THE ENDS OF A PRODUCT MANIFOLD

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ABSTRACT

THE ENDS OF A PRODUCT MANIFOLD

By

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The motivating problem behind this thesis was a characterization of the closed n-cell I^n in terms of point-set properties. This led to similar characterizations of the m-sphere S^m, the open m-cell E^m, and the annulus $[0,1) \times S^m$.

The techniques employed are a combination of recently developed algebraic results and the almost geometric study of the ends of a product.

We do not rely on the Poincaré Conjecture in dimension 3 or 4. All of the results are valid in each of the three categories TOP, PL, or DIFF,

The results include the following and some generalizations.

I^m is the only compact product manifold with sphere boundary. $[0,1) \times S^m$ is the only non-compact product manifold with sphere boundary. S^m is the only closed manifold M such that M-point is a product space. E^m is the only open manifold M such that M-point is a product space. I^m is the only manifold M with a point P in its boundary such that $M-P$ is a product space with one open factor, unless the other factor is $[0,1)$.

Dennis C. Hass

A compact manifold M, which is a suspension space, is a sphere if its boundary is empty or a cell if not.

These results follow from more general results such as this. A product of open generalized manifolds is k-connected at infinity only if it is k-connected. One consequence of this type of theorem is this characterization of E^m. Let M be a smooth open product m-manifold with one end E. If E is both tame and $[(m-1)/2]$ - connected, then M is E^m, $m \geq 6$.

Generalizing this to the homology level, we find that a product manifold with homology sphere boundary is a homology cell.

Finally, we prove that a certain condition is sufficient for a closed manifold to be S-cobordant to S^m.

THE ENDS OF A PRODUCT MANIFOLD

By

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DEDICATION

It is said beauty is in the eye of the beholder. This work is dedicated to those who will see the beauty to which it points. This includes Professor Patrick H. Doyle who sees deeply into everything, and Barbara who sees deeply into me.

TABLE OF CONTENTS

CHAPTER 1

Introduction

In this paper, we will be studying manifolds, their ends, and their factorizations. Most of the ends will be formed by deleting a point from a manifold, but the results often generalized to cellular sets, and to manifolds with sphere boundaries.

The main theorem concerns an m-manifold M without boundary and a point P in M. If $M-P$ is a product space, then M is either a sphere S^m, if M is compact, or Euclidean space E^m if M is not compact.

A direct consequence is that a product manifold M with sphere boundary is either a closed cell I^m, if M is compact, or an annulus $S^m \times [0,1)$, if M is not compact.

If P is in the boundary of a compact m-manifold M, such that $M-P$ is a product space with one open factor, then M is I^m.

Other similar conditions will be considered and the hypotheses of these theorems will be weakened, in chapter 4.

Suspension spaces and cone spaces which happen to be manifolds will be classified in chapter 5. For example, if a compact m-manifold with boundary is a cone, then it is a closed m-cell I^m.

A number of generalizations of these theorems are tested, proven or disproven in chapter 7. For example a product manifold with a homology sphere boundary will be shown acyclic.

In chapter 8, we examine the concept of cobordism very briefly and observe that the sufficiency of a certain condition to get an S-cobordism follows from our earlier results. This is sort of a half-cobordism theorem depending on the presence of one nice boundary component.

Many of these results have deceptively simple wording, and their hypotheses are traditional conditions from point-set topology; they could have been understood long ago. Although they sound intuitively appealing, their demonstrations rest on deep results due to many men. Many tools in their proofs have become available only during this past decade. For example, we will use such powerful algebraic results from the topology of manifolds as the Poincaré conjecture now known to be true for all dimensions except 3 and 4, Siebenmann's open collar theorem, and the technique of standard decompositions of manifolds due to Doyle and Hocking. We will not use the unproven Poincaré conjecture in dimensions 3 and 4 for main results, only for comments and examples.

In summary, this paper provides point-set characterizations of important manifolds such as spheres and open and closed cells.

CHAPTER 2

Preliminaries

There are some theorems so fundamental, and used so many times, in this paper, that we assemble them here for convenience and assume that the reader knows them throughout the sequel. It is important, too, to state some definitions explicitly.

In particular, since the original goal of this research was a characterization of the Euclidean closed m-cell I^m, this space should be carefully defined. If m is 1, then I^m is I, the closed interval $[0,1]$ on the real line, or any space homeomorphic to it. For $m \geq 2$, $I^m = I \times I^{m-1}$ is the Cartesian product. Most of the time we will write "equal" between spaces which are merely homeomorphic.

Manifolds will be separable metric spaces and may or may not have boundaries [1 Greenberg, p. 181]. The interior and boundary of a manifold M are written Int M and bM, respectively. An open manifold is a non-compact manifold with empty boundary. A closed manifold is a compact manifold with empty boundary. The n-sphere S^m is $b(I^{m+1})$, and Euclidean space E^m is Int I^m.

Throughout this paper we will be concerned with the category TOP of manifolds and continuous maps. All of the results, however, remain valid for either the category PL of combinatorial manifolds and piecewise-linear maps or the category DIFF of smooth manifolds and smooth maps.

The Cartesian product $W = M \times N$ of two manifolds is a manifold. The interior of W is $\text{Int } W = \text{Int } M \times \text{Int } N$, the product of the interior. The boundary of W is given by the union $bW = (M \times bN) \cup (bM \times N)$.

If the boundary of M has a component B which is a sphere, then the quotient space M/B is the manifold formed by adjoining a cell along B [2 Brown, p. 74].

We will be using algebraic results and techniques from both homotopy and homology theory. For all integers k, the homotopy group $\pi_k(M \times N) = \pi_k M \times \pi_k N$ is a direct product of groups [3 Spanier, p. 419], while the homology group $H_k(M \times N) = \bigoplus_{i=0}^{k} (H_i M \otimes H_{k-i} N)$ is a direct sum. This latter result is the Kunneth formula. [1 Greenberg, p. 198]. The Hurewicz theorem states that for path connected spaces M, the first non-trivial homotopy and homology groups are isomorphic and occur in the same dimension k, with the exception of $k = 1$ [3 Spanier, p. 398]. If $\pi_j M = 0$ for $j \leq k$, then M is k-connected. By the Hurewicz theorem, it suffices, in order to show M is k-connected, to check that M is 1-connected and $H_j M = 0$ for $j \leq k$. The Whitehead theorem states that if M is k-connected for all k, then M is contractible [3 Spanier, p. 406]. A synonym for 1-connected is simply connected. We will frequently refer to the Lefschetz duality theorem for compact m-manifolds M with boundary B.

By that theorem, $H_k M = H_{m-k}(M,B)$ where we are using mod 2 homology [1 Greenberg, p. 186].

A concept familiar to many is that of an ideal point at infinity corresponding to arbitrarily large concentric circles in the Euclidean plane. This point is called an end. Similarly the unit interval $(0,1)$ has two ends; the interval $[0,1)$ has only one.

For a more careful presentation of ends I recommend Chapter I in the thesis of L. C. Siebenmann [4]. From definition 1.2. of that thesis, an end of a Hausdorff space X is a collection E of subsets of X which is maximal with respect to the properties.

1) Each set in E is a connected open non-empty set with compact frontier.
2) Contained in the intersection of any two members of E is a third member of E.
3) The intersection of the closures of all members of E is empty.

By property 3) a compact Hausdorff space has no ends.

