ON THE WEDDERBURN PRINCIPLE THEOREM FOR COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

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ABSTRACT

ON THE WEDDERBURN PRINCIPLE THEOREM FOR
COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

by Robert Louis Hemminger

Let A be a strictly power-associative algebra with radical N and such that the difference algebra $A - N$ is separable. Then we say that A has a Wedderburn decomposition if A has a subalgebra $S \cong A - N$ with $A = S + N$ (vector space direct sum).

The so-called Wedderburn Principle Theorem for associative algebras can be stated as follows: If $A - N$ is separable for an associative algebra A then A has a Wedderburn decomposition. The analogue of this theorem for alternative and Jordan algebras has also been proved. This thesis investigates this theorem for the commutative strictly power-associative algebras.

Our first result of primary importance is an example of a commutative power-associative algebra which does not have a Wedderburn decomposition. Since the base field in this example only has the restriction that it have characteristic not $2, 3, 5$ we cannot even hope to prove the Wedderburn Principle Theorem for commutative strictly power-associative algebras by only restricting the base field.

On the other hand we show that large classes of commutative strictly power-associative algebras do have

Wedderburn decompositions by proving the following two theorems.

(a) If A is a commutative strictly power-associative algebra of characteristic not 2 such that $A - N = B_1 \oplus \dots \oplus B_t$ is separable such that each B_i is simple and has three pairwise orthogonal idempotents then A has a Wedderburn decomposition.

(b) Let $\mathfrak{P}$ be the class of commutative strictly power-associative algebras of characteristic not 2 that satisfy a property P such that A in $\mathfrak{P}$ implies that every subalgebra of A is in $\mathfrak{P}$. Then every algebra in $\mathfrak{P}$ has a Wedderburn decomposition if and only if every algebra in $\mathfrak{P}$ that has at most two pairwise orthogonal idempotents has a Wedderburn decomposition.

This last result is used to show that every stable commutative power-associative algebra over an algebraically closed field F of characteristic zero has a Wedderburn decomposition.

In the associative, alternative, and Jordan cases the proof was accomplished in two stages; namely, $N^2 = 0$ and $N^2 \neq 0$. For $N^2 = 0$ an actually Wedderburn decomposition was constructed while for $N^2 \neq 0$ a nil ideal M with $0 \subset M \subset N$ was constructed in terms of N and a Wedderburn decomposition was established by a simple induction argument.

In our case we didn't encounter the case $N^2 = 0$ but our proofs did bear some resemblance to the case $N^2 \neq 0$. This similarity is reflected in the following result which was our basic tool in establishing (a) and (b) above. If M is any ideal of A with $M \neq 0, N, A$ then A has a Wedderburn decomposition. Using this result repeatedly for various ideals we were able to reduce A sufficiently to be able to construct a Wedderburn decomposition for it.

ON THE WEDDERBURN PRINCIPLE THEOREM FOR COMMUTATIVE POWER-ASSOCIATIVE ALGEBRAS

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1. Introduction

Let A be a strictly power-associative algebra with radical N and such that the difference algebra $A - N$ is separable. Then we say that A has a Wedderburn decomposition if A has a subalgebra $S \cong A - N$ with $A = S + N$ (vector space direct sum).

As a matter of terminology, by an algebra we shall always mean a finite dimensional vector space on which there is a multiplication defined which satisfies both distributive laws. The radical of a strictly power-associative algebra is the unique maximal nil ideal and a non-nil algebra with zero radical is said to be semi-simple. A simple algebra is a non-nil algebra with no proper ideals. An algebra A is power-associative if $x^{\alpha}x^{\beta} = x^{\alpha+\beta}$ for all positive integers α and β, and every x in A. An algebra A over a base field F is strictly power-associative if $x^{\alpha}x^{\beta} = x^{\alpha+\beta}$ for all positive integers α and β, and every x in A_K where K is any scalar extension of F. The characteristic of an algebra is the characteristic of its base field. If the characteristic is not 2, 3, or 5 then strict power-associativity is equivalent to power-associativity [7, pp. 364]. An algebra is separable if it is semi-simple over every scalar extension of the base field. The

elements of the difference algebra $A - N$ are the classes $[a]$, defined for every a in A, where $[a] = [b]$ if and only if $a - b$ is in N, $[a] + [b] = [a + b]$, and $[a][b] = [ab]$.

The basic structure theory of commutative power-associative algebras of characteristic not 2, 3, or 5 was given by Albert in [4]. Most of these results were carried over to commutative strictly power-associative algebras of characteristics 3 and 5 by Kokoris in [7]. Any reference to [4] will thus be understood to imply a reference to the corresponding result in [7].

Most of the results on commutative strictly power-associative algebras depend on an idempotent decomposition where an element e in A is idempotent if $e^2 = e \neq 0$. For the idempotent e we have $A = A_e(1) + A_e(1/2) + A_e(0)$ where x is in $A_e(\lambda)$ if and only if $ex = \lambda x$ for $\lambda = 0, 1/2, 1$. Moreover $A_e(1)$ and $A_e(0)$ are orthogonal subalgebras of A and for $\lambda = 0, 1$ we have

$$A_e(\lambda)A_e(1/2) \subseteq A_e(1/2) + A_e(1 - \lambda)$$

and

$$A_e(1/2)A_e(1/2) \subseteq A_e(1) + A_e(0)$$

(the product BC of two subspaces B and C of the algebra A is the set of all finite sums $\sum bc$, b in B and c in C; in particular $B^2 = BB$ and $B^m = BB^{m-1}$

for $m \geq 2$). For x in A we will frequently use this idempotent decomposition of A to express x uniquely in the form $x = x_1 + x_{1/2} + x_0$ where x_λ is in $A_e(\lambda)$ for $\lambda = 0, 1/2, 1$. Every semi-simple commutative strictly power-associative algebra of characteristic not 2 has a unity element and can be expressed uniquely as a direct sum of simple algebras. These results are all contained in [4].

The characterization of the simple, and hence semi-simple, commutative strictly power-associative algebras is now essentially complete [see 10] so it is desirable to see if a Wedderburn decomposition can be given for them. The example in §2 shows that this is not possible in general. The purpose of this thesis is to show that a large class of the commutative strictly power-associative algebras do have Wedderburn decompositions and to point out what one might expect in those that do not have a Wedderburn decomposition.

In §4 we show that if A is a commutative power-associative algebra with characteristic not 2, $A - N$ is separable, and $A - N = B_1 \oplus B_2 \oplus \dots \oplus B_t$ where each B_i is simple and contains three pairwise orthogonal idempotents then A has a Wedderburn decomposition.

In §6 we show that if $\mathfrak{P}$ is the class of commutative strictly power-associative algebras having a property P

then every algebra in $\mathfrak{P}$ has a Wedderburn decomposition if and only if every algebra in $\mathfrak{P}$ having at most two pairwise orthogonal idempotents has a Wedderburn decomposition.

This result is applied in §7 to the class of stable algebras over algebraically closed fields of characteristic zero.

The so-called Wedderburn Principle Theorem for associative algebras can be stated as follows: If $A - N$ is separable for an associative algebra A then A has a Wedderburn decomposition. A proof of this can be found in [1, Theorem 23, pp. 47]. This theorem was generalized to alternative algebras by Schafer [12] and its analogue for Jordan algebras was proved by Penico [11]. Previous to that Albert had proved it for an important class of Jordan algebras [2]. In all of these cases the method was basically the following. For $N^2 = 0$ a subalgebra isomorphic to $A - N$ was actually constructed and for $N^2 \neq 0$ a nil ideal $M \neq 0$, N was constructed in terms of N and the theorem obtained by the induction argument we have given for the proof of Lemma 2.1. In each case the construction of M depended on knowing that an ideal is nilpotent if and only if it is nil (M is nilpotent if $M^n = 0$ for some positive integer n while M is nil if each element of M is nilpotent, that is, for each x in M there is a positive integer n, depending on

x, such that $x^n = 0$). But it is unknown if this is the case in commutative strictly power-associative algebras or not. This difficulty is mainly circumvented by Lemma 2.2 for according to that result if M is any ideal of A we can assume $M = 0$, N, or A. By repeated use of Lemma 2.2 we are able to reduce A sufficiently to actually construct a Wedderburn decomposition for it. This is done in the proof of Theorem 2. Since the latter part of this proof requires some preliminary material and is quite long we have put it in a separate section.

We will always let N represent the radical of the algebra A and we assume $N \neq 0$, A since otherwise A has a trivial Wedderburn decomposition. Unless otherwise specified we will understand that the generic symbol A represents a commutative strictly power-associative algebra of characteristic not two with $A - N$ separable.

2. Example

Let A be the 6-dimensional commutative algebra with basis e_{11}, e_{12}, e_{21}, e_{22}, m, n and multiplication table $e_{11}^2 = e_{11}$, $e_{22}^2 = e_{22}$, $e_{11}e_{12} = e_{22}e_{12} = 1/2e_{12}$, $e_{11}e_{21} = e_{22}e_{21} = 1/2e_{21}$, $e_{11}n = e_{12}m = n$, $e_{22}m = e_{21}n = m$, $e_{12}e_{21} = 1/2(e_{11} + e_{22} + m + n)$, and all other products zero.

The algebra A is commutative by definition. If we

restrict A to have a base field F of characteristic not 2, 3, or 5 and let x be a general element of A (expressed in terms of the basis elements) then by computation we find that $x^2x^2 = (x^2x)x$. So by [3, Lemma 4, pp. 554] A is power-associative.

For an algebra B of characteristic not 2, B^+ is the algebra with the same additive group as B but the multiplication of B^+ is defined by $xy = 1/2(x \circ y + y \circ x)$ where $x \circ y$ is the product of x and y in B.

