} \tag{3.18}$$

where

$$C = \lambda_x \dot{x} + \lambda_y \dot{y} + \lambda_u \dot{u} + \lambda_v \dot{v} \, .$$

First, it is noted that λu and λv satisfy Eq. (3.18) by the noted solutions (analogous to x and y which are continuous on a coasting trajectory). Next, parameters

λx and λy satisfy this corner condition because λx is a constant by Eq. (3.7), and from Eq. (3.8)

$$\lambda y = -3\omega^2 \int \lambda v \, dt \tag{3.20}$$

and so is also continuous. In order for Eq. (3.14) to hold, it is necessary that $\lambda\xi = 0$, since $\xi(t) \neq 0$ by definition. Hence, from Eq. (3.12)

$$\lambda m = (c/m)(\lambda u \operatorname{Cos} \alpha + \lambda v \operatorname{Sin} \alpha) \tag{3.21}$$

and, since $c = 0$ immediately before and after an impulse, $\lambda m_- = \lambda m_+$. Thus, all of the corner conditions of Eq. (3.18) are shown to be satisfied.

The corner condition specified by Eq. (3.19) furnishes one of a set of equations which can be solved to determine a stationary three impulse rendezvous trajectory. In total, twenty-seven equations can be written relating thirty-six parameters. From Eqs. (3.1-3.4) with zero thrust and from Eqs. (3.15 and 3.16), explicit solutions can be determined for the following twenty parameters:

x_i, x_T	λu_i, λu_T
y_i, y_T	λv_i, λv_T
$\dot{x}_{i-}$, $\dot{x}_{T-}$	$\dot{\lambda} u_i$
$\dot{y}_{i-}$, $\dot{y}_{T-}$	$\dot{\lambda} v_i$
$\dot{u}_{i-}$, $\dot{u}_{i+}$	λx_i
$\dot{v}_{i-}$, $\dot{v}_{i+}$	λy_i .

Six equations remain to be defined.

At each application of impulsive thrust the direction of the thrust vector must coincide with the added velocity vector. Hence, Eq. (3.17) applied at each impulse results in the three equations

$$\frac{\lambda v_k}{\lambda u_k} = \frac{\Delta \dot{y}_k}{\Delta \dot{x}_k} = \frac{\dot{y}_{k+} - \dot{y}_{k-}}{\dot{x}_{k+} - \dot{x}_{k-}} ; \quad k = 0, i, T . \qquad (3.22)$$

The three other equations, which have been shown to be necessary conditions for an optimal trajectory,[5] are

$$[(\lambda u_k)^2 + (\lambda v_k)^2]^{1/2} = 1; \quad k = 0, i, T . \qquad (3.23)$$

These equations state that λu and λv are not merely proportionsl to the thrust components, but in fact, are the direction cosines of the thrust vector.

The desired unknowns of impulse direction, magnitude, and timing can be determined as functions of the eight terminal conditions and total transfer time by solving the set of twenty-seven equations described above.

In order to see that no more than three impulses can be utilized for optimum transfer it will be recalled that Eqs. (3.19 and 3.22) must be satisfied at points of impulse application by the parameters λu and λv. Since these two parameters are described by second order differential equations between $t = 0$ and $t = T$, the four initial conditions λu_0, λv_0, $\dot{\lambda u}_0$, and $\dot{\lambda v}_0$ are just sufficient to satisfy the conditions specified by these equations. Additional conditions that would be

introduced by another corner could not, in general, be satisfied.

In summarizing the application of the calculus of variations to the problem of rendezvous fuel minimization, the following items are noted:

(1) Insofar as the linearized equations describe the relative motion of a rendezvous vehicle and target, it has been shown that—

(a) No arc flown with thrust between the maximum and minimum values can exist on a minimum energy trajectory.

(b) The minimum energy transfer is via either two or three impulses. No criteria was developed to determine which of these two modes would provide the minimum.

(2) Since the linearized equations have been shown to be reasonably accurate, it would not be expected that there would be any significant difference in the above results as applied to the actual rendezvous maneuver. It should be called to attention that the linearized x-y equations are based upon an axis system whose origin moves in a circular orbit, but that the target need not. It is necessary only that the target position be predictable in the reference frame at time T.

CHAPTER IV

ENERGY ANALYSIS BY TRAJECTORY PERTURBATION

In many respects this chapter presents the most important results obtained from the rendezvous energy study. The methods of analysis which were developed are described and a means for measuring the potential of a third impulse is formulated. The results of a parametric study to determine the conditions under which a potential exists are also presented.

PERTURBATION METHOD

Description

The perturbation method, as applied to the rendezvous energy study, determines the effect of perturbing a vehicle which would otherwise move along a nominal trajectory joining two space terminals. Let it be assumed that the origin of an axis system as shown in Fig. 4.1 moves along a nominal trajectory between the space terminals P_1 and P_2 as if it were attached to an undisturbed mass. Further, let the position and velocity of the origin be P_1, $\overline{V}_{c1}$ at $t = 0$, and P_2, $\overline{V}_{c2}$ at $t = T$. Then, if the absolute velocities of a vehicle immediately prior to P_1 and after P_2 are denoted by $\overline{V}_o$ and $\overline{V}_f$, these velocities relative to the coasting reference frame are determined by the vector differences

$$\overline{v}_o = \overline{V}_o - \overline{V}_{c1}$$
$$\overline{v}_f = \overline{V}_f - \overline{V}_{c2} \ .$$

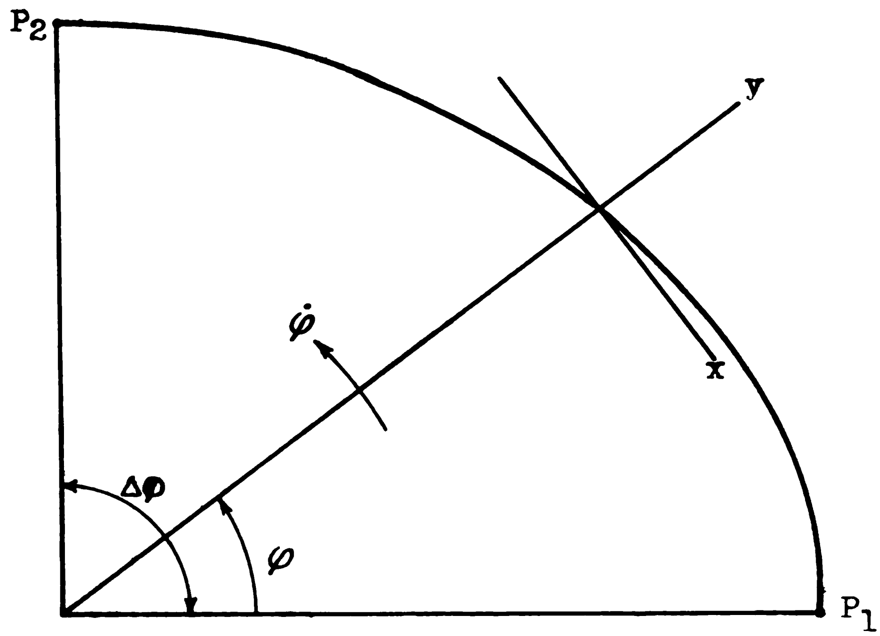


FIG. 4.1—AXIS SYSTEM ON NOMINAL COASTING TRAJECTORY

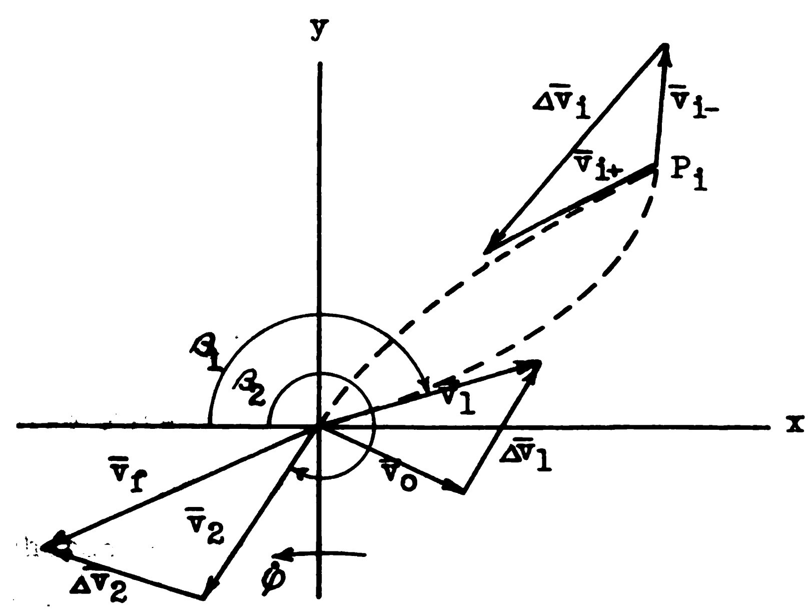


FIG. 4.2—RELATIVE TRAJECTORY OF PERTURBED VEHICLE

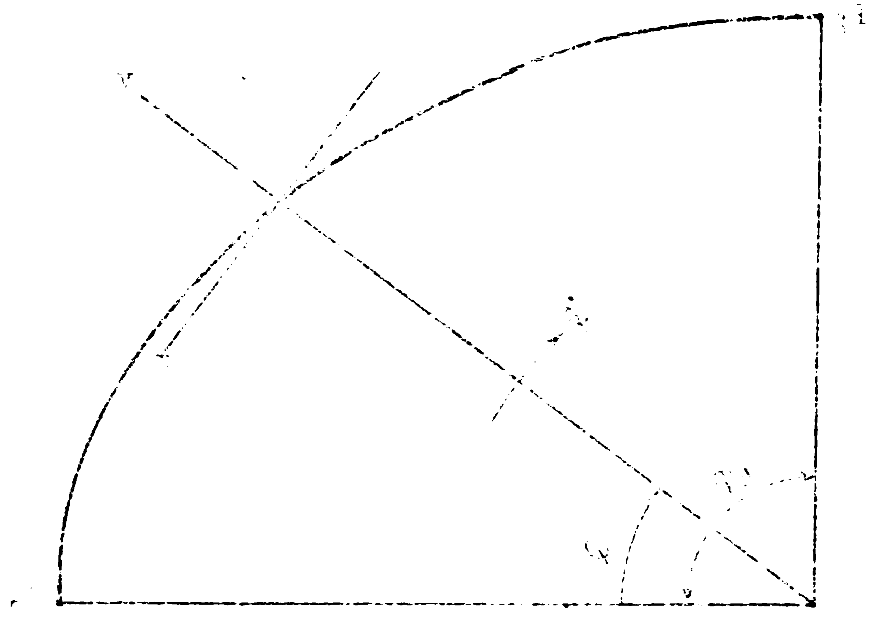


FIG. 4.1—[illegible]

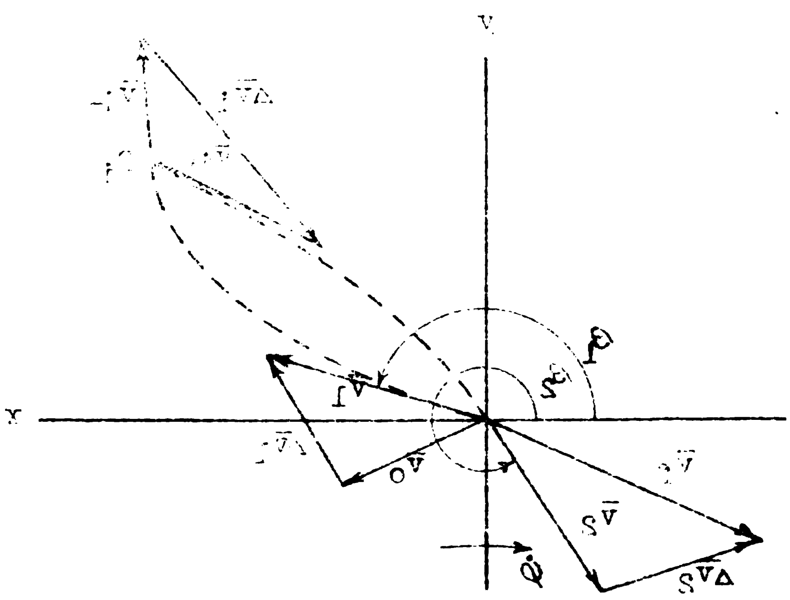


FIG. 4.2—RELATIVE TRAJECTORY OF [illegible]

(Upper case letters are used to denote absolute velocities while relative velocities are denoted by lower case letters.) If impulses were applied according to the equations

$$\bar{v}_1 = -\bar{v}_o$$
$$\bar{v}_2 = -\bar{v}_f,$$

the vehicle would follow the coasting trajectory and remain at the origin of the reference frame.

On the other hand, suppose that the relative velocity at P_1 is not entirely cancelled, but that a residual velocity, v_1, is allowed to remain. In this case the mass would drift away from the frame origin. However, at an intermediate time, $t = t_i$, another velocity impulse, Δv_i, could be applied that would bring the mass back to the origin at the moment P_2 is reached. Relative trajectories similar to those shown in Fig. 4.2 would result. The energy expended would be the sum of the three impulses according to the equation

$$\Delta V = |\bar{v}_1 - \bar{v}_o| + |\Delta v_i| + |\bar{v}_f - \bar{v}_2|.$$

<u>Third Impulse Potential</u>

In the case of a two-impulse rendezvous, the required characteristic velocity is determined by the equation

$$\Delta V = |\Delta\bar{v}_1| + |\Delta\bar{v}_2| = |\bar{v}_o| + |\bar{v}_2|.$$

Hence, if a three impulse transfer is to require less

energy than would be required by the use of only two, it is necessary that the energy denoted by Eq. (4.1) be less than that denoted by Eq. (4.2). Hence, it is necessary that

$$|\bar{v}_1 - \bar{v}_0| + |\Delta v_1| + |\bar{v}_f - \bar{v}_2| < |v_0| + |v_f| \quad (4.3)$$

or

$$|\Delta v_1| < |v_0| - |\bar{v}_1 - \bar{v}_0| + |v_f| - |\bar{v}_f - \bar{v}_0|. \quad (4.4)$$

Considering v_0, v_1, and $(\bar{v}_1 - \bar{v}_0)$ as three sides of a triangle, it may be seen that

$$|v_1| > |v_0| - |\bar{v}_1 - \bar{v}_0| \quad (4.5)$$

and likewise

$$|v_2| > |v_f| - |\bar{v}_f - \bar{v}_2|. \quad (4.6)$$

Upon substituting inequalities (4.5) and (4.6) into (4.4), it is found that

$$|\Delta v_1| < |v_1| + |v_2|.$$

This expression is more convenient when written as the equality

$$\delta v = |v_1| + |v_2| - |\Delta v_1|.$$

The quantity δv was used as a criterion for determining whether a third impulse is potentially useful for reducing rendezvous energy. A positive value signifies that a potential exists. However, in order to realize the full potential, $\bar{v}_1$ and $\bar{v}_2$ must be parallel to $\bar{v}_0$ and $\bar{v}_f$ respectively. At times, the

initial and final velocities may be such that the sum of the impulses required at P_1 and P_2 is greater than the potential corresponding to the selected magnitude and direction of v_1, i.e. the quantity

$$|\Delta v_1| + |\Delta v_2| = |\bar{v}_1 - \bar{v}_0| + |\bar{v}_f - \bar{v}_2| > \delta v \, .$$

In this case a potential would exist that could not be realized due to the directions and magnitudes of the initial and final velocities.

<u>Determination of Nominal Coasting Trajectory</u>

The trajectory of an object coasting in space follows a conical path which can be either an ellipse, parabola, or hyperbola. However, the subject investigation was restricted to a consideration of motion along an elliptical path. (Circular paths are included in this category.) The term "transfer ellipse" is used to designate the entire orbit of which the coasting transfer trajectory is a part. Thus, a transfer trajectory may be described by the ephemeris of the transfer ellipse.

Unless some special orientation is assumed for the transfer ellipse, the equations of Appendix A alone are not sufficient to determine the ellipse elements if the known information consists of only R_1, R_2, $\Delta\varphi$ and T. However, this information and the direction of orbit rotation does specify a unique trajectory. The orbit elements may be determined as follows:[18]

Let

$$a_1 = \frac{(\mu)^{1/2}\ T}{[2(R_1R_2)^{1/2}\ \mathrm{Cos}\ (\Delta\varphi/2)]^{3/2}}$$

$$b_1 = \frac{R_1 + R_2}{4(R_1R_2)^{1/2}\ \mathrm{Cos}\ (\Delta\varphi/2)} - 1/2$$

$$\Delta E = E_2 - E_1; \quad \Delta\varphi = \varphi_2 - \varphi_1$$

where E_1 and E_2 are the values of the eccentric anomaly at P_1 and P_2. Then, the parameter $\Delta E/2$ can be determined from one of the following equations which are derived in Appendix B:

$$\pm\, a_1 = [b_1 + \mathrm{Sin}^2\ (\Delta E/4)]^{1/2} + \frac{(\Delta E - \mathrm{Sin}\ \Delta E)}{\mathrm{Sin}^3\ (\Delta E/2)} [b_1 + \mathrm{Sin}^2\ (\Delta E/4)]^{3/2} \tag{4.9}$$

where the sign preceding a_1 is taken as

+ for $\Delta\varphi < 180^\circ$, and - for $\Delta\varphi > 180^\circ$;

and, for the case in which $\Delta\varphi = 180^\circ$

$$(\mu)^{1/2}\ T[2/(R_1 + R_2)]^{3/2} = \frac{\Delta E - \mathrm{Sin}\,\Delta E}{\mathrm{Sin}^3\ (\Delta E/2)}. \tag{4.10}$$

These equations can be solved by trial and error, and with ΔE known, other elements can be computed by the following equations:

$$p = \frac{2R_1R_2\ \mathrm{Sin}^2\ (\Delta\varphi/2)}{R_1 + R_2 - 2(R_1R_2)^{1/2}\ \mathrm{Cos}\ (\Delta\varphi/2)\ \mathrm{Cos}\ (\Delta E/2)} \tag{4.11}$$

$$a = \frac{R_1R_2\ \mathrm{Sin}^2\ (\Delta\varphi/2)}{p\ \mathrm{Sin}^2\ (\Delta E/2)} \tag{4.12}$$

$$e = \frac{(a - p)^{1/2}}{a} \tag{4.13}$$

Terminal coasting velocities can be determined from equations (A.14 and A.17).

