THEOREMS OF BARTH-LEFSCHETZ TYPE AND MORSE THEORY ON THE SPACE OF PATHS IN HOMOGENEOUS SPACES

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ABSTRACT

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Homotopy connectedness theorems for complex submanifolds of homogeneous spaces (sometimes referred to as theorems of Barth-Lefshetz type) have been established by a number of authors. Morse Theory on the space of paths lead to an elegant proof of homotopy connectedness theorems for complex submanifolds of Hermitian symmetric spaces. In this work we extend this proof to a larger class of compact complex manifolds namely quotients of complex orthogonal, unitary groups and exceptional groups.

Dedicated to my loving and supporting parents and my darling sister.

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Chapter 1

Introduction

In the 1920s Lefschetz [Le] stated the following theorem now known as the Lefschetz theorem on hyperplane sections. Let $H \subset \mathbb{P}^v$ be a connected complex submanifold of complex dimension n. Let H be a hyperplane and $N \cap H$ be a non singular hyperplane section. Then the relative cohomology groups satisfy:

$$H^j(N, N \cap H, \mathbb{C}) = 0 \qquad j \leqslant n-1$$

Fifty years later Barth [B] generalized Lefschetz' theorem: Let $M, N \in \mathbb{P}^v$ be complex submanifolds of complex dimensions m, n respectively. If M and N meet properly, then

$$H_j(N, N \cap M) = 0 \text{ , } j \leq min(n+m-v; 2m-v+1).$$

Generalizations of Barth's results to homotopy groups were first obtained by Larsen [La], Barth-Larsen [B-L] and later by Sommese. Sommese [S1][S2] and Goldstein [G] generalized these results to submanifolds of generalized flag manifolds, i.e manifolds of the form G^c/P where G^c is a semi-simple complex Lie group and P a parabolic subgroup.

In 1961 T. Frankel [F] proved a connectedness theorem for complex submanifolds of a Kähler manifold of positive holomorphic sectional curvature: Let V be a complete Kähler manifold of positive holomorphic sectional curvature and of complex dimension v. Let

$M, N \subset V$ be compact complex submanifolds of dimensions m and n, respectively. If $m+n \geq v$ then M and N must intersect. Later [K-W]and [S-W] expanded on this idea to prove the Barth-Lefschetz theorems on a class of generalized flag manifolds, namely Hermitian Symmetric Spaces and hence reproduced the results of [S1], [S2] and [G].

In this thesis we extend the theorem of [S-W] and [K-W] to a larger class of generalized flag manifolds. In the main theorem of the work we deal with the case when G^c is simple and later, state how to deal with the semisimple case.

Theorem 1.1. *Let G^c be a simple complex Lie group and P be a parabolic subgroup. Let V be the complex homogeneous space G^c/P with dimension v. Let $M, N \subset V$ be compact complex submanifolds dimension m and n respectively. Then there exists a number ℓ and a number $\lambda_0 = m + n - (v - \ell) - v$ such that*

$$\iota_* : \pi_j(N, N \cap M) \to \pi_j(V, M)$$

is an isomorphism for $j \leq \lambda_0$ and a surjection for $j = \lambda_0 + 1$.

(i) If $G^c = SL_{r+1}(\mathbb{C})$ then $\ell = r$

(ii) If $G^c = SO_{2r}(\mathbb{C})$ then $\ell = 2r - 3$

(iii) If $G^c = SO_{2r+1}(\mathbb{C})$ then $\ell = 2r - 2$

(iv) If $G^c = Sp_r(\mathbb{C})$ then $\ell = r$

(v) If $G^c = E_6$,E_7 ,E_8 then $\ell = 11, 17$ and 29 respectively

(vi) If $G^c = F_4$ the $\ell = 8$

(vii) If $G^c = G_2$ the $\ell = 3$

Corollary 1.2. *Suppose that V, M, and N satisfy the same hypotheses as Theorem 1.1 and ℓ remains the same then*

(a) If $j \leq 2m - v - (v - \ell) + 1$ then $\pi_j(V, M) = 0$

(b) If $j \leq min(2m - v - (v - \ell) + 1, n + m - v - (v - \ell))$ then

$\pi_j(N, N \cap M) = 0$

In part (iii) of Theorem 1.1, the case where $G^c = SO_{2r+1}(\mathbb{C})$ and P is the parabolic corresponding to the painted Dynkin diagram with all long roots painted, the result can be improved. By a Theorem of [O] the corresponding homogeneous space can be written in the form $SO_{2r+2}(\mathbb{C})/P$ where P is a parabolic subgroup of $SO_{2r+2}(\mathbb{C})$, in this case ℓ can be improved to $2r - 1$.