The radical N of A is spanned by m and n, $N^2 = 0$, and in the notation of the last paragraph we see that $A - N \cong F_2^+$ with basis $[e_{11}]$, $[e_{12}]$, $[e_{21}]$, $[e_{22}]$ where F_2 is the algebra of all 2 by 2 matrices over F. Suppose A had a subalgebra $S \cong A - N$. Then S would have the usual matrix basis g_{11}, g_{12}, g_{21}, g_{22} for F_2^+ and there would be an automorphism σ of $A - N$ such that $\sigma([e_{ij}]) = [g_{ij}]$. But this is a change of basis for the 2 by 2 matrices so there is a nonsingular element $[y] = \alpha[e_{11}] + \beta[e_{12}] + \gamma[e_{21}] + \delta[e_{22}]$ in $A - N$, with $\Delta = \alpha\delta - \beta\gamma \neq 0$, such that $[g_{ij}] = [y] \circ [e_{ij}] \circ [y]^{-1}$ (note that this multiplication takes place in F_2). But $[y]^{-1} = \Delta^{-1}(\delta[e_{11}] - \beta[e_{12}] - \gamma[e_{21}] + \alpha[e_{22}])$ so computing $[g_{ij}] = [y] \circ [e_{ij}] \circ [y]^{-1}$ we have

$$g_{11} = \Delta^{-1}(\alpha\delta e_{11} - \alpha\beta e_{12} + \gamma\delta e_{21} - \beta\gamma e_{22} + \epsilon_1 m + \epsilon_2 n)$$

$$g_{12} = \Delta^{-1}(-\alpha\gamma e_{11} + \alpha^2 e_{12} - \gamma^2 e_{21} + \alpha\gamma e_{22} + \theta_1 m + \theta_2 n)$$

$$g_{21} = \Delta^{-1}(\beta\delta e_{11} - \beta^2 e_{12} + \delta^2 e_{21} - \beta\delta e_{22} + \lambda_1 m + \lambda_2 n)$$

$$g_{22} = \Delta^{-1}(-\beta\gamma e_{11} + \alpha\beta e_{12} - \gamma\delta e_{21} + \alpha\delta e_{22} + \pi_1 m + \pi_2 n) \quad .$$

Equating coefficients of m and n in the products $g_{ij}g_{k\ell}$ (for example the coefficients of m and n in $g_{11}g_{12}$ and $1/2\ g_{12}$ are equal since $g_{11}g_{12} = 1/2\ g_{12}$) yields equations in $\alpha, \beta, \gamma, \delta, \epsilon_1, \epsilon_2, \ldots, \pi_1, \pi_2$ which force $\Delta = 0$. But this is a contradiction so A has no subalgebra $S \cong A - N$ and hence A has no Wedderburn decomposition.

This example of course shows we can not prove the Wedderburn Principle Theorem for the class of all commutative strictly power-associative algebras. Moreover it shows we can not even hope to prove it by only restricting the base field for in our example the base field is arbitrary other than the restriction that the characteristic not be 2, 3, or 5.

An algebra is called stable with respect to an idempotent e if $A_e(\lambda)A_e(1/2) \subseteq A_e(1/2)$ for $\lambda = 0, 1$ and it is called stable if it is stable with respect to each of its idempotents.

From the multiplication table for A above $ne_{21} = m$ so A is not stable with respect to e_{11}

(or e_{22}). Now by Theorem 2 of [9, pp. 698] if f is any other idempotent of A (and $f \neq 1 = e_{11} + e_{22}$) then $f = 1/2(1 + w)$ where $w^2 = 1$ and $w = \mu(e_{11} - e_{22}) + w_{12} + w_1 + w_2$ where $w_{12} \neq 0$ is in $A_{e_{11}}(1/2)$ and w_λ is in $A_{e_{11}}(\lambda) \cap N$ for $\lambda = 0, 1$. Computing the general element w with these properties we find that A is not stable with respect to $f = 1/2(1 + w)$ either. That is, our example is not stable with respect to any idempotent.

Looking at the other side of the coin, we show in §7 that A has a Wedderburn decomposition if it is stable over an algebraically closed base field of characteristic zero.

3. Pairwise orthogonal idempotents

In this section we will assume the algebra A has the element 1 as a unity. Based upon and related to the decomposition of A by a single idempotent Albert has given in [4, §5] a decomposition of A relative to a set of pairwise orthogonal idempotents $e_1, e_2, \ldots, e_t$ for which $1 = e_1 + e_2 + \ldots + e_t$. It is shown that we can write A in a vector space direct sum $A = \sum_{i \leq j} A_{ij}$ for $i, j = 1, 2, \ldots, t$ where $A_{ii} = A_{e_i}(1)$ and $A_{ij} = A_{ji} = A_{e_i}(1/2) \cap A_{e_j}(1/2)$ when $i \neq j$. Moreover if $g = e_i + e_j$ for $i \neq j$ then g is an idempotent

with $A_g(1) = A_{ii} + A_{ij} + A_{jj}$, $A_g(1/2) = \sum_{k \neq i,j} (A_{ik} + A_{jk})$, and $A_g(0) = \sum_{k,\ell \neq i,j} A_{k\ell}$. We will have occasion to use these only for $t = 3$. In that case $A = A_{11} + A_{22} + A_{33} + A_{12} + A_{13} + A_{23}$ and with $i, j\ k$ distinct and $g = e_i + e_j$ we have $A_g(1) = A_{ii} + A_{ij} + A_{jj}$, $A_g(1/2) = A_{ik} + A_{jk}$, and $A_g(0) = A_{kk}$. For i, j, k, ℓ distinct we have

$$A_{ii}^2 \subseteq A_{ii}$$

$$A_{ii}A_{ij} \subseteq A_{ij} + A_{jj}$$

$$A_{ii}A_{jj} = A_{ij}A_{k\ell} = A_{ii}A_{k\ell} = 0$$

$$A_{ij}A_{jk} \subseteq A_{ik}$$

$$A_{ij}^2 \subseteq A_{ii} + A_{jj} \ .$$

Since these relations are basic to much of our work we will generally use them without specific reference.

Also related to pairwise orthogonal idempotents we have the following lemma.

<u>Lemma 1</u>: Let $[u_1], [u_2], \ldots, [u_t]$ be pairwise orthogonal idempotents in $A - M$, M a nil ideal of A, and let $u = u_1 + u_2 + \ldots + u_t$. Then there exists an idempotent e and pairwise orthogonal idempotents

$e_1, e_2, \ldots, e_t$ in $A_e(1)$ such that $e = e_1 + e_2 + \ldots + e_t$, $[e] = [u]$, and $[e_i] = [u_i]$ for $i = 1, 2, \ldots, t$. Moreover if A has 1 as a unity element and $[1] = [u]$ then $e = 1$.

Proof: The proof of the first part of the lemma is by induction and the case $t = 1$ is Lemma 1 of [2, pp. 1]. Here $u = u_1$. Now $[u]^k = [u^k] = [u]$ so u cannot be nilpotent. Hence the associative algebra of all polynomials in u, denoted by $F[u]$, is not nilpotent and thus contains an idempotent $e = f(u)$ for f in $F[u]$. Then $[e] = [f(u)] = \alpha[u]$ where $\alpha = f(1)$ is in F. Thus $\alpha[u] = [e] = [e]^2 = \alpha^2[u]^2 = \alpha^2[u]$. Since e is an idempotent it is not in M so $\alpha[u] \neq 0$, $\alpha = 1$, and $[u] = [e]$ as desired.

Let $w = u_1 + u_2$ for $t \geq 2$. Then $u = w + u_3 + \ldots + u_t$ for pairwise orthogonal idempotents $[w], [u_3], \ldots, [u_t]$. By the induction hypothesis there exists an idempotent e and pairwise orthogonal idempotents $f, e_3, \ldots, e_t$ in $A_e(1)$ such that $e = f + e_3 + \ldots + e_t$, $[e] = [u]$, $[f] = [w]$, and $[e_i] = [u_i]$ for $i = 3, \ldots, t$. In particular f is an idempotent of A such that $[f] = [w]$ with $w = u_1 + u_2$, where u_1 and u_2 are orthogonal idempotents (note that this is essentially the case $t = 2$ only in that case f would have been obtained from the case $t = 1$ rather than from the

induction hypothesis). Then $[f][u_1] = [w][w - u_2] =$
$[w] - [u_2] = [u_1]$. If $[f][x] = [x]$ for x in A
we can write $x = x_1 + x_{1/2} + x_o$ and have
$[x_1] + [x_{1/2}] + [x_o] = [x] = [f][x] = [x_1] + 1/2[x_{1/2}]$
so $[x] = [x_1]$. Now $[f][u_1] = [u_1]$ so there is an element x_1 in $A_f(1)$ such that $[x_1] = [u_1]$. Moreover x_1 is not nilpotent since $[u_1]$ isn't. Hence the associative algebra $F[x_1]$ is not nilpotent and thus contains an idempotent e_1 which is in $A_f(1)$ since $A_f(1)$ is a subalgebra. Then just as in the case $t = 1$ we have $[e_1] = [x_1]$ and so $[e_1] = [u_1]$. Now $e_2 = f - e_1$ is an idempotent in $A_f(1)$, $e_2e_1 =$
$(f - e_1)e_1 = e_1 - e_1 = 0$, $[e_2] = [f - e_1] = [f] - [e_1] =$
$[w] - [u_1] = [u_2]$, and since e_1 and e_2 are in $A_f(1)$ they are orthogonal to e_i for $i = 3, \ldots, t$. Thus $e = f + e_3 + \ldots + e_t = e_1 + e_2 + e_3 + \ldots + e_t$ where the e_i are pairwise orthogonal idempotents with $[e] = [u]$ and $[e_i] = [u_i]$ for $i = 1, 2, \ldots, t$. Since f is in $A_e(1)$ we have $A_f(1) \subseteq A_e(1)$ so e_1 and e_2 are also in $A_e(1)$ which completes the proof of the first part of the lemma.