In subsequent references, the above described method of solution and the computer routine which was developed to solve the system of equations will be referred to as the "coasting routine".

<u>Calculation of Third Impulse Potential</u>

After having determined a nominal coasting trajectory, $\bar{v}_1$ and $\bar{v}_2$ can be computed for an arbitrarily selected value of $\bar{v}_1$. Two methods of solution were developed. One makes use of the coasting routine described in the previous section plus an integration of the orbital equations of motion. In the second method, the exact relative equations of motion are integrated using an iterative procedure to determine the sought variables. The majority of computations were made using the coasting routine and orbital equation technique because it required considerably less computer time. For the average solution only about one-tenth as much time was required. However, the relative equation technique, which is described in Appendix C, was found to give highly accurate solutions and was used to check the accuracy of the other method.

Following is a description of the coasting routine-orbital equation method for determining the quantities comprising δv. After $\bar{v}_{c1}$ and $\bar{v}_{c2}$ are computed by the coasting routine, a selected value of $\bar{v}_1$ is added to

$\overline{V}_{c1}$ to give a new absolute velocity at P_1, which is shown in Fig. 4.3 along with the other vectors of interest. The next step in the solution is to determine the radial and angular velocities at P_1, which can be computed from the equations

$$\dot{R}_1 = (\overline{V}_{c1} + \overline{v}_1) \sin \gamma_1 \tag{4.14}$$

$$\dot{\varphi}_1 = [(\overline{V}_{c1} + \overline{v}_1) / R_1] \cos \gamma_1 . \tag{4.15}$$

With the initial conditions at P_1 known, the equations

$$\ddot{R} - R\dot{\varphi}^2 + (\mu/R^2) = 0 \tag{4.16}$$

$$R\ddot{\varphi} + 2\dot{R}\dot{\varphi} = 0 \tag{4.17}$$

are integrated to determine the position $\overline{R}_i$ and velocity $\overline{V}_i$ at the intermediate time, t_i. Following this step, the coasting routine is used once again to determine a coasting trajectory that will traverse the distance between P_i and P_2 in the time remaining. Inputs to the routine this second time are denoted by primes and are

$$R_1' = R_i$$

$$R_2' = R_2$$

$$\Delta\varphi' = \Delta\varphi - |\varphi_i - \varphi_1|$$

$$T' = T - t_i .$$

The velocities $\overline{V}_i$ and $\overline{V}_2$ are determined at the terminals of the new coasting trajectory, and the sought parameters are given by the vector differences

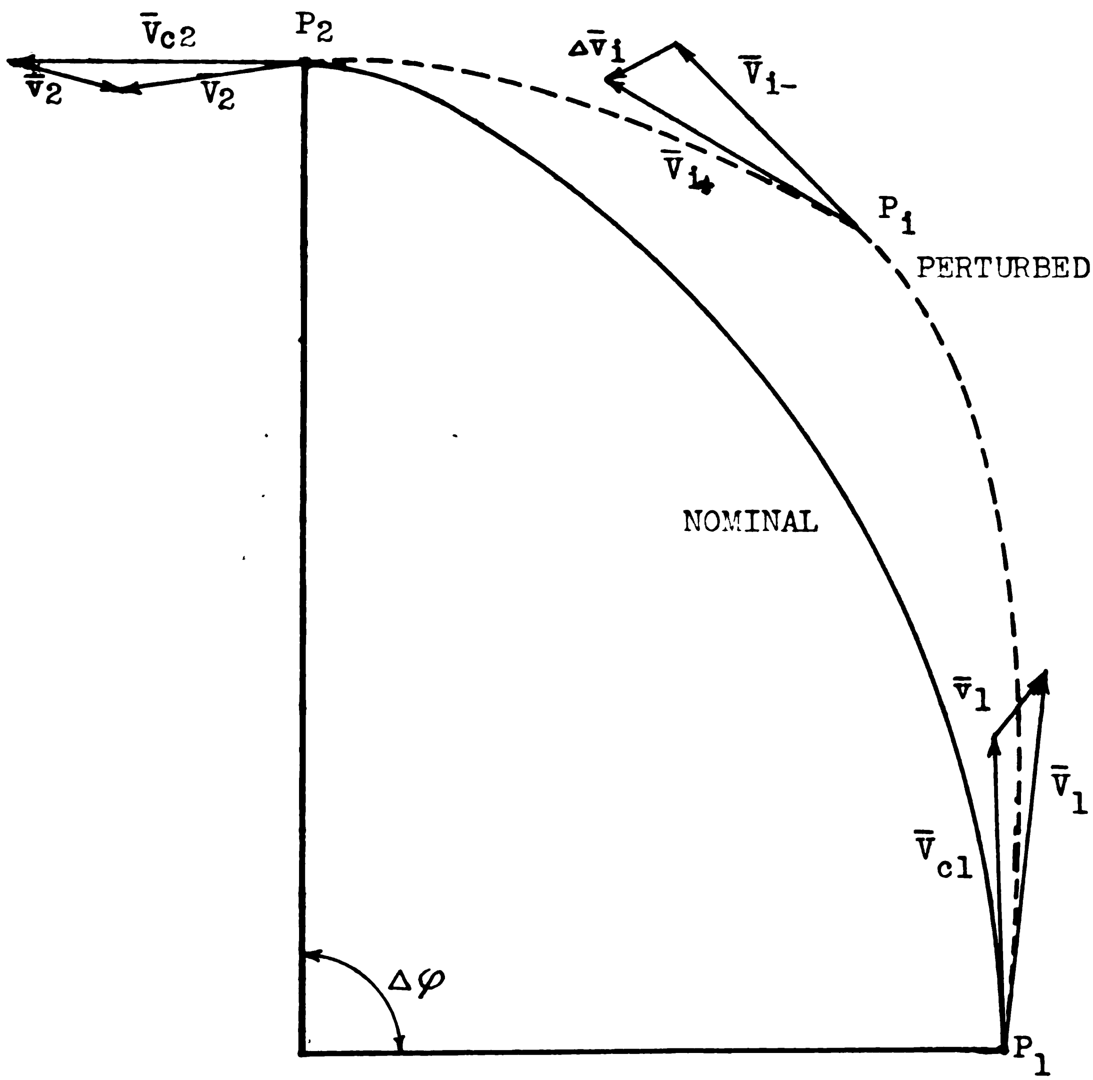
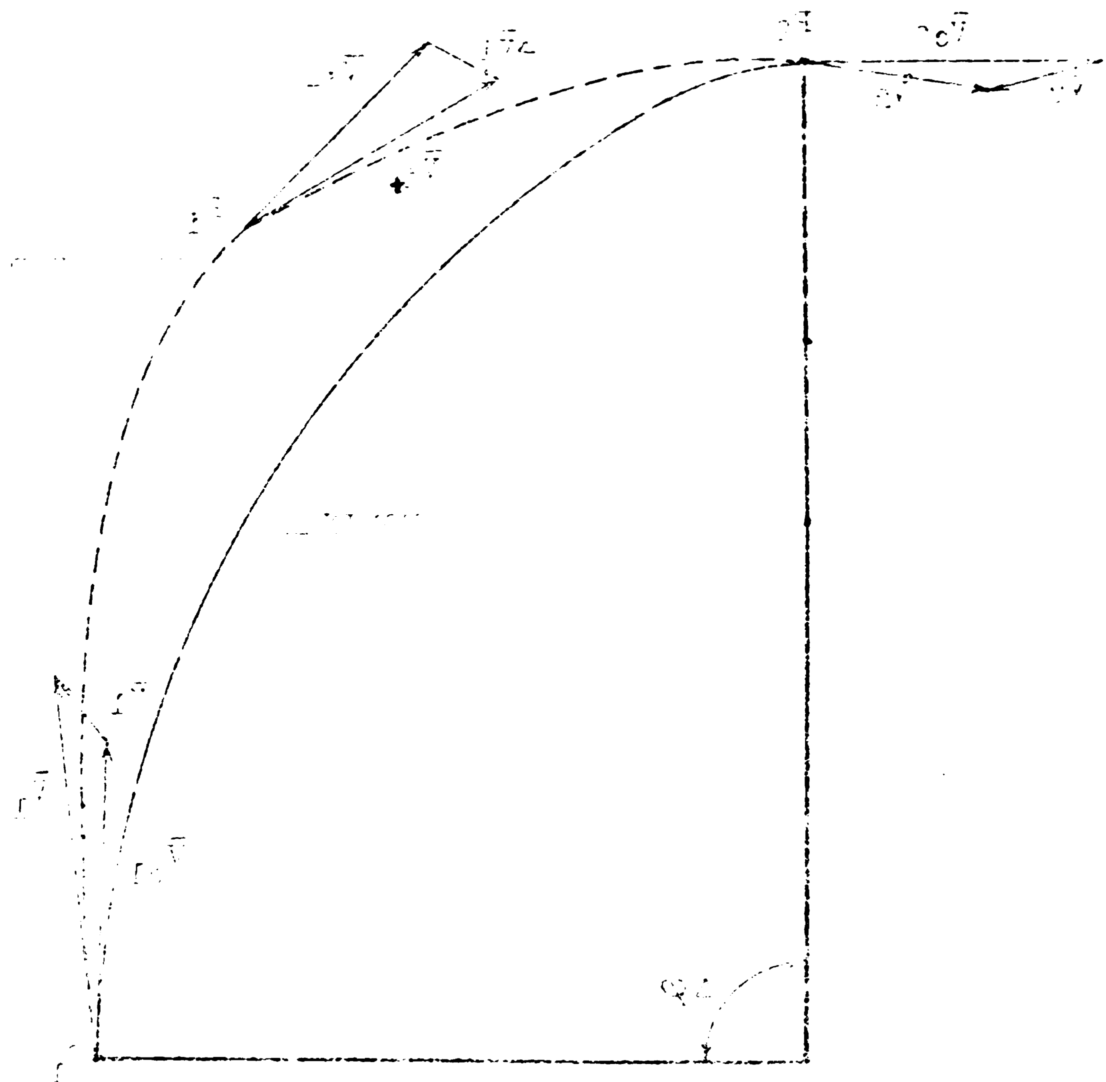


FIG. 4.3—INERTIAL TRAJECTORIES AND VELOCITIES



$$v_i = |\bar{V}_{i+} - \bar{V}_{i-}|$$

$$v_2 = |\bar{V}_2 - \bar{V}_{c2}| .$$

The total computer time required to determine a single value of δv, including use of the coasting routine for arcs P_1-P_2 and P_i-P_2, and integration over arc P_1-P_i, was slightly less than two seconds. The accuracy obtained was limited by round-off errors in the computer (an eight digit retention routine was used). Ellipse elements p and a were determined with errors less than 1 ft; and, although round-off velocity errors as large as 3 ft/sec were possible, the usual noted error was in the order of 0.2 to 0.5 ft/sec.

PARAMETRIC STUDY

The perturbation procedure described in the previous section was used to make a parametric study of the third-impulse potential, δv. The following parameters were varied during the study:

R_2/R_1 = ratio of final to initial radius

T = transit time

$\Delta\varphi$ = angle between R_1 and R_2

t_i = time of application of intermediate impulse

v_1 = magnitude of the residual velocity relative to coasting frame at P_1

β_1 = direction of v_1 (Fig. 4.2).

It was found that the parameters could be divided into two groups—those having primary effect and those evidencing only slight or secondary effect.

Secondary Effect Parameters

Small variations of R_2/R_1 and T from nominal values were found to have only a slight effect upon the output parameters comprising δv, i.e. Δv_1, v_2 and its direction β_2. In addition, it was found that the ratios $\Delta v_1/v_1$ and v_2/v_1 are independent of the magnitude of v_1 as is the velocity direction β_2.

Variations of R_2/R_1 were made with respect to a nominal R_1 corresponding to an orbit altitude of 300 statute miles. This altitude was selected on the basis that it is above the high atmospheric drag region ($\approx$200 s mi) and below the earth's radiation belt ($\approx$400 s mi). No variations in R_1 were made since reasonable changes, from a rendezvous consideration, would probably be less than 200 s mi. This variation would change the total radius by only a small percentage. Orbital motion is, of course, affected by the total radius rather than altitude.

The selection of data presented in Table I shows the insignificant effect upon $\Delta v_1/v_1$, v_2/v_1, and β_2 of varying R_2/R_1 and v_1 over the noted range. With respect to the nominal initial altitude of 300 s mi, the ratios 1.005, 1.010, and 1.015 correspond to increases in altitude of 21.3, 42.6, and 64 s mi. Although these variations are relatively small, the results obtained do indicate the lack of sensitivity to changes in R_2/R_1.

TABLE I—RESULTS OBTAINED WITH VARIOUS RADIUS RATIOS AND RESIDUAL VELOCITIES

$\Delta\varphi = 90^{\circ};\ \beta_1 = 0^{\circ};\ t_i/T = 1/4$

	$(v_2/v_1)\angle\beta_2$			$\Delta v_i/v_1$		
v_1 \ R_2/R_1	1.005	1.010	1.015	1.005	1.010	1.015
100	$.284\angle 253^{\circ}$	$.287\angle 254^{\circ}$	$.286\angle 254^{\circ}$	1.297	1.300	1.300
200	$.285\angle 253^{\circ}$	$.284\angle 254^{\circ}$	$.286\angle 254^{\circ}$	1.299	1.298	1.300
400	$.282\angle 253^{\circ}$	$.285\angle 254^{\circ}$	$.286\angle 254^{\circ}$	1.300	1.300	1.301

$\Delta\varphi = 180^{\circ};\ \beta_1 = 80^{\circ};\ t_i/T = 1/2$

	$(v_2/v_1)\angle\beta_2$			$\Delta v_i/v_1$		
v_1 \ R_2/R_1	1.005	1.010	1.015	1.005	1.010	1.015
100	$2.128\angle 47^{\circ}$	$2.124\angle 47^{\circ}$	$2.124\angle 47^{\circ}$	3.480	3.474	3.474
200	$2.123\angle 47^{\circ}$	$2.122\angle 47^{\circ}$	$2.120\angle 47^{\circ}$	3.474	3.473	3.470
400	$2.116\angle 48^{\circ}$	$2.116\angle 47^{\circ}$	$2.115\angle 47^{\circ}$	3.469	3.467	3.466

$\Delta\varphi = 270^{\circ};\ \beta_1 = 140^{\circ};\ t_i/T = 3/4$

	$(v_2/v_1)\angle\beta_2$			$\Delta v_i/v_1$		
v_1 \ R_2/R_1	1.005	1.010	1.015	1.005	1.010	1.015
100	$7.804\angle 217^{\circ}$	$7.842\angle 217^{\circ}$	$7.882\angle 217^{\circ}$	11.004	11.049	11.099
200	$7.836\angle 217^{\circ}$	$7.833\angle 217^{\circ}$	$7.813\angle 216^{\circ}$	11.045	11.044	11.127
400	$7.898\angle 216^{\circ}$	$7.828\angle 216^{\circ}$	$7.978\angle 216^{\circ}$	11.110	11.035	11.219

Further, the variation would, in many cases, include the altitude changes made during actual rendezvous.

Since the trajectories under consideration are coasting arcs in a conservative field, a trajectory from P_1 to P_2 is the reverse equivalent of the trajectory from P_2 to P_1. This is illustrated by Fig. 4.4 in which the reverse quantities are denoted by primes. The following equalities are noted:

$$v_1' \underline{/\beta_1'} = v_2 \underline{/-\beta_2}$$

$$\Delta v_1' = \Delta v_1$$

$$v_2' \underline{/\beta_2'} = v_1 \underline{/-\beta_1} \,.$$

Thus, except for the secondary effect resulting from a slight change in the reference (initial) radius, the data obtained for radius ratios greater than one also furnish data for the reciprocal ratios less than one.

