DEVELOPMENT OF A METHOD OF SOLUTION UTILIZING VOLTAGE PHASE ANGLES AS CONTROLLING VARIABLES FOR THE POWER SYSTEM ECONOMIC DISPATCH PROBLEM

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AN ABSTRACT

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ABSTRACT

A problem of major concern to the electrical utility industry is that of specifying for an operating power system the generating station power outputs so as to supply the load requirements at the lowest possible production costs to the system. Two primary factors which influence the production costs are the variations in generating costs at the different stations in the system and the variations in losses occurring as energy is transferred from the different stations to the loads over the transmission system. In this thesis a new approach to the solution of this problem is set forth. This approach is new in terms of the variables used and in terms of the application of a minimizing process directly to the production cost function while maintaining the identity of the individual loads. The node (bus)voltage phase angles are utilized as the controlling variables in the theoretical development and the required generating station power outputs are then determined in terms of these phase angles and other previously specified variables. While it is required that a certain set of system operating data be supplied for each computation, changes in these data are allowable as system operating conditions change; assumptions regarding certain variables remaining constant at their base load values, as are involved in the developments of some other methods, are not included.

Chapter I includes a general discussion of the problem considered and a summary of some of the techniques previously developed for its solution. In Chapter II the new sets of equations are established in terms of the new variables; the solutions of these equations specify values of the variables which satisfy necessary conditions for minimum system

production cost. The sets of resulting equations are non-linear involving products of variables and trigonometric functions of other variables. In Chapter III it is shown that the Newton-Raphson iterative technique for obtaining solutions to systems of non-linear equations is applicable to this problem and the sets of linear equations which are required to be solved successively as specified by the Newton-Raphson technique are then established. Use of a digital computer in the successive solution of these equations is next considered and a general computer flow diagram is indicated.

The general technique is applied to a particular example problem in Chapter IV. Results obtained on the MISTIC (Michigan State University Digital Computer) in determining solutions to the equations for this example system using a set of initial approximations to the variables (the voltage phase angles), computing new approximations to these variables, and continuing the successive solution process are shown for various system operating conditions. The generating station output powers, station production costs, and system production costs corresponding to the computed values of the phase angles are also calculated. Finally, verification of the determination of a minimum of the production cost function by the method of this thesis is shown by curves in which the system production cost for this example is plotted as a function of the phase angles, the phase angles being determined such that the specified individual load power requirements are satisfied.

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CHAPTER I

INTRODUCTION

1.1 The Problem in General Terms

A primary axiom of electrical power system operation is that the combined outputs of the generating stations be such as to supply the total load power requirements plus the losses in the system. However, whenever these total load requirements including the losses are less than the total capacity of the generating stations in operation, some choice exists as to how the load is to be divided among the various stations. The resulting problem, one of major importance to the operators of a power system, is that of specifying the individual generating station power outputs so as to minimize the overall system production costs under the specified load conditions. This is the so-called economic dispatch problem. In this thesis a method different from the methods used heretofore for solution of this problem is set forth. The method is different in terms of the variables used and in terms of the application of a minimizing process directly to the production cost function while maintaining the identity of the individual loads. The variables determined, with other previously specified variables, are sufficient to determine the required generating station power outputs. The general method is developed in detail with a selected set of specified parameters, a technique for solving the resulting system of non-linear equations is presented, and the method is then applied to a particular example system. Specifically, the procedure developed applies to systems in which it is desired to control on an economical basis, the operation of thermal (fuel-burning) generating stations; scheduling of hydroelectric plants and tie-lines is not considered.

The major items contributing to the system production cost include the costs of fuel, labor, maintenance, supplies, and water. By means of measurements made on the boiler, turbine, and generator units at a thermal generating station its fuel cost in dollars per hour can be reasonably accurately determined as a function of station power output. However, it is usually not possible to express the costs of labor, maintenance, and supplies at a station as functions of the power output of the station; these may, in some applications, be included as a fixed percentage of the fuel costs. Depending primarily on the geographic location of the generating stations, water may or may not constitute a significant part of the production costs. On the basis of these and other considerations, the production costs are, for purposes of specifying station power outputs, in many cases represented by only the fuel costs. Herein, the term system production cost is used to designate all parts of the total production cost which can be expressed as a function of the power outputs of the various generating stations. While these parts are made up primarily of fuel costs, inclusion of other costs is entirely possible if these costs can be expressed as functions of the generating station output powers.

The general economic dispatch problem can be stated in more detail as follows: given information concerning an electrical power system including the transmission lines in operation, the generating stations in operation, various characteristics of these lines and stations, and certain specifications regarding the loads on the system, determine how the required total generated power should be allocated among the various generating stations so as to minimize the overall system production cost.

At this point it should be noted that there are other considerations in the operation of a power system, related to operation at minimum production costs, but of equal or greater importance. These basic system requirements include:

(1) Provisions that the total capacity of the generating units in operation is greater than the sum of the load requirements plus the system losses.

(2) Provisions that the units in operation are operating at a level equal to the sum of the load requirements plus the system losses, whether or not the distribution of load is on an economic basis.

(3) Provisions for maintaining the system frequency at a specified value.

(4) Provisions for maintaining specified voltage levels at various points in the system.

The first item above is a matter of scheduling; once a set of generating units are selected, certain of the production cost parameters are then established. Establishment and regulation of the total generated power level and maintenance of the system frequency are controlled by a so-called load frequency control system. In general, this system operates in response to changes in system frequency and to changes in tie-line power measurements so as to adjust the total generating station power output to match the total load requirement and maintain a constant frequency; it is in conjunction with such a load frequency control system that a unit for determining conditions for economic operation performs. Information relative to load changes is operated on in such a way as to determine how the load changes should be allocated among the various stations so as to minimize the system production cost.

While papers were published relating to the general subject of economic operation of power systems as early as 1922 [1], it is only in recent years that effects of transmission system losses and other factors have been included and more nearly correct results obtained.

In order to provide background for the method presented in this thesis, some of the important work done previously on this subject is considered next with particular reference to the assumptions made in the developments.

1.2 Summary of Previous Work

1.2.1 Coordination Equations and Transmission Loss Formula with B Constants

For many years the scheduling of generating station power outputs was carried out on a so-called equal incremental rate basis in which the transmission losses were neglected, at least analytically. While an approximate consideration of these losses was sometimes included in the actual practical scheduling of station power outputs, no method was available which allowed their inclusion in an at all accurate manner, particularly on systems involving a transmission network interconnecting several generating stations. The book by Steinberg and Smith [2] published in 1943 considers much of the theory developed up to that time.

The general economic dispatch problem, with or without transmission losses included, is one which can be classified as a problem in Constrained Minima since it involves determination of values of variables so as to minimize a function while simultaneously satisfying at least one auxiliary equation. The classical method for solution of such problems is that of Lagrangian multipliers [3]. The first paper in which the Lagrangian multiplier method was applied to this problem was published in 1949 [4]. Previously, another method had been used for systems in which the transmission losses were neglected; the same conclusions were reached by application of this method as by application of the Lagrangian multiplier method. The expressions resulting from applying the Lagrangian multiplier method to a system in which the individual loads are combined into a single equivalent load are developed by Kirchmayer [5]. The development shows that if transmission losses are neglected, solution of the following simultaneous equations (1.2.1) and (1.2.2) for P_1, P_2, ... , P_N yield values which satisfy necessary conditions for a minimum of F_t, the total input to the system in dollars per hour.

$$\frac{dF_x}{dP_x} = \lambda \quad x = 1, 2, \ldots, N \tag{1.2.1}$$

$$\sum_{x=1}^{N} P_x + P_R = 0 \tag{1.2.2}$$

In equations (1.2.1) and (1.2.2)

$F_x = f_x(P_x)$ = input to station x in dollars per hour.

P_x = output power of station x.

P_R = a specified total load power.

λ = a Lagrangian multiplier.

N = the total number of generating stations.

If transmission losses are included, the equations take the following form,

$$\frac{dF_x}{dP_x} + \lambda \frac{\partial P_L}{\partial P_x} = \lambda, \quad x = 1, 2, \ldots, N \tag{1.2.3}$$

$$\sum_{x=1}^{N} P_x + P_L + P_R = 0 \tag{1.2.4}$$

where P_L = total transmission system losses.

This last set of equations has been given the name Economic Coordination Equations, to indicate that the production costs and transmission losses are both being considered, hence "coordinated." In order to utilize these Economic Coordination Equations in the form shown above it is necessary to have a relation expressing the transmission system losses in terms of the generator station power outputs. The first relation which expressed the transmission system losses in this way was proposed by George in 1943 [6]. It was in a quadratic form as shown in equations (1.2.5) and (1.2.6) below. The constants in the equation have since become known as B constants and the methods based on use of this type of equation as B constant methods. The total transmission loss, P_L, is given as

$$P_L = \sum_m \sum_n P_m B_{mn} P_n \qquad (1.2.5)$$

or, in matrix form,

$$P_L = \mathcal{P}_G^T \mathcal{B}_{mn} \mathcal{P}_G \qquad (1.2.6)$$

where $\mathcal{P}_G = \begin{bmatrix} P_1 \\ P_2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_n \end{bmatrix}$, $\mathcal{B}_{mn} = \begin{bmatrix} B_{11} & B_{12} & B_{13} \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & \cdot \\ B_{31} & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ \cdot & & & \\ B_{n1} & \cdots & \cdots & B_{nn} \end{bmatrix}$

and $\mathcal{B}_{mn}$ is symmetric.

In order to determine a formula of this form, a number of assumptions are necessary. A primary requirement is that the individual loads be replaced by a single load which can be shown to be equivalent to the individual loads under certain assumed conditions. Many developments [7, 8, 9, 10, 11, 12] of such a formula have involved essentially the following assumptions.

1. Each equivalent load current is under all operating conditions a constant complex fraction of the total load current, i.e., $I_k = m_k I_L$, where I_k is an equivalent load current and I_L is the total equivalent load current. The equivalent load current at a bus is defined as the sum of the line-charging, synchronous condenser, and load currents at that bus.

The fraction m_k is determined from a load flow study of a "base case" for which the loads are near the average of the expected range for the system.

2. The voltage magnitudes and angles at each generator bus are constant at their values as determined in the base case.

3. The generator Q/P ratios remain constant at their values determined in the base case.

If, as indicated above, the B constants are determined for only one base case condition and then applied for conditions which are different than for this base case, some errors are inherent in all further calculations which utilize these data. In an attempt to decrease this error, B constants are sometimes calculated for two or more different loading conditions and an average of the constants computed for each different condition is used in the final loss formula.

In any case, once some set of B constants are established and thereby a relation for the power loss in the transmission system in terms of the generating station outputs, these station power outputs can then be determined by solution of the set of N + 1 non-linear equations (1.2.3) and (1.2.4) with the expression in (1.2.5) substituted for P_L in (1.2.3) and (1.2.4). If values of λ are specified, equations (1.2.3) can be solved for P_1, P_2, . . . P_N and then the corresponding total load power, P_R, calculated from (1.2.4). This method has been used to pre-calculate station power output curves as functions of the total load [13, 14]. Also, analog computers have been constructed to solve the non-linear equations on a real time basis simultaneously with changes in load [15, 16]. More recently, digital computers have been used to solve the equations [5].

Loss formulas including additional terms, as shown in equation (1.2.7) below, have also been developed [17]. This form of the loss formula permits more flexibility in the manner in which the loads are assumed to vary.

$$P_L = \sum_m \sum_n P_m B_{mn} P_n + \sum P_n B_{no} + B_{oo} \qquad (1.2.7)$$

The constants of this type of equation have been determined empirically from data obtained from a number of load flow studies at both peak load and minimum load conditions each with different distributions of generating station power outputs. The assumptions used in this development are as follows:

1. Each load varies between its peak and minimum values linearly with total system load.
2. Each source bus voltage magnitude varies between its value at peak load and its value at minimum load linearly with total system load.
3. The power factor of each load varies from its value at system peak load to its value at system minimum load linearly with total system load.
4. The generation of reactive power at each source is that required to supply the load and maintain the source bus voltages.

In a later paper [18], another development of this expanded form of a loss formula was presented along with a method of determining its constants which requires less network analyzer data than in the previous development. The assumptions used in this development are the same as those used in the development of the shorter form of the loss formula except for a change in the assumption regarding the load currents. In this case it is assumed that the individual load currents are linear complex functions of the total load current, as given by

$$I_j = I_j^0 + m_j I_{L_T} \qquad (1.2.8)$$

where I_j^0 = value of j^{th} load current when $I_{L_T} = 0$.

I_{L_T} = total load current

m_j = complex rate of change of j^{th} load current with respect to I_{L_T}

1.2.2 Another Set of Coordination Equations and the Brownlee Phase Angle Method for Losses

As shown by Ward [19] and others [5, 20], an alternate form of Coordination Equations can be written as

$$\frac{dF_x}{dP_x} \quad \frac{1}{\left(1 - \dfrac{dP_{L_{x,\ ref}}}{dP_x}\right)} = \frac{dF_{ref}}{dP_{ref}} \qquad (1.2.9)$$

where $\dfrac{dF_{ref}}{dP_{ref}}$ = incremental production cost at a reference station.

$\dfrac{dF_x}{dP_x}$ = incremental production cost at station x.

$\dfrac{dP_{L_{x,\ ref}}}{dP_x}$ = rate of change of transmission loss with respect to the output of station x when changes in power output are made only by the reference station and station x.

Use of equation (1.2.9) requires an expression for $\dfrac{dP_{L_{x,\ ref.}}}{dP_x}$

Glimm et al. [14] developed an expression for this term using some of the same approximations as used in development of the loss formula in terms of B constants. Another expression for this term was developed in terms of voltage phase angles by Brownlee [20]. The assumptions inherent in the use of this latter expression are as follows:

1. The voltage magnitude at each bus remains constant.
2. The reactive power over the system is such as to maintain the constant voltages.

The relation for a two machine system without intermediate loads or generating stations can then be written as

$$\frac{dP_{L_{x,\ ref}}}{dP_x} = \frac{2 \tan \theta_{x,\ ref}}{K + \tan \theta_{x,\ ref}} \qquad (1.2.10)$$

where $\theta_{x,\ ref}$ = difference in voltage phase angle between bus x and the reference.

K = ratio of reactance to resistance of impedance between bus x and the reference bus with all other sources and loads open circuited.

Brownlee also proposed that the effect of generating stations located between bus x and the reference bus could be approximated by the expression

$$\frac{dP_{L_{x,\ ref}}}{dP_x} = \frac{4K \tan \frac{1}{2}\, \theta_{x,\ ref}}{[K + \tan \frac{1}{2}\theta_{x,\ ref}\,]^2} \qquad (1.2.11)$$

Cahn [21] developed further the approximate relations presented by Brownlee, investigated the errors involved and showed that these errors are small in many practical systems. Using the relations in terms of phase angles, Cahn developed formulae for both incremental losses and total losses in terms of the generating station power outputs. The incremental loss formula, which is the one used in economic scheduling studies, required further assumptions. These are as follows:

1. Each load power remains a fixed fraction of the total load power.
2. Q/P ratio of each load remains fixed.

1.2.3 Calvert and Sze Approach to Loss Minimization

In 1958 [22] Calvert and Sze presented a new approach to loss minimization in electrical power systems and in 1959 [23] presented some applications of this technique. The technique starts with a set of data specifying the load conditions and determines corresponding generator operating conditions so as to satisfy necessary conditions for minimizing a defined total loss function without requiring further approximations. The total loss function is the sum of a set of loss functions of which there is one for each element representing a generator or a load. The loss function for each element representing a generator is an expression which is used to relate the losses in the generating stations to losses in the transmission network; for each element representing a load, the loss function is equal to the power function of that element (a specified parameter) and is of opposite sign to the power function of a generator element. In detail, these relations are given by Sze and Calvert [22] as

"$\mathcal{L}_x(P_x)$ = actual loss at station x, watts

$F_x(P_x) = C \;\; \mathcal{L}_x(P_x)$ = equivalent loss at station x: the network watts costing the same amount per hour as does $\mathcal{L}_x$; hence C_x is an adjustable constant.

$\mathcal{P}_x = P_x + F_x(P_x)$ = equivalent primary power input for station x.

$\mathcal{P} = \sum_{x=1}^{n} \mathcal{P}_x$ = total equivalent loss in all generating stations plus network loss. Note that at loads $F_x(P_x)$ is zero and P_x is negative."

In the paper from which the above quotation is taken, P_x designates the power input to the network at node x and the nodes of the network are numbered 1 through n, hence the last summation above is taken over all nodes. The final function, $\mathcal{P}$, is the one which is minimized. It is asserted that conditions so determined as to satisfy requirements for minimization of the total "equivalent" loss expression are identical with conditions for minimum cost.

In the Calvert and Sze technique, the operating conditions (restrictions) required at the beginning of the problem are, in general, the real and reactive power and either the voltage or current in both magnitude and relative phase angle at all loads. The variables determined in the solution of the problem are the voltages (or currents) at the generator busses, these variables being determined both in magnitude and phase angle. The generator real and reactive power outputs are then determined as supplementary data. The Lagrangian multiplier technique is used to determine the necessary conditions for a minimum of the total loss function.

CHAPTER II

DEVELOPMENT OF CRITERIA FOR ECONOMIC SYSTEM OPERATION

2.1 Introduction

Consideration of the presently used and the proposed methods directed toward solution of the problem of determining conditions for economic operation of a power system, as summarized in Section 1.2, makes apparent the almost universal use of considerable numbers of approximations and assumptions in the developments. For example, the B constants used in the relation for power loss in a transmission system (Equation (1.2.5)) are determined from sets of data from particular base (average) system operating conditions but the formula for power loss is used in the Economic Coordination equations over wide ranges of system operating conditions. The Coordination equations are then not entirely accurate when used under conditions different from the base conditions. Several of the other procedures utilize similar assumptions.