For 1 in A, $1 - (e_1 + e_2 + \ldots + e_t)$ is either zero or an idempotent of A. But $[1] = [u_1] + \ldots +$
$[u_t] = [e_1] + \ldots + [e_t]$ means that $1 - (e_1 + \ldots + e_t)$ is in M, so it is nilpotent. Hence it is zero and

$1 = e_1 + e_2 + \dots + e_t$ as desired.

As a consequence of Lemma 1 we immediately have Corollary 1.

Corollary 1: If M is a nil ideal of A then A has t pairwise orthogonal idempotents if and only if $A - M$ has t pairwise orthogonal idempotents.

4. Classes of algebras with Wedderburn decompositions

Let $\mathfrak{A}$ be the class of all commutative strictly power-associative algebras A that have a Wedderburn decomposition and for which $A - N$ is simple.

We are using $B + C$ to mean the vector space direct sum of the subspaces B and C. In particular this means that $B \cap C = 0$. If in addition B and C are subalgebras of A such that $BC = 0$ then we write $B \oplus C$. This is called the direct sum of the subalgebras B and C.

Theorem 1: Let A be a commutative strictly power-associative algebra of characteristic not two. Then it is known that $A - N = B_1 \oplus \dots \oplus B_t$ where B_i is simple and has a unity element $[u_i]$. Let e_i be as in Lemma 1. Then A has a Wedderburn decomposition if $A_{e_i}(1)$ is in $\mathfrak{A}$ for $i = 1, 2, \dots, t$.

Proof: The proof is by induction on t. Let

$e = e_1 + e_2 + \ldots + e_t$ as in Lemma 1 and let $A_1 = A_{e_1}(1)$, $A_{12} = A_{e_1}(1/2)$, and $A_2 = A_{e_1}(0)$. Also let R_i be the radical of A_i and $N_i = N \cap A_i$ for $i = 1, 2$.

Remark: When B is a subspace of A then $B - N$ is the subspace of $A - N$ consisting of all classes [b] for b in B. When B is a subalgebra of A then $B - N$ is a subalgebra of $A - N$ and is isomorphic to $B - N_b$ where $N_b = N \cap B$.

From the above we clearly have $A_1 - N \cong B_1$, $A_{12} - N = 0$, and $A_2 - N \cong B_2 \oplus \ldots \oplus B_t$. So $A_{12} \subsetneq N$. Let $M = R_1 + A_{12} + R_2$. By the definitions above R_i is a nil ideal of A_i and $N \subsetneq M$ so $AM = (A_1 + A_{12} + A_2)(R_1 + A_{12} + R_2) \subsetneq A_1R_1 + A_1A_{12} + A_{12}M + A_2A_{12} + A_2R_2 \subsetneq M$ and M is an ideal of A. Moreover if x is in M then $x = a + n + b$ for $a \in R_1$, $n \in N$, and $b \in R_2$. So x^2 is in $a^2 + b^2 + N$, x^3 is in $a^3 + b^3 + N$, and by induction, x^k is in $a^k + b^k + N$ for every positive integer k. But a and b are nilpotent elements so for k sufficiently large we have x^k in N so x^k is nilpotent, x is nilpotent, and M is a nil ideal of A. Thus $M \subsetneq N$ so $M = N$ and $R_i = N_i$ for $i = 1, 2$.

Now for $t = 1$, $A_2 - N = 0$ so $A_{12} + A_2 \subsetneq N$ and since A_1 is in $\mathfrak{A}$ it has a Wedderburn decomposition,

say $A_1 = S_1 + N_1$. Then $A = S_1 + N$ is a Wedderburn decomposition for A.

If $t > 1$ then $A_2 - N \cong A_2 - N_2 \cong B_2 \oplus \dots \oplus B_t$ where $[u_i] = [e_i]$ is the unity element of B_i for $i = 2, \dots, t$. Moreover $(A_2)_{e_i}(1) = A_{e_i}(1)$ is in $\mathfrak{A}$ so by the induction hypothesis A_2 has a Wedderburn decomposition, say $A_2 = S_2 + N_2$. Then $A = (S_1 \oplus S_2) + N$ is a Wedderburn decomposition of A.

Before we can put much confidence in the value of Theorem 1 we must at least know that the class $\mathfrak{A}$ is of sufficient size to have some importance. That is the purpose of the next two theorems.

<u>Theorem 2</u>: Let A be a commutative strictly power-associative algebra with a unity element and of characteristic not 2 such that A has three pairwise orthogonal idempotents and $A - N$ is simple. Then A has a Wedderburn decomposition.

<u>Proof</u>: The proof is by induction on n, the dimension of A. Then $n \geq 3$ since A has three pairwise orthogonal idempotents. The theorem is trivial if $n = 3$ so assume every algebra of dimension less than n and of the type described in the theorem has a Wedderburn decomposition.

<u>Remark</u>: For a nil ideal M of A, $A - M$ is

semi-simple if and only if $M = N$.

For $M \subseteq N$ since N is the maximal nil ideal of A. Thus $N - M$ is a nil ideal of $A - M$. So for $A - M$ semi-simple, $N - M = 0$ and $N = M$. Conversely if M' is a nil ideal of $A - N$ then there is an ideal M in A such that $N \subseteq M$ and $M - N = M'$. But if b is in M then $[b]^s = 0$ for some positive integer s so b^s is in N and $(b^s)^t = 0$ for some positive integer t. Thus M is a nil ideal of A so $M = N$ and $M' = 0$ as desired.

The major tool in the proof of Theorem 2 is the important Lemma 2.2. We first prove a special case of it.

Lemma 2.1: If M is a nil ideal of A with $M \neq 0, N$ then A has a Wedderburn decomposition.

Proof: For convenience we will write $d(B)$ for the dimension of a subspace B.

By the homomorphism theorems $A - N \cong (A - M) - (N - M)$ so by the Remark above $N - M$ is the radical of $A - M$. Now $(A - M) - (N - M)$ is simple since $A - N$ is simple and $A - M$ has a unity element since A has one. And by Corollary 1, $A - M$ has three pairwise orthogonal idempotents. But M is a proper ideal of A and we have $3 \leq d(A - M) < n$ so by the induction hypothesis $A - M$ has a subalgebra C_0 such that

$C_o \cong (A - M) - (N - M) \cong A - N$. Again by the homomorphism theorems A has a subalgebra $C_1 \neq 0, A$ such that $M \subset C_1$ (that is $M \subseteq C_1$ and $M \neq C_1$) and $C_o \cong C_1 - M$. Thus we have a proper subalgebra C_1 of A such that $C_1 - M \cong A - N$. Similar to the considerations for $A - M$ above we see that M is the radical of C_1, $C_1 - M$ is simple, C_1 has three pairwise orthogonal idempotents, and $3 \leq d(C_1) < n$. So by the induction hypothesis C_1 has a subalgebra $C \cong C_1 - M$. Thus C is a subalgebra of A such that $C \cong A - N$. But $C \cap N$ is a nil ideal of C so $C \cap N = 0$ since $C \cong A - N$ which is simple. Therefore $C + N$ is a subspace of A with $d(C + N) = d(C) + d(N) = d(A - N) + d(N) = d(A)$. So $A = C + N$ and this is a Wedderburn decomposition for A.

Lemma 2.2: If M is any ideal of A with $M \neq 0, N,$ or A then A has a Wedderburn decomposition.

Proof: By Lemma 2.1 we can assume $M \not\subseteq N$ and $N \not\subseteq M$ since $A - N$ is simple. If $M \cap N \neq 0$ then it is a nil ideal of A different from 0 and N so by Lemma 2.1 A has a Wedderburn decomposition. If $M \cap N = 0$ then $M + N$ is an ideal of A and $(M + N) - N$ is a non-zero ideal of $A - N$. But $A - N$ is simple so $A = M + N$ with $M \cap N = 0$. This is a Wedderburn decomposition of A and completes the proof of Lemma 2.2.

The remainder of the proof of Theorem 2 involves repeated applications of Lemma 2.2 to various ideals of A. By this method we are able to reduce the algebra A to one for which we can construct a Wedderburn decomposition. This is done in the next section. But for the moment let us assume that Theorem 2 is proved. We can then prove a more general result.

<u>Theorem 3</u>: Let A be a commutative strictly power-associative algebra of characteristic not two such that, in the canonical representation, $A - N = B_1 \oplus \cdots \oplus B_t$ where each B_i has three pairwise orthogonal idempotents. Then A has a Wedderburn decomposition.

<u>Proof</u>: Let $[u_i]$ be the unity element for B_i and let e_i be the pairwise orthogonal idempotents in A as in Lemma 1. Fix i and let $f = e_i$. Then just as was done for e_1 in Theorem 1 we have $A_f(1) - N_f \cong A_f(1) - N \cong B_i$ where $N_f = N \cap A_f(1)$ is the radical of $A_f(1)$. But f is the unity element for $A_f(1)$ and by Corollary 1 $A_f(1)$ has three pairwise orthogonal idempotents. So by Theorem 2, $A_f(1)$ is in $\mathfrak{A}$. But this is true for each $i = 1, 2, \ldots, t$ so by Theorem 1 A has a Wedderburn decomposition.