In order to study the effect of variations in transit time, a nominal time, T_n, was determined and variations were taken with respect to this time. The nominal time was determined as the time corresponding to the case in which the perigee of the transfer ellipse coincided with the initial position. For this selected orientation, the transfer ellipse elements are determined by the orbital equations of Appendix A. Substituting R_1 and R_2 into Eq. (A.9) for the case when $\varphi_1 = 0$ and $\varphi_2 = \Delta\varphi$ results in two equations, from which, the semi-latus rectum is determined as

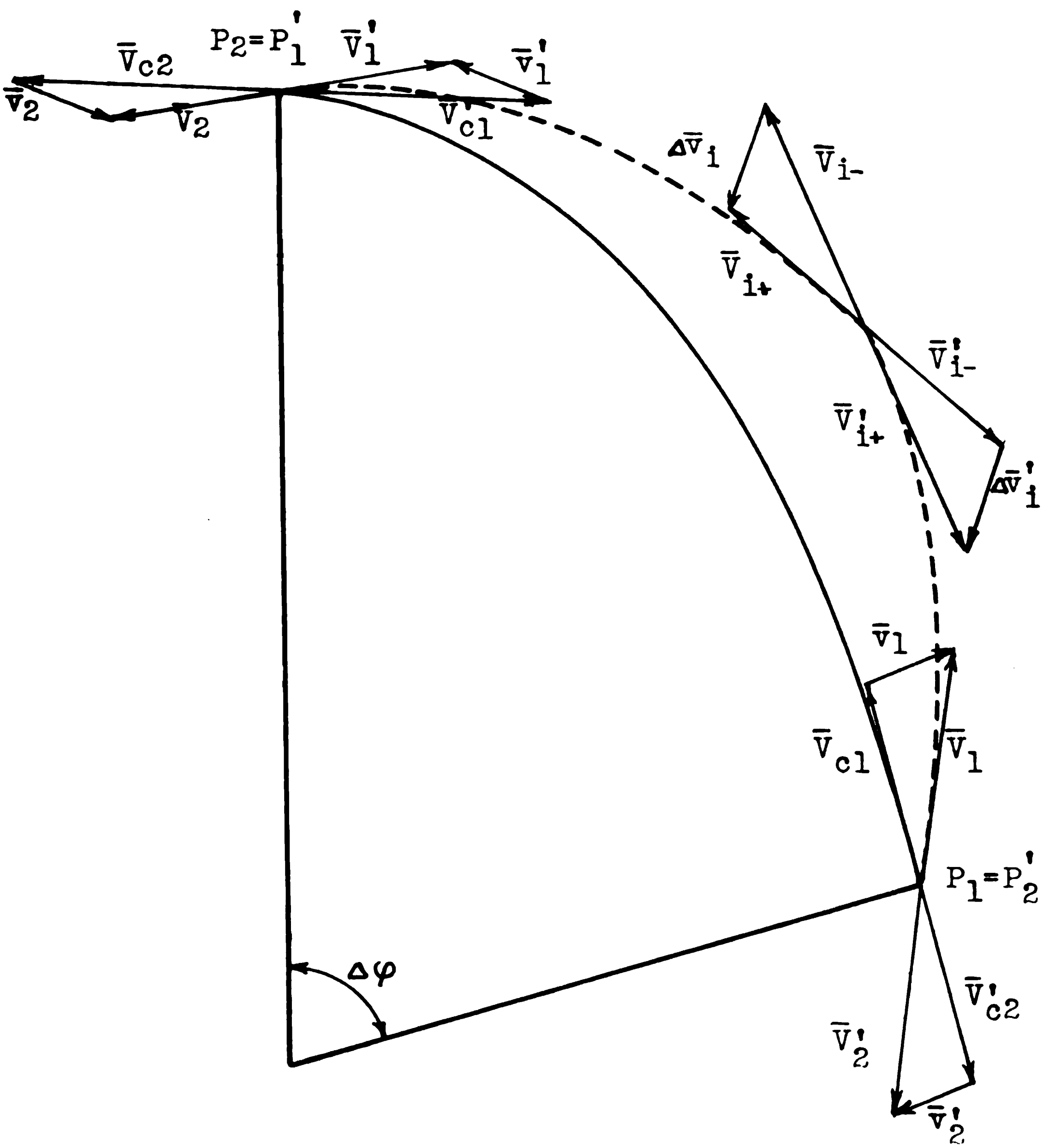
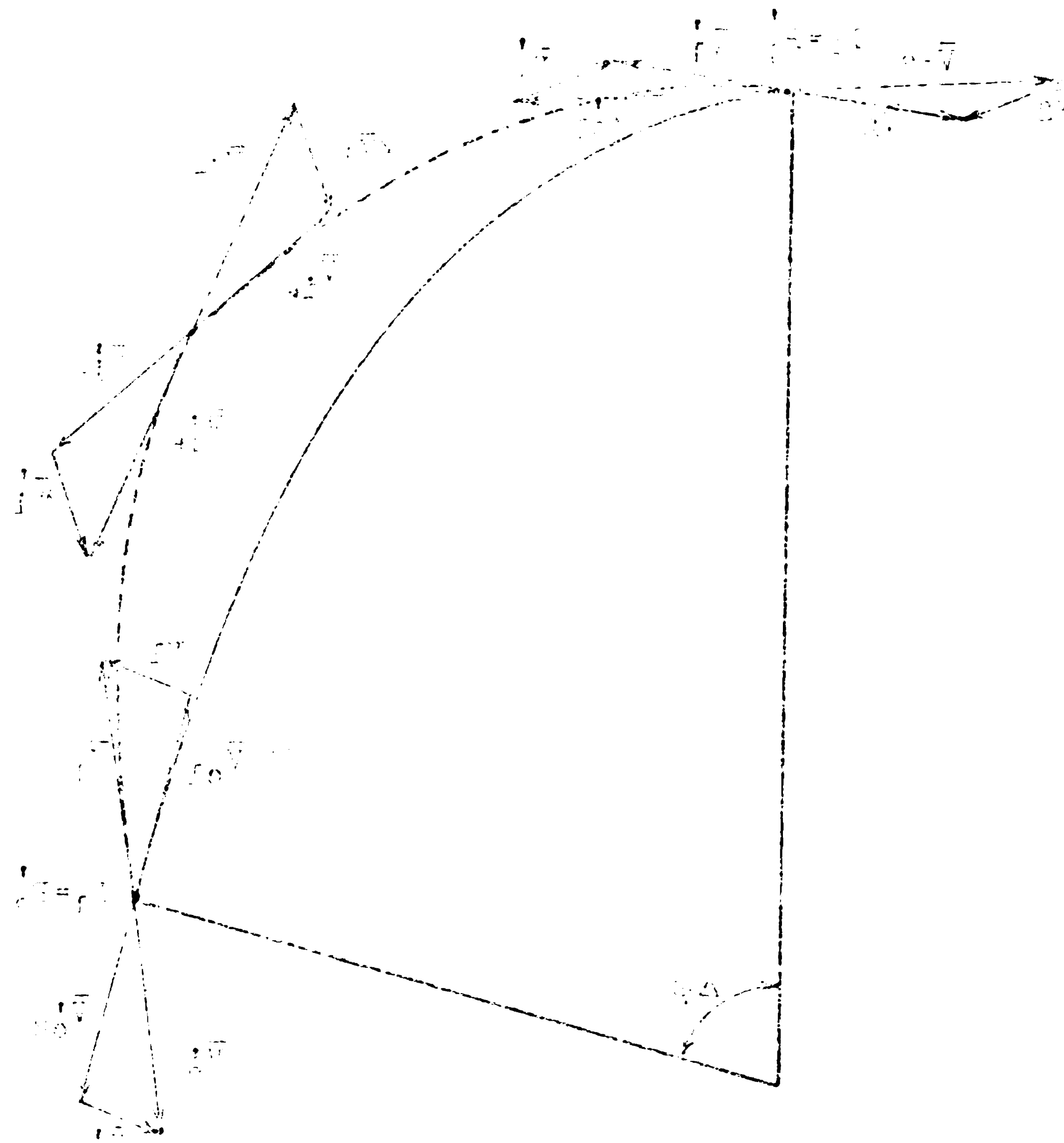


FIG. 4.4—ILLUSTRATING THE REVERSE EQUIVALENCE OF A TRAJECTORY



$$p = \frac{R_1R_2(1 - \cos \Delta\varphi)}{R_1 - R_2 \cos \Delta\varphi} . \tag{4.20}$$

After evaluating the other relevant ellipse elements from the orbital equations of Appendix A, T_n is determined according to Eq. (A.11) by setting $T_p = 0$, which gives

$$T_n = (E - e \sin E)/n. \tag{4.21}$$

Variations in transfer time up to

$$T = T_n \pm 0.2\ T_n$$

were investigated. This magnitude of change in T is comparatively large. Transfer times of 0.8 T_n, T_n, and 1.2 T_n for each of the transfer angles $\Delta\varphi = 90^o$, 180^o, and 270^o are shown in Table II. In the case of $\Delta\varphi = 90^o$, it may be noticed that the variations in the orientation of the transfer ellipses are such that the perigee is at P_1 for $T = T_n$, the perigee is between P_1 and P_2 for $T = 0.8\ T_n$, and the apogee is between P_1 and P_2 for $T = 1.2\ T_n$. Table III shows the effect of transit time variation. Although results certainly do show the effect of time variations, fluctuations are relatively small for the time variations involved. The values of $\Delta\varphi$, t_i/T, and β_1, selected for the preparation of Table III were selected to present a wide spread of data.

Primary Effect Parameters

The variables having primary effect were found

TABLE II—TRANSFER ELLIPSE ELEMENTS CORRESPONDING TO DIFFERENT TRANSFER TIMES

$\Delta\varphi = 90^{\circ}$

	$T = T_n$	$T = 0.8\ T_n$	$T = 1.2\ T_n$
T, sec	1417.2	1133.7	1700.6
p, ft	22,736,110	28,661,458	19,302,997
a, ft	22,738,383	33,427,124	20,172,621
e	0.0100	0.3776	0.2076
φ_1, deg	0	317.0	133.5
φ_2, deg	90	47.0	223.0

$\Delta\varphi = 180^{\circ}$

	$T = T_n$	$T = 0.8\ T_n$	$T = 1.2\ T_n$
T, sec	2849.2	2279.3	3419.0
p, ft	22,622,994	22,622,996	22.622,993
a, ft	22,623,554	23,499,807	23,029,011
e	0.00498	0.19316	0.13278
φ_1, deg	0	88.5	87.8
φ_2, deg	180	268.5	267.8

$\Delta\varphi = 270^{\circ}$

	$T = T_n$	$T = 0.8\ T_n$	$T = 1.2\ T_n$
T, sec	4324.6	3459.7	5189.6
p, ft	22,736,110	20,491,523	24,409,270
a, ft	22,738,383	20,862,765	24,718,922
e	0.0100	0.1334	0.1119
φ_1, deg	0	228.0	41.2
φ_2, deg	270	138.0	311.2

TABLE III—RESULTS OBTAINED WITH VARIOUS TRANSFER TIMES

$R_2/R_1 = 1.010;\ t_i/T = 1/4;\ v_1 = 100$ ft/sec; $\beta_1 = 0$

	$(v_2/v_1)\angle\beta_2$			$\Delta v_i/v_1$		
$\Delta\varphi$ \ T/T_n	0.8	1.0	1.2	0.8	1.0	1.2
90°	$.292\angle 257^\circ$	$.287\angle 254^\circ$	$.283\angle 250^\circ$	1.305	1.300	1.296
180°	$.210\angle 26^\circ$	$.220\angle 27^\circ$	$.244\angle 32^\circ$	1.867	1.943	2.025
270°	$.661\angle 71^\circ$	$.795\angle 71^\circ$	$.919\angle 71^\circ$	2.307	2.481	2.630

$R_2/R_1 = 1.010;\ t_i/T = 1/2;\ v_1 = 100$ ft/sec; $\beta_1 = 80^\circ$

	$(v_2/v_1)\angle\beta_2$			$\Delta v_i/v_1$		
$\Delta\varphi$ \ T/T_n	0.8	1.0	1.2	0.8	1.0	1.2
90°	$1.158\angle 353^\circ$	$1.214\angle 354^\circ$	$1.267\angle 355^\circ$	2.664	2.843	2.217
180°	$1.804\angle 47^\circ$	$2.122\angle 47^\circ$	$2.285\angle 46^\circ$	3.214	3.474	3.722
270°	$2.183\angle 70^\circ$	$2.388\angle 69^\circ$	$2.584\angle 69^\circ$	2.275	2.414	2.603

$R_2/R_1 = 1.010;\ t_i/T = 3/4;\ v_1 = 100$ ft/sec; $\beta_1 = 140^\circ$

	$(v_2/v_1)\angle\beta_2$			$\Delta v_i/v_1$		
$\Delta\varphi$ \ T/T_n	0.80	1.0	1.2	0.8	1.0	1.2
90°	$2.605\angle 44^\circ$	$2.510\angle 42^\circ$	$2.431\angle 39^\circ$	3.176	2.940	2.722
180°	$3.152\angle 169^\circ$	$3.075\angle 169^\circ$	$2.950\angle 171^\circ$	6.640	6.589	6.522
270°	$7.401\angle 214^\circ$	$7.841\angle 217^\circ$	$8.261\angle 219^\circ$	10.443	11.049	11.600

to be $\Delta\varphi$, β_1 and t_1/T. A number of computer runs were made to determine the effect of these variables on the parameters $\delta v/v_1, \Delta v_1/v_1$, v_2/v_1, and β_2. The somewhat surprising discovery was made that the results showed a periodicity of 180° with respect to β_1. This is seen in Figs. 4.5-4.8 which show the following relationships:

$$\delta v/v_1 \text{ at } (\beta_1 + 180^\circ) = \delta v/v_1 \text{ at } \beta_1$$

$$\Delta v_1/v_1 \text{ at } (\beta_1 + 180^\circ) = \Delta v_1/v_1 \text{ at } \beta_1$$

$$v_2/v_1 \text{ at } (\beta_1 + 180^\circ) = v_2/v_1 \text{ at } \beta_1$$

$$\beta_2 \text{ at } (\beta_1 + 180^\circ) = (\beta_2 \text{ at } \beta_1) + 180^\circ \ .$$

Figures 4.9-4.13 show the effect of varying $\Delta\varphi$ and t_1/T. It is seen from the curves relating $\delta v/v_1$ and β_1 that a third impulse is potentially useful for reducing the transfer energy in the case of each transfer angle for which data is shown. However, on extrapolating plots of peak values of δv versus $\Delta\varphi$ (Fig. 4.14) it is found that the curves intersect the abscissa at a value of $\Delta\varphi$ approximately equal to 30°, suggesting that no potential exists whenever the transfer angle is smaller than this value. Verification of this point would require more data than was obtained. In general, the potential increases with the transfer angle and becomes substantial at the larger angles.

It is also noticed from Fig. 4.13 that the potential

FIG. 4.5—THIRD IMPULSE POTENTIAL

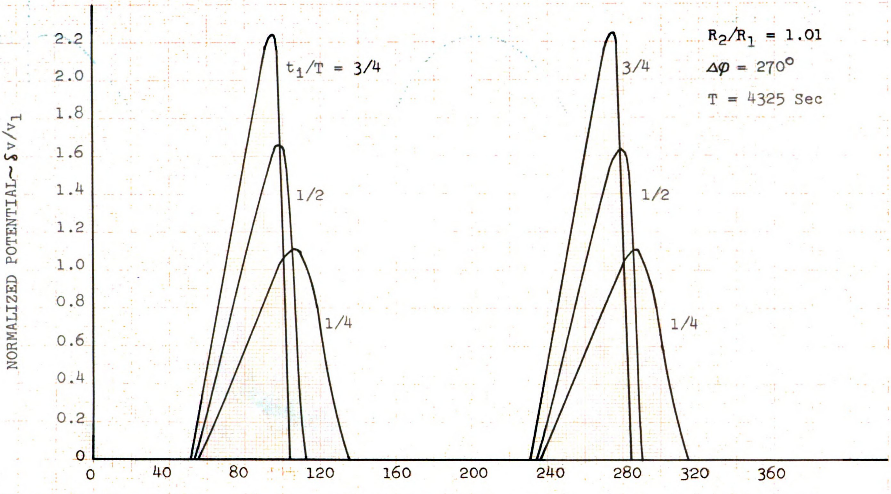


FIG. 4.5—THIRD IMPULSE POTENTIAL

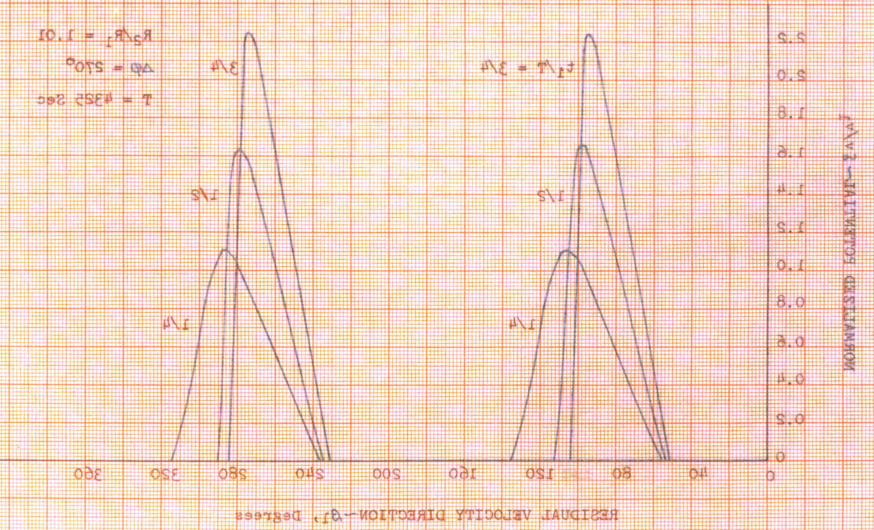
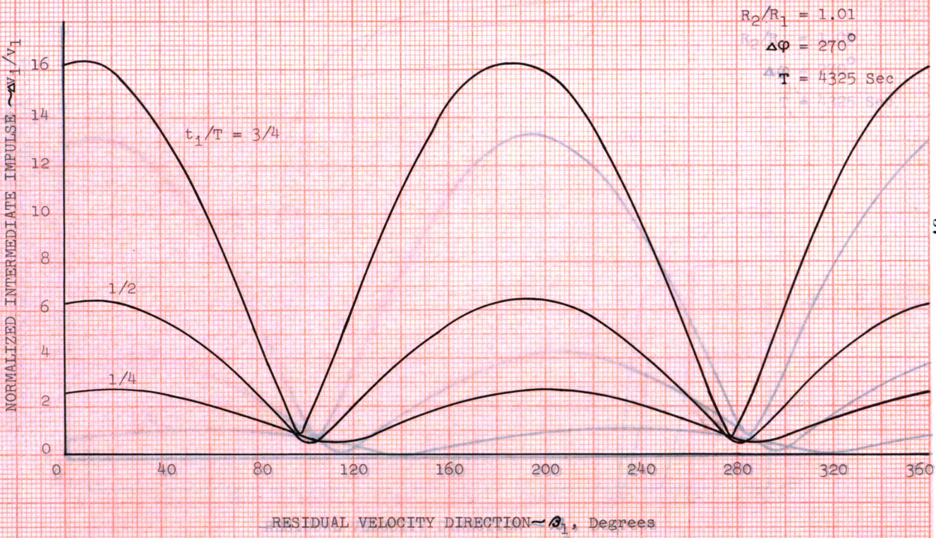