In the method proposed by Calvert and Sze [22], discussed in Section 1.5, a loss function is defined which, it is asserted, allows the losses in the generating stations to be replaced by network losses. This is accomplished for each generating station by multiplying its loss expressed in terms of the power output of the station by a constant equal to the ratio of the cost per kilowatt hour lost in the station to the cost per kilowatt hour lost in the network. The minimizing process is then applied to this so-called "equivalent" loss function rather than to the actual system production cost function. In every method approximations are used in some aspects, as for example, the generating station input-output curves are assumed to be of such a nature that they can be adequately approximated by polynomials, and, in all practical applications, these polynomials are taken to be of the second degree so that their derivatives are linear functions.

Although proponents of the general methods utilizing B constants in determining expressions for transmission system power loss assert that those methods give results which lead to significant savings in the production costs as compared with methods in which transmission losses are not considered, the need still exists for a more precise method and one which is more readily adaptable to changing conditions in the power transmission system. In this thesis the technique developed is such that after a set of specifications are given for certain of the variables of the system, as for example, power at each load, bus voltage magnitudes, etc., another set of variables are selected so as to satisfy necessary conditions for minimization of the production cost for the system. Assumptions regarding polynomial approximation of the generating station input-output curves of other methods are retained here; however, assumptions regarding certain variables remaining constant over wide ranges of system operation are not required, nor are "equivalent" loss functions involved. The development is new in the choice of variables used (the voltage phase angles) and in application of the Lagrangian multiplier method to the cost function while maintaining the identity of the individual loads.

In the sections following, the general procedure used in establishing the desired set of equations is outlined, followed by a discussion of possible sets of specified variables. Sets of equations are then developed for two different sets of specified variables. In the first case, the specified variables are the voltage magnitudes at the nodes (busses) in the system and the power of each of the elements representing the loads.

The node voltage relative phase angles are utilized as the controlling variables for which values are to be determined. In the second case for which equations are developed, the specified variables are the real and reactive power for each of the load elements and the magnitudes of the voltages at the generators. The controlling variables are the node voltage phase angles and the magnitudes of the voltages at the loads. In each case the resulting equations are non-linear involving products of variables and trigonometric functions of other variables. Because of these non-linearities, an iterative method of solution is necessary; the Newton-Raphson technique [24] is shown to be applicable to the problem, the relations required in the solution by this method are developed, and a suitable flow diagram for a digital computer for use in obtaining solutions to the set of non-linear equations is shown. The equations derived for the general case with the voltage phase angles as the controlling variables are then applied to a particular example problem. Results obtained using the MISTIC (Michigan State University Digital Computer) for carrying out the computations for this example for several specified load conditions are shown. Finally, effects on the system costs of variations in the phase angles while each load power is maintained at its specified value are investigated. Curves are shown which verify the existence of minimum production cost at the values of the phase angles determined by the method developed herein with the other parameters as specified.

2.2 A Summary of Steps Involved in Development of Equations

In brief form, the steps involved in this method of establishing a system of equations which upon solution determine a set of necessary conditions for economic operation of the power system are:

1. A power system network diagram is constructed in which the transmission system is represented by an equivalent n + 1 vertex network in which one vertex (the ground vertex)

is a reference and each other vertex corresponds to a bus at which either a load or a generating station is connected. A network junction at which neither a load nor a generating station is connected is included as a load vertex at which the load power is zero. The elements of the transmission system network are determined using the equivalent π representations of the transmission lines. The generators and loads of the power system are represented by corresponding elements on the network diagram.

2. The total production cost for the system is represented by a function designated as F_t. This F_t function is the sum of each of the cost per hour versus power output functions of the various generating stations.
3. The power functions of the elements representing the generating stations are expressed in terms of some desired set of interdependent variables.
4. A set of auxiliary equations are written, corresponding to specified load conditions.
5. The Lagrangian multiplier method is used to establish a system of equations, the solution of which determines values of the variables which correspond to necessary conditions for a minimum of F_t and which also satisfy the auxiliary equations.

2.3 Specified Variables

Factors to be considered in a study of sets of specified variables are: (1), the number of equations and number of unknowns for various possible sets of specified variables such that solutions are not made impossible due to excessive numbers of specified variables; (2), the topological aspects of the network graph which affect the allowable numbers and kinds of specified variables; (3), the directly related desirable practical requirements in actual power systems; and (4), the relative complexity of the resulting systems of equations which are to be solved. These factors are considered in this section. As an aid in referring to different types of variables, it is convenient to make the following definitions:

Possible Variables: The possible variables of a problem include all parameters which appear in the equations relating to the problem and which are not fixed by some previously specified set of conditions.

In the problem considered in this thesis, Possible Variables include real powers, reactive powers, voltage magnitudes, and voltage phase angles. Parameters which are fixed by some previously specified set of conditions are the admittances of the network elements representing the transmission lines of the power system.

Specified Variables: The specified variables are those variables of the set of Possible Variables for which values are assigned rather than determined by calculation.

To-be-determined Variables: The to-be-determined variables are those variables of the set of Possible Variables not included in the set of Specified Variables.

Control Variables: The control variables are those variables of the set of To-be-determined Variables which are evaluated so as to satisfy necessary conditions for minimizing the system production cost.

The equations involved in this problem (Equations (2.3.3) following) are of a form such that the real power, P, and reactive power Q, of the elements representing generators and loads are expressed as explicit functions of the voltage magnitudes and angles. Moreover, the equations cannot be solved directly for the voltage magnitudes and angles in terms of the P and Q functions. Because of these facts, and since the production costs are expressed explicitly in terms of the power functions, the Control Variables are selected from among the voltage magnitudes and angles.

Next, consider in detail the general $n + 1$ vertex network (representing the power system) and equations which relate the Possible Variables of certain elements of this network. The transmission lines of the power system are represented by their equivalent π networks and these are interconnected as the system is interconnected so as to form a basic n + 1 vertex network in which the vertices are numbered 1 through n + 1 as follows. The vertices which correspond to busses at which loads are connected are numbered 1, 2, . . . , a - 1 and are designated as load vertices; the vertices which correspond to busses at which generators are connected are numbered a, a + 1, . . ., n and are designated as generator vertices; the reference vertex (ground) is numbered n + 1. A network junction at which neither a load or a generating station is connected is represented as a load vertex and the corresponding load power (a Specified Variable) is specified as zero. Network elements representing loads are inserted between each load vertex and the reference and elements representing generators are inserted between each generator vertex and the reference. Node equations [25] for these elements representing the loads and generators can be written as

$$-\begin{bmatrix} I_1 \\ I_2 \\ \cdot \\ \cdot \\ \cdot \\ I_n \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \ldots & Y_{1n} \\ Y_{21} & Y_{22} \ldots & Y_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ Y_{n1} & Y_{n2} \ldots & Y_{nn} \end{bmatrix} \begin{bmatrix} E_1 \\ E_2 \\ \cdot \\ \cdot \\ \cdot \\ E_n \end{bmatrix} \qquad (2.3.1)$$

where E_k is the voltage of the generator or load element incident to vertex k, the element being oriented toward the reference vertex.

I_k is the current of the same element

Y_{kk} is the sum of the admittances of the network elements (other than the element representing the load or generator) incident to vertex k.

Y_{kj} is the negative of the sum of the admittances of the network elements incident to both vertex k and vertex j.

Then, since $P_k + jQ_k = E_k I_k^*$

where P_k it the real power function of element k

Q_k is the reactive power function of element k

and I_k^* is the conjugate of I_k,

equations for the real and reactive power of all the elements representing generators or loads can be written as follows:

$$
-\begin{bmatrix} P_1 + jQ_1 \\ P_2 + jQ_2 \\ \cdot \\ \cdot \\ \cdot \\ P_n + jQ_n \end{bmatrix} = \begin{bmatrix} E_1 & 0 & \cdot & \cdot & 0 \\ 0 & E_2 & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot \\ \cdot & & & \cdot & 0 \\ 0 & \cdot & \cdot & & 0E_n \end{bmatrix} \begin{bmatrix} Y_{11}^* & Y_{12}^* & \cdots & Y_{1n}^* \\ Y_{21}^* & \cdot & \cdots & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \cdot \\ \cdot & & & \\ Y_{n1}^* & & \cdots & Y_{nn}^* \end{bmatrix} \begin{bmatrix} E_1^* \\ E_2^* \\ \vdots \\ \\ E_n^* \end{bmatrix} \qquad (2.3.3)
$$

If equations (2.3.3) are separated into their real and imaginary parts there are then a total of 2n non-linear equations and the complete set of Possible Variables are the P's and Q's, each n in number, the voltage magnitudes, $|E|$'s, n in number, and the voltage phase angles, ϕ's, n - 1 in number, one being specified as a reference angle. The Y's, the admittances of the network elements representing the transmission system, are assumed to be previously specified and hence are not a part of the set of Possible Variables. The total number of Possible Variables is then 4n - 1. It is to be noted that because of the form of the equations, upon specification or calculation of all the voltage magnitudes and phase angles, each of the P's and Q's is then uniquely determined.

The Lagrangian multiplier method is herein used to determine values for certain variables such that necessary conditions for a minimum of the system production cost function are satisfied; this method allows determination of the values of these variables so as to satisfy simultaneously a set of auxiliary equations. In this case the auxiliary equations specify power functions of the load elements. Although criteria are not available for general systems of non-linear equations which allow one to specify which variables of a set of Possible Variables can be uniquely determined in terms of the remaining variables, or even that a solution will always be possible, some observations can be made in this regard for Equations (2.3.3) as follows:

(1) In any one equation of the set, the number of Specified Variables must be at least one less than the number of Possible Variables in that equation. In this regard, proper consideration must be given to the effect of any zero terms in the admittance matrix in essentially removing Possible Variables from the equation.

(2) In order to exclude possible multiple solutions after applying the Lagrangian multiplier method, the total number of variables for which values are still to be determined cannot exceed the total number of equations remaining.

Other factors which affect the placement of variables in the various categories are the factors related directly to the operation of the actual power system. In all power system operation, it is axiomatic that the generating system supply the power as required by the loads. In addition, it is often the case that the bus voltage magnitudes are regulated. With the condition of regulated voltage magnitudes in mind, the investigations of this thesis are centered about utilization of relative voltage phase angles as criteria for specification of generating station power outputs for minimum production cost. That is, the bus voltage phase angles relative to a reference are in the set of Control Variables. Correspondingly, two possible sets of Specified Variables are as follows:

(1) The voltage magnitude, $|E|$, at each vertex (bus) in the system and the real power, P, for each element representing a load.

(2) The voltage magnitude, $|E|$, at each generator vertex (bus) and both the real power P and reactive power Q for each element representing a load.

Consider further the immediately preceding set (1). As reference to equations (2.3.3) indicates, the Q values, n in number, are uniquely determined once all the node voltages are determined in both magnitude and phase. Use of this set of Specified Variables then requires the assumption that on the actual power system, reactive power capacity is such as to establish the computed values. However, with the voltage magnitudes in the set of Specified Variables and the voltage phase angles constituting the Control Variables, and since Q values are not specified, in this case only the n equations for the P's, the real parts of (2.3.3), need be examined in regard to numbers of variables and equations. In these n equations for the real powers, the To-be-determined Variables include the generator element P values, n-a+1 in number, and the phase angles of the bus voltages, n-1 in number, making a total of 2n-a To-be-determined Variables and n equations. Since n > a and hence 2n-a > n in all cases of interest (i.e. there is always more than one generator), there are always more unknowns than equations at this point. If the n-1 phase angles are determined such that necessary conditions for a minimum of the system production cost function are satisfied and so as to satisfy simultaneously the a-1 equations for the P's of the elements representing the loads, there remain n-a+1 equations and n-a+1 unknown variables, the P's of the elements representing the generators, and values for these P's are uniquely determined in terms of the now known voltage magnitudes and angles.

In certain systems, it is possible to specify additional variables and thereby reduce the number of equations to be solved; however this reduction in number of equations may be accompanied by an increased system production cost as compared with the condition in which all equations are considered, so should be considered carefully in any practical application. However, as an example, consider the voltage phase angles at the load vertices. If it is assumed that one of these angles is the reference angle for the system, there remain a-2 others. If these a-2 phase angles are added to the set of Specified Variables, the To-be-determined Variables are reduced from 2n-a to 2(n-a+1) and the number of unknowns is still greater than the number of equations as long as 2(n-a+1) > n. This expression can be simplified to n > 2(a-1). In words, then, if the total number of vertices (not including the reference vertex) is greater than twice the number of load vertices, the voltage phase angles at the load vertices can be specified in addition to the variables of set (1) and the number of unknowns is still greater than the number of equations so that, from this aspect at least, some choice exists as to the values of the remaining unknowns. In the example considered in Section 4.2, there are a total of 5 vertices of which two are load vertices, hence it is possible to include the phase angles at the load vertices in the set of Specified Variables.

Now consider set (2), a second possible set of Specified Variables, similarly. In this case all 2n equations (both real and imaginary parts) of (2.3.3) must be included, since both the P and the Q values for the elements representing loads are among the Specified Variables. The To-be-determined Variables include the generator element P values, n-a+1 in number, the generator element Q values, n-a+1 in number, the phase angles of the bus voltages, n-1 in number, and the magnitudes of the voltages at the load vertices, a-1 in number, making a total of 3n-a To-be-determined Variables. Again, since $n > a$ in all cases of interest, there are always more unknowns than equations at this point. If the phase angles, n-1 in number, and the voltages at the load vertices, a-1 in number, are determined such that necessary conditions for a minimum of the system operating cost function are satisfied and so as to simultaneously satisfy the 2(a-1) equations for the real and reactive power of the elements representing the loads, there remain 2(n-a+1) equations and the same number of unknowns so that all unknowns can be determined with this possible set of Specified Variables also.

A third consideration in selecting sets of Specified Variables is the relative complexity of the systems of equations which result when the Lagrangian multiplier method is applied. The equations which result under the conditions of the Specified Variables being those of Set (1) are developed in Section 2.4.1 following; those which result under the conditions of the Specified Variables being those of Set (2) are developed in Section 2.4.2. The equations corresponding to Set (1) are significantly less in number and of less complexity than those corresponding to Set (2).

2.4 Development of Equations

In this section sets of equations are developed, the solution of which determines values of the Control Variables corresponding to necessary conditions for a minimum of the function representing the system production cost. The two possible sets of Specified Variables discussed in Section 2.3 are considered separately.

2.4.1 Development of Equations, Case 1

Consider the case in which the set of Specified Variables consists of the voltage magnitude, $|E|$, at each bus in the system, and the real power, P, for each element which represents a load, and in which the Control Variables are the relative phase angles of the bus voltages. The power system is represented by the $n + 1$ vertex network discussed in Section 2.3. It is to be noted again that use of this set of Specified Variables requires the assumption that on the actual power system, the reactive power capacity is such as to establish the values for reactive power Q, resulting from solution of equations (2.3.3) with the node voltage magnitudes as specified and the phase angles as determined.

First, let the production cost at generating station x be represented by a function F_x as

$$F_x = f_x(P_x) \text{ for } a \leq x \leq n \qquad (2.4.1.1)$$

where F_x is the input to generating station x in dollars per hour and P_x is the power output of generating station x. Then, F_t, the total input to the system in dollars per hour, can be written as

$$F_t = \sum_{x=a}^{n} F_x = \sum_{x=a}^{n} f_x(P_x) \qquad (2.4.1.2)$$

With the voltage magnitude at each vertex in the set of Specified Variables, each P_x and hence F_t can be expressed as a function of the ϕ_x's $(x = 1, 2, \ldots, n)$ where ϕ_x is the relative phase angle of the voltage between vertex x and the reference vertex. The problem then is to determine the set of ϕ_x's which correspond to a minimum F_t subject to certain restrictions on the ϕ_x's. That is, since P_x is to be specified for the elements representing loads (where $1 \leq x \leq a - 1$) and since P_x is a function of the ϕ_x's, the ϕ_x's must satisfy a set of auxiliary equations. The method of Lagrangian multipliers can be applied to this problem as follows.

First, consider expressing the P_x functions for each of the generator and load elements in terms of the ϕ_x variables. Using the notation of Section 2.3, the P_x function for each of these elements can be written as

$$P_x = \tfrac{1}{2}(E_x I_x^* + E_x^* I_x) \qquad (2.4.1.3)$$

From the node equations (2.3.1),

$$I_x = - \sum_{y=1}^{n} Y_{xy} E_y. \qquad (2.4.1.4)$$

But $$E_y = |E_y| \epsilon^{j\phi_y}, \text{ where } |E_y| \text{ is specified,} \qquad (2.4.1.5)$$

and $$Y_{xy} = |Y_{xy}| \epsilon^{j\alpha_{xy}} \qquad (2.4.1.6)$$

so that $$I_x = - \sum_{y=1}^{n} |Y_{xy}|\, |E_y| \epsilon^{j(\phi_y + \alpha_{xy})} \qquad (2.4.1.7)$$

then, $$P_x = -\tfrac{1}{2}\Big[\sum_{y=1}^{n} |E_x|\, |E_y|\, |Y_{xy}| \epsilon^{j(\phi_x - \phi_y - \alpha_{xy})} + \sum_{y=1}^{n} |E_x|\, |E_y|\, |Y_{xy}| \epsilon^{-j(\phi_x - \phi_y - \alpha_{xy})}\Big] \qquad (2.4.1.8)$$

or, $$P_x = - |E_x| \sum_{y=1}^{n} |E_y|\, |Y_{xy}| \cos(\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.1.9)$$

for all $1 \leq x \leq n$.

