EXPRESSIVE COMPLETENESS

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ABSTRACT

EXPRESSIVE COMPLETENESS

by Herbert E. Hendry

In this essay an attempt is made to explicate a concept of expressive completeness for first-order extensional languages. The explication is intended to fill a gap in our present understanding of such systems. We are often concerned to determine whether a system is consistent, is complete (in some definite sense), or has an independent set of axioms. These concepts as they are usually understood are all explained with reference to certain relevant features of statements of the system. But with regard to the terms of a system we have only a well-defined notion of independence. Certain symmetries between statements on the assertory side of a system and terms on its conceptual side suggest that we look for corresponding concepts of consistency and completeness. Expressive completeness, as it is here explained, is intended to be a candidate for the latter.

The locus of this explication is an axiom system. Very roughly, a first-order extensional language is said to be expressively complete if it can define terms whose extensions exhaust the subclasses of, and relations on, its universe. The bulk of the work is devoted to refining and justifying this account and to developing its consequences. Among the

more interesting results are that for any finite universe an expressively complete first-order extensional language can be constructed, that no language is expressively complete if its universe is infinite, and that no language, whether its universe be finite or infinite, is complete if everything is in its universe.

Expressive completeness is compared with a number of other completeness concepts. They are all found to differ in rather essential ways. All, save one, are obviously unsuitable to fill the above-mentioned gap in our understanding. The exception is Tarski's notion of the completeness of concepts. A close examination of Tarski's development of this notion shows that although it isolates an important concept, it cannot be regarded as an acceptable analysis of expressive completeness. Relating these two concepts, it is shown that under certain minimal conditions any system that is expressively complete contains a set of sentences that is complete in the sense of Tarski. It is observed that the converse does not hold. That is, there are first-order extensional languages that are complete in the sense of Tarski but not expressively complete.

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1. INTRODUCTION

There is, in one important respect, a striking similarity between the roles played by the terms and statements of an axiomatic language.[1] We distinguish the primitive (undefined) terms of such a language from its derivative (defined) terms. Similarly, we distinguish between primitive statements (axioms or postulates) and derivative statements (theorems). These distinctions are vital for the study of axiomatic languages. For systematicity is of the very nature of such languages; and these

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[1] I distinguish between axiomatic systems and axiomatic languages. An axiomatic language is one kind of axiomatic system. An axiomatic system, of course, is axiomatic. It is a system of signs or expressions governed by syntactical rules. An axiomatic system, however, need not be interpreted. A language, and hence an axiomatic language, must be interpreted to the extent of having terms and statements. The _fundamentum divisionis_ here is their mode of meaning. To be a term is to have an extension. To be a statement is to have truth value. The taxonomy of systems is presently not very well developed. Consequently, the above pronouncements must be regarded with considerable caution. My sole motive for including them is to make it clear that only such systems as are languages are under consideration. Cf. Church, sec. 07.

distinctions point to two important sources of systematicity: definition and implication.[2]

The concepts of independence, consistency, and completeness have received much deserved attention in the field of axiomatics. They are well understood. Yet, ordinarily understood, these concepts are explained with reference to statements. The parallel roles of terms and statements suggest that we look for corresponding concepts whose explanation would involve reference to terms. Indeed, we do not have to look far to find a corresponding notion of independence.[3] It is now commonplace to inquire whether a set of terms is independent, that is, whether any one of them is definable in terms of the others.[4] But

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[2]Frege wrote that the "aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. . . . The further we pursue these enquiries [into the foundations of arithmetic], the fewer become the primitive truths to which we reduce every thing; and this simplification is in itself a goal worth pursuing." Frege, p. 2. But Frege does not tell us why simplification is worth pursuing. Perhaps, in line with the doctrines of neo-pragmatism, it is that simplicity (as a measure of systematicity) is in part constitutive of truth.

[3]For relevant literature by Padoa, Beth, Tarski, and Lindenbaum see Tarski(1).

[4]The importance of independence should not be minimized. In the extreme case every accepted statement of an axiomatic language might be counted as one of its axioms. Here there is no systematicity. If, on this intuitive level, we may speak of degrees of independence, we might consider it to be a law that, other things being equal, systematicity varies directly in proportion to degree of independence.

corresponding concepts of consistency and completeness do not seem to have received careful attention.

The primary aim of this essay is to provide an explication of <u>expressive completeness</u> for first-order extensional languages. Expressive completeness, as it is here explicated, is intended to be a property a language has (or fails to have) in virtue of certain relevant features of its terms. If successful, then, the essay will constitute a significant first step towards filling a lacuna in semantic theory.

2. A CRITERION FOR ADEQUACY

Antecedent to the explication of a concept it is desirable to have some relatively clear and objective criterion by which to determine whether the proposed explication is adequate. Subject to explanation and justification in the next section the criterion adopted in this essay is the following. An explication of expressive completeness will be considered adequate if, and only if, it fulfills the condition that:

> A language $\underline{L}_1$ is expressively complete if, and only if, there is no language $\underline{L}_2$ such that (i) $\underline{L}_1$ and $\underline{L}_2$ have the same universe of discourse, (ii) $\underline{L}_1$ and $\underline{L}_2$ have the same meaning base, and (iii) there is a term $\underline{T}_2$ of $\underline{L}_2$ and there is no term $\underline{T}_1$ such that $\underline{T}_1$ is definable in $\underline{L}_1$ and $\underline{T}_1$ has the same meaning as $\underline{T}_2$.[1]

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[1]The criterion is that it fulfills this condition. The formulation of the condition might be regarded as a defintion, albeit a very poor one. Here I speak of a criterion for adequacy not in the sense of Tarski(1), Tarski(2) and Carnap(1) but in the sense of Carnap(5). In the latter we find Carnap writing of a similar formulation (for an explication of L-truth) that "it is an informal formulation of a condition which any proposed definition of L-truth must fulfill in order to be adequate as an explication for our explicandum." He adds that it has "merely an explanatory and heuristic function." Carnap(5), p. 10.

3. EXPLANATION AND JUSTIFICATION OF THE CRITERION

In reverse order, each of the three parts of the criterion will be explained and justified.

Although a precise formulation of (iii) is difficult, its intent is relatively clear. The point is that a language is expressively complete if any term added to its base would be synonymous with some term already definable in the language. Any term added to an expressively complete language is redundant in the language. And to an expressively incomplete language there is always a term which could be added without redundancy. Were this part of the criterion violated, one of two absurd consequences would follow. Either (a) a language could at once be expressively complete and yet unable to formulate everything that there is to formulate about its subject matter, or (b) a language could be expressively incomplete even though it could formulate everything there is to formulate about its subject matter.

By the very nature of the case, the sort of justification just outlined suffers from crudeness. Its crudeness issues from its lack of precision. But imprecision always accompanies criteria for an adequate explication. (This is especially true when the concept to be explicated--expressive completeness in this instance--is not firmly entrenched in pre-existent usage.) A demand for complete precision is a demand that the concept to be explicated be precise. Were this the case there would be no need for analysis.

There are, however, several further difficulties with the formulation. But their resolution would demand the solution to a host of rather difficult problems. Among these are the problems associated with the nature of language and the disposition of definability. In connection with the first difficulty it will be noticed that the ensuing analysis relies on an uncritical quantification over first-order extensional languages. Thus it is committed to their existence. If it be asked what precisely <u>are</u> the values of these variables, that is, what sort of thing is such a language, the honest reply is that there is no fully acceptable answer.[1] I am inclined to

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[1]Quine's "Language is a social art" (Quine(5), p. ix.) and its ilk are of little help in this connection, nor, perhaps, are they intended to be. Yet even less picturesque accounts are only a little more satisfactory. Mates, for example, writes that "by 'language', in its most general sense, I wish to denote any aggregate of objects which are themselves meaningful or else are such that certain combinations of them are meaningful." Mates, p. 201. And Chomsky writes in the same vein that he considers "a language to be a set (finite or infinite) of sentences, each finite in length and constructed out of a finite set of elements." Chomsky, p. 13. But both of these understandings have the somewhat awkward consequence that the unit set of any sentence is a language and that any set of

identify two languages only if (a) they have the same expressions, (b) they have the same meaningful expressions, and (c) a meaningful expression of one language has the same meaning in the other language.[2] But entities that satisfy such conditions are not easily found.

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sentences, even from diverse languages, is a language. Carnap is a little more helpful. He writes that a language is a "system of signs, or rather of the habits of producing them, for the purposes of communicating with other persons, i.e., of influencing their actions, thoughts, etc." Carnap (1), p. 3. Elsewhere he gives a set-theoretical analysis of language. There a language is taken to be an ordered pair $\langle \underline{a}, \Sigma \rangle$ where $\underline{a}$ is the set of signs of the language and Σ is the set of its sentences. Contrary to the position of section 1, this account allows for uninterpreted languages. In the same place Carnap offers an analysis of an interpreted language which seems to me to be more appropriate as an analysis of language. He construes an interpreted language as an ordered triple $\langle \underline{a}, \Sigma, D \rangle$ where $\underline{a}$ and Σ are as before and D is the relation which assigns values to the sentences of the language. Carnap(2), pp. 102 f.

[2]These conditions, of course, are not sufficient. Throughout the unaxiomatized portions of the analysis further conditions are supposed. For example, in section 1 it is supposed that languages have terms and statements, and in section 17 it is supposed that the alphabet of a language is countable.

What conditions must be satisfied if a language is to be a first-order language is still another question. Unfortunately, I cannot provide a precise answer. The only thing that seems clear is that quantification is restricted to "individual variables." I do not require that such languages have statement connectives, singular terms, or functors. Apparatus for quantification seems to be essential, but I think that good arguments to the contrary could probably be advanced. In any event we shall assume an infinite stock of variables and the apparatus for both universal and existential quantification. For a discussion of first-order languages and their classification see Church, sec. 30.

What it means for a language to be extensional is explained in n. 4 of this section.

The second difficulty is in part the quite general problem of dispositional concepts. But it is not merely that some languages leave undefined some of the terms that they could define. The more specific and, for present purposes, crucial difficulty is that there are languages that do not even have the apparatus for constructing definitions. Moreover, the terms that are said to be definable in such languages may not even occur among their well-formed expressions. One might say that a term is definable in such a language if its addition to the language would occasion a redundancy. But it is unclear what it means to say that a term is added to a language. It is for this reason that the above criterion for adequacy is formulated comparatively. It is designed to take into account the quite plausible claim that if there is a new term there is a new language.[3] This whole matter is one that the analysis will have to skirt carefully. Its further consideration will be postponed for section 6.

When it is said, as in (ii), that two languages have the same meaning base it is intended that they are both extensional, or both intensional, or etc.[4] The expression

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[3] Cf. Church, p. 48, n. 111 and p. 50, n. 118.

[4] Roughly, a language is extensional if its coextensive expressions can be interchanged _salva_ _veritate_. A language is intensional if (1) it is not extensional, and (2) its cointensive expressions can be interchanged _salva veritate_. For a fuller and more precise account see Carnap(5), secs. 11-16.

"or etc." is intended to take into account languages whose strongest meaning relation is such that they are neither extensional nor intensional. We might count Lewis' equivalence in analytic meaning[5] or Carnap's intensional isomorphism[6] as such meaning relations. It is even conceivable that there be languages with other kinds of "ultra-intensional" bases. But it is presumed that an explication of expressive completeness for any of these sorts of languages would have to satisfy the above criterion for adequacy if it is to be acceptable. Thus, although the concept to be explicated in this essay is quite narrow in scope, the criterion for adequacy is quite general.

The point of (ii), then, is this. We can compare two languages with respect to expressive completeness only if they are comparable. By the nature of the case, an intensional language will have terms for which an extensional language can provide no <u>intensional</u> synonym. Similarly, a hyper-intensional language will have terms for which an intensional language can provide no <u>hyper-intensional</u> synonym. There is, perhaps, a sense in which an intensional (or hyper-intensional) language is expressively more powerful than an extensional language. (This

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[5]Lewis, pp. 245 f.

[6]Carnap(5), secs. 13-16.

added power, of course, has its cost in point of economy and ontology.) But further consideration of this topic would involve an extended study of the issues between extensionalism and intensionalism.

It is perhaps now relatively clear what is intended by 'has the same meaning' as it occurs in (iii). Two terms have the same meaning if either they have the same extension and belong to some language or other which has an extensional base, or they have the same intension and belong to some language or other which has an intensional base, or etc.[7] This is not intended to throw light on the difficult and interesting problem of synonymy for linguistic forms of ordinary language. But it has, I fear, consequences which would revolt the mildest of intensionalists. For example, in many languages for arithmetic the terms '5-1' and '6-2' are (as are other coextensive pairs of terms) interchangeable _salva veritate_. Such languages are, of course, extensional. It follows from the above understanding that they have the same meaning.

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[7] I regard terms within the same language as having the same meaning if and only if they are interchangeable _salva veritate_. I find such terms semantically indistinguishable. So to speak, terms which are indistinguishable insofar as they affect the truth values of statements in which they occur are indistinguishable in point of meaning. The above interlinguistic concept of synonymy is intended as a natural extension of this intralinguistic conception.

But the intensionalist will deny that they are synonymous and reject the account. He errs, however. His mistake is, I suspect, that he misidentifies terms and languages. In English the terms '5-1' and '6-2' are not interchangeable _salva veritate_. But this simply shows that we have a different pair of terms and that these terms are not synonymous.[8] Since they have the same extension (for English), it also shows that English is not an extensional language. The source of this confusion is, one might speculate, twofold. First, although we have only one pair of "types," we have two pairs of terms. That is, '5-1' and '6-2' are ambiguous. Second, we often do arithmetic in two languages, English and some one or another of several formal languages, never keeping clearly in mind just which language is being used.

The rationale of (i) parallels that of (ii). We can compare two languages with respect to expressive completeness only if they are comparable. There is no basis for so comparing two languages unless their subject matter is the same. It may be anticipated, however, that there are conditions under which two languages offer interesting comparisons (with respect to expressive power) even though

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[8]Occurrences of terms of the same language are counted as occurrences of the same term if they are of the same type and have the same meaning. Cf. Leonard, sec. 17.1.

their universes are nonidentical. Such comparisons will be considered in detail in section 14.

Despite the shortcomings of the criterion's formulation, it will prove to be serviceable. Moreover, the ensuing analysis should shed considerable light on the sources of its defects.