FIG. 4.6—INTERMEDIATE IMPULSE REQUIRED TO RETURN VEHICLE TO NOMINAL TRAJECTORY



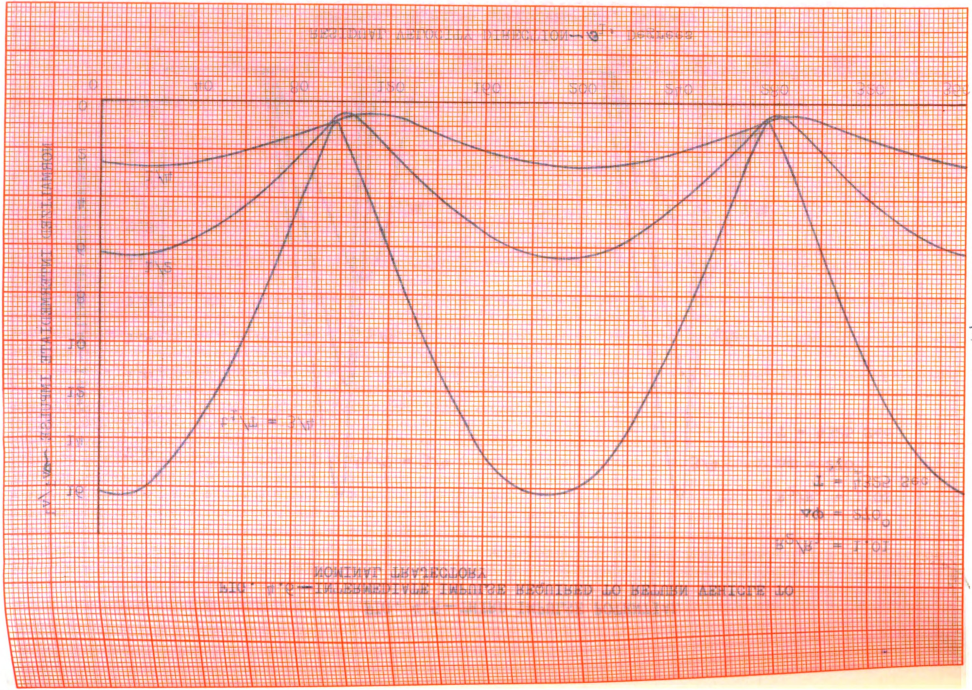
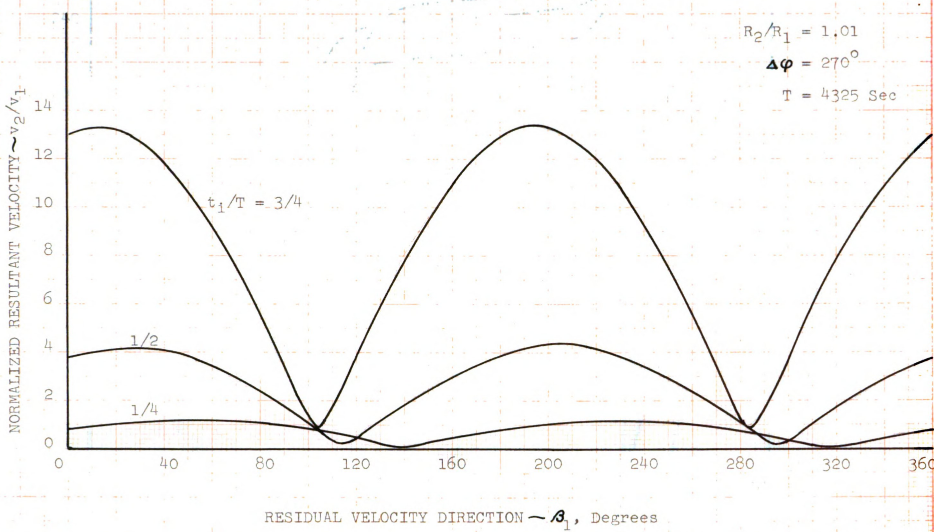


FIG. 4.7 RESULTANT VELOCITY AT END OF TRANSFER

FIG. 4.7—RESULTANT VELOCITY AT END OF TRANSFER



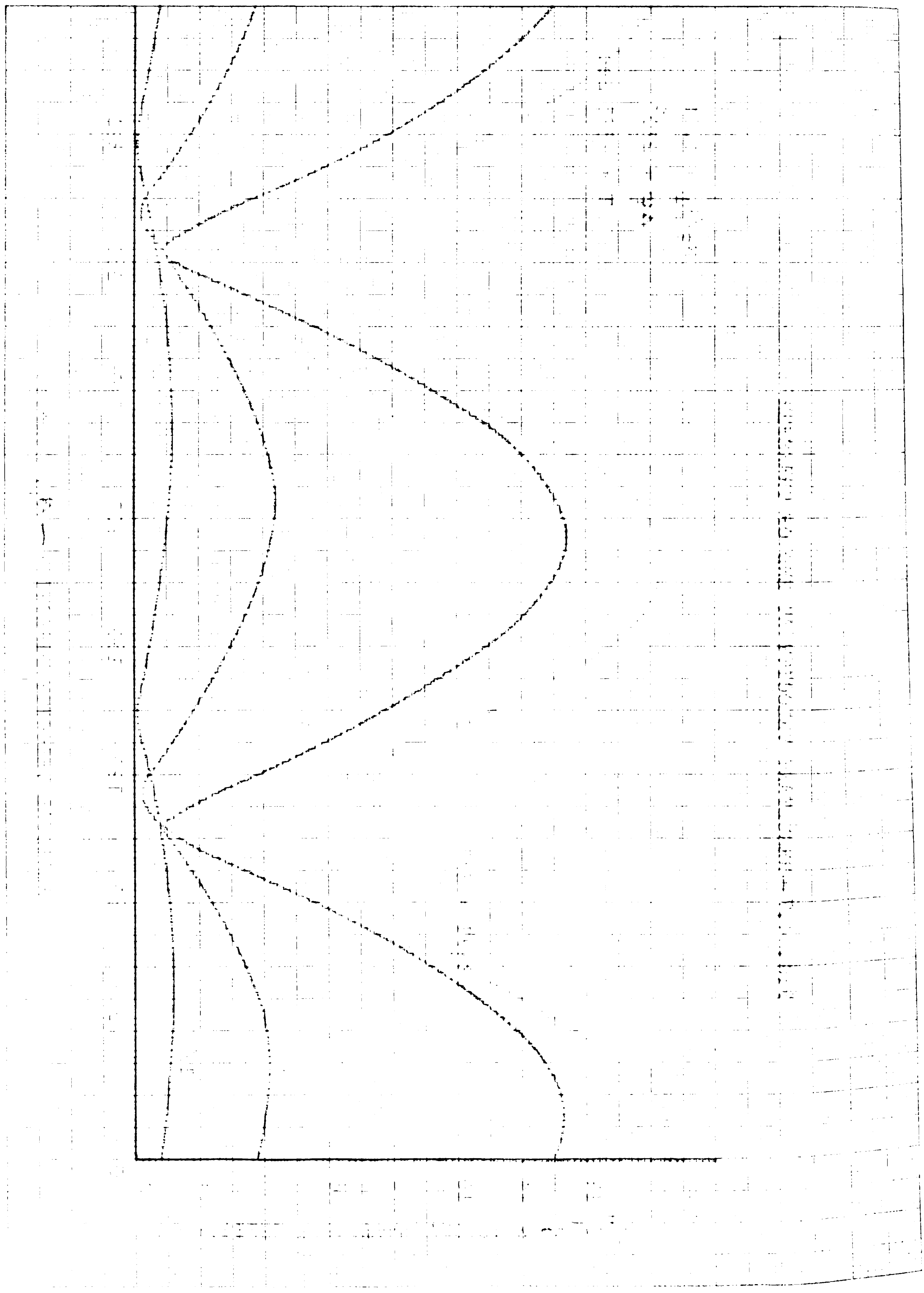
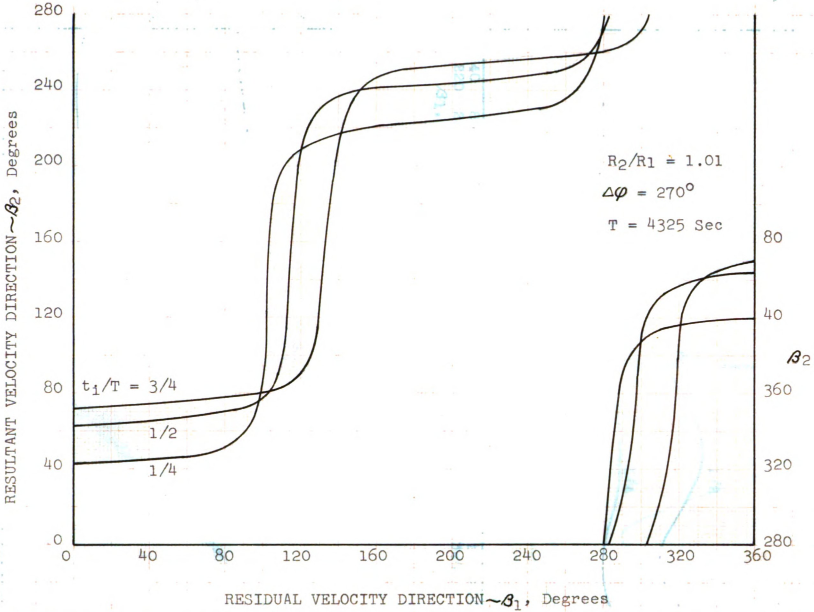


