OBSTRUCTION AND EXISTENCE FOR TWISTED KÄHLER-EINSTEIN METRICS AND CONVEXITY

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A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Mathematics–Doctor of Philosophy

2013

ABSTRACT

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Let $L \to X$ be an ample holomorphic line bundle over a compact Kähler manifold (X, ω_0) with $c_1(L)$ represented by the Kähler form ω_0. Given a semi-positive real $(1,1)$ form η representing $-c_1(K_X \otimes L)$, one can ask whether there exists a Kähler metric $\omega \in c_1(L)$ that solves the equation $Ric(\omega) - \omega = \eta$. We study this problem by twisting the Kähler-Ricci flow by η , that is evolve along the flow $\dot{\omega}_t = \omega_t + \eta - Ric(\omega_t)$ starting at ω_0. We prove that such a metric exists provided $\omega_t^n \geq K\omega_0^n$ for some $K > 0$ and all $t \geq 0$. We also study a twisted version of Futaki's invariant, which we show is well-defined if η is annihilated under the infinitesimal action of $\eta(X)$, in particular η is $Aut_0(X)$ invariant. Finally, using Chens ϵ-geodesics instead, we give another proof of the convexity of $\mathcal{L}_\omega$ along geodesics, which plays a central in Berman's proof of the uniqueness of critical points of $\mathcal{F}_\omega$.

ACKNOWLEDGMENTS

I would like to thank my advisor Prof. Xiaodong Wang for all the help I received over the years.

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Introduction

The study of extremal Kähler metrics has generated a lot of work. A major theme centers around the equivalence of special metrics and stability in various senses. In the Kähler setting the manifold version, conjectured by [Y1], [Tian00] and [Do] has taken time to handle. In one direction, when $Aut(X)_0$ is trivial, a refinement of an argument, due to [Sto] of [Do5] shows existence of a constant scalar curvature metric in $c_1(L)$ implies (X, L) is K-stable. See [Ber12] for generalizations to Fano varieties and other improvements: conditions on the group of automorphisms Aut(X) is dropped while the constant scalar curvature assumption is strengthened to admitting Kähler-Einstein metric. For the correct notion of stability there have been several candidates.

While existence of Kähler-Einstein metrics in the $C_1(-K_X)$ positive case comes with obstructions, existence has been shown to be equivalent to properness of J_ω functional on $\mathcal{H}_\omega$, this is analytic stability [Ti97]. Various notions of stability have been introduced by Yau, Tian, Donaldson and others, and progress to various degrees have been made. [Do3] has introduced a notion of B-stability from which existence of Kähler-Einstein metrics was deducible granted additional hypotheses some of which are removable. In [Sz1] on a Fano manifold X under the additional assumptions that both Riemann curvature tensor and Mabuchi functional are bounded below along Kähler-Ricci flow on X, it is shown K-polystability is enough to obtain the existence of a Kähler-Einstein metric. Recently [CDS] and [Ti13] have given solutions of Yau-Tian-Donaldson conjecture for Kähler-Einstein metrics.

Study of the twisted case appear in various settings see [Sto09], [SzCo], [Kel]. This was preceded by [Fi]. See also [Ber10] and [Bo] for more recent work. In [Sto09] a moment map description of constant scalar curvature equation (cscK) $S(\omega)-\overline{S}=\Lambda_\omega\alpha$ is available when α is a symplectic form, and its shown there is an obstruction and a stability condition in terms of the Ross-Thomas polynomial for the equation. [Do4] showed scalar curvature comes up as an equivariant moment map for the action of $Symp(B,\omega)$ on the space of integrable complex structures $\mathcal{J}$ .The second term $\Lambda_\omega\alpha$ in the twisted equation can be viewed as an equivariant moment map of the action of $Symp(B,\omega)$ on the space of diffeomorphism $f : B \to M$, $\mathcal{M}$, due to [Do7]. The full twisted cscK equation comes as a moment map for the diagonal action of $Symp(B)$ on a new space $\mathcal{S}\subset\mathcal{M}\times\mathcal{J}$ [Sto09]. From these considerations Stoppa is motivated to introduce twisted K-energy a quantity which we use in arguments below. Further in this connection, provided that η is annihilated under the infinitesimal action of $\eta(X)$, the Futaki type invariant for the twisted equation that we study below, can be shown to be well-defined. So in certain situations the equation in our setting does come with classically motivated obstructions.

We also study existence in the twisted setting below, and inspired by [Pa], we establish an existence result in the twisted setting under similar assumptions:

Proposition 1. *Let $L \to X$ be holomorphic ample line bundle polarizing (X,ω) ($c_1(L)=\omega$). Prescribe $0\leq\eta\in -c_1(K_X\otimes L)$ then there is a Kähler metric such that $\omega^{'}\in c_1(L)$ solving the twisted Kähler Einstein equation $Ric(\omega)-\omega=\eta$ provided ω_t deforms according to the twisted Kähler-Ricci flow starting at ω while satisfying the uniform estimate $\frac{\omega_t^n}{\omega^n}\geq K$ for $t\in[0,\infty)$ and some constant $K>0$.*

Recalling that $\eta(X)$ is the lie algebra of holomorphic vector fields on X, the following holds

Proposition 2. *When η is annihilated under the infinitesimal action of $\eta(X)$ the corresponding Futaki-type invariant for the twisted equation is well defined.*

So as one expects from the corresponding monge-ampere equations, which are not solvable in general, the twisted Kähler-Einstein equation comes with obstructions.

Moving in another direction, the problem of existence of smooth geodesics in the space of Kähler metrics and their properties are useful for the study of special metrics [Do6]. Using various methods: continuity method [Chen00], quantization [PS] only the existence of geodesics with $C^{1,1}$ regularity have been established. On the other hand one can obtain C^0 regularity geodesics directly by an envelope construction see [Bo]. Even with this weaker result progress on Bando-Mabuchi like theorems can still be made see [Ber10a], [Bo]. Despite the existence of smooth geodesics in the space of Kähler metrics being considered a dubious problem(they don't exist in general see [LV] and $C^{1,1}$ regularity is the best you can expect in general see [TL]), it morally clarifies the role of convexity in the infinite dimensional analysis. In fact with these considerations various generalized Moser Trudinger type inequalities are obtainable. [PSSW] have verified a conjecture of Tian that on a Kähler-Einstein manifold (X, ω_{KE}) properness of $F_{\omega_{KE}}$ on $\mathcal{H}_{\omega_{KE}}(X)$ can be upgraded to coerciveness with respect to $J_{\omega_{KE}}$. In a similar direction on a integral Kähler manifold with fixed smooth volume form using geodesics, Bergmann kernel asymptotics and convexity properties of $\log K_{\phi_t}$ a moser trudinger inequality conjectured by Aubin is established in [BeBo12], although in the Kähler-Einstein setting this is the Moser Trudinger inequality first proved by [DT]. It might be worth checking if the Moser-Trudinger inequality corresponding to the coercive estimate can be obtained using these considerations in the Kähler-Einstein setting, and naturally the next step would be to see if the quantitative versions holds beyond the Kähler-Einstein setting.

An important feature in analyzing the Bando Mabuchi type theorem in [Ber10a], [Bo09], [Bo] is the convexity of the $\mathcal{L}$ functional. Below we study this in a special case and then in general and obtain that it is convex along geodesics (in the sense of X.X Chen) using methods from complex geometry and ϵ-geodesics. In [Ber10a], [Bo09] estimates involved rely on the Hormander $\overline{\partial}$ estimates and the setup is more sophisticated. We also study the uniqueness issue, but from an elementary point of view provided the geodesics are smooth. So in particular $i\partial\overline{\partial}u_t > 0$. Again we rely on a complex geometry inequality crucial to obtain convexity and analyze the equality case. In the setting when $L = -K_X$, [Bo] shows the scope of the result can be improved by establishing uniqueness using only sub-geodesics and bypasses difficulties introduced by the degeneracies $i\partial\overline{\partial}u_t \geq 0$. Because of its relevance we describe this but suppress his bundle theoretic set-up in the discussion.

Chapter 1

Convexity of some functionals and consequences

1.1 Description of Functionals

Let $(L, h_0) \longrightarrow X$ be a hermitian holomorphic line bundle over a compact complex manifold X so that L is ample. A Kähler form ω_0 is given by the curvature (1,1) form, that is, set $\omega_0 = -(2\pi)^{-1}\sqrt{-1}\bar{\partial}\partial \log h_0$ in $c_1(L)$. Write $\omega_0 = (2\pi)^{-1}\sqrt{-1}\partial\bar{\partial}\psi_0$, where locally the background hermitian structure h_0 is represented as $h_0 = e^{-\psi_0}$. This data is manufactured by using an embedding determined from $H^0(L^k)$ $(k >> 0)$ pulling back the Fubini study metric on $\mathcal{O}(1) \to \mathbb{P}^n$ and taking k-th roots gives a hermitian metric on L with the required properties. Now set $V = \int_X dvol_g$. Consider the functional defined on the Kähler potentials, the open convex subset $\mathcal{H}_{\omega_0} = \{u \in C^\infty(X) | \omega_u = \omega_0 + \sqrt{-1}\partial\bar{\partial}u > 0\} \subset C^\infty(X)$, as

$$\mathcal{E}_{\omega_0}(u) = \frac{1}{(n+1)!\,V}(\sum_{i=0}^{n}\int_X u\omega_u^i \wedge \omega_0^{n-j})$$

Proposition 1. *The functional $\mathcal{E}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$ has differential*

$$(d\mathcal{E}_{\omega_0})_u(v) = \frac{1}{n!}\int_X v\omega_u^n$$

for $u \in \mathcal{H}_{\omega_0}$ *and* $v \in C^\infty(X)$

Proof.

$$\begin{aligned}
(d\mathcal{E}_{\omega_0})_u &= \frac{d}{dt}(\mathcal{E}_{\omega_0}(u+tv))|_{t=0} \\
&= \frac{d}{dt}(\frac{1}{(n+1)!\,V}(\sum_{i=0}^{n}\int_X (u+tv)\omega_{u+tv}^i \wedge \omega_0^{n-i}))|_{t=0} \\
&= \frac{1}{(n+1)!\,V}[\sum_{j=0}^{n}\int_X (v\omega_{u+tv}^j \wedge \omega_0^{n-j} + j(u+tv)\omega_{u+tv}^{j-1} \wedge dd^c v \wedge \omega_0^{n-j})]|_{t=0} \\
&= \frac{1}{(n+1)!\,V}[\sum_{i=0}^{n}\int_X v\omega_u^i \wedge \omega_0^{n-i} + \sum_{i=1}^{n}\int_X iu\omega_u^{i-1} \wedge dd^c v \wedge \omega_0^{n-i}] \\
&= \frac{1}{(n+1)!\,V}[\sum_{j=0}^{n}\int_X v\omega_u^i \wedge \omega_0^{n-i} + \sum_{i=1}^{n}\int_X iv\omega_u^{i-1} \wedge dd^c u \wedge \omega_0^{n-i}] \\
&= \frac{1}{(n+1)!\,V}[\sum_{j=0}^{n}\int_X v\omega_u^i \wedge \omega_0^{n-i} + \sum_{i=1}^{n}\int_X iv\omega_u^i \wedge \omega_0^{n-i} \\
&\quad - \sum_{i=1}^{n}\int_X iv\omega_u^{j-1} \wedge \omega_0^{n+1-j}] \\
&= \frac{1}{(n+1)!\,V}[\sum_{j=0}^{n}\int_X v\omega_u^i \wedge \omega_0^{n-i} + \sum_{i=1}^{n}\int_X iv\omega_u^i \wedge \omega_0^{n-i} \\
&\quad - \sum_{i=0}^{n-1}\int_X (i+1)v\omega_u^i \wedge \omega_0^{n-i}] \\
&= \frac{1}{(n+1)!\,V}[\sum_{i=0}^{n}\int_X v\omega_u^i \wedge \omega_0^{n-i} - \sum_{i=1}^{n-1}\int_X v\omega_u^i \wedge \omega_0^{n-i} + \int_X v(n)\omega_u^n - \int_X v\omega_0^n] \\
&= \frac{1}{(n+1)!\,V}[\int_X v\omega_0^n + \int_X v\omega_u^n + \int_X v(n)\omega_u^n - \int_X v\omega_0^n] \\
&= \frac{1}{n!\,V}\int_X v\omega_u^n
\end{aligned}$$

□

Remark 1. *This is true in much lower regularity settings in fact* $\mathcal{E}_{\omega_0}$ *extends to* $C^0(X)$ *by composing with a nonlinear projection. The extension is gateaux differentiable and has the* *same differential with $u, v \in C^0(X)$ see [Ber10a].*

A similar calculation can be made for a path $u_t \in \mathcal{H}_{\omega_0}$ that is at time t, v corresponds to $\dot{u_t}$.

Note that $-\mathcal{E}_{\omega_0}(u) = \frac{1}{n!V}(J_{\omega_0}(u) - \int_X u\omega_0^n) = \frac{1}{n!V}\mathcal{F}^0_{\omega_0}$ where J_{ω_0} is Aubin's energy functional. Note whereas $\mathcal{F}^0_{\omega_0}$ is convex, $\mathcal{E}_{\omega_0}$ is concave. Recall $J_{\omega_0}(u)$ has derivative $-\frac{1}{V}\int_X \dot{u}(\omega_u^n - \omega_0^n)$ since this induces a closed 1-form on $\mathcal{H}_{\omega_0}$ its primitive is taken to be J_{ω_0} after fixing the correct normalization. The differential $d\mathcal{E}_\omega$ can also be calculated in terms of the differential of J_ω. Also given a constant c, $\mathcal{E}_{\omega_0}(u+c) = \mathcal{E}_{\omega_0}(u) + c$ follows from the formula. Since $\mathcal{E}_{\omega_0} = -\frac{1}{n!V}\mathcal{F}^0_{\omega_0}$ the functional has the same cocycle property that $\mathcal{F}^0_{\omega_0}$ has.

Recall $\langle s, s\rangle_{\psi_0} = i^{n^2}\int_X s \wedge \overline{s}e^{-\psi_0}$ for $s \in H^0(X, L \otimes K_X)$.

Another functional of importance is

$$\mathcal{L}_{\omega_0}(u) := -\frac{1}{N}\log\det(T(u))$$

where $T(u) = [\langle s_i, s_j\rangle_{\psi_0+u}]$ and $\{s_i\}$ is a basis of $H^0(X, L\otimes K_X)$ orthogonal with respect to $\langle\cdot,\cdot\rangle_{\psi_0}$. This is independent of the $\{s_i\}$ basis orthogonal with respect to $\langle\cdot,\cdot\rangle_{\psi_0}$, since any two bases of this type are related by a unitary transformation and the effect of the change is to conjugate $T[u]$ by this. So $\mathcal{L}_{\omega_0}$ remains unchanged. The property $\mathcal{L}_{\omega_0}(u+c) = \mathcal{L}_{\omega_0}(u)+c$ follows from the definition.

Finally define $\mathcal{F}_{\omega_0} := \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$. This is a functional on $\mathcal{H}_{\omega_0}$. Recall the natural action of $\mathbb{R}^*$ on hermitian metrics $e^{-u}h_0$ on L by multiplication by e^{-c} descends to additive action of $\mathbb{R}$ on $\mathcal{H}_\omega$ i.e by addition by c. Under the action the functionals $\mathcal{E}_{\omega_0}, \mathcal{L}_{\omega_0}$ have values translated by c it follows $\mathcal{F}_{\omega_0}$ is constant under this action so that it descends to a functional on the space of all Kähler metrics in $c_1(L)$.

Also note that the natural action of $Aut_0(X,L)$ on the space of metrics on L corresponds to the action $(u,F) \to v := F^*(\psi_{0+u}) - \psi_0$ so that $\omega_v = F^*\omega_u$. So a statement P holds for ω_1 in the space of Kähler metrics up to automorphism means that P also holds for any Kähler forms ω in the orbit of ω_1 under the action of $Aut_0(X,L)$.

Proposition 2. *Given any orthogonal basis $\{s_i\}$ of $(H^0(X, L\otimes K_X), \langle\cdot,\cdot\rangle_{\psi_0})$*

- *$\mathcal{L}_{\omega_0}$ is a well defined functional on $\mathcal{H}_{\omega_0}$. The differential takes the form*

$$(d\mathcal{L}_{\omega_0})_u(v) = \frac{-1}{N} i^{n^2} \sum_{i=1}^{N} \int_X v s_i \wedge \overline{s_i} e^{-(\psi_0+u)}$$

  in any basis $\{s_i\}$ orthonormal with respect to $\langle\cdot,\cdot\rangle_{\psi_0+u}$ with u, v as before.

- *The functional $\mathcal{F}_{\omega_0} = \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$ defined on $\mathcal{H}_{\omega_0}$ is translation invariant and the differential at u is given by the second equality in (1.1) in any basis $\{s_i\}$ orthogonal with respect to $\langle\cdot,\cdot\rangle_{\psi_0+u}$. In particular critical points of $\mathcal{F}_{\omega_0}$ are smooth solutions of (1.2).*

Proof. The first item is essentially the discussion above and the differential $(d\mathcal{L}_{\omega_0})_u$ may be computed similarly as in the previous proposition: take $\{s_i\}$ orthonormal with respect to $\langle\cdot,\cdot\rangle_{\psi_0+u}$ then

$$\begin{aligned}
\frac{d}{dt}\mathcal{L}_{\omega_0}(u+tv)|_{t=0} &= \frac{d}{dt}\left(\frac{-1}{N}\log\det(\langle s_i, s_j\rangle_{\psi_0+u+tv})\right)|_{t=0} \\
&= [\frac{-1}{N}(T^{ij}\dot{T}_{ij})(u+tv)]|_{t=0} \\
&= \frac{-1}{N}Tr[\dot{T}] = \frac{-1}{N} i^{n^2} \sum_{i=1}^{N} \int_X v s_i \wedge \overline{s_i} e^{-(\psi_0+u)}
\end{aligned}$$

□

Critical points u of $\mathcal{F}_{\omega_0}$ are smooth solutions of

$$0 = (d\mathcal{F}_{\omega_0})_u = \int_X v(\frac{1}{n!\,V}\omega_u^n + \frac{i^{n^2}}{N}\sum_{i=1}^{N} s_i \wedge \overline{s_i} e^{-(\psi_0+u)}) \quad v \in C^\infty(X) \tag{1.1}$$

Equivalently u is a smooth solution of the monge-ampere equation

$$\frac{1}{n!\,V}(dd^c u + \omega_0)^n = -\frac{i^{n^2}}{N}\sum_{i=1}^{N} s_i \wedge \overline{s_i} e^{-(\psi_0+u)} \tag{1.2}$$

1.2 Geodesic Equation

The space of Kähler potentials associated to a kähler manifold (X, ω) is

$$\mathcal{K}_\omega = \{\omega_\phi | \omega_\phi = \omega + i\partial\overline{\partial}\phi > 0, \phi \in C^\infty(X)\} \tag{1.3}$$

This may be identified with $\mathcal{H}_\omega = \{\phi | \phi \in SPSH(X, \omega) \cap C^\infty(X)\}$ which is open in $C^\infty(X)$.

For each point of $\mathcal{H}_\omega$ one can associate to it a measure on X, $d\mu_\phi = \frac{\omega_\phi^n}{n!}$. With this the metric on $\mathcal{H}_\omega$ is given by specifying the L^2 norm on functions i.e.

$$||\psi||_\phi^2 = \int_X \psi^2 d\mu_\phi$$

where $\psi \in T_\phi\mathcal{H}_\omega \cong C^\infty(X)$.

So for a path $\phi(t)$ in $\mathcal{H}_{\omega_0}$, parametrized by the unit interval, length is given by

$$l(\phi) := \int_0^1 \sqrt{\int_X (\dot{\phi(t)})^2 d\mu_{\phi(t)}} dt \tag{1.4}$$

By taking the first variation of the energy functional $\int_0^1 \int_X |\dot{\phi(t)}|^2 d\mu_{\phi(t)} dt$, since the critical points define geodesics in the space of kähler metrics, the smooth geodesic equation is

$$\ddot{\phi}(t) - \frac{1}{2}|\nabla \dot{\phi(t)}|^2_{\phi(t)} = 0 \tag{1.5}$$

A path u_t in $\mathcal{H}_{\omega_0}$ is viewed as a function U on $X \times [0,1]$. The following was first noticed by Donaldson and Semmes.

Proposition 3. *A path u_t satisfying the geodesic equation is the same as looking for solutions of $\Omega_U^{n+1} = 0$ with $U(\cdot,1) = u_0$ and $U(\cdot, e) = u_1$, where $\Omega_U = (dd^c U + \pi_X^* \omega_0)$.*

Proof.

$$\begin{aligned} 0 &= (dd^c U + \pi_X^* \omega_0)^{n+1} \\ &= ((\partial_X \partial_{\bar{t}} + \partial_t \overline{\partial}_X + \partial_t \partial_{\bar{t}} + \partial_X \overline{\partial}_X) u_t + \pi_X^* \omega_0)^{n+1} \\ &= ((\partial_X \partial_{\bar{t}} + \partial_t \overline{\partial}_X + \partial_t \partial_{\bar{t}}) u_t + \omega_{u_t})^{n+1} \\ &= P(\omega_{u_t}) \end{aligned}$$

where P is a polynomial of degree at most $n+1$ i.e

$$P(\omega_t) = \sum_{i=0}^{n+1} a_{(n+1-i)} \wedge \omega_{u_t}^i$$

Clearly we may assume $a_0 = 0$ since $\omega_{u_t}^{n+1} = 0$ does not contribute. From binomial expansion the terms a_{n+1-i} for $i \geq 3$ are terms with forms containing at least three "dt's"($dt \wedge dt \wedge d\bar{t}$

etc), so they can also be assumed to vanish. For similar reasons we may assume

$$
\begin{aligned}
a_{n+1-1} = a_n &= (n+1)\partial_t \partial_{\bar{t}} u_t \quad \text{since } (\partial_X \partial_{\bar{t}} u_t) \wedge \omega_{u_t}^n = 0 \\
a_{n+1-2} = a_{n-1} &= \frac{(n+1)n}{2} 2\partial_X \partial_{\bar{t}} u_t \wedge \partial_t \overline{\partial}_X u_t \\
\implies 0 &= \frac{1}{(n+1)!}(dd^c U + \pi_X^* \omega_0)^{n+1} \\
&= \partial_t \partial_{\bar{t}} u_t \wedge \frac{\omega_{u_t}^n}{n!} - \frac{1}{(n-1)!} \partial_X \partial_{\bar{t}} u_t \wedge \overline{\partial}_X \partial_t u_t \wedge \omega_{u_t}^{n-1}
\end{aligned}
$$

A local calculation shows

$$
\partial \phi \wedge \overline{\partial} \phi \wedge \omega^{n-1} = \frac{1}{2n} |\nabla \phi|_g^2 \omega^n
$$

It follows

$$
0 = \frac{1}{n!}(\ddot{u}_t - \frac{1}{2} |\nabla \partial_{\bar{t}} u_t|_{u_t}^2) \omega_{u_t}^n \wedge dt \wedge d\bar{t}
$$

□

1.3 ϵ-Geodesics

By considering the boundary value problem involving a degenerate Monge-Ampere equation instead in [Chen00] $C^{1,1}$ geodesics are found i.e. solutions of

$$0 = \Omega_\phi^{n+1} \tag{1.6}$$

$$= \det(g_{\alpha\overline{\beta}} + \phi_{\alpha\overline{\beta}}) \text{ on } X \times R \tag{1.7}$$

$$\phi_0 = \phi \text{ on } \partial(X \times R) \tag{1.8}$$

where R is a riemann surface with boundary which can be taken to be a cylinder.

Solutions are extracted by running a continuity method. Adjustments at $t = 1$ are made so the corresponding equation is elliptic. In other words its solution defines a Kähler potential on $V \times R$ not just on each slice $V \times \{w\}$, $w \in R$. C^0 estimates can be obtained using the boundary data and the maximum principle. An application of Yau's C^2 estimate yields the alternative that either the laplacian is uniformly bounded from above or the maximum occurs on the boundary. So to obtain a pointwise bound on the maximum of the laplacian in terms of maximum of the gradient a boundary estimate is needed. This is achieved from the maximum principle applied to a barrier function construction and the structure of the equation over the continuity path. So uniform C^2 bounds for $t > 0$ are obtained by obtaining point-wise bounds on the gradient. This is done through a blowup analysis. This furnishes a sequence of regular solutions ϕ_i to elliptic equations corresponding to a sequence $t_i \searrow 0$. Since at $t = 0$ the equation is degenerate, one passes only to a subsequence extracting a weak $C^{1,1}$ solution. An application of maximum principle shows this limit is unique. The details are the main content of [Chen00].

As a consequence the following also holds. Another application of the maximum principle is required to get estimates on solutions with respect to the s parameter again see [Chen00] for details.

Lemma 1 (Geodesic Approximation Lemma)**.** *Let $C_i : \phi_i(s) : [0,1] \to \mathcal{H}$ $(i = 1, 2)$ be smooth curves in $\mathcal{H}$. For $\epsilon_0 > 0$ sufficiently small there exist two parameter smooth family of curves $C(s, \epsilon) : \phi(t, s, \epsilon) : [0,1] \times [0,1] \times (0, \epsilon_0] \to \mathcal{H}$ satisfying*

1. *For any fixed s, ϵ, there is an epsilon approximate geodesic $C(s, \epsilon)$ joining $\phi_1(s)$ and $\phi_2(s)$ i.e. $\phi(z, t, s, \epsilon)$ solves*

$$\det(g_{\alpha\overline{\beta}} + \phi_{\alpha\overline{\beta}}) = \epsilon \det(g) \quad V \times R \tag{1.9}$$

$$\phi(z', 0, s, \epsilon) = \phi_1(z', s) \tag{1.10}$$

$$\phi(z', 1, s, \epsilon) = \phi_2(z', s) \tag{1.11}$$

   where $z_{n+1} = t + \sqrt{-1}\theta$ and ϕ has trivial dependence on θ.

2. *There exists a uniform constant C(which depends only on ϕ_1, ϕ_2) satisfying*

$$|\phi| + |\partial_s \phi| + |\partial_t \phi| < C \tag{1.12}$$

$$0 \leq \partial_t^2 \phi < C \tag{1.13}$$

$$\partial_s^2 \phi < C \tag{1.14}$$

3. *For fixed s let $\epsilon \to 0$, then the convex curve $C(s, \epsilon)$ converges to the unique geodesic between $\phi_1(s)$ and $\phi_2(s)$ in the weak $C^{1,\eta}$ topology($0 < \eta < 1$).*

4. The energy element along $C(s,\epsilon)$ is given by

$$E(t,s,\epsilon) = \int_V |\partial_t \phi|^2 dg(t,s,\epsilon) \tag{1.15}$$

where $g(t,s,\epsilon)$ is the Kähler metric corresponding to $\phi(t,s,\epsilon)$. Then there exists a uniform constant C such that

$$\max_{t,s} |\partial_t E| \leq \epsilon \cdot C \cdot M \tag{1.16}$$

So the energy/length element converges to a constant along each convex curve as $\epsilon \to 0$.

1.4 Convexity and eigenvalue estimate

In this section we study convexity of $\mathcal{L}_{\omega_0}$ along smooth geodesics in the setting when $\dim H^0(X, K_X \otimes L) = 1$ and $L \otimes K_X$ is globally generated. So $L = -K_X$. For example this happens when $X = \mathbb{P}^1$ since $L \cong \mathcal{O}_{\mathbb{P}^1}(m)$ for some $m \in \mathbb{Z}$ from an application of a theorem of Grothendieck. So $m = 2$ since $L \otimes K_{\mathbb{P}^1} = \mathcal{O}(0)$.

Let (X, L, ω_0) be given as in §1 with the above restrictions. Since $N = 1$ for any $s \in H^0(X, K_X \otimes L)$, $\det(T(u))$ simplifies to

$$i^{n^2} s \wedge \bar{s} e^{-\psi_0} = e^{\theta_{\omega_0}} \omega_0^n$$

which is basically (using $L = -K_X$)

$$\frac{e^{-\psi_0}}{\det g} = e^{\theta_{\omega_0}} \tag{1.17}$$

We may write $s_{|U_\alpha} = \phi \otimes t$ locally where $\phi_\alpha \in \Gamma(U_\alpha, K_X)$ and $t_\alpha \in \Gamma(U_\alpha, L)$ holomorphic. Note that

$$\phi_\alpha \otimes t_\alpha = \phi_\beta \det(\psi_{\alpha\beta})^{-1} \otimes t_\beta \det(\psi_{\alpha\beta}) = \phi_\beta \otimes t_\beta \tag{1.18}$$

where $\{\psi_{\alpha\beta}, U_{\alpha\beta} := U_\alpha \cap U_\beta\}$ is the cocycle determining $\mathcal{T}_X$. Denote by $||\cdot||^2$ the fiber length induced by hermitian metric h_0. Let $\theta_\alpha : L_{U_\alpha} \cong U_\alpha \times \mathbb{C}$ be the associated trivialization induced by t_α so $|\theta_\alpha(t_\alpha)| = 1$.

$$(i^{n^2} s \wedge \overline{*s})_{|U_\alpha} = i^{n^2} \phi_\alpha \wedge \overline{\phi_\alpha} ||t_\alpha||^2 = i^{n^2} \phi_\alpha \wedge \overline{\phi_\alpha} |\theta(t_\alpha)|^2 e^{-\psi_0} = i^{n^2} \phi_\alpha \wedge \overline{\phi_\alpha} e^{-\psi_0} \tag{1.19}$$

from (1.18) we may glue the local versions together and view $i^{n^2} s \wedge \bar{s} e^{-\psi_0}$ as a global section of $K_X \otimes \overline{K_X}$ just like ω^n and so $\frac{i^{n^2} s \wedge \bar{s} e^{-\psi_0}}{\omega^n}$ defines a global function. Note also that from the last equality in (1.19), after shrinking U_α to a coordinate chart, with respect to the coordinates there choosing $\phi_\alpha = dz_1 \wedge \ldots \wedge dz_n$ gives that the global function is of the form given in (1.17).

Remark 2. *$L = -K_X$ is not necessary all thats needed is $0 \neq s \in H^0(K_X \otimes L)$ i.e holomorphic global sections of it. That is $i^{n^2} s \wedge \bar{s} e^{-\psi_0}$ transforms as sections of $K_X \otimes \overline{K}_X$ just as volume forms do so the ratio is a global function. To see this note that two trivialization θ, θ' are related by $\theta' = g\theta$ on the overlap and similarly $\psi_0' = \psi_0 + \log|g|^2$ on the overlap.*

So $\theta_{\omega_0} \in C^\infty(X)$. Applying $\partial\overline{\partial}log$ to (1.17)

$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log(e^{\theta_{\omega_0}}) = \frac{\sqrt{-1}}{2}(\partial\overline{\partial}\log(e^{-\psi}) - \partial\overline{\partial}\log\det(g))$$
$$\frac{\sqrt{-1}}{2}\partial\overline{\partial}\theta_{\omega_0} = Ric(\omega_0) - \omega_0 \quad (*)$$

Equation $*$ above is essentially

$$[F_{\nabla_{L\otimes K_X}}] = c_1(L \otimes K_X) = c_1(\mathcal{O}(0)) = 0$$

with $c_1(K_X)$ represented by the negative of the ricci form and $c_1(L)$ by the curvature of a hermitian metric given locally by $e^{-\psi_0}$. So it follows that $\omega_t - Ric(\omega_t) = \frac{\sqrt{-1}}{2}\partial\overline{\partial}(-\theta_{\omega_t})$. The negative sign is benign and chosen only to suit our conveniences.