Also the case $G^c = Sp_r(\mathbb{C})$ and for a special type of parabolic subgroup P, G^c/P is biholomorphic to CP^{2r-1} ([O]) so the number ℓ can be improved to $2r - 1$. The parabolic P is a maximal parabolic containing a copy of $Sp_{r-1}(\mathbb{C})$.

The results obtained from Theorem 1.1 also follow from the work of [S1] ,[S2] and [G]. In [S1] and [S2] Sommese shows that the number ℓ can be replaced by the co-ampleness of the respective homogeneous space, and in [G] the co-ampleness of the respective spaces are calculated. In the case where $G^c = Sp_r(\mathbb{C})$, F_4 or G_2 and P is a parabolic such that the corresponding painted Dynkin diagram contains a long root, the approach of [S1] ,[S2] and [G] leads to stronger results. All other cases treated in this work, the results are same as those obtained in [S1], [S2] and [G]. To prove these connectedness Theorems [S1] and [S2] apply Morse Theory locally on the ambient space but in this work we apply Morse theory on the space of paths following the work of [F],[S-W],[K-W],[N-W],[C],[FM] and [W].

The basic idea of [S-W] and [K-W] is to demonstrate that the index of the critical points in the space of paths joining two submanifolds has the appropriate index for a chosen Morse function. The Morse function that they choose on the space of paths is the energy function with respect to the Kähler metric. To compute a lower bound of the index at the critical points, variational vector fields are constructed along these geodesics and used in the second variation formula.

In this work we generalize the idea of [S-W] and [K-W]. Their argument cannot be generalized to non-symmetric homogeneous spaces as [M1] has shown that other homogeneous spaces that are not covers of products of Hermitian Symmetric Spaces don't posses Kähler metrics with non-negative curvature. But complex G^c/P, being a quotient of a compact Lie group, does posses a metric induced by the standard bi-invariant metric of the compact Lie group. This metric, which is commonly referred to as the 'normal metric' is what we use in this paper. This metric has non-negative curvature and is Kähler only in the case of a Hermitian Symmetric Space. Using this metric allows us to naturally generalize the work of [S-W] and [K-W].

Using the canonical connection we construct a new connection referred here as the complex-hat connection. We use this connection to form variational vector fields along geodesics. This connection is invariant and is compatible with the complex structure. The connection is also amenable towards the root structure, as a result most of the computations follow naturally. We also use certain types of linear combinations of these variational vector fields and show the existence of a quaternionic structure on the linear combinations. To demonstrate a lower bound on the index, we take an average using this quaternionic structure.

An outline of this thesis is as follows. In Chapter 2 we review Morse theory on the space of paths and its relation to homotopy theory. In Chapter 3 we review the basic properties of reductive homogeneous spaces. In Chapter 4 we construct the complex-hat connection, describe its properties, and describe its relation with the second variation formula. In Chapter 5 we establish a lower bound on the index of geodesics in terms on an invariant of a Lie algebra. In the final Chapter we compute this invariant, thereby proving Theorem 1.1.

Chapter 2

Morse Theory and Homotopy groups

In this chapter we follow [S-W] and talk about Morse Theory on the space of paths, and relate the index of geodesics with the vanishing of relative homotopy groups.

2.1 Morse Theory

Let V be a complete Riemannian manifold and let M and N be closed submanifolds with M compact and N a closed subset of V. We let $P(V; M, N)$ denote the set of continuous paths $\gamma : [0; 1] \to V$ such that $\gamma(0) \in M$ and $\gamma(1) \in N$ and let $\Omega(V; M, N)$ denote the set of piecewise smooth paths. To learn about the topology of the path space $P(V, M, N)$ it suffices to look at the space $\Omega(V; M, N)$, as the natural inclusion $i^* : \Omega(V; M, N) \to P(V; M, N)$ is a homotopy equivalence [M]. To study the topology of $\Omega(V; M, N)$ we use a Morse function following [M]. We will denote $\Omega(V; M, N)$ by simply Ω.

In [M], Milnor approximates the path space by finite-dimensional manifolds and employs techniques from finite-dimensional Morse Theory. However the problem that Milnor deals with is $\Omega(V; p, q)$, but the proofs given in [M] apply to the general case with only minor changes that can easily be made. Accordingly, in this section, we will describe the general set-up, state the results we will need and give the appropriate references to [M].