FIG. 4.8—DIRECTION OF RESULTANT VELOCITY AT END OF TRANSFER



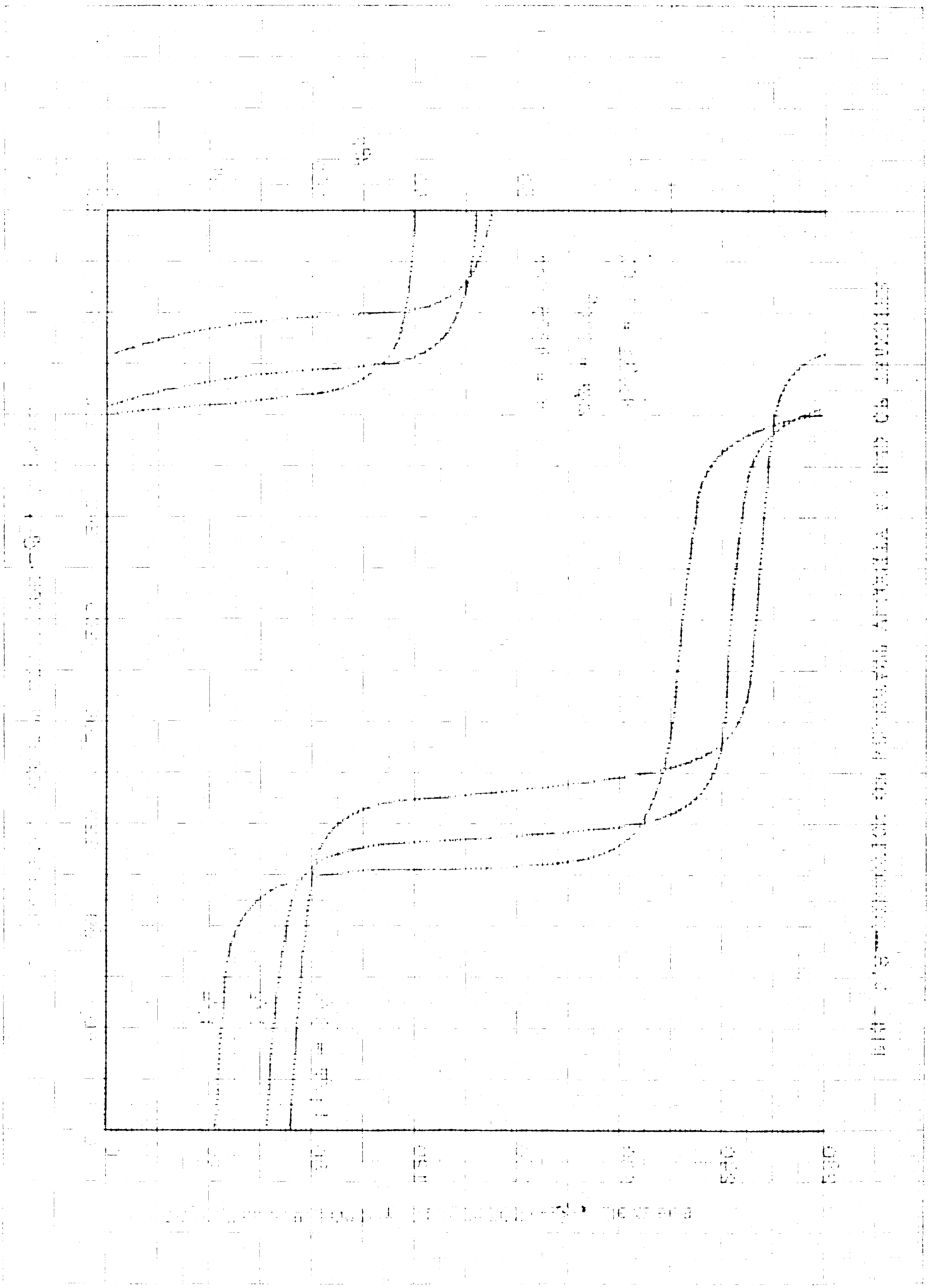


FIG. 4.9—THIRD IMPULSE PARAMETERS

$\Delta\varphi = 45^\circ$; $R_2/R_1 = 1.01$; $T = 700$ Sec

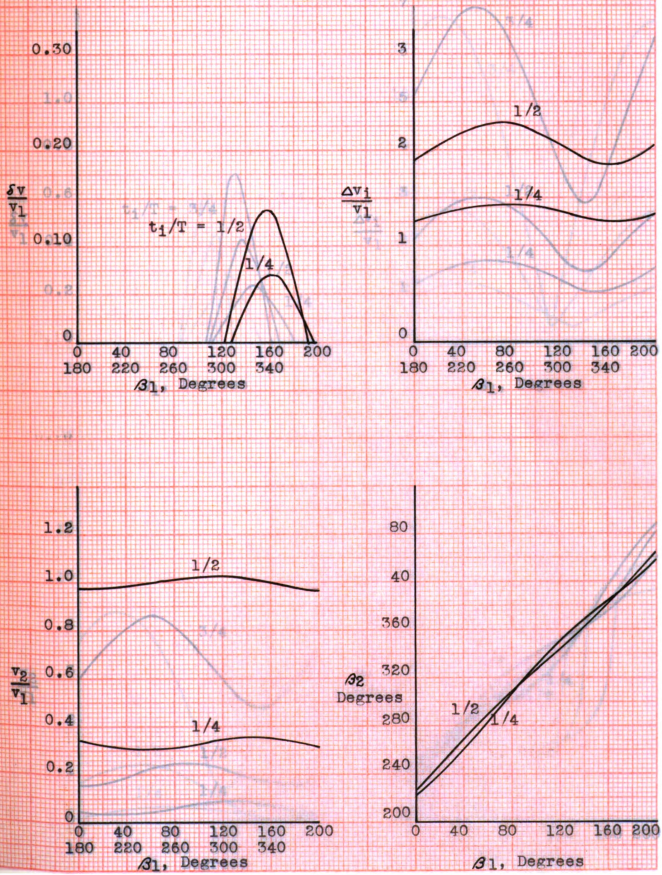


FIG. 4.10—THIRD IMPULSE PARAMETERS

$\Delta\varphi = 90^\circ$; $R_2/R_1 = 1.01$; T = 1417 Sec

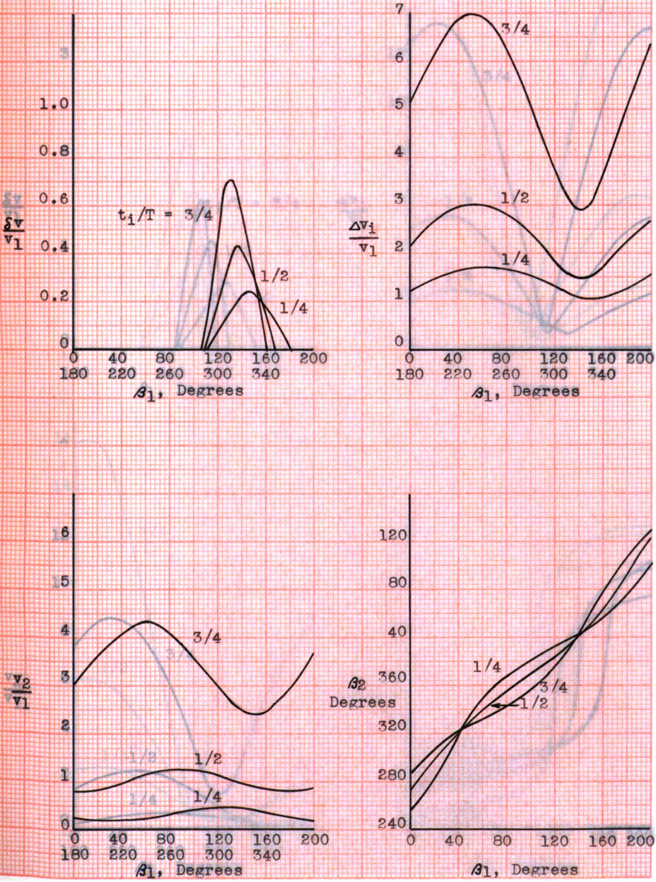


FIG. 4.10—THIRD IMPULSE PARAMETERS

$\Delta\phi = 90^\circ$; $R_2/R_1 = 1.01$; $T = 1417$ Sec

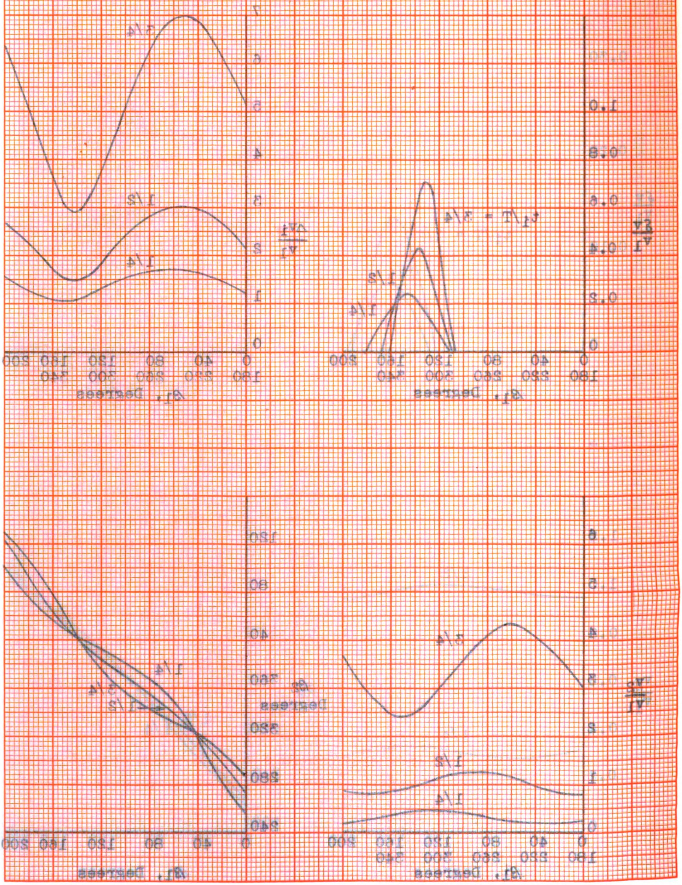


FIG. 4.11—THIRD IMPULSE PARAMETERS

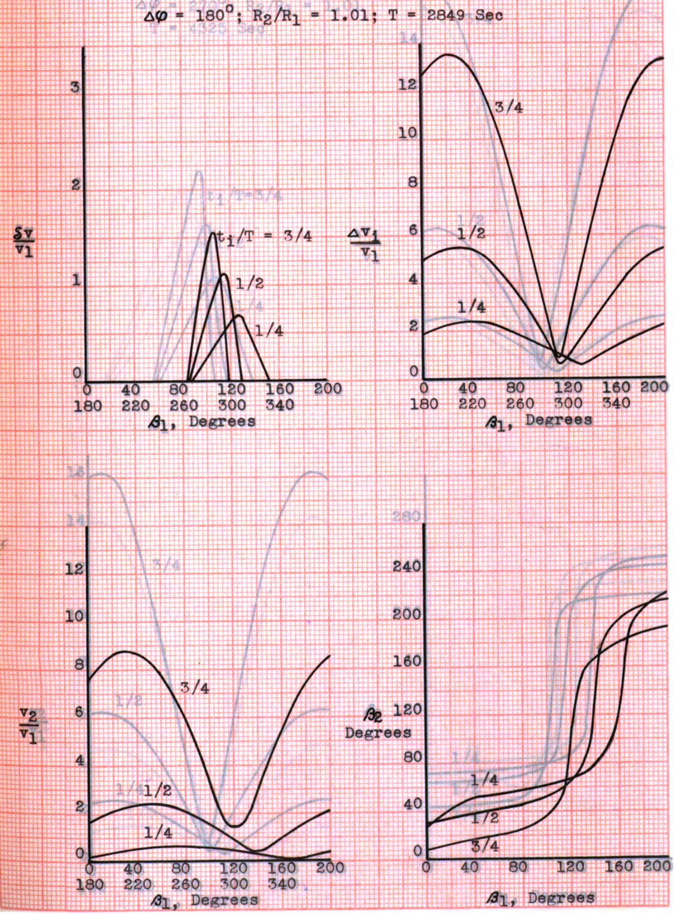


FIG. 4.11—THIRD IMPULSE PARAMETERS

$\Delta\varphi = 180°$; $R_2/R_1 = 1.01$; $T = 2849$ Sec

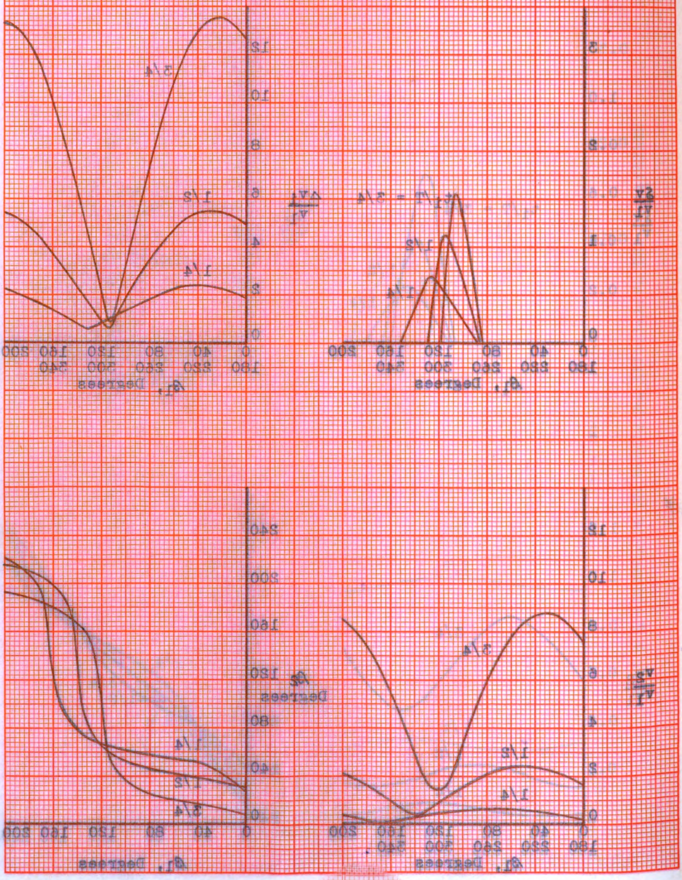


FIG. 4.12—THIRD IMPULSE PARAMETERS

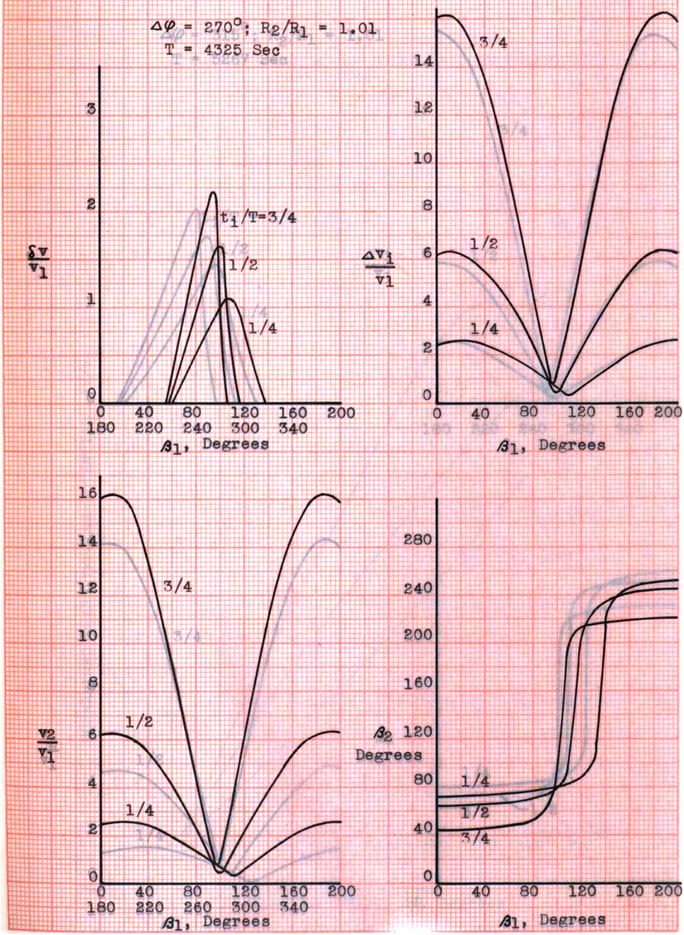


FIG. 4.12—THIRD IMPULSE PARAMETERS

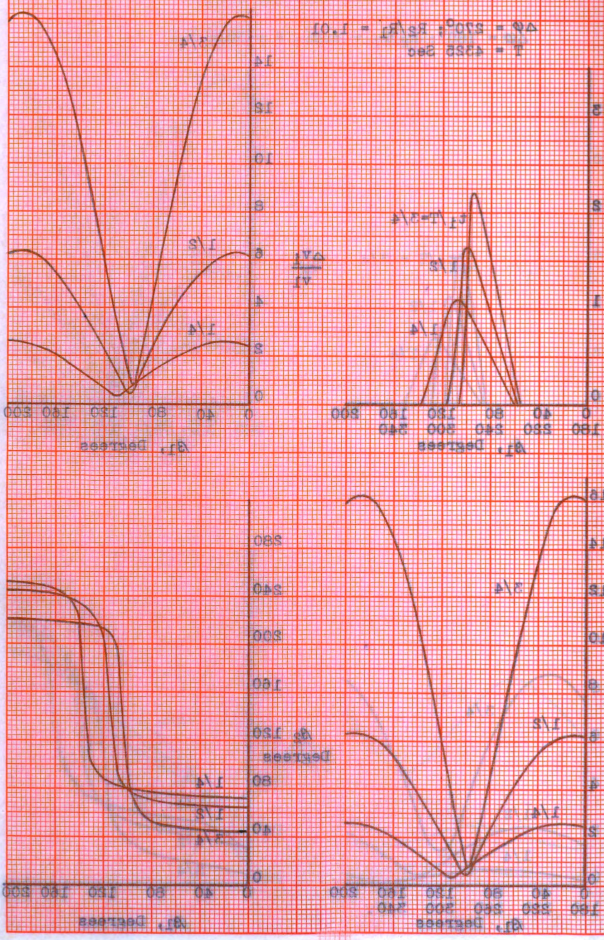
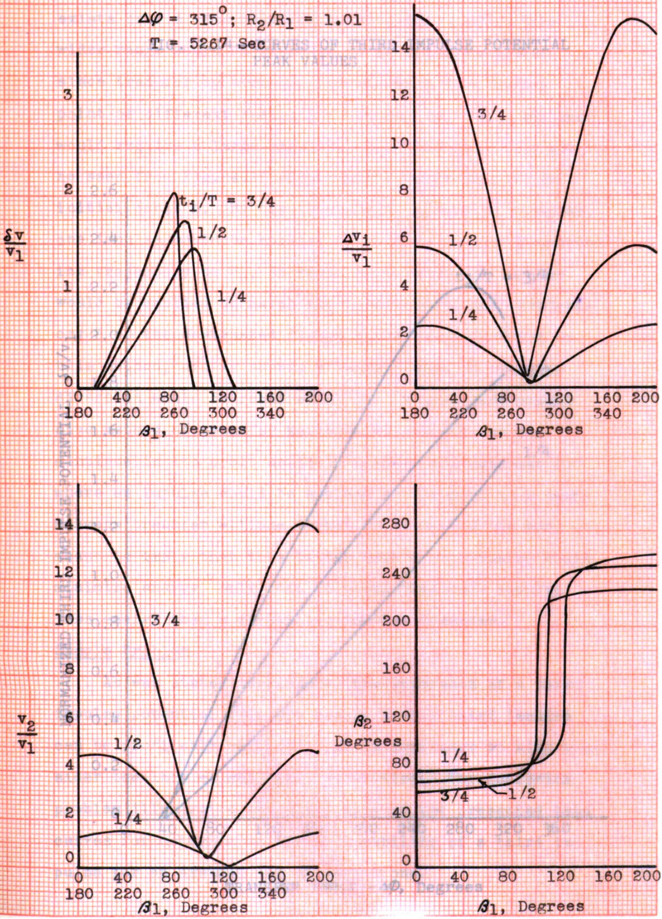


FIG. 4.13—THIRD IMPULSE PARAMETERS



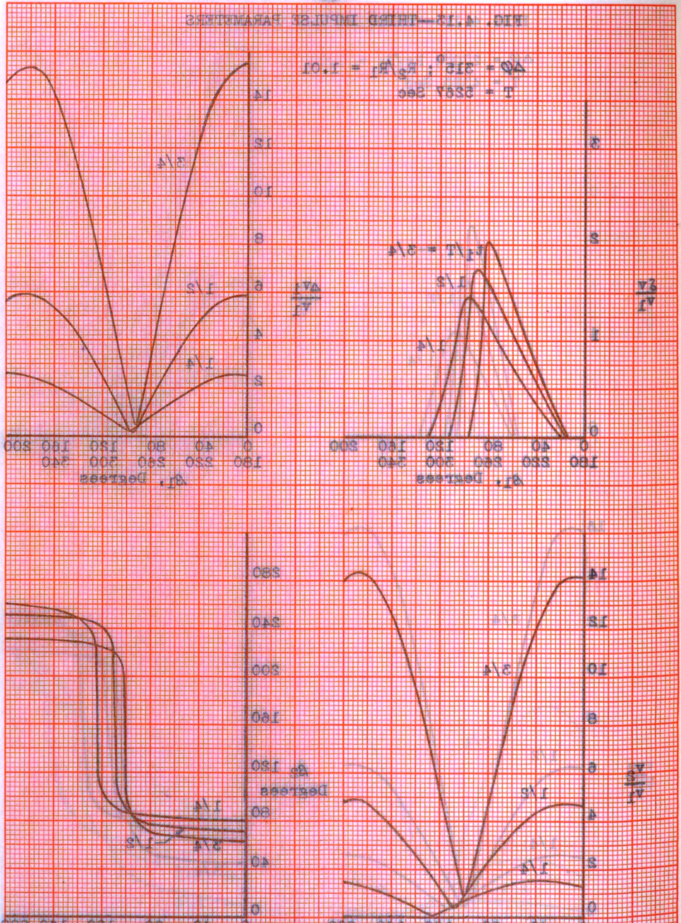


FIG. 4.15—THIRD IMPULSE PARAMETERS
Δφ = 315°; R2/R1 = 1.01
T = 5267 Sec
t1/T = 3/4
1/2
1/4
3/4
1/2
1/4
Δv1/v1
v2/v1
Δφ Degrees
θ1, Degrees
θ1, Degrees
θ1, Degrees
θ1, Degrees

FIG. 4.14—CURVES OF THIRD IMPULSE POTENTIAL PEAK VALUES

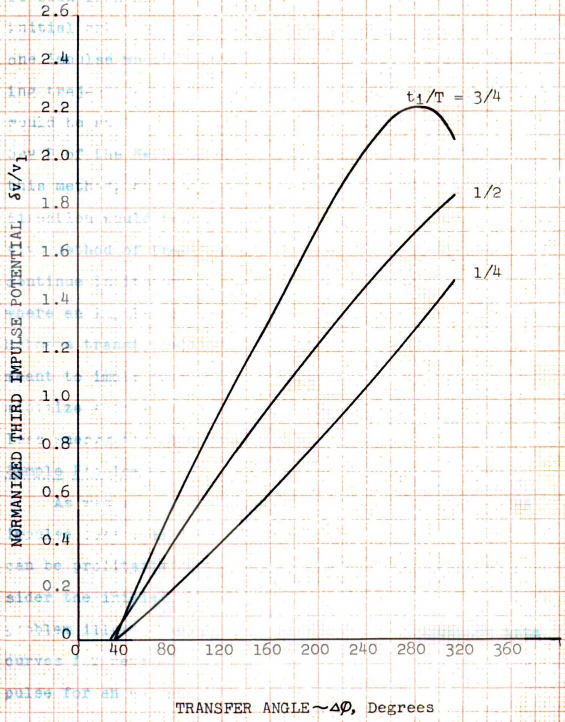
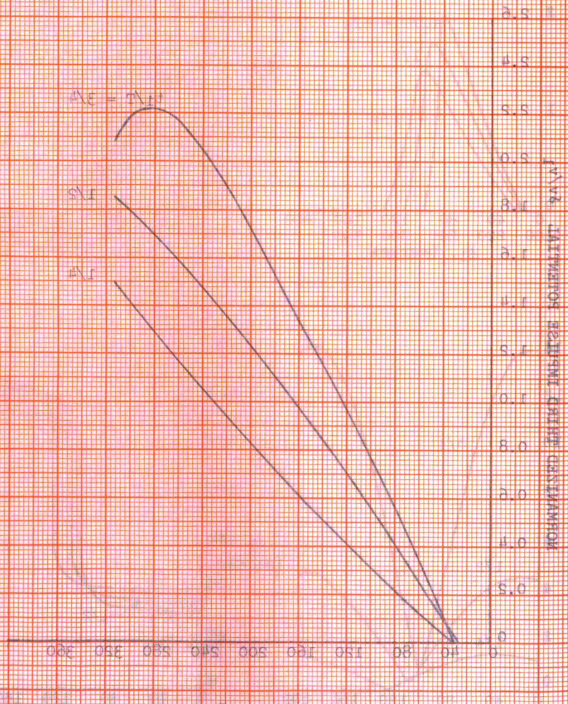


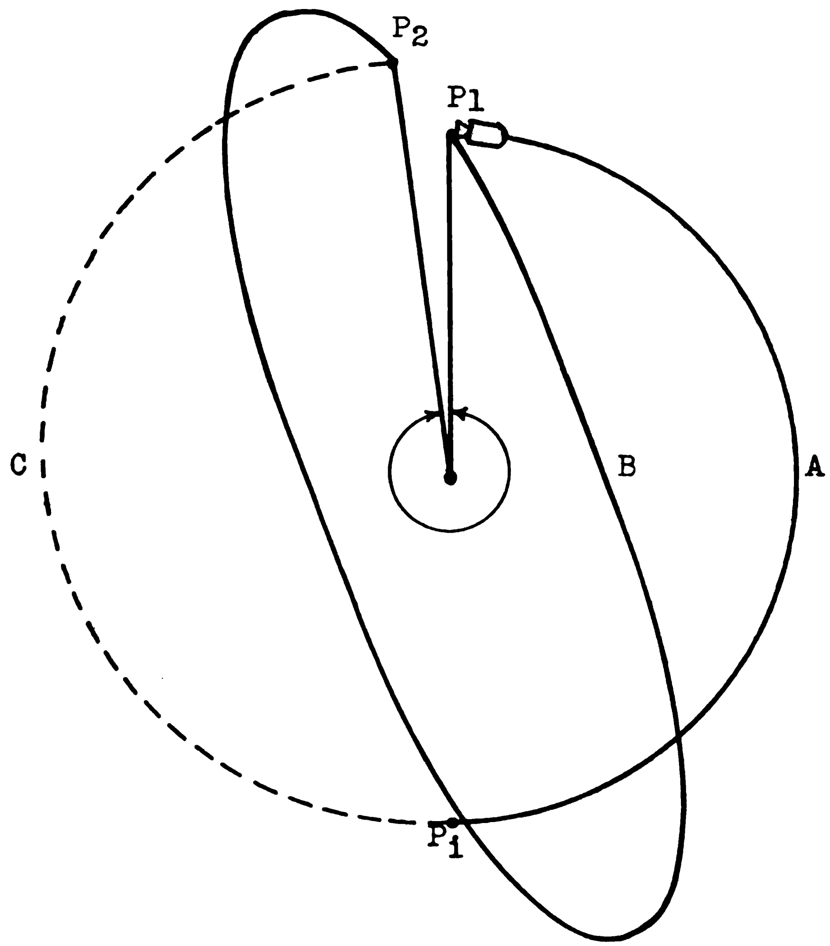
FIG. 4.14—CURVES OF THIRD IMPULSE POTENTIAL PEAK VALUES



exists over a wide range of β_1 when $\Delta\varphi = 315^\circ$. This wider range would be expected as $\Delta\varphi$ approaches 360°, since transfering through a large angle to a nearby point of different radius by applying only two impulses would require a large expenditure of energy. This may be seen from Fig. 4.15 which depicts a vehicle in an initial orbit A. By the two-impulse scheme of transfer, one impulse would be applied at P_1 to establish a coasting trajectory to P_2, at which point a second impulse would be applied to match the target velocity. Trajectory B of the designated figure illustrates transfer by this method, and as shown, a large change in velocity direction would be required at P_1. A much less expensive method of transfer would be to let the vehicle continue in its orbit until it reached a point near P_1 where an impulse could be applied to place it on a near Hohmann transfer ellipse (orbit C) to P_2. It is not meant to imply that this latter method of transfer would minimize energy, only that it would obviously require less energy than a direct two-impulse transfer.