By theorem 1.6 of that thesis every non-compact connected manifold has at least one end.

The same applies to generalized manifolds to be discussed below.

Chapter III of that thesis makes precise the concept of the fundamental homotopy group of an end. Higher homotopy groups are defined analogously and satisfy a very important property. One simple example of an end of a manifold is the one created by the removal of a compact boundary component. The homotopy groups of that end are identical to the corresponding groups of the missing boundary component. This is a direct consequence of the collar theorem [5 Brown, p 89].

In general, an end E may not have a fundamental group. If it does, we denote it by $\pi_1 E$.

While we are discussing the ends of a product space, it will be important to understand the structure of certain neighborhoods of an end. We assume each of the factors which is not compact has ends, and in particular, each is a connected separable metric space: Let $M = A \times B$, where A is not compact. If B is compact, there is a correspondence between the ends of M and the ends of A.

Let the set G of open connected sets G_i determine E. Then the set $G \times B$ of open connected sets $G_i \times B$ determines an end of M; we just check the definition. If B is not-compact, then M has one end. Let C_i, D_i be increasing sequences of compact sets such that $\cup C_i = A$, $\cup D_i = B$. Let $G_i = A \times (B-D_i) \cup (A-C_i) \times B$. Then the sequence G_i determines an end for M, by the definition. Furthermore, each G_i is connected.

Although we may assume the product $M = A \times B$ is a manifold, we will not know that A and B are manifolds [6 Bing, p. 399]. Fortunately for this paper, however, they are almost manifolds and have sufficient manifold-like properties for us to proceed.

According to F. Raymond, the factors of a manifold are generalized manifolds in the sense of Wilder [7 Wilder, chapter 8]. Generalized manifolds have interiors and boundaries analogous to those of manifolds and in particular the product formulas given earlier are shown to hold [8 Raymond, p. 7].

If B is a closed generalized b-manifold then the Poincaré duality theorem holds [9 Borel, p. 227]. That is, $H_k B = H_{b-k} B$, with mod 2 homology. We will also need to know that a 1-or 2-dimensional generalized manifold is a manifold [7 Wilder, chapter 9].

The Poincaré conjecture, that a closed manifold with the homotopy groups of a sphere S^m is a sphere, has been established for $m \geq 5$ [10 Newman, p. 555]. By Poincaré duality, then, a closed $[m/2]$-connected m-manifold is a sphere, $m \geq 5$. We shall rely heavily on this result and the corresponding characterization of E^m. E^m is the only contractible open manifold which has one simply connected end, $m \geq 5$.

CHAPTER 3

Product Manifolds Without Boundary

In this chapter, we will study the ends of a product space, with particular concern for certain generalized manifolds. We will apply these results to characterizations of the Euclidean n-spaces E^n and the Euclidean spheres S^n in this chapter. In later chapters, we will apply them to characterizations of closed n-cells I^n and the half-open product $[0,1) \times S^n$.

It may be helpful to the reader to recall a property of the 2-sphere S^2. The complement in S^2 of the north pole is homeomorphic to the Euclidean plane E^2. This homeomorphism is commonly given by stereographic projection [11 Ahlfors, p. 19]. The significance to us is that E^2 is a product space, $E^2 = E^1 \times E^1$.

Although most of our work will concern manifolds, we begin in a more general setting.

Suppose M is a path-connected, locally compact product space with dimension m greater than one which has at least one isolated end E. Let $M = A \times B$ with A not compact.

<u>Lemma 1</u>:

If E has a k-connected neighborhood S, then B is k-connected.

Proof:

Let A be a non-compact factor of $M = A \times B$. We will show that there is a point a in A such that $a \times B$ is contained in S. If B is compact, then there is an end E_1 of A such that $E = E_1 \times B$. That is, there is a compact set C in A such that one component of $(A-C) \times B$ is contained in S. Let a be in that component. If B is not compact, then there are sufficiently large compact sets C and D in A and B, respectively, such that $(A-C) \times B \cup A \times (B-D)$ is contained in S. Since this union is connected, we have shown that M has only one end. Let a be in $A-C$. In either case $a \times B$ is contained in S.

Now, the composite of inclusions j: $a \times B \subset S \subset M = A \times B$ induces a monomorphism $j_{\#}: \pi_i B \to \pi_i S \to \pi_i A \times \pi_i B$ for each integer i. In particular, if $i \leq k$ then $\pi_i S = 0$ because S is k-connected. Thus $j_{\#} = 0$ and $\pi_i B = 0$ for $i \leq k$. This means that B is k-connected. Unless B is compact, A and M are k-connected.

Corollary 2:

If M is k-connected at E, then B is k-connected.

Proof:

This extra assumption provides arbitrarily small S of the desired type.

In the following lemma, let M be a path-connected generalized m-manifold with the further property that for every point P of M, there is an open set N containing P such that $N-P$ is $(m-2)$-connected, and the closure of N is compact.

Lemma 3:

If $M-P$ is a product space, M is $(m-2)$-connected.

Proof:

The open set N provides the required neighborhood $S = N-P$ for the isolated end P of $M-P$. Thus $M-P$ is $(m-2)$-connected. Using a Mayer-Vietoris sequence [3 Spanier, p. 186] and the Seifert-Van Kampen theorem [12 Massey, p. 113], one can show that $\pi_k(M-P) = \pi_k M$, if $k \leq m-2$. For compact B , we use Poincaré duality as in theorem 5 below.

Corollary 4:

If M is an m-manifold such that $M-P$ is a product space, then M is $(m-2)$-connected.

Proof:

Every point P in M has an m-cell neighborhood N and $N-P$ is $(m-2)$-connected. Thus lemma 3 applies.

Theorem 5:

If M is a non-compact m-manifold without boundary, such that $M-P$ is a product space, then M is E^m.

Proof:

Let $M-P = A \times B$. Then by corollary 4, A, B, M and M-P are each (m-2)-connected. Since M-P is a non-compact manifold, A and B are generalized manifolds, one of which, say A, is non-compact. If B is non-compact, then by the proof of lemma 1, M-P has only one end, contradicting that M is non-compact. Thus B is compact. Now, $H_k(B;Z) \neq 0$, $k = \dim B$, by Poincaré Duality [7 Wilder, p.252]. Since B is (m-2)-connected, and since $\dim B \leq m-1$, the only choice for dim B is m-1. Thus $\dim A = 1$, and $A = E^1$. It is known that since $M = E^1 \times B$, M has precisely two identical ends. Since we formed one of these ends by removing a point P from a manifold, we must be able to consider the other end in the same way. That is, there is a closed manifold N such that $M = N-Q$ for some point Q in N. Since M is (m-2)-connected, so is N. If m is greater than 4, $N = S^n$ by the truth of the Poincaré Conjecture. Thus $M = N-Q$ is E^m, if m is greater than 4. If m is less than 4, then dim B is less than 3, and B is S^{m-1}. Thus, $M-P = E^1 \times S^{m-1}$ and $M = E^m$, if m is less than 4. Our proof in the remaining case, $m = 4$, is largely due to P. H. Doyle. Let $N = E^4 \cup C$ be a standard decomposition of the closed manifold N into a disjoint union of an open 4-cell and a compact residual set C, with P and Q not in C [13 Doyle and Hocking, p. 469]. According to this paper by P.H. Doyle and J.G. Hocking, if there is an embedding of a neighborhood of C in N disjoint from C itself, then $N = S^4$. Since this gives $M = E^4$, as we wanted, we proceed to construct such an embedding. Since C is contained in $E^1 \times B$, we can find an integer n sufficiently large, that C is contained in $[-n,n] \times B$. Define a homeomorphism $h: E^1 \times B \to E^1 \times B$ by $h(t,b) = (t + 3n,b)$. Since $h(c)$ and c are disjoint, we are done.