Recall $Ric(\omega_0) = -\frac{\sqrt{-1}}{2}\partial\overline{\partial}\log\det(g)$. Also

$$\omega_t = \omega_0 + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_t$$

where $\phi_t \in \mathcal{H}_{\omega_0}$ is an arbitrary path. So

$$\begin{aligned}
\frac{\sqrt{-1}}{2}\partial\overline{\partial}\theta_{\omega_t} =& Ric(\omega_t) - \omega_t \\
=& Ric(\omega_t) - Ric(\omega) + \omega_0 + \sqrt{-1}\partial\overline{\partial}\theta_{\omega_0} - (\omega_0 + \frac{\sqrt{-1}}{2}\partial\overline{\partial}\phi_t) \\
=& \frac{\sqrt{-1}}{2}(\log(\frac{\det(g)}{\det(g')}) + \theta_{\omega_0} - \phi_t) \quad (*_t)
\end{aligned}$$

and

$$\theta_{\omega_t} = \log(\frac{\det(g)}{\det(g')}) + \theta_{\omega_0} - \phi_t + c_t \tag{1.20}$$

θ_{ω_t} is clearly globally defined. To pin down the constant we choose normalization

$$\int_X e^{-\phi_t + c_t + \theta_{\omega_0}} \omega_0^n = 1 \tag{1.21}$$

Henceforth we abuse notation and denote $\phi_t + c_t$ by ϕ_t. We conclude after exponentiating (1.20)

$$e^{-\phi_t + \theta_{\omega_0}} \omega_0^n = e^{-\phi_t + \theta_{\omega_0}} \frac{\omega_0^n}{\omega_t^n} \omega_t^n = e^{\theta_{\omega_t}} \omega_t^n \tag{1.22}$$

That is ϕ_t moves along continuity path (1.22). Restricting further to smooth geodesics we have

Proposition 4. *When $L \otimes K_X$ is globally generated and* $\dim(H^0(X, L \otimes K_X)) = 1$ *the functional $\mathcal{L}_{\omega_0}$ is convex along smooth geodesics.*

To check convexity along these geodesics use second variation of the $\mathcal{L}_\omega$ functional. This simplifies to $-\log \int_X e^{-\phi_t + \theta_{\omega_0}} \omega_0^n$ from the discussion above. So taking t derivatives two times we obtain the quantity

$$\frac{(\int_X e^{-\phi_t + \theta_{\omega_0}} \omega_0^n) \int_X ((\dot{\phi})^2 - \ddot{\phi}) e^{-\phi_t + \theta_{\omega_0}} \omega_0^n - (\int_X e^{\theta_{\omega_t}} (\dot{\phi}_t) \omega_t^n)^2}{(\int_X e^{-\phi_t + \theta_{\omega_0}} \omega_0^n)^2}$$

from which convexity of $\mathcal{L}_\omega$ is determined provided the following inequality($\mathcal{L}_\omega$ has a negative sign):

$$(\int_X e^{-\phi_t+\theta_{\omega_0}}\omega_0^n)\int_X e^{-\phi_t+\theta_{\omega_0}}((\dot{\phi}_t)^2-(\ddot{\phi}_t))\omega_0^n-(\int_X e^{-\phi_t+\theta_{\omega_0}}(\dot{\phi}_t)\omega_0^n)^2\leq 0$$

Which simplifies using (1.22) to

$$(\int_X e^{\theta_{\omega_t}}\omega_t^n)\int_X e^{\theta_{\omega_t}}((\dot{\phi}_t)^2-(\ddot{\phi}_t))\omega_t^n-(\int_X e^{\theta_{\omega_t}}(\dot{\phi}_t)\omega_t^n)^2\leq 0$$

We may without loss assume $\int_X e^{\theta_{\omega_t}}(\dot{\phi}_t)\omega_t^n=0$ which follows from differentiating the normalization condition chosen. Hence the inequality desired is

$$\begin{aligned}\int_X e^{\theta_{\omega_t}}((\dot{\phi}_t)^2-(\ddot{\phi}_t))\omega_t^n\leq 0\\ \Rightarrow\int_X e^{\theta_{\omega_t}}((\dot{\phi}_t)^2-\frac{1}{2}(|\nabla\dot{\phi}_t|_{g(t)}^2))\omega_t^n\leq 0\\ \Rightarrow\int_X e^{\theta_{\omega_t}}(|\nabla\dot{\phi}_t|_{g(t)}^2)\omega_t^n\geq 2\int_X e^{\theta_{\omega_t}}((\dot{\phi}_t)^2)\omega_t^n\end{aligned}$$

So we need to show whenever $Ric(\omega)-\omega=\frac{\sqrt{-1}}{2}\partial\overline{\partial}\theta_\omega$ and $\int_X fe^{\theta_\omega}\omega^n=0$

$$\int_X e^{\theta_\omega}(f)^2\omega^n\leq\int_X\frac{1}{2}(|\nabla f|_g^2))e^{\theta_\omega}\omega^n$$

$$\Rightarrow\lambda_1(-\Delta-\nabla\theta_\omega\cdot)\geq 2 \tag{1.23a}$$

holds.

The first eigenvalue estimate in (1.23a) translates to, in the Kähler case,

$$\begin{aligned}\int_X \frac{1}{2}|\nabla f|^2 e^{\theta_\omega}\omega^n &= \int_X g^{\alpha\overline{\beta}} f_\alpha f_{\overline{\beta}} e^{\theta_\omega}\omega^n \\ &= -\int_X (\Box f + g^{\alpha\overline{\beta}} f_\alpha (\theta_\omega)_{\overline{\beta}}) e^{\theta_\omega} f\omega^n\end{aligned}$$

That is the corresponding first eigenvalue estimate (1.23a) in this setting is $\mu_1(-\Box - \frac{1}{2}\langle\nabla\cdot, \nabla\theta_\omega\rangle) \geq 1$.

Remark 3. *Just as with eigenvalue estimates for $\mu_1(-\Box)$ we may similarly consider using*

1. $R_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} + \theta_{\alpha\overline{\beta}}$
2. $\frac{1}{2}\Box|\nabla f|^2 = |f_{\alpha\beta}|^2 + |f_{\alpha\overline{\beta}}|^2 + g^{\alpha\overline{\beta}}(f_\alpha(\Box f)_{\overline{\beta}} + (\Box f)_\alpha f_{\overline{\beta}}) + R_{\alpha\overline{\beta}} f^\alpha f^{\overline{\beta}}$
3. $-\Box f = \mu_1 f + \frac{1}{2}\langle\nabla\theta_\omega, \nabla f\rangle$

However this point view encounters problematic pure type terms which cannot be controlled by (1)*.*

Instead consider $u : X \to \mathbb{C}$. Set $\langle u, v\rangle_\theta = \int_X u\overline{v}e^{\theta_\omega}\omega^n$ hermitian weighted scalar product. Using an orthonormal frame we have

$$\begin{aligned}\int_X |\overline{\partial u}|^2 e^{\theta_\omega}\omega^n &= \int_X u_{\overline{\alpha}}\overline{u_{\overline{\alpha}}} e^{\theta_\omega}\omega^n \\ &= \int_X -(u_{\alpha\overline{\alpha}} + \theta_\alpha u_{\overline{\alpha}})\overline{u}e^\theta\omega^n \\ &= \int_X -(\Box u + \theta_\alpha u_{\overline{\alpha}})\overline{u}e^\theta\omega^n = -\langle Lu, u\rangle_\theta\end{aligned}$$

where $Lu := \Box u + \theta_\alpha u_{\overline{\alpha}}$.

Applying to first eigenfunctions u with eigenvalue μ_1 obtain $\mu_1||u||^2_\theta > 0$ so $\mu_1 > 0$.

In the case where $Ric(\omega) \geq \omega$ it follows $\lambda_1(\Box) \geq 1$(using $\langle \cdot, \cdot \rangle_0$ restricted to real valued functions u):

$$\begin{aligned} 0 \leq \int |u_{\alpha\beta}|^2 \omega^n &= \int u_{\alpha\beta} u_{\overline{\alpha}\overline{\beta}} \omega^n \\ &= -\int u_{\beta\alpha,\overline{\beta}} u_{\alpha} \omega^n \\ &= -\int (u_{\beta\overline{\beta},\alpha} u_\alpha + R_{\alpha\overline{\beta}} u_{\overline{\alpha}} u_\beta) \omega^n \end{aligned}$$

Using $Ric(\omega) \geq \omega$ it follows that $\mu_1(\Box) \geq 1$.

Similarly consider the quantity $\int_X |u_{\overline{\alpha}\overline{\beta}}| e^\theta \omega^n \geq 0$ in the weighted setting. Also note that since the operator L is essentially $\Box$ up to lower order terms so it is elliptic: symbol is determined by highest order terms. We wish to apply the following lemma to the first eigenfunction which is a priori smooth.

Lemma 2. *Let $u \in C^\infty(X, \mathbb{C})$ then*

$$\int_X (-(Lu)_{\overline{\alpha}} \overline{u}_\alpha - |\overline{\partial} u|^2) e^\theta \omega^n \geq 0$$

Proof. Following the discussion above

$$\begin{aligned} 0 \leq \int_X |u_{\overline{\alpha}\overline{\beta}}| e^\theta \omega^n &= \int_X u_{\overline{\alpha}\overline{\beta}} \overline{u}_{\alpha\beta} e^\theta \omega^n \\ &= \int_X -(u_{\overline{\beta}\overline{\alpha}\beta} \overline{u}_\alpha + u_{\overline{\alpha}\overline{\beta}} \theta_\beta \overline{u}_\alpha) e^\theta \omega^n \\ &= \int_X (-(\Box u)_{\overline{\alpha}} u_\alpha - R_{s\overline{\beta}\overline{\alpha}\beta} \overline{u}_\alpha u_{\overline{s}} - u_{\overline{\alpha}\overline{\beta}} \theta_\beta \overline{u}_\alpha) e^\theta \omega^n \\ &= \int_X (-(Lu)_{\overline{\alpha}} \overline{u}_\alpha + \theta_{\beta\overline{\alpha}} u_{\overline{\beta}} \overline{u}_\alpha \\ &\quad - R_{s\overline{\alpha}} \overline{u}_\alpha u_{\overline{s}} \quad + \theta_\beta u_{\overline{\alpha}\overline{\beta}} \overline{u}_\alpha - u_{\overline{\alpha}\overline{\beta}} \theta_\beta \overline{u}_\alpha) e^\theta \omega^n \end{aligned} \tag{1.24}$$

Since $Ric(\omega) - \omega = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\theta$ in coordinates is equivalent to $R_{\alpha\overline{\beta}} - \theta_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$, it follows

$$(R_{\beta\overline{\alpha}} - \theta_{\beta\overline{\alpha}})u_{\overline{\beta}}\overline{u}_{\alpha} = u_{\overline{\alpha}}\overline{u}_{\alpha} = |\overline{\partial}u|^2$$

So (1.24) simplifies to

$$\int_X (-(Lu)_{\overline{\alpha}}\overline{u}_{\alpha} - |\overline{\partial}u|^2)e^{\theta}\omega^n \geq 0$$
$$\Rightarrow \int_X -(Lu)_{\overline{\alpha}}\overline{u}_{\alpha}e^{\theta}\omega^n \geq \int_X |\overline{\partial}u|^2 e^{\theta}\omega^n$$

□

proof of proposition. Suppose u is a first eigenfunction of the operator L with eigenvalue μ_1 (i.e. $-Lu = \mu_1 u$) then by the lemma

$$\mu_1 \int_X u_{\overline{\alpha}}\overline{u}_{\alpha}e^{\theta}\omega^n \geq \int_X |\overline{\partial}u|^2 e^{\theta}\omega^n$$
$$\implies \mu_1 \geq 1$$

□

1.5 Convexity of $\mathcal{L}_{\omega_0}$

Let u_t be a path in $\mathcal{H}_{\omega_0}$. Recall $N := \dim H^0(X, K_X \otimes L) < \infty$ by hodge theory. Also recall that $\mathcal{L}_{\omega_0}(u_t) = \frac{-1}{N}\log\det(T(u_t))$ where $T(u_t) = [\langle s_i, s_j\rangle_{\psi_0+u_t}]$.

Proposition 5. *When $u_t \in \mathcal{H}_{\omega_0}$ is an arbitrary smooth path, the first and second variations of $\mathcal{L}_{\omega_0}$ are given by*

$$\frac{d}{dt}\mathcal{L}_{\omega_0}(u_t) = \frac{-1}{N}(T^{ij}\dot{T}_{ij}) \tag{1.25}$$
$$\frac{d^2}{dt^2}\mathcal{L}_{\omega_0}(u_t) = \frac{-1}{N}[Tr(-(T^{-1}\dot{T})^2) + Tr(T^{-1}\ddot{T})] \tag{1.26}$$

Proof. This follows by direct computation recalling that for square matrices

$$\frac{d}{dt}\det(T(t)) = \det T(t) Tr(T^{-1}(t)\dot{T(t)})$$

□

Let $s \in H^0(X, \Omega^p_X \otimes L) \cong H^{p,0}(X, L)$ using resolution of $\Omega^p_X \otimes L$ by sheaves $\mathcal{A}^{p,\cdot}(L)$ which are acyclic (this is Dolbeaut's theorem asserting Dolbeaut cohomology is isomorphic to sheaf cohomology of holomorphic differential forms).

In this setting, via Hodge theory using hodge decomposition for holomorphic hermitian vector bundles on compact hermitian manifolds and type considerations in this range we obtain the decomposition

$$\mathcal{A}^{p,0}(X, L) = \overline{\partial}^* \mathcal{A}^{p,1}(X, L) \oplus \mathcal{H}^{p,0}(X, L)$$

Note that $\overline{\partial}^* \mathcal{A}^{p,1}$ is orthogonal to $Ker\overline{\partial}$. Let $\alpha \in \mathcal{A}^{p,1}$, then since

$$(\overline{\partial}\overline{\partial}^* \alpha, \alpha) = ||\overline{\partial}^* \alpha||^2 > 0$$

we can specialize to $p = n$ to obtain that $H^{n,0}(X, L) \cong \mathcal{H}^{n,0}(X, L)$. Hence $s \in H^0(X, K_X \otimes$

$L) \cong \mathcal{H}^{n,0}(X, L)$ is harmonic. So for $s \in \mathcal{A}^0(X, \Omega_X^n \otimes L)$ satisfying $s \perp H^0(X, K_X \otimes L)$ we have by hodge decomposition that $s = \overline{\partial}^* \sigma + \overline{\partial}\beta$ where $\sigma \in \mathcal{A}^{n,1}$ and $\beta \in \mathcal{A}^{n,-1} = 0$.

Lemma 3. *With s as above, orthogonal to global holomorphic sections of $K_X \otimes L$, the following estimate holds*

$$||\overline{\partial}s||^2 \geq ||s||^2 \tag{1.27}$$

Remark 4. *Restricted to the orthogonal complement of $H^0(X, K_X \otimes L)$, $\overline{\partial}_{K_X \otimes L}$ operator has no kernel (on the orthogonal complement, where $\overline{\partial}^*_{K_X \otimes L} = 0$, $\overline{\partial}_{K_X \otimes L}$ is a restriction of the elliptic $\overline{\partial} + \overline{\partial}^*$ operator) so should be invertible and thus satisfy an inequality of the type (1.27) with perhaps better constants.*

Proof. Since $s = \overline{\partial}^* \sigma$, (1.27) is equivalent to

$$||\overline{\partial}\overline{\partial}^* \sigma||^2 \geq \qquad\qquad ||\overline{\partial}^* \sigma||^2$$

An application of cauchy-schwartz gives that

$$||\overline{\partial}^* \sigma||^2 = \langle \overline{\partial}\overline{\partial}^* \sigma, \sigma \rangle \leq ||\overline{\partial}\overline{\partial}^* \sigma|| ||\sigma|| \tag{1.28}$$

Thus it suffices to show

Claim 1.

$$||\sigma||^2 \leq ||\overline{\partial}^* \sigma||^2 \tag{1.29}$$

The Hodge decomposition $\sigma = \alpha + \overline{\partial}\beta + \overline{\partial}^* \gamma$ simplifies to $\sigma = \overline{\partial}\beta$ where $\beta \in \mathcal{A}^{n,0}(L)$.

Since $0 = \overline{\partial}^* \alpha = \overline{\partial}^* \overline{\partial}^* \gamma$

$$s = \overline{\partial}^* \sigma = \overline{\partial}^* \beta$$

so we may take $\sigma = \overline{\partial} \beta$.

A version of the Bochner-Kodaira-Nakano identity simplifies using $[\Lambda, \theta(L)] = [\Lambda, \omega] = [\Lambda, L]$ to

$$\square_{D''} = \square_{D'} + (p + q - n) \cdot I \tag{1.30}$$

So we may obtain the L^2 identity from (1.30) applied to σ

$$\begin{aligned} ||\overline{\partial}\sigma||^2 + ||\overline{\partial}^*\sigma||^2 =& ||D'\sigma||^2 + ||(D')^*\sigma||^2 + (n + 1 - n)||\sigma||^2 \\ \implies ||\overline{\partial}\sigma||^2 + ||\overline{\partial}^*\sigma||^2 \geq& ||\sigma||^2 \end{aligned} \tag{1.31}$$

But σ is a holomorphic section so we obtain

$$||\overline{\partial}^* \sigma|| \geq ||\sigma||$$

So the claim follows and hence the lemma. $\square$

Now given $||\overline{\partial} s|| \geq ||s||$ for $s \perp H^0(X, K_X \otimes L)$ we can determine the shape of the inequality for $s \in \mathcal{A}^{n,0}(L)$. To do this consider

$$P : \mathcal{A}^{n,0}(L) \to H^0(X, K_X \otimes L)$$

the projection of $s \in \mathcal{A}^{n,0}(L)$ to its holomorphic part that can be viewed as a holomorphic global section of $K_X \otimes L$.

Proposition 6. *Given $s \in \mathcal{A}^{n,0}(L)$*

$$||\overline{\partial} s||^2 \geq ||s||^2 - ||P(s)||^2 \tag{1.32}$$

Proof. This follows immediately from the lemma applied to $s - P(s)$ which is orthogonal to $H^0(X, K_X \otimes L)$, and using that the holomorphic projection P is an orthogonal projection. □

Now having obtained the inequality (1.32) we can proceed to the the main business of this section

Proposition 7. *Assume $H^0(X, K_X \otimes L) \neq 0$ then $\mathcal{L}_{\omega_0}$ is convex along smooth geodesics (when they exist).*

Remark 5. *We could have replaced the assumption with the globally generated condition appearing in [Ber10a] but that assumption is really cooked up for the critical points equation; so that it is elliptic.*

Proof. Let h be an arbitrary metric on L deformed from the background metric on L, h_0, related by $h = h_0 e^{-\phi}$. Taking a basis $\{s_i\} \subset H^0(X, K_X \otimes L)$ orthogonal basis with respect to $\langle \cdot, \cdot \rangle_h$, applied to $\dot{\phi} s_i$ we have in (1.26) that $T^{-1} = Id$. So

$$\begin{aligned} Tr(-(T^{-1}\dot{T})^2) &= -\sum_{ij} (\int_X \dot{\phi}(s_i, s_j)_h)^2 \\ Tr(T^{-1}\ddot{T}) &= -\sum_i \int_X (\frac{|\nabla \dot{\phi}|^2}{2} - \dot{\phi}^2)(s_i, s_i)_h \end{aligned} \tag{1.33}$$

Also note

$$
\begin{aligned}
||P(\dot{\phi}s_i)||^2 &= \sum_j \big(\int_X \dot{\phi}(s_i, s_j)_h\big)^2 \\
||\overline{\partial}(\dot{\phi}s_i)||^2 &= \int_X \frac{|\nabla\dot{\phi}|^2}{2}(s_i, s_i)_h
\end{aligned} \tag{1.34}
$$

Applying (1.32) to sections $\dot{\phi}s_i$ with respect to the metric h on L and summing over $1 \leq i \leq \dim(H^0(K_X \otimes L))$ gives

$$
\begin{aligned}
\sum_i ||\overline{\partial}(\dot{\phi}s_i)||^2 &\geq \sum_i (||\dot{\phi}s_i||^2 - ||P(\dot{\phi}s_i)||^2) \\
\Rightarrow \sum_i \int_X \big(\frac{|\nabla\dot{\phi}|^2}{2} - \dot{\phi}^2\big)(s_i, s_i)_h &\geq -\sum_{ij} \big(\int_X \dot{\phi}(s_i, s_j)_h\big)^2
\end{aligned} \tag{1.35}
$$

The proposition follows. □

Together with ϵ-geodesics it can be shown that $\mathcal{L}_{\omega_0}$ is convex along $C^{1,1}$ geodesics. This is verified in §7.

1.6 Maximizers using ϵ-geodesics

Recall from section §3 there is a smooth path u_t , an ϵ-geodesic, connecting a critical point u_0 to another point u_1 of $\mathcal{H}_{\omega_0}$. It satisfies

$$
\begin{aligned}
(\ddot{u}_t - \frac{|\nabla \dot{u}_t|^2_{g(t)}}{2}) \det g(t) &= \epsilon \det g > 0 \\
\implies \ddot{u}_t &= \frac{\epsilon \det g}{\det g(t)} + \frac{|\nabla \dot{u}_t|^2_{g(t)}}{2}
\end{aligned}
$$

where $g(t) = g_{\alpha\overline{\beta}} + (u_t)_{\alpha\overline{\beta}}$ ($1 \leq \alpha, \beta \leq n$)

From proposition 1 along smooth paths $u_t \in \mathcal{H}_{\omega_0}$

$$
\begin{aligned}
d\mathcal{E}_{\omega_0}(u_t) &= \frac{1}{V}\int_X \dot{u}_t \frac{\omega_{u_t}^n}{n!} \\
\Longrightarrow \frac{d^2}{dt^2}\mathcal{E}_{\omega_0}(u_t) &= \frac{1}{V}\int_X (\ddot{u}_t - \frac{|\nabla \dot{u}_t|^2_{g(t)}}{2})\frac{\omega_{u_t}^n}{n!}
\end{aligned}
$$

so along an ϵ-geodesic it follows

$$
\frac{d^2}{dt^2}\mathcal{E}_{\omega_0}(u_t) = \frac{\epsilon}{V}\int_X \frac{\det g}{\det g(t)}\frac{\omega_{u_t}^n}{n!} = \epsilon \tag{1.36}
$$

Take an orthonormal basis $\{s_i\} \subset H^0(X, K_X \otimes L)$ with respect to $\langle \cdot, \cdot \rangle_h$ and using the ϵ-geodesic equation we get

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{L}_{\omega_0}(u_t) = \frac{-1}{N}[-\sum_{ij}(\int_X \dot{u}_t \langle s_i, s_j\rangle_h)^2 \\
+ \sum_i \int_X (\dot{u}_t{}^2 - \frac{|\nabla \dot{u}_t|^2_{g(t)}}{2} - \frac{\epsilon \det g}{\det g(t)})(s_i, s_i)_h] \\
\geq \frac{1}{N}\sum_i \int_X \frac{\epsilon \det g}{\det g(t)}(s_i, s_i)_h \geq 0
\end{aligned} \tag{1.37}
$$

The last inequality follows from (1.35). In other words

$$
-\sum_{ij}(\int_X \dot{u}_t \langle s_i, s_j\rangle_h)^2 + \sum_i \int_X (\dot{u}_t{}^2 - \frac{|\nabla \dot{u}_t|^2_{g(t)}}{2})(s_i, s_i)_h \leq 0
$$

The right hand side of (1.37) is positive and by item 3 of lemma 1 ϵ-geodesics converge in $C^{1,1}$ topology to the $C^{1,1}$ geodesic connecting the points. In particular sending $\epsilon \to 0$, we obtain $f(t) := \mathcal{L}_{\omega_0}(t)$ the function obtained by restricting $\mathcal{L}_{\omega_0}$ along $C^{1,1}$ geodesic, is the uniform limit of similarly defined functions $f_\epsilon(t)$ that are convex. So $f(t)$ is convex. In summary we have

Proposition 8. *If* $\dim H^0(X, K_X \otimes L) \geq 1$ *then* $\mathcal{L}_{\omega_0}$ *is defined and is convex along* $C^{1,1}$ *geodesics.*

Proof. See discussion above. □

Similarly $e(t) := \mathcal{E}_{\omega_0}(t)$ restricted to the $C^{1,1}$ geodesic is a uniform limit of functions $e_\epsilon(t)$, those function that arise from restricting $\mathcal{E}_{\omega_0}$ to ϵ geodesics. Recall $\mathcal{E}_{\omega_0}$ is continuous under uniform limits in $\overline{\mathcal{H}}_{\omega_0} \cap C^0(X)$ (the $L^1(X, \omega_0)$ closure), a result of Bedford-Taylor, in particular with respect to uniform limits of ϵ-geodesics. From (1.36) the second derivative of $\mathcal{E}_{\omega_0}$ along ϵ-geodesics goes to zero. Since the second derivative of $\mathcal{E}_{\omega_0}$ is the integral of the geodesic equation in the sense of Donaldson and Semmes, the convergence to zero is in the sense of Chen. Thus $e(t)$ is affine.

Corollary 1. *The functional*

$$\mathcal{F}_{\omega_0} := \mathcal{E}_{\omega_0} - \mathcal{L}_{\omega_0}$$

is concave along $C^{1,1}$ *geodesics.*

Proof. This follows since $\mathcal{E}_{\omega_0}$ is affine while $-\mathcal{L}_{\omega_0}$ is concave along $C^{1,1}$ geodesics. □

The result of Berman [Ber10a] follows:

Corollary 2. *Critical points (when they exist) are maximizers of* $\mathcal{F}_{\omega_0}$

Proof. Let u_0 be a critical point of $\mathcal{F}_{\omega_0}$ and $u_1 \in \mathcal{H}_{\omega_0}$ any other point connected by the $C^{1,1}$ geodesic u_t. Since $\mathcal{F}_{\omega_0}$ is concave along u_t, $d_t\mathcal{F}_{\omega_0}$ decreases along u_t. It follows u_0 is a maximum in $\mathcal{H}_{\omega_0}$ for $\mathcal{F}_{\omega_0}$. So an absolute maximum of $\mathcal{F}_{\omega_0}$ on $\mathcal{H}_{\omega_0}$ obtains at any critical point (when it exists). □

Also for future use we record

$$\frac{d^2}{dt^2}\mathcal{F}_{\omega_0}(u_t) \leq \epsilon(1 - \frac{1}{N}\sum_i \int_X \frac{\det g}{\det g(t)}(s_i, s_i)_h) \leq \epsilon$$

along ϵ-geodesics.

1.7 Uniqueness smooth case

Let u_t be a smooth geodesic in $\mathcal{H}_{\omega_0}$ so $\omega_{u_t} > 0$ connecting critical points of $\mathcal{F}_{\omega_0}$. A $(1,0)$ vector field V_t can be defined by

$$\omega_{u_t}(V_t, \cdot) = \overline{\partial}\dot{u}_t$$

We abuse language and call V_t a gradient vector field of $\dot{u}$(otherwise we need to fit the fixed complex structure J into expressions when making the reference).

The main objective in this section is

Proposition 9. *When there is a smooth geodesic connecting critical points $\omega_0 := \omega_{u_0}, \omega_{u_1}$ of $\mathcal{F}_{\omega_0}$ in $\mathcal{H}_{\omega_0}$ the critical points are related by an automorphism ϕ of (X, L) i.e $\omega_{u_1} = \phi_1^*\omega_0$.*

Remark 6. *Really there is always a $C^{1,1}$ geodesic connecting the critical points. The proposition applies with this path smooth.*

In this direction note that expression for a gradient vector field in the adjoint setting can be written as

Lemma 4. *For $s \in \mathcal{A}^{n,0}(L)$, $\omega_{u_t} > 0$ and V_t the gradient vector field of $\dot{u}_t$*

$$-\overline{\partial}\dot{u}_t \wedge s = \omega_{u_t} \wedge (V_t \rfloor s) = L_{\omega_{u_t}}(V_t \rfloor s) \tag{1.38}$$

where $L_{\omega_{u_t}}$ is the lefschetz operator defined by ω_{u_t}.

Proof. Simply wedge the equation defining the gradient vector field with $s \in \mathcal{A}^{n,0}(L)$. Use that $\omega_{u_t} \wedge s$ is of type $(n+1, 1)$ and therefore vanishes. Then conclude using the elementary calculation

$$\begin{aligned} 0 &= \omega_{u_t} \wedge s \\ 0 &= V_t \rfloor (\omega_{u_t} \wedge s) \\ &= (V_t \rfloor \omega_{u_t}) \wedge s + \omega_{u_t} \wedge (V_t \rfloor s) \\ \implies -\overline{\partial}\dot{u}_t \wedge s &= \omega_{u_t} \wedge (V_t \rfloor s) \end{aligned}$$

□

Corollary 3.

$$-\Lambda_{\omega_t}((\overline{\partial}\dot{u}_t) \wedge s) = \Lambda_{\omega_t} L_{\omega_{u_t}}(V_t \rfloor s) \tag{1.39}$$

Proof. Just take traces of (1.38). That is, act on it by $\Lambda_{\omega_{u_t}}$. □

Lemma 5. *As above all operators are defined with respect to $\omega_{u_t} > 0$*

$$-D''(V_t \rfloor s) = \Lambda D''(\overline{\partial}\dot{u}_t \wedge s) + i(D')^*(\overline{\partial}\dot{u}_t \wedge s) \tag{1.40}$$

Proof. To see this recall in the Kähler setting

$$[L, \Lambda] = H$$

where L, Λ, H are the Lefschetz, dual Lefschetz, and counting operator which act on the form part of the section.

Note that $L\Lambda(V_t \rfloor s) = 0$ since $V_t \rfloor s$ is an L valued $(n-1, 0)$ form and the action of Λ reduces type by $(1, 1)$.

Claim 2.

$$\Lambda L(V_t \rfloor s) = (V_t \rfloor s)$$

Proof. Indeed,

$$\Lambda L(V_t \rfloor s) = -[L, \Lambda](V_t \rfloor s) = -H(V_t \rfloor s) = -(n-1-n)I(V_t \rfloor s) = (V_t \rfloor s)$$

□

From the Kähler identity

$$[\Lambda, D^{''}] = -i(D^{'})^* \tag{1.41}$$

noting that the left side of (1.41) is a commutator it follows directly that

$$D^{''}\Lambda(\overline{\partial}\dot{u}_t \wedge s) = \Lambda D^{''}(\overline{\partial}\dot{u}_t \wedge s) + i(D^{'})^*(\overline{\partial}\dot{u}_t \wedge s) \tag{1.42}$$

Now (1.40) follows immediately from the corollary and claim. □

Proposition 10. *V_t is holomorphic for each fixed t.*

Proof. It is enough to show by lemma 5 (1.40) that $D^{''}(\Lambda(\overline{\partial}\dot{u}_t) \wedge s) = 0$ ($\overline{\partial} = D^{''}$ for unitary connections compatible with the holomorphic structure). This follows because from holomorphicity of s we obtain V_t is away from the zero's of the vector field, and conclude V_t is holomorphic by Riemanns extension theorem(V_t smooth).

Start by analyzing the equality case. Since $\mathcal{F}_{\omega_0}$ is concave it is affine along smooth geodesics connecting any two of its critical points. $\mathcal{E}_{\omega_0}$ is affine along geodesics so $\mathcal{L}_{\omega_0}$ is too. By smoothness this means (1.26) and identities (1.33), (1.34) yield the equalities

$$0 = \frac{d^2}{dt^2}\mathcal{L}_{\omega_0}(u_t) = \frac{-1}{N}(\sum_i(||\dot{u}_t s_i||^2 - ||P(\dot{u}_t s_i)||^2 - ||\overline{\partial}(\dot{u}_t s_i)||^2)) \geq 0 \tag{1.43}$$

which is equivalent to

$$||\overline{\partial}(\dot{u}_t s_i - P(\dot{u}_t s_i))||^2 = ||\dot{u}_t s_i - P(\dot{u}_t s_i)||^2 \tag{1.44}$$

Recalling (1.28), (1.29) and from the discusion in section §5 solving $\overline{\partial}^* \sigma_i = \dot{u}_t s_i - P(\dot{u}_t s_i)$ with σ_i holomorphic we have

$$||\overline{\partial}^* \sigma_i||^2 \leq ||\overline{\partial}\overline{\partial}^* \sigma_i|| ||\sigma_i|| \leq ||\overline{\partial}\overline{\partial}^* \sigma_i|| ||\overline{\partial}^* \sigma_i|| \leq ||\overline{\partial}^* \sigma_i|| ||\overline{\partial}^* \sigma_i|| \tag{1.45}$$

The last inequality is a consequence (1.44) from which one obtains $||\overline{\partial}\overline{\partial}^* \sigma_i|| = ||\overline{\partial}^* \sigma_i||$. Since (1.45) is really a string of equalities it follows that the inequality (1.29) is an equality $||\sigma_i|| = ||\overline{\partial}^* \sigma_i||$. Since (1.31) is equality precisely when the terms $||D' \sigma_i||, ||(D')^* \sigma_i||$ vanish in the Bochner Kodaira type identity we have

$$\Box_{D'} \sigma_i = 0, \quad \overline{\partial} \sigma_i = 0 \tag{1.46}$$

Claim 3.

$$\overline{\partial} \dot{u}_t \wedge s_i = \sigma_i \tag{1.47}$$

Proof. This a consequence of the equality case :

$$\begin{aligned}\overline{\partial}\dot{u}_t \wedge s_i &= \overline{\partial}(\dot{u}_t s) \\ &= \overline{\partial}(\dot{u}_t s_i - P(\dot{u}_t s_i)) \\ &= \overline{\partial\partial}^* \sigma_i \\ &= \sigma_i\end{aligned}$$

The last equality follows from type considerations and equations in(1.46) (essentially the content of (1.30), (1.31)). So

$$\overline{\partial\partial}^* \sigma_i = \square_{D''}\sigma_i = (\square_{D'} + I)\sigma_i = \sigma_i$$

□

As a consequence of the claim and (1.46)

$$D''\overline{\partial}\dot{u}_t \wedge s_i = D''\overline{\partial}(\dot{\phi} s_i) = 0; \quad (D')^*\overline{\partial}\dot{u}_t \wedge s_i = (D')^*\sigma_i = 0$$

So by the lemma 5 (1.40) V_t is holomorphic for each fixed t. □

Proposition 11. *V_t is static.*

Pass to co-ordinates and make the local calculation for $V_t \rfloor \omega_{u_t} = \overline{\partial} \dot{u}_t$

$$\begin{aligned} \sqrt{-1} g_{m\overline{n}} V_t^m X^{\overline{n}} &= (\dot{u}_t)_{\overline{q}} X^{\overline{q}} \\ V_t^p &= g^{p\overline{q}} g_{m\overline{q}} V_t^m \\ &= -\sqrt{-1} g^{p\overline{q}} (\dot{u}_t)_{\overline{q}} \end{aligned}$$

The following is well known see [Bo09], except here we operate directly on the manifold.