5. Proof of Theorem 2

a. Preliminaries

Since A is strictly power-associative we have $x^2x^2 = (x^2x)x$ for each x in A. The linearization of this identity gives

$$\begin{aligned} &4[(xy)(zw) + (xz)(yw) + (xw)(yz)] \\ (1) \qquad &= x[y(zw) + z(wy) + w(yz)] + y[x(zw) + z(wx) + w(xz)] \\ &+ z[x(yw) + y(wx) + w(xy)] + w[x(yz) + y(zx) + z(xy)]. \end{aligned}$$

We will also make use of some of the results of Albert on commutative strictly power-associative algebras; namely, results (5) and (8) of [4, pp. 505-506]. We state them as

$$(2) \qquad [w_{1/2}(x_1y_1)]_{1/2} = [(w_{1/2}x_1)y_1 + (w_{1/2}y_1)x_1]_{1/2}$$

$$(3) \qquad [w_{1/2}(x_1y_1)]_0 = 2[(w_{1/2}x_1)y_1 + (w_{1/2}y_1)x_1]_0$$

$$(4) \qquad [(w_{1/2}y_1)x_0]_1 = 1/2[(w_{1/2}x_0)y_1]_1$$

where z_λ, $\lambda = 0, 1/2, 1$, is the $A_e(\lambda)$ component of z; e an idempotent.

Before continuing we need to explain some new notation we will use. We have already commented on the vector space direct sum $B + C$. In the remainder of this thesis we will no longer require that $B + C$ indicate the <u>direct</u> sum of vector spaces; only the sum. That is we will now use $B + C$ on occasions where $B \cap C \neq 0$. But in §7 we will sometimes want to explicitly indicate that we are using the vector space direct sum. In that case we will write $B \dotplus C$.

We have also used the product BC previously. But it is too restrictive for our purposes now so we introduce a new product, $B \circ C$, of the subspaces B and C. Since A has a unity element, denoted by 1, and three pairwise orthogonal idempotents we can write $1 = e_1 + e_2 + e_3$ where the e_i are pairwise orthogonal idempotents. Then as in §3 A has a corresponding decomposition as $A = \sum_{i \leq j} A_{ij}$, $i, j = 1, 2, 3$. We define $B \circ C = \sum_{i \leq j} (BC)_{ij}$ where x is in $(BC)_{ij}$ if and only if there exists an element y in BC, $y = \sum_{i \leq j} y_{ij}$, such that $x = y_{ij}$. We write $B \circ B = B^{(2)}$. Evidently $BC \subseteq B \circ C$ but it may happen that $B \circ C \not\subseteq BC$. But if BC is an ideal of A then $BC = B \circ C$ (this can easily be seen by making appropriate linear combinations and multiplications by the e_i; for example $e_1(2e_1 y - y) = y_{11}$). Since we are only interested in using the product of subspaces to construct ideals we will use the product $B \circ C$ since it is easier to work with

and may in fact be an ideal even though BC isn't.

Lemma 2: For i, j, k distinct we have

(a) $A_{ii}(A_{ij} \circ A_{jk}) \subsetneq (A_{ii}A_{ij}) \circ A_{jk}$

(b) $A_{ik}A_{ij}^{(2)} \subsetneq (A_{ik}A_{ij}) \circ A_{ij}$

(c) $A_{ij}(A_{ik} \circ A_{jk}) \subsetneq A_{ik}^{(2)} + A_{jk}^{(2)}$

(d) $A_{ii}A_{ij}^{(2)} \subsetneq (A_{ii} \circ A_{ij}) \circ A_{ij}$.

Proof: Let $g = e_i + e_j$. Then $A_g(1) = A_{ii} + A_{ij} + A_{jj}$, $A_g(1/2) = A_{ik} + A_{jk}$, and $A_g(0) = A_{kk}$ as in §3. By (2) we have $[w_{jk}(x_{ii}y_{ij})]_{1/2} = [(x_{ii}w_{jk})y_{ij} + x_{ii}(y_{ij}w_{jk})]_{1/2} = [x_{ii}(y_{ij}w_{jk})]_{1/2}$ since $x_{ii}w_{jk} = 0$. From (3) we get $[w_{jk}(x_{ii}y_{ij})]_o = 2[x_{ii}(y_{ij}w_{jk})]_o$. So $x_{ii}(y_{ij}w_{jk}) = [w_{jk}(x_{ii}y_{ij})]_{ik} + [w_{jk}(x_{ii}y_{ij})]_{jk} + 1/2[w_{jk}(x_{ii}y_{ij})]_{kk}$ which is in $A_{jk} \circ (A_{ii}A_{ij})$. But $A_{ij}A_{jk} \subsetneq A_{ik}$ so $A_{ij}A_{jk} = A_{ij} \circ A_{jk}$ which proves (a).

We note that $(A_{ik}A_{ij})A_{ij} \subsetneq A_{jk}A_{ij} \subsetneq A_{ik}$ so using (2) and (3) as before we have $w_{ik}(x_{ij}y_{ij}) = [(w_{ik}x_{ij})y_{ij} + (w_{ik}y_{ij})x_{ij}]_{ik}$ which is in $(A_{ik}A_{ij}) \circ A_{ij}$. Moreover $A_{ik}A_{ij}^{(2)} = A_{ik}A_{ij}^2$ since $A_{ij}^2 \subsetneq A_{ii} + A_{jj}$ and $A_{ik}A_{jj} = 0$. That proves (b).

To prove (c) take x_{ij}, y_{ik}, w_{jk}, and e_j in (1) to obtain $x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik}) = y_{ik}(x_{ij}w_{jk}) + e_j[x_{ij}(y_{ik}w_{jk}) + w_{jk}(x_{ij}y_{ik})]$ as a result of simplifying

and noting that $e_j[y_{ik}(x_{ij}w_{jk})] = 0$. Multiplying this by e_i gives $e_i[x_{ij}(y_{ik}w_{jk})] = e_i[y_{ik}(x_{ij}w_{jk})]$. Interchanging the roles of i and j and of y and w in this gives $e_j[x_{ij}(y_{ik}w_{jk})] = e_j[w_{jk}(x_{ij}y_{ik})]$. Adding the last two equations we have $x_{ij}(y_{ik}w_{jk}) = e_i[y_{ik}(x_{ij}w_{jk})] + e_j[w_{jk}(x_{ij}y_{ik})]$ which is in $e_iA_{ik}^2 + e_jA_{jk}^2 \subseteq A_{ik}^{(2)} + A_{jk}^{(2)}$. Now $A_{ik}A_{jk} \subseteq A_{ij}$ so $A_{ik}A_{jk} = A_{ik} \circ A_{jk}$ and we have (c).

If we substitute x_{ii}, y_{ij}, w_{ij}, and e_i in (1) we get $x_{ii}(y_{ij}w_{ij}) = -1/2[y_{ij}(x_{ii}w_{ij}) + w_{ij}(x_{ii}y_{ij})] + e_i[y_{ij}(x_{ii}w_{ij}) + w_{ij}(x_{ii}y_{ij})] + y_{ij}[e_i(x_{ii}w_{ij})] + w_{ij}[e_i(x_{ii}y_{ij})]$ which is in $A_{ij} \circ (A_{ii} \circ A_{ij})$. Now $A_{ij}^2 \subseteq A_{ii} + A_{jj}$ and $A_{ii}A_{jj} = 0$ so $A_{ii}A_{ij}^2 = A_{ii}(A_{ij}^2)_{ii} = A_{ii}A_{ij}^{(2)}$ and we have proved (d).

Let e be an idempotent of A and define $B_e = \{x \text{ in } A_e(1) : xA_e(1/2) \subseteq A_e(0)\}$ and $C_e = \{x \text{ in } A_e(1) : xA_e(1/2) = 0\}$. Obviously $C_e \subseteq B_e \subseteq A_e(1)$. Moreover by [4, Lemma 1, pp. 506] C_e is an ideal of A, B_e is an ideal of $A_e(1)$, $B_e^2 \subseteq C_e$, and $A_e(1) - B_e$ is a Jordan algebra.

Let $f = e_1 + e_2$, $h = e_1 + e_3$, and $k = e_2 + e_3$. These are idempotents and if $g = e_i + e_j$ is one of them then, as just noted, B_g is an ideal of $A_g(1) = A_{ii} + A_{ij} + A_{jj}$ and C_g is an ideal of A. We will use the notation $B_{gi} = B_g \cap A_{ii}$, $B_{gij} = B_g \cap A_{ij}$,

$C_{gi} = C_g \cap A_{ii}$, and $C_{gij} = C_g \cap A_{ij}$.

Clearly $C_{gij} \subseteq B_{gij}$. On the other hand if b_{ij} is in $B_{gij} \subseteq A_{ij}$ then $b_{ij}A_g(1/2) = b_{ij}(A_{ik} + A_{jk}) = b_{ij}A_{ik} + b_{ij}A_{jk} \subseteq A_g(1/2)$ so $b_{ij}A_g(1/2) = 0$ and $B_{gij} = C_{gij}$.