Theorem 6:

If M is a compact m-manifold, such that $M-P$ is a product space, then $M = S^m$.

Proof:

Let $M-P = A \times B$. In the proof of theorem 5, we showed that for B to be compact it is necessary that dim $B = m-1$ and $A = E^1$. Since $M-P = E^1 \times B$ would have two ends as before, and since $M-P$ has only one end, we conclude that B is not compact. Thus B and similarly A and therefore $M-P$ are each acyclic and $(m-2)$-connected. If $m=2$, $B = E^1$. If m is greater than 2, then B is simply connected. By the Hurewicz and Whitehead theorems, B and similarly A, and therefore $M-P$, are each contractible. This implies, as in lemma 3, that M is $(m-1)$-connected. If m is not 3 or 4, then M is S^m. If $m = 3$, then $A = E^1$ and $B = E^2$. Thus $M-P = E^3$ and $M = S^3$. The final case, $m = 4$, is similar to our earlier work. If dim $B = 2$, then $A = E^2$, $B = E^2$, $M-P = E^4$, and $M = S^4$. If dim $B = 3$, $A = E^1$, as before. Let $M = E^4 \cup C$ be a standard decomposition with P not in the compact residual set C. Then C is contained in $E^1 \times B$ and the homeomorphism from theorm 5 provides an embedding of C in M disjoint from C. Thus $M = S^4$.

Corollary 7:

If M is a closed m-manifold with a cellular set S, such that $M-S$ is a product space, then $M = S^m$.

Proof:

Using theorem 1 from [2 Brown, p. 74] one can show that M/S is homeomorphic to M. Then theorem 6 applies.

CHAPTER 4

Product Manifolds with Boundary

In this chapter we will characterize closed Euclidean n-cells in a variety of ways. Two of the conditions we might impose on a manifold M are familar to us from the chapter preceeding this one. M may be a product space itself or perhaps M-P is a product space. Each of these conditions is useful to us in this chapter, but the second condition is ambiguous until we specify that the point P is either in the interior of M or in the boundary of M. Thus we have three conditions to choose from in this chapter. The other assumptions we shall employ deal with the kind of boundary M has.

We begin by investigating product spaces. Unless the four dimensional Poincaré conjecture is false, our first lemma is not different from our second, but we make the most general statement possible.

Lemma 1:

If the compact m-manifold M has a homotopy-sphere boundary B, and the interior of M is a product space, then the interior of M is E^m, $m \neq 4$.

Proof:

Since the interior of M has one end E and is $(m-2)$-connected at E, M is $(m-2)$-connected by corollary 2 of the last chapter. If $m = 2,3$ then the factors must be $A = E^1$, and $B =$ either E^1 or E^2. If m is greater than 4, then M is m-connected by Lefschetz duality [1 Greenberg, p. 186]. Thus M is contractible by the Hurewicz and Whitehead theorems. Now, the interior of M is a contractible open manifold, which is 1-connected at infinity. Thus it is E^m.

Lemma 2:

If M is a compact connected m-manifold with sphere boundary B, and the interior of M is a product space, then $M = I^m$.

Proof:

Since B is a sphere, the quotient space M/B is a closed manifold. Since the complement of $P = B/B$ is a product, namely, it is the same as the interior of M, then M/B is S^m by theorem 6 of the last chapter.

Theorem 3:

If M is a product space, and bM is a sphere, then $M = I^m$.

The following theorem, which prompted most of this thesis, was suggested by Professor Doyle and is now easily established using what we have already proven.

Theorem 4:

If the compact connected m-manifold M is a product space, and $bM-P$ is a connected product space, for P in bM, then $M = I^m$.

Proof:

Since bM is a closed manifold, $bM = S^{m-1}$, by theorem 6 of the last chapter. Thus $M = I^m$ by theorem 3.

Although this discussion covers manifolds of all dimensions, there is an alternate approach for low dimensional products of manifolds, m less than 5. As an example we include the following.

Theorem 5:

If the compact connected 4-manifold M is a product of manifolds and has a sphere boundary, then $M = I^4$.

Proof:

Let $M = X \times Y$ where X and Y are (necessarily) compact manifolds, and where dim X is less than 3. First, X is not closed, since if it were, $S^3 = bM$ is $X \times bY$. Thus, if dim $X = 1$, then $X = I$. Further, Y is not closed, since $I \times Y$ would then have two boundary components. Thus $H_3M = 0$, since $H_3(I \times Y) = H_3Y = 0$, or $H_3(X^2 \times Y^2) = 0$, either by the Kunneth formula, using Z_2 coefficients. From Lefschetz duality, we have $H_1 M = H_3M = 0$, so $H_1X = 0 = H_1Y$, by the Kunneth formula. Thus, if X and Y are both two dimensional, then they each equal I^2. Thus, M is I^4. If Y is three dimensional, then $bM = b(I \times Y)$, which is the topological double of Y [1 Greenberg, p. 184]. Using a Mayer-Vietoris sequence for the double of Y, we find that bY is connected and that $H_1 bY = 0$ [3 Spanier, p. 186]. Thus bY is S^2. From the same sequence, $H_2 Y = 0$. Thus Y is acyclic. Using the Seifert-Van Kampen theorem [12 Massey, p. 113] on the boundary of M, we find Y is simply connected because each of $bY = S^2$ and $bM = S^3$ is simply connected. By the Hurewicz and Whitehead theorems, Y is contractible. That the interior of Y is contractible, and even simply connected at infinity (because bY is S^2), may not be sufficient to prove $\text{Int } Y = E^3$. According to C. H. Edwards Jr. [14] however, we can make this claim because of one additional condition present in our case. Since $S^3 = bM$ contains two copies of $\text{Int } Y$, we know that every compact subset of $\text{Int } Y$ embeds in E^3. Thus, $\text{Int } Y = E^3$, $Y = I^3$ and $M = I^4$.

Continuing, after our digression, we examine the non-compact analog of lemma 2.

<u>Lemma 6</u>:

If M is a non-compact m-manifold with sphere boundary, such that the interior of M is a product space, then $M = [0,1) \times S^{m-1}$.

Proof:

Let $B = bM$; then the quotient space M/B is a manifold, and by theorem 5 of the last chapter, it is E^m.

For the next few lemmas, we replace the condition that M is a product by the condition that $M-P$ is a product, where P is a point in the interior of M.

Lemma 7:

If M is a compact connected m-manifold with non-empty boundary B such that $\operatorname{Int} M-P$ is a product space, then the interior of M is E^m, and B is a homotopy sphere.

Proof:

It is a direct consequence of theorem 5 of the last chapter, that $\operatorname{Int} M = E^m$. B is a homology sphere by Lefschetz duality, and is simply connected, because E^m is simply connected at infinity.

Lemma 8:

If M is a compact connected m-manifold with non-empty boundary, such that $\operatorname{Int} M-P$ is a product space, then $M = I^m$, if m is not 4 or 5.