The auxiliary equations can be written as

$$\psi_x = 0 \qquad 1 \leq x \leq a - 1 \tag{2.4.1.10}$$

where $\psi_x = P_{x_s} - P_x(\phi_1, \phi_2, \ldots, \phi_n)$ (2.4.1.11)

and P_{x_s} is specified. Substituting (2.4.1.9) in (2.4.1.11), ψ_x can be written as

$$\psi_x = P_{x_s} + |E_x| \sum_{y=1}^{n} |E_y|\,|Y_{xy}| \cos(\phi_x - \phi_y - \alpha_{xy}) \tag{2.4.1.12}$$

The Lagrangian function L can then be written as

$$L = F_t + \sum_{x=1}^{a-1} \lambda_x \psi_x \tag{2.4.1.13}$$

and at a minimum value of F_T, if it is assumed that ϕ_1 is the reference phase angle, the following equations must be satisfied.

$$\begin{aligned} \frac{\partial F_t}{\partial \phi_2} + \frac{\partial}{\partial \phi_2} \sum_{x=1}^{a-1} \lambda_x \psi_x &= 0 \\ &\vdots \\ \frac{\partial F_t}{\partial \phi_n} + \frac{\partial}{\partial \phi_n} \sum_{x=1}^{a-1} \lambda_x \psi_x &= 0 \end{aligned} \tag{2.4.1.14}$$

$$\begin{aligned} \psi_1 &= 0 \\ &\vdots \\ \psi_{a-1} &= 0 \end{aligned} \tag{2.4.1.15}$$

Note that (2.4.1.14) is a set of n - 1 equations and (2.4.1.15) is a set of a - 1 equations. In total there are n + a - 2 equations in the same number of unknowns, the unknowns being $\phi_2, \phi_3, \ldots, \phi_n$ and $\lambda_1, \lambda_2, \ldots, \lambda_{a-1}$.

In order to examine these equations in more detail, consider the following. Using (2.4.1.2) and (2.4.1.1),

$$\frac{\partial F_t}{\partial \phi_j} = \frac{\partial}{\partial \phi_j} \sum_{x=a}^{n} F_x = \sum_{x=a}^{n} \frac{dF_x}{dP_x} \frac{\partial P_x}{\partial \phi_j} \qquad (2.4.1.16)$$

since each F_x is a function only of its own P_x. There is then introduced into the equations another set of n-(a-1) = n-a+1 variables, the $\frac{dF_x}{dP_x}$'s where $a \leq x \leq n$. The evaluation of these variables is discussed following equation (2.4.1.28) of this section. Now, let $\frac{dF_x}{dP_x} \doteq F_x'$, then (2.4.1.16) is

$$\frac{\partial F_t}{\partial \phi_j} = \sum_{x=a}^{n} F_x' \frac{\partial P_x}{\partial \phi_j} \qquad (2.4.1.17)$$

Since λ_x is a constant, the other terms in (2.4.1.14) can be written as

$$\frac{\partial}{\partial \phi_j} \sum_{x=1}^{a-1} \lambda_x \Psi_x = \sum_{x=1}^{a-1} \lambda_x \frac{\partial \Psi_x}{\partial \phi_j} . \qquad (2.4.1.18)$$

Now consider the terms $\frac{\partial P_x}{\partial \phi_j}$, $2 \leq j \leq n$. Using P_x as in (2.4.1.9),

$$\left.\frac{\partial P_x}{\partial \phi_j}\right|_{j \neq x} = -|E_x|\,|E_j|\,|Y_{xj}| \sin(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.1.19)$$

and

$$\frac{\partial P_x}{\partial \phi_x} = |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y|\,|Y_{xy}| \sin(\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.1.20)$$

Also, using Ψ_x as in (2.4.1.12),

$$\left.\frac{\partial \Psi_x}{\partial \phi_j}\right|_{j \neq x} = |E_x|\,|E_j|\,|Y_{xj}| \sin(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.1.21)$$

and,

$$\frac{\partial \Psi_x}{\partial \phi_x} = -|E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y|\,|Y_{xy}| \sin(\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.1.22)$$

The j^{th} equation of set (2.4.1.14) is then

$$- \sum_{\substack{x = a \\ x \neq j}}^{n} F_x' |E_x| |E_j| |Y_{xj}| \sin(\phi_x - \phi_j - \alpha_{xj})$$

$$+ F_j' |E_j| \sum_{\substack{y = 1 \\ y \neq j}}^{n} |E_y| |Y_{jy}| \sin(\phi_j - \phi_y - \alpha_{jy})$$

$$+ \sum_{\substack{x = 1 \\ x \neq j}}^{a-1} \lambda_x |E_x| |E_j| |Y_{xj}| \sin(\phi_x - \phi_j - \alpha_{xj})$$

$$- \lambda_j |E_j| \sum_{\substack{y = 1 \\ y \neq j}}^{n} |E_y| |Y_{jy}| \sin(\phi_j - \phi_y - \alpha_{jy}) = 0 \qquad (2.4.1.23)$$

where the last term occurs only for $1 \leq j \leq a-1$.

Since α_{mn} is a constant in each case, (2.4.1.23) can be rewritten using the following

$$\sin(\phi_m - \phi_n - \alpha_{mn}) = \cos\alpha_{mn} \sin(\phi_m - \phi_n) - \sin\alpha_{mn} \cos(\phi_m - \phi_n) \qquad (2.4.1.24)$$

$$\cos(\phi_m - \phi_n - \alpha_{mn}) = \cos\alpha_{mn} \cos(\phi_m - \phi_n) + \sin\alpha_{mn} \sin(\phi_m - \phi_n) \qquad (2.4.1.25)$$

and

$$|Y_{mn}| \cos\alpha_{mn} = g_{mn} \qquad (2.4.1.26)$$

$$|Y_{mn}| \sin\alpha_{mn} = b_{mn} \qquad (2.4.1.27)$$

Then, (2.4.1.23), the j^{th} equation of set (2.4.1.14), can be written as

$$- \sum_{\substack{x = a \\ x \neq j}}^{n} F_x' |E_x| |E_j| [g_{xj} \sin(\phi_x - \phi_j) - b_{xj} \cos(\phi_x - \phi_j)]$$

$$+ F_j' |E_j| \sum_{\substack{y = 1 \\ y \neq j}}^{n} |E_y| [g_{jy} \sin(\phi_j - \phi_y) - b_{jy} \cos(\phi_j - \phi_y)]$$

$$+ \sum_{\substack{x=1 \\ x \neq j}}^{a-1} \lambda_x |E_x| |E_j| [g_{xj} \sin(\phi_x - \phi_j) - b_{xj} \cos(\phi_x - \phi_j)]$$

$$- \lambda_j |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy} \sin(\phi_j - \phi_y) - b_{jy} \cos(\phi_j - \phi_y)] = 0 \tag{2.4.1.28}$$

Another form results if substitutions are made as

$$\sin(\phi_m - \phi_n) = \sin\phi_m \cos\phi_n - \cos\phi_m \sin\phi_n \tag{2.4.1.29}$$

$$\cos(\phi_m - \phi_n) = \cos\phi_m \cos\phi_n + \sin\phi_m \sin\phi_n \tag{2.4.1.30}$$

Then, (2.4.1.28), the j^{th} equation of set (2.4.1.14) is

$$- \sum_{\substack{x=a \\ x \neq j}}^{n} F_x' |E_x| |E_j| [g_{xj} (\sin\phi_x \cos\phi_j - \cos\phi_x \sin\phi_j) - b_{xj}(\cos\phi_x \cos\phi_j + \sin\phi_x \sin\phi_j)]$$

$$+ F_j' |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy} (\sin\phi_j \cos\phi_y - \cos\phi_j \sin\phi_y) - b_{jy} (\cos\phi_j \cos\phi_y + \sin\phi_j \sin\phi_y)$$

$$+ \sum_{\substack{x=1 \\ x \neq j}}^{a-1} \lambda_x |E_x| |E_j| [g_{xj} (\sin\phi_x \cos\phi_j - \cos\phi_x \sin\phi_j) - b_{xj} (\cos\phi_x \cos\phi_j + \sin\phi_x \sin\phi_j)]$$

$$- \lambda_j |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy} (\sin\phi_j \cos\phi_y - \cos\phi_j \sin\phi_y) - b_{jy} (\cos\phi_j \cos\phi_y + \sin\phi_j \sin\phi_y)] = 0 \tag{2.4.1.31}$$

P_x and ψ_x can be written in similar forms. Considering P_x as in (2.4.1.9) and making use of (2.4.1.24), (2.4.1.25), (2.4.1.26), and (2.4.1.27), P_x can be written as:

$$P_x = - |E_x| \sum_{y=1}^{n} |E_y| [g_{xy} \cos(\phi_x - \phi_y) + b_{xy} \sin(\phi_x - \phi_y)] \tag{2.4.1.32}$$

or, using (2.4.1.29) and (2.4.1.30)

$$P_x = - |E_x| \sum_{y=1}^{n} |E_y| \left[g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y) \right] \qquad (2.4.1.33)$$

And, considering ψ_x as in (2.4.1.12) and making use of the same relations as used for P_x, ψ_x can be written as

$$\psi_x = P_{xs} + |E_x| \sum_{y=1}^{n} |E_y| \left[g_{xy}\cos(\phi_x - \phi_y) + b_{xy}\sin(\phi_x - \phi_y) \right] \qquad (2.4.1.34)$$

or,

$$\psi_x = P_{x_s} + |E_x| \sum_{y=1}^{n} |E_y| \left[g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y) \right] \qquad (2.4.1.35)$$

In certain cases of interest, it is possible to include additional variables in the set of Specified Variables. For example, as discussed in Section 2.3, it may be possible to specify ϕ_x for each load element $(1 \le x \le a-1)$. Then, in set (2.4.1.14) the number of equations and ϕ_j variables is reduced from (n-1) to (n-a+1). Under such conditions, (2.4.1.31) the j^{th} equation of set (2.4.1.14), can be written as, where $a \le j \le n$,

$$- \sum_{\substack{x=a \\ x \ne j}}^{n} F_x' |E_x| |E_j| \left[g_{xj}(\sin\phi_x \cos\phi_j - \cos\phi_x \sin\phi_j) - b_{xj}(\cos\phi_x \cos\phi_j + \sin\phi_x \sin\phi_j) \right]$$

$$+ F_j' |E_j| \sum_{y=1}^{a-1} \left[g_{jy}(E_{ry} \sin\phi_j - E_{iy}\cos\phi_j) - b_{jy}(E_{ry}\cos\phi_j + E_{iy}\sin\phi_j) \right] + F_j' |E_j| \sum_{\substack{y=a \\ y \ne j}}^{n} |E_y| \left[g_{jy}(\sin\phi_j \cos\phi_y - \cos\phi_j \sin\phi_y) \right.$$

$$-b_{jy}(\cos\phi_j \cos\phi_y + \sin\phi_j \sin\phi_y)]$$

$$+ \sum_{x=1}^{a-1} \lambda_x |E_j| [g_{xj}(E_{ix}\cos\phi_j - E_{rx}\sin\phi_j) \tag{2.4.1.36}$$

$$- b_{xy}(E_{rx}\cos\phi_j + E_{ix}\sin\phi_j) = 0$$

where
$$E_{rx} = \mathcal{R}e\, E_x \epsilon^{j\phi_x}$$
$$E_{ix} = \mathcal{I}_m\, E_x \epsilon^{j\phi_x}$$

Similarly, P_x, from (2.4.1.33), for $a \leq x \leq n$, if ϕ_j is specified for $1 \leq j \leq a-1$, is $P_x = -|E_x| \sum_{y=1}^{a-1} [g_{xy}(E_{ry}\cos\phi_x + E_{iy}\sin\phi_x)$

$$+ b_{xy}(E_{ry}\sin\phi_x - E_{iy}\cos\phi_x)]$$

$$- |E_x| \sum_{\substack{y=a \\ y \neq x}}^{n} |E_y| [g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y)$$

$$+ b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y)] - g_{xx}|E_x|^2$$

(2.4.1.37)

And Ψ_x. from (2.4.1.35), if ϕ_j is specified for $1 \leq j \leq a-1$, is

$$\Psi_x = P_{x_s} + g_{xx}|E_x|^2 + \sum_{\substack{y=1 \\ y \neq x}}^{a-1} [g_{xy}(E_{rx}E_{ry} + E_{ix}E_{iy})$$

$$+ b_{xy}(E_{ix}E_{ry} - E_{rx}E_{iy})] + \sum_{y=a}^{n} |E_y| [g_{xy}(E_{rx}\cos\phi_y + E_{ix}\sin\phi_y)$$

$$+ b_{xy}(E_{ix}\cos\phi_y - E_{rx}\sin\phi_y)] \tag{2.4.1.38}$$

Immediately preceding equation (2.4.1.17) it is noted that an additional set of n-a+1 variables, denoted as F_x' ($a \leq x \leq n$), had been introduced into the equations. Simultaneous evaluation of these variables, along with the others, is required. Consider $F_x = f_x(P_x)$, the input to station x in dollars per hour as a function of the output power of that station. In general, this function can be adequately approximated over various ranges by appropriately chosen polynomials. For any particular case, this function would be so chosen as to approximate measured values to a desired degree of accuracy. Here, for purposes of avoiding further complication, it is assumed that an equation of the second degree is satisfactory. That is, let

$$F_x = k_{0x} + k_{1x}P_x + k_{2x}P_x^2 \qquad (2.4.1.39)$$

and then

$$F_x' = \frac{dF_x}{dP_x} = k_{1x} + 2k_{2x}P_x \qquad (2.4.1.40)$$

It is to be noted that because of the use of the convention of orienting both **E** and **I** in the same direction for each generator element and each load element, $\mathbf{P_x}$ will be a negative number for elements representing generating stations (and a positive number for the elements representing loads). Then, in (2.4.1.39), if $k_0 > 0$, $k_1 < 0$, and $k_2 > 0$, $\mathbf{F_x}$ is positive but $\mathbf{F_x'}$ (as given by (2.4.1.40) is negative.

Another set of n-a+1 equations can then be added to the previous sets (2.4.1.14) and (2.4.1.15) making a total, in the general case, of 2n-1 equations in the same number of unknowns, or, if the phase angles at the load vertices are included in the set of Specified Variables, there are a total of 2n-a+1 equations in the same number of unknowns. Each of these added equations is of the form

$$F_x' - k_{1x} - 2k_{2x}P_x = 0 \qquad (2.4.1.41)$$

where $\mathbf{P_x}$ is as given by (2.4.1.33).

In summary then, in this case in which the set of Specified Variables consists of the voltage magnitudes at each bus in the power system and the power for each element representing a load, the equations to be solved are as follows:

$$\sum_{x=a}^{n} F_x' \frac{\partial P_x}{\partial \phi_2} + \sum_{x=1}^{a-1} \lambda_x \frac{\partial \psi_x}{\partial \phi_2} = 0$$

$$\vdots$$

$$\sum_{x=a}^{n} F_x' \frac{\partial P_x}{\partial \phi_n} + \sum_{x=1}^{a-1} \lambda_x \frac{\partial \psi_x}{\partial \phi_n} = 0 \qquad (2.4.1.42)$$

$$\psi_1 (\phi_2, \phi_3, \ldots, \phi_n) = 0$$

$$\vdots$$

$$\psi_{a-1} (\phi_2, \phi_3, \ldots, \phi_n) = 0$$

$$F_a' - k_{1a} - 2k_{2a} P_a = 0$$

$$\vdots$$

$$F_n' - k_{1n} - 2k_{2n} P_n = 0$$

There are n-1 equations of the first form, a-1 equations of the second form and n-a+1 equations of the third form. The unknowns are as follows:

ϕ_j	$2 \leq j \leq n$	a total of n-1
λ_x	$1 \leq x \leq a-1$	a total of a-1
F_y'	$a \leq y \leq n$	a total of n-a+1.

The j^{th} equation of the first form is indicated in detail in (2.4.1.31), or, if ϕ_j is specified for $1 \leq j \leq a-1$, in (2.4.1.36). A typical equation of the second form is indicated in (2.4.1.35), or, if ϕ_j is specified for $1 \leq j \leq a-1$, in (2.4.1.38). The form of P_x useful in the last set of equations is indicated in (2.4.1.33), or, if ϕ_j is specified for $1 \leq j \leq a-1$, in (2.4.1.37).

2.4.2 Development of Equations, Case 2

Another set of equations results if the Specified Variables are those of Set (2) of Section 2.3. The development of these equations is presented here, primarily as an illustration of an alternate approach; solution of the equations or further investigation of this case is not included.

This case is one in which the set of Specified Variables includes the real power, P, and reactive power, Q, for each element which represents a load, and the voltage magnitude, $|E|$, at each generator vertex and in which the set of Control Variables is made up of the relative phase angles of the bus voltages, except for the one designated as reference, and the voltage magnitudes at the load vertices.

Some of the relations of the previous section apply here as well, however those necessary to this development are repeated here for completeness. The relation which it is desired to minimize is again

$$F_t = \sum_{x=a}^{n} F_x = \sum_{x=a}^{n} f_x \, (P_x) \qquad (2.4.2.1)$$

where the symbols have the same meaning as previously.
Also, the P_x function can be expressed again as

$$P_x = \tfrac{1}{2} \, (E_x I_x^* + E_x^* I_x) \qquad (2.4.2.2)$$

or, as in (2.4.1.9), as

$$P_x = - |E_x| \sum_{y=1}^{n} |E_y| \, |Y_{xy}| \cos (\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.2.3)$$

Also, Q_x, the reactive power function for each of the generator and load elements can be written as

$$Q_x = \frac{-1}{2j} \, (E_x^* I_x - E_x I_x^*) \qquad (2.4.2.4)$$

and, using (2.4.1.4) through (2.4.1.7), this can be written as

$$Q_x = -|E_x| \sum_{y=1}^{n} |E_y| \, |Y_{xy}| \sin (\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.2.5)$$

The equations of constraint can be written as

$$\psi_{p_x} = 0 \quad \text{and} \quad \psi_{q_x} = 0, \qquad 1 \leq x \leq a-1 \tag{2.4.2.6}$$

where

$$\psi_{p_x} = P_{x_s} - P_x(\phi_1, \phi_2, \ldots, \phi_n) \tag{2.4.2.7}$$

and

$$\psi_{q_x} = Q_{x_s} - Q_x(\phi_1, \phi_2, \ldots, \phi_n) \tag{2.4.2.8}$$

where P_{x_s} and Q_{x_s} are specified.