Let $B_i = B_{e_i}$ and $C_i = C_{e_i}$ for $i = 1, 2, 3$. We now show that $B_{gij} = 0$ implies $B_g = B_i + B_j$. First we note that it is always true that $B_i \subseteq B_g$ for if x is in $B_i \subseteq A_{ii}$ then $x(A_{ij} + A_{ik}) \subseteq A_{jj} + A_{jk} + A_{kk}$. Thus $xA_g(1/2) = x(A_{ik} + A_{jk}) \subseteq A_g(0)$ since $xA_{jk} = 0$ and so $B_i \subseteq B_g$. Now let x be in $B_g = B_{gi} + B_{gj}$ and write $x = x_i + x_j$, x_i in B_{gi}, x_j in B_{gj}. Then x_i is in A_{ii} such that $x_i(A_{ik} + A_{jk}) \subseteq A_{kk}$. But B_g is an ideal of $A_g(1)$ so $x_iA_{ij} \subseteq B_{gj} \subseteq A_{jj}$. Therefore $x_i(A_{ij} + A_{ik}) \subseteq A_{jj} + A_{kk}$ and x_i is in B_i. Likewise x_j is in B_j so $B_g = B_i + B_j$.

b. <u>Completion of the proof</u>

Let $B = B_f + B_h + B_k$. We show B is an ideal of A as in [4, pp. 510]. As noted in §5a the subalgebra $A_h(1)$ has the property that $A_h(1) - B_h$ is a Jordan algebra. Since a Jordan algebra is stable, it follows that $A_{11}A_{13} \subseteq A_{13} + B_h$. But $B_{f1}A_{13} \subseteq A_f(0) = A_{33}$ and so $B_{f1}A_{13} \subseteq B_{h3}$. Evidently $B_{f1}A_{23} = 0$. By symmetry $B_{f2}A_{23} \subseteq B_{k3}$ and so $B_fA_f(1/2) \subseteq B$. Now $B_fA_f(0) = 0$ and since B_f is an ideal of $A_f(1)$

we have $B_f A_f(1) \subseteq B_f$. Therefore $B_f A \subseteq B$ and by symmetry $B_h A \subseteq B$ and $B_k A \subseteq B$ so B is an ideal of A.

Remark: At the time [11] was published the simple Jordan algebras of degree one and dimension greater than one were unknown. In [6] Jacobson shows they are isomorphic to the base field. This completed the classification of the simple Jordan algebras and since no new type appeared the proof in [11] is valid for all Jordan algebras of characteristic not two.

By Lemma 2.2 we can assume $B = 0$, N, or A. For $B = 0$ Albert proved in [4, Theorem 1, pp. 512-514] that A is a Jordan algebra. So by the results of Penico in [11] A has a Wedderburn decomposition.

Let $B = A$ and suppose the ideals C_f, C_h, and C_k are all nil. Then $A_{11} = B_{f1} + B_{h1}$ since $B_k \cap A_{11} = 0$. But we know that B_{f1} is an ideal of A_{11} since B_f is an ideal of $A_f(1)$ and $A_{11} \subseteq A_f(1)$. Moreover $B_{f1}^2 \subseteq B_f^2 \subseteq C_f$ so B_{f1} is a nil ideal of A_{11}. Likewise B_{h1} is a nil ideal of A_{11}. But then $A_{11} = B_{f1} + B_{h1}$ is nil which is a contradiction since e_1 is in A_{11}. Thus one of the ideals C_f, C_h, or C_k is a proper non-nil ideal of A and by Lemma 2.2 A has a Wedderburn decomposition. Thus we can assume $B = N$.

The above indicates our method of proof. Since we will make a few more such reductions we will label some of the cases to make it easier to follow the argument.

The following outline covers the remaining possibilities.

(A) $N = B_f = C_f$. This comes from assuming $C_g \neq 0$ where g is one of f, h, or k and without loss of generality we assume $g = f$. Clearly $C_f \neq A$ so by Lemma 2.2 we can assume $C_f = N$. So $N = C_f \subseteq B_f \subseteq B = N$ and $N = B_f = C_f$ as stated.

(B) $C_f = C_h = C_k = 0$, $B = N$.

Case (A) has two subcases

(A.1) $I_f = 0$ where $I_f = \left\{ \sum (y_0 w_{1/2})_1 : y_0 \text{ in } A_f(0) \text{ and } w_{1/2} \text{ in } A_f(1/2) \right\} = (A_{13}A_{33})_{11} + (A_{23}A_{33})_{22}$.

(A.2) $I_f = N$. This comes from $I_f \neq 0$. For clearly $I_f \neq A$ so by Lemma 2.2 we can assume $I_f = N$.

(A) $N = B_f = C_f$. Let $I = I_f + N$ where I_f is defined in (A.1) above. If x_1 is in I_f then by (4) we have $x_1(y_0 w_{1/2})_1 = [x_1(y_0 w_{1/2})]_1 = 2[(x_1 w_{1/2})y_0]_1 = 2[(x_1 w_{1/2})_{1/2} y_0]_1$ which is in I_f so I_f is an ideal of $A_f(1)$ and I is an ideal of $A_f(1)$. Since $N = C_f$ we have $NA_{13} = NA_{23} = 0$. Combining these results we get $AI = A_f(1)I + A_f(1/2)I + A_f(0)I \subsetneq I + A_{13}I + A_{23}I = I + A_{13}I_f + A_{23}I_f$. Now $A_{13}I_f = A_{13}I_{f1} + A_{13}I_{f2} = A_{13}I_{f1} = A_{13}(A_{13}A_{33})_{11}$ where $I_{f1} = I_f \cap A_{11}$. By Corollary 1 $A - N$ has three pairwise orthogonal idempotents since A has, so by [4, Theorem 1, pp. 512] $A - N$ is a Jordan algebra since $A - N$ is simple. Moreover $N = C_f \subsetneq A_f(1)$ so $A_{13}A_{33} \subsetneq A_{13} + N_1$. Therefore $A_{13}(A_{13}A_{33})_{11} \subsetneq A_{13}N_1 \subsetneq N \subsetneq I$. In the same manner we have $A_{23}I_f \subsetneq I$ and I is an ideal of A.

But $I \neq 0$ since $N \neq 0$ and $I \neq A$ since e_3 is not in I so by Lemma 2.2 we can assume that $I = N = C_f$. Thus $I_f \subsetneq N$ and $AI_f = A_f(1)I_f + A_f(1/2)I_f + A_f(0)I_f \subsetneq I_f + A_f(1/2)I_f = I_f + A_f(1/2)C_f = I_f$ so I_f is a nil ideal of A. This brings us to cases (A.1) and (A.2).

(A.1) $I_f = 0$. Hence $A_{33}A_{13} \subsetneq A_{13}$ and $A_{33}A_{23} \subsetneq A_{23}$. As noted in case (A), $A - N$ is a Jordan algebra and hence is stable. But $N = C_f \subsetneq A_f(1)$ so we have $A_{11}A_{13} \subsetneq A_{13}$ and $A_{22}A_{23} \subsetneq A_{23}$. Since $A_{11}A_{23} = A_{22}A_{13} = 0$ we can combine these results into

(5) $\qquad A_{ii}A_{j3} \subsetneq A_{j3}$ for $i = 1, 2, 3$ and $j = 1, 2$.

These relations enable us to construct another ideal.

Let $H_f = A_f(1/2) + [A_f(1/2)]^{(2)} = A_{13} + A_{23} + A_{13} \circ A_{23} + A_{13}^{(2)} + A_{23}^{(2)}$. Using (5) and Lemma 2 we now show that the subspace H_f is an ideal of A.
$A_{ii}(A_{13} + A_{23}) \subsetneq A_{13} + A_{23} \subsetneq H_f$ by (5). By (5) and (a) of Lemma 2 we obtain $A_{11}(A_{13} \circ A_{23}) \subsetneq (A_{11}A_{13}) \circ A_{23} \subsetneq A_{13} \circ A_{23} \subsetneq H_f$. By symmetry $A_{22}(A_{13} \circ A_{23}) \subsetneq H_f$. Also $A_{33}(A_{13} \circ A_{23}) \subsetneq A_{33}A_{12} = 0$. Now $A_{11}A_{13}^{(2)} \subsetneq (A_{11} \circ A_{13}) \circ A_{13}$ by (d) of Lemma 2. But by (5) $A_{11} \circ A_{13} = A_{11}A_{13} \subsetneq A_{13}$ so $A_{11}A_{13}^{(2)} \subsetneq H_f$. By the same argument $A_{33}A_{13}^{(2)} \subsetneq H_f$, $A_{22}A_{23}^{(2)} \subsetneq H_f$, and $A_{33}A_{23}^{(2)} \subsetneq H_f$. $A_{22}A_{13}^{(2)} \subsetneq A_{22}(A_{11} + A_{33}) = 0$ and similarly $A_{11}A_{23}^{(2)} = 0$. Thus we have $A_{ii}H_f \subsetneq H_f$ for $i = 1, 2, 3$.

Now $A_{12}(A_{13} + A_{23}) \subsetneq A_{23} + A_{13} \subsetneq H_f$ and by (c) of Lemma 2, $A_{12}(A_{13} \circ A_{23}) \subsetneq A_{13}^{(2)} + A_{23}^{(2)} \subsetneq H_f$. By (b) of Lemma 2, $A_{12}A_{13}^{(2)} \subsetneq (A_{12}A_{13}) \circ A_{13} \subsetneq A_{23} \circ A_{13} \subsetneq H_f$ and by symmetry $A_{12}A_{23}^{(2)} \subsetneq H_f$ so $A_{12}H_f \subsetneq H_f$.

Clearly $A_{13}(A_{13} + A_{23}) \subsetneq H_f$ and $A_{13}(A_{13} \circ A_{23}) \subsetneq A_{13}A_{12} \subsetneq A_{23} \subsetneq H_f$. By (5), $A_{13}A_{13}^{(2)} \subsetneq A_{13}(A_{11} + A_{33}) \subsetneq A_{13} \subsetneq H_f$ and $A_{13}A_{23}^{(2)} \subsetneq A_{13}(A_{22} + A_{33}) \subsetneq A_{13} \subsetneq H_f$ so $A_{13}H_f \subsetneq H_f$. But H_f is symmetric in A_{13} and A_{23} so we also have $A_{23}H_f \subsetneq H_f$.

Thus H_f is an ideal of A and by Lemma 2.2 we can assume $H_f = 0$, N, or A. $H_f = 0$ or N implies that $A_f(1/2) = 0$ since $N \subsetneq A_f(1)$. Thus $A = A_f(1) \oplus A_f(0)$,

$A_f(0)$ is an ideal of A with $A_f(0) \neq 0$, N, or A, so by Lemma 2.2 A has a Wedderburn decomposition.