4. AN INFORMAL SKETCH OF THE ANALYSIS

The universe of discourse (or as we shall say the *primitive domain*) of a language is constituted by the things about which the language has something to say. This universe uniquely determines certain classes. First, it determines those classes which are members of its power set. These are the *primitive selection classes* of the language. It is among these classes that an absolute term for the language will find its extension. Second, the universe determines ordered *n*-ads of its members and thereby classes of such *n*-ads. A class of ordered *n*-ads whose members are all of the same degree is an *auxiliary selection class* of the language. It is among these classes that the extension of a relative term for the language will be found. A class which is either a primitive selection class or an auxiliary selection class is said to be a *selection class* of the language. Thus, the extension of any term for the language will be one of its selection classes.

The following account of expressive completeness quite naturally suggests itself. A first-order extensional language is expressively complete if, and only if, it can define terms whose extensions exhaust its selection classes.

In the next section it will be shown that if the analysis just outlined can be effected, it will satisfy the criterion for adequacy. Subsequent sections fill in the details of the outline and develop some of its more important consequences.

The ensuing analysis of expressive completeness is intended to be axiomatic. That is, its locus is an axiomatic system. Unfortunately, this system is set-theoretical. A more satisfactory analysis would be conducted with greater parsimony. It would be developed within a metalanguage whose structure does not drag along a commitment to the rather "unlovely" ontology of classes. The theory of virtual classes seems to offer an avenue of escape. But, because of special difficulties (which will not be discussed here) in formulating an adequate theory of relations within such a rubric, this avenue has not been pursued.

Another misfortune of the metalanguage--a misfortune which besets all languages of its type--is that it is not known to be consistent. To steer clear of this latter difficulity, I have tried to confine the exposition to set-theoretical concepts and principles whose credentials are, as much as can be hoped, of established repute.

The metalanguage is many-sorted, but it is not type-theoretical. It employs three sorts of variables. The variables '$\underline{m}$', '$\underline{n}$', etc. are to take natural numbers as their values. The variables 'L', 'L_1', etc. are to take first-order extensional languages as their values. And, finally, the variables 'x', 'y', 'z', 'u', 'v', 'w', 'x_1', etc. are to take as their values the following sorts of things: (i) expressions of first-order extensional languages, (ii) members of the universes of such languages, and (iii) classes of these objects, classes of such classes, etc.[1]

The analysis will use but a single[2] primitive semantical concept, that of being an extension. This is understood as a triadic relation obtaining between a referring expression, a class, and a language. (See sections 6 and 7.) It also employs a primitive syntactical concept, that of being a relative referring expression. (See section 9.) Besides these a number of mathematical or logical concepts are used. These, however, are not formally examined.

The analysis is built upon three axioms. Roughly, they are the following. (i) Something has at most one extension for a language (see section 7). (ii) A relative referring expression for a language has an extension for

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[1]Hereafter, unless context dictates otherwise, 'language' is to be regarded as short for 'first-order extensional language'. Similar adjustments are required for other terms. For example, by 'expressively complete' will usually be intended 'expressively complete first-order extensional language'.

[2]Unless we count the use of 'L', 'L_1'. etc. as surrogate for a second semantical primitive.[1] But, cf. sec. 7, n. 1.

that language (see section 9). And (iii) the coordinates of the members of the extension of a relative referring expression for a language are members of the extension of some absolute referring expression for the language (see section 10). In addition to these a number of mathematical or logical principles are used without being formally examined.

5. ADEQUACY OF THE PROPOSED ANALYSIS

If the analysis just outlined can be effected, it can be readily seen to satisfy the criterion for adequacy.

First, if L_1 is expressively complete in the above sense, then for any L_2 if the primitive domain (i.e., universe of discourse) of L_2 is the same as that of L_1, there is no term of L_2 for which L_1 does not have a synonym. Suppose that L_1 is expressively complete. Then the terms (or referring expressions) of L_1 have extensions which exhaust its selection classes. Let L_2 have the same primitive domain as L_1. It follows that L_1 and L_2 have the same selection classes. But the extension of any term for L_2 is a selection class of L_1. Hence, for any term of L_2 there is a term of L_1 with which it is synonymous. That is, there is no term of L_2 for which L_1 does not have a synonym.

Second, if L_1 is expressively incomplete, then there is some L_2 such that the primitive domain of L_2 is the same as that of L_1, and there is a term of L_2 for which L_1 has no synonym. Suppose that L_1 is expressively incomplete. Then there is at least one of its selection classes (say) x which is the extension of none of its terms. Consider, then, L_2 which is to be constructed in the following way. L_2 is to have but two terms in its base, T_1 and T_2. T_1 is to be explained in such a way that (a) it is a universal term of L_2, and (b) it is true of just those things in the primitive domain of L_1. This guarantees that L_1 and L_2 have the same primitive domain. Thus x is also a selection class of L_2. T_2 can now be explained as having x as its extension. But L_1, then, has no term which is a synonym of T_2. Hence, L_2 has the same primitive domain as L_1 and a term for which L_1 has no synonym.

Subsequent sections fill in the details of the outlined analysis. Hence, they provide an adequate explication of expressive completeness.

6. THE SEMANTICAL PRIMITIVE

The sole semantical primitive concept of the analysis is that of being an extension of a referring expression for a language.[1] Expressions of the sort 'x is an extension of y for L' will often be abbreviated by corresponding expressions of the sort 'Ext(L,x,y)'. The concept of being a referring expression can be explained directly in terms of the primitive. A referring expression is simply something that has an extension. It will thus prove useful to make reference to both of these notions in the extrasystematic explanation of the primitive. (Axioms will shortly guarantee that each referring expression has exactly one extension; hence, we are justified in speaking of the extension of a referring expression.)

The notion of a referring expression approximates quite closely the ordinary understanding of 'term'. Like terms, referring expressions have exactly one extension. And, like terms, referring expressions can be classified in traditional ways as being either general or singular and either relative or absolute.

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[1]Unless confusion threatens, we will often speak simply of the extension of a referring expression, omitting the strictly required reference to its language.

The extension of an absolute referring expression (general or singular) is to be understood as the class of all and only those things of which it is true (or to which it refers). Thus, the extension of a singular referring expression is always either a unit class or the null class. The extension of a general (absolute) referring expression may have any number of members. It is to be understood, of course, that a member of the extension of an absolute referring expression for a language is also a member of the primitive domain of the language.

The extension of a relative referring expression is to be taken as a class of ordered _n_-ads. It will have as members all and only those ordered _n_-ads for which the referring expression holds true. It is to be understood that each member of the extension of an _n_-place ($1<n$) referring expression is an ordered _n_-ad. (This will be elaborated in section 8.)

There are only two departures from apparently popular conventions in the way in which the primitive is to be understood. First, some reputable authors understand 'extension' in such a way that sentences or statements have extensions. This understanding is not adopted in the present essay. (As a consequence of this decision, statements will not be regarded as terms.)

A second departure from convention relates to the notion of a _predicate_ that Quine develops in his _Methods_ *of Logic*.[2] Closed (quinean) predicates will be regarded as referring expressions of a language even though they may not be the result of concatenating atomic expressions (or signs) of the language. Thus a referring expression of a language need not be one of its expressions.

Roughly, closed quinean predicates are to be conceived as images of open sentences. They differ from open sentences only in having occurrences of circled numerals '①', '②', etc. for occurrences of free variables. For example, if '(Ex)Fxyz' is an open sentence of L, then '(Ex)Fx①②', '(Ex)Fx②①', and '(Ex)Fx①①' are to be regarded as referring expressions for L. The extension of the first is to be the class of all ordered pairs <u,v> such that (Ex)Fxuv. The extension of the second is to be the class of all ordered pairs <u,v> such that (Ex)Fxvu. And the extension of the third is to be the class of all things u such that (Ex)Fxuu.

In general, if a circled numeral 'ⓝ' occurs in a predicate each circled numeral 'ⓜ' such that $0<m<n$ occurs in the predicate. But they may occur in any order. Circled numerals can be thought of as being (or representing) the places of a predicate in which they occur. Then, an *n*-place predicate will have *n* circled numerals (to any number of occurrences). The extension of a 1-place closed predicate

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[2]Quine(3), secs. 23 and 25.

'. . . ① . . .' is the class of all things u such that . . .u. . . .Where *n*>1, the extension of an *n*-place closed predicate,

. . . ① . . . ② . . ., . . ., . . . ⓝ . . .

is the class of all ordered *n*-ads $\langle u_1, u_2, \ldots, u_n \rangle$ such that

. . . u_1. . . u_2. . ., . . ., . . . $u_{\underline{n}}$. . . .[3]

The reason behind this second departure is that the predicates of a language far outstrip its terms. If one attended solely to the terms of a language in assessing its expressive power, he would be woefully misled. Consider, for example, two languages, L_1 and L_2. Suppose that L_1 counts only 'F' and 'G' among its terms. Suppose, further, that L_2 is like L_1 except for having the term 'H' where:

$$Hx \leftrightarrow . \ Fx \vee Gx$$

defines 'H' in L_2. Clearly, L_2 does not have an advantage over L_1 in point of expressive power. The advantage of L_2, if any, is its greater notational convenience. Thus, in assessing the expressive power of a language we want to attend to both its terms and its predicates. Thus we will want to count

F① v G①

among the referring expressions of L_1.

The notion of a predicate affords a partial inroad on one of the earlier mentioned problems of definability.

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[3]See Quine's *Methods of Logic* for a fuller treatment. *Ibid*.

We can safely say that a term is definable within a language if, and only if, the language has a referring expression which is synonymous with that term.

7. REFERRING EXPRESSIONS AND EXTENSIONS

As anticipated, it is assumed that something has at most one extension for a language. That is,

A1 $\quad Ext(L,y,x).Ext(L,z,x). \rightarrow y=z.$

It was also anticipated that the concept of being a referring expression would be explained directly in terms of the semantical primitive. A referring expression for a language is simply something which has an extension for that language. Thus, where 'RefExp(L,x)' abbreviates 'x is a referring expression for L'.

D1 $\quad RefExp(L,x) \leftrightarrow (Ey)Ext(L,y,x).$[1]

An almost immediate consequence of D1 and A1 is that a referring expression has exactly one extension, and conversely. That is,

T1 $\quad RefExp(L,x) \leftrightarrow (Ey)[Ext(L,y,x).(z)(Ext(L,z,x) \rightarrow z=y)].$

Thus, we are justified in speaking of <u>the</u> extension of a referring expression for a language. Accordingly, we contextually introduce the functor 'ext'. Where 'ext(L,x)'

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[1]It is tempting at this point to explain a language as something which has something with an extension. Thus (where 'L' abbreviates 'language')

$$Lx \leftrightarrow (Ey)(Ez)Ext(x,y,z).$$

But rather than clarify the concept of language, this would only serve to emphasize the vagueness of our primitive.

may be read as 'the extension of x for L':

D2 ...ext(L,x)... ↔ (Ey)(Ext(L,y,x)....y...).

In D2 '...y...' represents any sentence which contains at least one free occurrence of 'y' and no bound occurrences of 'x'; and '...ext(L,x)...' represents the result of replacing each free occurrence of 'y' by 'ext(L,x)' in such sentences.

Since we have from D1 that:

RefExp(L,x) ↔ (Ey)(Ez)(Ext(L,y,x).y=z),

we have by D2 that:

T2 RefExp(L,x) ↔ (Ez)z=ext(L,x).

D2 and T2 will later allow for a simple statement of otherwise cumbersome definitions and theorems.

We will sometimes find it convenient to speak of the extensions of a language without explicit reference to the referring expressions of the language of which they are extensions. The justification for such talk is D3. Where 'Exten(L,x)' abbreviates 'x is an extension for L':

D3 Exten(L,x) ↔ (Ey)Ext(L,x,y).

8. ORDERED *n*-ads

Although the proposed analysis will employ but a single semantical primitive, it will use several extra-semantical concepts which will not be formally examined. These concepts, germane though they be, are not peculiar to the present subject matter. They belong more properly to the province of mathematics or logic. In this section we digress to informally discuss one of the more important of these concepts, the concept of an ordered *n*-ad.

The expression '$\{x_1,x_2,\ldots,x_n\}$' will be used to refer to the class whose members are x_1, x_2, . . ., and x_n. The expression '$\langle x_1,x_2,\ldots,x_n\rangle$' will be used to refer to the ordered *n*-ad whose first coordinate is x_1, whose second coordinate is x_2, . . ., and whose *n*th coordinate is x_n. The motive for speaking of x_1, x_2, . . . , and x_n as coordinates rather than as members of $\langle x_1,x_2,\ldots,x_n\rangle$ should be obvious. Under standard accounts the members of an ordered *n*-ad are not to be found among its coordinates.

The primary aim of this section is to explain what might be meant by 'ordered *n*-ad' where '*n*' is a genuine variable. The standard accounts are of little avail; they end too soon. Rather than providing a definition of 'ordered *n*-ad' they give us a recipe for constructing infinitely many definitions, a definition of 'ordered 2-ad', a definition of 'ordered 3-ad', and so on, but not a definition of 'ordered *n*-ad' where '*n*' is a genuine variable. All too frequently discussions of this concept and related ones conclude with the words 'and so on'. Quine's discussion in *Mathematical Logic* is paradigmatic. In connection with the theory of relations he writes:

> Relations in the sense here considered are known, more particularly, as *dyadic* relations; they relate elements in pairs. The relation of giving (*y* gives *z* to *w*) or betweeness (*y* is between *z* and *w*), on the other hand, is triadic; and the relation of paying (*x* pays *y* to *z* for *w*) is tetradic. But the theory of dyadic relations provides a convenient basis also for the treatment of such polyadic cases. A triadic relation among elements *y*, *z*, and *w* might be conceived as a dyadic relation borne by *y* to *z*;*w*. . . . Tetradic relations could be handled on the basis of triadic relations in a similar fashion, ... Similarly for pentadic relations, hexadic ones, and so on.[1]

Thus Quine gives us a recipe for constructing definitions for '*n*-adic relation' where '*n*' has a definite value but no definition of '*n*-adic relation' where '*n*' is a variable.