Proof:

The homotopy sphere boundary may not be a sphere in dimensions 3 and or 4.

Theorem 9:

If M is a compact connected m-manifold with sphere boundary, such that $M-P$ is a product space, then $M = I^m$.

Proof:

The assumption that bM is a sphere removes the dimensional restrictions necessary in lemma 8, although it is redundant in other dimensions. Lemma 7 applies, because $\text{Int } (m-P) = \text{Int } M-P$.

The third kind of condition which we impose on the manifold M is that $M-P$ is a product, but this time P is in the boundary - not the interior of M. This condition, however, has to be further strengthened. Since $M-P$ is a product manifold, the factors are generalized manifolds with or without generalized boundaries. We impose the extra assumptions that one of these generalized manifolds has an empty boundary - We shall call this one the factor without boundary -, and that the dimension of the other factor is at least 2.

Theorem 10:

If M is a compact connected m-manifold, with connected boundary B such that, for P in B, $M-P$ is a product with one factor without boundary, then M is I^m.

Proof:

Let $M-P = X \times Y$ while X has no boundary. Then, since $B-P = X \times bY$ is a product space, $B = S^{m-1}$ by the theorem 6 of the last chapter. Since the interior of M is a product space, $M = I^m$ by lemma 2.

We notice that it is possible to weaken the hypotheses of theorem 10. First, we can show M to be compact without assuming so. Secondly, we can weaken the dimensional restriction on Y. We continue to let P be in B, and $M-P = X \times Y$ with bX empty. We let M be possibly non-compact, however, and let dim Y be ≥ 1.

Lemma 11:

If $M-P = X \times Y$ and bX is empty, then M is compact and the double $2M$ is S^m.

Proof:

The double $2M$ of M is an m-manifold without boundary. Since P is in B, P is a well defined point in the interior of $2M$. Because bX is empty, $2M-P = 2(M-P) = 2(X \times Y) = X \times (2Y)$ [15 Dugundji,p.252]. Since $2M-P$ is a product space, if $2M$ is not compact, then $2M$ is E^m, by theorem 5 of chapter 3. From the proof of that theorem, moreover, we know that X is a compact generalized homotopy (m-1)-sphere and that $2Y$ is E^1. Thus $Y = [0,1)$ and $M-P = [0,1) \times X$. Since P is in the boundary of M, $B-P = b(M-P) = X$. Since $B-P$ is not compact and X is compact, we have reached a contradiction. Thus $2M$ is compact. By theorem 6 of chapter 3, $2M$ is S^m. Thus M is compact.

Theorem 12:

Let M be an m-manifold with boundary B such that for a point P in B, $M-P$ is a product space with one factor without boundary. Then M is I^m, unless the other factor is $[0,1)$ and m is 4 or 5.

Proof:

Let $M-P = X \times Y$ with bX empty. By lemma 11, we may assume that M is compact and that the double $2M$ is S^m. By theorem 10 we may assume that dim Y is 1. Since $E^m = S^m-P = 2M-P = X \times (2Y)$, each of X and $2Y$ is contractible. Thus Y is $[0,1)$. Since $B-P = b(M-P) = X$ is contractible, B is a homotopy $(m-1)$-sphere. Since Int $M = E^1 \times X$ is a product space, M is I^m by lemma 2 unless m is 4 or 5. Further, Int $M^5 = E^5$ by lemma 1.

Corollary 13:

If $M-P = [0,1) \times X$ for any manifold X, then $2M$ is S^m. Moreover, M is I^m, unless m is 4 or 5.

Proof:

$M-P = [0,1) \times X$ is topologically the same as $[0,1) \times \text{Int } X$. Now we may use theorems 11 and 12.

CHAPTER 5

Suspension Manifolds

We will, next, extend our previous results to classify manifolds which are suspensions or cones. Then, we will seek sufficiency conditions for a manifold to be the cone over its boundary.

Definition 1:

Let M be a topological space. The cone CM over M is the quotient space $M \times I/M \times 0$. The suspension SM over M is the quotient space $CM/M \times 1$ [15 Dugundji, P. 127].

Asking the significance of the assumption that SM is a manifold, we find, first, the following necessary condition on M.

Lemma 2:

If SM is an m-manifold, then M is a generalized (m-1)-manifold.

Proof:

Delete the two vertex points from SM. The resulting manifold, by definition is homeomorphic to $E^1 \times M$.

Theorem 3:

If SM is a closed m-manifold, then SM is S^m.

Proof:

Let P and Q be the two vertex points of the suspension. Then $SM-Q$ is a non-compact manifold such that removing the point P leaves a product space. By theorem 5 of Chapter 3, $SM-Q$ is E^m. Thus $SM = S^m$.

Theorem 4:

If SM is a compact m-manifold with boundary B, then SM is I^m.

Proof:

By lemma 2, M is a generalized $(m-1)$-manifold. Since $M \times 1/2$ is a closed subset of SM, M is also compact. If the generalized boundary bM of M is empty, then the boundary of SM is empty. Thus bM is not empty. Since $B = S(bM)$ is a closed $(m-1)$-manifold, $B = S^{m-1}$ by theorem 3. SM is a compact m-manifold with sphere boundary such that its interior is a product space, $E^1 \times \text{Int } M$. Thus, $SM = I^m$, by lemma 2 of chapter 4.

Next we study manifolds which are cones.

Theorem 5:

If CM is a compact m-manifold, then CM is I^m, for $m \neq 4,5$.

Proof:

The cone point P is in the interior of CM, and $CM-P = [0,1) \times M$ is a product space. By lemma 8 of chapter 4, $CM = I^m$.

In the piecewise-linear category, theorem 5 is more easily demonstrated, even for dimensions 4 and 5. Since M is the link of vertex P in the obvious triangulation, M is S^{m-1}, and $CM = I^m$ [16 Hudson, p. 26].

Now we investigate sufficiency conditions on a compact manifold M, for M to be the cone over its boundary B.

Lemma 6:

If P is in the interior of M, and $M-P$ is a product space, then $M = CB$.

Proof:

It follows from the proof of lemma 7 of chapter 4, that $M-P = [0,1) \times B$. Thus $M = CB$.

Theorem 7:

If M is simply connected and the inclusion $B \subset M-P$ is a homotopy equivalence, $m \geq 5$, then $M = CB$.

Proof:

Since $M-P$ is simply-connected at the end P, we may apply the open collar theorem [21 Siebenmann, p. 8]. Thus $M-P = B \times [0,1)$. By lemma 6, $M = CB$. Of course, if $m \geq 6$, then $M = I^m$ by theorem 5.

In lower dimensions, more work is required.

Lemma 8:

If M is a 3-manifold such that the inclusion $B \subset M-P$ is a homology equivalence, then $B = S^2$.

Proof:

Let W be M with an open 3-cell neighborhood of P deleted. Then bW consists of B and a disjoint copy of S^2. Moreover the inclusion $B \subset W$ is a homology equivalence. Thus $H_2W = H_2B = Z_2$. Further, $H_2W = H_1bM \oplus Z_2$, by Lefschetz duality. Thus $H_1 B = 0$, and $B = S^2$.

This result leaves us far from showing that the manifold in lemma 8 is a cone. For example, if M is a contractible 3-manifold with 2-sphere boundary B, but which is not I^3, then $B \subset M-P$ is even a homotopy equivalence, but M is not a cone. Thus we shall need an extra hypothesis to avoid the Poincaré conjecture.