Then, substituting (2.4.2.3) in (2.4.2.7) and (2.4.2.5) in (2.4.28), the last two relations can be written as

$$\psi_{p_x} = P_{x_s} + |E_x| \sum_{y=1}^{n} |E_y|\,|Y_{xy}| \cos(\phi_x - \phi_y - \alpha_{xy}) \tag{2.4.2.9}$$

and

$$\psi_{q_x} = Q_{x_s} + |E_x| \sum_{y=1}^{n} |E_y|\,|Y_{xy}| \sin(\phi_x - \phi_y - \alpha_{xy}) \tag{2.4.2.10}$$

The Lagrangian function for this case can be written as

$$L = F_t + \sum_{x=1}^{a-1} \lambda_{px} \psi_{px} + \sum_{x=1}^{a-1} \lambda_{qx} \psi_{qx} \tag{2.4.2.11}$$

Then, at an extreme value of F_t, the following equations must be satisfied

$$\frac{\partial F_t}{\partial \phi_2} + \sum_{x=1}^{a-1} \lambda_{px} \frac{\partial \psi_{p_x}}{\partial \phi_2} + \sum_{x=1}^{a-1} \lambda_{qx} \frac{\partial \psi_{q_x}}{\partial \phi_2} = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\frac{\partial F_t}{\partial \phi_n} + \sum_{x=1}^{a-1} \lambda_{px} \frac{\partial \psi_{p_x}}{\partial \phi_n} + \sum_{x=1}^{a-1} \lambda_{qx} \frac{\partial \psi_{q_x}}{\partial \phi_n} = 0$$

$$\frac{\partial F_t}{\partial |E_1|} + \sum_{x=1}^{a-1} \lambda_{p_x} \frac{\partial \psi_{p_x}}{\partial |E_1|} + \sum_{x=1}^{a-1} \lambda_{q_x} \frac{\partial \psi_{q_x}}{\partial |E_1|} = 0$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad (2.4.2.12)$$

$$\frac{\partial F_t}{\partial |E_{a-1}|} + \sum_{x=1}^{a-1} \lambda_{p_x} \frac{\partial \psi_{p_x}}{\partial |E_{a-1}|} + \sum_{x=1}^{a-1} \lambda_{q_x} \frac{\partial \psi_{q_x}}{\partial |E_{a-1}|} = 0$$

$$\psi_{p_1} = 0 \qquad\qquad \psi_{q_1} = 0$$

$$\vdots \qquad\qquad \vdots$$

$$\psi_{p_{a-1}} = 0 \qquad\qquad \psi_{q_{a-1}} = 0$$

In total, the above consists of 3a+n-4 equations in the same number of unknowns, the unknowns being ϕ_2, ϕ_3, . . . , ϕ_n,

$|E_1|$, $|E_2|$, . . . , $|E_{a-1}|$, λ_{p_1}, λ_{p_2}, . . . , $\lambda_{p_{a-1}}$ and λ_{q_i}, λ_{q_1}, . . . , $\lambda_{q_{a-1}}$.

The partial derivatives are as follows:

Using (2.4.2.1)

$$\frac{\partial F_t}{\partial \phi_j} = \frac{\partial}{\partial \phi_j} \sum_{x=a}^{n} F_x = \sum_{x=a}^{n} \frac{dF_x}{dP_x} \frac{\partial P_x}{\partial \phi_j} \qquad (2.4.2.13)$$

and

$$\frac{\partial F_t}{\partial |E_j|} = \sum_{x=a}^{n} \frac{dF_x}{dP_x} \frac{\partial P_x}{\partial |E_j|} \qquad (2.4.2.14)$$

As before, writing $\frac{dF_x}{dP_x} = F_x'$, these equations can be written as

$$\frac{\partial F_t}{\partial \phi_j} = \sum_{x=a}^{n} F_x' \frac{\partial P_x}{\partial \phi_j} \qquad (2.4.2.15)$$

$$\frac{\partial F_t}{\partial |E_j|} = \sum_{x=a}^{n} F_x' \frac{\partial P_x}{\partial |E_j|} \qquad (2.4.2.16)$$

Further relations required are

$$\left.\frac{\partial P_x}{\partial \phi_j}\right|_{j \neq x} = - |E_x| \; |E_j| \; |Y_{xj}| \; \sin(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.2.17)$$

$$\frac{\partial P_x}{\partial \phi_x} = |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \; |Y_{xy}| \; \sin(\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.2.18)$$

$$\left.\frac{\partial P_x}{\partial |E_j|}\right|_{j \neq x} = - |E_x| \; |Y_{xj}| \; \cos(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.2.19)$$

$$\left.\frac{\partial \psi_{p_x}}{\partial \phi_j}\right|_{j \neq x} = |E_x| \; |E_j| \; |Y_{xj}| \; \sin(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.2.20)$$

$$\frac{\partial \psi_{p_x}}{\partial \phi_x} = - |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \; |Y_{xy}| \; \sin(\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.2.21)$$

$$\left.\frac{\partial \psi_{q_x}}{\partial \phi_j}\right|_{j \neq x} = -|E_x| \; |E_j| \; |Y_{xy}| \; \cos(\phi_x - \phi_j - \alpha_{xy}) \qquad (2.4.2.22)$$

$$\frac{\partial \psi_{q_x}}{\partial \phi_x} = + |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \; |Y_{xy}| \; \cos(\phi_x - \phi_y - \alpha_{xy}) \qquad (2.4.2.23)$$

$$\left.\frac{\partial \psi_{p_x}}{\partial |E_j|}\right|_{j \neq x} = |E_x| \; |Y_{xj}| \; \cos(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.2.24)$$

$$\frac{\partial \psi_{p_x}}{\partial |E_x|} = \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \; |Y_{xy}| \; \cos(\phi_x - \phi_y - \alpha_{xy}) + 2 \; |E_x| \; |Y_{xx}| \cos \alpha_{xx} \qquad (2.4.2.25)$$

$$\left.\frac{\partial \psi_{q_x}}{\partial |E_j|}\right|_{j \neq x} = |E_x| \; |Y_{xj}| \; \sin(\phi_x - \phi_j - \alpha_{xj}) \qquad (2.4.2.26)$$

$$\frac{\partial \psi_{q_x}}{\partial |E_x|} = \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \; |Y_{xy}| \; \sin(\phi_x - \phi_y - \alpha_{xy}) - 2 \, |E_x| \, |Y_{xx}| \; \sin \alpha_{xx} \qquad (2.4.2.27)$$

Combining these relations as required, the j^{th} equation of the first form of set (2.4.2.12) is

$$- \sum_{\substack{x=a \\ x \neq j}}^{n} F_x' |E_x| |E_j| |Y_{xj}| \sin(\phi_x - \phi_j - \alpha_{xj}) + F_j' |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| |Y_{jy}| \sin(\phi_j - \phi_y - \alpha_{jy})$$

$$+ \sum_{\substack{x=1 \\ x \neq j}}^{a-1} \lambda_{px} |E_x| |E_j| |Y_{xj}| \sin(\phi_x - \phi_j - \alpha_{xj}) - \lambda_{pj} |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| |Y_{jy}| \sin(\phi_j - \phi_y - \alpha_{jy})$$

$$- \sum_{\substack{x=1 \\ x \neq j}}^{a-1} \lambda_{qx} |E_x| |E_j| |Y_{xj}| \cos(\phi_x - \phi_j - \alpha_{xj}) - \lambda_{qj} |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| |Y_{jy}| \cos(\phi_j - \phi_y - \alpha_{jy}) = 0$$

(2.4.2.28)

where λ_{pj} and λ_{qj} occur only for $1 \leq j \leq a-1$.

Similarly, the m^{th} equation of the second form of set (2.4.2.12) is

$$- \sum_{x=a}^{n} F_x' |E_x| |Y_{xm}| \cos(\phi_x - \phi_m - \alpha_{xm}) + \sum_{\substack{x=1 \\ x \neq m}}^{a-1} \lambda_{px} |E_x| |Y_{xm}| \cos(\phi_x - \phi_m - \alpha_{xm})$$

$$+ \lambda_{pm} \sum_{\substack{y=1 \\ y \neq m}}^{n} |E_y| |Y_{my}| \cos(\phi_m - \phi_y - \alpha_{xy}) + \lambda_{pm} 2 |E_m| |Y_{mm}| \cos \alpha_{mm}$$

$$+ \sum_{\substack{x=1 \\ x \neq m}}^{a-1} \lambda_{qx} |E_x| |Y_{xm}| \sin(\phi_x - \phi_m - \alpha_{xm}) + \lambda_{qm} \sum_{\substack{y=1 \\ y \neq m}}^{n} |E_y| |Y_{my}| \sin(\phi_m - \phi_y - \alpha_{my}) - \lambda_{qm} 2 |E_m| |Y_{mm}| \sin \alpha_{mm} = 0$$

(2.4.2.29)

These last two equations can be written in forms similar to those of Section (2.4.1) using (2.4.1.24) through (2.4.1.27) and (2.4.1.29) and (2.4.1.30).

Then, (2.4.2.28), the j^{th} equation of the first form of set (2.4.2.12), is

$$
\begin{aligned}
&- |E_j| \sum_{\substack{x=a \\ x \neq j}}^{n} F_x' |E_x| \left[g_{xj}(\sin \phi_x \cos \phi_j - \cos \phi_x \sin \phi_j) - b_{xj}(\cos \phi_x \cos \phi_j + \sin \phi_x \sin \phi_j) \right] \\
&+ F_j' |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| \left[g_{jy}(\sin \phi_j \cos \phi_y - \cos \phi_j \sin \phi_y) - b_{jy}(\cos \phi_j \cos \phi_y + \sin \phi_j \sin \phi_y) \right] \\
&+ |E_j| \sum_{\substack{x=1 \\ x \neq j}}^{a-1} \lambda_{p_x} |E_x| \left[g_{xj}(\sin \phi_x \cos \phi_j - \cos \phi_x \sin \phi_j) - b_{xj}(\cos \phi_x \cos \phi_j + \sin \phi_x \sin \phi_j) \right] \\
&- \lambda_{p_j} |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| \left[g_{jy}(\sin \phi_j \cos \phi_y - \cos \phi_j \sin \phi_y) - b_{jy}(\cos \phi_j \cos \phi_y + \sin \phi_j \sin \phi_y) \right] \\
&- |E_j| \sum_{\substack{x=1 \\ x \neq j}}^{a-1} \lambda_{q_x} |E_x| \left[g_{xj}(\cos \phi_x \cos \phi_j + \sin \phi_x \sin \phi_j) + b_{xj}(\sin \phi_x \cos \phi_j - \cos \phi_x \sin \phi_j) \right] \\
&+ \lambda_{q_j} |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| \left[g_{jy}(\cos \phi_j \cos \phi_y + \sin \phi_j \sin \phi_y) + b_{jy}(\sin \phi_j \cos \phi_y - \cos \phi_j \sin \phi_y) \right] = 0
\end{aligned}
\tag{2.4.2.30}
$$

and (2.4.2.29), the m^{th} equation of the second form of set (2.4.2.12), is

$$- \sum_{x=a}^{n} F_x' |E_x| [g_{xm}(\cos\phi_x \cos\phi_m + \sin\phi_x \sin\phi_m) + b_{xm}(\sin\phi_x \cos\phi_m - \cos\phi_x \sin\phi_m)]$$

$$+ \sum_{\substack{x=1 \\ x \neq m}}^{a-1} \lambda_{p_x} |E_x| [g_{xm}(\cos\phi_x \cos\phi_m + \sin\phi_x \sin\phi_m) + b_{xm}(\sin\phi_x \cos\phi_m - \cos\phi_x \sin\phi_m)]$$

$$+ \lambda_{p_m} \sum_{\substack{y=1 \\ y \neq m}}^{n} |E_y| [g_{my}(\cos\phi_m \cos\phi_y + \sin\phi_m \sin\phi_y) + b_{my}(\sin\phi_m \cos\phi_y - \cos\phi_m \sin\phi_y)]$$

$$+ \sum_{\substack{x=1 \\ x \neq m}}^{a-1} \lambda_{q_x} |E_x| [g_{xm}(\sin\phi_x \cos\phi_m - \cos\phi_x \sin\phi_m) - b_{xm}(\cos\phi_x \cos\phi_m + \sin\phi_x \sin\phi_m)]$$

$$+ \lambda_{q_m} \sum_{\substack{y=1 \\ y \neq m}}^{n} |E_y| [g_{my}(\sin\phi_m \cos\phi_y - \cos\phi_m \sin\phi_y) - b_{my}(\cos\phi_m \cos\phi_y + \sin\phi_m \sin\phi_y)]$$

$$+ 2 |E_m| [\lambda_{p_m} g_{mm} - \lambda_{q_m} b_{mm}] = 0 \qquad (2.4.2.31)$$

P_x, Ψ_{p_x}, Q_x, Ψ_{q_x} can all be written in forms similar to the last two equations. That is,

$$P_x = - |E_x| \sum_{y=1}^{n} |E_y| [g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y)] \qquad (2.4.2.32)$$

$$Q_x = - |E_x| \sum_{y=1}^{n} |E_y| [g_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y) - b_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y)] \qquad (2.4.2.33)$$

$$\Psi_{p_x} = P_{x_s} + |E_x| \sum_{y=1}^{n} |E_y| [g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y)] \qquad (2.4.2.34)$$

$$\Psi_{qx} = Q_{x_s} + |E_x| \sum_{y=1}^{n} |E_y| \, [g_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y)$$

$$- b_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y)\,] \quad (2.4.2.35)$$

It is noted that the equations of this section, as compared with those of Section (2.4.1), are of equal or greater complexity and are greater in number. Since the formulation offers no distinct advantage in terms of requirements on the power system or otherwise, and, as a practical matter it is usually necessary that the magnitude of the voltages at the loads be at a specified level and not varied over wide ranges as could be required here, the remainder of this thesis is devoted to further investigation of the developments of Section (2.4.1).

2.5 Conditions Sufficient for Minimum of F_t

Returning to the conditions considered in Section 2.4.1, in this section sets of conditions sufficient for a minimum of the production cost function, F_t, are determined. First, this function, F_t, is expanded by Taylor's theorem [25] about the point at which the necessary conditions for a minimum are met. In terms of the node voltage phase angles, which are subject to the restrictions of equations (2.4.1.15), this expansion can be written as follows:

$$F_t(\phi_1 + \Delta\phi_1, \phi_2 + \Delta\phi_2, \ldots, \phi_n + \Delta\phi_n) = F_t(\phi_1, \phi_2, \ldots, \phi_n)$$

$$+ dF_t(\phi_1, \phi_2, \ldots, \phi_n) + \frac{1}{2} d^2F_t(\phi_1, \phi_2, \ldots, \phi_n)$$

$$+ \ldots + \frac{1}{n!} d^nF_t(\phi_1, \phi_2, \ldots, \phi_n) + R_n \quad (2.5.1)$$

At the point at which the necessary conditions for a minimum are met,

$$dF_t(\phi_1, \phi_2, \ldots, \phi_n) = 0 \quad (2.5.2)$$

and, with the remainder of the third order,

$$F_t(\phi_1 + \Delta\phi_1, \phi_2 + \Delta\phi_2, \ldots, \phi_n + \Delta\phi_n) - F_t(\phi_1, \phi_2, \ldots, \phi_n)$$

$$= \frac{1}{2} d^2F_t(\phi_1, \phi_2, \ldots, \phi_n) + \epsilon \rho^2 \qquad (2.5.3)$$

where $\rho^2 = (d\phi_1)^2 + (d\phi_2)^2 + \ldots + (d\phi_n)^2$ and ϵ tends to zero with ρ. Hence the sign of the functional difference is determined essentially by the sign of d^2F_t and the problem is to determine conditions such that d^2F_t is positive in a neighborhood of the point at which the necessary conditions for a minimum are met. In order to determine an expression for $d^2F_t(\phi_1, \phi_2, \ldots, \phi_n)$, F_t, from (2.4.1.2), is written as

$$F_t = \sum_{x=a}^{n} F_x = \sum_{x=a}^{n} f_x(P_x) \qquad (2.5.4)$$

then, in terms of the dependent variable P_x,

$$dF_t = \sum_{x=a}^{n} \frac{dF_x}{dP_x} dP_x = \sum_{x=a}^{n} F_x' \ dP_x \qquad (2.5.5)$$

and [27],

$$d^2F_t = \sum_{x=a}^{n} F_x' \ d^2P_x + \sum_{x=a}^{n} F_x''(dP_x)^2 \qquad (2.5.6)$$

where

$$F_x'' = \frac{d^2F_x}{dP_x^2}$$

Under the assumption that each F_x can be adequately represented by a polynomial of the second degree as

$$F_x = k_0 + k_1 P_x + k_2 P_x^2 \qquad (2.5.7)$$

where k_0, $k_2 > 0$, $k_1 < 0$ and $P_x < 0$, then

$$F_x' = k_1 + 2k_2P_x < 0 \qquad (2.5.8)$$

and

$$F_x'' = 2k_2 > 0 \qquad (2.5.9)$$

The second summation of (2.5.6) is then always positive since each term is a product of two positive terms, one being a squared term.