The difficulty in explaining 'ordered *n*-ad' is specifying what the values of '*n*' are numbers of. For the number of coordinates of an ordered *n*-ad may be *m* ($1<m<n$). For example, $\langle a_1, a_1, a_1 \rangle$ is an ordered triad with but one coordinate, its first, second, and third coordinates being

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[1]Quine(2), p. 201.

identical. It is sometimes said that *n* is the number of "positions" or "places" of an ordered *n*-ad. But this is simply to provide a name for the problem. For we want, then, to know what positions or places are.

The following paragraphs (which rely heavily on standard accounts) provide a recursive definition of 'ordered *n*-ad' and outline definitions of 'position' and 'coordinate'.

First, we introduce the (binary) operator '$< \; , \; >$'.

D4 $\quad <x,y> = \{\{x\},\{x,y\}\}$

This definition is due to Kuratowski. It can be shown to satisfy the condition that

$$<x,y> = <u,v> \rightarrow . \; x=u . y=v.$$

Thus '$<x,y>$' can be regarded as referring to the ordered dyad whose first coordinate is x and whose second coordinate is y, and we can define 'ordered 2-ad' (abbreviated 'Ord2-ad') by:

D5 $\quad \text{Ord2-ad}(x,y) \leftrightarrow (Eu)(Ev)(x=<u,v>.y=\{u,v\}).$

It is to be noticed that being an ordered dyad (2-ad) and, shortly, being an ordered *n*-ad (for some *n*) are here construed as binary relations obtaining between two classes.[2] The reason for this is that an ordered *n*-ad (for some *n*)

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[2]Thus, being an ordered . . .-ad is a ternary relation obtaining between a number and two classes. When this relational character becomes important we will use such locations as 'is an ordered *n*-ad of' or 'is an ordered *n*-ad relative to

is generally explained as an ordered dyad in accordance with the schema:

S $\quad\quad <x_1, x_2, \ldots, x_n> = <x_1, <x_2, \ldots, x_n>>$.

Thus (e.g.) $<a_1, a_2, a_3, a_4>$ is an ordered dyad relative to $\{a_1, <a_2, a_3, a_4>\}$, an ordered triad relative to $\{a_1, a_2, <a_3, a_4>\}$ and an ordered tetrad relative to $\{a_1, a_2, a_3, a_4\}$.

We have in S the makings of a recursive definition of 'ordered *n*-ad'. For, a few moments reflection on S in conjunction with D5 is enough to convince us that

$$\text{Ord}\underline{n}\text{-ad}(x,y).(Ez)(u=<z,x>.v=y\cup\{z\}): \rightarrow \text{Ord}\underline{n}+1\text{-ad}(u,v),$$

and where $2<\underline{n}$ that

$$\text{Ord}\underline{n}\text{-ad}(u,v) \rightarrow (Ex)(Ey)(Ez)(\text{Ord}\underline{n}-1\text{-ad}(x,y).u=<z,x>.v=y\cup\{z\}).$$

But the conjunction of these two truths yields the equivalence that where $2<\underline{n}$

D5' $$\text{Ord}\underline{n}\text{-ad}(u,v) \leftrightarrow (Ex)(Ey)(Ez)(\text{Ord}\underline{n}-1\text{-ad}(x,y).u=<z,x>.v=y\cup\{z\}).$$

We can then take D5 and D5' as jointly constituting a recursive definition for 'Ord*n*-ad'; for together they lay down a necessary and sufficient condition for determining whether one class is an ordered *n*-ad of another.

We turn now to a consideration of what might be meant when we speak of the positions and coordinates of an ordered *n*-ad. The coordinates of an ordered *n*-ad relative to a class are, of course, simply the members of that class. But we want to know more than this. Just as we want to be able to explain 'jth position of the ordered n-ad x of y' so we want to be able to explain 'jth coordinate of the ordered n-ad x of y'.

A few preliminary definitions will prove useful. The concept of being an ordered dyad has been explained as a binary relation. At this point it is necessary to make use of a related (absolute) concept. For this concept we adopt the term 'ordered pair' (abbreviated 'OrdPr'). It is to be explained as follows:

D6 OrdPr(x) ↔ (Ey)Ord2-ad(x,y).

That is, an ordered pair is an ordered dyad of some class or other. Notice now that we can easily explain what is intended by the first coordinate of an ordered pair.

D7 FirstCoord(x,y) ↔ (Eu)(Ev)(y=<u,v>.x=u).[3]

Similarly we can explain what is to be understood by the second coordinate of an ordered pair.

D8 SecndCoord(x,y) ↔ (Eu)(Ev)(y=<u,v>.x=v)

One further definition is needed. We will understand by the ancestors of a class all of the entities generated from that class by the ancestral of classial membership. Thus, the ancestors of a class will include the class itself, its

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[3] Strictly, this explains 'a first coordinate of'. But uniqueness is easily established. A similar remark also holds for D8.

members, members of its members, and so on.

D9 $\quad \text{Ancestor}(x,y) \leftrightarrow (z)[y \varepsilon z.(u)(v)(u \varepsilon z.v \varepsilon u. \rightarrow v \varepsilon z). \rightarrow x \varepsilon z]$

Let A, $\langle a_1, a_2, \ldots, a_n \rangle$, be an ordered *n*-ad of B, $\{a_1, a_2, \ldots, a_n\}$. Consider then the following *m* (*m*=*n*-1) ancestors of A.[4]

$$\langle a_1, a_2, \ldots, a_n \rangle$$
$$\langle a_2, \ldots, a_n \rangle$$
$$\vdots$$
$$\langle a_n-1, a_n \rangle$$

Let the class whose members are these *m* ancestors of A be C. Notice that C's members are all ordered pairs. We can uniquely arrange these members of C in such a way that one succeeds another only if the former is the second coordinate of the latter. Above, C's members are exhibited as arranged by this rule. Clearly we can speak in this special sense of the first member of C, the second member of C, . . . , and the *m*th member of C. Now, we can easily explain what is meant by the first coordinate of A. It is the first coordinate of the first member of C. Similarly, the *j*th (*j*<*m*) coordinate of A (relative to B) is the first coordinate of the *j*th member of C. The *n*th coordinate of A (relative to B) can be explained as the second coordinate of the *m*th member of C.

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[4] See n. 6 of this section.

We can identify the positions of A relative to B as follows. Its first position is the first member of C, its second position is the second member of C, . . . , and its *n*th position is C. An explanation of "occupying" a position is also forthcoming. For, x occupies the *j*th position of an ordered *n*-ad y of z if, and only if, x is the *j*th coordinate of y of z.

We have been rather dogmatic in the last several paragraphs. The account stands in need of both refinement and justification.[5] But since the proposed analysis of expressive completeness can be exposited without reference to the coordinates and positions of ordered *n*-ads, this task will be postponed.

In this same dogmatic vein we list without justification some relevant truths which are consequences of the above definitions.

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[5]The only difficult problem is to define a predicate which will isolate the *m* members of C. If we assume that no member of B is a class, we can say that x is a member of C if, and only if, x is an ancestor of A and x is an ordered pair. But there are good reasons for not making this assumption. A more adequate account would make use of the ancestral of being the second coordinate of an ordered pair. For this relation let us adopt the term 'second coordinate ancestor'. Now C can be explained as follows: x is a member of C if, and only if, x is a second coordinate ancestor of A and x is not a (membership) ancestor of B. Here, the only assumption required is that A itself is not an ancestor of B. This assumption seems plausible, but its appraisal requires a more careful examination of the set-theoretical structure of our metalanguage than is here appropriate.

1. $\mathrm{Ord}\underline{n}\text{-}\mathrm{ad}(x,y) \rightarrow (Ez)z \varepsilon x$

2. $\mathrm{Ord}\underline{n}\text{-}\mathrm{ad}(x,y) \rightarrow (Ez)\mathrm{Ord2}\text{-}\mathrm{ad}(x,z)$

3. $(Ey)\mathrm{Ord}\underline{n}\text{-}\mathrm{ad}(x,y) \leftrightarrow \mathrm{OrdPr}(x)$

4. $\mathrm{Ord}\underline{n}\text{-}\mathrm{ad}(x,y).\mathrm{Ord}\underline{m}\text{-}\mathrm{ad}(u,v).x=u.y=v: \rightarrow \underline{m}=\underline{n}$

5. $\mathrm{Ord}\underline{n}\text{-}\mathrm{ad}(x,y).\mathrm{Ord}\underline{m}\text{-}\mathrm{ad}(u,v).x=u.\underline{m}=\underline{n}: \rightarrow y=v$

6. $\mathrm{Ord}\underline{n}\text{-}\mathrm{ad}(x,y).\mathrm{Ord}\underline{m}\text{-}\mathrm{ad}(u,v).x=u: \rightarrow .y=v \leftrightarrow \underline{n}=\underline{m}$

9. ABSOLUTE AND RELATIVE REFERRING EXPRESSIONS

The syntactical primitive adopted for the analysis is the concept of being a relative referring expression for a language. It is to be understood in such a way that all _n_-place (_n_>1) predicates and all relative terms are relative referring expressions, and nothing else is. It is, of course, assumed that all relative referring expressions have an extension. This assumption is made explicit in the axiom:

A2 RelRefExp(L,x) → (Ey)Ext(L,y,x)

where 'RelRefExp(L,x)' is short for 'x is a relative referring expression of L'.

A2 connects the syntactical and semantical primitives. In conjunction with D1 it allows us to infer that:

T3 RelRefExp(L,x) → RefExp(L,x),

that is, that all relative referring expressions are referring expressions. From T2 and T3 we have that:

T4 RelRefExp(L,x) → (Ey)y=ext(L,x)

that is, something is identical with the extension of a relative referring expression for a language.

In terms of the syntactical primitive we can explain the concept of being an absolute referring expression for a language. Where 'AbsRefExp(L,x)' abbreviates 'x is an absolute referring expression for L',

D10 AbsRefExp(L,x) $\leftrightarrow$.RefExp(L,x).~RelRefExp(L,x).

That is, an absolute referring expression is a referring expression which is not a relative referring expression.[1]

From D7 and T2 we have that:

T5 AbsRefExp(L,x) $\rightarrow$ (Ey)y=ext(L,x).

That is, something is identical with the extension of an absolute referring expression. And from T3 and D7 it follows that:

T6 RefExp(L,x) $\leftrightarrow$. AbsRefExp(L,x) v RelRefExp(L,x).

That is, something is a referring expression if, and only if, it is either an absolute or a relative referring expression. From T6 and D1 it follows that:

T7 (Ey)Ext(L,y,x) $\leftrightarrow$. AbsRefExp(L,x) v RelRefExp(L,x).

That is, something has an extension if, and only if, it is either an absolute or relative referring expression.

It might be thought that the syntactical primitive could be done away with. For the members of the extension of a relative referring expression are (as we shall later make explicit) all ordered _n_-ads. Thus, one might suppose, a relative referring expression can be explained as a referring expression whose extension has only ordered _n_-ads as

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[1] Where there is no danger of ambiguity the strictly required reference to the language for which a referring expression is a referring expression will be omitted.

members. This suggestion, however, is not viable; for, it has unwanted consequences. First, it would count some referring expressions which are ordinarily thought of as absolute as relative. Any term with a null extension would be relative. Second, there is no guarantee that some of the languages we are considering do not have ordered _n_-ads within their primitive domain. Thus, even on this score we would be forced to call absolute terms relative.

10. PRIMITIVE DOMAINS AND PRIMITIVE SELECTION CLASSES

We associate two domains with a language, its primitive domain (more commonly, its universe of discourse) and its auxiliary domain. The primitive domain of a language is the class of all objects which are members of the extension of any of the absolute referring expressions for the language.[1] Thus, where 'PrimDom(L)' is short for 'the primitive domain of L',

D11 $\quad PrimDom(L)=\hat{x}(Ey)(AbsRefExp(L,y).x\varepsilon ext(L,y).$

As an immediate consequence we have that

T8 $\quad x\ PrimDom(L) \leftrightarrow (Ey)(AbsRefExp(L,y).x\varepsilon ext(L,y).$

Perhaps a word of justification is needed. One might argue that the universe of a language has as its members all those objects which satisfy at least one place of at least one of the language's referring

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[1] I am fully aware that this imposes restrictions under which we can say things like 'Let the universe of L be the class of F's'. One must first establish that each F is a member of the extension of some absolute referring expression or other of L. There is no problem if L has a universal predicate whose extension is the class of F's. But there are difficult cases. G. J. Massey has suggested the following one. The universe of L is to be the set of natural numbers. L is to have variables, grouping indicators, quantifers, one connective (for conjunction), and one predicate, 'P' understood as '① is a prime'. Thus, only prime numbers are members of extensions for L. (And '(x)Px' is a true sentence of L.) But here I would want to say that we cannot <u>let</u> the universe of L be the class of natural numbers.

expressions.[2] This is correct, but it would be a mistake to conclude that 'RefExp' need replace 'AbsRefExp' in D8 and T8. For, it can easily be seen that if an object *a* satisfies the *i*th place of an *n*-place referring expression *E* of L, there is some absolute referring expression *E*' of L which has *a* as a member of its extension. If *n*=1, we can let *E*' be *E*. If *n*>1, then *E* is either (i) an *n*-place predicate or (ii) a relative term.

(i) *E* is an *n*-place predicate. Construct E'_1 as follows. Replace all except the *i*th numeral of *E* by distinct variables (not occurring in *E*); and replace the *i*th circled numeral by '①'. Then let *E*' be any of the existential closures of E'_1. *E*', then, is an absolute referring expression of L which has *a* as a member of its extension.

(ii) *E* is a relative term. There is an *n*-place predicate *E*'' which has the same extension as *E*. (Example:

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[2]In *The Structure of Appearance* Goodman writes: "In founding a system, we must not only choose the primitives but also determine the range of individual variables--the realm of individuals recognized by the system. . . . The individuals recognized include all that satisfy at least one place of at least one of these primitive predicates, and in addition . . . all sums of such individuals." Goodman(2), pp. 85-86. (Goodman is here concerned with systems which incorporate the "logic" of the part-whole relation; this accounts for the addition of all sums of such individuals.) I would prefer this sort of treatment. But although satisfaction is an intuitively straightforward concept, it does impose rather difficult theoretical problems. Cf. Tarski(1), p. 371, n. 15 and Tarski(2), pp. 190-193.

if E is 'between', E'' is '① is between ② and ③'.) But E' can be obtained from E'' as it was obtained from E'_1 in (i). Thus E' is an absolute referring expression with a as a member of its extension.[3]

The effect of the above argument is that:

T9 $RelRefExp(L,x).y\varepsilon ext(L,x) .\rightarrow (En)(Eu)$ $\{Ordn\text{-}ad(y,u).(z)[z\varepsilon u \rightarrow (Ew)(AbsRefExp(L,w).$ $z\varepsilon ext(L,w))]\}$.