Sample Problem

As was stated previously, the existence of a third impulse potential does not ensure that a third impulse can be profitably utilized. It is necessary to also consider the initial and final velocities. The following problem illustrates the usefulness of the presented data curves for determining the applicability of a third impulse for an assumed set of conditions.



P_1-P_2: TWO IMPULSE TRANSFER

P_1-P_i-P_2: THREE IMPULSE TRANSFER

FIG. 4.15—COMPARISON OF TRAJECTORIES FOR LARGE ANGLE TRANSFER

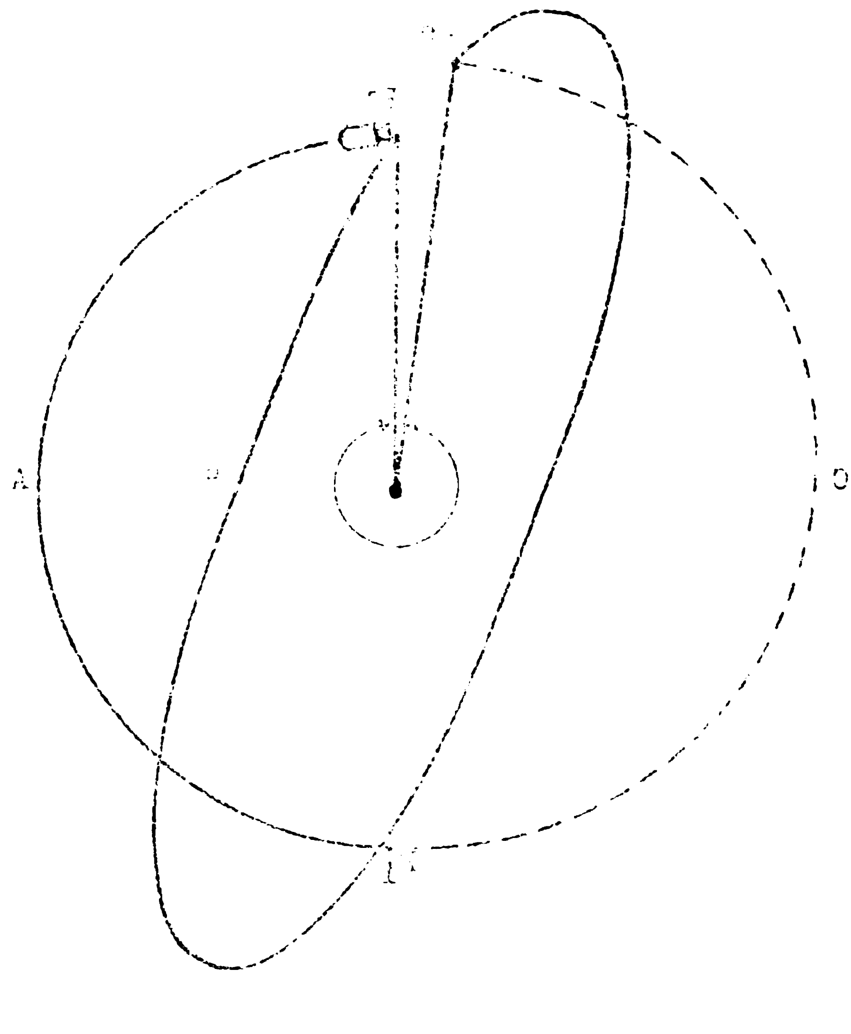


FIG. 4.11 —

Let it be assumed that a vehicle is to be transferred from P_1 to P_2 where the conditions of transfer are as follows:

$R_1 = 22.511 \times 10^6$ ft (300 s mi altitude)

$R_2 = 22.73611 \times 10^6$ ft (342.6 s mi altitude)

$\Delta\varphi = 270^\circ$

$T = 4324.63$ Sec $= 72.06$ min

$v_o = 211$ ft/sec at 120°

$v_f = 253$ ft/sec at 102.2°.

The velocities v_o and v_f are given relative to a coasting axis system as described earlier. A coasting trajectory joining P_1 and P_2 with transit time as noted above would require absolute velocities at these points as follows:

V_1 (Tangential) $= 25{,}132$ ft/sec

V_1 (Radial) $= 0$

V_2 (Tangential) $= 24{,}884$ ft/sec

V_2 (Radial) $= -249$ ft/sec.

A two-impulse rendezvous could be achieved by simply applying impulses to cancel v_o and v_f and would require a characteristic velocity of

$$\Delta V = v_o + v_f = 464 \text{ ft/sec.}$$

In order to examine the applicability of a third impulse, plots of v_i/v_1, v_2/v_1, and β_2 versus β_1 were used to determine the characteristic velocity required for rendezvous via three impulses. Only the case in which the magnitude of v_1 is equal to that of v_o was

examined. Thus, for $v_1 = v_0 = 211$ ft/sec and for various values of β_1, the quantities Δv_1, v_2, and β_2 were found and the characteristic velocity determined from the equation

$$\Delta V = |v_1 - v_0| + |\Delta v_1| + |v_f - v_2| .$$

The solid curves of Fig. 4.16 are plots of ΔV versus β_1 for various times of application of the intermediate impulse. Minimum values of these curves are joined by the dashed curve. On extrapalating by means of this dashed curve, the minimum characteristic velocity required for a three-impulse rendezvous is found to be 200 ft/sec for an intermediate impulse applied at a time between 3/8 and 1/2 T. Thus, by using three instead of two impulses for rendezvous under the stated problem conditions, characteristic velocity is reduced from 464 to 200 ft/sec.

FIG. 4.16—THREE IMPULSE CHARACTERISTIC VELOCITY CURVES FOR SAMPLE PROBLEM

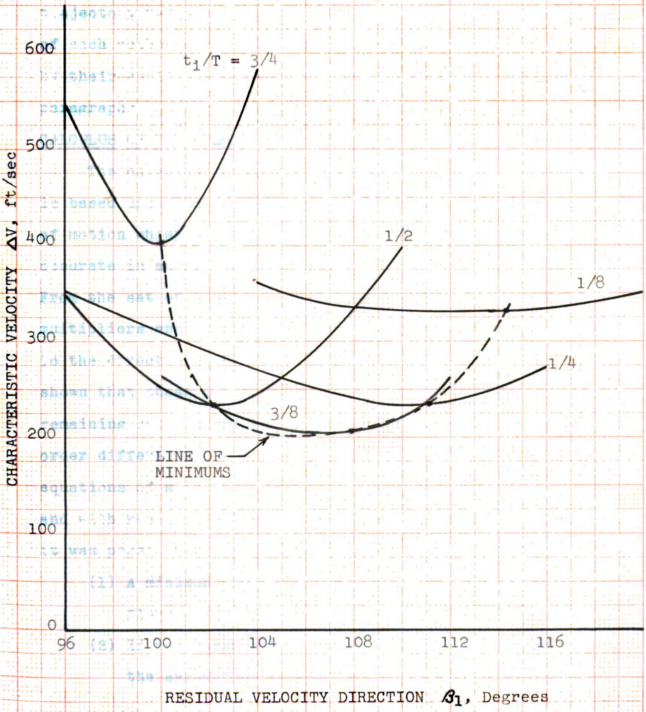
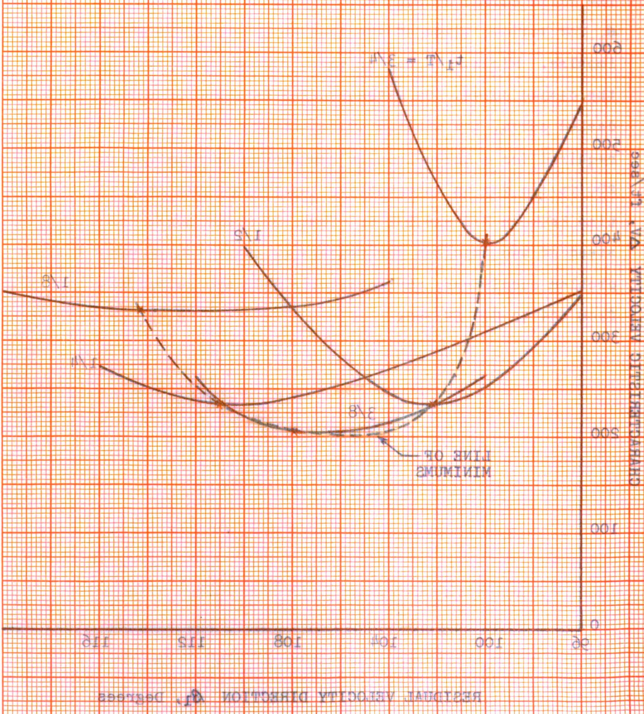


FIG. 4.16—THREE IMPULSE CHARACTERISTIC VELOCITY CURVES FOR SAMPLE PROBLEM



CHAPTER V

SUMMARY AND CONCLUSIONS

The problem of minimizing the energy required to perform a space rendezvous has been analyzed by two different methods: the calculus of variations, and a trajectory perturbation technique. A brief description of each method and a summary of the results obtained by their application are presented in the following paragraphs.

CALCULUS OF VARIATIONS

The calculus of variations analysis (Chapter III) is based upon the use of linearized relative equations of motion which are shown in Appendix A to be reasonably accurate in accounting for the acceleration forces. From the set of Euler-Lagrange equations, two Lagrange multipliers are determined which are identically equal to the direction cosines of the thrust vector. It is shown that these multipliers are independent of the remaining set and can be expressed in terms of second order differential equations identical in form to the equations of motion. As a consequence of this results, and with reservations according to the assumptions made, it was possible to reach the following conclusions:

(1) A minimum energy trajectory contains no arc flown with an intermediate level of thrust.

(2) If the upper bound of thrust is large so that the assumption of velocity impulses is valid,

a minimum energy trajectory is achieved with either two or three impulses. No criterion was established that would determine which of these modes should be used.

TRAJECTORY PERTURBATION

The perturbation technique is based upon determining the effect of perturbing a vehicle relative to a nominal coasting trajectory. If the vehicle were to move along the nominal trajectory it would traverse a path between space terminals P_1 and P_2 in a time T. However, by the perturbation technique, it is given a velocity impulse at P_1 which causes it to deviate from the nominal trajectory. Another impulse is applied at an intermediate time that brings the vehicle back onto the trajectory at P_2 when $t = T$. A third impulse potential is defined as the sum of the relative velocities at P_1 and P_2 (measured with respect to an axis system whose origin is restrained to move along the nominal trajectory) minus the intermediate impulse.

The following results obtained from a parametric study pertaining to the third impulse potential, δv, are noted:

(1) Small variation of the radii ratio R_2/R_1 and transfer time T were found to have only slight effect on δv.

(2) The intermediate impulse Δv_i and the terminal velocity v_2 were found to be directly proportional to the initial perturbing velocity, v_1. Also, it was found that β_2, the direction of v_2, is

independent of v_1.

(3) The primary effect parameters were found to be β_1 (the direction of the disturbing velocity), t_i (time of the intermediate impulse), and $\Delta\varphi$ (the transfer angle). Variation effects are shown in Figs. 4.5-4.14.

(4) A third impulse potential was found to exist for a wide range of conditions, with the greatest potential occurring at large values of $\Delta\varphi$. However, the results suggest that no potential exists for values of $\Delta\varphi$ less than 30^o. For a target in a circular, 300 s mi orbit this would correspond to a rendezvous time of approximately eight minutes.

The investigation reported in this thesis did not determine a complete answer to the rendezvous energy problem by any means. However, it is felt that much insight has been gained and that tools for further investigation have been developed.

CHAPTER VI

SUGGESTIONS FOR FUTURE STUDIES

EXTENSION OF DATA

The data presented in Chapter IV relative to the conditions under which a third impulse potential exists need to be expanded. In particularly, it would be desirable to obtain data for larger values of R_2/R_1. Insofar as transfer time is concerned, it is suggested that the times corresponding to the four transfer ellipse orientations obtained by placing the perigee and apogee alternately at P_1 and P_2 might be of interest.

PRACTICAL COMPUTATIONS

The method used to solve the sample problem of Chapter IV, although illustrating the principal of three impulse application, would not be practical for use in actual satellite interception. A method subject to rapid solution by digital computer would be needed. One method that would meet this requirement can be derived by determining equations to describe the curves relating $\Delta v_i/v_1$, v_2/v_1, and β_2 to β_1, and making use of the ordinary method of maxima-minima determination.

The parameters $\Delta v_i/v_1$, v_2/v_1, and β_2 are described with reasonable accuracy by equations of the form

$$(v_2/v_1),\ (\Delta v_i/v_1) = \frac{A + B \operatorname{Sin}^2 (\beta_1 + \zeta_1)}{C + \operatorname{Sin}^2 (\beta_1 + \zeta_1)} \qquad (6.1)$$

$$\text{Tan}\ (\beta_2 + \zeta_2) = \frac{D\ \text{Sin}\ (\beta_1 + \zeta_1)}{0.5 + \text{Cos}\ (\beta_1 + \zeta_1)} \tag{6.2}$$

where A, B, C, and D are constants for a given value of $\Delta\varphi$ and t_1/T, and the ζ are phase angles. The accuracy of the describing equations is shown in Fig. 6.1 for the case in which $\Delta\varphi = 270^\circ$, and $t_1/T = 1/2$. The solid curves are actual values while the dashed curves were determined by Eqs. (6.1 and 6.2). For a fixed value of $\Delta\varphi$, the constants and phase angles in these equations would depend upon t_1/T; and their mode of dependency should be determinable from plots of the quantities versus t_1/T. It will be assumed that the relationships could be determined in an appropriate form.

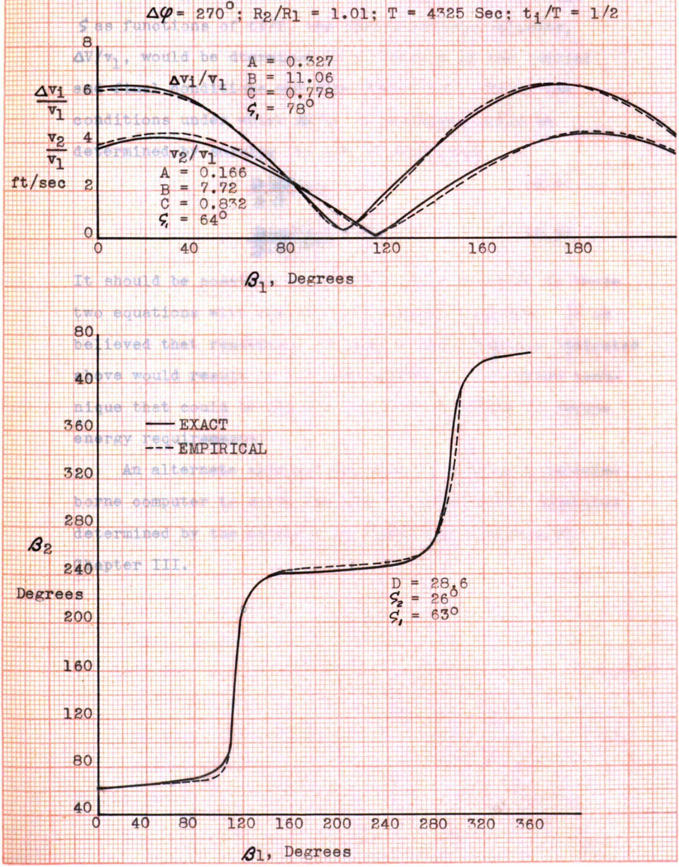
The characteristic velocity required for a three-impulse rendezvous is determined by Eq. (4.1), which in algebraic form is

$$\begin{aligned} \Delta V = {} & [v_o^2 + v_1^2 - 2v_o v_1\ \text{Cos}\ (\beta_o - \beta_1)]^{1/2} \\ & + v_1 + [v_2^2 + v_f^2 - 2v_2 v_f\ \text{Cos}\ (\beta_2 - \beta_f)]^{1/2}. \end{aligned} \tag{6.3}$$

On normalizing with respect to v_1 this equation takes the form

$$\begin{aligned} (\Delta V/v_1) = {} & [(v_o/v_1)^2 + 1 - 2v_o\ \text{Cos}\ (\beta_o - \beta_1)]^{1/2} \\ & + \Delta v_1/v_1 \\ & + [(v_2/v_1)^2 + (v_f/v_1)^2 \\ & \quad - (2v_2 v_f/v_1)\ \text{Cos}\ (\beta_2 - \beta_f)]^{1/2}. \end{aligned} \tag{6.4}$$

FIG. 6.1—CURVES COMPARING EXACT TO EMPIRICAL EQUATION VALUES



By substituting Eqs. (6.3 and 6.4) into this equation after having determined the quantities A, B, C, D and ζ as functions of ti/T, the characteristic velocity, $\Delta V/v_1$, would be dtermined as a function of the initial and final conditions and time ratio ti/T. Thus, the conditions under which $\Delta V/v_1$ is minimized could be determined by solving the set of equations

$$\frac{\partial(\Delta V/v_1)}{\partial \beta_1} = 0 \qquad (6.5)$$

$$\frac{\partial(\Delta V/v_1)}{\partial (t_1/T)} = 0. \qquad (6.6)$$

It should be possible to obtain a rapid solution to these two equations with the aid of a digital computer. It is believed that rendezvous studies in the direction indicated above would result in the development of a guidance technique that could be applied in actual practice to reduce energy requirements.