Thus we can assume $H_f = A$. Then $A_{11} = (A_{13}^2)_{11}$ and $A_{22} = (A_{23}^2)_{22}$, so $N_1A_{11} = N_1(A_{13}^2)_{11} \subsetneq N_1A_{13}^{(2)} \subsetneq (N_1 \circ A_{13}) \circ A_{13}$ by (d) of Lemma 2 where $N_i = N \cap A_{ii}$ and $N_{ij} = N \cap A_{ij}$ for $i, j = 1, 2, 3$. But $N_1 = C_{f1}$ so by (5) $N_1 \circ A_{13} = N_1A_{13} = 0$. Thus $N_1A_{11} = 0$. But e_1 is in A_{11} so $N_1 = e_1N_1 = 0$. In the same manner we obtain $N_2 = 0$ so $N = N_{12} \subsetneq A_{12}$. Then by (b) of Lemma 2, $N_{12}A_{11} \subsetneq N_{12}A_{13}^{(2)} \subsetneq (N_{12}A_{13}) \circ A_{13} = 0$ again because $N = C_f$. But this gives $N = N_{12} = e_1N_{12} = 0$ which is a contradiction. That completes the proof in case (A.1).

Before taking up case (A.2) we need a lemma.

<u>Lemma 2.3</u>: If $N = B_1 + B_2 + B_3$ and $H_g = A_g(1/2) + [A_g(1/2)]^{(2)}$ where $g = f$, h, or k then $H_g + N$ is a non-zero ideal of A.

<u>Proof</u>: As noted in case (A) $A - N$ is a Jordan algebra and hence it is stable. This and having $N \subsetneq A_{11} + A_{22} + A_{33}$ gives

$$A_{ii}A_{ij} \subsetneq A_{ij} + B_j \text{ for } i \neq j;\ i, j = 1, 2, 3 \quad . \tag{6}$$

Without loss of generality we can assume $g = f$. Then the proof that $H_f + N$ is an ideal of A is

essentially the same as the proof in case (A.1) that H_f was an ideal of A. We only indicate this by considering two of the relations that need to be checked.

By (6) and (a) of Lemma 2 we have $A_{11}(A_{13} \circ A_{23}) \subsetneq (A_{11}A_{13}) \circ A_{23} \subsetneq (A_{13} + B_3) \circ A_{23} \subsetneq A_{13} \circ A_{23} + N \subsetneq H_f + N$. By (6) and (d) of Lemma 2 we have $A_{11}A_{13}^{(2)} \subsetneq (A_{11} \circ A_{13}) \circ A_{13} \subsetneq (A_{13} + B_3) \circ A_{13} \subsetneq A_{13}^{(2)} + N \subsetneq H_f + N$. In this fashion we find that $H_f + N$ is an ideal of A.

<u>Corollary 2</u>: If $A = H_f + N = H_h + N = H_k + N$ then H_f is a subalgebra of A. If we also have $C_1 = C_2 = C_3 = 0$ then $A = H_f + N$ is a Wedderburn decomposition for A.

<u>Proof</u>: From the hypothesis we immediately have $A_{ii} = (A_{ij}^2)_{ii} + B_i = (A_{ik}^2)_{ii} + B_i$ and $A_{ij} = A_{ik}A_{jk}$ for i, j, k distinct, $i, j, k = 1, 2, 3$. Hence by (c) of Lemma 2 $(A_{13}^2)_{33} = [A_{13}(A_{12}A_{23})]_{33} \subsetneq (A_{12}^{(2)} + A_{23}^{(2)})_{33} = (A_{23}^{(2)})_{33} = (A_{23}^2)_{33}$. Similarly $(A_{23}^2)_{33} \subsetneq (A_{13}^2)_{33}$ so $(A_{23}^2)_{33} = (A_{13}^2)_{33}$. Denote these as S_3. By the same type of argument we also have $S_2 = (A_{23}^2)_{22} = (A_{12}^2)_{22}$ and $S_1 = (A_{12}^2)_{11} = (A_{13}^2)_{11}$.

The proof that $A_{12}H_f \subsetneq H_f$ given in case (A.1) is valid here since (5) wasn't used. Therefore $(A_{13} \circ A_{23})H_f = A_{12}H_f \subsetneq H_f$.

Clearly $A_{13}(A_{13} + A_{23}) \subseteq H_f$ and $A_{13}(A_{13} \circ A_{23}) = A_{13}A_{12} = A_{23} \subseteq H_f$. Also $A_{13}A_{23}^{(2)} \subseteq (A_{13}A_{23}) \circ A_{23} = A_{12} \circ A_{23} = A_{13} \subseteq H_f$ by (b) of Lemma 2. Similarly $A_{13}A_{12}^{(2)} \subseteq A_{13} \subseteq H_f$. From these and the relations for S_1 and S_3 we have $A_{13}A_{13}^{(2)} = A_{13}[(A_{13}^2)_{11} + (A_{13}^2)_{33}] = A_{13}[(A_{12}^2)_{11} + (A_{23}^2)_{33}] \subseteq A_{13}(A_{12}^{(2)} + A_{23}^{(2)}) \subseteq H_f$. Thus $A_{13}H_f \subseteq H_f$ and by symmetry $A_{23}H_f \subseteq H_f$.

By symmetry and what we have just checked of H_fH_f it remains to show that $A_{13}^{(2)}A_{13}^{(2)}$ and $A_{13}^{(2)}A_{23}^{(2)}$ are subsets of H_f. But $A_{13}^{(2)}A_{13}^{(2)} = A_{13}^{(2)}[(A_{13}^2)_{11} + (A_{13}^2)_{33}] = A_{13}^{(2)}[(A_{12}^2)_{11} + (A_{23}^2)_{33}] \subseteq A_{13}^{(2)}A_{12}^{(2)} + A_{13}^{(2)}A_{23}^{(2)}$. These summands are handled in the same manner so we will only consider the latter. We note that $A_{13}^{(2)}A_{23}^{(2)} = A_{13}^2A_{23}^2$ since $A_{ii}A_{jj} = 0$ for $i \neq j$. Taking $x_{13}, y_{13}, z_{23}, w_{23}$ in (1) gives $4(x_{13}y_{13})(z_{23}w_{23}) + 4(x_{13}z_{23})(y_{13}w_{23}) + 4(x_{13}w_{23})(y_{13}z_{23}) = x_{13}[y_{13}(z_{23}w_{23}) + z_{23}(y_{13}w_{23}) + w_{23}(y_{13}z_{23})] + y_{13}[x_{13}(z_{23}w_{23}) + z_{23}(x_{13}w_{23}) + w_{23}(x_{13}z_{23})] + z_{23}[x_{13}(y_{13}w_{23}) + y_{13}(x_{13}w_{23}) + w_{23}(x_{13}y_{13})] + w_{23}[x_{13}(y_{13}z_{23}) + y_{13}(x_{13}z_{23}) + z_{23}(x_{13}y_{13})]$ which is in $A_{13}[A_{13}A_{23}^2 + A_{23}(A_{13}A_{23})] + A_{23}[A_{13}(A_{13}A_{23}) + A_{23}A_{13}^2] \subseteq A_{13}H_f + A_{23}H_f \subseteq H_f$ by our previous results. Also $4(x_{13}z_{23})(y_{13}w_{23}) + 4(x_{13}w_{23})(y_{13}z_{23})$ is in $(A_{13}A_{23})^2 \subseteq A_{12}^2 \subseteq H_f$. So $A_{13}^{(2)}A_{23}^{(2)} = A_{13}^2A_{23}^2 \subseteq H_f$ and

H_f is a subalgebra of A.

Now assume we also have $C_1 = C_2 = C_3 = 0$. Evidently $S_1A_{12} = (A_{13}^2)_{11}A_{12} \subseteq A_{12}A_{13}^{(2)} \subseteq (A_{12}A_{13}) \circ A_{13} \subseteq A_{23} \circ A_{13} \subseteq A_{12}$ by (b) of Lemma 2. Likewise $S_1A_{13} \subseteq A_{13}$. So for x in S_1 we get $x(A_{12} + A_{13}) \subseteq A_{12} + A_{13}$ while x in B_1 implies that $x(A_{12} + A_{13}) \subseteq B_2 + B_3 \subseteq A_{22} + A_{33}$. Therefore $S_1 \cap N = S_1 \cap B_1 \subseteq C_1 = 0$. Similarly $S_2 \cap N = S_3 \cap N = 0$ so $A = H_f + N$ is a Wedderburn decomposition of A and that completes the proof of the corollary.

(A.2) $I_f = N$. Then $N = B_f \subseteq A_{11} + A_{22}$, $B_{f12} = 0$, and $N = B_1 + B_2$ as in §5a. So by Lemma 2.3 $H_f + N$, $H_h + N$, and $H_k + N$ are non-zero ideals of A. If one of them, say $H_f + N$ for example, is N then $H_f \subseteq N$, $A_f(1/2) = 0$, $A = A_f(1) \oplus A_f(0)$, $A_f(0)$ is a proper non-nil ideal of A and A has a Wedderburn decomposition by Lemma 2.2. Thus we can assume $A = H_f + N = H_h + N = H_k + N$.

If $C_1 = C_2 = 0$ (we already have $C_3 \subseteq B_3 = 0$) then by Corollary 2 A has a Wedderburn decomposition. Therefore we assume, without loss of generality, that $C_1 \neq 0$. Clearly $C_1 \neq A$ so by Lemma 2.2 we can further assume that $C_1 = N$.