But this rather cumbersome statement is not derivable from the foregoing axioms. Let us supplement our axioms so that it is.

We will assume, roughly, that the members of the extension of a relative referring expression are all ordered n-ads of some subclass or other of the primitive domain of its language. More precisely,

A3 $RelRefExp(L,x).y=ext(L,x):\rightarrow (En)(u)[u\varepsilon y \rightarrow (Ev)$ $(Ordn\text{-}ad(u,v).v\subset PrimDom(L))]$.

This is a rather prodigal axiom. And in a less inelegant formalization it would appear as a derivative statement. The difficulty here is that we have not made a fine enough analysis of semantical and syntactical distinctions. Such an account should probably find it advantageous to

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[3]The argument requires that first-order extensional languages have the apparatus for universal quantification and at least as many variables as places in any of its referring expressions.

take as its primitive concept satisfaction understood as a relation between an n-place referring expression and an "n-place" sequence of objects. Being the extension of a referring expression would then be a derivative concept. But this strategy has difficulties of its own, and their resolution seems rather remote from the concerns of this essay.

It should be noted in passing that D8 is not intended to correspond to the way we ordinarily determine the primitive domain of a language. Generally the universe of a language is most conveniently specified by a metalinguistic statement, e.g., 'The universe of discourse for L is . . . .', or 'The values of L's variables are . . . .', or 'T is a universal term for L'. But this in no way conflicts with the definition. For the aim of the definition is to explain what the universe of a language is, not how we might most conveniently determine what it is.[4]

We shall speak of subclasses of the primitive domain of a language as primitive selection classes of the language. Thus (where 'PrimSel(L,x)' abbreviates 'x is a primitive selection class of L'):

D12 PrimSel(L,x) $\leftrightarrow$ x$\subset$PrimDom(L).

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[4]See n. 1 of this section.

Obviously, the extension of any absolute referring expression for a language is a primitive selection class of the language. That is,

T10 AbsRefExp(L,y).x=ext(L,y) .→ PrimSel(L,x).

But we cannot make the broader claim that the extension of a referring expression for a language is a primitive selection class of the language. For, generally, the extension of a relative referring expression is not a primitive selection class of its language. Nor do we have the converse of T10. The primitive selection classes of a language need not be exhausted by the extensions of its absolute referring expressions. When they are not so exhausted, we have seen, the language is expressively incomplete. But it would be premature to explain expressive completeness at this point, for not enough has been said about relative referring expressions.

11. AUXILIARY DOMAINS AND AUXILIARY SELECTION CLASSES

Suppose we were to understand by the auxiliary domain (or AuxDom) of a language the class all of whose members are ordered n-ads of some one or another of the primitive selection classes of the language. That is,

$$\text{AuxDom}(L)=\hat{x}(Ey)(E\underline{n})(\text{PrimSel}(L,y).\text{Ord}\underline{n}\text{-ad}(x,y)).$$

As an immediate consequence we should then have that:

(1) $x\varepsilon\text{AuxDom}(L) \leftrightarrow (Ey)(E\underline{n})(\text{PrimSel}(L,y).\text{Ord}\underline{n}\text{-ad}(x,y))$.

We should also have that the extension of any relative referring expression for a language is a subclass of the auxiliary domain of the language. That is,

(2) $\text{RelRefExp}(L,x) \rightarrow \text{ext}(L,x)\subset\text{AuxDom}(L)$.

We would not have, however, that the subclasses of the auxiliary domain of a language are all extensions of relative referring expressions for the language. There are two reasons. First, the language may fail to be expressively complete. Second, some of these subclasses cannot even be appropriately regarded as extensions of relative referring expressions. These are classes whose members (relative to the primitive selection classes of which they are ordered n-ads) do not have the same number of positions. That is, these are classes that are not "relations" of members of the primitive domain of the language. Consequently, the explanation of 'auxiliary selection class' cannot be modeled after the explanation of 'primitive selection class'. A better course is to first introduce the concept of an auxiliary selection class and, then, explain the auxiliary domain of a language as the class of all its auxiliary selection classes.

The auxiliary selection classes of a language are those classes all of whose members are, for some *n*, ordered *n*-ads relative to some primitive selection class or other of the language. Or, where 'AuxSel(L,x)' is short for 'x is an auxiliary selection class for L',

D13 AuxSel(L,x) $\leftrightarrow$ (E*n*)(u)[uϵx $\rightarrow$ (Ev)(PrimSel(L,v). Ord*n*-ad(u,v))].

In analogy to (2), then, we have as a theorem that:

T11 RelRefExp(L,x) $\rightarrow$ AuxSel(L,ext(L,x)).

We can now explain the auxiliary domain of a language as the class of all its auxiliary selection classes. That is,

D14 AuxDom(L)=$\hat{x}$AuxSel(L,x).

The relation between an auxiliary selection class of a language and the auxiliary domain of the language does not correspond to the relation between primitive selection classes and primitive domains. The former is the class membership relation; the latter is the relation of classial inclusion.

As an immediate consequence of D11 we have that:

T12 $x \varepsilon \text{AuxDom}(L) \leftrightarrow \text{AuxSel}(L,x)$.

Hence we have that:

T13 $\text{RelRefExp}(L,x) \rightarrow \text{ext}(L,x)\varepsilon\text{AuxDom}(L)$,

that is, that the extension of a relative referring expression for a language is a member of its auxiliary domain. We do not have, however, that the members of the auxiliary domain of a language (i.e., its auxiliary selection classes) are all extensions of some relative referring expression or other of the language. The reason, of course, is just that the language may be expressively incomplete.

We will understand by a selection class of a language any class which is either a primitive or auxiliary selection class of the language. Thus, where 'Sel' is short for 'selection class',

D15 $\text{Sel}(L,x) \leftrightarrow. \text{PrimSel}(L,x) \vee \text{AuxSel}(L,x)$.

We have as a theorem, then, that:

T14 $\text{RefExp}(L,x).y=\text{ext}(L,x) \;.\!\rightarrow \text{Sel}(L,y)$.

That is, the extension of a referring expression for a language is a selection class for the language.

The domain (abbreviated 'Dom') of a language can now be explained as the class whose members are the selection classes of the language.

D16 $\text{Dom}(L)=\hat{x}\text{Sel}(L,x)$.

As an immediate consequence we have that:

T15 $x \varepsilon Dom(L) \leftrightarrow Sel(L,x)$.

Thus we have that the extension of a referring expression for a language is a member of the domain of the language. That is,

T16 $RefExp(L,x).y=ext(L,x) \;.\!\rightarrow y \varepsilon Dom(L)$.

Parallel to T14 and T16 we have, of course, that extensions for a language are selection classes for the language and members of its domain. That is,

T17 $Exten(L,x) \rightarrow Sel(L,x)$,

and

T18 $Exten(L,x) \rightarrow x \varepsilon Dom(L)$.

It will be observed that the converses of T14, T16, T17, and T18 do not hold. It is just in this case that we want to say that a language is expressively incomplete.

We are now in a position to explain expressive completeness. But before doing so, let us note a few consequences of A3 and the chain of definitions leading to the introduction of 'Dom(L)'. One such consequence is that two languages with the same auxiliary domain have the same primitive domain. That is,

T19 $AuxDom(L_1)=AuxDom(L_2) \rightarrow PrimDom(L_1)=PrimDom(L_2)$.

The converse of T19 holds as a matter of set theory (and defintions). Hence, we have that

T20 $AuxDom(L_1)=AuxDom(L_2) \leftrightarrow PrimDom(L_1)=PrimDom(L_2)$.

We also have that languages with the same primitive domain have the same domain.

T21 $\text{PrimDom}(L_1)=\text{PrimDom}(L_2) \rightarrow \text{Dom}(L_1)=\text{Dom}(L_2)$

I have a strong suspicion, but as yet have been unable to prove, that the converse of T21 is a theorem. That is, that

T22* $\text{Dom}(L_1)=\text{Dom}(L_2) \rightarrow \text{PrimDom}(L_1)=\text{PrimDom}(L_2)$

is a theorm. T22* can be shown to hold, however, provided L_1 and L_2 satisfy either of three conditions. The first is that both languages have finite primitive domains. The second is that no relation (i.e., set of ordered _n_-ads) is a member of either of their primitive domains. The third is that neither has a class as a member of their primitive domain. The second condition is a consequence of the third and is, of course, weaker. The first condition is independent of the others. Under either of these conditions we also have that

T23* $\text{PrimDom}(L_1)=\text{PrimDom}(L_2) \leftrightarrow \text{Dom}(L_1)=\text{Dom}(L_2)$.

An asterisk following a theorem name is to be taken as a signal that something less than theoremhood is claimed for the formulas they precede. For T22* and T23* the theorems in question are conditionals with suitable formulations of either one of the three conditions or their alternation. But from the standpoint of this essay the third, and possibly strongest, condition isolates an interesting class of languages. In section 14 a number of claims for such languages will be made. Since these claims cannot be made for languages which satisfy only the first or second, the antecedent of those theorems is to be some suitable formulation of the third condition.

12. EXPRESSIVE COMPLETENESS

The explanation of expressive completeness is now quite simple. A language is expressively complete if its extensions exhaust its selection classes. That is, where 'ExpComp(L)' is short for 'L is expressively complete',

D17 $\quad$ ExpComp(L) $\leftrightarrow$ (y)(Sel(L,y) $\rightarrow$ Exten(L,y)).

We then have as theorems that:

T24 $\quad$ ExpComp(L) $\leftrightarrow$ (y)(Sel(L,y) $\leftrightarrow$ Exten(L,y)),

and

T25 $\quad$ ExpComp(L) $\leftrightarrow$ (y)(yεDom(L) $\leftrightarrow$ Exten(L,y)).

(We also have as corollaries of T17 and T18 the results of replacing their occurrence of 'Exten(L,y)' by occurrences of '(Ez)Ext(L,yz)'.)

We have often had occasion to use the expression 'expressively incomplete.' It was used in the sense that a language is expressively incomplete if it is not expressively complete. Thus, where 'ExpInComp(L)' is short for 'L is expressively incomplete',

D18 $\quad$ ExpInComp(L) $\leftrightarrow$ $\sim$ExpComp(L).

Then, as theorems, perhaps too obvious to mention, we have:

T26 $\quad$ ExpInComp(L) $\leftrightarrow$ (Ey)(Sel(L,y).$\sim$Exten(L,y)),

and

T27 $\quad$ ExpInComp(L) $\leftrightarrow$ (Ey)(yεDom(L).$\sim$Exten(L,y)).

(Again we have as corollaries the result of replacing 'Exten(L,y)' by '(Ez)Ext(L,y,z)'.)

13. AN APPARENT ANOMALY

There is a consequence of the preceeding analysis which may appear to be anomalous. This apparent anomaly can be presented and dispelled informally.

Consider, for example, a language whose primitive domain is mankind. Suppose that the language is expressively complete. None of the statements which can be formulated within this language should, from the vantage point of semantics, be surprising. It can say that Socrates married Xanthippe, that Socrates is a bachelor, that Xanthippe is a shrew, etc. What is surprising, perhaps, are some of the things which the language cannot formulate. It cannot say, for example, that Socrates drank hemlock, that Socrates lived in Egypt, that Socrates is mortal and that Socrates is not an elephant. The explanation is quite simple. These latter statements, normally understood, concern (among other things) not only men but also hemlock, Egypt, mortals and elephants. Each statement has in the extension of one of its terms something which is not a man and, hence, something which does not belong to the primitive domain of the language under consideration. The source of the apparent anomaly is now clear. Objects within the primitive domain of a language may belong to classes which are not included in that domain.[1]

There are three points which tend to mitigate the force of the anomaly.

Note, first, that we are not distressed by the fact that a formal language whose primitive domain is the class of all numbers cannot formulate such statements as '5 is blue', '3 is colorless', etc. Yet we would not, at least on this account, condemn such languages to expressive incompleteness. I am aware that there are those who would regard such statements as meaningless. But this is simply not the case. Take as a typical example Russell's familiar 'Laziness drinks procrastination'. Consider then the following argument.

> Only animate objects drink anything.
> Laziness is an abstract object.
> No abstract object is a concrete object.
> Animate objects are all concrete.
> Therefore, laziness does not drink procrastination.

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[1]This paragraph requires a much more careful formulation than I am now able to give. For example, I do not mean that such a language could formulate a statement synonymous with 'Socrates is a bachelor' in the sense of "expressing the same proposition." That is, the synonymy is not intensional but rather extensional. But by the extensional synonymy of statements I do not understand mere agreement of truth value. I think a concept of extensional isomorphism would provide a good avenue of approach to the solution of this problem. This concept would be explained in a manner parallel to Carnap's notion of intensional isomorphism. Carnap(5), secs. 14-15.

The premises of this argument are meaningful. Further, they are true. Since the argument is valid, its conclusion is also true. But the conclusion is the negation of 'Laziness drinks procrastination'. Hence, the latter statement is false. Hence it is meaningful. We might speculate that what seems "funny" about statements of this type is that they are patently false. Given any such statement it seems that one can always construct an agrument whose premises are incontrovertibly true and whose conclusion is the negation of that statement.

The above discussion must be regarded with a proper air of suspicion. For surely the semantical and logical paradoxes have their origin (in part) in an overeagerness to grant meaningfulness to linguistic forms. Yet the examples[2] with which I have been concerned do not appear to be of the type which so readily lend themselves to the

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[2]I would be hard put to specify principles which would isolate the cases I have in mind. It is not simply that they are patently false; for 'Cats are all dogs' is patently false and patently not a case in point. As further examples I would count Chomsky's 'Colorless green ideas sleep furiously', 'Sincerity admires John', and 'John frightens sincerity'. Chomsky, p. 15 and p. 42. Ryle's 'There exist prime numbers and Wednesdays and public opinions and navies' (Ryle, p. 23) is problematic; for if it is false, it is not patently false. Cf. White, c. IV. But I would definitely want to exclude Carnap's 'Pirots karulize elatically' (Carnap(3), p. 2) and Lewis Carroll's familiar 'Twas brillig, and the slithy toves did gyre and gimble in the wabe'. These latter examples contain purported terms which are in themselves meaningless. Whether Hempel's 'The Absolute is perfect' (Hempel, p. 51) is in this category is, of course, a matter of some debate.

paradoxes. That is, none of the terms employed need be explained impredicatively.