The set $\Omega(V; M, N)$ can be topologized as follows. Let ρ denote the Riemannian distance function on V. Let $\gamma_1, \gamma_2 \in \Omega(V; M, N)$. Define the distance $d(\gamma_1, \gamma_2)$ by :

$$d(\gamma_1, \gamma_2) = \max_{0 \le t \le 1} \rho(\gamma_1(t), \gamma_2(t)) + \int_0^1 (|\dot{\gamma_1}(t)| - |\dot{\gamma_2}(t)|)^2 dt$$

The energy of a path, given by

$$E(\gamma) = \int_0^1 |\dot{\gamma}(t))|^2 dt,$$

defines a continuous map from $\Omega(V; M, N) \to \mathbb{R}$.

Define the 'tangent space' of Ω at γ, $T_\gamma \Omega$, to be the vector space of piecewise smooth vector fields W along γ such that $W(0) \in T_{\gamma(0)}\Omega$ and $W(1) \in T_{\gamma(1)}\Omega$. A standard computation in [C-E] shows that the first variation of E in the direction of $W \in T_\gamma \Omega$ is given by :

$$\frac{1}{2} E_*(W) = \langle W, \dot{\gamma} \rangle |_0^1 - \sum_t \langle W(t), \Delta_t \dot{\gamma} \rangle - \int_0^1 \langle W, \frac{D\dot{\gamma}}{dt} \rangle dt$$

where $\Delta_t \dot{\gamma} = \dot{\gamma}(t^+) - \dot{\gamma}(t^-)$, is the discontinuity of $\dot{\gamma}$ at t. It follows that γ is a critical point of E if :

(a) γ is a smooth geodesic.

(b) γ is normal to M and N at $\gamma(0)$ and $\gamma(1)$, respectively.

Let $W_1, W_2 \in T_\gamma \Omega$. If γ is a critical point of E then the second variation of E along γ, is given by:

$$\frac{1}{2} E_{**}(W_1, W_2) = \sum_t \langle W_2(t), \Delta_t \frac{DW_1}{dt} \rangle - \int_0^1 \langle W_2, \frac{D^2 W_1}{dt^2} + R(\dot{\gamma}, W_1)\dot{\gamma} \rangle \tag{2.1}$$

Let Ω_c denote the closed subset $E^{-1}([0, c]) \subset \Omega$ and let Ω_c^* denote the open subset $E^{-1}([0, c))$. We construct a finite dimensional approximation of Ω_c. Choose some subdivision $0 = t_0 < t_1 < ...... < t_k = 1$ of $[0,1]$. Let $\Omega(t_0, ...t_k)$ be the subspace of Ω consisting of paths $\gamma : [0,1] \to V$ such that :

(a) $\gamma(0) \in M$ and $\gamma(1) \in N$

(b) $\gamma|_{[t_{i-1},t_i]}$ is a geodesic for each $i = 1, ...k$

Define the subspace $\Omega_c^*(t_0, ...t_k) = \Omega_c^* \cap \Omega(t_0, ..., t_k)$

Theorem 2.1. *Let V be a complete Riemannian manifold and let M and N be sub-manifolds with M compact and N a closed subset of V. Let c be a fixed positive number such that $\Omega_c \neq \phi$. Then for all sufficiently fine subdivisions $0 = t_0 < t_1 < ......t_k = 1$ of $[0,1]$ the set $\Omega_c^*(t_0, ...t_k)$ can be given the structure of a smooth finite dimensional manifold.*

Proof. [M] Sect 16. □

Denote the manifold $\Omega^*(t_0, ...t_k)$, of broken geodesics by B. Let $E' : B \to \mathbb{R}$ denote the restriction to B of the energy function.

Theorem 2.2. *$E' : B \to \mathbb{R}$ is a smooth map. For each $a < c$ the set $B_A = (E_B)^{-1}([0,a])$ is compact and is a deformation retract of the set Ω_a. The critical points of E' are precisely the same as the critical points of E in Ω_c^*, that is the smooth geodesics from M to N, intersecting M and N orthogonally, and with energy less than c. The index of the hessian of E' at each such critical point γ is equal to the index of E_{**} at γ.*

Proof. [M] Sect. 14 and Sect 16.

□

Now suppose that every nontrivial critical point γ of E on Ω has index $\lambda > \lambda_0 \geq 0$. We remark that this implies that $N \cap M \neq \emptyset$, otherwise there exists a nontrivial minimizing geodesic from M to N and the index of such a geodesic must be zero. It follows that if every nontrivial critical point γ on Ω has index $\lambda > \lambda_0 \geq 0$ then the space Ω_0 of minimal (i.e trivial) geodesics can be identified with the subspace $N \cap M \subset \Omega$.

Proposition 2.3. *Suppose N intersects M transversely and that every non-trivial critical point of E on Ω has index $\lambda > \lambda_0 \geq 0$. Then the relative homotopy groups $\pi_j(\Omega, \Omega_0)$ are zero for $0 \leq j \leq \lambda_0$*

The essential element in the proof of the proposition is the following simple lemma whose proof can be found in [M], about functions on finite-dimensional manifolds. Let X be a smooth manifold and $f : X \to \mathbb{R}$ be a smooth real-valued function with minimum value 0 such that each $X_c = f^{-1}([0, c])$ is compact.