In order to determine the signs of the terms in the first summation of (2.5.6), it is necessary to obtain an expression for d^2P_x. Symbolically, in terms of the node voltage phase angles,

$$P_x = f(\phi_1, \phi_2, \ldots, \phi_n) \tag{2.5.10}$$

Then,

$$dP_x = \sum_{k=1}^{n} \frac{\partial P_x}{\partial \phi_k} d\phi_k \tag{2.5.11}$$

and

$$d^2P_x = \sum_{i=1}^{n} \frac{\partial^2 P_x}{\partial \phi_i^2} (d\phi_i)^2 + \sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} \frac{\partial^2 P_x}{\partial \phi_i \partial \phi_j} d\phi_i \, d\phi_j \tag{2.5.12}$$

Now, to obtain the partial derivatives needed, using P_x as in (2.4.1.33), rewritten here for convenience,

$$P_x = -|E_x| \sum_{y=1}^{n} |E_y| \, [g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y)] \tag{2.5.13}$$

$$\left.\frac{\partial P_x}{\partial \phi_i}\right|_{i \neq x} = -|E_x| \, |E_i| \, [g_{xi}(-\cos\phi_x \sin\phi_i + \sin\phi_x \cos\phi_i) + b_{xi}(-\sin\phi_x \sin\phi_i - \cos\phi_x \cos\phi_i)] \tag{2.5.14}$$

$$\frac{\partial P_x}{\partial \phi_x} = -|E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \, [g_{xy}(-\sin\phi_x \cos\phi_y + \cos\phi_x \sin\phi_y) + b_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y)] \tag{2.5.15}$$

$$\left.\frac{\partial^2 P_x}{\partial \phi_i^2}\right|_{i \neq x} = -|E_x| \, |E_i| \, [g_{xi}(-\cos\phi_x \cos\phi_i - \sin\phi_x \sin\phi_i) + b_{xi}(-\sin\phi_x \cos\phi_i + \cos\phi_x \sin\phi_i)] \tag{2.5.16}$$

$$\frac{\partial^2 P_x}{\partial \phi_x^2} = - |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \left[g_{xy}(-\cos\phi_x \cos\phi_y - \sin\phi_x \sin\phi_y) + b_{xy}(-\sin\phi_x \cos\phi_y + \cos\phi_x \sin\phi_y) \right] \quad (2.5.17)$$

$$\frac{\partial^2 P_x}{\partial \phi_z \partial \phi_j} = 0 \qquad \begin{matrix} z \neq j \\ z \neq x \end{matrix} \quad (2.5.18)$$

$$\frac{\partial^2 P_x}{\partial \phi_i \partial \phi_x} = \frac{\partial^2 P_x}{\partial \phi_x \partial \phi_i} = - |E_x| \, |E_i| \left[g_{xi}(\sin\phi_x \sin\phi_i + \cos\phi_k \cos\phi_i) + b_{xi}(-\cos\phi_x \sin\phi_i + \sin\phi_x \cos\phi_i) \right] \quad (2.5.19)$$

Then, substituting these relations into (2.5.12), the result is

$$\begin{aligned} d^2 P_x = & \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_x| \, |E_y| \left[g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y) \right] (d\phi_y)^2 \\ & + |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \left[g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y) \right] (d\phi_x)^2 \\ & - \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_x| \, |E_y| \left[g_{xy}(\sin\phi_x \sin\phi_y + \cos\phi_x \cos\phi_y) + b_{xy}(-\cos\phi_x \sin\phi_y + \sin\phi_x \cos\phi_y) \right] (d\phi_x)(d\phi_y) \\ & - \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_x| \, |E_y| \left[g_{xy}(\sin\phi_x \sin\phi_y + \cos\phi_x \cos\phi_y) + b_{xy}(-\cos\phi_x \sin\phi_y + \sin\phi_x \cos\phi_y) \right] (d\phi_y)(d\phi_x) \end{aligned} \quad (2.5.20)$$

or

$$d^2 P_x = |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \left[g_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y) + b_{xy}(\sin\phi_x \cos\phi_y - \cos\phi_x \sin\phi_y) \right] (d\phi_x - d\phi_y)^2 \quad (2.5.21)$$

or, in another form,

$$d^2P_x = |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y| \, [g_{xy}\cos(\phi_x - \phi_y) + b_{xy}\sin(\phi_x - \phi_y)](d\phi_x - d\phi_y)^2 \qquad (2.5.22)$$

In order that d^2F_t (as given by (2.5.6)) be positive, a sufficient condition is that for $a \leq x \leq n$, d^2P_x (as given by 2.5.22)) be negative for all $d\phi_i$, $i = 1, 2, \ldots, n$, since in the expression for d^2F_t, (2.5.6), d^2P_x is multiplied by a negative quantity, F_x' and, as is noted immediately following (2.5.9), the second summation of (2.5.6) is always positive. The expression for d^2P_x as given by (2.5.22) is a quadratic form which can be written as

$$d^2P_x = a_{11_x}(d\phi_1)^2 + a_{22_x}(d\phi_2)^2 + \ldots + a_{xx_x}(d\phi_x)^2 + \ldots +$$

$$a_{nn_x}(d\phi_n)^2 + 2a_{12_x}d\phi_1 d\phi_2 + 2a_{13_x}d\phi_1 d\phi_3 + \ldots +$$

$$2a_{1x_x}d\phi_1 d\phi_x + \ldots + 2a_{1n_x}d\phi_1 d\phi_n + 2a_{23_x}d\phi_2 d\phi_3 +$$

$$\ldots + 2a_{2x_x}d\phi_2 d\phi_x + \ldots + 2a_{2n_x}d\phi_2 d\phi_n + \ldots +$$

$$2a_{n-1,n_x}d\phi_{n-1}\,d\phi_n \qquad (2.5.23)$$

where

$$a_{ii_x}\Big|_{i \neq x} = |E_x|\,|E_i|\,[g_{xi}\cos(\phi_x - \phi_i) + b_{xi}\sin(\phi_x - \phi_i)]$$

$$a_{xx_x} = |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y|\,[g_{xy}\cos(\phi_x - \phi_y) + b_{xy}\sin(\phi_x - \phi_y)]$$

$$a_{ij_x} = 0 \qquad i \neq x,\ j \neq x$$

and

$$a_{ix_x} = -\,|E_x|\,|E_i|\,[g_{xi}\cos(\phi_x - \phi_i) + b_{xi}(\sin\phi_x - \phi_i)]$$

A quadratic form which is negative for all values of the variables, as is desired here, is said to be negative definite. A necessary and sufficient condition for (2.5.23) to be a negative definite quadratic form is that [29]

$$a_{11_x} < 0, \quad \begin{vmatrix} a_{11_x} & a_{12_x} \\ a_{21_x} & a_{22_x} \end{vmatrix} > 0 \quad \begin{vmatrix} a_{11_x} & a_{12_x} & a_{13_x} \\ a_{21_x} & a_{22_x} & a_{23_x} \\ a_{31_x} & a_{32_x} & a_{33_x} \end{vmatrix} < 0, \text{ etc.} \qquad (2.5.24)$$

Where

$$a_{ij_x} = a_{ji_x}$$

Hence, a set of conditions which are sufficient for a minimum of F_t at the point at which the necessary conditions are met, with F_x' given as in (2.5.8) and F_x'' as in (2.5.9), are that (2.5.24) be satisfied at this point. It is to be noted that these conditions, while sufficient, are considerably stronger than is necessary; the conditions are such as to make each term in the first summation of (2.5.6) be positive, while the actual requirement is rather that the sum be positive. This sum can be written as in (2.5.25) below, using (2.5.23) and combining the corresponding coefficients

$$\sum_{x=a}^{n} F_x' d^2 P_x = (F_a' a_{11_a} + F_{a+1}' a_{11_{a+1}} + \ldots + F_n' a_{11_n})(d\phi_1)^2$$

$$+ (F_a' a_{22a} + F_{a+1}' a_{22a+1} + \ldots + F_n' a_{22n})(d\phi_2)^2$$

$$+ \ldots + (F_n' a_{nn_a} + F_n' a_{nn_{a+1}} + \ldots + F_n' a_{nn_n})(d\phi_n)^2$$

$$+ 2(F_a' a_{12_a} + F_{a+1}' a_{12_{a+1}} + \ldots + F_n' a_{12n})\, d\phi_1 d\phi_2$$

$$+ 2(F_a' a_{13_a} + F_{a+1}' a_{13_{a+1}} + \ldots + F_n' a_{13n})\, d\phi_1 d\phi_3$$

$$+ \ldots + 2(F_a' a_{n-1,n_a} + F_{a+1}' a_{n-1,n_{a+1}} + \ldots + F_n' a_{n-1,n_n})$$

$$d\phi_{n-1}\, d\phi_n \qquad (2.5.25)$$

or,

$$\sum_{x=a}^{n} F_x' d^2 P_x = b_{11}(d\phi_1)^2 + b_{22}(d\phi_2)^2 + \ldots + b_{nn}(d\phi_n)^2 + 2b_{12}\, d\phi_1 d\phi_2$$

$$+ 2b_{13}\, d\phi_1 d\phi_3 + \ldots + 2b_{1n} d\phi_1 d\phi_n + 2b_{23}\, d\phi_2 d\phi_3 + \ldots + 2b_{2n}\, d\phi_2 d\phi_n$$

$$+ \ldots + 2b_{n-1,\,n}\, d\phi_{n-1}\, d\phi_n \qquad (2.5.26)$$

where

$$b_{ii} = (F_a' a_{ii_a} + F_{a+1}' a_{ii_{a+1}} + \ldots + F_n' a_{ii_n})$$

$$b_{ij} = b_{ji} = (F_a' a_{ij_a} + F_{a+1}' a_{ij_{a+1}} + \ldots + F_n' a_{ij_n})$$

The requirement is now that (2.5.26) be a positive definite quadratic form which will be true if [29]

$$b_{11} > 0, \quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix} > 0, \quad \begin{vmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{vmatrix} > 0, \text{ etc.} \qquad (2.5.27)$$

Hence a second set of conditions which if satisfied are sufficient to insure a minimum of F_t at the point at which the necessary conditions are met are that (2.5.27) be satisfied at this point. These conditions are somewhat less strict than those given by (2.5.24), however, there are a number of further calculations required in determining the coefficients of the quadratic form in this second case. In a practical application of this method of determining conditions for economic operation of a power system, it is likely that prior results on an operating system along with variations in generating station power outputs, if necessary, would serve to assure that the determined operating condition is a minimum as readily as would determination of the computations specified by (2.5.24) or (2.5.27).

CHAPTER III

A METHOD OF SOLUTION OF THE EQUATIONS; USE OF A DIGITAL COMPUTER

3.1 Introduction

The equations to be solved, represented symbolically in (2.4.1.42), are non-linear involving products of variables and trigonometric functions of other variables. There are several important considerations relative to obtaining solutions to such a system of equations. These considerations include: (1), conditions as to whether or not a solution exists; (2), choice of iterative technique of solution; (3), conditions on convergence of the iterative technique; (4), determination of a set of initial approximations; and (5), possibilities of multiple solutions. These considerations are examined in order in the following.

Under the assumption that the function representing the system production cost, F_t, has a minimum, the existence of a solution to the equations (2.4.1.42) is assured since by the Lagrangian multiplier rule [37] these equations, (2.4.1.42), must be satisfied at any minimum value of F_t (more generally, at any extreme value). The choice of a general technique of solution lies between a Seidel-type method in which each equation is solved for one of the unknowns in terms of functions of the variables involved in the equation and a method of functional iteration which involves partial derivatives of the equations considered as functions of the variables. Since these equations involve trigonometric functions of some of the unknowns, solution for these unknowns in terms of the others is not readily possible, hence a method of functional iteration, the Newton-Raphson method [24], was selected. This method consists of the following steps: (1), each equation, considered as a function of the variables involved, is expanded in a truncated Taylor's series in which the second and higher order terms are neglected; (2), using a set of initial approximations to the desired values of the variables, the set of linear equations in incremental values for the variables resulting from the Taylor's series expansions are solved; (3), the incremental values so found are added to the initial approximations to obtain a new set of approximations and the process is repeated until successively computed values of the variables differ by less than some selected precision index. The iteration converges, provided primarily that the initial approximations are sufficiently close to the desired values. However, it is not possible to define the limits on the region of n-space in which the initial approximation must be located so as to guarantee convergence. Faced with such a condition, a reasonable approach is to select the set of initial approximations based on physical aspects of the problem such that engineering judgment indicates that the initial approximations are sufficiently close to the desired values. This point is discussed further with respect to this particular problem in Section 3.2, following development of the required system of linear equations. In regard to the possibilities of multiple solutions similar considerations apply. That is, criteria are not available which allow one to state whether or not a general system of non-linear equations possesses a unique solution or multiple solutions. Directly related is the accompanying problem that if multiple solutions do exist, how is one assured that a solution obtained (by any method) is the desired solution? One method of obtaining such assurance in the problem of this thesis is to require satisfaction of a set of sufficient conditions for minimization of the production cost function as well as satisfaction of the necessary conditions. Development of a set of sufficient conditions is the subject of Section 2.5.

It is interesting to note that solutions to the so-called power flow problem of electric power systems which is directly concerned with obtaining a solution to a system of non-linear equations (Equations 2.3.3 of this thesis, in fact) in which certain variables are specified and others are to be found, repeatedly have been determined and the methods discussed, in the literature, without, as far as this writer can determine, criteria which specify allowable regions for a set of initial approximations or which exclude possible solutions other than the desired one. Further reference is made to this point in Section 5.2.

Before investigating the equations of Section 2.4 it is desirable to consider the Newton-Raphson technique for the case of two equations in two unknowns. Let the equations be written in functional form as

$$f(x, y) = 0 \qquad (3.1.1)$$

$$g(x, y) = 0 \qquad (3.1.2)$$

Assume that values $x_0 + \Delta x$ and $y_0 + \Delta y$ satisfy these equations, where x_0 and y_0 are a set of initial approximations. Then the functions $f(x, y)$ and $g(x, y)$ can be expanded using Taylor's Theorem as

$$f(x_0 + \Delta x, y_0 + \Delta y) = 0 = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)\,\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + 0\,(\Delta^2) \qquad (3.1.3)$$

$$g(x_0 + \Delta x, y_0 + \Delta y) = 0 = g(x_0, y_0) + \frac{\partial g}{\partial x}(x_0, y_0)\Delta x + \frac{\partial g}{\partial y}(x_0, y_0)\Delta y + 0\,(\Delta^2) \qquad (3.1.4)$$

where $0(\Delta^2)$ indicates the higher order terms in the series. If these higher order terms are neglected, the resulting linear equations may be solved for Δx and Δy. Then a second set of approximations to the desired values can be found as

$$x_1 = x_0 + \Delta x \qquad (3.1.5)$$

$$y_1 = y_0 + \Delta y \qquad (3.1.6)$$

and the process repeated until successive values of x and y differ by less than some prescribed precision index.

3.2 Equations for this Problem Using the Newton-Raphson Technique

Referring to (2.4.1.31), (2.4.1.35), (2.4.1.41), and (2.4.1.33), upon application of the Newton-Raphson technique, the linear equations in the incremental values of the variables are as follows, where each partial derivative and the function is to be evaluated at the values corresponding to the set of initial approximations. Consider (2.4.1.31) first and let this be represented as follows.

$$f_j(\phi_2, \phi_3, \ldots, \phi_n, F_a', F_{a+1}', \ldots, F_n', \lambda_1, \lambda_2, \ldots, \lambda_{a-1}) = 0 \qquad (3.2.1)$$

The corresponding linear equation in the incremental values is

$$\sum_{k=2}^{n} \frac{\partial f_j}{\partial \phi_k} \Delta\phi_k + \sum_{r=a}^{r=n} \frac{\partial f_j}{\partial F_r'} \Delta F_r' + \sum_{m=1}^{m=a-1} \frac{\partial f_j}{\partial \lambda_m} \Delta \lambda_m + f_j = 0 \qquad (3.2.2)$$

The individual terms in this equation are

$$\begin{aligned} \left.\frac{\partial f_j}{\partial \phi_k}\right|_{k \neq j} = & -F_k' |E_k| \, |E_j| \, [g_{kj}(\cos\phi_k \cos\phi_j + \sin\phi_k \sin\phi_j) \\ & - b_{kj} (\cos\phi, \sin\phi_j - \sin\phi_k \cos\phi_j)] \\ & - F_j' |E_j| \, |E_k| \, [g_{jk} (\sin\phi_j \sin\phi_k + \cos\phi_j \cos\phi_k) \\ & + b_{jk}(\sin\phi_j \cos\phi_k - \cos\phi_j \sin\phi_k)] \\ & + \lambda_k |E_k| \, |E_j| \, [g_{kj}(\cos\phi_k \cos\phi_j + \sin\phi_k \sin\phi_j) \\ & - b_{kj} (\cos\phi_k \sin\phi_j - \sin\phi_k \cos\phi_j)] \\ & + \lambda_j |E_j| \, |E_k| \, [g_{jk} (\sin\phi_j \sin\phi_k + \cos\phi_j \cos\phi_k) \\ & + b_{jk} (\sin\phi_j \cos\phi_k - \cos\phi_j \sin\phi_k)] \end{aligned} \qquad (3.2.3)$$

$$\frac{\partial f_j}{\partial \phi_j} = \sum_{\substack{x=a \\ x \neq j}}^{n} F_x' |E_x| |E_j| [g_{xj}(\sin\phi_x \sin\phi_j + \cos\phi_x \cos\phi_j) + b_{xj}(\sin\phi_x \cos\phi_j - \cos\phi_x \sin\phi_j)]$$

$$+ F_j' |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy}(\cos\phi_j \cos\phi_y + \sin\phi_j \sin\phi_y) - b_{jy}(\cos\phi_j \sin\phi_y - \sin\phi_j \cos\phi_y)]$$

$$+ \sum_{x=1}^{a-1} \lambda_x |E_x| |E_j| [g_{xj}(\sin\phi_x \sin\phi_j + \cos\phi_x \cos\phi_j) + b_{xj}(\sin\phi_x \cos\phi_j - \cos\phi_x \sin\phi_j)]$$

$$- \lambda_j |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy}(\cos\phi_j \cos\phi_y - \sin\phi_j \sin\phi_y) - b_{jy}(\cos\phi_j \sin\phi_y - \sin\phi_j \cos\phi_y)] \quad (3.2.4)$$

$$\left.\frac{\partial f_j}{\partial F_r'}\right|_{r \neq j} = -|E_r| |E_j| [g_{rj}(\sin\phi_r \cos\phi_j - \cos\phi_r \sin\phi_j) - b_{rj}(\cos\phi_r \cos\phi_j + \sin\phi_r \sin\phi_j)] \quad (3.2.5)$$

$$\frac{\partial f_j}{\partial F_j'} = |E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy}(\sin\phi_j \cos\phi_y - \cos\phi_j \sin\phi_y) - b_{jy}(\cos\phi_j \cos\phi_y + \sin\phi_j \sin\phi_y)] \quad (3.2.6)$$

$$\left.\frac{\partial f_j}{\partial \lambda_m}\right|_{m \neq j} = |E_m| |E_j| [g_{mj}(\sin\phi_m \cos\phi_j - \cos\phi_m \sin\phi_j) - b_{my}(\cos\phi_m \cos\phi_j + \sin\phi_m \sin\phi_j)] \quad (3.2.7)$$

and

$$\frac{\partial f_j}{\partial \lambda_j} = -|E_j| \sum_{\substack{y=1 \\ y \neq j}}^{n} |E_y| [g_{jy}(\sin\phi_j \cos\phi_y - \cos\phi_j \sin\phi_y) - b_{jy}(\cos\phi_j \cos\phi_y + \sin\phi_j \sin\phi_y)] \quad (3.2.8)$$