Second, it is only in an intensional sense that the above language cannot formulate the statements that Socrates drank hemlock, that Socrates lived in Egypt, etc. It can formulate, for example, the statement that Socrates is a man-who-drank-hemlock. And this statement is equivalent to the statement that Socrates drank hemlock. Clearly, if Socrates is a man-who-drank-hemlock, he drank hemlock. And if Socrates drank hemlock, since he is by hypothesis a man, he is a man-who-drank-hemlock. Thus, Socrates is a man-who-drank-hemlock if, and only if, he drank hemlock. This argument is easily generalized.

Third, to say that a language such as the one contemplated above is expressively incomplete is to condemn every language to expressive incompleteness. The source of the supposed expressive incompleteness was that things in the primitive domain of the language were members of classes which are not subclasses of that domain. An expressively complete language would have to have the universal class as its primitive domain. But, as will be shown in section 16, such a language cannot be expressively complete.

If the reader is not convinced that the paradox is dispelled, all is not lost. He can take expressive completeness to be what we shall shortly call 'universal expressive completeness.' There are two noteworthy consequences of this move. The first has already been noted. No language will be expressively complete. The second consequence is that the criterion of adequacy formulated in section 2 and defended in section 3 must be rejected.

14. EXPRESSIVE POWER

One outcome of the analysis of expressive completeness is that languages can be compared with respect to their expressive power.

A language will be said to be expressively complete relative to another if the selection classes of the latter are all extensions of the former. Where 'ExpCompRel(L_1,L_2)' is short for 'L_1 is expressively complete relative to L_2',

D19 $\quad \mathrm{ExpCompRel}(L_1,L_2) \leftrightarrow (x)(\mathrm{Sel}(L_2,x) \rightarrow \mathrm{Exten}(L_1,x))$.

It is a theorem, then, that:

T28 $\quad \mathrm{ExpCompRel}(L_1,L_2).\mathrm{ExpCompRel}(L_2,L_3) \;.\rightarrow \mathrm{ExpCompRel}(L_1,L_3)$.

That is, the relation is transitive. If it is granted that

$$(\mathrm{E}L_1)(\mathrm{E}L_2)(\mathrm{ExpComp}(L_1).\mathrm{ExpInComp}(L_2).\mathrm{Dom}(L_1)=\mathrm{Dom}(L_2)),$$ [1]

the relation is readily seen to be nonsymmetrical and nonreflexive. It is provable that two languages which stand in this relation to one another are both expressively complete. That is,

T29 $\quad \mathrm{ExpCompRel}(L_1,L_2).\mathrm{ExpCompRel}(L_2,L_1) \;.\rightarrow. \mathrm{ExpComp}(L_1).\mathrm{ExpComp}(L_2)$.

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[1] Such languages are easily "constructed"; but this statement does not follow from our axioms.

The converse is not provable. But we do have that:

T30 $\text{ExpCompRel}(L_1,L_2).\text{ExpCompRel}(L_2,L_1) .\leftrightarrow. \text{ExpComp}(L_1).\text{ExpComp}(L_2).\text{Dom}(L_1)=\text{Dom}(L_2)$.

That is, languages with the same domain are expressively complete if, and only if, they are expressively complete relative to one another. Consequently,

T31 $\text{ExpComp}(L) \leftrightarrow \text{ExpCompRel}(L,L)$.

That is, a language is expressively complete if, and only if, it is expressively complete relative to itself.

A language is said to be as expressively powerful as another if all of the extensions of the latter are also extensions of the former. Where '$\text{AsExpPow}(L_1,L_2)$' abbreviates 'L_1 is as expressively powerful as L_2',

D20 $\text{AsExpPow}(L_1,L_2) \leftrightarrow (x)(\text{Exten}(L_2,x) \rightarrow \text{Exten}(L_1,x))$.

This relation is transitive and reflexive. That is,

T32 $\text{AsExpPow}(L_1,L_2).\text{AsExpPow}(L_2,L_3) .\rightarrow \text{AsExpPow}(L_1,L_3)$,

and

T33 $\text{AsExpPow}(L,L)$.

But the relation is nonsymmetrical.[2] It is provable that if the domain of a language is included within the domain of an expressively complete language, then the former language is expressively complete if it is as expressively powerful as the latter. That is,

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[2]That this is so is not a theorem; but languages can be constructed which show that 'AsExpPow' is neither symmetrical nor asymmetrical.

T34 $Dom(L_1) \subset Dom(L_2) . ExpComp(L_2) . AsExpPow(L_1,L_2) . \rightarrow$
$ExpComp(L_1)$.

A language is said to have greater expressive power than another just in case it is as expressively powerful as the other, and the other is not as expressively powerful as it. Thus, where '$GrExpPow(L_1,L_2)$' abbreviates 'L_1 has greater expressive power than L_2',

D21 $GrExpPow(L_1,L_2) \leftrightarrow . AsExpPow(L_1,L_2) . \sim AsExpPow(L_2,L_1)$.

This relation is transitive, irreflexive and asymmetrical. That is,

T35 $GrExpPow(L_1,L_2) . GrExpPow(L_2,L_3) . \rightarrow GrExpPow(L_1,L_3)$,

and

T36 $\sim GrExpPow(L,L)$,

and

T37 $GrExpPow(L_1,L_2) \rightarrow \sim GrExpPow(L_2,L_1)$.

It is provable that a language whose domain includes the domain of a language with greater expressive power is expressively incomplete. That is,

T38 $Dom(L_1) \subset Dom(L_2) . GrExpPow(L_1,L_2) . \rightarrow ExpInComp(L_2)$.

A language is said to have less expressive power than another if the other has greater expressive power than it. Thus, where '$LsExpPow(L_1,L_2)$' is short for 'L_1 has less expressive power than L_2',

D22 $LsExpPow(L_1,L_2) \leftrightarrow GrExpPow(L_2,L_1)$.

This relation is also transitive, irreflexive and asymmetrical. That is, T39 $LsExpPow(L_1,L_2).LsExpPow(L_2,L_3) \ .\rightarrow LsExpPow(L_1,L_3)$

and

T40 $\sim LsExpPow(L,L)$,

and

T41 $LsExpPow(L_1,L_2) \rightarrow \sim LsExpPow(L_2,L_1)$.

Languages which have as much expressive power as one another are said to have equivalent expressive power. Where, '$EqExpPow(L_1,L_2)$' abbreviates 'L_1 and L_2 have equivalent expressive power',

D23 $EqExpPow(L_1,L_2) \leftrightarrow. AsExpPow(L_1,L_2).AsExpPow(L_2,L_1)$.

This relation is transitive, reflexive and symmetrical. Thus,

T42 $EqExpPow(L_1,L_2).EqExpPow(L_2L_3) \ .\rightarrow EqExpPow(L_1,L_3)$,

and

T43 $EqExpPow(L,L)$

and

T44 $EqExpPow(L_1,L_2) \rightarrow EqExpPow(L_2,L_1)$.

It is provable that if two languages with the same domain are equivalent in expressive power, then either both or neither are expressively complete (or incomplete). That is,

T45 $Dom(L_1)=Dom(L_2).EqExpPow(L_1,L_2) \ .\rightarrow. ExpComp(L_1) \leftrightarrow ExpComp(L_2)$,

and

T46 $Dom(L_1)=Dom(L_2).EqExpPow(L_1,L_2) \ .\rightarrow. ExpInComp(L_1) \leftrightarrow ExpInComp(L_2)$.

We also have that if two languages are equivalent in expressive power, they have the same extensions, the same selection classes, and the same domain. That is,

T47 $\text{EqExpPow}(L_1,L_2) \rightarrow (x)(\text{Exten}(L_1,x) \leftrightarrow \text{Exten}(L_2,x))$,

T48* $\text{EqExpPow}(L_1,L_2) \rightarrow (x)(\text{Sel}(L_1,x) \leftrightarrow \text{Sel}(L_2,x))$,

and

T49* $\text{EqExpPow}(L_1,L_2) \rightarrow \text{Dom}(L_1)=\text{Dom}(L_2)$.[3]

Of these, only the converse of T47 holds. Thus,

T50* $\text{EqExpPow}(L_1L_2) \leftrightarrow (x)(\text{Exten}(L_1,x) \leftrightarrow \text{Exten}(L\ ,x))$.

Further, if a language is equivalent in expressive power to an expressively complete language, then it is itself expressively complete.

T51* $\text{EqExpPow}(L_1,L_2).\text{ExpComp}(L_2) .\rightarrow \text{ExpComp}(L_1)$

Similarly, a language equivalent in expressive power to an expressively incomplete language is expressively incomplete.

T52* $\text{EqExpPow}(L_1,L_2).\text{ExpInComp}(L_2) .\rightarrow \text{ExpInComp}(L_1)$.

Two expressively complete languages are equivalent in expressive power if, and only if, they have the same selection classes and the same domain. That is,

T53* $\text{ExpComp}(L_1).\text{ExpComp}(L_2) .\rightarrow. \text{EqExpPow}(L_1,L_2) \leftrightarrow$
$(x)(\text{Sel}(L_1,x) \leftrightarrow \text{Sel}(L_2,x))$,

T54* $\text{ExpComp}(L_1).\text{ExpComp}(L_2) .\rightarrow. \text{EqExpPow}(L_1,L_2) \leftrightarrow$
$\text{Dom}(L_1)=\text{Dom}(L_2)$.

The converses of T54* and T55* do not hold.

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[3]See the last paragraph of section 11.

Given two languages, it may be the case that neither stands in any of the above relations to the other. For this reason it is desirable to be able to speak of a language as being expressively complete (or incomplete) with respect to a class of objects regardless of whether the class is identical with the primitive domain of the language. The remainder of this section is devoted to showing how this is possible.

Just as we can speak of the domain of the universe of a language so we can speak of the domain of any class of objects. Here we give an analogous, if condensed, explanation of this broader concept. Where, 'dom(x)' is short for 'the domain of x',

D24 $dom(x)=\hat{y}\{y \subset x \vee (E\underline{n})\ (u)[u \varepsilon y \rightarrow (Ev)(v \subset x . Ord\underline{n}\text{-}ad(u,v))]\}$.

Then, a language is expressively complete with respect to a class (ExpComRes) just in case the extensions of the language exhaust the members of the domain of that class. That is,

D25 $ExpCompResp(L,x) \leftrightarrow (y)(y \varepsilon dom(x) \rightarrow Exten(L,y))$.

It is a theorem that:

T55 $ExpCompResp(L,PrimDom(L)) \leftrightarrow ExpComp(L)$.

That is, a language is complete with respect to its primitive domain if, and only if, it is expressively complete. Also, a language is expressively complete if, and only if, it is expressively complete with respect to all of its primitive selection classes. That is, T56 $\text{ExpComp}(L) \leftrightarrow (x)(\text{PrimSel}(L,x) \rightarrow \text{ExpCompResp}(L,x))$.

And in general,

T57 $\text{ExpCompResp}(L,x) \leftrightarrow (y)(y \subset x \rightarrow \text{ExpCompResp}(L,y))$.

That is, a language is expressively complete with respect to a class of objects if, and only if, it is expressively complete with respect to all of its subclasses.

It would perhaps be even more desirable to develop a metrical concept of expressive power. Such a concept would assign a numerical value to each language; this value would be its degree of expressive power. A concept of this sort would allow for the comparison of any two languages with respect to their expressive capacity. But how this concept is to be explicated is by no means evident. Perhaps the most natural attack is to assign each language a simple proportion, the ratio of the number of its extensions to the number of its selection classes. But this is not viable; for any language with a non-null domain has an infinite number of auxiliary selection classes and, hence, an infinite number of selection classes.

Further exploration of this material concept and the difficulties which are occasioned by its analysis fall outside the scope of the present essay.

We conclude by noting that there is a class with respect to which every language having at least one referring expression is expressively complete.

T58 (Ex)(L)[(Ey)RefExp(L,y) → ExpCompResp(L,x)].

The case in point is, of course, the null class. (The assumption that every language has at least one referring expression is rather modest. But to my knowledge it does not follow from the above axioms.)

15. THE ELIMINATION OF GENERAL TERMS

In this section it is shown that for any language L with a finite primitive domain, there is a language L* which (a) has the same primitive domain as L, (b) is as expressively powerful as L, and (c) has no extralogical general terms.[1]

Let the primitive domain of L and L* be $\{a_1,\ldots,a_n\}$. Thus, both L and L* have the same finite primitive domain. Let 'a_1', . . . , and 'a_n' be singular terms of L* whose respective extensions are the unit classes of a_1, . . . , and a_n. Let 'F' be any general term of L. Then 'F' is either an absolute or a relative term.

(i) 'F' is an absolute term. The extension of 'F' is either the null class, a unit class of either a_1, . . . , or a_n, (say $\{a_i\}$), or an $\underline{m}$-membered ($1<\underline{m}\leq\underline{n}$) subclass of $\{a_1,\ldots,a_n\}$ (say $\{a_{i_1},\ldots,a_{i_m}\}$). If the extension of 'F' is the null class, then 'Fx' can be explained as:

$$(Ey)(y\neq y.x=y).$$

If the extension of 'F' is $\{a_i\}$, then 'Fx' can be explained as: $x=a_i$.

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[1]This proof, as will be obvious, requires principles which are not guaranteed by the above axioms.