Theorem 9:

Let M be a (non-compact) 3-manifold with 2-sphere boundary B, such that the inclusion $B \subset M$ is a homotopy equivalence. Then, if M is 1-connected at infinity, and if every compact set in M embeds in E^3, then M is $S^2 \times [0,1)$, which is a cone less the vertex.

Proof:

Let W be the adjunction space W to I^3 identifying the sphere boundaries. Then there is a deformation retraction of W to I^3, relative to I^3, because of the homotopy equivalence [7 Brown and Crowell, p.445]. Since I^3 is contractible so is W. By the Edwards result used earlier [14 Edwards, p391], W is E^3 and $M = [0,1) \times S^2$.

Theorem 9 actually follows from a far stronger theorem due to Wall [18 Wall, p263] and is included as an illustration of the difficulties found in the lower dimensions.

The next theorem is due to Professor Doyle.

Let M be a compact m-manifold with point P in Int M and with boundary B.

Theorem 10:

If there is an isotopy of M-P into a collar of B, relative to B, then M = CB is a cone, $m \geq 5$.

Proof:

The double K = 2M is a closed m-manifold. Let Q be the mirror point of P in 2M. Using the Doyle and Hocking technique [13], we form a standard decomposition of K into an open m-cell and a compact residual set. From their proof, there is a manifold neighborhood N of the residual set which has a sphere boundary, and further more, we may choose N disjoint from P and Q. Using the isotopies on both copies of M-P simultaneously, we can embed N in $bM \times E^1$.

Since N is compact, there are infinitely many copies of N in M. By the Seifert-Van Kampen theorem $\pi_1(K) = \pi_1 N * \pi_1(K-N)$ is the free product. Since $\pi_1 K$ is finitely generated $\pi_1 N = 1$, by induction. For $i \leq m-2$, the homology groups $H_i N = 0$, for the same reason. For $m \geq 3$, $K = 2M$ is a homotopy m-sphere, and B is a homotopy $(m-1)$-sphere. For $m \geq 6$, $M = I^m$ is clearly a cone. If $m = 5$, $\mathrm{Int}\, M = E^5$, and $M = CB$.

CHAPTER 6

EQUIVALENCES

The two main theorems in chapter 3 were the characterizations of E^m and S^m. Although most of the succeeding results were presented as consequences of these two, some of them are actually equivalent to one or the other of these two important characterizations. We will re-examine two such pairs of equivalences and discuss the possibility of a third.

Lemma 1:

If M is a manifold without boundary and P is a point in M, then $M-P$ is the interior of a manifold with sphere boundary.

Proof:

Instead of deleting just P from M, delete an m-cell neighborhood of P. Call the resulting manifold W. The boundary of W is the boundary of the deleted m-cell which is an (m-1)-sphere B. The projection map $f: W \to W/B$ provides a homeomorphism from W/B to M since B bounds a cell. Also f: Int $W \to M-P$ is a homeomorphism.

Theorem 2:

The following statements are equivalent.

a) (Theorem 5 of chapter 3) E^m is the only open m-manifold M such that $M-P$ is a product space.

b) $[0,1) \times S^{m-1}$ is the only non-compact m-manifold M with sphere boundary such that Int M is a product space.

<u>Proof</u>:

That a) implies b) is lemma 6 of chapter 4.

Let M be an open m-manifold such that $M-P$ is a product space. We form a manifold W with an $(m-1)$-sphere boundary component B as in lemma 1. Then Int W is $M-P$ and $W/B = M$. According to b) we have $W = [0,1) \times S^{m-1}$. Thus $M = E^m$, proving a).

<u>Theorem 3</u>:

The following two statements are equivalent.

c) (Theorem 6 of chapter 3) S^m is the only closed m-manifold M such that $M-P$ is a product space.

d) I^m is the only compact m-manifold M with sphere boundary such that Int M is a product space.

<u>Proof</u>:

That c) implies d) is lemma 2 of chapter4.

Let M be a closed m-manifold where $M-P$ is a product space. Form a compact manifold W with an $(m-1)$-sphere boundary B as in lemma 1. According to d) we have $W = I^m$. Thus $M = S^m$, proving c).

Consider, also the following statement.

e) I^m is the only compact product m-manifold with sphere boundary.

By theorem 3 of chapter 4, e) follows from d). Under what conditions will d) follow from e)?

Let M be an m-manifold with boundary $bM = S^{m-1}$ and interior Int $M = A \times B$. We should try to establish directly that $M = X \times Y$ so that statement e) is applicable. If we could, then $M = I^m$ providing d).

Our first observation is that the X and Y which we are seeking may have very little relationship to the A and B which we are given. For example, it is too much to hope for, to always have that $A =$ Int X and $B =$ Int Y. This is the natural converse to the fact Int$(X \times Y) =$ Int $X \times$ Int Y, and it is the only way we can apply statement e), but we will see that it isn't true. If X is compact, it is called a completion for Int X.

According to D. R. McMillan, Jr. there are uncountably many topologically distinct open 3-manifolds B in E^3 such that if $A = E^1$, then $A \times B = E^4$. Let B be one of them other than E^3 itself, and let A be E^1. Although A has completion I, B has none. [19 McMillen, p. 100]

<u>Lemma 4</u>:

B has no completion.

Proof:

Let Y be a compact 3-manifold such that Int $Y = B$. Since $A \times B = E^4$, B and Y are contractible. By Lefschetz duality bY is a homology 2-sphere. Thus it is S^2, and B is 1-connected at infinity. By a result used in chapter 4 [14 Edwards,p.391], B is E^3.

Since not all open manifolds B do have completions, we might ask which ones do. Fortunately, this very deep problem was considered by L. C. Siebenmann. He requires that M be a smooth open manifold with finitely many tame ends and defines an algebraic obstruction, s. For $m \geq 6$, s vanishes if and only if B has a completion [4 Siebenmann, p.391]. It is also known that tame ends of open 2-manifolds can be completed [4 Siebenmann,p. iii]. With these as our major tools, the weakness of our approach is already apparent. First we don't know what to do with generalized manifold factors which are not manifolds. Secondly, we can't handle 3-,4-, or 5-manifolds.

In summary, we have seen that even when Int $M = A \times B$ has two open manifold factors, these factors need not have the completions which we need for statement e).

What if one of the factors were already compact? We know from our original proofs of statements c) and d) that this can't occur, but can we deny the possibility in another way? If we can find a completion X for the open factor A then, $S^{m-1} = bM$ and $bX \times B$ have the same homotopy groups.

Thus B is S^{m-1} and $X = [0,1)$. Since X is compact, this is already a contradiction. When is there a completion X for A?

This depends on the congruence class of m reduced modulo 4. We will need the following theorem due to L. C. Siebenmann first, in order to have tame ends to work with.

Theorem 5: [4 Siebenmann, p.66]

Let N be an open manifold with one and E. E is tame if and only if $N \times G$ has a completion for any closed manifold G, dim $G \geq 5$, with the Euler characteristic of G equal to zero. N and G are smooth.

Now, we are able to study the ends of A.

Theorem 6:

If Int $M = A \times B$ is a product of manifolds where B is closed, then A is an open manifold with one tame simply connected end.