Now consider (2.4.1.35). If this is represented as

$$\psi_x = g_x(\phi_2, \phi_3, \ldots, \phi_n) \tag{3.2.9}$$

The corresponding linear equation in the incremental values is

$$\sum_{k=2}^{n} \frac{\partial g_x}{\partial \phi_k} \Delta\phi_k + g_x(\phi_2, \phi_3, \ldots, \phi_n) = 0 \tag{3.2.10}$$

Where

$$\left.\frac{\partial g_k}{\partial \phi_k}\right|_{k \neq x} = -|E_x|\,|E_k|\,[g_{xk}(\cos\phi_x \sin\phi_k + \sin\phi_x \cos\phi_k)$$
$$- b_{xk}(\sin\phi_x \sin\phi_k + \cos\phi_x \cos\phi_k)] \tag{3.2.11}$$

and

$$\frac{\partial g_x}{\partial \phi_x} = |E_x| \sum_{\substack{y=1 \\ y \neq x}}^{n} |E_y|\,[g_{xy}(\cos\phi_x \sin\phi_y - \sin\phi_x \cos\phi_y)$$
$$+ b_{xy}(\cos\phi_x \cos\phi_y + \sin\phi_x \sin\phi_y)] \tag{3.2.12}$$

Finally, let (2.4.1.41) be represented as

$$h_x(F_x', \phi_2, \phi_3, \ldots, \phi_n) = 0 \tag{3.2.13}$$

and the corresponding linear equation in the incremental values is

$$\frac{\partial h_x}{\partial F_r'} \Delta F_x' + \sum_{k=2}^{n} \frac{\partial h_x}{\partial \phi_k} \Delta\phi_k + h_x(F_x', \phi_2, \ldots, \phi_n) = 0 \tag{3.2.14}$$

Where

$$\frac{\partial h_x}{\partial F_x'} = 1 \tag{3.2.15}$$

$$\left.\frac{\partial h_x}{\partial \phi_k}\right|_{k \neq x} = |E_x|\,|E_k|\,[g_{xk}(\sin\phi_x \cos\phi_k - \cos\phi_x \sin\phi_k)$$
$$- b_{xk}(\sin\phi_x \sin\phi_k + \cos\phi_x \cos\phi_k)] \tag{3.2.16}$$

and

$$\frac{\partial h_x}{\partial \phi_x} = |E_x| \sum_{\substack{y = 1 \\ y \neq x}}^{n} |E_y| \, [g_{xy}(\cos \phi_x \sin \phi_y - \sin \phi_x \cos \phi_y)$$

$$+ b_{xy}(\cos \phi_x \cos \phi_y - \sin \phi_x \sin \phi_y)] \qquad (3.2.17)$$

In summary, the complete set of linear equations in the incremental values consists of equations of the form of (3.2.2), (3.2.10) and (3.2.14). These equations can be written as

$$\begin{aligned}
&\sum_{k=2}^{n} \frac{\partial f_2}{\partial \phi_k} \Delta\phi_k + \sum_{r=a}^{n} \frac{\partial f_2}{\partial F_r'} \Delta F_r' + \sum_{m=1}^{a-1} \frac{\partial f_2}{\partial \lambda_m} \Delta \lambda_m + f_2 = 0 \\
&\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\sum_{k=2}^{n} \frac{\partial f_n}{\partial \phi_k} \Delta\phi_k + \sum_{r=a}^{n} \frac{\partial f_n}{\partial F_r'} \Delta F_r' + \sum_{m=1}^{a-1} \frac{\partial f_n}{\partial \lambda_m} \Delta\lambda_m + f_n = 0 \\
&\sum_{k=2}^{n} \frac{\partial g_1}{\partial \phi_k} \Delta\phi_k + g_1 = 0 \\
&\qquad \vdots \qquad\qquad \vdots \\
&\sum_{k=2}^{n} \frac{\partial g_{a-1}}{\partial \phi_k} \Delta\phi_k + g_{a-1} = 0 \\
&\sum_{k=2}^{n} \frac{\partial h_a}{\partial \phi_k} \Delta\phi_k + \frac{\partial h_a}{\partial F_a'} \Delta F_a' + h_a = 0 \\
&\qquad \vdots \qquad\qquad \vdots \\
&\sum_{k=2}^{n} \frac{\partial h_n}{\partial \phi_k} \Delta\phi_k + \frac{\partial h_n}{\partial F_n'} \Delta F_n' + h_n = 0
\end{aligned} \qquad (3.2.18)$$

As discussed in the Introduction to this Chapter, one of the major problems in connection with solving a system of non-linear equations is that of determining a satisfactory set of initial approximations. For the problem of this thesis it is expected that for any particular operating system the voltage phase angles existing on the system would serve as a satisfactory set of initial approximations. That is, it would be assumed that the system is operating under conditions sufficiently close to those specified for economic dispatch by the criteria of this thesis that the bus voltage phase angles existing on the system could be used as a set of initial approximations to the desired values. As an alternative it may be possible in many cases simply to use zero degrees as the initial approximation for each phase angle, since it is often true that the variations in phase angles over a system are relatively small. As a matter of interest, the power flow problem is ordinarily solved starting with zero degrees as the initial approximation for each bus voltage phase angle. In the particular example considered in Chapter 4, the equations are of such a form that approximations to the angles can be made from other considerations, as explained there. For this example computations were carried out with angles determined both from these other considerations and with each angle assumed initially as zero degrees. The results are exactly the same.

In any case, once a set of initial approximations are decided on for the phase angles, initial approximations for the other variables can be found as follows. The last n-a+1 equations of set (2.4.1.42) can be solved explicitly for values for F_{a_0}', F_{a+1_0}', . . . , F_{n_0}', using the set of initial approximations for the phase angles. The zero subscripts are used to denote the initial approximations. Values for λ_{1_0}, λ_{2_0}, . . . , λ_{a-1_0} can be found in turn by solution of sets of a-1 equations from the first n-1 equations of set (2.4.1.42) using the initial approximations for the phase angles and for the F_x' parameters.

Another way of determining initial approximate values for the F_x' variables is as follows [23]. Each of the last n-a+1 equations of set (2.4.1.42) can be written in the form $F_x' - k_{1x} - k_{2x} P_x = 0$. Then if the approximation is made that the losses in the transmission system are some small percentage, say five per cent, of the total load, another equation can be written as

$$P_{a_0} + P_{a+1_0} + \ldots + P_{n_0} = 1.05 \, P_{load} \tag{3.2.19}$$

If, in addition, it is assumed that

$$F_{a_0}' = F_{a+1_0}' = \ldots = F_0' \tag{3.2.20}$$

which would be the condition for economic dispatch if transmission losses were negligible, the resulting set of (n-a+2) linear equations can be solved for F_0' for any specified total load.

Assuming that a satisfactory set of initial approximations has been determined, the Newton-Raphson iteration can then be carried out in the following sequence. Each of the coefficients in (3.2.18) is evaluated using the set of initial approximations and the resulting linear equations solved for the incremental values of the variables. The incremental value for each variable is then added to the corresponding initial approximation value, the coefficients in (3.2.18) re-evaluated using the new approximations, and the linear equations solved for new incremental values. This process is repeated until some convergence criteria is satisfied. In the example problem considered in Chapter 4, the iteration was terminated when differences between successively computed values of the variables were less than a specified precision index.

3.3 Application of a Digital Computer

Successive solution of a system of equations such as (3.2.18), each solution being followed by calculations for new values of the variables, is practical only if done with automatic computing equipment. A flow diagram for carrying out these calculations on a digital computer is shown in Figure 3.3.1. This is the basic flow diagram used in solving the equations for the particular example system considered in Chapter 4.

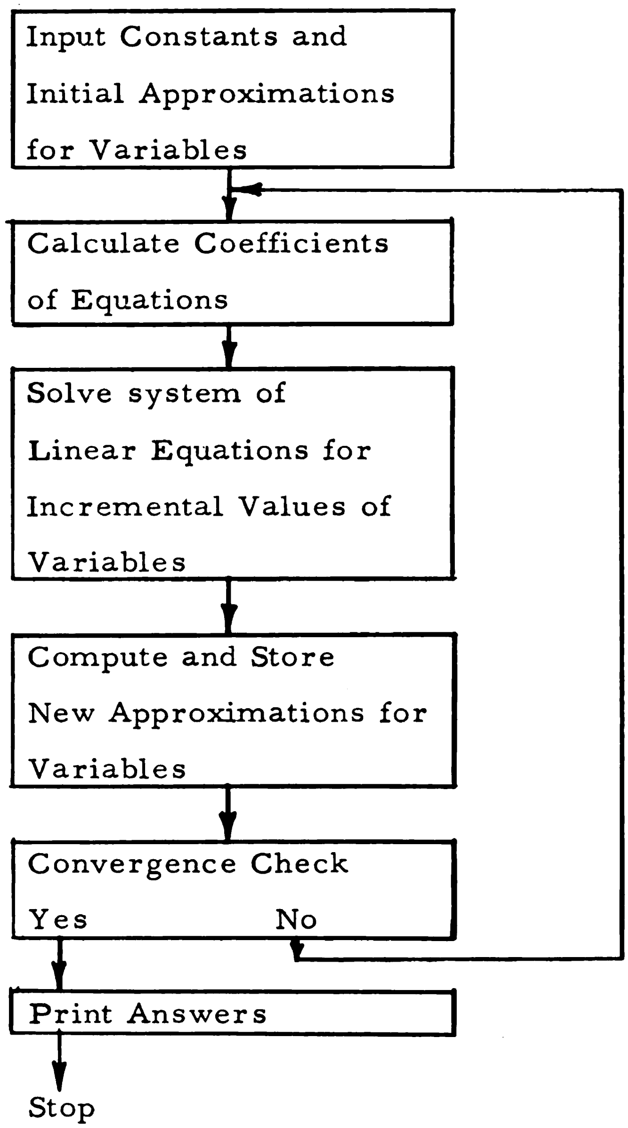

Fig. 3.3.1 Digital Computer Flow Diagram

It is to be noted that a major portion of the actual computations indicated in this flow diagram are those having to do with solution of the system of linear equations in the incremental values. It is expected that in any computing facility upon which solution of this problem were to be attempted, complete routines for obtaining solutions to such systems of equations would be readily available. The major sections remaining are those for calculating the coefficients and performing the convergence check with the calculation of coefficients involving routines which determine the sine and cosines of angles and the convergence check based on differences in the magnitudes of successively computed values of the variables being small.

CHAPTER IV

APPLICATION TO A SPECIFIC POWER SYSTEM

4.1 Introduction

In this section, in order to illustrate the computations required, the new method is applied to a particular power system and results are obtained under various operating conditions.

4.2 Power System Considered and Results Obtained

A network diagram of the power system chosen for the example is shown in Figure 4.2.1. Load flow study data for a base case were reported for this system by Dandeno [28]. The system was also considered in terms of a loss function and with different specified parameters by Sze, Garnett, and Calvert [23]. The system is one in which the total number of vertices (other than the reference vertex) is greater than twice the number of load vertices, hence, as discussed in Section 2.3, it is possible to include the phase angles of the voltages at the load vertices in the set of Specified Variables. If this is done, and equations (2.4.1.36), (2.4.1.38), (2.4.1.41) and (2.4.1.37) are used, the set of equations for this example are as follows:

$$- F_3'|E_3|[g_{32}(E_{r2}\sin\phi_3 - E_{i_2}\cos\phi_3) - b_{32}(E_{r_2}\cos\phi_3 + E_{i2}\sin\phi_3)]$$
$$+|E_3|\{\lambda_2[g_{23}(E_{i2}\cos\phi_3 - E_{r2}\sin\phi_3) - b_{23}(E_{r2}\cos\phi_3 + E_{i2}\sin\phi_3)]\} = 0$$
$$- F_4'|E_4|[g_{41}(E_{r1}\sin\phi_4 - E_{i1}\cos\phi_4) - b_{41}(E_{r1}\cos\phi_4 + E_{i1}\sin\phi_4)$$
$$+ g_{42}(E_{r_2}\sin\phi_4 - E_{i2}\cos\phi_4) - b_{42}(E_{r2}\cos\phi_4 + E_{i2}\sin\phi_4)]$$
$$+ |E_4|\{\lambda_1[g_{14}(E_{i1}\cos\phi_4 - E_{r1}\sin\phi_4) - b_{14}(E_{r1}\cos\phi_4 + E_{i1}\sin\phi_4)]$$
$$+ \lambda_2[g_{24}(E_{i2}\cos\phi_4 - E_{r2}\sin\phi_4) - b_{24}(E_{r2}\cos\phi_4 + E_{i2}\sin\phi_4)]\} = 0$$

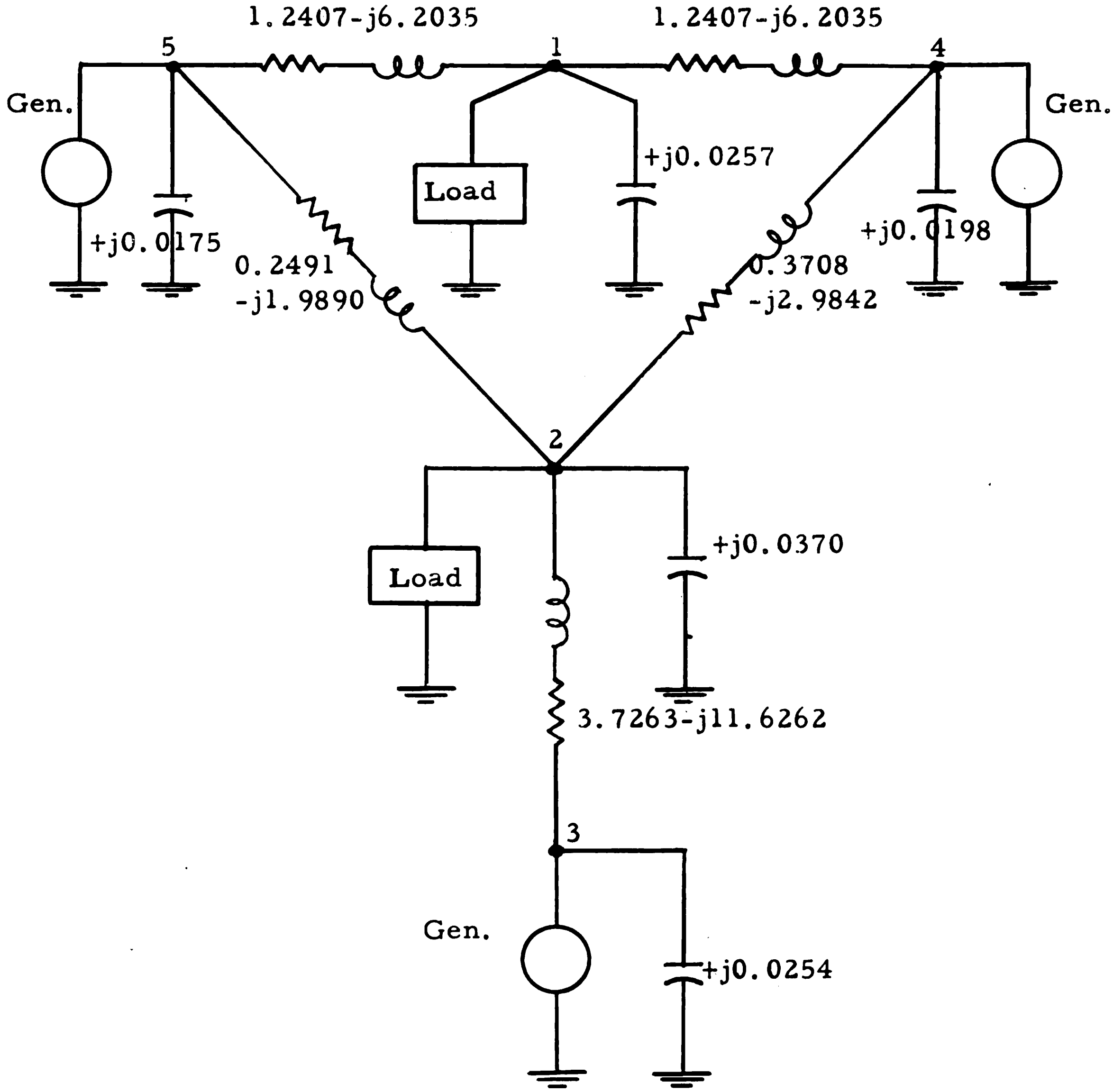

Fig. 4.2.1 Network Diagram of Example Power System with Admittances in per unit on 100 MVA., 110 KV. Base. (Data from Dandeno [28]).