And if the extension of 'F' is $\{a_{i_1},\ldots,a_{i_m}\}$, then 'Fx' can be explained as:

$$x=a_{i_1} \text{ v } \ldots \text{ v } x=a_{i_m}.$$

(ii) 'F' is an <u>k</u>-place relative term. The extension of 'F' is either the null class, one of L's 1-membered auxiliary selection classes (say $\{\langle a_{i_1},\ldots,a_{i_k}\rangle\}$), or one of L's <u>m</u>-membered ($1<m$) auxiliary selection classes (say $\{\langle a_1^1,\ldots,a_k^1\rangle,\ldots,\langle a_1^m,\ldots,a_k^m\rangle\}$). If the extension of 'F' is the null class, then '$Fx_1\ldots x_n$' can be explained as:

$$(Ey)(y\neq y.x_1=y. \ \ldots \ .x_k=y).$$

If the extension of 'F' is $\{\langle a_{i_1},\ldots,a_{i_k}\rangle\}$, then '$Fx_1\ldots x_k$' can be explained as

$$x_1=a_{i_1} \ . \ \ldots \ . \ x_n=a_{i_k}.$$

If the extension of 'F' is

$$\{\langle a_1^1,\ldots,a_m^1\rangle,\ldots,\langle a_1^m,\ldots,a_k^m\rangle\},$$

then '$Fx_1\ldots x_n$' may be explained as

$$x_1=a_1^1. \ \ldots \ .x_k=a_k^1.\text{v}.\ldots.\text{v}.x_1=a_1^m.\ldots.x_k=a_k^m.$$

The justification in each case is obvious.

There are several points that are worth making. (1) Unless L is expressively complete, L and L* are not equivalent in expressive power. For, L* is expressively complete; and according to T39 a language which is equivalent in expressive power to an expressively complete language is itself expressively complete.

(2) That L* is expressively complete is evident. In the proof that L* is as expressively powerful as L, (i) and

(ii) show that L* has referring expressions whose extensions exhaust the domain of $\{a_1, \ldots, a_n\}$. Hence, where x is a finite class, we may assert:

$$(EL)\text{ExpCompResp}(L,x).$$

That x is finite is essential. For, if x were infinite, it would have proper subclasses which are infinite. If such subclasses were to be extensions for L*, then, in the absence of extralogical general terms, L* would have to have referring expressions of infinite "length." (The class x offers no difficulties, for it is the extension of '① = ①' for L*.) Indeed, as already anticipated, it will be shown in section 17 that where x is infinite:

$$\sim(EL)\text{ExpCompResp}(L,x).$$

(3) It might be suggested by (2) that where x is finite, we can assert:

$$x\varepsilon\text{PrimDom}(L) \rightarrow \text{Exten}(L,\{x\}) \;.\!\rightarrow \text{ExpComp}(L).$$

That is, if the unit class of each member of the primitive domain of a language is an extension for the language, then the language is expressively complete. But this is not the case. It may fail for any of several reasons. For example, the language may not have a synonym for '='. Or it may not have suitable truth-functional apparatus for constructing the appropriate compound sentences.

(4) The above shows how the general <u>terms</u> of a language may be reduced to one, viz., '='. Such a "reduced" language still has a full range of general

referring expressions, absolute and relative. '①$=a_1$', for example, is an absolute referring expression of L*, and it was required for the proof of L*'s expressive completeness. Whether it is a logical or an extralogical referring expression, I must confess, I do not know. Indeed, even '=' has been spoken of as a logical term only through courtesy, not through understanding.

(5) In many instances the analogue in L* of a synthetic sentence of L is analytic for L*. Suppose that '(x)(Fx → Gx)' is a true synthetic sentence of L. Suppose further that the extension of 'F' is $\{a_1, \ldots, a_j\}$, and that the extension of 'G' is $\{a_1, \ldots, a_j, \ldots, a_m\}$. Then, the analogue in L* of '(x)(Fx → Gx)' is the sentence:

$$(x)(x=a_1 \lor \ldots \lor x=a_j . \rightarrow . \; x=a_1 \lor \ldots \lor x=a_j \lor \ldots \lor x=a_m)$$

which is analytic for L*. There are those for whom this will impugn the claim that L* is as expressively powerful as L. But there are also those for whom this will make the analytic and the synthetic all the more puzzling.

(6) Quine[2] has shown that the general terms of the vocabulary of language (or theory) can be replaced by a single dyadic predicate without loss of expressive power. Quine's reduction, however, requires (in many cases) that

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[2] Quine(4). Quine's results are not restricted to theories with finite universes.

the primitive domain of the language be extended to include a "modest fund of classes." But, as Goodman[3] has shown, these classes are "modest neither in number nor in complexity" and, consequently, Quine's reduction "effects no genuine gain in simplicity." The reduction of this section also shows that the general terms of a vocabulary can be replaced by a single dyadic predicate. And it does so without imposing an unwanted extension of the primitive domain. Yet, again, no claim to simplicity can be made. For what is gained by the elimination of general terms is lost by the introduction of singular terms.

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[3]Goodman(1).

16. UNIVERSAL EXPRESSIVE COMPLETENESS

The foregoing analysis allows for a precise statement of what it means to say of a language that it has universal expressive completeness. We should expect of such a language that it could formulate anything that is formulable in any (extensional) language. But to say this is simply to say that it is expressively complete and that its primitive domain is the universal class. Accordingly, where 'UniverExpComp(L)' is short for 'L has universal expressive completeness',

D26 UniverExpComp(L) $\leftrightarrow$. ExpComp(L).(x)xϵPrimDom(L).

Quine[1] has shown that no language can have universal expressive completeness. His argument runs somewhat as follows. Suppose that L has universal expressive completeness. Then everything is a member of its primitive domain. Hence, its referring expressions are members of its primitive domain. Consider, then, the class of L's referring expressions which are not members of their own extension. Let this class be a. Then a is one of L's selection classes. Since L is expressively complete there is a referring expression of L which has a as its extension. Let b be

[1]Quine(5), pp. 331-332. Of the argument Quine writes that it "is in principle Cantor's. The form I have given it is reminisent also of Grelling' paradox, and the use made of it is reminisent of Tarski." Quine(5), p. 332, n. 6.

such a referring expression. Then b is a member of a if, and only if, b is not a member of a. For, if b is a member of a, then, since no member of a is a member of its own extension, b is not a member of a. And if b is not a member of a, then, since b is not a member of its own extension, b is a member of a. But this is a contradiction. Hence, L does not have universal expressive completeness. That is,

T59 $\sim$UniverExpComp(L).

There are two comments which, perhaps, go without saying. First, the above argument establishes that L does not have universal expressive completeness, not that L has universal expressive completeness only on pain of inconsistency. The inconsistency arises in the metalanguaguage, not in L. Second, the argument is not a banner for mysticism. It does not establish something to be ineffable. Every language must leave something unsaid. But this is not to say that there is something which cannot be said in any language.

Quine's argument may be broadened. The argument suffices to show that no language is expressively complete if all of its referring expressions are members of its primitive domain. This shows that the difficulty is not that a language with universal expressive completeness must have everything in its primitive domain. Rather it is that all of its referring expressions are members of its primitive domain. And this is not peculiar to languages with universal primitive domains.

17. INFINITE PRIMITIVE DOMAINS

In this section it is shown that a language whose primitive domain is infinite is not expressively complete.[1]

Let us understand by the alphabet of a language the class of all its atomic expressions. These atomic expressions, the members of an alphabet, will be referred to as letters of the alphabet. The expressions of a language can be understood as finite sequences of letters of its alphabet. Let us say that an alphabet α is richer than an alphabet β just in case the cardinal number of the class of expressions generated by concatenation from α is greater than the cardinal number of the class of expressions generated by concatentation from β. Let the class whose two members are the numerals '0' and '1' be called the standard alphabet.

Lemma 1: No alphabet is richer than the standard alphabet.

Proof: Let α be any alphabet, and let ℓ_1, ℓ_2,... be the letters of α. Then the role of ℓ_1 in α is to be played by

[1]Once again principles are assumed which are not guaranteed by our axioms. The most important assumption is that the alphabet of a language is countable, i.e., either finite or denumerable.

'101' in the standard alphabet; the role of ℓ_2 is to be played by '1001'; and, generally, the role of ℓ_i is to be played by '10...01' where there are _i_ occurrences of '0' in '0...0'. Expressions generated from the standard alphabet can thus be mirrored after expressions generated from α.

Expressions generated from the standard alphabet as above ('101', '1001', etc., but not '011', '100', etc.) and sequences of such expressions will be referred to as _standard expressions_.

Lemma 2: The set of standard expressions has a cardinal number less than or equal to that of the set of natural numbers.

Proof: With each standard expression a unique natural number can be correlated. Associate with each standard expression the natural number to which it conventionally refers. For example, associate with the standard expression '101' the number 101; associate with '1000001100110001' the number 1,000,001,100,110,001.

Lemma 3: The cardinal number of the set of natural numbers is less than or equal to that of the set of standard expressions.

Proof: It is well known that we have several alphabets rich enough to generate names for each of the natural numbers. (E.g., the alphabet whose only letters are the first ten numerals, or the alphabet whose only letters are the accent and '0' are sufficient.) But these alphabets, in accordance with Lemma 1, are no richer than the standard alphabet. From this and the fact that each expression generated from the standard alphabet can be represented by a standard expression the lemma follows.

Lemma 4: The set of standard expressions and the set of of natural numbers are equinumerous.

Proof: This follows immediately from lemma 2 and lemma 3 in accordance with the well-known Schröder-Bernstein theorem.[2]

Theorem: If the primitive domain of L is infinite, then L is expressively incomplete.

Proof: Suppose that the primitive domain of L is infinite. Then the domain of L is nondenumerable, for a subset (e.g., the power set of L's primitive domain) of that domain is nondenumerable. Hence, L has nondenumerably many selection classes. But from lemma 1 and lemma 4 it follows that L has at most denumerably many expressions. Hence, there are selection classes of L for which there are no referring expressions.

It is perhaps of some interest that an almost immediate consequence of the above theorem is that any language for arithmetic is expressively incomplete.

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[2]Actually something stronger has been shown. Notice that _any_ non empty alphabet can generate denumerably many expressions. Thus we might take the unit set of '1' to be our standard alphabet. Then '1...1' (with _i_ occurrences of '1') could be paired with ℓ_i. But this account would tend to blur certain distinctions. For example, there is no way to reflect the distinction between the letter ℓ_2 and the result of concatenating ℓ_1 with itself.

18. OTHER CONCEPTS OF COMPLETENESS

In this section we briefly explain several important concepts of completeness that are frequently encountered in the literature. They are compared with the concept of expressive completeness and are found to differ in essential ways. Further, we observe that although allusions to expressive completeness and related concepts are not uncommon in the literature, these concepts do not appear to have received careful analysis.

The following four concepts of completeness are isolated as being especially important and typical.

(i) A system is <u>complete</u> if for every sentence S either $\vdash$S or $\vdash$n(S).[1]

(ii) A system θ is <u>absolutely complete</u> if for every sentence S either $\vdash$S or θ+{S} is absolutely inconsistent.[2] (A system is said to be <u>absolutely inconsistent</u> if for every sentence S, $\vdash$S.)

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[1]Read '$\vdash$S' as 'S is a theorem', and read 'n(S)' as 'the negation of S'. These notions as well as the notion of a sentence are, of course, relative to a system in question. But for ease of exposition we leave this implicit.

[2]I follow Church, sec. 18 and Tarski(1), chap. v in this terminology. θ+{S} is to be understood as the result of adding the sentence S as an axiom to θ.

(iii) A system θ is <u>negation-complete</u> if for every sentence S either ⊢S or θ+{S} is negation-inconsistent. (And a system is <u>negation-inconsistent</u> if there is a sentence S such that ⊢S and ⊢n(S).)[3]

(iv) A system is complete relative to a class of sentences Σ if for any sentence S if S∊Σ then ⊢S.[4]

Obviously completeness in the sense of either (ii) or (iv) has a more general character than in the sense of either (i) or (iii). For completeness in the latter senses presupposes a concept of negation and is non-vacuously applicable only to those systems to which the concept applies. But completness in the former senses makes no reference to negation and has no such limitation on its applicability.

Earlier it was claimed that completness concepts as they are ordinarily explained have received much attention

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[3]Church formulates an interesting generalization of this concept of completeness: completeness with respect to a transformation. A system θ is <u>t-complete</u> if for every sentence S either ⊢S or θ+{S} is <u>t</u>-inconsistent. And a system is <u>t-inconsistent</u> if there is a sentence S such that ⊢S and ⊢$\overline{t(S)}$.

[4]Alternatively, but certainly not equivalently, we might say that a system is complete relative to a class of sentences Σ if for every sentence S if S∊Σ then ⊢S or ⊢n(S). Still other concepts of completeness might be introduced. For example, Church (sec. 18) introduces completeness in the sense of Post. A system θ is <u>complete in the sense of Post</u> if for every sentence S either ⊢S or θ+{S} is inconsistent in the sense of Post. A system is inconsistent in the sense of Post if there is some "propositional variable" S such that ⊢S.

and are well understood.[5] This is certainly evidenced by the solid knowledge that is available concerning their applicability to various sorts of systems. For example, standard systems of the sentential calculus are known to be complete in senses (ii) and (iii) if they employ a rule of substitution (rather than axiom schemata). Further they are known to be complete in sense (iv) if Σ is the class of truth-functionally valid sentences (i.e., tautologies). (This holds whether or not a rule of substitution is employed.) But they are not complete in sense (i).[6] Concerning standard systems of first-order logic we know that they are complete in neither sense (i), (ii), nor (iii). But they are complete in sense (iv) if Σ is the class of valid sentences (i.e., sentences true under every interpretation of their nonlogical signs in every non empty universe).[7]

We also have in our possession fundamental knowledge concerning the applicability of these concepts to first-order theories.[8] For example, Lindenbaum has proved that

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[5]Sec. 1.

[6]For justification and valuable historical information see Church secs. 18 and 29.

[7]For justification see Church secs. 32 and 44.

[8]By a first-order theory is understood a theory whose "underlying logic" is a first-order logic. Here for the sake of convenience we shall assume that the logic is standard. In this case the first three senses of completeness

if θ is a consistent first-order theory, Σ is the set of sentences of θ , and Σ is finitely axiomatizable, then there exists a consistent complete extension of θ.[9] And, of course, it would be unforgivable if we failed to mention Gödel's famous incompleteness theorem (as it applies to first-order arithmetic): If θ is a first-order arithmetic, then θ is complete if, and only if, θ is inconsistent. (This result may also be stated in the terminology of (iv). Let θ be a first-order arithmetic, let Σ_1 be the set of theorems of θ. Then Gödel's theorem says, in effect, that $\Sigma_1 \neq \Sigma_2$.) If we accept that arithmetic can be translated into the notation of set theory, then, of course, the corresponding result holds for first-order set theory.

Clearly our knowledge of these several completeness concepts is extensive. Unfortunately this claim cannot be extended to cover expressive completeness. For, as should be quite clear, expressive completeness is to be sharply distinguished from these other concepts.