Lemma 6. *Along smooth geodesics*

$$\dot{V}_t \rfloor \omega_{u_t} = \overline{\partial}(\ddot{u}_t - \frac{1}{2}|\nabla \dot{u}_t|^2_{\omega_{u_t}}) = 0 \tag{1.48}$$

Proof. Differentiate $V_t \rfloor \omega_{u_t} = \overline{\partial} \dot{u}_t$ to obtain

$$\dot{V}_t \rfloor \omega_{u_t} + V_t \rfloor \partial_t \omega_{u_t} = \overline{\partial} \ddot{u}_t \tag{1.49}$$

$$\dot{V}_t \rfloor \omega_{u_t} = \overline{\partial} \ddot{u}_t - V_t \rfloor \partial_t \omega_{u_t} \tag{1.50}$$

Using $V_t \rfloor \omega_{u_t} = \overline{\partial} \dot{u}_t$ again obtain

$$\sqrt{-1} \partial V_t \rfloor \omega_{u_t} = \sqrt{-1} \partial \overline{\partial} \dot{u}_t = \partial_t \omega_{u_t}$$

Equivalently (1.50) is

$$\dot{V}_t \rfloor \omega_{u_t} = \overline{\partial} \ddot{u}_t - \sqrt{-1} V_t \rfloor (\partial (V_t \rfloor \omega_{u_t}))$$

Computing $\sqrt{-1}V_t\partial(V_t\rfloor\omega_{u_t})$ locally:

$$\begin{aligned}\sqrt{-1}V_t\rfloor\partial((\dot{u}_t)_{\overline{j}}dz^{\overline{j}}) &= \sqrt{-1}V_t\rfloor((\dot{u}_t)_{i\overline{j}}dz^i\wedge dz^{\overline{j}})\\ &= \sqrt{-1}V_t^i(\dot{u}_t)_{i\overline{j}}dz^{\overline{j}}\\ &= g^{i\overline{s}}(\dot{u}_{\overline{s}})(\dot{u}_{i\overline{j}})dz^{\overline{j}}\\ &= \overline{\partial}\frac{1}{2}|\nabla\dot{u}_t|^2_{\omega_{u_t}} - \overline{\partial}(V_t^i)(\dot{u}_t)_i\\ &= \overline{\partial}(\frac{1}{2}|\nabla\dot{u}_t|^2_{\omega_{u_t}})\end{aligned}$$

□

proof of prop. 11. By the lemma

$$\dot{V}_t\rfloor\omega_{u_t} = \overline{\partial}(\ddot{u}_t - \frac{1}{2}|\nabla\dot{u}_t|^2_{\omega_{u_t}}) = 0$$

So conclude $\delta V_t = 0$ □

proof of prop. 9. Now we are ready to conclude with proposition 8. $\frac{-V+\overline{V}}{2} = -ImV$ generates the flow ϕ_t (ϕ_t biholomorphism). Since V above is static so is iV. Abusing notation denote this by V and then ϕ_t is the flow generated by ReV. Locally $V\rfloor\partial\overline{\partial}(u_t+\psi_0) = \overline{\partial}\dot{u}_t$. Virtues of compactness grant a uniform $r>0$, with $u_t+\psi_0$ given on some $B_{\omega_0}(r,p_{i_0})$ and $\phi_t(B_{\omega_0}(\frac{r}{2},p_i))\subset B_{\omega_0}(r,p_i)$ for $|t|<\frac{r}{2}$ and all i. A consequence of chain rule is

$$((u_t+\dot{\psi_0})\cdot\phi_t) = \frac{1}{2}(V\rfloor\partial(u_t+\psi_0) + \overline{V\rfloor\partial(u_t+\psi_0)})\cdot\phi_t + (u_t\dot{+}\psi_0)\cdot\phi_t \tag{1.51}$$

taking $\sqrt{-1}\partial\overline{\partial}$ of (1.51) obtain

$$\sqrt{-1}\partial\overline{\partial}((u_t + \dot{\psi_0}) \cdot \phi_t) = \sqrt{-1}\partial\overline{\partial}\frac{1}{2}(V\rfloor\partial(u_t + \psi_0) + \overline{V\rfloor\partial(u_t + \psi_0)}) \cdot \phi_t$$
$$+ \sqrt{-1}\partial\overline{\partial}(u_t \dot{+} \psi_0) \cdot \phi_t$$
$$\frac{d}{dt}\phi_t^*\omega_{u_t} = \frac{1}{2}\phi_t^*(\sqrt{-1}\partial\overline{\partial}(V\rfloor\partial(u_t + \psi_0) + \overline{V\rfloor\partial(u_t + \psi_0)}))$$
$$+ \phi_t^*(\sqrt{-1}\partial\overline{\partial}(u_t \dot{+} \psi_0))$$
$$\frac{d}{dt}\phi_t^*\omega_{u_t} = \frac{1}{2}\phi_t^*(\partial(\sqrt{-1}V\rfloor\overline{\partial}\partial(u_t + \psi_0)) + \overline{\partial}(-\sqrt{-1}\overline{V\rfloor\overline{\partial}\partial(u_t + \psi_0)}))$$
$$+ \phi_t^*(\sqrt{-1}\partial\overline{\partial}(u_t \dot{+} \psi_0))$$
$$= \frac{1}{2}\sqrt{-1}\phi_t^*(-\partial\overline{\partial}(u_t \dot{+} \psi_0) + \overline{\partial}\partial(u_t \dot{+} \psi_0)) + \phi_t^*(\sqrt{-1}\partial\overline{\partial}(u_t \dot{+} \psi_0)) = 0$$

So $\frac{d}{dt}\phi_t^*(\omega_{u_t})_{|B(\frac{r}{2},p_{i_0})} = \partial\overline{\partial}((u_t + \dot{\psi_0}) \cdot \phi_t) = 0$. Although the proposition now follows almost directly, we may also conclude by partitioning $[0,1]$ into sufficiently small sub-intervals depending on the cover and the fact that the time one map is a composition of the maps corresponding to each subinterval.

Remark 7. *Note the above is essentially a manifestation of*

$$\frac{d}{dt}\phi_{-t}^*\omega_{u_t} = \phi_{-t}^*\frac{d}{dt}\omega_{u_t} + \phi_{-t}^*\mathcal{L}_{-\frac{(V-\overline{V})}{2}}\omega_{u_t}$$

where V is the gradient vector field originally defined and ϕ_t is the flow generated by ImV. Use Cartans formula to obtain cancellations.

Since $\phi_{-t}^*\omega_{u_t} - \omega_{u_0} = 0$ this means at the level of potentials $\phi_{-t}^*(u_t + \psi_0) - \psi_0 = C_t$.

V can be lifted to a holomorphic vector fields on the total space L so that action by ϕ_t is induced from $Aut_0(M, L)$ by lemma 13 in [Ber10a]. □

1.8 Berndtsson argument setup

Recall (X, ω) is Kähler and that $\mathcal{F}_\omega$ is concave. Another advantage is that it behaves nicely in the low regularity setting. In fact, for low regularity purposes, when $L = -K_X$ the functional simplifies to negative of the Ding-Tian functional and this has better regularity properties than the Mabuchi functional . Berndtsson shows by a direct envelope construction critical points of the Ding-Tian functional can be connected by a C^0 sub-geodesic (see §11, [Bo] and [BerDe]). These two inputs (Ding-Tian functional and C^0 sub-geodesics) can be used to obtain the Bando-Mabuchi uniqueness theorem.

More precisely Berndtsson obtains Bando-Mabuchi uniqueness theorem by deducing

Proposition 3. *Let $L = -K_X$ be semi-positive and assume $H^{n,1}(X) = 0$. Let ϕ_t be C^0 sub-geodesic such that ϕ_t does not depend on Imt, then $L(t) = -\log i^{n^2} \int_X e^{-\phi_t} dz \wedge d\bar{z}$ is convex. Further if $L(t)$ is affine in a neighborhood of 0 then there is holomorphic vector field V (perhaps time dependent) on X with flow F_t such that $F_t^* \partial\bar{\partial}\phi_t = \partial\bar{\partial}\phi_0$.*

As in the previous section one needs to analyze the smooth case. The final stage involves approximation.

Let u_t smooth but $i\partial\bar{\partial}u_t \geq 0$. Then consider solvability of

$$\partial^{u_t} v = \pi_\perp(\dot{u}_t s) =: \eta \tag{1.52}$$

$s \in H^0(X, \Omega_X^n \otimes L)$ as before. From the consequence of the Lefschetz decomposition that $\Lambda^{n-1}V \cong \Lambda^{n+1}V$ write $\alpha = L_\omega v = v \wedge \omega$ where $v \in \mathcal{A}^{n-1,0}(X)$ when $\alpha \in \mathcal{A}^{n,1}(X, L)$. Then

Proposition 12. *Solvability for v in (1.52) is equivalent to solvability of*

$$\overline{\partial}^*_{u_t}\alpha = \eta$$

for $\alpha \in \mathcal{A}^{n,1}(X, L)$ when η is orthogonal to $H^0(X, K_X \otimes L)$.

Proof. The equivalence is a calculation using the following facts:

- Note that ∂_{u_t} is the $(1, 0)$ part of the Chern connection i.e

$$\nabla^{(1,0)} := \partial + \partial \log(e^{-u_t}) = \partial - \partial u_t \wedge = \partial_{u_t}$$

- Recall we have for $\alpha \in P^k$(primitive elements of Λ^k)

$$*L^j \alpha = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j} \mathbb{I}(\alpha)$$

  $*, L$ determined by the structure from (X, ω).

- $[\overline{\partial}^*, L] = i\partial$. For $L_{|U} \cong \mathcal{O}_U$ with hermitian structure depending on u_t as above, the adjoint operator is given by

$$\overline{\partial}^*_t = -\overline{*}\overline{\partial_{u_t}}\overline{*} = - * \partial_{u_t} *$$

then

$$
\begin{aligned}
\overline{\partial}^*_{u_t}\alpha &= \overline{\partial}^*_{u_t} v \wedge \omega \\
&= -*(\partial - \partial u_t \wedge *(v \wedge \omega)) \\
&= [\overline{\partial}^*, L]v + *\partial u_t * Lv
\end{aligned}
$$

But v is primitive and in $v \in P^{n-1}$. It follows $*Lv = (-1)^{\frac{(n-1)n}{2}} i^{n-1} v$, and using that $\partial\phi \wedge v \in P^n$ we get

$$
\begin{aligned}
\overline{\partial}^*_{u_t}\alpha = \overline{\partial}^* Lv + *\partial u_t * Lv &= [\overline{\partial}^*, L]v + (-1)^{\frac{(n-1)n}{2}} i^{n-1} * \partial u_t \wedge v \\
&= i\partial v + (-1)^{\frac{(n-1)n}{2}} i^{n-1} * \partial u_t \wedge v \\
&= i\partial v + (-1)^{\frac{(n-1)n}{2}} i^{n-1} (-1)^{\frac{n(n+1)}{2}} i^n \partial u_t \wedge v \\
&= i\partial v + (-1)^{n^2} (-1)^{n-1} i\partial u_t \wedge v \\
&= i(\partial - \partial u_t \wedge)v \\
&= i\partial_{u_t} v \\
\implies \overline{\partial}^*_{u_t}\alpha = \eta &= \partial_{u_t} v
\end{aligned}
$$

□

The solvability of these equations also comes with estimates.

Fact 1. *Recall for* $\overline{\partial}$ *and its adjoint (Von Neuman's sense is the relevant one)*

- $Ker\overline{\partial}^* = (Im\overline{\partial})^\perp$
- $(Ker\overline{\partial})^\perp = \overline{Im\overline{\partial}^*}$

In particular when $\overline{\partial}$ *is a surjection,* $\overline{\partial}^*$ *is injective. In fact when* $H^{n,1}(X) = 0$ *we have a surjection on* $\overline{\partial}$ *closed forms and the adjoint is injective. When* $\overline{\partial}$ *has closed range the adjoint has closed range equal to* $(Ker\overline{\partial})^{\perp}$ *. In this setting closedness of range boils down to the estimates.*

If $\overline{\partial}$ *has closed range then solutions to the equation* $\overline{\partial}f = \alpha$ *comes with the estimate*

$$||f|| \leq C||\alpha|| \tag{1.53}$$

From here it follows that solvability of $\partial_{u_t}v = \eta$ *comes with the estimates*

$$||v|| \leq C||\eta||$$

To elaborate, (1.53) is similar to §6 where solving $\sigma_1 = \overline{\partial}\sigma_0$ for σ_0 orthogonal to holomorphic $(n, 1)$ L-valued sections comes with estimate $||\overline{\partial}\sigma_0|| \geq C_0||\sigma_0||$, $C_0 = 1$. In §6 we specialized the hermitian metric to the one whose curvature is $-i\omega$ to conclude. However, in the Kähler setting the Akizuki-Nakano identity applies for any unitary connection compatible with the holomorphic structure. So more generally $||\overline{\partial}\sigma_0|| \geq C||\sigma_0||$ (adjusting to an auxiliary hermitian metric h changes the constant arising from $\int_X \langle [\theta_h, \Lambda]\sigma_0, \sigma_0\rangle dV$). So for our purposes proceed trivially obtaining the estimate $||\overline{\partial}^*_{u_t}\sigma_0||^2 + ||\overline{\partial}\sigma_0||^2 \geq C||\sigma_0||^2$, adding the extra nonnegative term $||\overline{\partial}^*_{u_t}\sigma_0||^2$. From here conclude through a functional analysis argument (see [De]) to obtain the estimate $||\overline{\partial}^*_{u_t}\sigma_0|| \geq \tilde{C}||\sigma_0||$.

In particular we obtain $\eta = \overline{\partial}^*_{u_t}\alpha$ and $\alpha = \overline{\partial}f$

$$||\overline{\partial}^*_{u_t}\alpha|| \geq \tilde{C}||\alpha||$$

Next we claim

Claim 4. *For $\alpha = v \wedge \omega$ as above, $||\alpha|| = ||v||$.*

Proof.

$$\langle \alpha, \alpha \rangle = \langle Lv, Lv \rangle = \langle v, \Lambda Lv \rangle$$

$$\Lambda Lv = -[L, \Lambda]v = -Hv = -(n-1-n)v = v$$

$$\implies \langle \alpha, \alpha \rangle = \langle v, v \rangle$$

□

As a consequence

$$||\overline{\partial}^*_{u_t}\alpha||\tilde{C} \geq ||\alpha|| \iff ||\partial_{u_t} v|| \geq \tilde{C}||v||$$

It follows that if v_t solves

$$\partial_{u_t} v_t = \pi_\perp(\dot{u}s)$$

the following estimate holds

$$||v_t|| \leq ||\pi_\perp(\dot{u}_t s)|| \leq ||\dot{u}_t s|| \leq \frac{1}{\tilde{C}}||\dot{u}_t||||s|| \tag{1.54}$$

From the properties of C^0 sub-geodesics i.e its Lipschitz, it is known that $||\dot{u}_t||$ is bounded.

1.9 Generalized Gradient Vector Field

Since $L \otimes K_X \cong \mathcal{O}_X$ we can take $1 = s \in H^0(X, L \otimes K_X)$ and define v_t by $\partial^{u_t} v_t = \pi_\perp(\dot{u}_t s)$. In turn, since s does not vanish, define V_t by

$$v_t = -V_t \rfloor s \tag{1.55}$$

Recall that curvature of a hermitian connection on a holomorphic vector bundle has no $(0, 2)$ part so it is given by

$$[D_{u_t}, \overline{\partial}] = \partial^{u_t}\overline{\partial} + \overline{\partial}\partial^{u_t} = \partial\overline{\partial}u_t \tag{1.56}$$

Lemma 7. *For solutions $\partial_{u_t} v_t = \pi_\perp(\dot{u}_t s)$ the following identity holds*

$$\partial\overline{\partial}u_t \wedge v = \overline{\partial}(\dot{u}_t s) + \partial^{u_t}\overline{\partial}v \tag{1.57}$$

Proof. Using (1.56) and the definition of $\partial^{u_t} v_t$

$$\partial\overline{\partial}u_t \wedge v_t = [D_{u_t}, \overline{\partial}]v_t = \partial^{u_t}\overline{\partial}v_t + \overline{\partial}\pi_\perp(\dot{u}_t s)$$

□

From (1.55) obtain

$$\begin{aligned}\partial\overline{\partial}u_t \wedge v &= \partial\overline{\partial}u_t \wedge (-V_t \rfloor s)\\ &= (V_t \rfloor \partial\overline{\partial}u_t) \wedge s\end{aligned}$$

Lemma 8. *If $\overline{\partial} v = 0$ then V_t defined in (1.55) satisfies*

$$\overline{\partial}\dot{u}_t = V_t \rfloor \omega_{u_t} \tag{1.58}$$

Proof. For s as given above

$$\begin{aligned} \overline{\partial}\dot{u}_t \wedge s &= (V_t \rfloor \partial\overline{\partial} u_t) \wedge s \\ \implies i\overline{\partial}\dot{u}_t &= V_t \rfloor i\partial\overline{\partial} u_t \\ &= V_t \rfloor \omega_{u_t} \end{aligned}$$

□

Remark 8. *V_t as defined above is a generalized gradient vector field(referred to as a gradient vector field for convenience since it behaves similarly).*

Note that

$$\mathcal{F}(t) := -\log \int_X e^{-u_t} = -\log ||s||^2 \tag{1.59}$$

where $||s||^2 = \int_X c_n s \wedge \overline{s} e^{-u_t}$.

Remark 9. *$||\tilde{s}||^2$ can be viewed as integration along fibers. Exactly as Berndtsson considers Kähler fibrations with compact fibers, $p : \tilde{X} \to Y$. Here, $0 \in U^0 = U = Y \subset \mathbb{C}$, the fibers are copies $X_t := X = p^{-1}(t)$ and we may think of $\tilde{X} = U \times X$ (we suppress the other structures since we wish to discuss this naively. Involved is the introduction of a structure $E := \cup_{t \in U}\{t\} \times E_t$ with $E_t := H^0(X_t, L|_{X_t} \otimes K_{X_t})$. $E \to U$ is naturally a holomorphic vector bundle from semi-positivity of L and that X is Kähler using an Ohsawa-Takegoshi extension type theorem: an element of E_t needs to extend to a section of E locally in a* *holomorphic way with estimates. Elements of E_t which are L valued $(n,0)$ forms on X_t can be viewed as sections taking values in $K_{\tilde{X}}$ over X_t(by wedging with dt) or as the restriction to X_t of sections over $\tilde{X}$ with values in $K_{\tilde{X}}$ see [Bo09] and [Bo07] for details. Granted elements of E_t extend to local sections of E, take a basis of E_t that extends to a local frame of E. These can be viewed as a collection of $(n,0)$ forms over the preimage under the projection p of an open set in U in the base. Denote one such by u. Its restriction to each X_t defines an element of E_t. u defines holomorphic section of E if it defines a holmorphic section of $K_{\tilde{X}}$ i.e $u \wedge dt$ is a holomorphic section of $K_{\tilde{X}}$, that is $\overline{\partial} u \wedge dt = 0$. Finally E has a naturally defined hermitian metric coming from that on L(fiber-wise this is the usual hermitian metric on $L \otimes K_X$) allowing to define the Chern connection operator on E.)*

Recall

$$||s||^2 = \pi_*(c_n s \wedge \overline{s} e^{-u_t})$$

by definition of integration along fibers obtain

Claim 5. *Let $v \in \mathcal{A}^{n-1,0}(X)$ and s as above. Set $\tilde{s} = s - dt \wedge v$ then*

$$\partial\overline{\partial}_t ||\tilde{s}||^2 = \partial\overline{\partial}_t ||s||^2$$

This can also be calculated directly viewing $\int_X$ as integration along fibers since the fibration is trivial.

First note

$$\tilde{s} \wedge \tilde{s} = s \wedge \overline{s} - s \wedge \overline{dt \wedge v} - dt \wedge v \wedge \overline{s} + dt \wedge v \wedge \overline{dt \wedge v}$$

Proof. In the following calculation the type I forms have no contribution so

$$\begin{aligned}
\partial\overline{\partial}_t||\tilde{s}||^2 &= \int_X c_n\partial\overline{\partial}_t(\tilde{s}\wedge\overline{\tilde{s}}e^{-u_t}) \\
&= \int_X c_n\partial\overline{\partial}_t(s\wedge\overline{s} - s\wedge\overline{dt\wedge v} - dt\wedge v\wedge\overline{s} + dt\wedge v\wedge\overline{dt\wedge v})e^{-u_t} \\
&= -\int_X c_n s\wedge\overline{s}\wedge\partial\overline{\partial}_t u_t e^{-u_t} + \int_X (c_n s\wedge\overline{s}\wedge\partial_t u_t\wedge\overline{\partial}_t u_t e^{-u_t}) \\
&= \int_X c_n\partial\overline{\partial}_t(s\wedge\overline{s}e^{-u_t}) = \partial\overline{\partial}_t||s||^2
\end{aligned}$$

□

Proposition 13. *Given $\tilde{s}$ as above we have*

$$\begin{aligned}
\partial\overline{\partial}_t||\tilde{s}||^2 =& (-1)^n(\int_X c_n\partial^{u_t}\tilde{s}\wedge\overline{\partial^{u_t}\tilde{s}}e^{-u_t} + \int_X c_n\overline{\partial}\tilde{s}\wedge\overline{\overline{\partial}\tilde{s}}e^{-u_t}) \\
&+ \int_X c_n\tilde{s}\wedge\overline{\tilde{s}}\wedge\partial\overline{\partial}u_t e^{-u_t} &(1.60)\\
=& (-1)^n\int_X c_n\overline{\partial}\tilde{s}\wedge\overline{\overline{\partial}\tilde{s}}e^{-u_t} + \int_X c_n\tilde{s}\wedge\overline{\tilde{s}}\wedge\partial\overline{\partial}u_t e^{-u_t} &(1.61)\\
=& (||\overline{\partial}v||^2 dt\wedge d\overline{t} + \pi_*(c_n\partial\overline{\partial}u_t\wedge\tilde{s}\wedge\overline{\tilde{s}}e^{-u_t})) &(1.62)
\end{aligned}$$

where $\int_X$ is interpreted as integration along fibers.

Proof. (1.60) simplifies to (1.61) since $\partial^{u_t}\tilde{s}$ vanishes:

$$\begin{aligned}
\partial^{u_t}v_t &= \pi_\perp(\dot{u}s) \\
\implies dt\wedge\partial^{u_t}v_t &= dt\wedge\pi_\perp(\dot{u}_t s) \\
&= dt\wedge(\dot{u}_t s + h) \\
&= \dot{u}_t dt\wedge s + dt\wedge h
\end{aligned}$$

$$= \pi_{\perp}(\partial u_t \wedge s)$$

$$= -\pi_{\perp}((\partial - \partial u_t)s)$$

$$= -\partial^{u_t} s$$

Where in the third line h subtracts out the holomorphic part of $\dot{u}_t s$. The last line follows since $Im\partial^u \subseteq \overline{Im\overline{\partial}^*} = Ker\overline{\partial}^{\perp}$. So $\partial^{u_t}\tilde{s} = 0$.

(1.62) follows since $\tilde{s} = s - dt \wedge v$ so

$$\overline{\partial}\tilde{s} = dt \wedge \overline{\partial} v$$

$$\implies \overline{\partial}\tilde{s} \wedge \overline{\overline{\partial}\tilde{s}} = (-1)^n \overline{\partial} v \wedge \overline{\overline{\partial} v} \wedge dt \wedge \overline{dt}$$

To obtain (1.60) some pre-computation is necessary. Observe (1.63), (1.64), and (1.65) hold:

$$\int_X c_n \overline{\partial}_t \tilde{s} \wedge \overline{\tilde{s}} e^{-u_t} = 0 \tag{1.63}$$

since this involves integration along fibers of a type I form

$$\int_X c_n \overline{\partial}_t \tilde{s} \wedge \overline{\tilde{s}} e^{-u_t} = \int_X c_n dt \wedge \overline{\partial}_t v \wedge \overline{\tilde{s}} e^{-u_t} = 0$$

Similarly

$$\int_X c_n \tilde{s} \wedge \overline{\overline{\partial}\tilde{s}} e^{-u_t} = 0 \tag{1.64}$$

from (1.63) and (1.64) it follows:

$$0 = \overline{\partial_t} \int_X c_n \tilde{s} \wedge \overline{\overline{\partial}\tilde{s}} e^{-u_t} = \int_X c_n \overline{\partial}\tilde{s} \wedge \overline{\overline{\partial}\tilde{s}} e^{-u_t} + (-1)^n \int_X c_n \tilde{s} \wedge \overline{\partial^{u_t}\overline{\partial}\tilde{s}} e^{-u_t} \tag{1.65}$$

(1.60) follows from the following computation:

$$\begin{aligned}
\partial\overline{\partial}_t||\tilde{s}||^2 &= (-1)^n\partial_t\pi_*(c_n\tilde{s}\wedge\overline{\partial_t^{u_t}\tilde{s}e^{-u_t}}) \\
&= (-1)^n\pi_*(c_n\partial_t^{u_t}\tilde{s}\wedge\overline{\partial_t^{u_t}\tilde{s}}e^{-u_t}) + \pi_*(c_n\tilde{s}\wedge\overline{\overline{\partial}\partial_t^{u_t}\tilde{s}}e^{-u_t}) \\
&= (-1)^n\pi_*(c_n\partial^{u_t}\tilde{s}\wedge\overline{\partial^{u_t}s}e^{-u_t}) + \pi_*(c_n\tilde{s}\wedge\overline{\overline{\partial}\partial^{u_t}\tilde{s}}e^{-u_t}) \\
&= (-1)^n\pi_*(c_n\partial^{u_t}\tilde{s}\wedge\overline{\partial^{u_t}\tilde{s}}e^{-u_t}) + \pi_*(c_n\tilde{s}\wedge\overline{\partial\overline{\partial}u_t\wedge\tilde{s}}e^{-u_t}) \\
&\quad - \pi_*(c_n\tilde{s}\wedge\overline{\partial^{u_t}\overline{\partial}\tilde{s}}e^{-u_t})
\end{aligned}$$

Further, applying (1.65)

$$\begin{aligned}
\partial\overline{\partial}_t||\tilde{s}||^2 &= (-1)^n(\pi_*(c_n\partial^{u_t}\tilde{s}\wedge\overline{\partial^{u_t}\tilde{s}}e^{-u_t}) \\
&\quad + (-1)^n\pi_*(c_n\overline{\partial}\tilde{s}\wedge\overline{\overline{\partial}\tilde{s}}e^{-u_t})) + \pi_*(c_n\tilde{s}\wedge\overline{\partial\overline{\partial}u_t\wedge\tilde{s}}e^{-u_t})
\end{aligned}$$

and the proposition follows. □

Proposition 14.

$$||s||^2\partial\overline{\partial}_t\mathcal{F}(t) = \langle\theta_t s, s\rangle = -\partial\overline{\partial}_t||s||^2$$

where θ_t is the curvature of E.

Proof. Recall $||s||^2 = ||\tilde{s}||^2$. For the second equality see [Bo09]. In the first equality note that

$$\partial_{\overline{t}}||\tilde{s}||^2 = \langle\overline{\partial}_t\tilde{s},\tilde{s}\rangle + (-1)^n\langle\tilde{s},\partial^{u_t}\tilde{s}\rangle = 0$$

so

$$\partial_t \partial_{\bar{t}} \mathcal{F}(t) = -\frac{\partial \overline{\partial}_t ||\tilde{s}||}{||s||^2} + \frac{\partial_{\bar{t}} ||\tilde{s}|| \partial_t ||\tilde{s}||}{||s||^4} = -\frac{\partial \overline{\partial}_t ||\tilde{s}||}{||s||^2} = -\frac{\partial \overline{\partial}_t ||s||}{||s||^2}$$

where the last equality follows from the claim. □

Remark 10. *Granted the necessary regularity, from convexity of $\mathcal{F}$ along u_t connecting two Kähler-Einstein metrics $\partial \overline{\partial}_t \mathcal{F} \equiv 0$ i.e $\mathcal{F}$ is linear on u_t. (1.62) and the subsequent proposition obtain $||\overline{\partial} v|| = 0$. It also follows $\partial \overline{\partial} u_t \wedge \tilde{s} \wedge \overline{\tilde{s}} = 0$ from which $\partial \overline{\partial} u_t \wedge \tilde{s} = 0$ since $i \partial \overline{\partial} u_t \geq 0$. Since $\overline{\partial} v = 0$ we see from (1.58) that V_t as defined is a gradient vector field.*

Lemma 9. *Suppose u_t is smooth and $\partial \overline{\partial} u_t \wedge \tilde{s} = 0$ then*

$$\left(\frac{\partial u_t}{\partial t \partial \bar{t}} - \partial_X \left(\frac{\partial u_t}{d\bar{t}}\right)(V_t)\right) = 0 \tag{1.66}$$

Proof. Since $\partial \overline{\partial} u_t \wedge \tilde{s} = 0$ it follows that the coefficient of $dt \wedge d\bar{t}$ vanishes:

$$0 = dt \wedge d\bar{t}\left(\frac{\partial u_t}{\partial t \partial \bar{t}} \wedge s\right) - \partial_X \partial_{\bar{t}} u_t \wedge dt \wedge v \tag{1.67}$$

but since $v_t = -V_t \rfloor s$

$$\partial_X \partial_{\bar{t}} u \wedge dt \wedge v = \partial_X \frac{\partial u_t}{\partial \bar{t}} dt \wedge d\bar{t} \wedge V_t \rfloor s \tag{1.68}$$

and

$$\begin{aligned} 0 &= \partial_X \left(\frac{\partial u_t}{\partial \bar{t}}\right) \wedge s \\ \implies 0 &= V_t \rfloor \partial_X \left(\frac{\partial u_t}{\partial \bar{t}}\right) \wedge s - \partial_X \left(\frac{\partial u_t}{\partial \bar{t}}\right) \wedge V_t \rfloor s \end{aligned} \tag{1.69}$$

So from (1.68) and (1.69)

$$\partial_X \partial_{\bar{t}} u \wedge dt \wedge v_t = dt \wedge d\bar{t} \wedge s(V_t \rfloor \partial_X (\frac{\partial u_t}{d\bar{t}}))$$
$$\implies 0 = dt \wedge d\bar{t} \wedge s(\frac{\partial u_t}{\partial t \partial \bar{t}} - \partial_X (\frac{\partial u_t}{d\bar{t}})(V_t))$$

This concludes the calculation. □

Set $\mu := (\frac{\partial u_t}{\partial t \partial \bar{t}} - \partial_X (\frac{\partial u_t}{d\bar{t}})(V_t))$.

Remark 11. *When $i\partial\overline{\partial} u_t > 0$, we have $0 = \mu = c(\phi)$ satisfies the geodesic equation because $\partial \dot{\phi}(V_t) = |V_t|_t^2$ with V_t the gradient vector field.*

Since V_t satisfies the equation $v_t = -V_t \rfloor s$, the condition that V_t is static translates to

$$0 = \frac{\partial v_t}{\partial \bar{t}} = -\frac{\partial V_t}{\partial \bar{t}} \rfloor s$$

since s does not vanish.