$$- F_5' |E_5| [g_{51}(E_{r_1} \sin \phi_5 - E_{i_1} \cos \phi_5) - b_{51}(E_{r_1} \cos \phi_5 + E_{i_1} \sin \phi_5)$$
$$+ g_{52}(E_{r_2} \sin \phi_5 - E_{i_2} \cos \phi_5) - b_{52}(E_{r_2} \cos \phi_5 + E_{i_2} \sin \phi_5)]$$
$$+ |E_5| \{ \lambda_1 [g_{15}(E_{i_1} \cos \phi_5 - E_{r_1} \sin \phi_5) - b_{15}(E_{r_1} \cos \phi_5 + E_{i_1} \sin \phi_5)]$$
$$+ \lambda_2 [g_{25}(E_{i_2} \cos \phi_5 - E_{r_2} \sin \phi_5) - b_{25}(E_{r_2} \cos \phi_5 + E_{i_2} \sin \phi_5)] \} = 0$$

$$P_1 + |E_1|^2 g_{11} + |E_4| [g_{14}(E_{r_1} \cos \phi_4 + E_{i_1} \sin \phi_4) + b_{14}(E_{i_1} \cos \phi_4 - E_{r_1} \sin \phi_4)]$$
$$+ |E_5| [g_{15}(E_{r_1} \cos \phi_5 + E_{i_1} \sin \phi_5) + b_{15}(E_{i_1} \cos \phi_5 - E_{r_1} \sin \phi_5)] = 0$$

$$P_2 + |E_2|^2 g_{22} + |E_5| [g_{23}(E_{r_2} \cos \phi_3 + E_{i_2} \sin \phi_3) + b_{23}(E_{i_2} \cos \phi_3 - E_{r_2} \sin \phi_3)]$$
$$+ |E_4| [g_{24}(E_{r_2} \cos \phi_4 + E_{i_2} \sin \phi_4) + b_{24}(E_{i_2} \cos \phi_4 - E_{r_2} \sin \phi_4)]$$
$$+ |E_5| [g_{25}(E_{r_2} \cos \phi_5 + E_{i_2} \sin \phi_5) + b_{25}(E_{i_2} \cos \phi_5 - E_{r_2} \sin \phi_5)] = 0$$

$$F_3' - k_{1_3} - 2k_{2_3} \{ |E_3| [g_{23}(E_{r_2} \cos \phi_3 + E_{i_2} \sin \phi_3)$$
$$+ b_{23}(E_{r_2} \sin \phi_3 - E_{i_2} \cos \phi_3)] - |E_3|^2 g_{33} \} = 0$$

$$F_4' - k_{1_4} - 2k_{2_4} \{ |E_4| [g_{41}(E_{r_1} \cos \phi_4 + E_{i_1} \sin \phi_4) + b_{41}(E_{r_1} \sin \phi_4 - E_{i_1} \cos \phi_4)$$
$$+ g_{42}(E_{r_2} \cos \phi_4 + E_{i_2} \sin \phi_4) + b_{42}(E_{r_2} \sin \phi_4 - E_{i_2} \cos \phi_4)] - |E_4|^2 g_{44} \} = 0$$

$$F_5' - k_{1_5} - 2k_{2_5} \{ |E_5| [g_{51}(E_{r_1} \cos \phi_5 + E_{i_1} \sin \phi_5) + b_{51}(E_{r_1} \sin \phi_5 - E_{i_1} \cos \phi_5)$$
$$+ g_{52}(E_{r_2} \cos \phi_5 + E_{i_2} \sin \phi_5) + b_{52}(E_{r_2} \sin \phi_5 - E_{i_2} \cos \phi_5)] - |E_5|^2 g_{55} \} = 0$$

(4.2.1)

This set of equations was solved using the Newton-Raphson iterative method, as discussed in Chapter III, for several conditions of specified load variables. In order to accomplish the calculations required, a program was written corresponding to the general Computer Flow Diagram of Figure 3.3.1 and the computations carried out on the MISTIC (Michigan State University Digital Computer). Certain of the assumed values for load variables as used by Sze, Garnett and Calvert [23] were also used in this investigation. These conditions are given below where the voltage magnitudes and power values are in per unit.

Total System Load	100% base load	80%	60%	40%	20%
(p.u.)	5.2190	4.1752	3.1314	2.0876	1.0438
Load 1					
E_1	$1.15\epsilon^{j18.8^\circ}$	$1.15\epsilon^{j15.04^\circ}$	$1.15\epsilon^{j11.3^\circ}$	$1.15\epsilon^{j7.5^\circ}$	$1.15\epsilon^{j3.8^\circ}$
E_{r_1}	1.0886	1.1107	1.1278	1.1401	1.1476
E_{i_1}	0.3706	0.2984	0.2249	0.1505	0.0754
P_1	1.0560	0.8448	0.6336	0.4224	0.2112
Load 2					
E_2	$1.02\epsilon^{j0^\circ}$	$1.02\epsilon^{j0^\circ}$	$1.02\epsilon^{j0^\circ}$	$1.02\epsilon^{j0^\circ}$	$1.02\epsilon^{j0^\circ}$
E_{r_2}	1.02	1.02	1.02	1.02	1.02
E_{i_2}	0	0	0	0	0
P_2	4.1630	3.3304	2.4978	1.16652	0.8326

The expressions given by Sze, Garnett, and Calvert [23] for the equivalent primary power input for stations 3, 4, and 5 ($\mathcal{P}_3$, $\mathcal{P}_4$, and $\mathcal{P}_5$) were here assumed as F_3, F_4, and F_5. That is, the station production costs as functions of their power outputs were assumed as

$$F_3 = 0.6 - 1.5P_3 + 0.55P_3^2 \quad (4.2.2)$$

$$F_4 = 0.7 - 1.8P_4 + 0.40P_4^2 \quad (4.2.3)$$

$$F_5 = 0.8 - 2.1P_5 + 0.30P_5^2 \quad (4.2.4)$$

In these equations, the variables are both in p.u. on 100 unit bases where the base for F_x is 100$/hr and for P_x is 100 mws. For this example, it was assumed that these relations held over the ranges of station operation involved; it is noted that changes in the values of the coefficients over different ranges of operation and also minimum and maximum operating limits would necessarily have to be included in a complete practical application of this technique.

Then,

$$F_3' = -1.5 + 1.1P_3 \quad (4.2.5)$$

$$F_4' = -1.8 + 0.8P_4 \quad (4.2.6)$$

$$F_5' = -2.1 + 0.6P_5 \quad (4.2.7)$$

Additionally it was assumed that the generator voltage magnitudes were to be held at their value in the load flow study of Dandeno[28], that is,

$$|E_3| = 1.16, \ |E_4| = 1.18, \text{ and } |E_5| = 1.19.$$

For the 100% base load case, the numerical form of equations (4.2.1) is, where f_j, g_j, h_j correspond to the notation of Section 3.2,

$$F_3'(4.4090 \sin\phi_3 + 13.7561 \cos\phi_3) + \lambda_2(13.7561 \cos\phi_3 - 4.4090 \sin\phi_3) = f_3 = 0$$

$$F_4'(11.0177 \cos\phi_4 + 4.7527 \sin\phi_4) + \lambda_1(8.5111 \cos\phi_4 + 1.1191 \sin\phi_4)$$
$$+ \lambda_2(3.5917 \cos\phi_4 - 0.4463 \sin\phi_4) = f_4 = 0$$

$$F_5'(9.9032 \cos\phi_5 + 4.6453 \sin\phi_5) + \lambda_1(8.5833 \cos\phi_5 + 1.1286 \sin\phi_5)$$
$$+ \lambda_2(2.4143 \cos\phi_5 - 0.3023 \sin\lambda_5) = f_5 = 0$$

$$4.3376 + 1.1191 \cos\phi_4 - 8.5111 \sin\phi_4 + 1.1286 \cos\phi_5$$
$$- 8.5833 \sin\phi_5 = g_1 = 0$$

$$8.6847 - 4.4090 \cos\phi_3 - 13.7561 \sin\phi_3 - 0.4463 \cos\phi_4$$
$$- 3.5917 \sin\phi_4 - 0.3023 \cos\phi_5 - 2.4143 \sin\phi_5 = g_2 = 0$$

$$F_3' + 1.5 - 1.1(4.4090 \cos\phi_3 - 13.7561 \sin\phi_3 - 5.0141) = h_3 = 0$$

$$F_4' + 1.8 - 0.8(4.7527 \cos\phi_4 - 11.0179 \sin\phi_4 - 2.2439) = h_4 = 0$$

$$F_5' + 2.1 - 0.6(4.6459 \cos\phi_5 - 9.9030 \sin\phi_5 - 2.1097) = h_5 = 0 \quad (4.2.8)$$

The system of linear equations to be solved successively for the incremental values of the variables for this example, corresponding to equations (3.2.18), are shown as equations (4.2.9) below.

$$[F_3'(4.4090 \cos \phi_3 - 13.7561 \sin \phi_3) - \lambda_2(13.7561 \sin \phi_3 + 4.4090 \cos \phi_3)] \Delta\phi_3 + (4.4090 \sin \phi_3 + 13.7561 \cos \phi_3) \Delta F_3' + (13.7561 \cos \phi_3 - 4.4090 \sin\phi_3) \Delta\lambda_2 + f_3 = 0$$

$$[F_4'(4.7527 \cos \phi_4 - 11.0177 \sin \phi_4) + \lambda_1 (1.1191 \cos \phi_4 - 8.5111 \sin \phi_4) - \lambda_2(3.5916 \sin \phi_4 + .4463 \cos \phi_4)] \Delta\phi_4 + (11.0177 \cos \phi_4 + 4.7527 \sin \phi_4) \Delta F_4' + (8.5111 \cos \phi_4 + 1.1191 \sin \phi_4) \Delta\lambda_1 + (3.5916 \cos \phi_4 - 0.4463 \sin \phi_4) \Delta\lambda_2 + f_4 = 0$$

$$[F_5'(4.6453 \cos \phi_5 - 9.9032 \sin \phi_5) + \lambda_1 (1.1286 \cos \phi_5 - 8.5833 \sin \phi_5) - \lambda_2(2.4143 \sin \phi_5 + 0.3023 \cos \phi_5)] \Delta\phi_5 + (9.9032 \cos \phi_5 + 4.6453 \sin \phi_5) \Delta F_5' + (8.5833 \cos \phi_5 + 1.1286 \sin \phi_5) \Delta\lambda_1 + (2.4143 \cos \phi_5 - 0.3023 \sin \phi_5) \Delta\lambda_2 + f_5 = 0$$

$$- (1.1191 \sin \phi_4 + 8.5111 \cos \phi_4) \Delta\phi_4 - (1.1286 \sin \phi_5 - 8.5823 \cos \phi_5) \Delta\phi_5 + g_1 = 0$$

$$(4.4090 \sin \phi_3 - 13.7561 \cos \phi_3) \Delta\phi_3 + (0.4463 \sin \phi_4 - 3.5916 \cos \phi_4) \Delta\phi_4 + (0.3023 \sin \phi_5 - 2.4143 \cos \phi_5) \Delta\phi_5 + g_2 = 0$$

$$(4.8499 \sin \phi_3 + 15.1317 \cos \phi_3) \Delta\phi_3 + \Delta F_3' + h_3 = 0$$

$$(3.8022 \sin \phi_4 + 8.8143 \cos \phi_4) \Delta\phi_4 + \Delta F_4' + h_4 = 0$$

$$(2.7872 \sin \phi_5 + 5.9418 \cos \phi_5) \Delta\phi_5 + \Delta F_5' + h_5 = 0 \qquad (4.2.9)$$

As noted in Section 3.2, solutions to the equations for this example were obtained using two different sets of initial approximations. The first set used were obtained using the last method considered in Section 3.2. That is, equations (4.2.5), (4.2.6) and (4.2.7) were written as

$$F_0' + 1.5 - 1.1P_3 = 0$$
$$F_0' + 1.8 - 0.8P_4 = 0 \qquad (4.2.10)$$
$$F_0' + 2.1 - 0.6P_5 = 0$$

where F_3', F_4' and F_5' have each been replaced by a common value F_0'. Also, it was assumed that

$$P_3 + P_4 + P_5 = -1.05P_L \qquad (4.2.11)$$

where P_L represents the total load power.

Then, for a specified total load, equations (4.2.10) and (4.2.11) were solved for initial approximations for each F', as represented by F_0', and for P_3, P_4, and P_5. Since in this particular example, with the load variables as specified, P_3 is a function only of ϕ_3, P_4 is a function only of ϕ_4, and P_5 only of ϕ_5, initial approximations to the phase angles were then obtained corresponding to the initial approximations for P_3, P_4 and P_5. Initial approximations for λ_1 and λ_2 were found from the first three of the equations of set (4.2.8).

In the second set of initial approximations used, the phase angles were all taken as being zero degrees and then with the value for F_0' as found from equations (4.2.10) and (4.2.11), initial approximations for λ_1 and λ_2 were found as in the previous set, from the first three equations of set (4.2.8). Use of either set of initial approximations in the Newton-Raphson technique resulted in exactly the same answers.

Table 4.2.1 shows the results from the computer solution of equations (4.2.8) obtained using the second set of initial approximations. The results shown are for the full load case and it is noted that convergence to the selected degree of accuracy (change no greater than 1 in the fifth decimal place) was obtained in four iterations. Such convergence was obtained in 3 or 4 iterations in all cases considered. Equations for the generating station output powers for the case of 100% base load are given in (4.2.12) below. These relations are derived from (2.4.1.37) with the proper values substituted for the specified variables.

$$\begin{aligned} P_3 &= 4.4090 \cos \phi_3 - 13.7561 \sin \phi_3 - 5.0141 \\ P_4 &= 4.7527 \cos \phi_4 - 11.0179 \sin \phi_4 - 2.2439 \qquad (4.2.12) \\ P_5 &= 4.6459 \cos \phi_5 - 9.9030 \sin \phi_5 - 2.1097 \end{aligned}$$

The generating station power outputs calculated from these equations and the similar ones for the other load conditions using the voltage phase angles determined from the computer solution of equations (4.2.8) are shown in Table 4.2.2, and the corresponding production costs as calculated from equations (4.2.2), (4.2.3), and (4.2.4) are shown in Table 4.2.3.

TABLE 4.2.1

SOLUTIONS FOR VARIABLES FOR 100% BASE LOAD CASE

	Initial Approx.	After first Iteration	After 2nd Iteration	After 3rd Iteration	After 4th Iteration
ϕ_3	0.00	0.08961	0.09952	0.09954	0.09954
ϕ_4	0.00	0.36927	0.36739	0.36741	0.36741
ϕ_5	0.00	0.40106	0.40264	0.40262	0.40262
λ_1	2.9	2.97904	3.11043	3.10986	3.10986
λ_2	3.3	3.71060	3.93562	3.93752	3.93752
F_3'	-3.28	-3.52161	-3.69280	-2.69327	-3.69327
F_4'	-3.28	-3.04784	-3.21252	-3.21272	-3.21272
F_5'	-3.28	-2.96163	-2.12987	-3.12974	-3.12974

TABLE 4.2.2

GENERATING STATION POWER OUTPUTS FOR EXAMPLE PROBLEM

Load in % of Base Load	100%	80%	60%	40%	20%
Gen. Station Outputs					
P_3	-1.9939	-1.5549	-1.1393	-0.7434	-0.3656
P_4	-1.7659	-1.4491	-1.1342	-0.8212	-0.5013
P_5	-1.7162	-1.3600	-0.9992	-0.6343	-0.2826

TABLE 4.2.3

GENERATING STATION AND SYSTEM PRODUCTION COSTS FOR EXAMPLE PROBLEM

Load in % of Base Load	100%	80%	60%	40%	20%
Station Production Costs in 100$/hr					
F_3	5.7774	4.2621	3.0229	2.0191	1.2219
F_4	5.1189	4.1423	3.2561	2.4479	1.2219
F_5	5.2923	4.2109	3.1978	2.2527	1.4174
System Production Cost in 100$/hr	16.1886	12.6153	9.4768	6.7197	4.3422

4.3 Results With Other Phase Angles

In order to examine further the effects of variations in the voltage phase angles on the system production costs and to demonstrate that the method developed herein does in fact determine values of the variables which correspond to minimum production costs with the other parameters as specified, a number of further calculations were carried out on the digital computer for the example power system. First, combinations of phase angles were determined which, along with the specified voltage magnitudes, are such as to satisfy the specified load power requirements. Using these computed phase angles, the corresponding generator output powers and system costs were then calculated and plotted as functions of the phase angles.

The voltage phase angles were determined using the fourth and fifth equations of set (4.2.8). Symbolically, these equations can be written as

$$P_{1_S} - P_1(\phi_4, \phi_5) = 0 \qquad (4.3.1)$$

$$P_{2_S} - P_2(\phi_3, \phi_4, \phi_5) = 0 \qquad (4.3.2)$$

where P_{1_S} and P_{2_S} represent the specified powers for loads 1 and 2 respectively. With all specified parameters replaced by their values for 100% base load, the equations are as given in (4.3.3) and (4.3.4) below which are the same as in set (4.2.8).

$$4.3376 + 1.1191 \cos\phi_4 - 8.5111 \sin\phi_4 + 1.1286 \cos\phi_5 - 8.5833 \sin\phi_5 = 0 \qquad (4.3.3)$$

$$8.6847 - 4.4090 \cos\phi_3 - 13.7561 \sin\phi_3 - 0.4463 \cos\phi_4 - 3.5917 \sin\phi_4 - 0.3023 \cos\phi_5 - 2.4143 \sin\phi_5 = 0 \qquad (4.3.4)$$

Values of the three variables ϕ_3, ϕ_4, and ϕ_5 which satisfy these two equations were found by assigning values for one of the variables and then solving for the other two. Hence the values obtained are sets of voltage phase angles which, with the voltage magnitudes as previously specified, result in the load powers being as required. In the case for which values are assigned to ϕ_3, there remain two simultaneous non-linear equations in the two variables ϕ_4 and ϕ_5. These two equations were solved using the Newton-Raphson iterative technique as was also used for the set of 8 non-linear equations considered in Section 4.2. In the other two cases, for which values were assigned for ϕ_4 and ϕ_5, it was necessary to solve only one equation at a time. That is, with an assigned value for either ϕ_4 or ϕ_5, equation (4.3.3) was solved for the one of these two not assigned and then both of these values used in (4.3.4) to determine ϕ_3.

These computations were carried out over the range of each variable for which eash of the generator power outputs was of the proper sign; possible station operating limits were neglected. Outside the range covered, at least one of the power functions changes sign; that is, the element appears as a load rather than as a generator. Accordingly, ϕ_3 was varied from 0.089 to 0.109 radians in steps of 0.001 radians and ϕ_4 and ϕ_5 were varied from 0.25 to 0.55 radians in steps of 0.01 radians. For each case the corresponding generating station output powers, station production costs and system production costs were then calculated, using respectively (4.2.12) and (4.2.2), (4.2.3), and (4.2.4).