Proof:

Since bM is connected, Int M has one (tame) end. A is not compact, because Int M is not compact. Each distinct end E of A contributes a distinct end $E \times B$ of $A \times B$. Thus A has precisely one end E. Since Int M has a completion, Int $M \times S^5$ has a completion. Since Int $M \times S^5 = A \times B \times S^5$ has a completion, A has a tame end, by theorem 5. We are using $G = B \times S^5$ where the Euler characteristic is zero, because that of S^5 is zero. It is also known that $1 = \pi, bM = \pi, E \times \pi, B$. Thus A is 1-connected at E, and incidently B is 1-connected.

Lemma 7:

$$H_i M = H_{m-i} M, \quad 1 \leq i \leq m-1$$

Proof:

This depends on the sphere boundary and Lefschetz duality.

In many dimensions m, it is possible to prove that the Euler characteristic of M is 1 [1 Greenberg, p. 167], using M/B as a closed manifold. We will assume so for all m.

This attempt at an alternate proof has led us to the following result. Since it is an alternate approach, we will just outline briefly what is actually a tedious proof.

Theorem 8:

Let M be an open m-manifold such that $H_i M = H_{m-i} M$, $1 \leq i \leq m-1$, M has one simply connected end, and the Euler characteristic of M is one. If $M = A \times B$ is the product of two smooth manifolds, one of which is closed, then $m = (4k+1)r$, for $r = 3$ or 5 and $k \geq 1$, where $r = \dim A$.

Proof:

This is accomplished by a careful study of Euler characteristics and by use of the Siebenmann obstruction, to force $r = 3, 4,$ or 5, and $m = 4j+r$. Then for each of these three cases there is a recursive formula describing the ranks b_i of the mod 2 homology groups of B inductively. If $r = 3$,

$b_{3\ell} = 1$, $b_{3\ell+1} = 0$, $b_{3\ell+2} = 0$ for $\ell \geq 0$. If $r = 4$, $b_{i+1} = (a_3-a_1)b_i - b_{i-1} + b_{\overline{2(i+1)}}$ $i \geq 1$ where $\overline{2(i+1)}$ is the residue class of $2(i+1)$ mod 4. If $r = 5$, $b_i = 0$ unless $i = 5\ell$, $b_{5\ell} = 1$.

In case $r=4$, or unless $m = (4k+1)r$ in the other two cases, we can show inductively that the Euler characteristic of B is greater than 1. Since that of $M = A \times B$ is 1, this is a contradiction.

CHAPTER 7

GENERALIZATIONS

In any endeavor there comes a time for perspective, a time to measure that which has been done against that which remains. In mathematical endeavors this frequently suggests the interesting art of constructing counterexamples. We will now look at some of our major results in this light. We will either extend them to greater generality or discuss why this isn't possible.

On the successful side, we will find new characterizations of E^m, homology-cells, and certain 3-dimensional product manifolds.

It is well known that for a manifold M the concepts of connectivity and connectivity at infinity are relatively independent. Removing an m-cell from any closed m-manifold provides an (m-1)-connected end. Conversely, there are contractible manifolds in each dimension above 3 which are not even simply connected at infinity [20 Curtis, p. 819].

From chapter 3, however, we know that if M is a product space, there may be a relation between these two concepts.

According to corollary 2 of chapter 3, if the product M of two open manifolds is k-connected at infinity, then M is k-connected. For large n, the following lemma provides examples to show that M need not be more than k-connected.

Lemma 1:

The manifold $M = (E^1 \times S^{m+1}) \times E^{n+1}$ is m-connected and is k-connected at infinity where k is the minimum of m and n.

Proof:

M is the interior of $(I \times S^{m+1}) \times I^{n+1}$ which has boundary B equal to $S^{m+1} \times S^{n+1}$. B is k-connected.

For small n, this same lemma provides counterexamples to the converse statement. That is, M can be connected to any degree, depending only on m, while it is not even 1-connected at infinity for $n = 0$.

Lemma 3 of chapter 3 was presented in terms of certain generalized manifolds. The extra property required there is hard to replace. Even if we require M to be a locally contractible generalized manifold, our product conditions are not sufficient for M to be 1-connected. Let B be a 3-dimensional Poincaré space, i.e., a homology sphere which is not 1-connected [7 Wilder, p. 245]. The suspension SB of B is locally contractible. With one of the suspension points removed, it provides a locally contractible generalized 4-manifold M. If P is the other suspension point, then $M-P$ is a product manifold which is not simply connected.

Using our characterization of I^m from chapter 4 we can now go back and prove another characterization of E^m.

Theorem 2:

Let M be a smooth (or piecewise linear) open product m-manifold with one end E. If E is both tame and $\left[\frac{m-1}{2}\right]$-connected, then M is E^m, $m \geq 6$.

Proof:

Since M is simply connected at E, M is the interior of a compact m-manifold W with boundary B, by Siebenmann's main theorem [4 Siebenmann, p.36]. Since B is a closed $\left[\frac{m-1}{2}\right]$- connected (m-1)-manifold, $m-1 \geq 5$, B is a sphere. By lemma 2 of chapter 4, W is I^m and M is E^m.

This same lemma 2 of chapter 4 will be very difficult to generalize directly. One possible generalization is equivalent to the Poincaré conjecture.

Theorem 3:

The following two statements are equivalent.

a) The Poincaré conjecture holds in dimension 3.

b) I^4 is the only compact 4-manifold with simply connected boundary and product space interior.

Proof:

That a) implies b) is theorem 3 of chapter 4.

Let S be a homotopy 3-sphere, with point P in S. Then $S-P$ is the interior of a compact manifold W with 2-sphere boundary. Let $M = I \times W$. Then $bM = 2W$ is a homotopy 3-sphere. By b), M is I^4 and bM is S^3. Thus

$S-P = E^3$ by Edwards result [14 Edwards, p391]. Thus S is S^3.

Another possible generalization of this theorem is to choose a different boundary, say E^{m-1}. There are many m-manifolds M such that Int M is a product space and $bM = E^{m-1}$. Even in dimension 2, $M = [0,1) \times E^1$ and $M = [0,1) \times S^1 - (0,1)$ provide distinct examples. In fact, an E^{m-1} boundary can be adjoined to any open m-manifold. What if we require M itself to be a product? Then $M = X \times Y$ and $bM = E^{m-1}$. For simplification, let $bX = \emptyset$. Then $bM = X \times bY$ so X and bY are contractible. Let N^2 be one of the two 2-manifolds constructed just above. Then the formula $M = E^{m-2} \times$ provides a pair of examples in each dimension m. There are more. To any open (m-1)-manifold adjoin an E^{m-2} boundary, calling the result N. Then $M = E^1 \times N$ is an example. Let B be one of the certain contractible open (m-2)-manifolds in E^{m-2}, like the McMillan example in chapter 6. Then B can be adjoined to any open (m-1)-manifold, forming N. Then $M = E^1 \times N$ is another example, because $bM = E^1 \times B$ is E^{m-1}.

Our main characterization of I^m in chapter 4 has an exact analog at the homology level; We have a parallel characterization of homology cells. Due to its length and especially to its generality, this theorem must be considered one of our major results.

Let M^m be a compact connected m-manifold with boundary B such that $H_q B = H_q(S^{m-1})$ for all q. Let $M = Y^p \times X^{m-p}$

also be a non-trivial product space. Then Y, X are generalized p-and $(m-p)$-manifolds respectively. We will use mod 2 homology.

Theorem 4:

M is a homology cell.