Proposition 15. *If $H^{n,1}(X, L) = 0$(vanishes if $i\partial\overline{\partial} u_t > 0$) then $\frac{\partial v_t}{\partial \bar{t}} = 0$*

Proof. Recall we have

$$\partial^{u_t} v = \dot{u}_t s + h_t \tag{1.70}$$

where h_t is holomorphic for each fixed t.

Since $\partial^{u_t} = \partial - \partial u_t \wedge$

$$\frac{\partial}{\partial \bar{t}} \partial^{u_t} v = \partial^{u_t} \frac{\partial v}{\partial \bar{t}} - \partial_X \frac{\partial u_t}{\partial \bar{t}} \wedge v$$
$$= \partial^{u_t} \frac{\partial v}{\partial \bar{t}} + V_t \rfloor \partial_X (\frac{\partial u_t}{\partial \bar{t}}) s \tag{1.71}$$

from (1.69).

Similarly the right hand side of (1.70) becomes

$$\frac{\partial}{\partial \bar{t}}(\dot{u}_t s + h_t) = \frac{\partial^2 u_t}{\partial t \partial \bar{t}} s + \frac{\partial h_t}{\partial \bar{t}} \tag{1.72}$$

Combining (1.71), (1.72)

$$\partial^{u_t} \frac{\partial v}{\partial \bar{t}} = \mu s + \frac{\partial h_t}{\partial \bar{t}}$$

but $\partial^{u_t} \frac{\partial v}{\partial \bar{t}}$ is orthogonal to holomorphic forms so

$$\partial^{u_t} \frac{\partial v}{\partial \bar{t}} = \pi_\perp(\mu s + \frac{\partial h_t}{\partial \bar{t}}) = \pi_\perp(\mu s) = 0$$

since $\mu = 0$. Note $\frac{\partial v}{\partial \bar{t}} \wedge \omega$ is $\overline{\partial}_X$ closed. This entails $\frac{\partial v}{\partial \bar{t}} = 0$ because the assumption $H^{n,1}(X) = 0$ gives that $\overline{\partial}_X$ is surjective so the adjoint is injective i.e let v belong to the kernel of the adjoint. Then

$$\langle u, v \rangle = \langle \overline{\partial}_X \gamma, v \rangle = (-1)^n \langle \gamma, \partial^{u_t} v \rangle = 0 \implies v = 0$$

So the generalized gradient vector field as defined is static and the proposition follows. □

1.10 Non-smooth case

This section overviews the last part of Berndtssons argument. See [Bo] for further details.

In general a singular metric ϕ with $i\partial\overline{\partial}\phi \geq 0$ cannot be approximated by a decreasing sequence of smooth metrics with nonnegative curvature. However, this is possible if the line bundle has some smooth metric of strictly positive curvature.

In fact it is known that one can approximate a singular metric on L with nonnegative curvature by a decreasing sequence of smooth metrics such that

$$i\partial\overline{\partial}\phi^{\nu} > -\epsilon_{\nu}\omega \tag{1.73}$$

where ω is some Kähler form.

This proceeds by considering the line bundle $L + \epsilon F$ where F is positive. Then $L + \epsilon F$ admits hermitian metric $u_t + \epsilon\psi$ and this can be approximated with smooth metrics χ_{ν} of positive curvature (see [ZBSK]). Then $u_t^{\nu} = \chi^{\nu} - \epsilon\psi$ approximates u_t satisfying (1.73). Further the sequence may be arranged to be decreasing.

Recall given u_i where $i = 0, 1$ such that $i\partial\overline{\partial}u_i \geq 0$ there is a bounded geodesic u_t defined for the real part of $t \in [0, 1]$ where u_t is given by

$$u_t = \sup\{\psi_t\}$$

and the supremum is taken over all ψ_t with $\lim_{t\to i}\psi_t \leq u_i$. Note that the following barrier function participates in the supremum

$$\chi_t = \max\{\phi_0 - A\Re(t), \phi_1 + A(\Re(t) - 1)\}$$

for $A > 0$ sufficiently large because χ_t satisfies the boundary conditions and is plurisubharmonic.

So it suffices to restrict to competitors larger the χ. But then we have $-A \leq \lim_{t\to 0^+}\dot{\psi}$ and $\lim_{t\to 1^-}\dot{\psi} \leq A$. Since ψ is independent of the imaginary part of t, $i\partial\overline{\partial}\psi \geq 0$ gives that

ψ is convex hence

$$-A \leq \dot{\psi} \leq A \tag{1.74}$$

$$\implies \phi_0 - A\Re(t) \leq \psi \leq \phi_1 + A\Re(t)$$

So the same inequality holds for the majorant u_t and in fact its upper semicontinous regularization participates in the supremum so that u_t is plurisubharmonic. Since its maximal, it solves the monge-ampere equation with given boundary values. Inequality (1.74) gives that u_t is Lipschitz. Solutions u_t arising in this way are called C^0 sub-geodesics.

Obtain $\mathcal{F}_\nu$ from $\mathcal{F}$ in (1.59) by replacing u_t with u_t^ν approximating u_t as above. The loss in positivity $i\partial\overline{\partial}u_t^\nu \geq -\epsilon_\nu\omega$ is notational and one can instead proceed as if $i\partial\overline{\partial}u_t^\nu \geq 0$. Then $i\partial\overline{\partial}_t\mathcal{F}_\nu$ goes to zero weakly. Corresponding to the smooth metrics u_t^ν solutions of

$$\partial^{u_t^\nu} v_t^\nu = \pi_\perp(\dot{u_t^\nu})$$

satisfy

$$||v_t^\nu|| \leq C||\pi_\perp(\dot{u}_t^\nu s)|| \leq C||s||||\dot{u}_t^\nu|| \leq C^{'}A < \infty$$

So we may extract a subsequence of v_t^ν that weak converges to a form $v \in L^2$. Proposition 13 and 14 give that $||\overline{\partial}v_t^\nu|| \to 0$ on $X \times K$ where $K \subset \Omega$ compact. So weak converges to an element $w \in L^2$. This is $\overline{\partial}v$ in the distributional sense. It follows $\overline{\partial}v = 0$ since

$$\langle w, w\rangle = \lim\langle\overline{\partial}v_t^\nu, w\rangle \leq \liminf ||\overline{\partial}v_t^\nu||B = 0$$

from cauchy-schwartz and definition of weak convergence.

v also satisfies

$$\partial^{u_t} v = \pi_\perp(\dot{u_t} s)$$

in the weak sense i.e

$$\int_{X\times\Omega} dt \wedge d\bar{t} \wedge v \wedge \overline{\bar{\partial} W} e^{-u_t} = (-1)^n \int_{X\times\Omega} dt \wedge d\bar{t} \wedge \pi_\perp(\dot{u_t} s) \wedge \overline{W} e^{-u_t}$$

where W is a smooth form of appropriate degree.

When there are no nontrivial holomorphic vector fields then $v = 0$, and hence $\pi_\perp(\dot{u_t} s) = 0$. So $\dot{u_t}$ is holomorphic and constant since it depends only on the real part of t. Otherwise one needs to show $\partial_{\bar{t}} v = 0$ in a weak sense. Following the smooth case one needs to obtain a the distributional formulation of $\partial^{u_t} \frac{\partial v_t}{\partial \bar{t}} = \pi_\perp(\mu s)$ and then conclude using the cohomological assumption. There is some difficulty in doing this since care is needed taking limits because it is only known that $v_t \in L^2$. Some work is also required in deriving the distributional formulation. However the starting point is the use of proposition 13 and 14 to get that

$$\int_{X\times\Omega} i\partial\bar{\partial} u_t^\nu \wedge \hat{u} \wedge \bar{\hat{u}} e^{-u_t^\nu} \to 0$$

See the latest version of [Bo] for details.

Chapter 2

Obstruction and existence for twisted Kähler-Einstein equation

2.1 Twisted K.E scalar Equation

Let L be an ample holomorphic hermitian line bundle on a Kähler manifold X. Given $[\eta] = -c_1(K_X \otimes L) \in H^2(M, \mathbb{R}) \cap H^{1,1}(M, \mathbb{C})$ it is natural to seek a corresponding Kähler metric ω with $[\omega] = c_1(L)$ satisfying the twisted Kähler-Einstein equation

$$Ric(\omega) - \omega = \eta \tag{2.1}$$

In the Kähler-Einstein setting for Fano manifolds, where $\eta = 0 (L = -K_X)$, it is known this is not always solvable. Similarly extra conditions are needed here.

A flow version of (2.1) can be written as

$$\partial_t \tilde{g_{i\bar{j}}} = -\tilde{R_{i\bar{j}}} + \tilde{g_{i\bar{j}}} + \eta_{i\bar{j}} \tag{2.2}$$
$$\tilde{g_{i\bar{j}}}(0) = g_{i\bar{j}}$$

This flow is now known to be called Twisted Kähler-Ricci flow (Tkrf) (see [SzCo]).

Heuristically, if we had long time existence and convergence in C^∞ topology for time derivatives included, then as $t \to \infty$

$$0 = \lim \partial_t \tilde{g_{i\bar{j}}} = \lim(-\tilde{R_{i\bar{j}}} + \tilde{g_{i\bar{j}}} + \eta_{i\bar{j}})$$
$$= -(R_\infty)_{i\bar{j}} + (g_\infty)_{i\bar{j}} + \eta_{i\bar{j}}$$

In other words we obtain a metric g_∞ satisfying the twisted Kähler-Einstein equation. §20 makes this precise.

But to even consider the flow above, short time existence must be clarified when starting the Tkrf flow at any initial metric g_0. To do this it is enough to notice that this can be written as a scalar flow very similar to the Kähler-Einstein setting for Fano manifolds where scalar flow is a parabolic flow for which short time existence is well known.

Write $\tilde{g_{i\bar{j}}} = g_{i\bar{j}} + u_{i\bar{j}}$ where $u \in C^\infty(M \times [0,T))$, $0 < T < \infty$. Set $T_{i\bar{j}} = g_{i\bar{j}} + \eta_{i\bar{j}}$. Since $\frac{\sqrt{-1}}{2\pi} R_{i\bar{j}} dz^i \wedge dz^{\bar{j}}, \frac{\sqrt{-1}}{2\pi} T_{i\bar{j}} dz^i \wedge dz^{\bar{j}} \in C_1(M)$, $0 = [T - Ric] \in H^2(M,\mathbb{R}) \cap H^{1,1}(M,\mathbb{C})$, there is $f \in C^\infty(M)$ unique up to a constant such that $T_{i\bar{j}} - R_{i\bar{j}} = f_{i\bar{j}}$.

Proposition 16. *(2.2) can be written as the scalar equation*

$$\frac{\partial u}{\partial t} = \log \frac{\omega_u^m}{\omega^m} + u + f + \phi(t) \tag{2.3}$$

where $\phi(t)$ comes from the ambiguity in constant on each time slice that is fixed by the normalization

$$\int_M e^{\frac{\partial u}{\partial t} - (u+f)} dV = e^{\phi(t)} Vol(M) \tag{2.4}$$

Proof. Rewrite (2.2)

$$\begin{aligned}
\partial_t \tilde{g_{i\bar{j}}} &= -\tilde{R_{i\bar{j}}} + \tilde{g_{i\bar{j}}} + \eta_{i\bar{j}} \\
&= -\tilde{R_{i\bar{j}}} + (R_{i\bar{j}} - R_{i\bar{j}}) + \tilde{g_{i\bar{j}}} + \eta_{i\bar{j}} \\
&= -\tilde{R_{i\bar{j}}} + R_{i\bar{j}} + (-R_{i\bar{j}} + \tilde{g_{i\bar{j}}} + \eta_{i\bar{j}})
\end{aligned}$$

Using the formula for ricci curvatures obtain

$$\begin{aligned}
&= \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \log \frac{\omega_u^m}{\omega^m} + u_{i\bar{j}} + (-R_{i\bar{j}} + (g_{i\bar{j}} + \eta_{i\bar{j}})) \\
&= \frac{\partial^2}{\partial z^i \partial z^{\bar{j}}} \log \frac{\omega_u^m}{\omega^m} + u_{i\bar{j}} + (f_{i\bar{j}})
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&\partial\overline{\partial}\Big(\frac{\partial u}{\partial t} - \log \frac{\omega_u^m}{\omega^m} - (u + f)\Big) = 0 \\
\implies &\frac{\partial u}{\partial t} = \log \frac{\omega_u^m}{\omega^m} + u + f + \phi(t)
\end{aligned}$$

Recall $[\omega_u] = [\omega]$ so we obtain the normalizations in (2.4). □

Remark 12. *As a monge-ampere equation the twisted Kähler-Einstein equation is the same as that arising in the Kähler-Einstein Fano case i.e.*

$$Ae^{-\phi + h_\omega} \omega^m = (\omega')^m \tag{2.5}$$

where A is given by

$$Vol_g(M) = \int_M dV' = A \int_M e^{-\phi + h_\omega} dV \tag{2.6}$$

Using the following

1. $g'_{i\bar{j}} = g_{i\bar{j}} + \phi_{i\bar{j}}$
2. $Ric(\omega') - \omega' = \eta \Leftrightarrow R'_{i\bar{j}} - g'_{i\bar{j}} = \eta_{i\bar{j}}$
3. $(R_{i\bar{j}}), (\eta_{i\bar{j}} + g_{i\bar{j}})$ with associated forms in $C_1(M)$
4. $T_{i\bar{j}} - R_{i\bar{j}} = (-h_\omega)_{i\bar{j}}$

obtain

$$(R'_{i\bar{j}} - R_{i\bar{j}}) + R_{i\bar{j}} - g'_{i\bar{j}} = \eta_{i\bar{j}}$$
$$R'_{i\bar{j}} - R_{i\bar{j}} - \phi_{i\bar{j}} = T_{i\bar{j}} - R_{i\bar{j}} = (-h_\omega)_{i\bar{j}}$$

This is equivalent to

$$\partial\bar{\partial}(-\log \frac{(\omega')^m}{\omega^m} - \phi + h_\omega) = 0$$
$$\implies -\log \frac{(\omega')^m}{\omega^m} - \phi + h_\omega = -\log A \text{ where A } > 0$$
$$\implies Ae^{-\phi + h_\omega}\omega^m = (\omega')^m$$

Since $[\omega'] = [\omega]$, A is normalized to (2.6).

2.2 Various Estimates (Toy version)

Adjust the scalar equation (2.3) of the twisted Kähler-Ricci flow by dropping the term $\phi(t)$ (referred to c_t in subsequent sections) and study

$$\partial_t u = \log \frac{\omega_u^n}{\omega^n} + u + f \tag{2.7}$$

temporarily to gain experience ($c(t)$ adds complications so is relegated to later sections). Thanks to short time existence there is a $T > 0$. Choose $0 < \epsilon < T < \infty$.

Proposition 17. *C^2 estimates of u on $M \times [0, T - \epsilon]$ depend on oscillation of u and boundedness of $\partial_t u$. Explicitly the following estimates are available*

- $|\partial_t u| \leq \max_M |f|$
- *$n + \Delta u > 0$ and $\Delta u < C$. The reduction to estimates on Δu comes from well known point-wise calculation. See [Jo].*

Proof. (2.7) can be written as

$$e^t \partial_t (e^{-t} u) = \log \frac{\omega_u^n}{\omega^n} + f \tag{2.8}$$

Differentiate in time to obtain

$$\partial_t (e^t \partial_t (e^{-t} u)) = \Delta_{\tilde{g}} (\partial_t u)$$

$$e^t [\partial_t \partial_t (e^{-t} u) + \partial_t (e^{-t} u)] = \Delta_{\tilde{g}} (\partial_t u)$$

$$\partial_t [\partial_t (e^{-t} u) + e^{-t} u] = \Delta_{\tilde{g}} (e^{-t} \partial_t u) = \Delta_{\tilde{g}} (\partial_t (e^{-t} u) + e^{-t} u)$$

Parabolic maximum principle entails (really quantities need to be adjusted by $\pm\epsilon t$ and let ϵ run to zero.)

$$\max_{M\times[0,T-\epsilon]}(\partial_t + I)(e^{-t}u) = \max_M e^{-t}\partial_t u = \max_M \partial_t u \leq \max_M f$$

The last inequality follows from (2.8) at $t = 0$ (replace u with $-u$ to get the bound from below). So $\dot{u}$ is bounded.

Lemma 10. *Along Tkrf on* $[0, T-\epsilon]$, $\omega_u > 0$. *In particular* $n + \Delta u > 0$

Proof. Since $\dot{u}$ is bounded the monge-ampere equation corresponding to (2.7) $\omega_u^n = e^{\dot{u}-u-f}\omega^n$ gives $\omega_u^n > 0(\omega > 0)$ so it follows its eigenvalues are nonzero real. At a minimum point $p \in M$, $u_{\alpha\overline{\beta}}$ has positive eigenvalues so that ω_u has positive eigenvalues near p. By connectedness and covering appropriately this holds globally(change in sign of eigenvalues would require ω_u^n to degenerate somewhere). Really this just says that if ω_u is positive at a point of M_t it is positive everywhere on M_t. So $Tr_g(\omega_u) = n + \Delta u > 0$ on $[0, T-\epsilon]$. □

To get an upper bound on Δu use C^2 inequality obtained by Yau:

$$\begin{aligned}\Delta_{\tilde{g}}(e^{-C_0}(n + \Delta_g u)) \geq\ & e^{-C_0 u}(\Delta F - Cn^2) \\ & - C_0 e^{-C_0 u} n(n + \Delta_g u) \\ & + (C_0 + C)e^{-C_0 u}e^{-\frac{F}{n-1}}(n + \Delta_g u)^{\frac{n}{n-1}}\end{aligned} \tag{2.9}$$

where $C = \inf_{i\neq k} R_{i\bar{i}k\overline{k}}$ and $C_0 + C > 0$.

Rewriting (2.7) we set

$$F := \log(\frac{\omega_u^n}{\omega^n}) = \frac{\partial u}{\partial t} - u - f$$

Inserting into estimate (2.9) and evaluating at a maximum point of the quantity $e^{-C_0 u}(n+\Delta_g u)$, $(p_0, t_0) \in M \times [0, T-\epsilon]$ obtain

$$\begin{aligned} 0 \geq \Delta_{\tilde{g}}(e^{-C_0 u}(n+\Delta_g u)) \geq & -e^{-C_0 u}(\Delta_g f + Cn^2) \\ & - C_0 e^{-C_0 u} n(n+\Delta_g u) + e^{-C_0 u}\Delta_g(\frac{\partial u}{\partial t} - u) \\ & + (C_0 + C)e^{-C_0 u} e^{-\frac{(-u-f+\frac{\partial u}{\partial t})}{n-1}}(n+\Delta_g u)^{\frac{n}{n-1}} \end{aligned} \tag{2.10}$$

(2.10) becomes after multiplying by $e^{c_0 u}$ and rearranging

$$\begin{aligned} C' \geq \Delta_g f + Cn^2 \geq -C_0 n(n+\Delta_g u) - \Delta_g u + \Delta_g \frac{\partial u}{\partial t} \\ +(C_0 + C)e^{-(\frac{\frac{\partial u}{\partial t} - f}{n-1})} e^{\frac{u}{n-1}}(n+\Delta_g u)^{\frac{n}{n-1}} \end{aligned} \tag{2.11}$$

Claim 6. *At (p_0, t_0) when $0 \leq t_0 \leq T - \epsilon$*

$$\Delta_g \frac{\partial u}{\partial t} \geq C_0 \frac{\partial u}{\partial t}(n+\Delta_g u) \tag{2.12}$$

Proof. Indeed

$$\begin{aligned} 0 \leq \frac{\partial}{\partial t}(e^{-C_0 u}(n+\Delta_g u)) = -C_0 e^{-C_0 u}\frac{\partial u}{\partial t}(n+\Delta_g) + e^{-C_0 u}\Delta_g u \frac{\partial u}{\partial t} \\ \implies \Delta_g \frac{\partial u}{\partial t} \geq C_0 \frac{\partial u}{\partial t}(n+\Delta_g u) \end{aligned}$$

□

So (2.11) becomes

$$C^{'}(n) \geq -(C_0 n+1)(n+\Delta_g u)+C_0\frac{\partial u}{\partial t}(n+\Delta_g u) \\ +(C_0+C)e^{-(\frac{\frac{\partial u}{\partial t}-f}{n-1})}e^{\frac{u}{n-1}}(n+\Delta_g u)^{\frac{n}{n-1}} \tag{2.13}$$

at $t_0 = 0$ we have

$$e^{-C_0 u(p_0,0)}n \geq e^{-C_0 u(p_0,0)}(n+\Delta_g u(p_0,0)) \geq e^{-C_0 u}(n+\Delta_g u) \\ \Longrightarrow (n+\Delta_g u) \leq n e^{C_0(u-\inf_{M\times[0,T-\epsilon]} u)}$$

using that $e^{-C_0 u(p_0,t_0)} \leq e^{-C_0 \inf_{M\times[0,T-\epsilon]} u}$.

When $0 \leq t \leq T-\epsilon$ reduce (2.13) further

$$\begin{aligned} C^{'}(n) \geq & -(C_0 n+1-C_0\frac{\partial u}{\partial t})(n+\Delta_g u) \\ & +(C_0+C)e^{\frac{-|f|_\infty+f}{n-1}}e^{\frac{u}{n-1}}(n+\Delta_g u)^{\frac{n}{n-1}} \\ \geq & -C_0^{'}(n+\Delta_g u)+\tilde{C}e^{\frac{-||u||_{C^0}}{n-1}}(n+\Delta_g u)^{\frac{n}{n-1}} \end{aligned}$$

here $C_0^{'} = C_0 n+1+C_0|f|_\infty$, $\tilde{C} = e^{\frac{f-|f|_\infty}{n-1}}(C_0+C)$ and $n+\Delta_g u > 0$ holds.

So the inequality takes the form

$$C_2(1+n+\Delta_g u) \geq \tilde{C}e^{-\frac{||u||_{C^0}}{n-1}}(n+\Delta_g u)^{\frac{n}{n-1}} \tag{2.14}$$

Claim 7. *For $x > 0$ and positive constants a, b inequalities of the form*

$$(1+x)a < (x)^{\frac{n}{n-1}} b$$

hold whenever

$$x > (2^k)^{n-1}$$

provided $k > \log_2(\frac{a}{b}) + 1$ i.e for $x > (2\frac{a}{b})^{n-1}$

Proof. Clearly $x^{\frac{n}{n-1}}$ grows faster than x. Taking $x^{\frac{1}{n-1}} > 2^k$

$$\frac{(1+x)a}{x^{\frac{n}{n-1}} b} = \frac{a}{x^{\frac{n}{n-1}} b} + \frac{a}{x^{\frac{1}{n-1}} b} < \frac{a}{b}(\frac{1}{2^{nk}} + \frac{1}{2^k}) < \frac{a}{b2^{k-1}} < 1$$

for k an integer bigger than $\log_2(\frac{a}{b}) + 1$. □

By the claim, since (2.14) is the reverse inequality it follows that there is a $0 < C := (\frac{2C_2}{\tilde{C}})^{n-1}$ so that

$$\begin{aligned}
(n + \Delta_g u)(p_0, t_0) &\leq Ce^{||u||_{C^0}} \\
\implies 0 < e^{-C_0 u}(n + \Delta_g u) &\leq e^{-C_0 u(p_0,t_0)}(n + \Delta_g u(p_0, t_0)) \\
&\leq Ce^{||u||_0} e^{-C_0 u(p_0,t_0)} \\
\implies 0 < (n + \Delta_g u) &< Ce^{||u||_0} e^{C_0(u - u(p_0,t_0))} \\
&< Ce^{(\sup_{M\times[0,T-\epsilon]} u - \inf_{M\times[0,T-\epsilon]} u)(C_0+1)}
\end{aligned}$$

note that $u(0) = 0$ so

$$||u||_0 \leq \max\{\sup_{M\times[0,T-\epsilon]} u, -\inf_{M\times[0,T-\epsilon]} u\} \leq \sup_{M\times[0,T-\epsilon]} u - \inf_{M\times[0,T-\epsilon]} u$$

□

2.3 Twisted Mabuchi functional

In this section consider the K-energy functional in the twisted setting. This involves adjusting the K-energy functional so that its critical points satisfy the twisted Kähler-Einstein equation (2.1).

Definition 1. $\nu^{\eta}_{\omega}(\phi) = -\int_0^1 \int_M (Ric(\omega_\phi) - (\omega_\phi + \eta))\frac{\omega_\phi^{n-1}}{(n-1)!} \wedge dt$

Proposition 18. *The twisted Mabuchi functional defines a closed 1-form*

$$\tilde{B}_\phi(\psi) = \int_M \psi(\omega_\phi + \eta - Ric(\omega_\phi)) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!}.$$

where $\psi \in T_\phi \mathcal{H}_\omega$

Recall that

$$d\tilde{\beta}_\phi(u,v) = \delta_u \beta(v) - \delta_v \beta(u)$$

where $u, v \in T_\phi \mathcal{H}_\omega$. Mabuchi energy being understood, it suffices to study the differential of $\int \psi tr_\phi \eta \frac{\omega_\phi^n}{n!}$. This is given by

Claim 8.

$$\delta_u \int v tr_\phi \eta \frac{\omega_\phi^n}{n!} = \int v(tr_\phi \eta \Box_\phi u - \langle \nabla\overline{\nabla} u, \eta\rangle)\frac{\omega_\phi^n}{n!} \tag{2.15}$$

Proof. Clearly the first term in (2.15) comes from differentiating the volume form. The second term comes from differentiating the $tr_{\phi}\eta$ term which is locally given by

$$-g^{\alpha\bar{t}}g^{s\overline{\beta}}(u_{s\bar{t}})\eta_{\alpha\overline{\beta}} = -\langle\nabla\overline{\nabla}u, \eta\rangle$$

Really a factor is suppressed but its harmless. $\square$

The proposition follows from the next lemma.

Lemma 11.

$$\begin{aligned} d\tilde{\beta}(u,v) &= \delta_u \int vtr_{\phi}\eta\frac{\omega_{\phi}^n}{n!} - \delta_v \int utr_{\phi}\eta\frac{\omega_{\phi}^n}{n!} \\ &= \int [tr\eta(v\Box_{\phi}u - u\Box_{\phi}v) + u\langle\nabla\overline{\nabla}v, \eta\rangle - v\langle\nabla\overline{\nabla}u, \eta\rangle]\frac{\omega_{\phi}^n}{n!} = 0 \end{aligned} \tag{2.16}$$

Proof. Note the second equality follows from the claim. Involved is integration by parts to obtain cancellation to zero. We suppress all integrals and divergences involved and focus on the integrands.

By performing integration by parts on $tr\eta(v\Box_{\phi}u - u\Box_{\phi}v)$ in (2.16) we obtain four terms that are given by

$$\begin{aligned} (-tr\eta\nabla v\cdot\overline{\nabla}u - v\nabla tr\eta\cdot\overline{\nabla}u) + (tr\eta\overline{\nabla}u\cdot\nabla v + u\overline{\nabla}tr\eta\cdot\nabla v) \\ = -v\nabla tr\eta\cdot\overline{\nabla}u + u\overline{\nabla}tr\eta\cdot\nabla v \end{aligned} \tag{2.17}$$

The formulas may be verified through a local calculation. For example an integration by parts on a term with integrand $tr\eta v g^{p\overline{q}} u_{p\overline{q}}$ yields

$$-(tr\eta)_p v g^{p\overline{q}} u_{\overline{q}} - tr\eta v_p g^{p\overline{q}} u_{\overline{q}} = (-v\nabla tr\eta \cdot \overline{\nabla} u - tr\eta \nabla v \cdot \overline{\nabla} u)$$

similarly for the other term.

An integration by parts on the third and fourth terms in (2.16) gives

$$\begin{aligned} -g^{\alpha\bar{t}} g^{s\overline{\beta}} v_s u_{\bar{t}} \eta_{\alpha\overline{\beta}} - g^{\alpha\bar{t}} g^{s\overline{\beta}} v_s \eta_{\alpha\overline{\beta},\bar{t}} u + g^{\alpha\bar{t}} g^{s\overline{\beta}} v_s u_{\bar{t}} \eta_{\alpha\overline{\beta}} + g^{\alpha\bar{t}} g^{s\overline{\beta}} u_{\bar{t}} \eta_{\alpha\overline{\beta},s} v \\ = -g^{\alpha\bar{t}} g^{s\overline{\beta}} v_s \eta_{\alpha\overline{\beta},\bar{t}} u + g^{\alpha\bar{t}} g^{s\overline{\beta}} u_{\bar{t}} \eta_{\alpha\overline{\beta},s} v \end{aligned}$$

using that $d\eta = 0$ we obtain

$$-g^{\alpha\bar{t}} g^{s\overline{\beta}} v_s \eta_{\alpha\overline{\beta},\bar{t}} u = -g^{\alpha\bar{t}} g^{s\overline{\beta}} v_s \eta_{\alpha\bar{t},\overline{\beta}} u = -g^{s\overline{\beta}} (tr\eta)_{\overline{\beta}} v_s u = -u\overline{\nabla} tr\eta \cdot \nabla v \tag{2.18}$$

similarly

$$g^{\alpha\bar{t}} g^{s\overline{\beta}} u_{\bar{t}} \eta_{\alpha\overline{\beta},s} v = v g^{\alpha\bar{t}} (tr\eta)_\alpha u_{\bar{t}} = v\nabla tr\eta \cdot \overline{\nabla} u \tag{2.19}$$

Pairing up corresponding terms in (2.17) with (2.18) and (2.19) we obtain $d\tilde{\beta} = 0$. □

Recall the Cech 2-differential is given by

$$(\delta f)(\omega_1, \omega_2, \omega_3) = f(\omega_1, \omega_2) + f(\omega_2, \omega_3) + f(\omega_3, \omega_1)$$

where $f(\omega_\alpha, \omega_\beta) = -f(\omega_\beta, \omega_\alpha)$. When f defines a Cech cocycle we have

$$f(\omega_1, \omega_2) - f(\omega_3, \omega_2) = f(\omega_1, \omega_3)$$

Letting $\omega_1 = \omega$, $\omega_2 = \omega + i\partial\overline{\partial\phi}$, and $\omega_3 = \omega + i\partial\overline{\partial\psi} =: \omega^{'}$ we can recover how Tian writes the cocycle condition

$$f_\omega(\phi) - f_{\omega^{'}}(\phi^{'}) = f_\omega(\psi)$$

By the lemma $d\nu_\omega^\eta(\phi)$ defines a closed 1 form. Since $\mathcal{H}_\omega$ is contractible the one form is in fact exact and thus can be integrated to give the functional

$$\nu_\omega^\eta(\phi) = \nu_\omega(\phi) + \int_0^1 \int_X \dot{\phi}_t \eta \wedge \frac{\omega_t^{n-1}}{(n-1)!} \tag{2.20}$$

From the discussion in the previous paragraph ν_ω^η also satisfies the cocycle condition.

2.4 Twisted Futaki type invariant

For this section we denote the manifold by M so we may notate holomorphic vector fields by X. Set $G := Aut_0(X)$. Let $\eta(M)$ denote the lie algebra of holomorphic vector fields on M. Given a smooth differential form α we say that the infinitesimal action of $X \in \eta(M)$ annihilates α if $\mathcal{L}_X\alpha = 0$. We say $\eta(M)$ annihilates α under the infinitesimal action if $\mathcal{L}_X\alpha = 0$ for each $X \in \eta(M)$.

Here we see that the Futaki invariant can be adapted to the twisted setting under the condition that η is annihilated by $\eta(M)$. With this taken for granted it can be seen why the non collapsing condition introduced in subsequent sections guaranteeing existence cannot hold if the twisted Futaki invariant does not vanish. Though the following argument is an explicit calculation it seems possible also to conclude through using a moment map interpretation appearing in [Sto09].

Proposition 19. *Provided η is annihilated by $\eta(M)$, $\mathcal{F}_M^\eta : \eta(M) \to \mathbb{C}$ given by (2.21) is well-defined.*

$$\mathcal{F}_M^\eta(X, [\omega]) = -\int_M \theta_X(Ric(\omega) - \omega) \wedge \frac{\omega^{n-1}}{(n-1)!} + \int_M \theta_X \eta \wedge \frac{\omega^{n-1}}{(n-1)!} \tag{2.21}$$

where $\theta_X + \alpha = i_X\omega$, α a harmonic 1-form, and $X \in \eta(M)$.

Remark 13. *Since η is a real $(1,1)$ form the condition that it is annihilated by the infinitesimal action of $\eta(M)$ means, using that the lie algebra of G is generated by the real holomorphic vector fields of M, since $\mathcal{L}_{X+\overline{X}}\eta = 0$ for each $X \in \eta(M)$ we conclude that η is G-invariant.*

The first term above is the usual Futaki invariant F_X so is independent of the choice of metric in $[\omega]$. However the second term can potentially destroy the independence.