An alternate approach would be to develop a vehicle-borne computer to solve the set of twenty-seven equations determined by the calculus of variations analysis of Chapter III.

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APPENDIX A

EQUATIONS OF MOTION AND ORBITAL MECHANICS

The basic equations used in the rendezvous energy study are presented in this appendix. No derivations are presented since they may be found either in well known mechanics texts or in current literature. Derivations are presented in the noted references.

ORBITAL MOTION

The polar equations of motion of the center of mass of a satellite orbiting about a spherical earth are

$$\ddot{R} - R\dot{\phi}^2 = -\mu/R^2 + A_R \qquad \text{(A.1)}$$

$$R\ddot{\phi} + 2\dot{R}\dot{\phi} = A_T \qquad \text{(A.2)}$$

where $(-\mu/R^2)$ is the instantaneous gravity force per unit mass, and A_R and A_T are the radial and tangential accelerations due to thrusting forces. In the absence of thrusting forces, the center of mass will describe an ellipse, parabola, or hyperbola accordingly as the sum of the kinetic and potential energy is negative, zero, or positive.

The subject investigation was restricted to an analysis of elliptical motion. The various parameters shown in Fig. A.1 which are used to describe elliptical motion are related as follows:[18]

semimajor axis -

$$a = 1/2\ (R_{apogee} + R_{perigee}) \qquad \text{(A.3)}$$

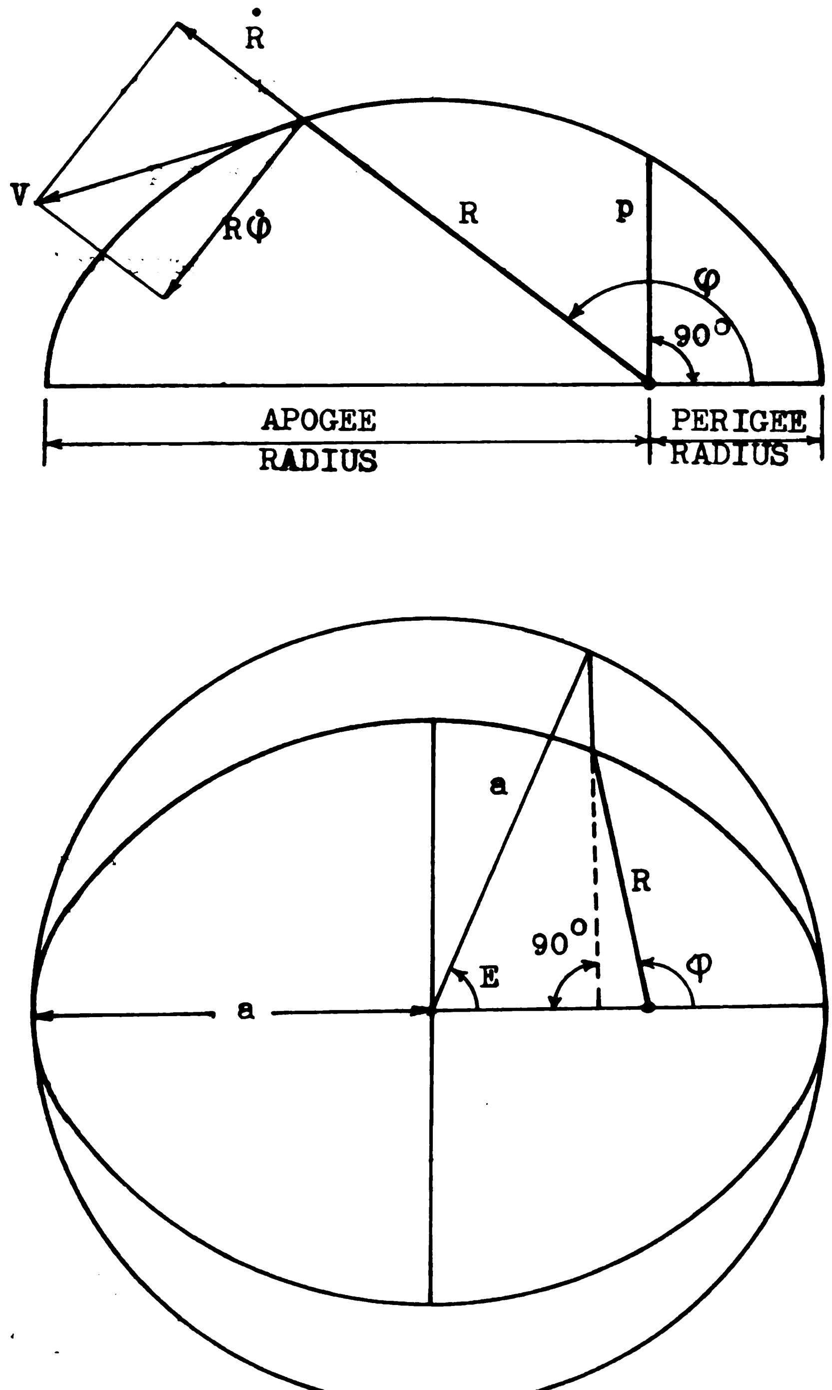


FIG. A.1—ELLIPTICAL ORBIT GEOMETRY

eccentricity -

$$e = \frac{R_{apogee} - R_{perigee}}{R_{apogee} + R_{perigee}} \qquad (A.4)$$

semilatus rectum -

$$p = a(1 - e^2) \qquad (A.5)$$

angular momentum -

$$J = (\mu p)^{1/2} \qquad (A.6)$$

average angular rate -

$$n = \frac{\mu}{a^{3/2}} \qquad (A.7)$$

orbit period -

$$P = 2\pi/n \qquad (A.8)$$

radius -

$$R = \frac{p}{1 + e \operatorname{Cos} \varphi} \qquad (A.9)$$

eccentric anomaly -

$$E = \operatorname{Cos}^{-1} [(a - R)/ea] \qquad (A.10)$$

time from perigee passage -

$$t - T_{perigee} = (E - e \operatorname{Sin} E)/n \qquad (A.11)$$

flight path angle -

$$\gamma = \operatorname{Tan}^{-1}[(e \operatorname{Sin} \varphi)/(1 + e \operatorname{Cos} \varphi)] \qquad (A.12)$$

total velocity -

$$V = [\mu(2/R - 1/a)]^{1/2} \tag{A.13}$$

radial velocity -

$$\dot{R} = V \sin \gamma \tag{A.14}$$

or

$$\dot{R} = (\mu/p)^{1/2} e \sin \varphi \tag{A.15}$$

tangential velocity -

$$V_T = V \cos \gamma \tag{A.16}$$

angular velocity -

$$\dot{\varphi} = J/R^2 \tag{A.17}$$

or

$$\dot{\varphi} = (V \cos \gamma)/R . \tag{A.18}$$

EQUATIONS OF RELATIVE MOTION

In order to facilitate the study of the space rendezvous problem, it is at times advantageous to use relative equations of motion. A convenient axis system is one which has its origin affixed to the orbiting target. Such an axis system is illustrated in Fig. A.2 which shows a right-handed rectilinear system with the negative y axis extending through the center of the earth and the x axis in the orbit plane.

The equations of relative motion are[19,20]

$$\ddot{x} - (y + R)\ddot{\varphi} - 2(\dot{y} + \dot{R})\dot{\varphi} - x\dot{\varphi}^2 + \mu x/\rho^3 = A_x \tag{A.19}$$

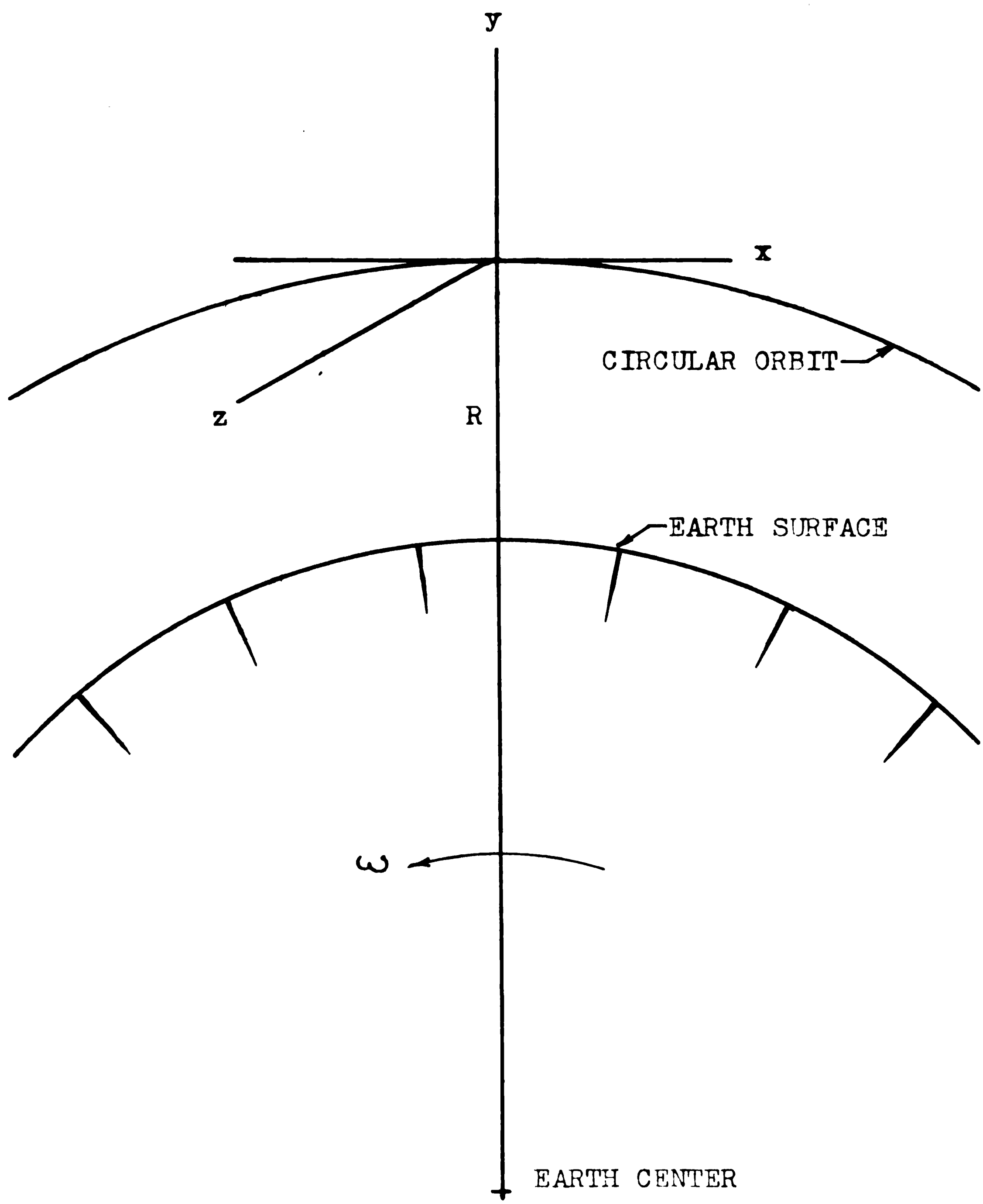
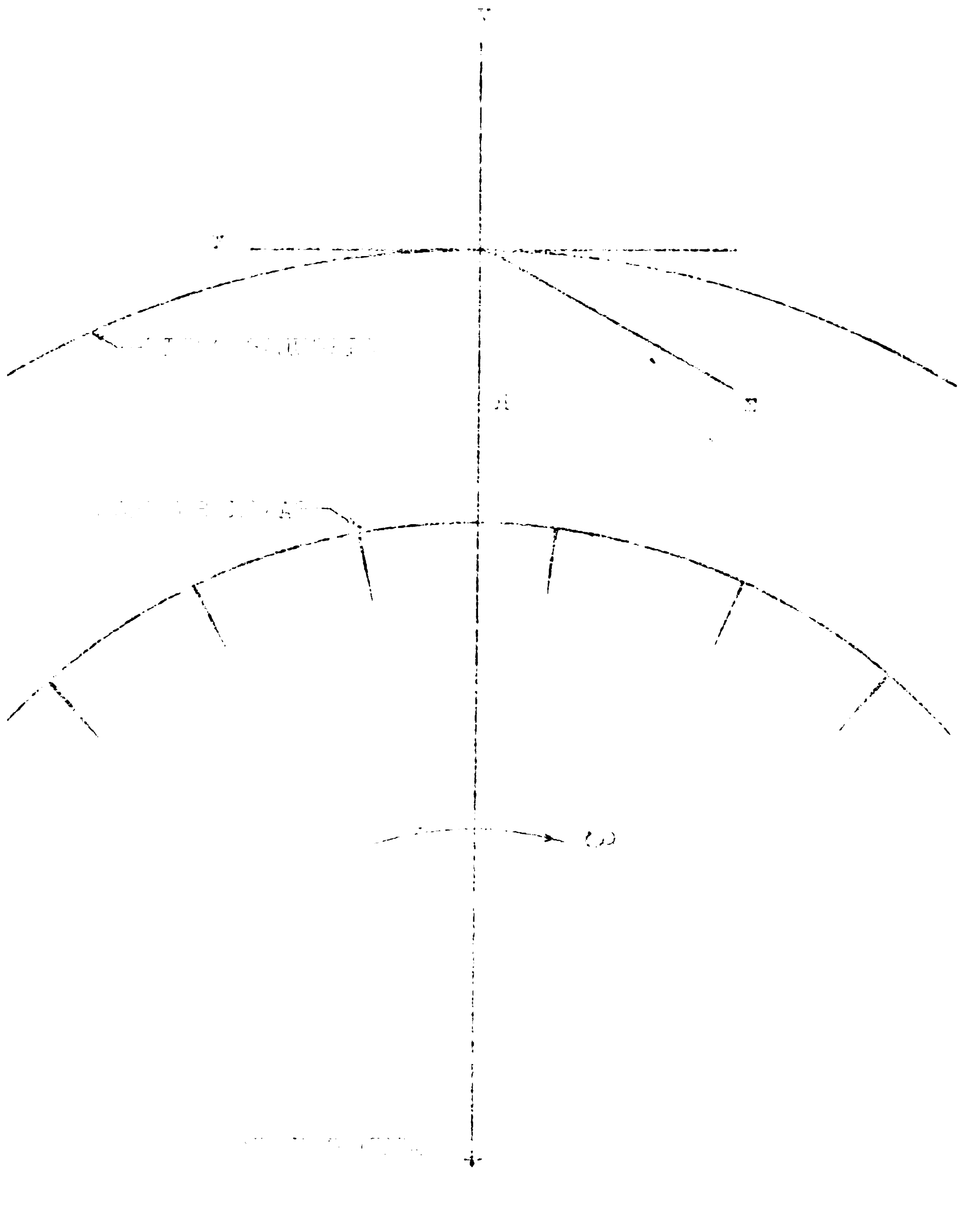


FIG. A.2—RECTANGULAR COORDINATE SYSTEM



$$\ddot{y} + x\ddot{\varphi} + 2\dot{x}\dot{\varphi} + \ddot{R} - (y + R)\dot{\varphi}^2 + \mu(y + R)/\rho^3 = A_y \tag{A.20}$$

$$\ddot{z} + \mu z/\rho^3 = A_z \tag{A.21}$$

where

$$\rho = [x^2 + (y + R)^2]^{1/2}.$$

The quantities A_x, A_y, and A_z are thrust accelerations.