The results of these computations are shown in Figures (4.3.1), (4.3.2), and (4.3.3) in which the power output of each generating station and the system production cost are shown as functions of ϕ_3, ϕ_4, and ϕ_5 respectively. For any values of the phase angles shown, the equations for the specified load powers, equations (4.3.3) and (4.3.4), are satisfied. It is noted that the phase angles have an important effect on the system production cost and a definite minimum value of the production cost function is apparent in each of the figures. The values of the phase angles corresponding to the minimum point as read from the curves are shown in Table 4.3.1 along with the values computed in Section 4.2 for this 100% base load case. Taking into account the accuracy with which the values from the curves can be obtained the values are essentially the same, thereby confirming that the results obtained in Section 4.2 do correspond to a minimum of the production cost function.

TABLE 4.3.1

PHASE ANGLES AT MINIMUM SYSTEM PRODUCTION COST AS DETERMINED BY METHOD OF THESIS AND BY VARYING THE ANGLES AND COMPUTING THE COST

Angle	By Method of Thesis	By Varying the Angles
ϕ_3	0.0995	0.0997
ϕ_4	0.3674	0.368
ϕ_5	0.4026	0.404

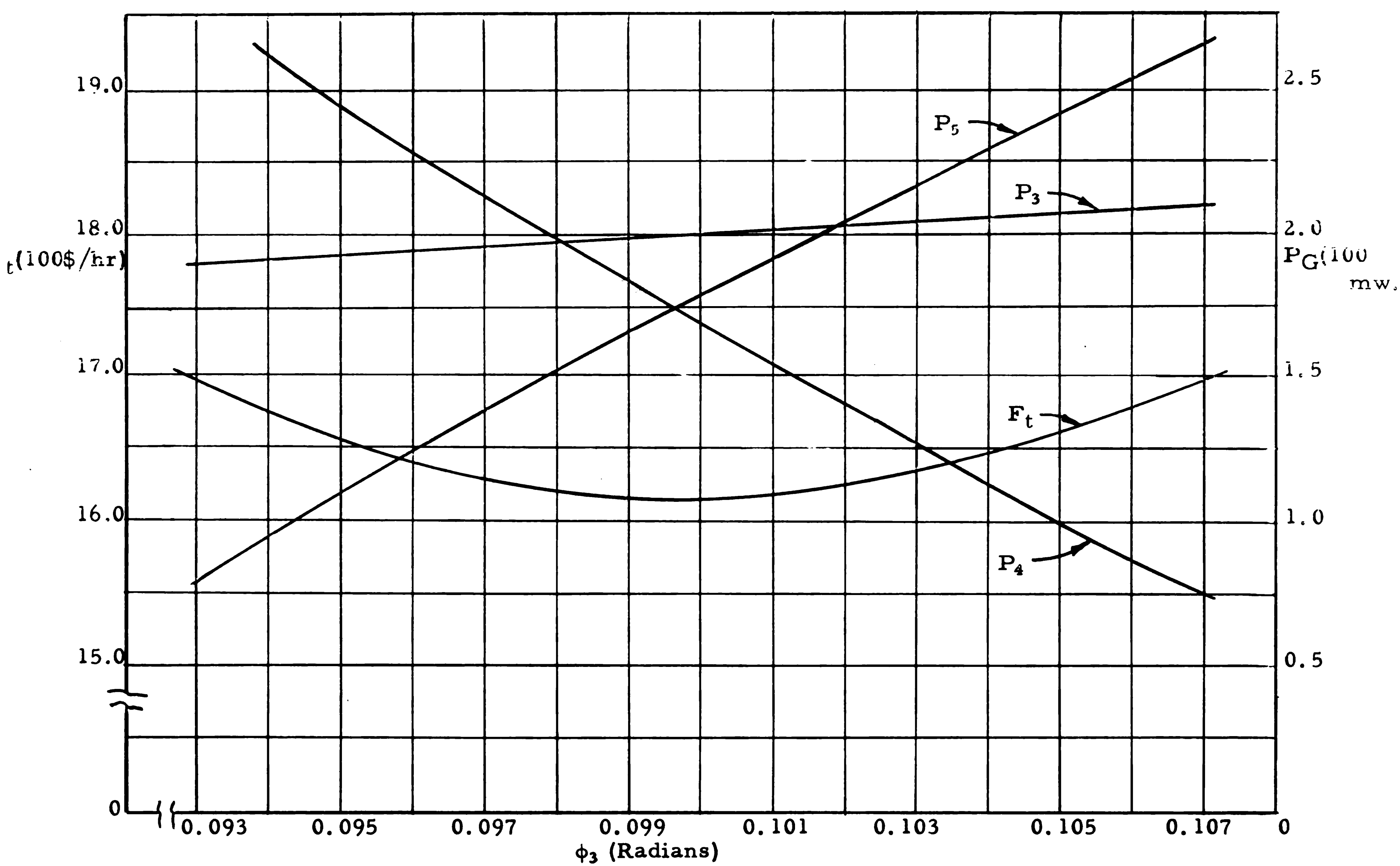

Fig. 4.3.1 System Production Cost and Generating Station Power Outputs as Functions of ϕ_3 with Constant Loads P_1 and P_2.

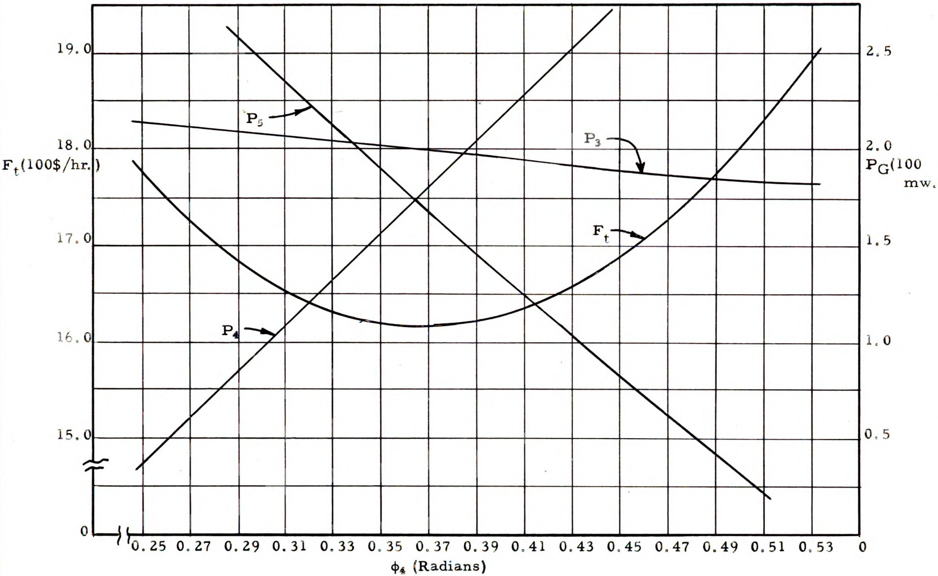

Fig. 4.3.2 System production Cost and Generating Station Power Outputs as Functions of ϕ_4 with Constant Loads P_1 and P_2.

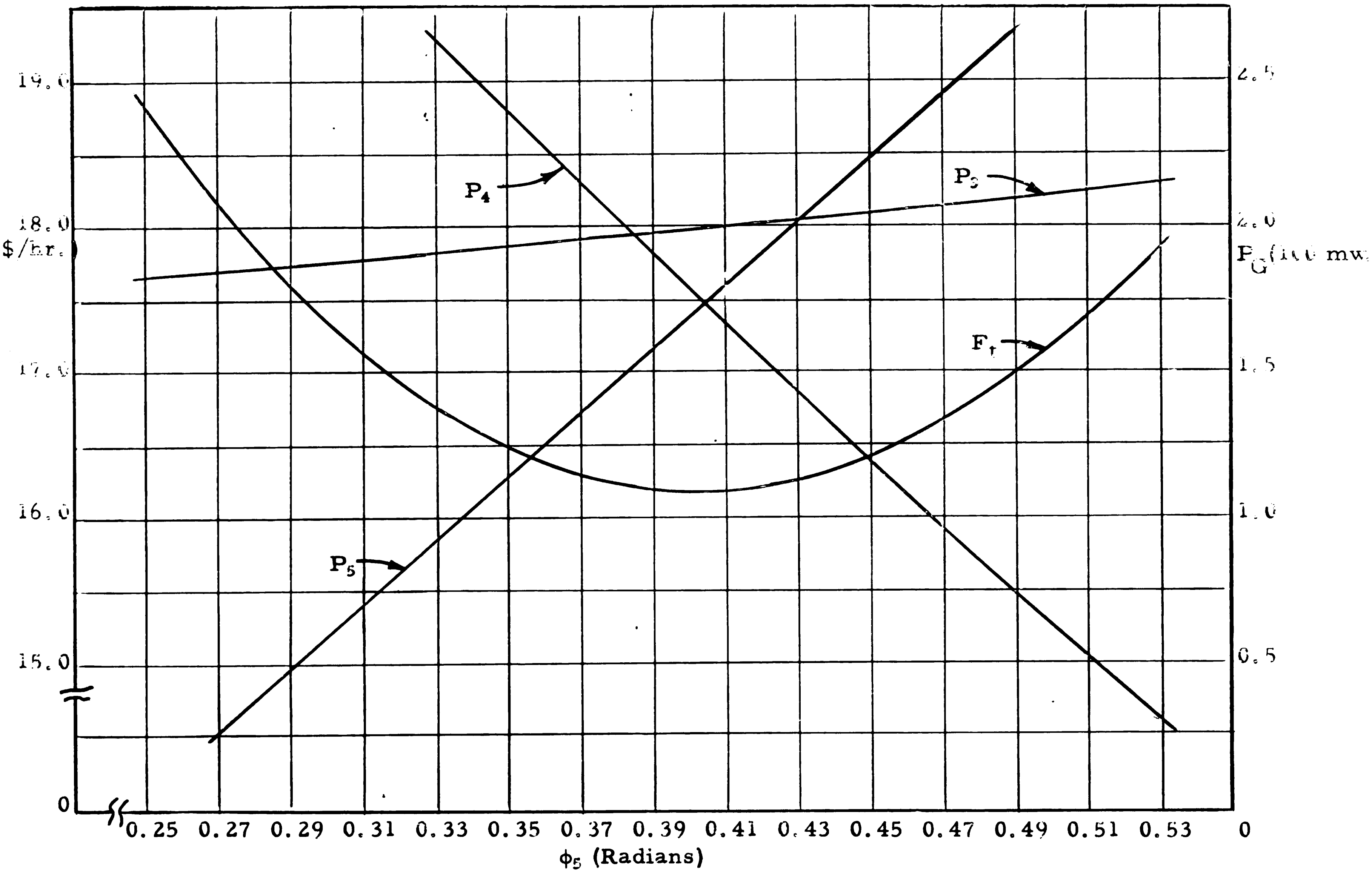

Fig. 4.3.3 System Production Cost and Generating Station Power Outputs as Functions of ϕ_5 with Constant Loads P_1 and P_2.

CHAPTER V

SUMMARY AND SUGGESTIONS FOR FURTHER RESEARCH

5.1 Summary

A new method in terms of utilizing node (bus) voltage phase angles as the controlling variables and in terms of application of a minimizing process to the production cost function while maintaining the identify of the individual loads is herein developed for determining a solution to the power system economic dispatch problem. A solution to the economic dispatch problem consists of a set of generating station power outputs which correspond to minimum system production cost for each specified load and accompanying set of system operating conditions. In the method developed in this thesis, because of the specified variables used, the generating station power outputs are uniquely determined once the phase angles are found.

A method used at the present time on certain power systems for determining optimum generating station power outputs involves an expression for power loss in the transmission system in terms of the so-called B constants. The usefulness of this technique is a matter of disagreement among various groups in the power industry. While the proponents of the method assert that significant savings result from its use as compared with methods in which transmission losses are not considered, others have not adopted the method primarily because of reservations in regard to its accuracy and lack of adaptability to changing conditions in the power system. A specific assumption generally made in determining the expression for power loss in terms of the B constants is that each "equivalent" load current remains under all operating conditions a constant complex fraction of the total "equivalent" load current, where the equivalent load current at a bus is defined as the sum of the line-charging, synchronous condenser, and load currents at that bus. This assumption is made so as to eliminate the individual load currents as variables of the problem and thereby to allow the individual loads to be represented by a single load. If the assumed proportional variation between each individual equivalent load current and the total equivalent load current is not maintained, some errors are introduced in all subsequent calculations using this single load representation. A number of other similar assumptions are also involved. In addition, determination of the B constants for a particular configuration of a transmission system is a considerable computational problem in itself, and is one which must be repeated in full in order to account for any significant changes occurring in the transmission system. There is then a definite need for a method of solution of the power system economic dispatch problem which first of all does not involve in its development assumptions which may not be realized in actual system operation, and secondly which is more easily adaptable to changes in the power transmission system.

Considered as a whole, the economic dispatch problem involves three different sets of equations; viz., (1), equations relating the currents and voltages of the system, e.g., the node system of equations; (2), auxiliary equations expressing specified restrictions on certain variables, e.g., the real power and/or reactive power for each load; and (3), the function which is to be minimized, the system production cost expressed as a function of the generating station power outputs. The problem is one of determining values of the variables which not only minimize the production cost function but which also satisfy the auxiliary equations; that is, the variables are not independent. A method of determining a solution for a problem of this nature is the method of Lagrangian multipliers. In the Lagrangian multiplier method additional variables are introduced in a manner such that the previously dependent variables can be considered to be independent variables in determining the desired solution.

The method first proposed by Brownlee [20] for determining incremental losses in a transmission system, summarized in Section 1.2.2 of this thesis, involves phase angles but in an entirely different manner from that considered herein. The method proposed by Calvert and Sze [23], summarized in Section 1.2.3 of this thesis, involves variation of generator voltage magnitudes as well as phase angles; also, in that method the minimization process is applied to a defined loss function rather than directly to the production cost function. The method of this thesis is new in the choice of the variables, relative phase angles of the node voltages, as well as in application of the Lagrangian multiplier method to the minimization of the production cost function while maintaining the identity of the individual load nodes. As developed herein, solution of the equations established by means of the Lagrangian multiplier method yields values of the phase angles which satisfy both the necessary conditions for a minimum of the production cost function and also the auxiliary equations which correspond to the specified load conditions. The phase angles have not been considered in this way in previously reported research. Also developed are a set of conditions, which, if satisfied at the point at which the necessary conditions are met, are sufficient to insure that the point so found is a minimum.

Since the magnitudes of the node voltages are specified at the beginning of the problem in addition to the admittances of the elements representing the transmission system, and the phase angles are determined as a part of the solution, the generating station power outputs are thereby uniquely determined. In addition, the reactive power functions for each of the elements representing the generating stations and loads are likewise determined. Hence, the assumption that the so-determined reactive power capacity is available is required. The results are then exact under the restrictions that the specified and computed values for the variables are maintained for a specified load condition. Changes in the power transmission system can more readily be entered into this method than in the B constant method. Since the node equations and hence the admittance matrix for the transmission system is involved directly in the equations to be solved, the entries in this admittance matrix are those of the transmission system at the corresponding time and can be changed from one computation to another by changing certain constants in the computations.

The equations established and for which a solution is required are non-linear but of a type to which the Newton- Raphson iterative technique can be applied. The corresponding system of linear equations in the incremental values of the variables is established in this thesis and methods of selecting initial approximations to the desired values of the variables are discussed. Use of a digital computer in the successive solution of the system of linear equations is a necessary part of the method presented here and an applicable computer flow diagram is shown. In order to illustrate the steps involved, the new method is applied to a particular power system and the node voltage phase angles are determined for this system so as to minimize the system production cost for several specified load conditions. The corresponding generating station power outputs and costs are also calculated. Finally, the effects on the system costs of variations in the phase angles while each load power and the voltage magnitudes are maintained at their specified values are investigated. It is shown that the phase angles are significant factors in the system production cost and also that the method developed herein does in fact determine the values of the phase angles which correspond to minimum cost with the other parameters as specified.

In conclusion, it can be stated that the method developed in this thesis results in determination of a set of conditions for the variables of the equations relating to a power system so as to minimize the system production cost without resort to extensive simplifying assumptions; rather, the results are exact under the restrictions imposed, these restrictions being that the specified and computed values for the variables are maintained for a specified load condition. The identity of the individual loads is maintained and equations specifying the power for each load are satisfied. Also, changes in the transmission system can be more readily accommodated than in most previously developed methods and, in any event, as readily as in any of the methods.

5.2 Suggestions for Further Research

The following topics related to the work reported in this thesis are suggested as being worthy of further investigation:

1. Possible integration of the power flow problem and the economic dispatch problem so as to obtain a common solution. Both of these problems start with much the same basic data, the major difference being that in the power flow problem the generating station power outputs are specified whereas in the economic dispatch problem this is the primary information to be determined. It appears reasonable to propose that, with further development, the general method set forth in this thesis for solution of the economic dispatch problem might be combined with a method for solution of the power flow problem so as to result in simultaneous solution of both problems. In terms of a digital computer solution such a combination might involve iterations on voltages as in the power flow problem, in inner computation loops, in addition to the iterative computations required in the method set forth herein.

2. Investigation of the possibilities of simplification of the method and the calculations required in the economic dispatch problem through

use of an all-vertex equivalent tree representation of the transmission system as developed by Reed et al. [30].

3. Application of the method set forth in this thesis to a larger example system so as to obtain further evaluation of its usefulness. In conjunction with such a study it would be desirable also to make economic dispatch calculations based on the B constant methods of determining transmission losses and then to compare the results, taking into account in both methods changes in all variables of the system.

4. Investigation of the development of useful criteria which for a system of non-linear equations would insure the existence of a solution and would yield information as to the existence of multiple solutions. In particular, the equations which arise in both the power flow and economic dispatch problems (Equations 2.3.3 of this thesis) should be investigated. First consideration might be given to the possibilities of multiple solutions and the resulting non-uniqueness of answers obtained in the usual power flow problems.

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