Following the classical argument there is no loss in assuming holomorphic vector fields satisfy

$$i_X\omega = \overline{\partial}\theta_X \tag{2.22}$$

since the harmonic piece has no contribution after an integration by parts (see [Tian00] and (2.26)).

In co-ordinates (2.22) reads

$$X^i = g^{i\overline{j}}(\theta_X)_{\overline{j}} = (\theta_X)^i \tag{2.23}$$

When the metrics vary over any family $\omega_t = \omega + \partial\overline{\partial}\phi_t$ in a fixed Kähler class

$$\theta_{X,t} = \theta_X + X(\phi_t) + c_t$$

since

$$\overline{\partial}(\theta_{X,t}) = i_X\omega_t = i_X(\omega + \partial\overline{\partial}\phi_t) = \overline{\partial}(\theta_X + X(\phi_t)).$$

Now deduce as in (2.23)

$$X^i = g_t^{i\overline{j}}(\theta_{X,t})_{\overline{j}} = \theta^i_{X,t}. \tag{2.24}$$

Using (2.22), that $X \in \eta(M)$ and the definition of Ric

$$\overline{\partial}\Box_t\theta_{X,t} = -i_X\mathrm{Ric}_t$$

see [Tian00] for details. Since $\eta \in -c_1(L \otimes K_M)$ we have

$$\mathrm{Ric}(\omega) - \omega = \eta + \partial\overline{\partial}\psi$$

for some $\psi \in C^\infty(M)$. Varying over the family $\{\omega_t\}$ we get

$$\mathrm{Ric}_t - (\omega_t + \eta) = \partial\overline{\partial}\xi_t \tag{2.25}$$

where $\mathrm{Ric}_t = \mathrm{Ric}(\omega_t)$.

proof of proposition. From (2.25) we may simplify to get

$$\mathcal{F}^\eta_M(X, \omega_t) = -\int_M \theta_{X,t}\partial\overline{\partial}\xi_t \wedge \frac{\omega_t^{n-1}}{(n-1)!} = \int_M X\xi_t\frac{\omega_t^n}{n!} \tag{2.26}$$

Since the space Kähler metrics is affine it is enough to check the variation over any family of metrics in the fixed Kähler class vanishes. So start by computing

$$\frac{d}{dt}\mathcal{F}^{\eta}_{M}(X,\omega_t) = \int_M (X\dot{\xi}_t + \Box_t\dot{\phi}_t X\xi_t)\frac{\omega_t^n}{n!} \tag{2.27}$$

Recall we obtain the deformation of the scalar curvature S_t by differentiating

$$S_t = -g_t^{k\bar{l}}\frac{\partial^2}{\partial z_k \partial z_{\bar{l}}}\log\det((g_t)_{i\bar{j}})$$

to get

$$\dot{S}_t = -\Box^2\dot{\phi} - R_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} \tag{2.28}$$

Tracing $\mathrm{Ric}_t - (\omega_t + \eta) = \partial\overline{\partial}\xi_t$ we obtain

$$S_t - n - tr_t\eta = \Box_t\xi_t \tag{2.29}$$

Differentiating (2.29) and applying (2.28) we obtain

$$-\Box_t^2\dot{\phi} - R_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} + \eta_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} = \dot{\Box}_t\xi_t + \Box\dot{\xi}_t \tag{2.30}$$

Recall that

$$\dot{\Box}_t\xi_t = -(\xi_t)_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} \tag{2.31}$$

Set

$$\tilde{R}_{\alpha\overline{\beta}} := R_{\alpha\overline{\beta}} - \eta_{\alpha\overline{\beta}} - \xi_{\alpha\overline{\beta}} = g_{\alpha\overline{\beta}} \tag{2.32}$$

then $\tilde{Ric}$ is harmonic since $\Lambda\tilde{Ric} = cnst$ and $\overline{\partial}^* = -i[\Lambda, \partial]$ on a Kähler manifold.

In terms of $\tilde{\text{Ric}}$ from (2.30), (2.31)

$$\begin{aligned}\Box\dot{\xi}_t &= -\Box_t^2\dot{\phi} - R_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} + \eta_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} + \xi_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} \\ &= -\Box_t^2\dot{\phi} - \tilde{R}_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} \end{aligned} \tag{2.33}$$

So we obtain integrating by parts and using the identity (2.33) for $\Box_t\dot{\xi}_t$

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_0^{\eta}(X,\omega_t) &= \int_M(-\theta_{X,t}\Box\dot{\xi}_t + \Box_t\dot{\phi}_t X\xi_t)\frac{\omega_t^n}{n!} \\ &= \int_M(\theta_{X,t}\Box_t^2\dot{\phi} + \theta_{X,t}\tilde{R}_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}} + X\xi_t\Box_t\dot{\phi})\frac{\omega_t^n}{n!} \\ &= \int_M(\Box_t\theta_{X,t} + X\xi_t)\Box\dot{\phi}\frac{\omega_t^n}{n!} + \int_M\theta_{X,t}\tilde{R}_{\alpha\overline{\beta}}\dot{\phi}^{\alpha\overline{\beta}}\frac{\omega_t^n}{n!} \end{aligned} \tag{2.34}$$

The first term in (2.34) simplifies to

$$\int_M -g_t^{\alpha\overline{\beta}}\dot{\phi}_\alpha((\Box_t\theta_{X,t})_{\overline{\beta}} + X^i(\xi_t)_{i\overline{\beta}})\frac{\omega_t^n}{n!} \tag{2.35}$$

using $\overline{\partial}\Box_t\theta_{X,t} = -i_X\text{Ric}_t$ and (2.32), (2.35) simplifies to

$$\begin{aligned}&\int_M g_t^{\alpha\overline{\beta}}\dot{\phi}_\alpha(\text{Ric}_{i\overline{\beta}}X^i - (\xi_t)_{i\overline{\beta}}X^i)\frac{\omega_t^n}{n!} \\ &\qquad = \int_M g_t^{\alpha\overline{\beta}}\dot{\phi}_\alpha(\tilde{R}_{i\overline{\beta}})X^i\frac{\omega_t^n}{n!} + \int_M g_t^{\alpha\overline{\beta}}\dot{\phi}_\alpha\eta_{i\overline{\beta}}X^i\frac{\omega_t^n}{n!}\end{aligned} \tag{2.36}$$

Performing by parts on the second term in (2.34) using (2.24) gives

$$= -\int_M\theta^i_{X,t}\tilde{R}_{i\overline{\beta}}g_t^{\alpha\overline{\beta}}\dot{\phi}_\alpha\frac{\omega_t^n}{n!} \tag{2.37}$$

Putting (2.36) and (2.37) together gives

$$
\begin{aligned}
&= \int_M g_t^{\alpha\overline{\beta}} \dot{\phi}_\alpha \tilde{R}_{i\overline{\beta}} (X^i - \theta^i_{X,t}) \frac{\omega_t^n}{n!} + \int_M g_t^{\alpha\overline{\beta}} \dot{\phi}_\alpha \eta_{i\overline{\beta}} X^i \frac{\omega_t^n}{n!} \\
&= \int_M g_t^{\alpha\overline{\beta}} \dot{\phi}_\alpha \eta_{i\overline{\beta}} X^i \frac{\omega_t^n}{n!} \qquad (2.38)
\end{aligned}
$$

Compressing notation in (2.38) write

$$\langle \partial\dot{\phi}, i_X \eta \rangle := \int_M g_t^{\alpha\overline{\mu}} \dot{\phi}_\alpha \eta_{i\overline{\mu}} X^i \frac{\omega_t^n}{n!}$$

To get a well defined invariant we need the last term to vanish. But $\eta(M)$ annihilates η so

$$
\begin{aligned}
0 = L_X \eta &= \partial i_X \eta \\
(\eta_{i\overline{\mu}} X^i)_\alpha dz^\alpha \wedge dz^{\overline{\mu}} &= 0 \\
\implies (\eta_{i\overline{\mu}} X^i)_\alpha &= 0 \qquad (2.39)
\end{aligned}
$$

From integration by parts and (2.39)

$$\langle \partial\dot{\phi}, i_X \eta \rangle = - \int_X g_t^{\alpha\overline{\mu}} \theta_{X,t} (\eta_{i\overline{\mu}} X^i)_\alpha \frac{\omega_t^n}{n!} = 0$$

□

2.5 Convexity of the Twisted Mabuchi functional

We restrict ourselves to the smooth setting and consider the twisted mabuchi functional. Write the differential as

$$d\nu_\omega^\eta(\phi) = dE_0(X, \omega) + \int_M \dot{\phi}\eta \wedge \frac{\omega^{n-1}}{(n-1)!}$$

where $\phi \in C^\infty \cap Psh(\omega, X)$. Here we actually mean strictly ω-psh so $\omega_\phi > 0$. It was observed in [Sto09]

Proposition 20. *Under the provision that $\eta \geq 0$ the twisted mabuchi functional is convex along smooth geodesics. The second variation of twisted mabuchi energy given by*

$$\frac{d^2}{dt^2}\mathcal{F}_0^\eta(X, \omega) = -\int_M (\ddot{\phi} - \frac{1}{2}|\nabla\dot{\phi}|_\phi^2)(Ric(\omega_\phi) - \omega_\phi - \eta) \wedge \frac{\omega^{n-1}}{(n-1)!}$$
$$+ (\partial\dot{\phi} \wedge \overline{\partial}\dot{\phi}, \eta) + ||L\dot{\phi}||^2$$

where the operator $L = \overline{\partial} \uparrow \overline{\partial}$ (also denoted $\mathcal{D}$).

No argument is given in the literature to the best of our knowledge so we suspect, although its straightforward, there is an easier way to see this than the argument given below.

Proof. Mabuchi computed the second variation for E_0 to be

$$-\int_M \ddot{\phi}(Ric(\omega_\phi) - \omega_\phi) \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} + ||L\dot{\phi}||^2$$

see [Mab]. So it suffices to compute the first variation of

$$\int_M \dot{\phi} \wedge \eta \wedge \frac{\omega_\phi^{n-1}}{(n-1)!}$$

Differentiating we obtain

$$\int_M \ddot{\phi}\eta \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} + \int_X \dot{\phi}\eta \wedge i\partial\overline{\partial}\dot{\phi} \wedge \frac{\omega_\phi^{n-2}}{(n-2)!} \tag{2.40}$$

Integrating by parts the second term may be written as

$$-\int_M \partial\dot{\phi} \wedge \overline{\partial}\dot{\phi} \wedge \eta \wedge \frac{\omega_\phi^{n-2}}{(n-2)!}$$

In the following we abuse notation $\eta \leftrightarrow \eta_\epsilon$ where $\eta_\epsilon = \eta + \epsilon\omega > 0$ since we can let ϵ run to zero without trouble.

At a point $p \in M$ by choosing normal co-ordinates we may arrange that

$$\eta = \eta_{i\bar{i}} dz_i \wedge dz_{\bar{i}}, \quad \omega_\phi = dz_j \wedge dz_{\bar{j}}$$

we omit the $\frac{\sqrt{-1}}{2}$ factor which is ultimately absorbed into $\frac{\omega^n}{n!}$.

Claim 9. *The (n,n) form in the second term of (2.40) at the point p is*

$$(-\dot{\phi}_\alpha \dot{\phi}_{\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}) \wedge (\eta_{i\bar{i}} dz_i \wedge dz_{\bar{i}}) \wedge \sum_{(p\overline{p} \neq q\overline{q})}{}' \frac{\omega_\phi^n}{n!\,(dz_p \wedge dz_{\overline{p}}) \wedge (dz_q \wedge dz_{\overline{q}})}$$
$$= (-\frac{1}{2}|\nabla\dot{\phi}|^2 + \dot{\phi}_p \dot{\phi}_{\overline{p}})\eta_{p\overline{p}} \frac{\omega_\phi^n}{n!}$$

Proof. This follows because

$$
\begin{aligned}
\eta_{i\bar{i}} dz_i \wedge dz_{\bar{i}} \wedge \sum_{(p\bar{p} \neq q\bar{q})'} [\ldots] \\
&= \sum_{p<q} \eta_{p\bar{p}} \frac{\omega_\phi^n}{n!\, dz_q \wedge dz_{\bar{q}}} + \sum_{p<q} \eta_{q\bar{q}} \frac{\omega_\phi^n}{n!\, dz_p \wedge dz_{\bar{p}}} \\
&= \sum_{p \neq q} \eta_{p\bar{p}} \frac{\omega_\phi^n}{n!\, dz_q \wedge dz_{\bar{q}}}
\end{aligned}
$$

Now

$$
\begin{aligned}
-(\partial\dot{\phi} \wedge \overline{\partial}\dot{\phi}) \wedge \sum_{p \neq q} [\ldots] &= -\sum_{p \neq q} \dot{\phi}_q \dot{\phi}_{\bar{q}} \eta_{p\bar{p}} \frac{\omega_\phi^n}{n!} \\
&= (-\frac{1}{2}|\nabla\dot{\phi}|^2 + \dot{\phi}_p \dot{\phi}_{\bar{p}}) \eta_{p\bar{p}} \frac{\omega_\phi^n}{n!}
\end{aligned}
$$

□

Remark 14. *Another point-wise calculation shows that*

$$
(\partial\dot{\phi} \wedge \overline{\partial}\dot{\phi}, \eta) = \int_M \dot{\phi}_\alpha \dot{\phi}_{\bar{\alpha}} \eta_{\alpha\bar{\alpha}} \frac{\omega_\phi^n}{n!}
$$

So it follows from the remark and point-wise computations that

$$
\begin{aligned}
\int_M \partial\dot{\phi} \wedge \eta \wedge i\partial\overline{\partial}\dot{\phi} \wedge \frac{\omega_\phi^n}{(n-2)!} \\
&= -\int_M \frac{1}{2} |\nabla\dot{\phi}|_\phi^2 tr_\phi \eta \frac{\omega_\phi^n}{n!} + (\partial\dot{\phi} \wedge \overline{\partial}\dot{\phi}, \eta)
\end{aligned}
$$

Modulo the second variation of Mabuchi energy, the second variation looks like

$$\int_M (\ddot{\phi} - \frac{1}{2}|\nabla|^2_\phi)\eta \wedge \frac{\omega_\phi^{n-1}}{(n-1)!} + (\partial\dot{\phi} \wedge \overline{\partial}\dot{\phi}, \eta)$$

so we obtain the second variation formula of Stoppa in untraced form along smooth geodesics:

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{F}_0^\eta(X,\omega) = & -\int_M (\ddot{\phi} - \frac{1}{2}|\nabla\dot{\phi}|^2_\phi)(Ric(\omega_\phi) - \omega_\phi - \eta) \wedge \frac{\omega^{n-1}}{(n-1)!} \\ & + (\partial\dot{\phi} \wedge \overline{\partial}\dot{\phi}, \eta) + ||L\dot{\phi}||^2\end{aligned}$$

Convexity along smooth geodesics follows immediately provided $\eta \geq 0$. □

2.6 Application of Twisted Mabuchi Energy: Existence

To attack the existence problem of solutions to the twisted ricci equation

$$Ric(\omega) - \omega = \eta \quad \eta \geq 0$$

we consider twisted Kähler-Ricci flow starting at some Kähler metric $\omega_0 \in c_1(L)$ satisfying the following condition along the flow

$$\omega_t^n \geq C\omega_0^n \quad C > 0 \text{ cnst} \quad \forall t \geq 0 \tag{2.41}$$

Remark 15. *Replacing the condition $\eta \geq 0$ with $\omega_0 + \eta$ represents a Kähler class allows η to be negative. Unfortunately reapplying arguments for $\eta \geq 0$ don't carry over in any obvious fashion in regards to the C^0 estimate and the maximum principles for Perelmans estimates.*

To establish existence one uses the parabolic formulation

$$\frac{d}{dt}\omega_t = T_t^{\eta} - Ric_t$$

where $T_t^{\eta} = \omega_t + \eta$. Henceforth, as is common this will be referred to twisted Kähler-Ricci flow (Tkrf). Establishing convergence as $t \to \infty$ is the goal. For this we use three ingredients the twisted mabuchi functional introduced above, a modified version of Perelmans estimates for Kähler-Ricci flow, and a potential theory based C^0 estimate of Tian-Zhu [Ti07].

The relevant theorems are described below. First the Tian-Zhu estimate.

Proposition 4. *Let (X, ω) be a compact Kähler manifold of complex dimension n. Let $\phi \in \mathcal{H}_\omega$ solve $\frac{\omega_\phi^n}{\omega^n} = f$. Then there is ϵ_0, δ_0 so that for $\epsilon \in (0, \epsilon_0)$ and $\delta \in (0, \delta_0)$ there exists constants $C, C' > 0$ depending only on $\omega, \epsilon_0, \delta_0$ such that*

$$Osc(\phi) \leq C(\frac{1}{\epsilon\delta})^{n+\delta}||f||^{\delta}_{L^{1+\epsilon}(X,\omega)} + C' \tag{2.42}$$

for non-negative $f \in L^{1+\epsilon}(X)$.

The twisted version version of Perelmans theorem can be stated by first setting $T_t^{\eta} := \omega_t + \eta$. Then in the situation η is chosen so that $T_t^{\eta} \geq 0$

Proposition 5. *Over a Fano manifold (X, ω) of complex dimension n, the twisted Kähler-Ricci flow*

$$\frac{d\omega_t}{dt} = T_t - Ric(\omega_t) = i\partial\overline{\partial}\dot{\phi}_t$$

with $\eta \geq 0$ satisfies uniform estimates on the following quantities

$$|\dot{\phi}|, |\nabla_t \dot{\phi}|_t, |\Delta_t \dot{\phi}_t|, Diam_t(X) \tag{2.43}$$

Provided $\frac{\omega_t^n}{\omega} \geq K_0, t \in [0,\infty)$ *we also have*

$$|Sc_t| \leq C \tag{2.44}$$

Here normalize the Ricci potential $\dot{\phi}_t$ *by the condition*

$$\frac{1}{V}\int_X e^{-\dot{\phi}}\omega_t^n = 1 \tag{2.45}$$

This proposition differs from the recent result [SzCo] only in that the extra condition on the density of the volume of the Tkrf allows to bound scalar curvature.

The twisted Kähler-Ricci functional is decreasing along the twisted Kähler-Ricci flow and there is a similar identity to the classical case from which we conclude this functional is uniformly bounded from below. Extracting the canonical metric is described in §20. The arguments follow similar lines as appearing in [Pa] .

Recall the at the level of potentials we have the equation

$$\dot{\phi}_t = \log\frac{\omega_{\phi_t}^n}{\omega^n} + \phi_t + f + c_t$$

where $f = -h_{\omega,\eta}$ and let $\hat{\phi} := \phi + c_t$.

Also the differential of the twisted mabuchi functional looks like

$$d\nu_\omega^\eta(\phi_t) = \int_X \dot{\phi}_t(T_{t,\eta} - Ric_t) \wedge \frac{\omega_t^{n-1}}{(n-1)!}$$

It is essential that our choice of functional enjoys the following property.

Proposition 21. *If* ϕ_t *evolves according to the Tkrf flow above then the twisted mabuchi* *functional decreases along the flow.*

Proof. Indeed,

$$\frac{d}{dt}\nu_{\omega}^{\eta}(\phi_t) = \int_X \dot{\phi}_t i\partial\overline{\partial}\dot{\phi}_t \wedge \frac{\omega_t^{n-1}}{(n-1)!}$$
$$= -\int_X |\partial\dot{\phi}_t|_t^2 \frac{\omega_t^n}{n!}$$

□

The following can be considered among the more important properties for the existence result. Since extracting the limit in §20 crucially depends on this.

Proposition 22. *The twisted mabuchi energy (2.20) is bounded along Tkrf provided the non collapsing estimate (2.41) holds. If the flow exists for all time then* $lim_{t\to\infty} v_{\omega}^{\eta}(\phi_t) < \infty$

Claim 10. *Along Tkrf the following identity for the twisted K-energy is available.*

$$\nu_{\omega}^{\eta}(\phi) = \frac{1}{V}\int_X \dot{\phi}_t\omega_t^n + J_{\omega}(\phi_t) - \frac{1}{V}\int_X \hat{\phi}_t\omega^n + \frac{1}{V}\int_X h_{\omega,\eta}\omega^n \tag{2.46}$$

The twisted ricci flow

$$\frac{d\omega_t}{dt} = T_{\omega_t,\eta} - Ric(\omega_t)$$

at the level of potential is exactly (2.3)(let $\hat{\phi} = \phi + c_t$), and is written as

$$\hat{\phi} = \dot{\phi}_t - \log\frac{\omega_t^n}{\omega^n} + h_{\omega,\eta} \tag{2.47}$$

The corresponding monge-ampere equation is

$$e^{h_{\omega,\eta}-\phi}\omega^n = e^{-\dot{\phi}}\omega_t^n \tag{2.48}$$

In the classical setting there is the following expression for mabuchi energy(see [Ru]):

$$\nu_\omega(\phi) = \frac{-1}{V}\int_X f_{\omega_\phi}\omega_\phi^n + \frac{1}{V}\int_X f_\omega \omega^n + J_\omega(\phi) - \frac{1}{V}\int_X \phi\omega^n - \log\frac{1}{V}\int_X e^{f_\omega - \phi}\omega^n \tag{2.49}$$

Adjusting the formula for twisted K-energy we obtain essentially the same formula.

Lemma 12. *Along Tkrf we have the following identity:*

$$\begin{aligned}\nu_\omega^\eta(\phi) = \frac{-1}{V}\int_X h_{\omega_\phi,\eta}\omega_\phi^n + \frac{1}{V}\int_X h_{\omega,\eta}\omega^n + J_\omega(\phi) - \frac{1}{V}\int_X \phi\omega^n \\ - \log\frac{1}{V}\int_X e^{h_{\omega,\eta}-\phi}\omega^n\end{aligned} \tag{2.50}$$

In fact the argument is very similar to the untwisted version.

Proof. The log term in (2.50) vanishes because of normalization (2.45) using (2.48). Recall the twisted ricci potential is given by $-\sqrt{-1}\partial\overline{\partial}\dot{\phi} = Ric - T_{t,\eta} = \sqrt{-1}\partial\overline{\partial}h_{\omega_t,\eta}$ which translates at the level of potentials

$$\begin{aligned} h_{\omega_t,\eta} =& \, -\log\frac{\omega_t^n}{\omega^n} - \phi_t + h_{\omega,\eta} \\ \implies \dot{h_{\omega_t,\eta}} =& \, -\Delta_t\dot{\phi}_t - \dot{\phi}_t \end{aligned}$$

Again since the log term vanishes $F_\omega(\phi) = J_\omega(\phi) - \int_X \phi\wedge\omega^n$. So $\frac{d}{dt}F_\omega(\phi) = -\int_X \dot{\phi}\omega_t^n = \int_X \dot{h_{\omega_t,\eta}}\omega_t^n$. The lemma follows from the following calculation using an integration by parts in the second line

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{V}\int_X h_{\omega_t,\eta}\omega_t^n &= \frac{1}{V}\int_X (\dot{h_{\omega_t,\eta}} + h_{\omega_t,\eta}\Delta_t\dot{\phi})\omega_t^n \\
&= \frac{d}{dt}F_\omega(\phi) + \frac{1}{V}\int_X \dot{\phi}(i\partial\overline{\partial}h_{\omega_t,})n \wedge \omega_t^{n-1} \\
&= \frac{d}{dt}F_\omega(\phi) + \frac{1}{V}\int_X \dot{\phi}(Ric(\omega_\phi) - T_{\omega_t,\eta}) \wedge n\omega_t^{n-1} \qquad (2.51)
\end{aligned}
$$

Notice that the second term in (2.51) is negative of the differential of twisted mabuchi k-energy. Integrating in t from 0 to 1 obtains (2.50). □

proof of claim. To obtain the claim note that $h_{\omega_t,\eta} = \dot{\phi} - c_t$ and since $\int_X \omega_\phi^n = V$ we may combine the c_t term as in the statement of the claim. □

proof of proposition. By hypothesis we have uniform noncollapsing estimate $\omega_t^n \geq k\omega^n$ for $t \in [0,\infty)$. Applying the uniform lower bound on the ratio of the volume forms $\frac{\omega_t^n}{\omega^n} \geq k$ to (2.47) translates into an upper bound

$$
\hat{\phi} = \phi_t + c_t \leq C
$$

using the Perelman type uniform estimate $|\dot{\phi}| \leq \tilde{c}$ in (2.43). With this and that $J_\omega(\phi) \geq 0$ on the Kähler potentials we have by the claim

$$
0 \leq J_\omega(\phi) = \nu_\omega(\phi_t) - \frac{1}{V}\int_X \dot{\phi}_t\omega_t^n + \frac{1}{V}\int_X \hat{\phi}_t\omega^n - \frac{1}{V}\int_X h_{\omega,\eta}\omega^n \leq \nu_\omega^\eta(\phi) + C \qquad (2.52)
$$

So

$$
\nu_\omega^\eta(\phi) > -C
$$

We can conclude since the twisted Kähler ricci functional decreases along the flow and is bounded from below $\lim_{t\to\infty} \nu_\omega^\eta(\phi_t) < \infty$. $\square$

Remark 16. *We note that the term* $\int_0^1 \int_X \dot{\phi}\eta \frac{\omega_t^n}{(n-1)!} dt$ *appearing in the definition of the twisted mabuchi K-energy can be bounded from below in terms of the scalar curvature.*

$$\begin{aligned}\int_0^1 \int_X \dot{\phi}_t \eta \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt = & \int_0^1 \int_X \dot{\phi} tr_{\omega_t} \eta \frac{\omega_t^n}{n!} dt \\ > & - C \int_0^1 tr_{\omega_t} \eta \frac{\omega_t^n}{n!} \\ = & - C \int_0^1 \int_X \eta \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt\end{aligned}$$

Since

$$tr_{\omega_t \eta} \geq 0, \quad |\dot{\phi}| < C$$

$$\int_0^1 \int_X \eta \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt = \int_0^1 \int_X (\eta - Ric(\omega_t)) \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt + \int_0^1 \int_X Ric(\omega_t) \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt$$

Since $\eta \in -c_1(K_X \otimes L)$, $\eta - Ric(\omega_t) \in -c_1(L)$ *we may write* $\eta - Ric(\omega_t) = -(\omega_t + \partial\overline{\partial})f$. *It follows*

$$\begin{aligned}\int_0^1 \int_X (\eta - Ric(\omega_t)) \wedge \frac{\omega_t^{n-1}}{(n-1)!} = & - \int_0^1 \int_X (\omega_t + \partial\overline{\partial} f) \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt \\ = & - nV - \int_0^1 \int_X \partial\overline{\partial} f \wedge \frac{\omega_t^{n-1}}{(n-1)!} dt = -nV\end{aligned}$$

where we used that

$$\int_X \partial\overline{\partial} f \wedge \frac{\omega_t^{n-1}}{(n-1)!} = 0$$

which follows from integration by parts since ω_t *is closed.*

Since we have that $|S(\omega_t)| < C$ along the flow

$$\int_0^1 \int_X Ric(\omega_t) \wedge \frac{\omega_t^{n-1}}{(n-1)!} = n \int_0^1 \int_X S(\omega_t) \frac{\omega_t^n}{n!} < nCV$$

Remark 17. $0 < T_{\omega_t} \in c_1(-K_X)$ *so that X is Fano.*

Proposition 23. *For all times $t \geq 0$ assume $\omega_t^n \geq k\omega^n$ along the twisted ricci flow(non collapsing estimate). Then this flow satisfies the uniform estimate $|\phi_t + c_t| \leq K_0$, where $K_0 > 0, k$ are independent of t.*

Details differ marginally from [Pa] but we provide the argument below for convenience.

Claim 11. *It suffices to obtain the estimate*

$$||\hat{\phi}_t||_{C^0(X)} \leq Osc(\hat{\phi}_t) + C$$

The propositions follows after applying the estimate in (2.42) and moser iteration.

Proof. The claim follows from

$$||h_{\omega,\eta} - \hat{\phi}_t|| \leq Osc(h_{\omega,\eta} - \hat{\phi}_t)$$

which clearly holds if $h_{\omega,\eta} - \hat{\phi}_t$ changes sign. But this is true since along the twisted flow the volume forms deform according to $e^{h_{\omega,\eta} - \phi}\omega^n = e^{-\dot{\phi}}\omega_t^n$ and the normalization condition on $\dot{\phi}_t$ gives that

$$\int_X e^{h_{\omega,\eta} - \phi}\omega^n = \int_X e^{-\dot{\phi}_t}\omega_t^n = V = \int_X \omega^n \tag{2.53}$$

So

$$\int_X (e^{h_{\omega,\eta}-\hat{\phi}_t} - 1)\omega^n = 0$$

so $h_{\omega,\eta} - \hat{\phi}_t$ changes sign. $\square$

From the estimate in (2.42) we need to bound $\int_X e^{(\dot{\phi}_t + h_{\omega,\eta} - \hat{\phi}_t)(1+\epsilon)}\omega^n$. In view of Perelman's estimate on $\dot{\phi}$ we can reduce to bounding

$$\int_X e^{-(1+\epsilon)\hat{\phi}_t}\omega^n$$

Claim 12. *The quantity* $\max_X \hat{\phi}_t$ *is bounded. So define* $\theta_t := \max_X \hat{\phi}_t - \hat{\phi}_t$. *It follows* $e^{-(1+\epsilon)\hat{\phi}_t}\omega^n \leq Ce^{\epsilon\theta_t}\omega_t^n$ *for* $t \geq 0$.

Proof. Since $\frac{\omega_t^n}{\omega^n} \geq K$ from (2.47) and Perelman's estimate $|\dot{\phi}| < C$ it follows that $\hat{\phi}_t < C_1$. So it suffices to show that $\max_X(\hat{\phi}_t - h_{\omega,\eta}) \geq 0$ to conclude $\max_X \hat{\phi}_t$ is bounded for $t \geq 0$. But note that if $0 > \max_X(\hat{\phi}_t - h_{\omega,\eta}) \geq \hat{\phi}_t - h_{\omega,\eta}$ then $e^{h_{\omega,\eta}-\hat{\phi}_t} > 1$. In which case we contradict $\int_X (e^{h_{\omega,\eta}-\hat{\phi}_t} - 1)\omega^n = 0$. So $\max \hat{\phi}_t$ is bounded and θ_t is well-defined. The inequality for the volume forms is an easy calculation which involves writing $-\hat{\phi}_t(1+\epsilon) = \epsilon\theta - \epsilon \max \hat{\phi}_t - \hat{\phi}_t$ and then use that $\max \hat{\phi}_t$ is bounded and $\hat{\phi}_t$ is bounded from above (see [Pa]) . $\square$

So from the volume estimate in the claim it follows it is sufficient to bound

$$\int_X e^{\epsilon\theta_t}\omega_t^n$$

By power series expanding $e^{\epsilon\theta_t}$ it suffices to get the bound

$$\int_X \theta_t^p \omega_t^n \leq C^p p!$$

for all integers $p \geq 1$ and then take $0 < \epsilon < \frac{1}{C}$ so that the geometric series converges.

Claim 13. *The estimates* $0 \leq \frac{1}{V}\int_X \theta_t \omega_t^n \leq C$ *and* $\int_X \theta_t^{p+1}\omega_t^n \leq C(p+1)\int_X \theta_t^p \omega_t^n$ *hold along the flow.*

Note that by iterating the second estimate and combining with the first estimate we may obtain the desired bound for all integers $p \geq 1$.

Proof. For the first estimate apply inequality (2.52) , which uses the hypothesis $\omega_t^n \geq k\omega^n$. Combined with the fact that the twisted mabuchi functional is decreasing along the flow, it follows that $J_\omega(\phi_t)$ is bounded along the flow.

So

$$0 < I_\omega(\phi_t) = \frac{1}{V}\int_X \hat{\phi}_t(\omega^n - \omega_t^n) \leq (n+1)J_\omega(\phi_t) \leq C \tag{2.54}$$

From (2.48) we have

$$\begin{aligned} V = \int_X e^{h_{\omega,\eta} - \hat{\phi}}\omega^n &\geq C' \int_X e^{-\hat{\phi}_t}\omega^n \\ \implies \frac{1}{V}\int_X e^{-\hat{\phi}_t}\omega^n &< C \\ \implies -\frac{1}{V}\int_X \hat{\phi}_t \omega^n &< C \end{aligned} \tag{2.55}$$

where the last line follows from Jensens inequality.