For the case in which the origin of the axis system moves in a circular orbit and relative distances are not too great, the above equations may be linearized. (Actually either of two modes of usage are possible: the origin is affixed to a target moving in a circular orbit, or neither the target nor vehicle are in circular orbits but their motion is expressed with respect to an x-y axis system whose origin moves in a circular orbit.) Assuming a circular orbit leads to a constant value for R and $\dot{\varphi} = \omega$ where, following custom, ω is used to denote a constant value of $\dot{\varphi}$. Then, in order to linearize the equations, ρ^{-3} is expanded as a power series and all terms of second order and higher are dropped to give

$$\rho^{-3} \approx (\mu/R^3)(1-3y/R)$$

Further, for a circular orbit

$$\mu/R^3 = \omega^2$$

so that

$$\rho^{-3} \approx \mu^2(1 - 3y/R) .$$

On substituting this expression into Eqs. (A.19-A.21) and dropping the terms containing y/R which do not cancel, there results

$$\ddot{x} - 2\omega\dot{y} = A_x \tag{A.22}$$

$$\ddot{y} + 2\omega\dot{x} - 3\omega^2 = A_y \tag{A.23}$$

$$\ddot{z} + \omega^2 z = A_z \ . \tag{A.24}$$

When the thrusting accelerations are zero these equations can be readily solved to obtain

$$x = 2[(2\dot{x}_0/\omega) - 3y_0] \text{ Sin } \omega T - (2\dot{y}_0/\omega) \text{ Cos } \omega T$$
$$+ [6y_0 - 3(\dot{x}_0/\omega)]\omega T + x_0 + 2\dot{y}_0/\omega \tag{A.25}$$

$$y = [(2\dot{x}_0/\omega) - 3y_0] \text{ Cos } \omega T + (\dot{y}_0/\omega) \text{ Sin } \omega T$$
$$+ 4y_0 - 2\dot{x}_0/\omega \tag{A.26}$$

$$z = z_0 \text{ Cos } \omega T + (\dot{z}_0/\omega) \text{ Sin } \omega T \ . \tag{A.27}$$

Equations (A.25-A.27) can be used to determine the velocity components that would be required at a given initial position in order to place a vehicle on a coasting path that would intercept a designated target after a specified time. Assuming the target to be at the origin of the axis system, the required components are

$$\dot{x}_0 = \frac{x_0 \text{ Sin } \omega T + y_0[6\omega T \text{ Sin } \omega T - 14(1 - \text{Cos } \omega T)}{\Delta/\omega} \tag{A.28}$$

$$\dot{y}_0 = \frac{2x_0(1 - \text{Cos } \omega T) + y_0(4 \text{ Sin } \omega T - 3\omega T \text{ Cos } \omega T)}{\Delta/\omega} \tag{A.29}$$

$$\dot{z}_0 = \frac{-z_0}{\text{Tan } \omega T} \tag{A.30}$$

where

$$\Delta = 3\omega T \text{ Sin } \omega T - 8(1 - \text{Cos } \omega T).$$

The linearized equations are reasonably accurate provided the relative distance is not too great. For example, assuming distances of $x = y = 50$ statue miles, the discarded terms would amount to an accelerating force of approximately 0.01 ft/sec^2 in the x and y directions and 0.0002 ft/sec^2 in the z direction.

It should be noted that the weak coupling between the z motion and motion in the x-y plane as seen in the exact equations no longer exists in the linearized equations. For this reason, many investigators have chosen to analyze only the more complicated x-y (coplanar) motion with the suggestion that the total motion be determined by analyzing the z motion separately and superposing the results on the x-y motion.

APPENDIX B

NOMINAL TRAJECTORY EQUATIONS

The method of Gauss can be used to determine the elliptical elements of an orbiting body when consecutive values of the radius and arc swept are known with respect to time. Following is a derivation of the necessary equations.[18]

Equation (A.17) can be rearranged to give the areal rate being swept by a radius vector as

$$R\left(R\,\frac{d\varphi}{dt}\right) = 2\,\frac{dA}{dt} = J = (\mu p)^{1/2} \qquad \text{(B.1)}$$

where dA/dt denotes areal rate. On integrating Eq. (B.1) over the time of observation the following equation is obtained

$$A_{sector} = \frac{[(\mu p)^{1/2}]\,T}{2}. \qquad \text{(B.2)}$$

The area of the triangle between radii R_1 and R_2 is given by the equation

$$A_{triangle} = \frac{R_1 R_2 \sin\Delta\varphi}{2} \qquad \text{(B.3)}$$

where $\Delta\varphi$ is the angle between the radii. The method of Gauss depends upon the ratio of these areas, which is

$$\eta = \frac{[(\mu p)^{1/2}]\,T}{R_1 R_2 \sin\Delta\varphi}. \qquad \text{(B.4)}$$

Upon substituting values of R_1, R_2, φ_1, and φ_2 into

Eq. (A.9), two equations result which can be solved to give

$$\frac{p(R_1 + R_2)}{R_1R_2} = 2 + 2e \cos\left(\frac{\varphi_2 + \varphi_1}{2}\right) \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right). \tag{B.5}$$

Through the use of Eqs. (A.9 and A.10), and after several equation manipulations, the relationships

$$e \cos(\varphi_2 + \varphi_1) = \frac{p}{(R_1R_2)^{1/2}} \cos\left(\frac{E_2 - E_1}{2}\right) - \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right) \tag{B.6}$$

$$e \cos\left(\frac{E_2 + E_1}{2}\right) = \cos\left(\frac{E_2 - E_1}{2}\right) - \frac{(R_1R_2)^{1/2}}{a} \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right) \tag{B.7}$$

can be obtained. Equations (B.5) and (B.6) combine to give

$$p = \frac{2R_1R_2 \sin^2(\Delta\varphi/2)}{R_1 + R_2 - 2(R_1R_2)^{1/2} \cos(\Delta\varphi/2) \cos(\Delta E/2)} \tag{B.8}$$

where

$$\Delta\varphi = \varphi_2 - \varphi_1$$

$$\Delta E = E_2 - E_1 \;.$$

On eliminating p from Eq. (B.5) and using the notations

$$a_1 = \frac{(\mu)^{1/2}\, T}{[2(R_1R_2)^{1/2} \cos(\Delta\varphi/2)]^{3/2}}$$

$$b_1 = \frac{R_1 + R_2}{4(R_1R_2)^{1/2} \cos(\Delta\varphi/2)} - 1/2$$

the following relationship can be obtained

$$\eta^2 = \frac{(a_1)^2}{b_1 + \text{Sin}^2\,(\Delta E/4)} . \tag{B.9}$$

Using Eqs. (A.7 and A.11), the expression

$$\frac{(\mu)^{1/2}\,T}{a^{3/2}} = \Delta E - 2e\,\text{Sin}\,(\Delta E/2)\,\text{Cos}\,\left(\frac{E_2 + E_1}{2}\right)$$

is obtained, and on eliminating $e\text{Cos}\,\left(\frac{E_2 + E_1}{2}\right)$ by the use of Eq. (B.6) it is found that

$$\frac{(\mu)^{1/2}\,T}{a^{3/2}} = \Delta E - \text{Sin}\,\Delta E + 2\frac{(R_1R_2)^{1/2}}{a}\,\text{Sin}\,(\Delta E/2)\,\text{Cos}\,(\Delta E/2). \tag{B.10}$$

Another equation involving "a" will be determined so that it can be eliminated. By Eq. (A.10)

$$R_1/a = 1 - e\,\text{Cos}\,E_1$$

$$R_2/a = 1 - e\,\text{Cos}\,E_2$$

from which

$$\frac{R_1 + R_2}{a} = 2 - 2e\,\text{Cos}\,(\Delta E/2)\,\text{Cos}\,\left(\frac{E_2 + E_1}{2}\right). \tag{B.11}$$

Again using Eq. (B.6) to eliminate the term $e\text{Cos}\,\left(\frac{E_2 + E_1}{2}\right)$ and rearranging terms, Eq. (B.11) can be written as

$$1/a = \frac{2\,\text{Sin}^2\,(\Delta E/2)}{R_1 + R_2 - 2(R_1R_2)^{1/2}\,\text{Cos}\,(\Delta E/2)\,\text{Cos}\,(\Delta\varphi/2)} . \tag{B.12}$$

Further, on combining Eq. (B.9) with this equation it is found that

$$1/a = \frac{[2\eta \text{ Sin } (\Delta E/2) \text{ Cos } (\Delta\varphi/2)]^2 R_1R_2}{\mu T^2} . \tag{B.13}$$

Eliminating "a" from Eqs. (B.10) and (B.13) gives

$$\frac{\eta}{(a_1)^2} - \frac{\eta}{(a_1)^2} = \frac{\Delta E - \text{Sin } \Delta E}{\text{Sin}^3 (\Delta E/2)} \tag{B.14}$$

and from Eq. (B.9)

$$\eta = \pm \frac{a_1}{[b_1 + \text{Sin}^2 (\Delta E/4)]^{1/2}} . \tag{B.15}$$

In order to determine which sign should be taken, it is noted that according to Eq. (B.4), η is positive for $\Delta\varphi < 180^o$, and negative for $\Delta\varphi > 180^o$. Therefore, Eqs. (B.14 and B.15) can be combined to give

$$\pm a_1 = [b_1 + \text{Sin}^2 (\Delta E/4)]^{1/2} + \frac{(\Delta E - \text{Sin } \Delta E)}{\text{Sin}^3 (\Delta E/2)} [b_1 + \text{Sin}^2 (\Delta E/4)]^{3/2} \tag{B.16}$$

where the sign preceding a_1 is taken as

$+$ for $\Delta\varphi < 180^o$

$-$ for $\Delta\varphi > 180^o$.

It can also be seen from Eq (B.4) that η is singular for $\Delta\varphi = 180^o$. A separate equation must be determined for this case. For $\Delta\varphi = 180^o$, Eqs. (B.10 and B.12) reduce to

$$a^{3/2} = \frac{(\mu)^{1/2} T}{\Delta E - \text{Sin } \Delta E} \tag{B.17}$$

and

$$a = \frac{R_1 + R_2}{2 \operatorname{Sin}^2 (\Delta E/2)} . \tag{B.18}$$

Whence

$$(\mu)^{1/2} T[2/(R_1 + R_2)]^{3/2} = \frac{\Delta E - \operatorname{Sin} \Delta E}{\operatorname{Sin}^3 (\Delta E/2)} . \tag{B.19}$$

APPENDIX C

TRAJECTORY PERTURBATIONS USING RELATIVE EQUATIONS

The method described in this appendix can be used to determine the parameters comprising the third impulse potential of Chapter IV, i.e. v_1, v_2, and β_2. The same type of perturbation technique is used to examine the effect of deviations from a nominal coasting trajectory due to a disturbing velocity, v_1, applied at the initial space terminal. However, relative equations are used instead of the orbital equations previously employed.

Equations (A.19 and A.20), which are exact, are used to express the relative motion. Since the coefficients of these equations are functions of R and φ, Eqs. (A.1 and A.2), with $A_R = A_T = 0$, are also needed.

Initial conditions $\dot{x}_0$ and $\dot{y}_0$ are determined as components of the selected value of $\bar{v}_1$, and since the vehicle is assumed to be at the origin before it is perturbed, $x_0 = y_0 = 0$. Initial conditions for the R-equations are determined from the nominal coasting trajectory as explained in Appendix B. Having determined the initial conditions, the equations can be integrated simultaneously from $t = 0$ to $t = t_1$ to determine the position and velocity just prior to the intermediate impulse. It is next necessary to determine values of

$\dot{x}$ and $\dot{y}$ which would reduce x and y to zero in the remaining time, $(T - t_1)$. The problem is a boundary value problem in which the initial and final positions are known and the initial velocity is sought.

The following described technique, employing a system of adjoint equations and iterative integration, is used to determine the initial velocity.

The second order x-y equations are replaced by a set of first order equations by making the substitutions

$$x = y_1; \quad y = y_2; \quad \dot{x} = y_3; \quad \dot{y} = y_4$$

in Eqs. (A.19 and A.20). The resulting set of first order equations are as follows:

$$\dot{y}_1 = y_3 \tag{C.1}$$
$$\dot{y}_2 = y_4 \tag{C.2}$$
$$\dot{y}_3 = (y_2 + R)\ddot{\varphi} + 2(y_4 + \dot{R})\dot{\varphi} + y_1\dot{\varphi}^2 - \mu y_1/\rho^3 \tag{C.3}$$
$$\dot{y}_4 = - y_1\ddot{\varphi} - 2y_3\dot{\varphi} - \ddot{R} + (y_2 + R)\dot{\varphi}^2 - \mu(y_2 + R)/\rho^3 \tag{C.4}$$

where $\rho = [y_1 + (y_2 + R)^2]^{1/2}$.

In order to clarify notations, a new variable τ is introduced to denote time; and a solution is obtained for the time span $\tau = 0$ to τ_f, where $\tau_f = T - t_1$. At the initiation of solution, the known initial and final conditions are

$$y_1(\tau=0) = y_1(t=t_1); \quad y_2(\tau=0) = y_2(t=t_1);$$
$$y_1(\tau=\tau_f) = y_2(\tau=\tau_f) = 0.$$

The immediate objective is to determine $y_3(\tau=0)$ and $y_4(\tau=0)$. As a first step, trial values are selected for these sought parameters. (It was found that a good first guess could be determined by Eqs (A.28 and A.29) which are solutions of the linearized x-y equations.) Denoting the trial solution of Eqs. (C.1-C.4) as y_1^*, let

$$y_i(\tau) = y_i(\tau) - y_i^*(\tau). \tag{C.5}$$

It is desired to make $\delta y_i(\tau_f)$ equal to or less than some selected value.

Taking the derivative of Eq. (C.5) and using the notation

$$\delta \dot{y}_i = g_i(y_j); \quad j = 1,2,3,4,$$

where the g_i denote the functions of Eqs. (C.1-C.4), gives

$$\delta \dot{y}_i = g_i(y_j) - g_i^*(y_j) = \delta g_i(y_j)$$

or

$$\delta \dot{y}_i = \delta g_i = \frac{\partial g_i}{\partial y_i} y_i; \quad i = 1,2,3,4 . \tag{C.6}$$

Using matrix notation, Eq. (C.6) becomes

$$\delta \dot{y}_i = A \, \delta y_i \tag{C.7}$$

where A has the form

$$\begin{pmatrix} a_{11} & \cdots & a_{14} \\ \vdots & & \vdots \\ a_{41} & \cdots & a_{44} \end{pmatrix}$$

The matrix coefficients are determined by carrying out the operations denoted by Eq. (C.6) and are

$$a_{11} = a_{12} = a_{14} = a_{21} = a_{22} = a_{23} = a_{33} = a_{44} = 0$$

$$a_{13} = a_{24} = 1$$

$$a_{31} = \dot{\varphi}^2 - \mu(\rho^{*2} - 3y_1^{*2})/\rho^{*5}$$

$$a_{32} = \ddot{\varphi} + 3\mu y_1^*(y_2^* + R)/\rho^{*5}$$

$$a_{34} = 2\dot{\varphi}$$

$$a_{41} = -\ddot{\varphi} + 3\mu y_1^*(y_2^* + R)/\rho^{*5}$$

$$a_{42} = \dot{\varphi}^2 - \mu[\rho^{*2} - 3(y_2^* + R)^2]/\rho^{*5}$$

$$a_{43} = -2\dot{\varphi}$$

where the star denotes evaluation along the trial trajectory.

The next step is to form the adjoint system of equations which are defined as

$$-\dot{x}_i = A' x_i \tag{C.8}$$

where A' is the transpose of A. Goodman and Lance[21] showed the application of Green's Theorem to the adjoint equations to derive the relationship

$$\sum_{i=1}^{4} x_i(\tau_f)\delta y_i(\tau_f) - \sum_{i=1}^{4} x_i(0)\delta y_i(0) = 0 \; . \tag{C.9}$$

This equation expresses a relationship between the end values of δy_i and the adjoint equation parameters x_i. No restrictions have been placed on the end conditions of the x_i. Thus, various sets of end conditions can be selected that will allow a set of equations to be derived from Eq. (C.9) that can be used to determine initial values of δy_i when the final values are known.

Taking first the initial conditions $[x_1(\tau_f) = 1, x_2(\tau_f) = x_3(\tau_f) = x_4(\tau_f) = 0]$, Eqs. (C.9) and the R-equations are integrated in reverse time to get $x_{31}(0)$ and $x_{41}(0)$. The second subscript denotes the value of $x_3(0)$ and $x_4(0)$ for the selected initial condition $x_1(\tau_f) = 1$. On substituting $x_{31}(0)$ and $x_{41}(0)$ into Eq. (C.9) and noting that $y_1(0) = y_2(0) = 0$, the following equation is obtained

$$\delta y_1(\tau_f) = x_{31}(0)\delta y_3(0) + x_{41}(0)\delta y_4(0) \quad . \tag{C.10}$$

In the same manner the set of conditions $[x_1(\tau_f) = 0, x_2(\tau_f) = 1, x_3(\tau_f) = x_4(\tau_f) = 0]$ are used to obtain

$$\delta y_2(\tau_f) = x_{32}(0)\delta y_3(0) + x_{42}(0)\delta y_4(0) \quad . \tag{C.11}$$

Equations (C.10 and C.11) are solved for $\delta y_3(0)$ and $\delta y_4(0)$ and these delta quantities are used to determine a better guess for the initial velocities of a new trial solution. New trial values are determined by Eq. (C.5) as

$$[y_i^*(0)]_{new} = [y_i^*(0)]_{old} + \delta y_i(0); \quad i = 3,4. \tag{C.12}$$

The process is repeated until the miss distance is equal to or less than the selected value. As applied to the rendezvous energy study, only one correction was found to be sufficient to reduce the miss distance to less than 10 ft in the usual case.

Upon determining an acceptable solution, the trajectory between P_1 and P_2 is known and the following

velocities have been determined:

$$\dot{x}(t=t_i)\,;\ \dot{y}(t=t_i)\,;\ \dot{x}(\tau=0)\,;\ \dot{y}(\tau=0)\,;\ \dot{x}(\tau=\tau_f)\,;\ \dot{y}(\tau=\tau_f)\,.$$

Thus, the intermediate impulse is found as

$$\Delta v_i = \{[\dot{x}(\tau=0) - \dot{x}(t=t_i)]^2 + [\dot{y}(\tau=0) - \dot{y}(t=t_i)]^2\}^{1/2}. \tag{C.13}$$

The final velocity is

$$v_2 = \{[\dot{x}(\tau_f)]^2 + [\dot{y}(\tau_f)]^2\}^{1/2} \tag{C.14}$$

and its direction is determined by the equation

$$\beta_2 = \mathrm{Tan}^{-1}[\dot{y}(\tau_f)/\dot{x}(\tau_f)]. \tag{C.15}$$