Proof:

We shall show $H^{\#}_{*} M = 0$, through a succession of lemmas.

Since B is non-empty, $H_m M = 0$. Since M is connected $H^{\#}_{o} M = 0$.

Lemma 5:

If $1 \leq q \leq m-1$, $H_{m-q} M = H_q M$.

Proof:

Since M is compact, this is a direct result of Lefschetz duality, using the fact that $H_q B = 0$ unless $q = m-1$.

Lemma 6:

B is not a product space.

Proof:

If it were, then $B = W \times V$ is a product of closed generalized manifolds. Let $q = \dim W$. Since $1 \leq q \leq m-2$, the Kunneth formula yields $0 = H_q B = (H_o W \otimes H_q V) \oplus \cdots \oplus (H_q W \otimes H_o V) \neq 0$. $H_q W = Z_2$, by Poincaré duality.

Corollary 7:

Neither Y nor X is a closed generalized manifold.

Proof:

If bY is empty, then Y is a factor of B. Thus, one of X and Y is B and the other is $[0,1)$ which is a non-compact factor of M.

Lemma 8:

$H_1 M = 0$.

Proof:

By lemma 5, $H_1 M = H_{m-1} M$. By the Kunneth formula, $H_{m-1} M = \bigoplus_{i=0}^{m-1} (H_i Y \otimes H_{m-1-i} X)$ is a direct sum. Each term in the sum, however, is zero. If $i \geq p$, $H_i Y = 0$. If $i \leq p - 1$, then $m-1-i \geq m-p$ and $H_{m-1-i} X = 0$.

Corollary 9:

$H_i Y = 0 = H_1 X$, by the Kunneth formula.

Define $n = [m/2]$, and fix an integer k $2 \leq k \leq n$. With no loss of generality, we may let $p \leq n \leq m-p$.

In order to show, eventually, that $H_k M = 0$, we make the inductive hypothesis that $H_j M = 0$ for all j, $1 \leq j \leq k-1$. There are two immediate consequences.

Lemma 10:

$H_j Y = 0 = H_j X$, for $1 \leq j \leq k-1$.

Proof:

By the inductive hypotheses and the Kunneth formula $0 = H_jM = \bigoplus_{i=0}^{j} (H_iY \otimes H_{j-i}X)$. Thus, for $i = 0$, since $H_0Y = Z_2$, we have $H_jX = 0$. Similarly, for $i = j$, $H_jY = 0$, because $H_0X = Z_2$.

Corollary 11.

$H_kM = H_kX \oplus H_kY$, by the Kunneth formula.

Lemma 12:

$H_{p-j}Y = H^{\#}_{j-1}bY$ and $H_{m-p-j}X = H^{\#}_{j-1}bX$ for $1 \leq j \leq k-1$ and $j \leq p$.

Proof:

We substitute $H_iY = 0 = H_iX$, for $1 < i \leq j$, in Lefschetz duality for Y and then for X.

Lemma 13:

$$\bigoplus_{\ell=0}^{k-2} (H^{\#}_{\ell}\, bY \otimes H^{\#}_{k-2-\ell}bX) = 0$$

Proof:

We recall that $2 \leq k \leq n$. Since we may represent B as a union of $Y \times bX$ and $bY \times X$ with $bY \times bX$ in common, we may apply the Mayer-Vietoris sequence for this union. Since $H_iB = 0$ $1 \leq i \leq m-1$, all we need is the Kunneth formula for $H_{k-2}Y \times bX$, $H_{k-2}bY \times X$, and $H_{k-2}bY \times bX$. The lemma follows by cancellation.

We are now ready to complete the induction.

Lemma 14:

$H_k M = 0 \quad 2 \le k \le n.$

Proof:

We need to say $H_{m-k}Y = 0$. This is clear, unless $m-k \le p-1$. This inequality implies $2n \le m \le p+k-1 \le 2n-1$, which cannot occur. Using the Kunneth formula and lemma 5, we have

$$H_k M = H_{m-k}M = \bigoplus_{i=0}^{m-k} (H_{m-k-i}Y \otimes H_i X)$$

$$= \bigoplus_{i=m-k-p+1}^{m-p-1} (H_{m-k-i}Y \otimes H_i X)$$

because $\dim X = m-p$ and $\dim Y = p$. Setting $\ell = i-m+k+p-1$ and substituting, that sum

$$= \bigoplus_{\ell=0}^{k-2} (H_{p-(\ell+1)}Y \otimes H_{m-p-(k-\ell-1)}X)$$

$$= \bigoplus_{\ell=0}^{k-2} (H^{\#}_{\ell}\, bY \otimes H^{\#}_{k-2-\ell}\, bX), \text{ by lemma 12}$$

$$= 0 \quad , \quad \text{by lemma 13.}$$

This completes the inductive step.

Theorem 4 now follows, because $H_k M = 0$ for $1 \le k \le n$ and $H_k M = H_{m-k}M = 0$ if $n+1 \le k \le m-1$.

Can we significantly weaken the hypotheses of theorem 4, getting a stronger characterization of a homology cell? One natural and historic generalization of our product hypothesis would be the assumption of a fibration. What if the manifold M is the total space of a fibration, and still has a homology sphere boundary? Since the Leray-Hirsch theorem is a fibration analog of the Kunneth formula for product spaces it seems this question is familiar. The result, however, is different, since M is not acyclic. The simplest counterexample is the Moebius band M [12 Massey, p. 3]. M is the total space of a fibration over S^1. The fiber I is also a manifold and the boundary of M is a sphere S^1.

The last major result in chapter 4, was theorem 12, the characterization of I^m as a manifold M such that $M-P = X \times Y$ where bX is empty and P is in bM. The dimensional restrictions in that theorem may or may not be necessary, but one hypothesis is indespensible. If both X and Y have boundaries, then we lose uniqueness for M.

Theorem 15:

If Y is a m-manifold with boundary, then $M = I \times Y$ has the property $M-P = I \times X$, for p in bM.

Proof:

Let Q be a point in bY, and let $P = (O,Q)$ in bM. Then $M-P = I \times Y - (O,Q) = I \times Y - I \times Q = I \times (Y-Q) = I \times X$, where $X = Y-Q$ is an m-manifold.

All of the examples of manifolds M provided in theorem 15 are product manifolds; this fact suggests we may be able to construct more examples. In dimension 3, however, we have already exhausted this method.

Theorem 16:

Let $M = X \times Y$ be a connected product 3-manifold with boundary B, such that for P in B, $M-P = W \times V$. Then I is a factor of M and also of $M-P$.

Proof:

Let V be the 1-dimensional factor of $M-P$. If bV is empty, then $M = I^3$ by theorem 12 of chapter 4. Thus, V is I or $[0,1)$. If V is $]0,1)$, then $M = I^3$ by corollary 13 of chapter 4. Thus V is I. Now $M-P = I \times W$. It remains to show that M can always be represented by $I \times X$.

Let Y be the 1-dimensional factor of M. Then Y is either E^1, S^1, or $[0,1)$. Let $M = E^1 \times X$ and consider $B-P = b(I \times W) = 2W$ as the topological double. Also $B = E^1 \times bX$. Thus B is either $E^1 \times E^1$ or $E^1 \times S^1$. In the latter case $B-P = 2W$ is a twice punctured plane.