Remark 18. *In fact (2.55) follows directly from (2.52)*

From equation (2.54) and (2.55) obtain

$$-\frac{1}{V}\int_X \hat{\phi}_t\omega_t^n = I_\omega(\phi_t) - \frac{1}{V}\int_X \hat{\phi}_t\omega^n =\leq C - \frac{1}{V}\int_X \hat{\phi}_t\omega^n < 2C \qquad (2.56)$$

Using (2.56) and that $\max_X \hat{\phi}_t$ is bounded, conclude

$$0 \leq \int_X \theta_t\omega_t^n \leq \frac{1}{V}\int_X (\max_X \hat{\phi}_t - \hat{\phi}_t)\omega_t^n < C$$

For the second estimate the starting point is the well known identity

$$\int_X \theta_t^p(\omega_t^n - \omega_t^{n-1}\wedge\omega) = -\int_X \theta_t^p\partial\overline{\partial}\theta_t\wedge\omega_t^{n-1} = \frac{4p}{n(p+1)^2}\int_X |\partial\theta_t^{\frac{p+1}{2}}|_t^2\omega_t^n$$

from which it follows (using $\omega_t^{n-1}\wedge\omega \geq 0$)

$$\int_X |\partial\theta_t^{\frac{p+1}{2}}|_t^2\omega_t^n \leq \frac{n(p+1)^2}{4p}\int_X \theta_t^p\omega_t^n \qquad (2.57)$$

□

In [Pa] it is shown that

Lemma 13. *On compact Kähler manifolds (X,ω) of complex dimension n for any $u, h \in C^\infty(X,\mathbb{R})$*

$$\begin{aligned}\int_X |\overline{\partial}\nabla^{1,0}u|_\omega^2 e^h\omega^n = &-\int_X \langle\partial\Delta_{\omega,h}u, \partial u\rangle_\omega e^n\omega^n \\ &-\int_X (Ric(\omega) - i\partial\overline{\partial}h)(\nabla_\omega u, J\nabla_\omega u)e^h\omega^n\end{aligned}$$

from which a poincare-type inequality obtains.

Corollary 4. *Let X be a Fano manifold of complex dimension n. Let $\omega \in c_1(L)$ be a Kähler metric, $0 \leq \eta \in -c_1(L \otimes K_X)$, and let $h_{\omega,\eta} \in C^\infty(X,\mathbb{R})$ satisfy $Ric(\omega) - T_{\omega,\eta} = i\partial\overline{\partial}h_{\omega,\eta}$. $V_{h_{\omega,\eta}} := \int_X e^{h_{\omega,\eta}}\omega$. Then for any $\phi \in C^\infty(X)$*

$$\int_X |\partial\phi|^2_{T_{\omega,\eta}} e^{h_{\omega,\eta}}\omega^n \geq \int_X \phi^2 e^{h_{\omega,\eta}}\omega^n - \frac{1}{V_{h_{\omega,\eta}}}(\int_X \phi e^{h_{\omega,\eta}}\omega^n)^2$$

Note that when the uniform (in t) estimate $\frac{\omega_t^n}{\omega^n} \geq K > 0$ is true

$$c\omega_t \geq T_{\omega_t,\eta} \geq \omega_t \tag{2.58}$$

The second inequality follow directly since $\eta \geq 0$, independent of the uniform estimate. The first inequality is a consequence of the C^2 estimate and is discussed in §7.

Remark 19. *Roughly the content of this lemma has appeared in §5. The identity in the lemma after dropping the nonnegative term leads to an inequality, which for first eigenfunctions simplifies to*

$$\begin{aligned} 2\int_X |\partial u|^2_\omega e^{h_{\omega,\eta}}\omega^n = \int_X |\nabla u|^2_\omega e^{h_{\omega,\eta}}\omega^n &\leq \int_X |\nabla u|^2_{T_{\omega,\eta}} e^{h_{\omega,\eta}}\omega^n \\ &\leq \lambda_1 \int_X |\partial u|^2_\omega e^{h_{\omega,\eta}}\omega^n \end{aligned}$$

Note that (2.58) is applied in the first inequality and the lemma is used for the second inequality. The variational characterization of the first eigenvalue

$$\inf \frac{\int_X |\partial u|^2_\omega e^{h_{\omega,\eta}}\omega^n}{\int_X u^2 e^{h_{\omega,\eta}}\omega^n} = \lambda_1 \geq 2$$

applied with $\phi - \frac{\int_X \phi e^{h_{\omega,\eta}}\omega^n}{V_{h_{\omega,\eta}}}$ *can be used to obtain the corollary.*

proof of proposition. Under the twisted Kähler-Ricci flow

$$\partial\overline{\partial}\dot{\phi} = T_{\omega_t,\eta} - Ric(\omega_t)$$

So $h_{\omega_t,\eta} = -\dot{\phi}_t$. Applying the corollary with metric ω_t and function $\theta_t^{\frac{p+1}{2}}$ gives

$$\int_X |\partial\theta_t^{\frac{p+1}{2}}|^2_{T\omega,\eta} e^{-\dot{\phi}}\omega_t^n \geq \int_X \theta_t^{p+1} e^{-\dot{\phi}_t}\omega_t^n - \frac{1}{V_{h_{\omega,\eta}}}(\int_X \theta_t^{\frac{p+1}{2}} e^{-\dot{\phi}}\omega_t^n)^2$$

Apply hölder's inequality to $\theta_t^{\frac{p+1}{2}} e^{-\dot{\phi}_t} = \theta_t^{\frac{1}{2}} e^{-\frac{\dot{\phi}_t}{2}} \theta_t^{\frac{p}{2}} e^{\frac{-\dot{\phi}_t}{2}}$ and using $T_{\omega_t,\eta} \geq \omega_t$ obtain

$$\int_X \theta_t^{p+1} e^{-\dot{\phi}}\omega_t^n \leq \int_X |\partial\theta_t^{\frac{p+1}{2}}|_t^2 e^{-\dot{\phi}_t}\omega_t^n + C\int_X \theta_t^p e^{-\dot{\phi}_t}\omega_t^n \int_X \theta_t e^{-\dot{\phi}_t}\omega_t^n$$

Apply inequality (2.57) with the uniform estimate $|\dot{\phi}_t| < C$ to obtain

$$\int_X \theta_t^{p+1} e^{-\dot{\phi}}\omega_t^n \leq Cp\int_X \theta_t^p\omega_t^n + C\int_X \theta_t^p e^{-\dot{\phi}_t}\omega_t^n = C(p+1)\int_X \theta_t^p\omega_t^n$$

and so the second estimate of the claim follows: $||\hat{\phi}_t||_{C^0(X)} \leq C$ for $t \geq 0$ with C independent of t. □

2.7 Perelman's estimates twisted setting

In proposition (2.43) we need to check Perelman's estimates carries over to the twisted setting, avoiding any circularity in verifying the first inequality in (2.58) . That is, twisted flows satisfying the non-collapsing estimate are uniformly equivalent to the initial metric.

Recall from the calculation

$$\begin{aligned}
\partial\overline{\partial}\dot{\phi}_t &= \frac{\partial}{\partial t}\omega_t \\
&= T_{t,\eta} - Ric(\omega_t) \\
&= T_{t,\eta} - T_{\omega_0,\eta} + Ric(\omega_0) - Ric(\omega_t) - Ric(\omega_0) + T_{\omega_0,\eta} \\
&= \partial\overline{\partial}(\phi_t + log\frac{\omega_t^n}{\omega_0^n} - h_{\omega_0,\eta})
\end{aligned}$$

the scalar equation for the potentials is

$$\dot{\phi}_t = \phi_t + c_t + \log\frac{\omega_t^n}{\omega_0^n} - h_{\omega_0,\eta} \tag{2.59}$$

Time differentiating and setting $u_t = \dot{\phi}_t$ we obtain the same equation appearing in the Kähler-Ricci flow setting

$$\Box_t u_t = u_t + a_t$$

where $a_t = \dot{c}_t$ and $\Box_t = \partial_t - \frac{1}{2}\Delta$. So apply the argument appearing in [Pa] directly to obtain that a_t is uniformly bounded in t.

Lemma 14. *The scalar curvature R is uniformly bounded from below along the twisted Kähler-Ricci flow.*

Note this follows from the maximum principle:

Proof.

$$\frac{\partial\Lambda_{\omega_t}(Ric-\eta)}{\partial t}(\omega_t) = |Ric_{\alpha\overline{\beta}} - \eta_{\alpha\overline{\beta}}|^2 + \Delta_t(\Lambda_{\omega_t}(Ric-\eta))(\omega_t) - \Lambda_{\omega_t}(Ric-\eta)(\omega_t) \tag{2.60}$$

This simplifies to

$$\frac{\partial e^t \Lambda_{\omega_t}(Ric - \eta)}{\partial t}(\omega_t) = e^t |Ric_{\alpha\overline{\beta}} - \eta_{\alpha\overline{\beta}}|^2 + e^t \Delta_t(\Lambda_{\omega_t}(Ric - \eta))(\omega_t)$$

from which we get

$$\Lambda_{\omega_t}(Ric - \eta)(\omega_t) \geq e^{-t}\Lambda_{\omega_0}(Ric - \eta)(\omega_0) \tag{2.61}$$

Since $\omega_t > 0$ and $\eta \geq 0$

$$\Lambda_{\omega_t}\eta = tr_t \eta \geq 0$$

we obtain

$$S(\omega_t) \geq e^{-t}(S(0) - tr_\omega \eta) \geq \min\{0, S(0) - tr_\omega \eta\}$$

To verify (2.60) note that the twisted Kähler-Ricci flow differs from the standard Kähler-Ricci flow in that Ric is replaced by $Q := Ric - \eta$. Note $\dot{Q} = \dot{Ric}$. So the check for the identity is no different from the Kähler-Ricci flow setting:

$$\begin{aligned}
\frac{d\Lambda_t Q}{\partial t} &= \frac{\partial}{\partial t} g_t^{\alpha\overline{\beta}} Q_{\alpha\overline{\beta}} = -g_t^{\alpha\overline{q}} g_t^{p\overline{\beta}} \dot{g_{p\overline{q}}} Q_{\alpha\overline{\beta}} + g_t^{\alpha\overline{\beta}} \dot{Q}_{\alpha\overline{\beta}} \\
&= -(g_t - Q)_{p\overline{q}}(Q)^{p\overline{q}} - g_t^{\alpha\overline{\beta}} \partial_\alpha \partial_{\overline{\beta}} g_t^{p\overline{q}} \dot{g_{p\overline{q}}} \\
&= -g_t^{p\overline{q}} Q_{p\overline{q}} + |Q_{p\overline{q}}|^2 - \Delta_t g_t^{p\overline{q}}(g_t - Q)_{p\overline{q}} \\
&= |Q_{p\overline{q}}|^2 - \Lambda_{\omega_t} Q + \Delta_t \Lambda_{\omega_t} Q
\end{aligned}$$

□

Corollary 5. *Δu is bounded from above.*

Proof.

$$\begin{aligned}\Delta u =& n - (S(\omega_t - Tr_t\eta))\\ \leq& n - e^{-t}(S(\omega_0) - Tr_0\eta)\\ \leq& n - \max\{0, S(\omega_0) - Tr_0\eta\}\\ <& C\end{aligned}$$

□

In the following lemma we sketch details in places since the argument is almost identical with the untwisted setting in [ST].

Lemma 15. *The function $u(t) := \dot{\phi}(t)$ is uniformly bounded from below.*

Proof. Along the twisted flow we have from (2.59)

$$\frac{du}{dt} = n - (R - tr_t\eta) + u + a \leq n + C + u \tag{2.62}$$

where the second inequality follows from the lower bound on $R - tr_t\eta$ obtained from the lemma. The argument proceeds by contradiction as in [ST].

Start off assuming there is a point and time (t_0, y_0) where u is very negative. Using (2.62) gives that $u(t)$ stays negative for $t \geq t_0$ in a neighborhood U of y_0. Obtain estimates

$$\begin{aligned}& u(t)(z) \leq e^{t-t_0}(C + u(t_0)) \leq -\tilde{C}e^t \quad t \geq t_0 \quad z \in U\\ & \phi(t)(z) \leq \phi(t)(z) - \tilde{C}e^{t-t_0} \leq -Ce^t \quad t >> 0 \quad z \in U\end{aligned} \tag{2.63}$$

The first item follows from integrating out the differential inequality $\frac{du}{dt} \leq C + u$ and using that $u(t_0)$ can be made very negative. For the second integrate one more time using $u = \dot{\phi}$.

Using the normalization $\frac{1}{V}\int_M e^{-u(t)} = 1$ obtain that $u(t)$ can't be everywhere too negative. In particular we have the uniform estimate

$$\max_M(u(t)) \geq -C \tag{2.64}$$

With (2.62) rewritten as

$$\frac{d}{dt}(u - \phi) < C \tag{2.65}$$

obtain the estimate

$$\max_M \phi(t) \geq -C - \tilde{C}t \tag{2.66}$$

after integrating and combining with (2.64). All constants are uniform.

We may obtain an upper bound on $\max_M \phi(t)$ using the Green's formula (see [ST]) applied to $\phi(t)$ and $-\Delta_0\phi(t) = -tr_0 g(t) + n < n$ and (2.63). Obtain

$$\max_M \phi(t) \leq \alpha \max_M \phi(t) - Ce^t + \tilde{C}$$

where $0 < \alpha < 1$ and $t \geq t_0$. Then $\alpha < 1$ gives the estimate

$$\max_M \phi(t) \leq -Ce^t + \tilde{C}$$

So taking large values of t it follows that $\max_M \phi(t)$ decays no slower than a linear function whereas the upper bound gives that it decays at least as fast as an exponential so a contradiction is obtained. $\square$

Remark 20. *By the corollary*

$$\Delta u < C(u + 2B)$$

for B sufficiently large, since u is bounded below by the lemma.

Proposition 24. *Under the twisted ricci flow we check the evolution of $|\nabla u|^2$ and Δu satisfy*

$$\square(\Delta u) = (\partial_t - \Delta)\Delta u = -|\nabla\overline{\nabla}u|^2 + \Delta u \tag{2.67}$$

$$\square(|\nabla u|^2) = \partial_t|\nabla u|^2 - \Delta|\nabla u|^2 = -|\nabla\nabla u|^2 - |\nabla\overline{\nabla}u|^2 + |\nabla u|^2 - \langle \eta, \nabla u\overline{\nabla}u\rangle_g \tag{2.68}$$

Proof. Check (2.67) by direct calculation using $\dot{g}_{i\bar{j}} = u_{i\bar{j}} = T_{i\bar{j}} - Ric_{i\bar{j}}$ and $\partial_t u = \Delta u + u + a$

$$\begin{aligned}
\partial_t \Delta u =& \partial_t g^{i\bar{j}} u_{i\bar{j}} \\
=& \Delta\dot{u} - 2g^{i\bar{q}}g^{p\bar{j}}\dot{g_{p\bar{q}}}u_{i\bar{j}} \\
=& \Delta(\Delta u + u + a) - 2g^{i\bar{q}}g^{p\bar{j}}u_{p\bar{q}}u_{i\bar{j}} \\
=& \Delta(\Delta u + u) - |\nabla\overline{\nabla}u|^2
\end{aligned}$$

and the identity follows.

For (2.68) we start with the Bochner formula

$$\Delta\frac{1}{2}|\nabla u|^2 = \frac{1}{2}|\nabla\nabla u|^2 + \frac{1}{2}|\nabla\overline{\nabla}u|^2 + R_{\alpha\overline{\beta}}u^{\alpha}u^{\overline{\beta}} + g^{\alpha\overline{\beta}}(u_{\alpha}(\Delta u)_{\overline{\beta}} + (\Delta u)_{\alpha}u_{\overline{\beta}})$$

Also note that

$$\partial_t \frac{1}{2}|\nabla u|^2 = g^{i\bar{j}}\dot{u}_i u_{\bar{j}} + g^{i\bar{j}} u_i \dot{u}_{\bar{j}} - g^{i\bar{t}} g^{s\bar{j}} u_{s\bar{t}} u_i u_{\bar{j}}$$
$$= g^{i\bar{j}}(\Delta u + u + a)_i u_{\bar{j}} + g^{i\bar{j}} u_i (\Delta u + u + a)_{\bar{j}} + g^{i\bar{t}} g^{s\bar{j}} (Ric - T)_{s\bar{t}} u_i u_{\bar{j}}$$

Putting this together

$$(\partial_t - \Delta)|\nabla u|^2 = -|\nabla\nabla u|^2 - |\nabla\overline{\nabla} u|^2 + |\nabla u|^2 - \eta^{i\bar{j}} u_i u_{\bar{j}}$$

The identity follows. □

Using these evolution equations applied to the same quantities appearing in [ST] obtain by application of the maximum principle the following estimates

Claim 14.

$$|\nabla u|^2 \leq C(u + c) \tag{2.69}$$

$$-\Delta u \leq C(u + C) \tag{2.70}$$

provided $\eta \geq 0$.

So uniform bounds on $|u|$ give uniform bounds on $|\nabla u|$ and $|\Delta u|$.

Remark 21. *After this is verified we follow arguments of [SzCo] where Proposition 7. in the exposition of Sesum-Tian is replaced by a twisted entropy functional more appropriate to the study of the twisted Ricci flow. At this point instead of following arguments in [ST] such as Claim* 8 *where upper bounds on* u, R *are obtained in terms of the diameter,* u *is analyzed just as the diameter is (by considering sub level sets of* u*) in the subsequent propositions. We* *outline briefly the remaining arguments for the sake of exposition. Roughly the arguments make essential use of the monotonicity properties of twisted Perelman entropy and that its coercive (in the sense of Tao), in that it provides a scale invariant geometric control on the flow known as κ-noncollapsing. Eventually to bound the scalar curvature we will see that the assumption $\omega_t^n \geq K\omega^n$ for $t \in [0, \infty)$ is used.*

Remark 22. *A priori its not clear what the effect of $\eta \geq 0$ is on applying the maximum principle as in original Kähler setting. We will see the effect is benign. However, if $\eta < 0$ there are complications.*

Proof. Just as in the case of Kähler Ricci flow we consider the quantity

$$H = \frac{|\nabla u|^2}{u + 2B}$$

an application of the maximum principle will yield (2.69).

Using $\partial_t u = \Delta u + u + a$

$$\partial_t \frac{|\nabla u|^2}{u + 2B} = \frac{\partial_t |\nabla u|^2}{u + 2B} - \frac{|\nabla u|^2 (\Delta u + u + a)}{(u + 2B)^2}$$

Similarly (we omit a factor of 2)

$$\begin{aligned}
\Delta \frac{|\nabla u|^2}{u + 2B} &= g^{p\bar{q}} \partial_{\bar{q}} \partial_p \left(\frac{|\nabla u|^2}{u + 2B}\right) \\
&= g^{p\bar{q}} \left(\frac{|\nabla u|_p^2}{u + 2B} - \frac{|\nabla u|^2 u_p}{(u + 2B)^2}\right)_{\bar{q}} \\
&= \frac{\Delta |\nabla u|^2}{u + 2B} - \frac{g^{p\bar{q}} |\nabla u|_p^2 u_{\bar{q}}}{(u + 2B)^2} - \frac{g^{p\bar{q}} |\nabla u|_{\bar{q}}^2 u_p + |\nabla u|^2 \Delta u}{(u + 2B)^2} + 2\frac{g^{p\bar{q}} |\nabla u|^2 u_p u_{\bar{q}}}{(u + 2B)^3}
\end{aligned}$$

So

$$\begin{aligned}\Box H =&\frac{\partial_t|\nabla u|^2}{u+2B} - \frac{|\nabla u|^2(u+a)}{(u+2B)^2} - \frac{\Delta|\nabla u|^2}{u+2B} + \frac{g^{p\bar{q}}|\nabla u|^2_p u_{\bar{q}}}{(u+2B)^2} + \frac{g^{p\bar{q}}|\nabla u|^2_{\bar{q}} u_p}{(u+2B)^2} - 2\frac{g^{p\bar{q}}|\nabla u|^2 u_p u_{\bar{q}}}{(u+2B)^3}\\ =&\frac{\Box|\nabla u|^2}{u+2B} - \frac{|\nabla u|^2(u+a)}{(u+2B)^2} + \frac{\langle\nabla|\nabla u|^2, \overline{\nabla}u\rangle + \langle\overline{\nabla}|\nabla u|^2, \nabla u\rangle}{(u+2B)^2} - 2\frac{|\nabla u|^4}{(u+2B)^3}\end{aligned}$$

using the evolution identity (2.68) obtain

$$\begin{aligned}\Box H =&\frac{-|\nabla\nabla u|^2 - |\nabla\overline{\nabla}u|^2 - \langle\eta, \nabla u\overline{\nabla}u\rangle_g}{u+2B} + \frac{|\nabla u|^2(2B-a)}{(u+2B)^2}\\ &+ \frac{\langle\nabla|\nabla u|^2, \overline{\nabla}u\rangle + \langle\overline{\nabla}|\nabla u|^2, \nabla u\rangle}{(u+2B)^2} - 2\frac{|\nabla u|^4}{(u+2B)^3}\end{aligned} \tag{2.71}$$

Note that

$$\begin{aligned}\nabla H =&g^{\alpha\overline{\beta}}\left(\frac{|\nabla u|^2_\alpha}{u+2B} - \frac{|\nabla u|^2 u_\alpha}{(u+2B)^2}\right)\\ \Longrightarrow& (2-\epsilon)\frac{\overline{\nabla}u\cdot\nabla H}{u+2B}\\ =&(2-\epsilon)\frac{\langle\overline{\nabla}u, \nabla|\nabla u|^2\rangle}{(u+2B)^2} - (2-\epsilon)\frac{|\nabla u|^4}{(u+2B)^3}\end{aligned} \tag{2.72}$$

Rewriting the last two terms in (2.71) using (2.72) obtain

$$\begin{aligned}2\frac{\langle\overline{\nabla}u, \nabla|\nabla u|^2\rangle}{(u+2B)^2} - 2\frac{|\nabla u|^4}{(u+2B)^3} =& (2-\epsilon)\frac{\overline{\nabla}u\cdot\nabla H}{u+2B} + \epsilon\frac{\langle\overline{\nabla}u, \nabla|\nabla u|^2\rangle}{(u+2B)^2}\\ &- \epsilon\frac{|\nabla u|^4}{(u+2B)^3}\end{aligned} \tag{2.73}$$

Using an orthonormal frame and an application of Cauchy-Schwartz gives

$$\begin{aligned} |\overline{\nabla} u \cdot \nabla |\nabla u|^2| =& \nabla_{\bar{i}} u \nabla_i (\nabla_j u \nabla_{\bar{j}} u) \\ =& \nabla_{\bar{i}} u (\nabla_i \nabla_j u) \nabla_{\bar{j}} u + \nabla_{\bar{i}} u \nabla_j u (\nabla_i \nabla_{\bar{j}} u) \\ \leq& |\nabla u|^2 (|\nabla \nabla u| + |\nabla \overline{\nabla} u|) \end{aligned} \tag{2.74}$$

Fix a constant $C \geq 1$. Then apply (2.74) to the term in (2.73) gives

$$\begin{aligned} \epsilon \frac{|\overline{\nabla} u \cdot \nabla |\nabla u|^2|}{(u+2B)^2} \leq& C\epsilon \frac{|\nabla u|^2 (|\nabla \nabla u|^2 + |\nabla \overline{\nabla} u|^2)}{(u+2B)^{3/2}(u+2B)^{1/2}} \\ \leq& \frac{\epsilon}{4} \frac{|\nabla u|^4}{(u+2B)^3} + \frac{C^2 \epsilon}{2} \frac{(|\nabla \nabla u|^2 + |\nabla \overline{\nabla} u|^2)}{u+2B} \end{aligned} \tag{2.75}$$

From (2.75) obtain

$$\begin{aligned} 2\frac{\langle \overline{\nabla} u, \nabla |\nabla u|^2 \rangle}{(u+2B)^2} - 2\frac{|\nabla u|^4}{(u+2B)^3} =& (2-\epsilon) \frac{\overline{\nabla} u \cdot \nabla H}{u+2B} + \epsilon \frac{\langle \overline{\nabla} u, \nabla |\nabla u|^2 \rangle}{(u+2B)^2} - \epsilon \frac{|\nabla u|^4}{(u+2B)^3} \\ \leq& (2-\epsilon) \frac{\overline{\nabla} u \cdot \nabla H}{u+2B} + 2C^2 \epsilon \frac{|\nabla \nabla u|^2 + |\nabla \overline{\nabla} u|^2}{u+2B} - \frac{3\epsilon}{4} \frac{|\nabla u|^4}{(u+2B)^3} \end{aligned}$$

Choose ϵ so that $2C^2\epsilon < \frac{1}{2}$. Applying this inequality to the expression for $\Box H$ obtain

$$\begin{aligned} \Box H =& \frac{-|\nabla \nabla u|^2 - |\nabla \overline{\nabla} u|^2 - \langle \eta, \nabla u \overline{\nabla} u \rangle_g}{u+2B} + \frac{|\nabla u|^2 (2B-a)}{(u+2B)^2} \\ &+ (2-\epsilon) \frac{\overline{\nabla} u \cdot \nabla H}{u+2B} + 2C^2 \epsilon \frac{|\nabla \nabla u|^2 + |\nabla \overline{\nabla} u|^2}{u+2B} - \frac{3\epsilon}{4} \frac{|\nabla u|^4}{(u+2B)^3} \end{aligned} \tag{2.76}$$

Note that since $\eta \geq 0$

$$\langle \eta, \nabla u \overline{\nabla} u \rangle_g \geq 0$$

This can be checked through point-wise calculation simultaneously diagonalizing η with respect g. In co-ordinates this looks like

$$\eta^{i\bar{j}}u_i u_{\bar{j}} = \lambda_{i\bar{i}}\delta_{i\bar{j}}u_i u_{\bar{j}} = \lambda_{i\bar{i}}|u_i|^2 \geq 0$$

So drop this term in (2.76) to obtain the inequality

$$\Box H \leq \frac{(2C^2\epsilon - 1)(|\nabla\nabla u|^2 + |\nabla\overline{\nabla} u|^2)}{u+2B} + \frac{|\nabla u|^2(2B-a)}{(u+2B)^2} + (2-\epsilon)\frac{\overline{\nabla} u \cdot \nabla H}{u+2B} - \frac{3\epsilon}{4}\frac{|\nabla u|^4}{(u+2B)^3}$$

Since $2C^2\epsilon - 1 < 0$ and $u + 2B > 0$ we may drop the first term on the right hand side to obtain

$$\Box H \leq \frac{|\nabla u|^2(2B-a)}{(u+2B)^2} + (2-\epsilon)\frac{\overline{\nabla} u \cdot \nabla H}{u+2B} - \frac{3\epsilon}{4}\frac{|\nabla u|^4}{(u+2B)^3} \tag{2.77}$$

So even with the extra term $\langle \eta, \nabla u \overline{\nabla} u\rangle_g$ we apply the maximum principle to the quantity H just as in the Kähler-Ricci flow case. At a point where H achieves it maximum we have $\nabla H = 0$ and $\Delta H \leq 0$. So

$$0 \leq \partial_t H_{\max} \leq \Box H_{\max} \leq \frac{|\nabla u|^2}{(u+2B)^2}(2B - a - \frac{3\epsilon}{4}\frac{|\nabla u|^2}{u+2B}) \tag{2.78}$$

If the inequality $H = \frac{|\nabla u|^2}{u+2B} \leq C$ fails we may produce a sequence of counter examples given by the data $\{(x_{\max}, t_n), C_n\}$ where $C_n \to \infty$. So that the parenthesis of the second inequality in (2.78) is negative on the sequence for $n >> 0$. Contradiction. (2.69) follows.

The second inequality $-\Delta u < C(u + 2B)$ obtains similarly. The dependency on η arises as follows

$$\Box(K+bH) = \frac{-b|\nabla\nabla u|^2 - (b-1)|\nabla\overline{\nabla u}|^2}{u+2B} + \frac{(K+bH)(2B-a)}{u+2B}$$
$$+2\frac{\overline{\nabla} u \cdot \nabla(K+bH)}{u+2B} - \frac{\langle \eta, \nabla u \overline{\nabla} u\rangle}{u+2B}$$

where $K = \frac{-\Delta u}{u+2B}$ and $b > 1$.

Since $\frac{\langle \eta, \nabla u \overline{\nabla} u\rangle}{u+2B} \geq 0$ we may drop the last term. Set $G = K + bH$. Then the evolution identity for $G_{\max}$ is

$$\frac{d}{dt} G_{\max} \leq -(b-1)\frac{|\nabla\overline{\nabla} u|^2}{u+2B} + \frac{G_{\max}(2B-a)}{u+2B}$$

This is also the inequality that one gets in the Kähler-Ricci flow case so the remaining part of this argument is identical. □

Tracing the twisted Kähler-Ricci flow equation obtain $\Delta u = Tr_{\omega_t} T_t - R$ then $R = Tr_{\omega_t} T_t - \Delta u$. Once we can establish

$$T_{\omega_t,\eta} \leq c\omega_t \tag{2.79}$$

where c is a constant independent of t, we can bound the scalar curvature (2.44) since

$$Tr_t T_{\omega_t,\eta} \leq cn$$

and we can obtain the bound $R < C(u + 2B)$.

Remark 23. *However, proceeding exactly as in [ST] does not work since the κ-noncollapsing* *property for the flow:*

$$Vol_t(B(x,1)) \geq \kappa$$

for any metric g satisfying

$$|R - tr_t\eta| \leq 1 \quad on\ B(x,1) \quad \partial B(x,1) \neq \emptyset$$

needs to be established.

Recently this has been verified in [SzCo]. This involves introducing the twisted entropy functional:

Definition 2. *On a compact Kähler manifold let η be a closed nonnegative (1,1) form. The twisted entropy functional*

$\mathcal{W} : Met \times C^{\infty}(\mathbb{R}) \times \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ *is given by*

$$\mathcal{W}^{\eta}(g, f, \tau) := \int_M (\tau(R - tr_g\eta + |\nabla f|_g^2) + f - 2n)(4\pi\tau)^{-n} e^{-f} dm$$

for the unnormalized twisted Kähler ricci flow

$$\dot{\omega} = -2(Ric(\omega) - \eta) \tag{2.80}$$

This is Perelman's entropy functional with R replaced by $R - tr_g\eta$; exactly the same adjustment needed to obtain the twisted Mabuchi functional. Similar to the twisted Mabuchi functional the monotonicity property of the entropy functional carries over to the twisted setting. The twisted entropy functional shares many other useful properties with the entropy functional.

Proposition 6. *For* $(g(t), f(t), \tau(t)) \in Met \times C^{\infty}(\mathbb{R}) \times \mathbb{R}$

$$\partial_t \mathcal{W}(g(t), f(t), \tau(t)) = \tau \int_M |Ric - \eta + Hess(f) - \frac{g}{2\tau}|_t^2 (4\pi\tau)^{-n} e^{-f} dm$$

where the triple $(g(t), f(t), \tau(t))$ *satisfies the usual system of PDE's with* R, Ric *replaced by* $R - tr_t\eta, Ric - \eta$.

Following a contradiction argument κ-noncollapsing in the formulation of [SzCo] is obtained by applying the twisted entropy to a test function. From its monotonicity properties and effective estimates one can conclude. The softer version in the spirit of [ST] works too. But first the flow needs to be reparametrized to work with the twisted entropy functional.

Claim 15. *The twisted Kähler ricci flow can be reparametrized to unnormalized twisted ricci flow*

$$\dot{\omega} = -2(Ric(\omega) - \eta) \tag{2.81}$$

Proof. Let $\tilde{g} = \psi(t)(g)$ denote the reparametrized metric with respect $t(s)$. To determine $t(s)$ we need to solve ode's:

$$\begin{aligned} \partial_t \tilde{g} &= \dot{\psi}(g) + \psi(\dot{g}) \\ &= \dot{\psi}(g) + \psi(-Ric + g + \eta) \\ &= (\dot{\psi} + \psi)(g) + \psi(\eta - Ric) \end{aligned}$$

So

$$\frac{2}{\psi}\partial_t \tilde{g} = (2\frac{\dot{\psi}}{\psi} + 2)(g) - 2Ric = -2(Ric - \eta)$$

provided $\frac{\dot{\psi}}{\psi}+1=0$. Since $\partial_s = \frac{\partial t}{\partial s}\partial_t = \frac{2}{\psi}\partial_t$ a choice of reparmetrization $t(s)$ can be obtained by solving

$$\frac{2\dot{\psi}}{\psi}+2=\frac{\partial\psi}{\partial s}+2=0$$
$$\frac{dt}{ds}=\frac{2}{\psi}$$

Solving we obtain $t(s)=-ln(C-2s)$ and if we enforce that $t(0)=0$ we can take $C=1$. □

Remark 24. *Reparametrization allows to transfer the κ-noncollapsing property for unnormalized flow to normalized twisted Kähler ricci flow. See [SzCo].*

From the discussion above adjusting u by a constant appropriately the following uniform estimates are in hand

$$|\Delta u|, |\nabla u|^2 < Ku \tag{2.82}$$

So it suffices to bound u from above.