There are at least two important points of difference. For one thing these other concepts all make an essential reference to theoremhood. And they are related

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become coextensive. We shall also assume that only formulas without free variables are counted as sentences. Otherwise the explanation of completeness in senses (i) and (iii) would have to be modified.

[9] For a proof see Tarski(1) chap. v.

in important ways only to those systems to which that concept applies, that is, to deductive systems. But whether a system is expressively complete is a matter wholly independent of the character of its theorems. Indeed a system may be expressively complete even though it has no theorems. And this is fully in accord with the motivation of the present study. We wanted to explicate a concept of completeness that could be explained not with reference to the sentences (or theorems) of a system but with reference to its terms.

There is another difference between the concept of expressive completeness and these other concepts. Expressive completeness is clearly a semantical concept. And it is applicable only to interpreted systems (i.e., to languages). Now it is by no means clear whether these other concepts are (or are not) semantical. But it is evident that they are applicable to uninterpreted as well as interpreted systems.

The question as to whether the above concepts of completeness are semantical can be resolved only after a careful classification of the concepts of sentence, negation of a sentence and theorem as semantical or syntactical. The classification is by no means an easy one. For although these concepts are usually explained syntactically they are susceptible to rather straightforward semantical explanations: to be a sentence is to have truth value;

to be a theorem of a system is to be a consequence of its axioms,[10] and to be a negation of a sentence is to be counted as true just in case it is counted as false. Moreover, the usual syntactical explanations of these concepts belong to special syntax not to general syntax. But these explanations are of little help in this present matter for we are concerned with (first-order) systems in general not with this or that system.[11]

In any event this much seems perfectly clear. Completeness as usually explained is applicable only to deductive systems. And it may be applicable whether or not they are interpreted. Expressive completeness, on the other hand, is applicable only to interpreted systems. And it may be applicable whether or not they are deductive.

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[10] A sentence is a consequence of a set of sentences if it is true in every model of that set of sentences.

[11] The above point is somewhat different from Church's when he writes ". . . the notion of completeness of a logistic system has a semantical motivation, consisting roughly in the intention that the system shall have all possible theorems not in conflict with the interpretation." (Church, p. 109.) The distinction between general and special branches of semiotic is from Carnap(1). The best essay with which I am acquainted that deals with these matters in general (rather than special) semiotic is chap. v. of Tarski(1). Unfortunately the primitive concepts of Tarski's treatment are the concepts of being a sentence and being a consequence of a set of sentences. Thus the essay is of limited value when it comes to the problem of classification. It must be mentioned, however, that Tarski's extrasystematic explanation of being a consequence of a set of sentences makes reference to "rules of inference." It would seem, then, that it is being treated by Tarski as a syntactical concept.

It was earlier noted that the concept of expressive completeness is not firmly entrenched in our linguistic behavior.[12] This is undoubtedly true; but this is not to say that it is wholly unentrenched. Indeed, this and directly related concepts are alluded to with moderate frequency. Some such allusions are so brief (and unpretentious) that they might easily escape our critical attention. As an example, we cite a passage from Carnap's Logical Syntax of Language. There he writes:

> Limited universal operators and regressively defined [functors] are not mere abbreviations, and if we were to renounce them, the expressive capacity of the language would be very considerably diminished. On the other hand to renounce the limited existential operator . . . and the symbols of conjunction and implication together with all explicitly defined [numerals, predicates and functors] would only succeed in rendering the language more clumsy without in the least diminishing the extent of the expressible.[13]

The reader will perhaps be reminded of many passages with the same general direction. Their claims are undoubtedly justified. But their precise sense is not always evident. Nonetheless, claims like these constitute a significant part of the hard core of what we know about expressive power. An analysis which radically altered their truth value would be unacceptable. But it should be obvious that the foregoing analysis gives them a precise sense and leaves their truth value unaltered.

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[12] Sec. 1.

[13] Carnap(3), p. 31. Emphasis added.

There are some passages in the literature which are obviously related to expressive completeness but too garbled to be regarded as making acceptable claims. As an example, we cite the following:

> It might be supposed that the invention of new symbolism is conclusive evidence of the inadequacy of the language to which it is adjoined, and that one would be justified in condemning such a language even before the need for further symbolism is felt. It is natural to view symbols of the form _a_ + _bi_, for example, as additions made to the existing language of numbers for the purpose of filling in gaps which the language previously had. The situation is the same with rudimentary languages which a highly developed language seems to complete. Thus, the present language of arithmetic in which we have the possibility of indefinitely writing numerals, seems to complete the language having the numerals from one to five and the word "many." That it is capable of expressing facts (e.g., that 6>5)which the rudimentary language cannot seems properly to be ascribed to the superiority of a complete language over a fragmentary one.
>
> Is this, however, a proper account of the difference between a simple and a highly developed language? And if it is, must we not be forced to say that for all we know every language is incomplete? A counterquestion is in order here: Incomplete with reference to what standard? Unless there exists a wider language of which a given symbolism is a part, we have no standard in relation to which it is incomplete. Further, even the fact that a symbolism L_1 is part of another, L_2, does not necessarily make L_1 incomplete, although it may be inadequate for certain purposes. The language of arithmetic can be said to be part of the language of real numbers; it lacks certain symbols and the rules for their usage. But although arithmetic is _inadequate_ for certain purposes, e.g., for solving algebraic equations, it is not an _incomplete_ arithmetic. No parts of it are _missing_, as there would be from a symbolism which purported to be our arithmetic but which lacked the operation 4x4. Taken by itself it is the whole language. It is completely unlike a dictionary with missing pages. Any inadequacy which at a given moment a _language_ comes to have is not due to incompleteness. The classification "incomplete" (and hence also the classification "complete") is not properly applicable

> to a language. A symbolism which purports to be a language but which has missing parts can be called incomplete, but a language L_1 does not become incomplete when it becomes a part of L_2, because it does not purport to be L_2. It is a whole even though additions are made to it, since these additions do not supply missing parts. To repine that, for all we know, every language may be incomplete is to indulge in the absurd complaint that a whole language is perhaps not a whole language. Furthermore, it sounds as though a remedy may be needed, whereas there is no completing what is already a whole. We can add to it; but we cannot complete it.[14]

The kindest thing we can say is that it is suggestive.

Some passages are more direct in their reference to expressive completeness. For example, within the context of a general discussion of formal systems, Copi writes:

> When the formal system has been constructed, the question naturally arises as to whether or not it is adequate to the formulations of all propositions it is intended to express. If it is, it may be said to be 'expressively complete' _with respect to that subject matter_. We are here discussing what can be _said_ in the system, not what can be proved. With respect to a given subject matter, a formal system is 'expressively complete' when it is possible to assign meanings to its undefined terms in such a way that every proposition about that subject matter can be _expressed_ as a formula of the system.[15]

It should be noted that Copi does not explain what he understands by such obviously crucial terms as 'meaning', 'proposition', 'expresses a proposition', 'proposition a system is intended to express', and 'subject matter'. He

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[14] Ambrose, pp. 30-31.

[15] Copi, pp. 178-179. Author's emphasis.

does, however, explain that the functional completeness of various sets of statement connectives is one kind of expressive completeness.[16]

Elsewhere, and along these same lines, we find Fraenkel and Bar-Hillel writing that

> the notion of <u>notational</u> (or <u>expressive</u>) <u>completeness with respect to a given subject matter</u> deserves at least to be mentioned. Its meaning should be clear. As an illustration, let us only mention that the propositional calculus, based upon '⊃' and '∼' as the only connectives, is notationally complete with respect to the truth functions: in short, is truth functionally complete, since it can easily be shown that all truth functions are expressible on this base.[17]

When the claim is made that "its meaning should be clear" it is not intended that its meaning should be clarified but that it <u>is</u> clear. If that claim is correct, this whole essay becomes pointless. But I think that it is not. There is only one place in the literature known to me where we have an analysis of what might be thought of as expressive completeness. The analysis is Tarski's.[18] He speaks not of expressive completeness but of the <u>completeness of concepts</u>. In the next section we offer a brief account of his analysis.

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[16] <u>Ibid</u>., p. 192.

[17] Fraenkel, p. 295, n. 3. Author's emphasis.

[18] Tarski(1), "Some methodological investigations on the definability of concepts," pp. 296-319.

19. TARSKI AND THE COMPLETENESS OF CONCEPTS

In this section we offer a brief explanation of Tarski's analysis of the completeness of concepts. It is argued that although his analysis isolates an important concept it does not provide an explication of expressive completeness. We conclude by showing that these two concepts, expressive completeness and completeness of concepts, are related in a rather definite way.

Let *c* be an extralogical constant and let β be a set of such constants. Further, let $\phi(v,\beta)$ be any open sentence which has *v* as its only free variable and no extralogical constants other than those of β. Then,

$$(v)(v=c \leftrightarrow \phi(v,\beta))$$[1]

is said to be a (possible) *definition of c in terms of β*.

Where Σ is a set of sentences, *c* is said to be *definable in terms of β on the basis of Σ*, if (1) *c* and each member of β occurs in some member or other of Σ, and (2) some definition of *c* in terms of β is derivable from the sentences of Σ.

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[1]Here '$\leftrightarrow$', etc. are used autonymously. Note that Tarski's account is directly applicable only to those sets of sentences containing these signs.

Where Σ_1 and Σ_2 are sets of sentences, let β_1 be the set of extralogical constants occurring in the members of Σ_1 and let β_2 be the set of extralogical constants occurring in the members of Σ_2. Then, Σ_1 *is essentially richer than* Σ_2 with respect to specific terms if (1) $\Sigma_2 \subset \Sigma_1$, and (2) there is some extralogical constant *c* such that $c \varepsilon \beta_1$, $\sim c \varepsilon \beta_2$, and *c* is not definable in terms of β_2 on the basis of Σ_1.

Finally, a set of sentences Σ_1 *is complete with respect to its specific terms* if there is no set of sentences Σ_2 such that Σ_2 is categorical, and Σ_2 is essentially richer than Σ_1 with respect to specific terms.[2]

Tarski does not argue the adequacy of his analysis. But categoricity is of great importance; and we must agree with Tarski when he writes that a "non-categorical set of sentences (especially if it is used as an axiom system for a deductive theory) does not give the impression of a closed and organic unity and does not seem to determine

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[2]Very roughly, a set of sentences is *categorical* if any two normal interpretations of the set are isomorphic. For brief but careful formulations of this concept see Carnap(2), pp. 173-174, Church, pp. 317-332, and Mendelson, pp. 90-91. An interpretation is *normal* if it assigns the identity relation of its universe to ' = '. The isomorphism of two interpretations (not to be confused with an isomorphism of their universes) can be explained as follows. Let c_1, c_2, . . . , and c_m be the terms occurring in the members of a set of sentences S. Consider two interpretations of S; I in the universe U and I* in the universe U*. Then I and I* are *isomorphic interpretations* of S if there is a function *f* which establishes an isomorphism between U and U* such that: for any c_i ($1 \leq i \leq m$), x, x_1, x_2, ..., and x_n, (i) if c_i is an absolute term, then $x \varepsilon ext(c_i)$ under I if and only if $f(x) \varepsilon ext(c_i)$ under I*, and (ii) if c_i is a relative term, then $\langle x_1, x_2, \ldots, x_n \rangle \varepsilon ext(c_i)$ under I if and only if $\langle f(x_1), f(x_2), \ldots, f(x_n) \rangle \varepsilon ext(c_i)$ under I*.

precisely the meaning of the concepts contained in it."[3] We must also agree with Carnap when he makes what seems to be the inverse claim that a categorical axiom system "specifies all the structural properties of its possible models."[4] There can be no doubt that Tarski's completeness of concepts is one (important) kind of completeness. But I do not think that it is the kind of completeness we want to speak of as expressive completeness. I have two reasons.

First, Tarski's concept is applicable to uninterpreted sets of sentences. But sentences of such sets do not express any thing. Thus it seems improper to speak of sets of these sentences as being expressively complete. An obvious, if inadequate, rejoinder is that sets of sentences which satisfy Tarski's conditions need only satisfy one further condition to be expressively complete. That condition is that they be interpreted. Thus, one might argue, if Tarski has not explicated expressive completeness, he has come very near to doing so. My second reason shows that this line of thought is mistaken.

If we may speak loosely, and use the terminology of Copi, a language (or theory) is not expressively complete unless it has the apparatus to express all the propositions about its subject matter. But Tarski has shown that there

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[3] Tarski(1), p. 311.

[4] Carnap(2), p. 174.

are postulate sets for the real numbers which are complete with respect to their specific terms.[5] Relative to such systems, as is well known, there are real numbers which cannot be designated. If r is such a real number, then that (Ex)x=r is a proposition which cannot be expressed within the system. Thus the language system cannot be expressively complete. Hence, expressive completeness and completeness with respect to concepts are distinct.

Let me remind the reader that Tarski's analysis is not being criticized as being inadequate or unimportant. My point is simply that it is not an adequate analysis of expressive completeness. And, so far as I know, Tarski has never claimed that it is.

It has been noted that the two analyses are concerned with different kinds of systems. Tarski is primarily concerned with axiom systems (though his concept is applicable to sets of sentences). Expressive completeness, however, was understood to be applicable only to language systems. And as was suggested earlier[6] only those axiom systems which are interpreted are to be regarded as languages, and only those languages with axioms are to be regarded as axiom systems.

We have seen that an interpreted axiom system which is complete with respect to its specific terms need not be

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[5]Tarski(1), pp. 313-314.

[6]Section 1.

expressively complete. It is quite natural at this point to inquire whether the converse holds, that is, whether an expressively complete axiom system is always complete with respect to its specific terms. It should be clear that this is not the case. Whether an axiom system is expressively complete is independent of the character of its axioms.[7] There is no logical restriction governing which of a system's sentences one selects as its axioms. A system's axioms may be as meager and uncategorical as one chooses.

But even if the converse does not hold we can establish the following, somewhat related, claim.

(1) If L is expressively complete, then there is an L' such that (i) L and L' are equivalent in expressive power, and (ii) there is a set Σ of true sentences of L' such that Σ is complete with respect to its specific terms.

In order to establish this it will be convenient to make use of a theorem proved by Tarski.[8]

(2) If Σ is a monotransformable set of sentences, then Σ is complete with respect to its specific terms.[9]

Thus, to establish (1) it will suffice to show that:

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[7] Cf. Sec. 1.