Using the Meyer-Vietoris sequence for the double, and checking out the cases, we find that W is either an annulus A with an E^1 boundary or a disk D with three E^1 boundary components. If $W = D$, then $\text{Int } M = E^1 \times \text{Int } D = E^3$ Now $\text{Int } X$ is E^2 and bX is S^1. Thus, X is I^2. $M = E^1 \times I^2$ satisfies the condusion and is our first example. If $W = A$, then $\text{Int } M = \text{Int}(M-P) = \text{Int}(I \times A) = E^1 \times E^1 \times S^1$, which embeds in E^3. Also $\text{Int } M = E^1 \times \text{Int } X$. Thus, $\text{Int } X$ has the homotopy groups of S^1. Since $E^1 \times$ (Moebius Band) does not embed in E^3, $\text{Int } X$ is $E^1 \times S^1$. Now $X = [0,1) \times S^1$, and $M = E^1 \times [0,1) \times S^1 = I \times [0,1) \times S^1$ satisfying the conclusion.

Looking for more examples, we may try $bX = E^1$. In this case, B is E^2 and $B-P$ is $E^1 \times S^1 = 2W$. Using the Meyer-Vietoris sequence for the double, we find that W is either the disk $D = I \times E^1$ or the annulus $A = S^1 \times [0,1)$. First, we show that W is not A. If it were, then $M-P = I \times [0,1) \times S^1 = E^1 \times [0,1) \times S^1$. Since $E^1 \times \text{Int } X = \text{Int } M = E^1 \times E^1 \times S^1$, just as above, we can show Int X is $E^1 \times S^1$. Thus, X is $[0,1) \times S^1$ with one boundary point removed. Since $M-P$ and $I \times W$ are homeomorphic, the inclusion of the boundary into the manifold should behave algebraically the same in each case. We find, however, that the generator of $\pi_1(B-P)$ is mapped by inclusion trivially into $\pi_1(M-P)$, while that of $\pi_1(2A)$ is mapped onto the generator of $\pi_1(M-P)$. Thus, W is D, not A. $M-P = I \times W = I \times I \times E^1$ implies that $E^1 \times \text{Int } X = \text{Int } M = \text{Int}(M-P) = E^3$. Thus Int X is E^2 and bX is E^1. This implies X is $[0,1) \times E^1$, and $M = E^1 \times [0,1) \times E^1$. Since M is homeomorphic to $I^2 \times [0,1)$, it satisfies the conclusion of the theorem.

In the second major case, we let $M = [0,1) \times X$ and $M-P = I \times W$. Here, $B = \text{Int } X$, and $B-P$ is $2W$. Again, we are able to discriminate on the basis of the inclusions of $B-P$ and $2W$ into $M-P$. $B \subset M$ is a homotopy equivalence and $B-P \subset M-P$ is the same, except that one extra generator (a loop in B around P) is mapped trivially. On the other hand, the inclusion $2W \subset M-P = I \times W$ maps each generator of $\pi_1 W$ in a 2 to 1 fashion, since there are two copies of W. As long as $\pi_1 W$ has a non-trivial generator, this is a contradiction. Thus W is simply-connected. Now Int W is E^2, and Int M is E^3. Thus Int X is E^2 and $M = [0,1) \times I^2$. This is one of the examples we found in case 1.

In the third major case, we let $M = S^1 \times X$ and $M-P = I \times W$. Now, $S^1 \times \text{Int } X = \text{Int } M = E^1 \times \text{Int } W$. Using the Kunneth formula on $H_2(\text{Int } M)$, we find $H_1 X = H_2 W = 0$. Thus, $\text{Int } X = E^2$, and $\text{Int } M = S^1 \times E^2$ embeds in E^3. Thus, $\pi_1 W = Z$. Since $E^1 \times \text{Int } W$ embeds in E^3, $\text{Int } W = S^1 \times E^1$ and not the Moebius band. Since $B-P = 2W$ is connected, so is $B = S^1 \times bX$. Thus, bX is either S^1 or E^1. If bX is E^1, B is an annulus, and B-P is a twice punctured plane. By the Mayer-Vietoris sequence on the double, W is an annulus with an E^1 boundary. $M = S^1 \times X = S^1 \times [0,1) \times E^1 = S^1 \times [0,1) \times I$ is one of our earlier examples. Finally, if $bX = S^1$, then X is I^2 and $M = S^1 \times I^2$ is another example.

Since all five of the examples we have discovered by this method ($E^1 \times I^2$, $[0,1) \times I^2$, $S^1 \times I^2$, $S^1 \times [0,1) \times I$, and of course $I \times I^2$) fit the old pattern, the theorem follows.

CHAPTER 8

Cobordism

We will take a brief look at cobordism, and then prove a sufficient condition for a closed m-manifold M to be s-cobordant to the m-sphere S^m. For $m \geq 5$, it follows that M is homeomorphic to S^m.

<u>Definition 1</u>:

A cobordism is a manifold W whose boundary has two components, M and N. M and N are said to be cobordant. For an h-cobordism W the inclusions $M \subset W$ and $N \subset W$ are homotopy equivalences. For our purposes, an S-cobordism is a simply connected h-cobordism, although this is a very special case of the more general situation |16 Hudson, p. 223|.

The following - at first surprising - result points out the need for some extra hypotheses.

<u>Theorem 2</u>:

Any two m-manifolds M and N are cobordant.

<u>Proof</u>:

We remove from the product space $M \times I$ all of $M \times \{1\}$ except one E^m neighborhood. After treating N similarly, we adjoin the resulting manifolds by a homeomorphism from one to E^m neighborhood to the other. The union W is the desired cobordism.

If M and N are h-cobordant, then they have the same homotopy type. Conversely, many pairs of M and N with the same homotopy type are h-cobordant.

Theorem 3:

If there is an m-manifold A such that A embeds, as a deformation retract, in each of M and N, then M and N are h-cobordant.

Proof:

This proof is the same as the preceeding one, except that we adjoin the products along the copies of A instead of along the E^n neighborhoods. The inclusions deform to the inclusions of A into $A \times I$. Thus they are homotopy equivalences and we have constructed an h-cobordism.

Corollary 4:

Any two contractible m-manifolds are h-cobordant.

Historically, the main interest in cobordisms has been in compact cobordisms between (necessarily) closed manifolds. It is well known that a closed manifold M which is s-cobordant to the m-sphere S^m is itself a sphere, for $m \geq 5$. It turns out that requiring two homotopy equivalences is not necessary.

Theorem 5:

If W is a cobordism between M and S^m, if the inclusion $S^m \subset W$ is a homotopy equivalence, and if M is simply connected, then W is an s-cobordism.

Proof:

This is well known. We attach an $(m+1)$-cell along S^m forming a compact contractible manifold with boundary M. By Lefschetz duality M is $(m-1)$-connected. Since W is $(m-1)$-connected, the inclusion is a homotopy equivalence, and W is an s-cobordism.

In sufficiently high dimensions we may choose the other inclusion and still get an s-cobordism.

Theorem 6:

If W is a cobordism between M and S^m, if the inclusion $M \subset W$ is a homotopy equivalence and if M is simply connected, then W is an s-cobordism, $m \geq 4$.

Proof:

We attach an $(m+1)$-cell along S^m forming a compact manifold X. For a point P in the cell the inclusion $M \subset X-P$ is a homotopy equivalence. By theorem 7 of chapter 5, X is the cone CM over M. Thus W is an s-cobordism.

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