It was observed in [SzCo] that by considering sublevel sets of the form

$$M(a,b)=\{x\in M|a<u<b\}$$

instead of geodesic annuli u can be bounded directly without requiring a diameter bound.

Remark 25. *If $a<b<c<d$ then $M(a,b)\cap M(c,d)=\emptyset$*

For the purpose of bounding u a contradiction argument needs to be made and one starts by assuming u grows without bound.

$V = Vol(M_t)$ is constant along the flow. Partition M using the range of u then

$$\sum_{i=1}^{N} Vol(M(2^{10^{i-1}k}, 2^{10^{i}k})) < V$$

for k sufficiently large (depending on u) and taking $N > \frac{V}{\epsilon}$, there is an $1 \leq i_0 \leq N$ for which

$$0 < Vol(M(2^{10^{i_0-1}k}, 2^{10^{i_0}k})) < \epsilon$$

Note that i_0 can be taken to be 1 at the cost of making k larger.

Like geodesic annuli considered in [ST], [SzCo] does the same for $M(a, b)$ instead. In particular:

Lemma 16. *There is a point $x \in M$ with $u(x) = a + 1$ and constants κ_1 such that if $b - 2 > a > K$ then*

$$Vol(M(a, b)) > \kappa_1 a^{-n}$$

Restricting (2.82) to $M(a, a + 2) \subset M(a, b)$ gives estimates necessary to apply the κ non-collapsing property to conclude.

After specifying a threshold that k above must exceed, since u is assumed unbounded we may assume there is k so that $Vol(M(2^k, 2^{10k}) < \epsilon < 1$ and we can always find an $x \in M$ so that $u(x) = 2^{5k} + 1$ say. Clearly for $k_1, k_2 \in [k, 10k]$, $Vol(M(2^{k_1}, 2^{k_2})) < \epsilon$ holds. Moreover, similar to Claim (10) in [ST]

Lemma 17. *Provided k exceeds the threshold* $\max\{\log_2(\kappa^{\frac{-1}{n}}), 2\}$ *and* $0 < \epsilon < 1$*, there exists*

$k_1, k_2 \in [k, 10k]$ *with* $k_2 > k_1 + 4$ *such that*

$$Vol(M(2^{k_1}, 2^{k_2})) < \epsilon$$

$$Vol(M(2^{k_1+2}, 2^{k_2-2})) > 2^{-3n} Vol(M(2^{k_1}, 2^{k_2}))$$

The second estimate above follows by iterating the reverse inequality starting with the sublevel set $M(2^k, 2^{9k+2})$. Finally to conclude one applies the previous lemma and uses the threshold to obtain a contradiction.

The penultimate step is similar to Lemma to 11 in [ST] with $-\Delta u = Tr_t(Ric - \eta) - n$ replacing scalar curvature and provided $k_2 > k_1 + 1$ then

Lemma 18. *There exists* $r \in [2^{k_1}, 2^{k_1+1}]$ *and* $r_2 \in [2^{k_2-1}, 2^{k_2}]$ *so that*

$$\int_{M(r_1,r_2)} (-\Delta u) dm < C Vol(M(2^{k_1}, 2^{k_2}))$$

As before one works with (2.82) on sublevel sets. An application of co-area formula allows to pass to estimates on some smooth sets $u = r_i$, $i = 1, 2$. Then conclude as in [ST].

Finally just as in Proposition 9 in [ST] we are in the setting of lemma (17) so

Proposition 25. *There is an* $\epsilon > 0$ *such that if* $k > \max\{log_2(\kappa_1^{\frac{-1}{n}}), 2\}$ *and* $Vol(M(2^{k_1}, 2^{k_2})) < \epsilon$ *then* u *is bounded.*

This proceeds by contradiction, when u grows without bound a cutoff function is constructed so that lemma's $17, 18$ may be used and fed into the twisted entropy functional just as in [ST]. In fact the same argument in [ST] with use of the twisted entropy function can be made. However [SzCo] proceeds using effective estimates to obtain the contradiction: roughly by lemma (17) there is k_1, k_2 such that $V := Vol(M(2^{k_1}, 2^k)) < \epsilon$, whereas via the twisted entropy functional we may obtain a choice for which $V > \epsilon$ a contradiction. So u is bounded from above.

To summarize, along the twisted Kähler ricci flow [SzCo] obtained

Proposition 7 (Sz-Co)**.** *Along TKRF with $g(0) = g_0$ there exist a constant C depending continuously on the C^3 norm of g_0(and a uniform lower bound of g_0) such that*

$$|u| + |\nabla u|_{g(t)} + |\Delta_{g(t)} u| \leq C$$

Now we are in a position to start bounding scalar curvature and also justify the first inequality in (2.58). For this it suffices to show for the twisted Kähler Ricci flow:

Lemma 19. *Suppose $\frac{\omega_t^n}{\omega^n} \geq K_0$ for all $t \in [0, \infty)$ where K_0 is a constant independent of t. Then there exist positive constants k_0, K independent of $t > 0$ such that for all $t > 0$ the following estimates hold*

$$0 < n + \Delta_\omega \phi_t < K \tag{2.83}$$

$$|\partial\overline{\partial}\phi_t|_\omega < k + \sqrt{n} \tag{2.84}$$

$$k_0^{-1}\omega < \omega_t < K\omega \tag{2.85}$$

Remark 26. *(2.83) appears as the parabolic analogue of Yau's C^2 estimate. But a little modification is needed. In Kähler-Ricci flow case [Pa] uses Perelman's estimate $|u| < C$. Recall in [ST] this depends on scalar curvature but thanks to [SzCo] this dependency can be removed.*

Proof. For (2.83) the ingredients remain the same as in [Pa]. Consider the quantity appearing in Kähler-Ricci flow

$$A := log(tr_\omega \omega_t) - k(\phi_t + c_t) \quad M \times [0, \infty)$$

Follow a computation similar to that in [BBEGZ]. Start with $\tau, \tau^{'}$ Kähler forms on a complex manifold. There is a lower bound $B > 0$ for $R_{i\bar{i}j\bar{j}}(\tau)$ so that

$$\Delta_{\tau^{'}} \log tr_\tau \tau^{'} \geq -\frac{tr_\tau Ric(\tau^{'})}{tr_\tau \tau^{'}} - Btr_{\tau^{'}}\tau$$

Apply this when $\tau = \omega$ and $\tau^{'} = \omega_t$. So we obtain when $\Box_t = -(\partial_t - \Delta_t)$

$$\begin{aligned}\Box_t \log(tr_\tau \tau^{'}) &\geq -\frac{\partial_t tr_\tau \tau^{'}}{tr_\tau \tau^{'}} - \frac{tr_\tau Ric(\tau^{'})}{tr_\tau \tau^{'}} - Btr_{\tau^{'}}\tau \\ &= -\frac{(\partial_t tr_\tau \tau^{'} + tr_\tau Ric(\tau^{'}))}{tr_\tau \tau^{'}} - Btr_{\tau^{'}}\tau\end{aligned}$$

Note

$$\partial_t \Delta_\tau \tau^{'} = \Delta_\omega \dot{\phi} = tr_\tau \tau^{'} - tr_\tau Ric(\tau^{'}) + tr_\tau \eta$$

Since $C \geq tr_\tau \eta \geq 0$ and $n \leq tr_\tau \tau^{'} tr_{\tau^{'}}\tau$ we have

$$\begin{aligned}\Box_t \log(tr_\tau \tau^{'}) &\geq -\frac{tr_\tau \tau^{'} + tr_t \eta}{tr_\tau \tau^{'}} - Btr_{\tau^{'}}\tau \\ &\geq -1 - \frac{C}{tr_\tau \tau^{'}} - Btr_{\tau^{'}}\tau \\ &\geq -1 - (\frac{C}{n} + B)tr_{\tau^{'}}\tau \\ &= -1 - Ctr_{\tau^{'}}\tau\end{aligned}$$

So choosing $k > C$ and using $tr_{\tau'}\tau = tr_t\omega = n - \Delta_t\phi = n - \Delta_{\tau'}\phi$

$$\Box_t A \geq -1 + (k - C)tr_{\tau'}\tau + k(\dot{\phi}_t - n + a_t) \tag{2.86}$$

Recall

$$tr_{\tau_1}\tau_2 \leq \frac{\tau_2^n}{\tau_1^n}(tr_{\tau_2}\tau_1)^{n-1}$$

so using $\frac{\omega_t^n}{\omega^n} = e^{h+\dot{\phi}-\hat{\phi}}$

$$\begin{aligned} tr_{\tau'}\tau \geq& \left(\frac{\tau^n}{(\tau')^n} tr_\tau \tau'\right)^{\frac{1}{n-1}} \\ =& e^{\frac{\hat{\phi}-h-\dot{\phi}}{n-1}} (tr_\omega \omega_t)^{\frac{1}{n-1}} \\ =& e^{\frac{A}{n-1}} e^{\frac{(k+1)\hat{\phi}-h-\dot{\phi}}{n-1}} \end{aligned}$$

Since estimates $|\dot{\phi}|, |\phi_t + c_t| < C$ are available obtain

$$tr_{\tau'}\tau \geq C_0 e^{\frac{A}{n-1}}$$

where $C > 0$.

So (2.86) becomes

$$\Box_t A \geq -C + C_0 e^{\frac{A}{n-1}}$$

It follows by applying the maximum principle that we have a uniform upper bounds on the maximum of $e^A < C^{n-1}$; since a maximum of A is a maximum of e^A. Using boundedness $\hat{\phi}$

on $[0, \infty)$ the upper bound of (2.83) follows:

$$n + \Delta_t \phi = tr_\omega \omega_t < K$$

The upper bound in (2.85) follows directly from here since

$$1 + \phi_{i\bar{i}} < tr_\omega \omega_t < K$$

so

$$\omega_t < K\omega$$

To get the first inequality in (2.85) use the uniform estimate to get

$$K_0 \leq \frac{\omega_t^n}{\omega^n} = \Pi_{j=1}^n (1 + \phi_{j\bar{j}}) < K^{n-1}(1 + \phi_{i\bar{i}})$$

Conclude $\omega_t > k_0^{-1}\omega$ for $k_0 := \frac{K_0}{K^{n-1}} > 0$. In particular, it follows that $T_{t,\eta}$ is uniformly equivalent to ω.

Another pointwise calculation yields (2.84). At a point p

$$\begin{aligned}
\langle \omega_t, \omega_t \rangle_g =& \sum_i (1 + \lambda_{i\bar{i}})^2 \\
<& (\sum_i (1 + \lambda_{i\bar{i}}))^2 \\
=& (tr_\omega \omega_t)^2 \\
<& K^2
\end{aligned}$$

Since $tr_\omega \omega_t > 0$ it follows $\sum_i (1+\lambda_{i\bar{i}}) > 0$ and so $-\sum_{i=1}^n \lambda_{i\bar{i}} < n$. It follows

$$\sum_i (\lambda_{i\bar{i}}^2 + \lambda_{i\bar{i}}) < \sum_i (1+\lambda_{i\bar{i}})^2 < K^2$$
$$\Longrightarrow |dd^c\phi|_g^2 = (\lambda_{i\bar{i}})^2 < K^2 - \sum_{i=1}^n \lambda_{i\bar{i}} < K^2 + n < (K+\sqrt{n})^2$$

□

2.8 Twisted Perelman entropy

We note, following [Tao], that this functional can be obtained by analyzing variations of known functionals. Temporarily replacing the volume form by a static measure, a critical quantity, which also happens to be a coercive quantity (in the sense of Tao) can be obtained due to Perelman. It is also monotone with special type of critical points. Denote the volume form by $d\mu$. Consider the functionals

$$E(f) = \frac{1}{2}\int_M |\nabla f|_g^2 d\mu \tag{2.87}$$

$$H(M,g) = \int_M R d\mu \tag{2.88}$$

Provided g is static the E functional deforms like:

$$\frac{d}{dt}E = -\int_M \Delta_g f \dot{f} d\mu$$

The first variation of the H functional is given by

$$\begin{aligned}
\frac{d}{dt}H = \frac{d}{dt}\int_M Rd\mu &= \int_M (\dot{R} + \frac{1}{2}tr_g(\dot{g}))d\mu \\
&= \int_M (-Ric^{\alpha\beta}\dot{g_{\alpha\beta}} - \Delta tr_g\dot{g} + \nabla^\alpha\nabla^\beta\dot{g}_{\alpha\beta} + \frac{1}{2}tr_g(\dot{g})_{\alpha\beta})d\mu \\
&= \int_M (-Ric^{\alpha\beta}\dot{g}_{\alpha\beta} + R\frac{1}{2}tr_g(\dot{g})_{\alpha\beta})d\mu
\end{aligned}$$

The gradient flow for the negative functional is known to be not parabolic in general($\dot{g} = Ric - Rg\ n \geq 3$). Replacing $d\mu$ by a static measure $dm = e^{-f}d\mu$ (so the potential f deforms like $\dot{f} = \frac{1}{2}tr_g(\dot{g})$) removes the contribution from $R\frac{1}{2}tr_g\dot{g}_{\alpha\beta}$ causing this issue. In this way obtain modified functionals H^{mod}, E^{mod}. H^{mod} deforms like

$$\begin{aligned}
\frac{d}{dt}H^{mod} &= \int_M \dot{R}dm \\
&= \int_M (-Ric^{\alpha\beta}\dot{g}_{\alpha\beta} - \Delta tr_g\dot{g} + \nabla^\alpha\nabla^\beta\dot{g}_{\alpha\beta})dm \\
&= \int_M (-Ric^{\alpha\beta}\dot{g}_{\alpha\beta} + (\Delta f - |\nabla f|_g^2)g^{\alpha\beta}\dot{g}_{\alpha\beta} \\
&\quad + \dot{g}_{\alpha\beta}(\nabla^\alpha f\nabla^\beta f) - \dot{g}_{\alpha\beta}(\nabla^\alpha\nabla^\beta f))dm
\end{aligned}$$

Note that in the deformation of H^{mod} the second term following the third equality $\Delta f tr_g\dot{g}$ above comes with an opposite sign to that of the variation of Dirichlet energy, which gives another motivation for the choice of E^{mod}. Its first variation is given by

$$\begin{aligned}
\frac{d}{dt}E^{mod} &= \int_M \frac{d}{dt}(g^{\alpha\beta}\nabla_\alpha f \nabla_\beta f)dm \\
&= \int_M (-\dot{g}_{\gamma\delta}\nabla^\gamma f \nabla^\delta f + g^{\gamma\delta}\dot{f}_\gamma f_\delta + g^{\gamma\delta} f_\gamma \dot{f}_\delta)dm \\
&= \int_M (-\dot{g}_{\gamma\delta}\nabla^\gamma f \nabla^\delta f + g^{\gamma\delta}(\frac{1}{2}tr(\dot{g}))_\gamma f_\delta + g^{\gamma\delta} f_\gamma \frac{1}{2}(tr_g \dot{g})_\delta)dm \\
&= \int_M (-\dot{g}_{\gamma\delta}\nabla^\gamma f \nabla^\delta f - (\Delta f - |\nabla f|_g^2)g^{\gamma\delta}\dot{g}_{\gamma\delta})dm
\end{aligned}$$

Define

$$\mathcal{F}_m(M,g) := H^{mod} + E^{mod} = \int_M (|\nabla f|_g^2 + R)dm$$

So $\mathcal{F}_m$ deforms along $\dot{g} = -(2Ric + 2Hessf)$ as

$$\partial_t \mathcal{F}_m(M,g,f) = 2\int_M |Ric(g) + Hess(f)|^2 dm$$

It follows $\mathcal{F}_m$ is non decreasing along the flow. Along this gradient flow using the relation $\dot{f} = \frac{1}{2}tr\dot{g}$ we see that the potential deforms according to $\dot{f} = -\Delta f - R$. Using $\mathcal{L}_{\nabla f} g_{\alpha\beta} = 2\nabla_\alpha \nabla_\beta f = 2Hessf$ and $\mathcal{L}_{\nabla f} f = |\nabla f|_g^2$ and $\partial_t \phi_t^* \omega_t = \phi_t^*(\mathcal{L}_X \omega_t + \dot{\omega}_t)$ it follows the gradient flow and the potential flow can be conjugated by a diffeomorphism to $\dot{g} = -2Ric(g)$ and $\dot{f} = -\Delta f - R + |\nabla f|_g^2$.

After conjugation f does not define a static measure but since $\mathcal{F}_m$ is invariant under diffeomorphism, its variation remains the same whether modified by a diffeomorphism or not. In particular under the modified flow induced by the diffeomorphism: $\dot{g} = -2Ric(g)$ and $\dot{f} = -\Delta f - R + |\nabla f|_g^2$, $\mathcal{F}_m$ is monotone non-decreasing.

From the monotonicity property it can be deduced the periodic solutions (ones for which

$\phi^*g(t_2) = g(t_1))$ are critical points i.e satisfy $Ric = -Hessf$. Similarly we can consider functionals with critical points solutions to $Ric + Hessf - \frac{1}{2\tau}g = 0$ (gradient shrinking solitons), which when $f = 0$ has positively curved Einstein metrics as critical points.

Note that

$$|Ric + Hessf - \frac{g}{2\tau^2}|_g^2 = |Ric + Hessf|_g^2 - \frac{1}{\tau}(R + \Delta f) + \frac{n}{4\tau^2} \tag{2.89}$$

With respect to Ricci flow scaling τ has dimension 2. So the derivative of the scale invariant quantity must have dimension -2. But each of the three terms have dimension -4. So we must consider a quantity like $2\tau \int_M |Ric + Hessf - \frac{g}{2\tau^2}|_g^2 dm$. Recalling the Nash entropy functional $N_m := \int_M \log \frac{dm}{d\mu} dm = -\int_M f dm$ deforms like (provided dm is static)

$$\frac{d}{dt} N_m = -\int_M \dot{f} dm = \int_M (\Delta F + R) dm = \int_M (|\nabla f|_g^2 + R) dm$$

we may integrate to obtain $\mathcal{W}_m(M, g, f, \tau)$.

Similarly

Proposition 26. *The twisted entropy functional given by*

$$\mathcal{W}_m^\alpha(M, g, \tau, f) = (4\pi\tau)^{-n} \int_M (\tau(R - Tr_g \alpha + |\nabla f|_g^2) + (f - 2n)) e^{-f} dm$$

deforms like

$$2\tau \int_M |Ric - \alpha + Hessf - \frac{g}{2\tau}|_g^2 dm$$

It is monotone provided $(g(t), f(t), \tau(t))$ *solves the coupled system*

$$\begin{aligned}\partial_g \dot{g} =& -2(Ric - \alpha)\\ \partial_t f =& -\Delta_g f + |\nabla f|_g^2 - R + Tr_g \alpha + \frac{n}{\tau}\\ \partial_t \tau =& -1\end{aligned}$$

on some interval $[0, T]$.

Proof. Following the same heuristics one obtains the twisted entropy functional. For the same reasons it will be both monotone and a critical quantity (in the sense of Tao).

Along $\dot{g} = -2(Ric - \alpha + Hessf)$ with dm static so $\dot{f} = -\Delta_g f - R + tr_g \alpha$, it follows

$$\frac{d}{dt}\int_M tr_g \alpha dm = \int_M -g^{s\alpha} g^{\beta t} \dot{g}_{\alpha\beta} \alpha_{st} dm = 2\int_M \langle Ric - \alpha + Hessf, \alpha \rangle dm$$

Just as in (2.89) we have

$$|Ric - \alpha + Hessf - \frac{g}{2\tau}|_g^2 = |Ric - \alpha + Hessf|_g^2 - \frac{1}{\tau}(R + \Delta f - tr_g \alpha) + \frac{n}{4\tau^2} \qquad (2.90)$$

Similarly we have that for $\mathcal{F}_m^\alpha := \int_M (R + |\nabla f|_g^2 - tr_g \alpha) dm$

$$\frac{d}{dt}\mathcal{F}_m^\alpha = -\int_M (Ric^{\alpha\beta} + \nabla^\alpha \nabla^\beta f - \alpha^{\alpha\beta}) \dot{g}_{\alpha\beta} dm$$

Likewise for $\mathcal{N}_m^\alpha := \int_M -f dm$. So if dm is static along $\dot{g} = -2(Ric - \alpha + Hessf)$ then

$$\begin{aligned}\frac{d}{dt}\mathcal{N}_m^\alpha &= \int_M -\dot{f} dm = \int_M (R + \Delta_g f - tr_g \alpha) dm \\ &= \int_M (R + |\nabla f|_g^2 - tr_g \alpha) dm\end{aligned}$$

It follows that

$$\frac{d}{dt}(\tau \mathcal{F}_m^\alpha - \mathcal{N}_m^\alpha - \frac{n}{2}\log\tau) = 2\tau \int_M |Ric - \alpha + Hessf - \frac{g}{2\tau}|_g^2 dm$$

normailized so that dm is a probability measure.

Write $e^{-f}d\mu = dm = (4\pi\tau)^{-\frac{n}{2}} e^{-\tilde{f}} d\mu$ so $f = \tilde{f} + \frac{n}{2}\log(4\pi\tau)$. Since dm is a probability measure up to an arbitrary constant we may write the functional as

$$\int_M (\tau(R + |\nabla \tilde{f}|_g^2 - Tr_g \alpha) + \tilde{f} - cnst)(4\pi\tau)^{-n} dm$$

Normalize the arbitrary constant to n so that in the euclidean setting, when also $\alpha = 0$, dm is gaussian measure and the expression vanishes.

Note that $\dot{\tilde{f}} = -\Delta_g f - R + tr_g \alpha + \frac{n}{2\tau}$ so conjugating by a diffeomorphism induced by the vector field ∇f obtains the coupled system

$$\begin{aligned}\dot{g} &= -2(Ric - \alpha) \\ \dot{\tilde{f}} &= (-\Delta_g \tilde{f} - R + Tr_g \alpha + \frac{n}{2\tau} + |\nabla \tilde{f}|_g^2)\end{aligned}$$

□

2.9 Extracting canonical metric

In this section we show how the canonical metric g_∞, solving (2.1), can be extracted. This follows along the same lines as [Pa] for the Kähler-Einstein case. Recall we have in hand the following (with the exception of the fourth bullet point)

- $\dot{\phi} = \log \frac{\omega_\phi^n}{\omega^n} + \phi + f + c_t$
- ν_ω^η is bounded below.
- ν_ω^η is decreasing along the twisted Kähler-Ricci flow.
- $|\hat{\phi}_t|_{C^0(X)} + |\partial\overline{\partial}\phi_t|_{C^0(X)} + |\nabla\partial\overline{\partial}\phi_t|_{C^0(X)} < C$

provided $\eta \geq 0$ and the non-collapsing estimate along the flow holds.

A consequence of the second and third bullet points is $\lim_{t\to\infty} \nu_\omega^\eta < \infty$. So for sequences $\{t_k\} \nearrow \infty$ we have

$$\lim_{k\to\infty} \int_X |\nabla \dot{\phi}_{t_k}|^2_{t_k} \omega^n_{t_k} = 0$$

Since otherwise we may integrate and contradict boundedness along the flow.

From the uniform C^2, C^3 estimates we obtain that the $(1,1)$ forms $\partial\overline{\partial}\phi_t$ are uniformly bounded in $C^{0,\alpha}(X)$ topology.

By differentiating the first bullet by $\zeta = \partial_{z_k}, \partial_{\overline{z_k}}$ we have from [Pa]

$$\Box_t(\zeta\phi_t) + \zeta\phi_t = (Tr_\omega - Tr_t)(L_\zeta\omega) + \zeta h_{\omega,\eta}$$

The laplacian term in $\Box_t$ and Tr_t contain g_t^{-1} so are bounded in $C^{0,\alpha}$ norm. By schauder regularity theory for parabolic equations we obtain $\zeta\phi_t$ is uniformly bounded in $C^{2,\alpha}$.

We recall that $C^{k,\alpha} \hookrightarrow C^k$ is a compact embedding. So a bounded sequence in $C^{k,\alpha}$ lies in a compact set in $C^k(\ k \geq 0, 0 < \alpha < 1)$ (see [Jo]).

Since $\zeta\phi_t \in C^{2,\alpha}$ is uniformly bounded, $\phi_t \in C^{3,\alpha}$ is uniformly bounded. Further, thanks to the C^0 uniform estimate $|\hat{\phi_t}| < C$ we have $\hat{\phi_t}$ lies in a bounded set in $C^{3,\alpha}$. So by the compact embedding we may arrange for a subsequence of ϕ_{t_k} so that $(\hat{\phi_{s_k}}, d\phi_{s_k}, \partial\overline{\partial}\phi_{s_k}, \nabla\partial\overline{\partial}\phi_{s_k})$ converges uniformly to $(\phi_\infty, d\phi_\infty, \partial\overline{\partial}\phi_\infty, \nabla\partial\overline{\partial}\phi_\infty)$.

The uniform estimate $\frac{\omega_t^n}{\omega^n} \geq K > 0$ gives that $\frac{\omega_{\phi_\infty}^n}{\omega^n} > K$. By (2.85) we have $\omega_{\phi_\infty} > 0$. The equation in the limit on this sequence then reads

$$\psi := \lim_{k\to\infty} \dot{\phi}_{s_k} = \log \frac{\omega_\infty^n}{\omega^n} + \phi_\infty - h_{\omega,\eta}$$

In particular ψ is C^1. Since $\phi_{s_k} \in C^{3,\alpha}$ converges in C^3

$$\lim_{k\to\infty} \partial\dot{\phi}_{s_k} = \lim_{k\to\infty} (\Delta_{s_k}\partial\phi_{s_k} + \partial\phi_{s_k} - \partial h_{\omega,\eta}) = \Delta_\infty\partial\phi_\infty + \partial\phi_\infty - \partial h_{\omega,\eta} = \partial\psi$$

The convergence being uniform gives that

$$0 = \lim_{k\to\infty} \int_X |\nabla\dot{\phi}_{s_k}|_{s_k}^2 \omega_{s_k}^n = \int_X |\partial\psi|_{\phi_\infty}^2 \omega_{\phi_\infty}^n$$

So ψ is a constant and from the normalization $\int_X e^{-\dot{\phi}_t}\omega_t^n = V$ we obtain that $\psi = 0$. So ϕ_∞ satisfies

$$0 = \log \frac{\omega_{\phi_\infty}^n}{\omega^n} + \phi_\infty - h_{\omega,\eta} =: F(\phi_\infty)$$

Recall that h_ω corresponds to the twisted ricci potential. Since $\omega_\infty > 0$ ellipticity follows from computing the linearization:

$$dF_{\phi_\infty}(v) = \Delta_{\phi_\infty} v + v$$

By Schauder regularity theory for elliptic equations we can conclude that ϕ_∞ is smooth. So ϕ_∞ is a desired solution to the equation $Ric(\omega_\infty) = \omega_\infty + \eta$.

2.10 C^3 Estimate

Following [PSS] work with the quantity $h^\alpha_\beta := \hat{g}^{\alpha\overline{k}} g_{\overline{k}\beta}$ ($\hat{g}^{-1}g$ is an endomorphism). We follow the notation $g_{\overline{k}\beta}$ since most quantities appearing in this formulation come as endomorphisms. Let $\hat{g}$ denote the initial metric. Note that $Trh = Tr_{\hat{g}}g$. Similarly $h^{-1} = g^{-1}\hat{g}$. Recall also that in the Kähler setting the connection looks like $g^{-1}\partial g$, is of pure type i.e. $\Gamma^k_{ij} = g^{k\overline{s}}\partial_j g_{i\overline{s}}$ (old notation) and the torsion free condition gives symmetry in permutation of i, j. Similarly the curvatures look like $R^{\cdot}_{\cdot\overline{k}\cdot} = \partial_{\overline{k}}\Gamma^{\cdot}_{\cdot\cdot}$.

The change in connection with respect to the initial metric can be written in terms of the quantity $(\nabla h)h^{-1}$.

$$(\nabla_m h)^k_\lambda h^\lambda_l = (\Gamma - \hat{\Gamma})^k_{ml}$$

so as not to distract from our main goal we only outline the calculation schematically:

$$
\begin{aligned}
(\nabla h)h^{-1} &= (\partial h - h\Gamma + h\Gamma)(h^{-1}) \\
&= (\partial h - \hat{g}^{-1}gg^{-1}\partial g + hg^{-1}\partial g)(h^{-1}) \\
&= (\partial h - \hat{g^{-1}}\partial g)(h^{-1}) + \Gamma \\
&= (\partial(\hat{g}^{-1}g) - \hat{g}^{-1}\partial g)h^{-1} + \Gamma \\
&= (\partial\hat{g}^{-1}gh^{-1} + \Gamma) \\
&= (-\hat{g}^{-1}\hat{g}^{-1}(\partial\hat{g})\hat{g} + \Gamma) \\
&= (-\hat{g}^{-1}\partial\hat{g} + \Gamma) \\
&= (-\hat{\Gamma} + \Gamma)
\end{aligned}
$$

Here in the first line the minus sign comes from the connection extended to forms (∇_m in the first line is the covariant derivative induced on endomorphisms). In passing from line 4 to 5 the torsion free property is used to cancel out $-\hat{g}^{-1}\partial g$ appearing in line 4.

On forms and vector fields the Levi-Cevita connection differs by a sign we have the following expression for the change in connection acting on forms and vector fields:

$$
\begin{aligned}
(\nabla_m - \hat{\nabla}_m)V_l &= -V_\alpha(\nabla_m hh^{-1})_l^\alpha \\
(\nabla_m - \hat{\nabla}_m)V^l &= (\nabla_m hh^{-1})_\alpha^l V^\alpha
\end{aligned}
$$

Similarly for curvature

$$(\hat{R} - R)^\alpha_{j\bar{k}\beta} = \partial_{\bar{k}}(\nabla_j hh^{-1})^\alpha_\beta \tag{2.91}$$

Also

$$\phi_{j\bar{k}m} = \hat{\nabla}_m \phi_{j\bar{k}} = -g_{\bar{k}\alpha}(\nabla_m h h^{-1})^{\alpha}_{j} \tag{2.92}$$

The minus sign here is attributed to following the convention that $g_{\bar{k}\alpha} = \hat{g}_{\bar{k}\alpha} + \phi_{\bar{k}\alpha}$.

To establish the C^3 estimate one considers the quantity

$$S = g^{j\bar{r}} g^{s\bar{k}} g^{m\bar{t}} \phi_{j\bar{k}m} \phi_{\bar{r}s\bar{t}}$$

This can be written as

$$S = g^{m\overline{\gamma}} g_{\overline{\mu}\beta} g^{l\overline{\alpha}} (\nabla_m h h^{-1})^{\beta}_{l} \overline{(\nabla_{\gamma} h h^{-1})^{\mu}_{\alpha}} = |\nabla h h^{-1}|^2$$

using (2.92).

We summarize the process involved in establishing C^3 estimates. The following obtains

$$\Delta S = g^{-1} g g^{-1} (\Delta(\nabla h h^{-1}) \overline{\nabla h h^{-1}} + (\nabla h h^{-1}) \overline{\overline{\Delta} \nabla h h^{-1}}) + |\overline{\nabla}(\nabla h h^{-1})|^2 + |\nabla(\nabla h h^{-1})|^2 \tag{2.93}$$

by direct calculation. The term with $\overline{\Delta} = g^{p\overline{q}} \nabla_q \nabla_{\overline{p}}$ can be written in terms of Δ by commuting derivatives and introducing curvatures. That is, with $(T^{\gamma}_{\alpha})_j = (\nabla_j h h^{-1})^{\gamma}_{\alpha}$ obtain

$$\overline{\Delta}(T^{\gamma}_{\alpha})_j = \Delta(T^{\gamma}_{\alpha})_j - R^{\gamma}_{\mu}(T^{\mu}_{\alpha})_j + R^{\mu}_{\alpha}(T^{\gamma}_{\mu})_j + R^{\mu}_{j}(T^{\gamma}_{\alpha})_{\mu}$$

Replacing the $\overline{\Delta}$ expression in (2.93) by this gives another expression for ΔS with three extra terms involving curvatures. See [PSS] for the formula. Note that ΔS is a fifth order term and the expression in [PSS] (2.43) contains terms with $\Delta(\nabla h h^{-1})$ which are also fifth order. In order to obtain an expression of the form $\Delta S \geq -C_1 S - C_2$ to apply a maximum principle we need to reduce the order of the terms appearing. At worst to fourth and third order terms, and of course the fourth order terms must come with favorable sign so they can be dropped.