From $N = I_f = B_f$ we notice that $N^2 = I_fB_f = 0$. For if b_1 is in $B_f \subseteq A_f(1)$ and $(y_ow_{1/2})_1$ is in $I_f \subseteq A_f(1)$ then $b_1w_{1/2}$ is in $A_f(0)$ so by (4)

$b_1(y_0w_{1/2})_1 = [b_1(y_0w_{1/2})]_1 = 2[(b_1w_{1/2})y_0]_1 = 0.$

But we also have $N = C_1 \subsetneq A_{11} = (A_{13}^2)_{11} + N$. This and (d) of Lemma 2 give $NA_{11} \subsetneq NA_{13}^{(2)} + N^2 = NA_{13}^{(2)} \subsetneq (N \circ A_{13}) \circ A_{13} = 0$. But e_1 is in A_{11} so $N = e_1N = 0$ which is a contradiction.

There remains case (B).

(B) $C_f = C_h = C_k = 0$, $B = N$. We saw in §5a that $B_{gij} = C_{gij}$ so $B_{gij} = 0$ for $g = f, h, k$; $i \neq j$; $i, j = 1, 2, 3$. This and the related results in §5a give $N = B_1 + B_2 + B_3$. In addition $C_i \subsetneq C_g$ for if x is in C_i then $x(A_{ij} + A_{ik}) = 0$. But x is in A_{ii} so $xA_{jk} = 0$. Therefore $xA_g(1/2) = x(A_{ik} + A_{jk}) = 0$ and x is in C_g. Thus $C_1 = C_2 = C_3 = 0$. We then proceed as in the first part of case (A.2) using Lemmas 2.2 and 2.3 and Corollary 2 to show that A has a Wedderburn decomposition. That completes the proof of case (B) and consequently of Theorem 2.

6. A reduction theorem

Theorem 4 is important not only because it simplifies the problem of showing that each algebra of certain classes of algebras has a Wedderburn decomposition, but also because it suggests where we can expect difficulties in general; namely, in the algebras with at most two pairwise orthogonal idempotents.

Let P be a property of algebras such that if A has property P then each of its subalgebras has property P. Let $\mathfrak{P}$ be the class of all commutative strictly power-associative algebras of characteristic not two having property P with $A - N$ separable for A in $\mathfrak{P}$.

Theorem 4: Every algebra in $\mathfrak{P}$ has a Wedderburn decomposition if and only if every algebra in $\mathfrak{P}$ that has at most two pairwise orthogonal idempotents has a Wedderburn decomposition.

Proof: The necessity of the condition is obvious so we assume that every algebra in $\mathfrak{P}$ that has at most two pairwise orthogonal idempotents has a Wedderburn decomposition. Thus if A is in $\mathfrak{P}$ and $n = d(A) = 1$ or 2 then A has at most two pairwise orthogonal idempotents and hence A has a Wedderburn decomposition. If $n \geq 3$ then assume that every algebra of $\mathfrak{P}$ with dimension less than n has a Wedderburn decomposition. Evidently we can assume A has three pairwise orthogonal idempotents. If $A - N$ is simple then A has a Wedderburn decomposition by Theorem 3, so we only need to show that we can assume $A - N$ is simple. This follows immediately from Lemma 4.1.

Lemma 4.1: If D is a non-nil ideal of A with

$D \neq 0, A$ then A has a Wedderburn decomposition.

Proof: D has an idempotent since it is non-nil. It is also well-known that this implies D has a principle idempotent, say e (e is principle if $A_e(0)$ is nil). Write $D = D_e(1) + D_e(1/2) + D_e(0)$ and let M be the radical of D. According to Albert [4, Theorem 7, pp. 524] $D_e(1/2) + D_e(0) \subseteq M$ since e is principle. We write $M = J + D_e(1/2) + D_e(0)$ where $J = M \cap D_e(1)$. We may also write $A = A_e(1) + A_e(1/2) + A_e(0)$ and it should be evident that $D_e(\lambda) = D \cap A_e(\lambda)$. However e is in D so $xe = \lambda x$ is in D for every x in $A_e(\lambda)$. By taking $\lambda = 1$ and $1/2$ we see that $A_e(1) + A_e(1/2) \subseteq D$, $A = D_e(1) + D_e(1/2) + A_e(0)$, and $D_e(0) \subset A_e(0)$ ($D_e(0) \neq A_e(0)$ since $D \neq A$). Moreover $D_e(0)$ is an ideal of $A_e(0)$ since D is an ideal of A and $A_e(0)$ is a subalgebra. Albert proceeded to show in [4, pp. 525] that M is an ideal of A and his proof is valid here since he did not use the simplicity of A for this result. Thus $M \subseteq N$.

Now $D \neq 0, A$ so $0 < d(D) < n$ and by the induction hypothesis $D = T + M$ where T is a semi-simple subalgebra of D (and hence of A) and $T \cap M = 0$. Thus $T \subseteq D_e(1)$ and $D_e(1) = T + J$ is a Wedderburn decomposition of $D_e(1)$. Likewise $D \neq 0, A$ means that $0 < d(A_e(0)) < n$ so $A_e(0) = S + N_o$ where S is a semi-simple subalgebra

of $A_e(0)$ (and hence of A), N_o is the radical of $A_e(0)$, and $S \cap N_o = 0$. Note that $D_e(0) \subsetneq N_o$ since $D_e(0)$ is a nil ideal of $A_e(0)$. Let $N_a = J + A_e(1/2) + N_o$. Then $N \subsetneq N_a$ and just as in the proof of Theorem 1 we find that N_a is nil so $N = N_a$, $J = N \cap A_e(1)$, and $N_o = N \cap A_e(0)$. But $S \subsetneq A_e(0)$ and $T \subsetneq A_e(1)$ are semi-simple subalgebras of A so $S \oplus T$ is a semi-simple subalgebra of A. Moreover $(S \oplus T) \cap N = 0$ since $S \cap N = S \cap N_o = 0$ and $T \cap N = T \cap J = 0$. Hence $A = A_e(1) + A_e(1/2) + A_e(0) = T + (J + A_e(1/2) + N_o) + S = (S \oplus T) + N$ is our desired Wedderburn decomposition of A.

7. An application

We are now able to apply Theorem 4 to the class of stable algebras (defined in §2).

Theorem 5: If A is a stable commutative power-associative algebra over an algebraically closed field F of characteristic zero then A has a Wedderburn decomposition.

Proof: Let P be the property of being stable and having an algebraically closed base field of characteristic zero. Then by Theorem 4 we can assume A has at most two pairwise orthogonal idempotents.

Since A is non-nil it has at least one idempotent and hence a principle idempotent, say e. Then by [4, Theorem 7, pp. 524] $A_e(1/2) + A_e(0) \subseteq N$. As in the proof of Theorem 1 we have $N_1 = A_e(1) \cap N$ is the radical of $A_e(1)$. So if $A_e(1)$ has a Wedderburn decomposition, say $A_e(1) = S + N_1$, then $A = S + N$ is a Wedderburn decomposition for A. So without loss of generality we can assume A has a unity element 1 to begin with.

Suppose that A does not have two orthogonal idempotents. Then 1 is a primitive idempotent (that is $1 \neq e_1 + e_2$ for orthogonal idempotents e_1 and e_2). For an algebraically closed field the degree of A is the maximum number of pairwise orthogonal idempotents whose sum is the unity element. Thus A is of degree one. Then as in [4, proof of Theorem 9, pp. 526-527] $A = 1 \cdot F + N$ which is a Wedderburn decomposition of A.

Thus we can assume A is of degree two. So $1 = u + v$ for primitive orthogonal idempotents u and v. Then $A = A_1 + A_{12} + A_2$ as in §3 where we are letting $A_{11} = A_1$ and $A_{22} = A_2$. Then A_1 and A_2 are of degree one and as above $A = uF + R_1$ and $A_2 = vF + R_2$ where R_1 is the radical of A_1. Let $N_1 = N \cap A_1$ and $N_{12} = N \cap A_{12}$ as usual.

Let x be an element of A_{12}. If x^2 is not in $R_1 + R_2$ then x is said to be non-singular and it is known [4, Lemma 10, pp. 517] that $x^2 = \alpha + g$ for g in $R_1 + R_2$ and α a non-zero element of F. If x^2 is in $R_1 + R_2$ then x is said to be singular.

Suppose every element in A_{12} is singular. If x, y are in A_{12} then $2xy = x^2 + y^2 - (x - y)^2$ which is in $R_1 + R_2$ so $A_{12}^2 \subseteq R_1 + R_2$. Let $M = R_1 + A_{12} + R_2$. Since A is stable we have $AM = (A_1 + A_{12} + A_2)$ $(R_1 + A_{12} + R_2) \subseteq A_1R_1 + A_{12} + A_{12}^2 + A_2R_2 \subseteq M$. Moreover M is nil for if not then M has an idempotent $f = f_1 + f_{12} + f_2$ with f_i in R_i and f_{12} in A_{12}. It is clear that $f_{12} \neq 0$. Computing $f^2 = f$ we obtain $f_1^2 + f_{12}^2 + f_2^2 + (f_1 + f_2)f_{12} = f_1 + f_{12} + f_2$. Equating the components in A_{12} we get $(f_1 + f_2)f_{12} = f_{12}$. Let T be the linear transformation given by $T(x) = xf_{12}$ for all x in $A_1 + A_2$. Then it is known [4, pp. 517] that T is nilpotent. But $(f_1 + f_2)f_{12} = f_{12}$ means that $T^k(f_1 + f_2) = f_{12}$ for every positive integer k. Thus $f_{12} = 0$ which is a contradiction. Therefore M is a nil ideal, $M \subseteq N$, $M = N$, $R_i = N_i$, $A_{12} \subseteq N$, and $A = (uF + vF) + N$ is a Wedderburn decomposition of A.