[8] Tarski(1), pp. 313-317.

[9] Roughly again, a set of sentences is monotransformable if there is at most one way of establishing an isomorphism between any two of its interpretations.

> If L is expressively complete, then there is an L' such that (i) L and L' are equivalent in expressive power, and (ii) there is a set, Σ, of true sentences of L' such that Σ is monotransformable.

Let us suppose that L is an expressively complete language. Then, by the reasoning of section 17, we know that L has a finite primitive domain. Let a_1, a_2, . . . , and a_n be the _n_ distinct members of that finite domain. Since L is expressively complete, L has referring expressions, A_1, A_2, . . . , and A_n, whose extensions are respectively $\{a_1\}$, $\{a_2\}$, . . . , and $\{a_n\}$. Let us now consider another language, L'. As will be shortly evident, L' is to have standard apparatus for quantification and truth-functional combination. The only terms of L' are to be '=', B_1, B_2, . . . , and B_n. '=' is to be understood as having the ordered pairs $\langle a_1,a_1\rangle$, $\langle a_2,a_2\rangle$, . . . , and $\langle a_n,a_n\rangle$ as the members of its extension. Further, the sentence '(x)x=x' is to be understood as true for L'. Thus, L and L' are to have the same primitive domain. Each B_i is to be explained in such a way that its extension is $\{a_i\}$. Reasoning in the manner of section 15, we see that L' is expressively complete. Since L and L' have the same primitive domain and are both expressively complete, (by theorem 5 of section 14)

(i) L and L' are equivalent in expressive power.

It is clear that the following are true sentences of L'.

I $(x)(B_1x \vee B_2x \vee \ldots \vee B_nx)$

$(Ex)[B_1x \,.\, (y)(B_1y \rightarrow x=y)]$
$(Ex)[B_2x \,.\, (y)(B_2y \rightarrow x=y)]$

II .
.
.
$(Ex)[B_nx \,.\, (y)(B_ny \rightarrow x=y)]$

$(x)[B_1x \rightarrow (\sim B_2x \,.\, \sim B_3x \,.\, \ldots \,.\, \sim B_nx)]$
$(x)[B_2x \rightarrow (\sim B_1x \,.\, \sim B_3x \,.\, \ldots \,.\, \sim B_nx)]$

III .
.
.
$(x)[B_nx \rightarrow (\sim B_1x \,.\, \sim B_2x \,.\, \ldots \,.\, \sim B_{n-1}x)]$

Let Σ be the set of these sentences. Suppose, now, that we have two normal interpretations of Σ, I and I*. If these interpretations render the sentences of S true, they are interpretations in n-membered universes (or primitive domains), say, U and U* respectively. Let β_i be the subset of U which is assigned (as extension) to B_i under I. Similarly, let β^*_i be the subset of U* which is assigned to B_i under I*. Clearly, β_i and β^*_i are unit sets. This is guaranteed by Group II above. Also, $B_j=B_k$ if, and only if $\beta_j=\beta_k$, and $B_j=B_k$ if, and only if, $\beta^*_j=\beta^*_k$. This is guaranteed by Group III above. Think now of b_i as the sole member of β_i and b^*_i as the sole member of β^*_i. It is obvious at this point that if a relation is to establish an isomorphism between I and I*, it can do so only by pairing b_i with b^*_i. Thus there is only one way of establishing an isomorphism between I and I*. That is,

Σ is monotransformable. Thus,

(ii) There is a set, Σ, of true sentences of L' such that Σ is monotransformable.

This completes the proof of (3) and therewith the proof of (1).

It is perhaps of interest to observe that just as the reasoning of section 15 shows how to construct an expressively complete language for a finite universe, so the above reasoning shows how to construct a complete theory for a finite universe.[10] But there can be little comfort in this. For just as the language prescribed is cumbersome and unwieldy, so the theory prescribed is inelegant and complex. It has, after all, more primitive terms and more primitive sentences than there are entities in its universe.[11]

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[10]Completeness, here, can be understood either as completeness with respect to specific terms or monotransformability. Tarski mentions as unsolved the problem whether only monotransformable sets of sentences are complete with respect to their specific terms, i.e., whether the converse of (2) holds.

[11]We can, of course, take a conjunction of sentences in Σ and trivially reduce the number of axioms to one. (Though this may require the addition of a new rule of inference.) A more genuine economy is effected simply by omitting any one (but not more than one) of the sentences in group III.

APPENDICES

AXIOMS

A1 $Ext(L,y,x).Ext(L,z,x). \rightarrow y=z$

A2 $RelRefExp(L,x) \rightarrow (Ey)Ext(L,y,x)$

A3 $RelRefExp(L,x).y=ext(L,x): \rightarrow (E\underline{n})(u)[u \varepsilon y \rightarrow (Ev)$

$(Ord\underline{n}\text{-}ad(u,v).v \subset PrimDom(L))]$

DEFINITIONS

D1 RefExp(L,x) ↔ (Ey)Ext(L,y,x)

D2 ...ext(L,x)... ↔ (Ey)(Ext(L,y,x)....y...)

D3 Exten(L,x) ↔ (Ey)Ext(L,x,y)

D4 <x,y> = {{x},{x,y}}

D5 Ord2-ad(x,y) ↔ (Eu)(Ev)(x=<u,v>.y={u,v})

D5' Ord<u>n</u>-ad(x,y) ↔ (Ex)(Ey)(Ez)(Ord<u>n</u>-1-ad(x,y).
u=<z,x>.v=y∪{z})

D6 OrdPr(x) ↔ (Ey)Ord2-ad(x,y)

D7 FirstCoord(x,y) ↔ (Eu)(Ev)(y=<u,v>.x=u)

D8 SecndCoord(x,y) ↔ (Eu)(Ev)(y=<u,v>.x=v)

D9 Ancestor(x,y) ↔ (z)[yεz.(u)(v)(uεz.vεu.→ vεz).→ xεz]

D10 AbsRefExp(L,x) ↔. RefExp(L,x).∼RelRefExp(L.x)

D11 PrimDom(L)=x̂(Ey)(AbsRefExp(L,y).xεext(L,y))

D12 PrimSel(L,x) ↔ x⊂PrimDom(L)

D13 AuxSel(L,x) ↔ (E<u>n</u>)(u)[uεx → (Ev)(PrimSel(L,v).
Ord<u>n</u>-ad(u,v))]

D14 AuxDom(L)=x̂AuxSel(L,x)

D15 Sel(L,x) ↔. PrimSel(L,x) v AuxSel(L,x)

D16 Dom(L)=x̂Sel(L,x)

D17 ExpComp(L) ↔ (y)(Sel(L,y) → Exten(L,y))

D18 ExpInComp(L) ↔ ∼ExpComp(L)

D19 ExpCompRel(L_1,L_2) ↔ (x)(Sel(L_2,x) → Exten(L_1,x))

D20 AsExpPow(L_1,L_2) ↔ (x)(Exten(L_2,x) → Exten(L_1,x))

D21 GrExpPow(L_1,L_2) ↔. AsExpPow(L_1,L_2).∼AsExpPow(L_2,L_1)

D22 LsExpPow(L_1,L_2) ↔ GrExpPow(L_2,L_1)

D23 EqExpPow(L_1,L_2) ↔. AsExpPow(L_1,L_2).AsExpPow(L_2,L_1)

D24 dom(x)=ŷ{y⊂x v (E<u>n</u>)(Eu)[uεy → (Ev)(v⊂x.Ord<u>n</u>-ad(u,v))]}

D25 ExpCompResp(L,x) ↔ (y)(yεdom(x) → Ext(L,y))

D26 UniverExpComp(L) ↔. ExpComp(L).(x)xεPrimDom(L)

THEOREMS

T1 RefExp(L,x) ↔ (Ey)[Ext(L,y,x).(z)(Ext(L,z,x) → z=y)]

T2 RefExp(L,x) ↔ (Ez)z=ext(L,x)

T3 RelRefExp(L,x) → RefExp(L,x)

T4 RelRefExp(L,x) → (Ex)y=ext(L,x)

T5 AbsRefExp(L,x) → (Ey)y=ext(L,x)

T6 RefExp(L,x) ↔. AbsRefExp(L,x) v RelRefExp(L,x)

T7 (Ey)Ext(L,y,x) ↔. AbsRefExp(L,x) v RelRefExp(L,x)

T8 xεPrimDom(L) ↔ (Ey)(AbsRefExp(L,y).xεext(L,y))

T9 RelRefExp(L,x).yεext(L,x) →. (En)(Eu){Ordn-ad(y,y).
(z)[zεu → (Ew)(AbsRefExp(L,w).zεext(L,w)))]}

T10 AbsRefExp(L,y).x=ext(L,y) .→ PrimSel(L,x)

T11 RelRefExp(L,x) → AuxSel(L,ext(L,x))

T12 xεAuxDom(L) ↔ AuxSel(L,x)

T13 RelRefExp(L,x) → ext(L,x)εAuxDom(L)

T14 RefExp(L,x).y=ext(L,x) .→ Sel(L,y)

T15 xεDom(L) ↔ Sel(L,x)

T16 RefExp(L,x).y=ext(L,x) .→ yεDom(L)

T17 Exten(L,x) → Sel(L,x)

T18 Exten(L,x) → xεDom(L)

T19 AuxDom(L_1)=AuxDom(L_2) → PrimDom(L_1)=PrimDom(L_2)

T20 AuxDom(L_1)=AuxDom(L_2) ↔ PrimDom(L_1)=PrimDom(L_2)

T21 PrimDom(L_1)=PrimDom(L_2) → Dom(L_1)=Dom(L_2)

T22* Dom(L_1)=Dom(L_2) → PrimDom(L_1)=PrimDom(L_2)

T23* PrimDom(L_1)=PrimDom(L_2) ↔ Dom(L_1)=Dom(L_2)

T24 ExpComp(L) ↔ (y)(Sel(L,y) ↔ Exten(L,y))

T25 ExpComp(L) ↔ (y)(yεDom(L) ↔ Exten(L,y))

T26 $ExpInComp(L) \leftrightarrow (Ey)(Sel(L,y).\sim Exten(L,y))$

T27 $ExpInComp(L) \leftrightarrow (Ey)(y \varepsilon Dom(L).\sim Exten(L,y))$

T28 $ExpCompRel(L_1,L_2).ExpCompRel(L_1,L_2) \ .\rightarrow$
$ExpCompRel(L_1,L_3)$

T29 $ExpCompRel(L_1,L_2).ExpCompRel(L_2,L_1) \ .\rightarrow.$
$ExpComp(L_1).ExpComp(L_2)$

T30 $ExpCompRel(L_1,L_2).ExpCompRel(L_2,L_1) \ .\leftrightarrow .$
$ExpComp(L_1).ExpComp(L_2).Dom(L_1)=Dom(L_2)$

T31 $ExpComp(L) \leftrightarrow ExpCompRel(L,L)$

T32 $AsExpPow(L_1,L_2).AsExpPow(L_2,L_3) \ .\rightarrow AsExpPow(L_1,L_3)$

T33 $AsExpPow(L,L)$

T34 $Dom(L_1) \subset Dom(L_2).ExpComp(L_2).AsExpPow(L_1,L_2) \ .\rightarrow$
$ExpComp(L_1)$

T35 $GrExpPow(L_1,L_2).GrExpPow(L_2,L_3) \ .\rightarrow GrExpPow(L_1,L_3)$

T36 $\sim GrExpPow(L,L)$

T37 $GrExpPow(L_1,L_2) \rightarrow \sim GrExpPow(L_2,L_1)$

T38 $Dom(L_1) \subset Dom(L_2).GrExpPow(L_1,L_2) \ .\rightarrow ExpInComp(L_2)$

T39 $LsExpPow(L_1,L_2).LsExpPow(L_2,L_3) \ .\rightarrow LsExpPow(L_1,L_3)$

T40 $\sim LsExpPow \ (L,L)$

T41 $LsExpPow(L_1,L_2) \rightarrow \sim LsExpPow(L_2,L_1)$

T42 $EqExpPow(L_1,L_2).EqExpPow(L_2,L_3) \ .\rightarrow EqExpPow(L_1,L_3)$

T43 $EqExpPow(L,L)$

T44 $EqExpPow(L_1,L_2) \rightarrow EqExpPow(L_2,L_1)$

T45 $Dom(L_1)=Dom(L_2).EqExpPow(L_1,L_2) \ .\rightarrow. ExpComp(L_1) \leftrightarrow$
$ExpComp(L_2)$

T46 $Dom(L_1)=Dom(L_2).EqExpPow(L_1,L_2)$.→. $ExpInComp(L_1)$ ↔
$ExpInComp(L_2)$

T47 $EqExpPow(L_1,L_2)$ → $(x)(Exten(L_1,x)$ ↔ $Exten(L_2,x))$

T48* $EqExpPow(L_1,L_2)$ → $(x)(Sel(L_1,x)$ ↔ $Sel(L_2,x))$

T49* $EqExpPow(L_1,L_2)$ → $Dom(L_1)=Dom(L_2)$

T50* $EqExpPow(L_1,L_2)$ ↔ $(x)(Exten(L_2,x)$ ↔ $Exten(L_1,x))$

T51* $EqExpPow(L_1,L_2).ExpComp(L_2)$.→ $ExpComp(L_1)$

T52* $EqExpPow(L_1,L_2).ExpInComp(L_2)$.→ $ExpInComp(L_1)$

T53* $ExpComp(L_1).ExpComp(L_2)$.→. $EqExpPow(L_1,L_2)$ ↔
$(x)(Sel(L_1,x)$ ↔ $Sel(L_2,x)$

T54* $ExpComp(L_1).ExpComp(L_2)$.→. $EqExpPow(L_1,L_2)$ ↔
$Dom(L_1)=Dom(L_2)$

T55 ExpCompResp(L,PrimDom(L)) ↔ ExpComp(L)

T56 ExpComp(L) ↔ (x)(PrimSel(L,x) → ExpCompResp(L,x))

T57 ExpCompResp(L,x) ↔ (y)($y \subset x$ → ExpCompResp(L,y))

T58 (Ex)(L)[(Ey)RefExp(L,y) → ExpCompResp(L,x)]

T59 ∿UniverExpComp(L)

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