An application of Bianchi identity gives

$$\Delta(\nabla h h^{-1}) = -\nabla . R^{\cdot}_{\cdot} + g^{-1} \nabla . \hat{R}^{\cdot}_{\cdot\cdot\cdot} \tag{2.94}$$

The first term in (2.94) and curvature terms $R_{\overline{\beta}\alpha}$ appearing in [PSS] (2.43) need to cancel out since these are not controlled.

Next $\dot{S}$ can be expressed in terms of $h^{-1}\dot{h}$. Noting

- $\dot{g} = \hat{g}\dot{h} = g(h^{-1}\dot{h})$
- $\dot{g^{-1}} = -\dot{g}^{-1} = -(h^{-1}\dot{h})g^{-1}$
- $\nabla h = g g^{-1} \partial g^{-1} g h = g^{-1} \partial (g h g^{-1}) g$

The appearance of g after the second equality in the first bullet corresponds to lowering the endomorphism $(h^{-1}\dot{h})$ to a (0,2) tensor. Similarly in the second bullet g^{-1} corresponds to raising it to a (2,0) tensor.

Some application of these bullet points give

1. $\dot{\nabla h} = -h^{-1}\dot{h} + (\nabla h)h^{-1}\dot{h} + \nabla(h^{-1}\dot{h}h) = \nabla(h^{-1}\dot{h})h + \nabla h(h^{-1}\dot{h})$
2. $(\nabla h \dot{h^{-1}}) = \nabla(h^{-1}\dot{h})$

Here the first equality in (1.) follows from time differentiating bullet three. With this the expression $\dot{S}$ can be computed. This gives rise to terms

$$g^{m\overline{\gamma}}g_{\overline{\mu}\beta}g^{l\overline{\alpha}}(\partial_t(\nabla_m hh^{-1})_l^\beta\overline{(\nabla_\gamma hh^{-1})_\alpha^\mu})$$

and time differentiating the $g^{l\overline{\alpha}}$ term in S using the second bullet and likewise $g_{\overline{\mu}\beta}$ using the first bullet boils down to replacing $g^{l\overline{\alpha}}$ by $-(h^{-1}\dot{h})^{l\overline{\alpha}}$ and $g_{\overline{\mu}\beta}$ by $(h^{-1}\dot{h})_{\overline{\mu}\beta}$. That is we get terms like:

$$-g^{m\overline{\gamma}}g_{\overline{\mu}\beta}(h^{-1}\dot{h})^{l\overline{\alpha}}(\nabla_m hh^{-1})_l^\beta\overline{(\nabla_\gamma hh^{-1})_\alpha^\mu}$$
$$g^{m\overline{\gamma}}(h^{-1}\dot{h})_{\overline{\mu}\beta}g^{l\overline{\alpha}}(\nabla_m hh^{-1})_l^\beta\overline{(\nabla_\gamma hh^{-1})_\alpha^\mu}$$

The formula for $\dot{S}$ appears in [PSS] as equation 2.47. Note that in 2.47 $h^{-1}\dot{h}$ is raised or lowered with the metric. So under the action of $\Delta - \partial_t$, S deforms as

$$\begin{aligned}(\Delta-\partial_t)S =&|\overline{\nabla}(\nabla hh^{-1})|^2 + |\nabla(\nabla hh^{-1})|^2 + g^{m\overline{\gamma}}g_{\overline{\mu}\beta}g^{l\overline{\alpha}}((\Delta-\partial_t)(\nabla_m hh^{-1})_l^\beta\overline{(\nabla_\gamma hh^{-1})_\alpha^\mu})\\ &+ g^{m\overline{\gamma}}g_{\overline{\mu}\beta}g^{l\overline{\alpha}}((\nabla_m hh^{-1})_l^\beta\overline{(\Delta-\partial_t)(\nabla_\gamma hh^{-1})_\alpha^\mu}) + ((h^{-1}\dot{h}+R)^{m\overline{\gamma}}g_{\overline{\mu}\beta}g^{l\overline{\alpha}}\\ &- g^{m\overline{\gamma}}(h^{-1}\dot{h}+R)_{\overline{\mu}\beta}g^{l\overline{\alpha}} + g^{m\overline{\gamma}}g_{\overline{\mu}\beta}(h^{-1}\dot{h}+R)^{l\overline{\alpha}})(\nabla_m hh^{-1})_l^\beta\overline{(\nabla_\gamma hh^{-1})_\alpha^\mu}\end{aligned} \tag{2.95}$$

Now finally we may begin the verification of the C^3 estimates for our problem.

Proposition 27. *Along the Tkrf the uniform estimate* $|\nabla\partial\overline{\partial}\phi_t|_{C^0(X)} < C$ *holds.*

Restrict to twisted Kähler-Ricci flow. Then

$$(h^{-1}\dot{h})_l^\beta = g^{\beta\overline{\alpha}}(g+\eta-R)_{l\overline{\alpha}} = (\delta+\eta-R)_l^\beta$$

It follows that

$$(h^{-1}\dot{h} + R)_l^\beta = (\delta + \eta)_l^\beta = (T_\eta)_l^\beta$$

With

$$c\omega_t \geq T_{\omega_t,\eta} \geq \omega_t$$

$$(\delta)_l^\beta \leq (h^{-1}\dot{h} + R)_l^\beta \leq c(\delta)_l^\beta \tag{2.96}$$

together with (2.94), 2. and (2.91) obtain

$$\begin{aligned}(\Delta - \partial_t)(\nabla_j hh^{-1})_m^l =& -\nabla_j R_m^l + \nabla^{\overline{p}}\hat{R}_{\overline{p}jm}^l - \nabla_j(h^{-1}\dot{h})_m^l \\ =& -\nabla_j R_m^l + \nabla^{\overline{p}}\hat{R}_{\overline{p}jm}^l - \nabla_j(\delta + \eta - R)_m^l \\ =& \nabla^{\overline{p}}\hat{R}_{\overline{p}jm}^l - \nabla_j(\eta)_m^l\end{aligned}$$

Using (2.96)

$$\begin{aligned}(\Delta - \partial_t)S \geq & |\overline{\nabla}(\nabla hh^{-1})|^2 + |\nabla(\nabla hh^{-1})|^2 \\ & + g^{m\overline{\gamma}}(\nabla^{\overline{p}}\hat{R}_{\overline{p}ml}^\beta - \nabla_m(\eta)_l^\beta)\overline{(\nabla_\gamma hh^{-1})_{\overline{\beta}}^{\overline{l}}}) \\ & + g^{m\overline{\gamma}}(\nabla_m hh^{-1})_{\overline{\mu}}^{\overline{\alpha}}\overline{(\nabla^{\overline{p}}\hat{R}_{\overline{p}\gamma\alpha}^\mu - \nabla_\gamma(\eta)_\alpha^\mu)}) \\ & + (2-c)g^{m\overline{\gamma}}g_{\overline{\mu}\beta}(g)^{l\overline{\alpha}}(\nabla_m hh^{-1})_l^\beta\overline{(\nabla_\gamma hh^{-1})_\alpha^\mu}\end{aligned} \tag{2.97}$$

Lemma 20. *The second and third terms in (2.97) are* $O(S + c_1\sqrt{S})$

Proof. We check this for $g^{m\overline{\gamma}}\nabla^{\overline{p}}\hat{R}_{\overline{p}ml}^\beta\overline{(\nabla_\gamma hh^{-1})_{\overline{\beta}}^{\overline{l}}}$, and terms involving the conjugate expression are similar.

$$\begin{aligned}\nabla^{\overline{p}}\hat{R}^{\beta}_{\overline{p}ml} &= g^{j\overline{p}}(\hat{\nabla}_j\hat{R}^{\beta}_{\overline{p}ml} - (\nabla_j hh^{-1})^{\alpha}_{m}\hat{R}^{\beta}_{\overline{p}\alpha l} \\ &\quad - (\nabla_j hh^{-1})^{\alpha}_{l}\hat{R}^{\beta}_{\overline{q}m\alpha} + (\nabla_j hh^{-1})^{\beta}_{\alpha}\hat{R}^{\alpha}_{\overline{q}ml}) \\ &= g^{j\overline{q}}(\hat{\nabla}_j\hat{R}^{\beta}_{\overline{q}ml} + O(\nabla hh^{-1}\hat{R}))\end{aligned}$$

Claim 16. *Along Tkrf the following estimates are available:*

$$\begin{aligned} g^{j\overline{q}}\hat{\nabla}_j\hat{R}^{\beta}_{\overline{q}ml}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} &\geq -|\hat{\nabla}\hat{R}||\nabla hh^{-1}| \geq -C\sqrt{S} \\ -g^{m\overline{\gamma}}g^{j\overline{q}}(\nabla_j hh^{-1})^{\alpha}_{m}\hat{R}^{\beta}_{\overline{q}\alpha l}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} &\geq -CS \\ g^{j\overline{q}}g^{m\overline{\gamma}}(\nabla_j hh^{-1})^{\alpha}_{l}\hat{R}^{\beta}_{\overline{q}m\alpha}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} &\geq -CS \\ g^{j\overline{q}}g^{m\overline{\gamma}}(\nabla_j hh^{-1})^{\beta}_{\alpha}\hat{R}^{\alpha}_{\overline{q}ml}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} &\geq -CS \end{aligned}$$

This follows by using $\hat{R}$ is bounded, so $\hat{R}^{\beta}_{\overline{q}\alpha l} \leq C\delta^{\beta}_{\alpha}g_{\overline{q}l}$ (by the C^2 estimate $g_{\cdot,\cdot}$ is equivalent to $\hat{g}$) and cauchy schwartz.

Similarly, but with less effort, $-g^{m\overline{\gamma}}\nabla_m\eta^{\beta}_{l}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}}$ can be handled. Since η is fixed there is a $C > 0$ so that $-C\delta^{\beta}_{\alpha} \leq \eta^{\beta}_{\alpha} \leq C\delta^{\beta}_{\alpha}$. So

$$\begin{aligned} -g^{m\overline{\gamma}}\nabla_m\eta^{\beta}_{l}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} &= -g^{m\overline{\gamma}}\hat{\nabla}_m\eta^{\beta}_{l}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} + g^{m\overline{\gamma}}(\nabla_m hh^{-1})^{\alpha}_{l}\eta^{\beta}_{\alpha}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} \\ &\quad - g^{m\overline{\gamma}}(\nabla_m hh^{-1})^{\beta}_{\alpha}\eta^{\alpha}_{l}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} \\ &\geq -\langle\eta, \nabla hh^{-1}\rangle - 2Cg^{m\overline{\gamma}}(\nabla_m hh^{-1})^{\beta}_{l}\overline{(\nabla_\gamma hh^{-1})^{\overline{l}}_{\overline{\beta}}} \\ &\geq -|\eta||\nabla hh^{-1}| - 2C|\nabla hh^{-1}|^2 \geq C_1\sqrt{S} - 2CS \end{aligned}$$

□

proof of proposition. Applying the lemma, (2.97) becomes

$$(\Delta - \partial_t)S \geq |\overline{\nabla}(\nabla h h^{-1})|^2 + |\nabla(\nabla h h^{-1})|^2 - \tilde{C}S - \tilde{C}_1\sqrt{S} \geq -C_1 S - C_2$$

To conclude, exactly the same argument as in [PSS] applies. That is, with A sufficiently large the following expression

$$(\Delta - \partial_t)(S + A\hat{\Delta}\phi) \geq C_3 S - C_4 \tag{2.98}$$

$C_3 > 0$ is available. An application of the maximum principles allows to conclude that S is bounded by a positive number. □

Chapter 3

Future direction

3.1 Coupled system

Assume $H^0(X, K_X \otimes L)$ is endowed with the natural L^2 inner product induced from the hermitian metric on the adjoint bundle $K_X \otimes L$. Consider the coupled system

$$(\omega_u)^n = (\sum_i s_i \wedge \overline{s_i} e^{-\phi}) e^{-u} = \sum_i |s_i|^2_{h,\omega_0} \omega_0^n e^{-u} := \mu_{\overline{s}} e^{-u} \tag{3.1}$$

$$\int_X h_{K_X \otimes L}(s_i, s_j) e^{-u} = \int_X i^{n^2} s_i \wedge \overline{s_j} e^{-(\phi+u)} = \langle s_i, s_j \rangle_u = C\delta_{ij} \tag{3.2}$$

We note that for the coupled system above solutions are balanced metrics solving the mean field equation (3.1).

Equation (3.1) is equivalent to the density of states condition

$$1 = i^{n^2} \frac{\sum_i s_i \wedge \overline{s_i} e^{-(\phi+u)}}{\omega_u^n} = i^{n^2} \sum_i |s_i|^2_{\omega,h} e^{-u} \frac{\omega_0^n}{\omega_u^n} \tag{3.3}$$

very much in the spirit of Donaldson's double quotient in [Do]. However the hermitian metric is defined on $K_X \otimes L$ and is a coupling of a hermitian metric on K_X and one on L with

$\omega_u \in c_1(L)$. The same effect is obtained by choosing a hermitian metric only from L:

$$1 = i^{n^2} \sum_i |s_i|^2_{\omega,h} e^{-u} \frac{\omega_0^n}{\omega_u^n} = i^{n^2} \sum_i |s_i|^2_{\omega,h} e^{-u + \log \frac{\omega_0^n}{\omega_u^n}}$$

So the hermitian metric on L giving (3.3) is determined by the weight $\phi + u - \log \frac{\omega_0^n}{\omega_u^n}$.

The orthogonality condition (3.2) on a basis of $H^0(X, K_X \otimes L)$ with respect to $\langle \cdot, \cdot \rangle_{\phi+u}$ can now be written as:

$$C\delta_{ij} = i^{n^2} \int_X s_i \wedge \overline{s}_j e^{-(\phi+u)} = i^{n^2} \int_X s_i \wedge \overline{s_j} e^{-(\phi + u + \log \frac{\omega_u^n}{\omega_0^n})} \frac{\omega_u^n}{\omega_0^n} \tag{3.4}$$

Note that for a solution of the mean field equation that (3.4) can also be written as

$$C\delta_{ij} = \int_X \frac{s_i \wedge \overline{s_j} e^{-\phi}}{\sum_k s_k \wedge \overline{s_k} e^{-\phi}} \omega_u^n \tag{3.5}$$

Definition 3. *Given an embedding into $\mathbb{CP}^N$, for some $N > 0$, induced by an L^2 orthonormal basis (s_i) of $H^0(X, K_X \otimes L^k)$ (with respect to $\langle \cdot, \cdot \rangle_{k\phi+ku}$) we say that embedding is balanced if (3.5) holds.*

Fix notation

$$B_{k(\phi+u)} e^{-k(\phi+u)} := \sum s_j \wedge \overline{s_j} e^{-k(\phi+u)}$$

then from [Bo09] the following asymptotics are available

$$\begin{aligned} B_{k(\phi+u)} e^{-k(\phi+u)} &= k^n (a_0 + \frac{a_1}{k} S + O(k^{-2})) \omega_u^n \\ &= k^n \omega_u^n (1 + O(k^{-1})) \end{aligned}$$

Applying this to right hand side of (3.5) obtain

$$\int_X \frac{s_i \wedge \overline{s_j} e^{-\phi}}{\sum_\gamma s_\gamma \wedge \overline{s_\gamma} e^{-k\phi} e^{-ku}} \omega_{ku}^n = C \int_X \frac{s_i \wedge \overline{s_j} e^{-k\phi}}{k^n \omega_u^n (1 + O(k^{-1}))} k^n \omega_u^n$$
$$= C \int_X s_i \wedge \overline{s_j} e^{-k(\phi+u)} (1 + O(k^{-1})) \tag{3.6}$$

where C is the constant appearing in (3.4).

From [BBEGZ] the mabuchi energy is introduced in a more general setting by defining it as

$$Mab_{\mu_{\overline{s}}}(\phi) := \int_X \log(\frac{\omega_\phi^n}{\mu_{\overline{s}}}) \omega^n + J(\phi) - I(\phi) \tag{3.7}$$

On $\mathcal{H}_\omega$ this restricts to the usual mabuchi functional $\nu_\omega(\phi)$. Also if Mab_μ is proper then the corresponding mean field equation $\omega_u^n = \mu_{\overline{s}} e^{-u}$ can be solved, see [BBEGZ]. Next we show properness is independent of the choice of $\underline{s}$.

Proposition 8. *If $Mab_{\mu_{\overline{s}}}$ is proper on $\mathcal{H}_\omega$ then so is $Mab_{\mu_{\overline{s}'}}$.*

Proof. Recall that the mabuchi functional satisfies the cocyle property [Tian00]

$$\nu_\omega(\phi) - \nu_{\omega'}(\phi') = \nu_\omega(\psi)$$

where $\omega' = \omega + \overline{\partial}\partial\psi$ and $\phi' = \phi - \psi$. This follows from computing the differential of the left hand side. Since it vanishes we obtain the left hand side above is constant. Setting $\phi = \psi$ determines the constant.

In the smooth case using the cocyle condition and properness as defined by Tian [Tian00] we get

$$\nu_{\omega'} \geq \mu(J_\omega(\phi)) - C$$

Since the function μ is increasing, it suffices to show

$$J_{\omega'}(\phi') \leq \alpha J_\omega(\phi) + \beta$$

where $\alpha, \beta > 0$ are constants. We have

$$\mu'(\cdot) = \mu(\frac{\cdot - \beta}{\alpha})$$

is increasing and then

$$\nu_{\omega'} \geq \mu'(J_{\omega'}(\phi')) - C$$

Begin by recalling

$$J_\omega(\phi) = \frac{1}{V} \sum_{i=0}^{n-1} \frac{i+1}{n+1} \int_X \partial\phi \wedge \overline{\partial\phi} \wedge \omega^i \wedge \omega_\phi^{n-i-1} \tag{3.8}$$

where $V = \int_X \omega^n$. Throughout we assume that $\phi \in \mathcal{H}_\omega$. This gives, through a simultaneous diagonalization argument that the integrands are non-negative so $J_\omega(\phi) \geq 0$.

Also recall that

$$\frac{n+1}{n^2} J_\omega(\phi) \leq I_\omega(\phi) \leq (n+1) J_\omega(\phi) \tag{3.9}$$

where

$$I_\omega(\phi) = \frac{1}{V}\int_X \phi(\omega^n - \omega_\phi^n) \tag{3.10}$$

Lemma 21. *With the notations above the following identity holds:*

$$\begin{aligned}
I_\omega(\phi) =& \frac{1}{V}\int_X \phi(\omega^n - (\omega^{'})^n) + \int_X \phi((\omega^{'})^n - \omega_\phi^n) \\
=& \int_X \partial\phi \wedge \overline{\partial\psi} \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega^{'} \ldots + (\omega^{'})^n) + \int_X \phi^{'}((\omega^{'})^n - \omega_\phi^n) \\
&+ \int_X \psi((\omega^{'})^n - \omega_\phi^n) \\
\geq& I_{\omega^{'}}(\phi^{'}) + \int_X \psi((\omega^{'})^n - \omega_\phi^n) + O(1)
\end{aligned} \tag{3.11}$$

To obtain inequality (3.11) first

Claim 17. $\int_X \partial\phi \wedge \overline{\partial\psi} \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega^{'} \ldots + (\omega^{'})^{n-1}) > -2CnV$

Rewrite the expression as

$$\begin{aligned}
&\int_X \partial\phi \wedge \overline{\partial\psi} \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega^{'} \ldots + (\omega^{'})^{n-1}) \\
&= -\int_X \psi(\partial\overline{\partial}\phi) \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega^{'} \ldots + (\omega^{'})^{n-1}) \\
&= -\int_X \psi\omega_\phi \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega^{'} \ldots + (\omega^{'})^{n-1}) \\
&\quad + \int_X \psi\omega \wedge (\omega^{n-1} + \omega^{n-2}\wedge\omega^{'} \ldots + (\omega^{'})^{n-1})
\end{aligned} \tag{3.12}$$

Note that $|\psi| < C$. Then

Claim 18. *On $\mathcal{H}_\omega$ with notations as above*

$$\omega_\phi \wedge ((\omega^{n-1} + \omega^{n-2} \wedge \omega^{'} \ldots + (\omega^{'})^{n-1})) \geq 0$$

At an arbitrary point $p \in X$

$$\omega_p = dz_i \wedge dz_{\bar{i}}, \quad \omega_p^{'} = (1+\lambda_i) dz_i \wedge dz_{\bar{i}} \quad \text{where } |\lambda_i| < 1$$

set $(1+\lambda)_J = \Pi_{k \in J}(1+\lambda)_k$. Then

$$\begin{aligned} \omega_\phi \wedge \omega^{n-1-i} \wedge (\omega^{'})_p^i =& \omega_\phi \wedge \sum_{|I|+|J|=n, I\cap J=\emptyset} (n-1-i)!\, i!\, (1+\lambda)_J dz^I \wedge dz^J \\ =& \sum_k (1+\phi_{k\bar{k}}) \sum_{I \subset \{1\ldots n\} \setminus \{k\}} |I|!\, |J|!\, (1+\lambda)_J \frac{\omega^n}{n!} \end{aligned}$$

Since each term in the inner sum is nonnegative there is an $\epsilon > 0$ satisfying

$$\sum_{I \subset \{1\ldots n\} \setminus \{k\}} |I|!\, |J|!\, (1+\lambda)_I > \frac{\epsilon}{n}$$

Therefore

$$\omega_\phi \wedge \omega^{n-1-i} \wedge (\omega^{'})_p^i > \epsilon \sum_k (1+\phi_{k\bar{k}}) \frac{\omega^n}{n!} = \epsilon tr_\omega \omega_\phi \frac{\omega^n}{n!} > 0$$

and the claim follows.

So the first term in the last equality in (3.12) becomes

$$
\begin{aligned}
-\int_X \psi\omega_\phi \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1})& \\
\geq - C\int_X \omega_\phi \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1})& \\
= - C\int_X \omega \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1})& \\
- C\int_X \partial\overline{\partial}\phi \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1})& \\
= - C\int_X \omega \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1})&
\end{aligned}
$$

This a consequence of

$$\int_X \partial\overline{\partial}\phi \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1}) = 0$$

obtained using integration by parts and stokes theorem. The claim follows from

$$
\begin{aligned}
&\int_X \partial\phi \wedge \overline{\partial\psi} \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1}) \\
&> - 2C\int_X \omega \wedge (\omega^{n-1} + \omega^{n-2} \wedge \omega' \ldots + (\omega')^{n-1}) \\
&> - 2CnV
\end{aligned}
$$

Next expand binomially

$$\omega^{n-i} \wedge (\omega')^i = \omega^{n-i} \wedge (\omega^i + c_{n,1}\omega^{n-1} \wedge \partial\overline{\partial}\phi + \ldots (\partial\overline{\partial}\phi)^i)$$

Integrating and combining with an application of by parts and stokes theorem obtain that

$$\int_X \omega^{n-i} \wedge (\omega^{'})^i = \int_X \omega^n = V$$

Now we can conclude

$$I_\omega(\phi) > I_{\omega^{'}}(\phi^{'}) + \int_X \psi((\omega^{'})^n - \omega_\phi^n) - 2CnV$$

Since $\omega^{'}, \omega_\phi$ represent the same cohomology class

$$\int_X c((\omega^{'})^n - \omega_\phi^n) = 0$$

So we may choose $c > 0$ so that $\psi + c < -1$. Then we have (suppressing the $-2CnV$ term since it is $O(1)$)

$$\begin{aligned} I_\omega(\phi) \geq & I_{\omega^{'}}(\phi^{'}) + \int_X (\psi + c)((\omega^{'})^n - \omega_\phi^n) \\ = & I_{\omega^{'}}(\phi^{'}) + \int_X (\psi + c)(\omega^{'})^n - \int_X (\psi + c)\omega_\phi^n \end{aligned}$$

Because

$$-\int_X (\psi + c)\omega_\phi^n > V > 0, \quad \int_X (\psi + c)(\omega^{'})^n = O(1)$$

we finally obtain

$$I_\omega(\phi) \geq I_{\omega^{'}}(\phi^{'}) + C$$

for some constant C. From (3.9) it follows

$$J_\omega(\phi) \geq \frac{1}{n+1} I_\omega(\phi) \geq \frac{1}{n+1} I_{\omega'}(\phi) + \frac{C}{n+1} \geq \frac{1}{n^2} J_{\omega'}(\phi) + \frac{C}{n+1}$$

So $\alpha = n^2$ and $\beta = -\frac{Cn^2}{n+1}$ and we obtain properness of $\nu_{\omega'}$ on $\mathcal{H}_\omega$ given that ν_ω is. □

We have seen from (3.6) that metrics in $c_1(L^k)$ solving the coupled system are approximately balanced for sufficiently large k. In view of the proposition we finally conclude by posing the following question:

Question 1. *Provided $Mab_{\mu_{\underline{S}}}$ is proper, do solutions $\omega_u \in c_1(L^k)$ to the coupled system for sufficiently large k exist?*

BIBLIOGRAPHY

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[Ber12] Berman, R.J: K-polystability of Q-Fano varieties admitting Kähler-Einstein metrics. Preprint.

[BeBo12] Berman, R.J.; Berndtsson, B: Moser-Trudinger type inequalities for complex Monge- Ampre operators and Aubins hypothse fondamentale. Preprint in 2011 at arXiv:1109.1263.

[BerBo] Berman, R.J.; Berndtsson, B: Real Monge-Ampre equations and Khler-Ricci solitons on toric log Fano varieties. Preprint.

[Ber10] R. Berman: A thermodynamical formalism for Monge-Amp'ere equations, Moser-Trudinger inequalities and Kähler-Einstein metrics. Preprint (2010) arXiv:1011.3976.

[Ber10a] Berman, R.J. Analytic torsion, vortices and positive Ricci curvature. arXiv:1006.2988

[BBEGZ] Berman R., S. Boucksom, P. Eyssidieux, V. Guedj A. Zeriahi Kähler-Einstein metrics and the Kähler-Ricci flow on log-Fano varieties, arXiv:1111.7158v2 (2011).

[Bo] Berndtsson, B: A Brunn-Minkowski type inequality for Fano manifolds and the Bando-Mabuchi uniqueness theorem. arXiv:1103.0923

[Bo09] Berndtsson, B: Positivity of direct image bundles and convexity on the space of Kähler metrics. J. Differential Geom. Volume 81, Number 3 (2009), 457-482. arxiv: math.CV/0608385.

[Bo07] B. Berndtsson, Curvature of vector bundles associated to holomorphic fibrations, math arXiv: mathCV/0511225.

[Chen00] Chen, Xiuxiong The space of Kähler metrics. J. Differential Geom. 56 (2000), no. 2, 189234.

[CDS] X.X Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds, I: approximation of metrics with cone singularities, arXiv:1211.4566, 2012.

[CDS1] X.X Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds, II: limits with cone angle less than 2 pi, arXiv:1212.4714, 2012.

[CDS2] X.X Chen, S. Donaldson, S. Sun, Kähler-Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2 and completion of the main proof, arXiv:1302.0282.

[SzCo] Collins, T.C., Szekelyhidi, G. The twisted Kahler-Ricci flow. Preprint.

[TL] Darvas T.; Lempert, L.:Weak Geodesics in the space of Kähler metrics, preprint.

[BerDe] Berman, R.; Demailly, J.-P.: Regularity of plurisubharmonic upper envelopes in big cohomology classes. Preprint (2009) arXiv:0905.1246.

[De] J. -P. Demailley: L^2 estimates for the $\overline{\partial}$-operator on complex manifolds, Notes de cours, Ecole d't de Mathmatiques (Analyse Complexe), Institut Fourier, Grenoble, Juin 1996.

[DT] Ding, W. and Tian, G.: The generalized Moser-Trudinger Inequality. Proceedings of Nankai International Conference on Nonlinear Analysis, 1993.

[Do4] S. K. Donaldson.: Remarks on gauge theory, complex geometry and 4-manifold topology. In Fields Medalists lectures, volume 5 of World Sci.Ser. 20th Century Math., pages 384403. World Sci. Publ., River Edge, NJ, 1997.

[Do7] Donaldson, S. K.: Moment maps and diffeomorphisms. Sir Michael Atiyah: a great mathematician of the twentieth century. Asian J. Math. 3 (1999), no. 1, 115.

[Do5] Donaldson, S. K.: Lower bounds on the Calabi functional. J. Differential Geom. 70 (2005), no. 3.

[Do] Donaldson, S.K.: Scalar curvature and projective embeddings. I. J. Differential Geom. 59 (2001), no. 3, 479-522.

[Do2] Donaldson, S.K.: Scalar curvature and stability of toric varities. J. Diff. Geom. 62 (2002), 289-349.

[Do3] Donaldson, S.K.: "Discussion of the Kähler-Einstein problem". Available at http://www2.imperial.ac.uk/ skdona/KENOTES.PDF

[Do6] Donaldson, S. K.: Symmetric spaces, Kähler geometry and Hamiltonian dynamics. Northern California Symplectic Geometry Seminar, 1333, Amer. Math. Soc. Transl. Ser. 2, 196, Amer. Math. Soc., Providence, RI, 1999.

[Fi] J. Fine, Constant scalar curvature Kähler metrics on fibered complex surfaces. J.Differential Geom. 68 (2004), no. 3, 397432, MR 2144537, Zbl 1085.53064.

[Jo] Joyce, Dominic D. (2007), Riemannian Holonomy Groups and Calibrated Geometry, Oxford Graduate Texts in Mathematics, Oxford: Oxford University Press, ISBN 978-0-19-921559-1.

[Kel] Keller, J: Twisted Balanced Metrics. Lie Groups : New research", Mathematics Research Developments , Editor Altos B. Canterra, 267–281, Nova publishers NJ.

[Mab] Toshiki Mabuchi, K-energy maps integrating Futaki invariants, Tohoku Math. J. (2) 38 (1986), no. 4, 575593. MR 867064 (88b:53060).

[Pa] Pali, Nefton: Characterization of Einstein-Fano manifolds via the Kähler-Ricci flow. Indiana Univ. Math. J. 57 (2008), no. 7, 32413274.

[PS] Phong, D. H.; Sturm, Jacob The Monge-Ampre operator and geodesics in the space of Kähler potentials. Invent. Math. 166 (2006), no. 1, 125149.

[PSSW] Phong, D.H: Song, J; Sturm, J; Weinkove, B: The Moser-Trudinger inequality on Kähler-Einstein manifolds. Amer. J. Math. 130 (2008), no. 4, 1067-1085, arXiv:math/0604076

[PSS] Phong, D.H., N. Sesum, and J. Sturm, Multiplier ideal sheaves and the Kähler-Ricci flow, Comm. Anal. Geom. 15 (2007), no. 3, 613632.

[LV] L. Lempert, L. Vivas, Geodesics in the space of Kähler metrics, arXiv:1105.2188.

[Ru] Rubinstein, Yanir:On energy functionals, Kähler-Einstein metrics, and the Moser-Trudinger-Onofri neighborhood, J. Funct. Anal. 255.

[ST] Sesum, N., Tian G.: Bounding scalar curvature and diameter along the Kähler Ricci Flow(after Perelman): Lecture note.

[Sto] J. Stoppa: K-stability of constant scalar curvature Kahler manifolds, Adv. Math. 221 (2009), no. 4, 13971408.

[Sto09] Stoppa, J: Twisted constant scalar curvature Khler metrics and Khler slope stability. J. Differential Geom. Volume 83, Number 3 (2009), 663-691.

[Sz1] Szekelyhidi, G.: The Kähler-Ricci flow and K-stability. arXiv:0803.1613.

[Tian00] Tian, G: Canonical Metrics in Kähler Geometry, Birkhauser, 2000.

[Ti07] G. Tian and X. Zhu, Convergence of the Kähler-Ricci flow, J. Amer. Math. Soc.20 (2007), no. 3, 675-699.

[Ti97] Tian, G: Kähler-Einstein metrics with positive scalar curvature. Invent. Math. 130 (1997), no. 1, 137.

[Ti13] Tian, G. K-stability and Kähler-Einstein metrics. arXiv:1211.4669.

[Tao] Terence Tao, Poincares legacies: pages from year two of a mathematical blog (Volume I, Volume II), American Mathematical Society, Volume I: ISBN-10 0-8218-4883-6, ISBN-13 978-0-8218-4883-8; Volume II: ISBN-10 0-8218-4885-2, ISBN-13 978-0-8218-4885-2.

[Y1] Yau, S.-T.: (1993). Open Problems in Geometry. Differential Geometry: partial differential equations on manifolds (Los, Angeles, CA, 1990), 1-28, Proc. Sympos. Pure Math., 54, AMS publications.

[ZBSK] Z. Blocki, S. Kolodziej:On Regularization of Plurisubharmonic functions on Manifolds, Proceeding of the AMS, Volume 134, Number 7, 2007.