Thus we can assume there is a non-singular element x in A_{12}. Let $M = R_1 + R_1A_{12} + R_2A_{12} + R_2$. Then the proof by Kokoris in [8] that M is an ideal of A is valid here since he only used the simplicity of A to conclude that A_{12} had a non-singular element. Moreover the proof that M is nil is just a duplication of the argument in the last paragraph so $M \subsetneq N$, $R_1 \subsetneq N_1$, and hence $R_1 = N_1$.

Before we continue with our argument we give two lemmas. We need parts of Lemmas 3 and 7 of [5] and we state them here as Lemma 5.1.

<u>Lemma 5.1</u>: If x is a non-singular element of A_{12} then there exists a quantity c in $F[x^2] \subsetneq A_1 + A_2$ such that $w^2 = 1$ for $w = cx$ in A_{12}. Moreover $A_{12} = wB \dot{+} G$ where $B = \{b$ in $A_1 + A_2 : w(wb) = b\}$ and $G = \{g$ in $A_{12} : gw = 0\}$.

<u>Remarks</u>: There are several comments that need to be made regarding Lemma 5.1.

First we would like to indicate briefly how we intend to use Lemma 5.1 to construct a Wedderburn decomposition for A. Let $w = w_1$. Then we will show that we can keep "breaking elements w_i out of G" where $w_i w_j = \delta_{ij}$ (the Kronecker delta) until what remains of G is a set of singular elements $G_{(m)} \subsetneq N_{12}$. From this we see that $A = (uF + w_1F + \dots + w_mF + vF) + N$ is a Wedderburn decomposition of A.

Next we note that $B = \{\alpha + b : \alpha$ in F and b in $N_1 + N_2$ such that $w(wb) = b\}$. For if x is in B then $x = \alpha u + \beta v + b$ with α, β in F and b in $N_1 + N_2$. Thus $x = w(wx) = w[w(\alpha u + \beta v + b)] = w[1/2(\alpha + \beta)w + wb] = 1/2(\alpha + \beta) + w(wb)$ so $\alpha = \beta = 1/2(\alpha + \beta)$, $b = w(wb)$, and $x = \alpha + b$. Conversely if $x = \alpha + b$ as above it is clear that $w(wx) = x$ so x is in B. In particular this means that $wB = \{\alpha w + wb : \alpha$ in F and b in $N_1 + N_2$ such that $w(wb) = b\}$. The importance in this for us is that $wB \subseteq wF \dot{+} N_{12}$.

Lemma 5.1 holds for any stable idempotent $u \neq 1$. But we are assuming A is stable so Lemma 5.1 holds for any idempotent $u \neq 1$.

Let $e = 1/2(1 + w)$. Then e is an idempotent and for x in A, $ex = 1/2(1 + w)x = 1/2x$ if and only if $wx = 0$. Therefore w is in the annihilator of $A_e(1/2)$. In particular, taking x in A_{12} gives $G = A_{12} \cap A_e(1/2)$. And since A is stable it is evident that $[A_f(1/2)]^{2m-1} \subseteq A_f(1/2)$ for any idempotent f and every positive integer m. Thus $G^{2m-1} \subseteq G$ for every positive integer m.

If z is a non-singular element in G then according to Lemma 5.1 there is a quantity c in $F[z^2]$ such that $y^2 = 1$ for $y = cz$. But then $y = \alpha_1 z + \alpha_2 z^3 + \dots + \alpha_k z^{2k-1}$ and by the last paragraph z^{2m-1} is in G for

every positive integer m so y is in G and $wy = 0$. Applying Lemma 5.1 with respect to u and then with respect to e we can write $A_{12} = yB_y \dot{+} G_y$ and $A_e(1/2) = yB_{y1} \dot{+} G_{y1}$ where $B_y = \{b \text{ in } A_1 + A_2 : y(yb) = b\}$, $G_y = \{g \text{ in } A_{12} : gy = 0\}$, $B_{y1} = \{b \text{ in } A_e(1) + A_e(0) : y(yb) = b\}$, and $G_{y1} = \{g \text{ in } A_e(1/2) : gy = 0\}$.

<u>Lemma 5.2</u>: For y in G with $y^2 = 1$ we know that every element h in $A_e(1/2)$ has a unique representation in the form $h = yb + g$ for yb in yB_{y1} and g in G_{y1}. But for h in G we also have yb in yB_y and g in $G \cap G_y$.

<u>Proof</u>: $G_{y1} \subseteq A_e(1/2)$ so $gw = 0$ as noted above. But we have h in A_{12} so $(yb)_1 + (yb)_2 + g_1 + g_2 = 0$ where the subscripts refer to the subspaces A_1, A_2, and A_{12}. Examining the $A_1 + A_2$ component of the equation $0 = wg = w(g_1 + g_2) + wg_{12}$ we have $wg_{12} = 0$ since A is stable. Similarly $yg_{12} = 0$. Thus g_{12} is in $G \cap G_y$.

Since A is stable $(yb)_{12} = [y(b_1 + b_{12} + b_2)]_{12} = y(b_1 + b_2)$. Thus $b_1 + b_2$ is in $(A_1 + A_2) \cap (A_e(1) + A_e(0))$ such that $y[y(b_1 + b_2)] = b_1 + b_2$, so $y(b_1 + b_2)$ is in $yB_y \cap yB_{y1}$. Therefore $h = (yb)_{12} + g_{12}$ where $(yb)_{12}$ is in yB_{y1} and g_{12} is in G_{y1}.

But h has a unique representation in that form; namely, $h = yb + g$ so we must have $yb = (yb)_{12}$ in yB_y and $g = g_{12}$ in $G \cap G_y$ which proves the lemma.

Previous to Lemma 5.1 we had gotten to the point where A_{12} had a non-singular element and $N_i = R_i$. We can now put the intermediate pieces together by induction to give a Wedderburn decomposition for A.

By Lemma 5.1 A_{12} contains an element w_1 such that $w_1^2 = 1$ and $A_{12} = w_1B_1 \dotplus G_1$ where $B_1 = \{\alpha + b :$ α in F and b in $N_1 + N_2$ such that $w_1(w_1b) = b\}$ and $G_1 = \{g$ in $A_{12} : gw_1 = 0\}$.

If every element of G_1 is singular then let $M_1 = N + G_1$. For $x = n + g$ in M_1, $x^2 = n^2 + 2ng + g^2$ which is in N so x^2 is nilpotent, x is nilpotent, and M_1 is nil. In particular for x, y in G_1 we have $2xy = x^2 + y^2 - (x - y)^2$ in N so $G_1^2 \subseteq N$. Thus $A_{12}M_1 = A_{12}(N + G_1) \subseteq N + A_{12}G_1 \subseteq N + (w_1F + N + G_1)G_1 \subseteq N + G_1^2 \subseteq N \subseteq M_1$ and $A_1M_1 \subseteq u(N + G_1) + N_1(N + G_1) \subseteq N + G_1 = M_1$. Likewise $A_2M_1 \subseteq M_1$ so M_1 is a nil ideal of A. Hence $G_1 \subseteq N$ and $A = (uF + w_1F + vF) + N$ is a Wedderburn decomposition of A. Thus we can continue by assuming G_1 has a non-singular element.

For notation in the general case we will have w_i in A_{12} with $w_i^2 = 1$ and will write $A_{12} = w_iB_i \dotplus G_i$ by Lemma 5.1 where B_i and G_i are defined in terms

of w_i as in the case $i = 1$ above. Let $e_i = 1/2(1 + w_i)$.

Assume that $A_{12} = w_1F + \dots + w_{m-1}F + N_{12} + G_{(m-1)}$ where $G_{(m-1)} = \bigcap_{i=1}^{m-1} G_i$, $G_{(m-1)}$ has a non-singular element x, and $w_iw_j = \delta_{ij}$ for $i, j = 1, 2, \dots, m - 1$.

From Lemma 5.1 as before there is an element w_m in $G_{(m-1)}$ such that $w_m^2 = 1$ and $w_mw_i = 0$ for $i = 1, 2, \dots, m - 1$. Let $G_{(m)} = G_m \cap G_{(m-1)}$. Then we wish to show that we can write $A_{12} = w_1F + \dots + w_mF + N_{12} + G_{(m)}$.

Let h be in $G_{(m-1)}$. Then h is in G_i for each $i = 1, 2, \dots, m - 1$ so taking $G = G_i$ and $y = w_m$ in Lemma 5.2 the element h has a unique representation in the form $h = w_mb_i + g_i$, $i = 1, 2, \dots, m - 1$, where w_mb_i is in w_mB_m and g_i is in $G_i \cap G_m$. But by Lemma 5.1 h also has the unique representation $h = w_mb + g$ for w_mb in w_mB_m and g in G_m. Thus $g_i = g$ for $i = 1, 2, \dots, m - 1$ so g is in $G_{(m)}$ as desired. For if a is in A_{12} we have $a = \alpha_1w_1 + \dots + \alpha_{m-1}w_{m-1} + n_{m-1} + h$ where n_{m-1} is in N_{12} and h is in $G_{(m-1)}$. But by our last result we can write this as $a = \alpha_1w_1 + \dots + \alpha_{m-1}w_{m-1} + n_{m-1} + (\alpha_mw_m + n_m + g) = \alpha_1w_1 + \dots + \alpha_{m-1}w_{m-1} + \alpha_mw_m + n + g$, with n in N and g in $G_{(m)}$ as desired.

This inductive process cannot continue indefinitely

since $d(G_{(m)}) < d(G_{(m-1)})$ so for some m, $G_{(m)}$ must consist of singular elements. Then as before $G_{(m)} \subseteq N$ and $A = (uF + w_1F + \dots + w_mF + vF) + N$ is a Wedderburn decomposition of A.

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