Lemma 2.4. *If the set X_0 of minimal points has a neighborhood U with retraction $r : U \to X_0$ and if every critical point in $X \backslash X_0$ has index $> \lambda_0$ then*

$$\pi_j(X, X_0) = 0 \textit{ for } 0 \leq j \leq \lambda_0$$

For the proof of Proposition 2.3 we refer back to [S-W].

2.2 Homotopy Theory and Index

In this section we will apply Morse Theory to the path spaces $\Omega(V; M, N)$, in the spirit of [S-W]. Here we cite Theorem 1.5 of [S-W] which is an improvement of Proposition 2.3. The assumption of transversality is dropped.

Theorem 2.5. *Let V be a complete complex manifold. Let $M, N \subset V$ be closed complex submanifolds and suppose that M is compact and N is a closed subset of V. If every nontrivial* *critical point of* E *on* Ω *has index* $\lambda > \lambda_0 \geq 0$*, then the relative homotopy groups* $\pi_j(\Omega, \Omega_0) = 0$ *for* $0 \leq j \leq \lambda_0$.

As we can identify Ω_0 with $M \cap N$ we have $\pi_j(\Omega, M \cap N) = 0$ for $0 < j < \lambda_0$. This observation, along with the long exact sequence of the pair $(\Omega, N \cap M)$ implies that the homomorphism induced by the inclusion:

$$\iota_* : \pi_j(N \cap M) \to \pi_j(\Omega) \tag{2.2}$$

is an isomorphism when $j < \lambda_0$ and is a surjection $j = \lambda_0$.

Now consider the fibration

$$\Omega(V; M, x) \to \Omega(V; M, N) \xrightarrow{e} N$$

where e is the evaluation map $e : \gamma \to \gamma(1)$ and $x \in N$. The long exact homotopy sequence of the fibration is:

$$...\pi_{j+1}(N) \to \pi_j(\Omega(V; M, x)) \to \pi_j(\Omega) \xrightarrow{e_*} \pi_j(N) \to \pi_{j-1}(\Omega(V; M, x))... \tag{2.3}$$

It is well known that the homotopy groups of the fiber $\Omega(V; M, x)$ satisfy.

$$\pi_j(\Omega(V; M, x)) \simeq \pi_{j+1}(V, M) \tag{2.4}$$

for all j and hence the sequence (2.3) becomes

$$...\pi_{j+1}(N) \to \pi_{j+1}(V, M) \to \pi_j(\Omega) \xrightarrow{e_*} \pi_j(N) \to \pi_j(V, M)... \tag{2.5}$$

Theorem 2.6. *Let V be a complete complex manifold. Let $M, N \subset V$ be complex submanifolds and suppose that M is compact and N is a closed subset of V. If every nontrivial critical point of E on Ω has index $\lambda > \lambda_0 \geq 0$, then the homomorphism induced by the inclusion.*

$$\iota_* : \pi_j(N, N \cap M) \to \pi_j(V, M)$$

is an isomorphism for $j \leq \lambda_0$ and a surjection for $j = \lambda_0 + 1$.

Proof. Consider the following diagram :

$$\begin{array}{ccccccccc}
...\pi_{j+1}(N) & \longrightarrow & \pi_{j+1}(V, M) & \longrightarrow & \pi_j(\Omega) & \longrightarrow & \pi_j(N) & \longrightarrow & \pi_j(V, M)... \\
\simeq \uparrow & & \uparrow & & \uparrow & & \simeq \uparrow & & \uparrow \\
...\pi_{j+1}(N) & \longrightarrow & \pi_{j+1}(N, N \cap M) & \longrightarrow & \pi_j(N \cap M) & \longrightarrow & \pi_j(N) & \longrightarrow & \pi_j(N, N \cap M)..
\end{array}$$

The first horizontal line is just equation 2.5, the second is the long exact sequence for the pair $(N, N \cap M)$ and the vertical arrows are the inclusion maps. For $j < \lambda_0$ the middle map is an isomorphism (2.2) and hence by the five lemma $\iota_* : \pi_j(N, N \cap M) \to \pi_j(V, M)$ is an isomorphism. For $j = \lambda$ (2.2) is onto so the corresponding map is onto via the five lemma.

□

Chapter 3

Homogeneous Spaces

In this Chapter we introduce and set up notation to objects associated to compact complex homogeneous spaces, including the canonical connection and the Levi-Civita connection. The canonical connection associated to a reductive homogeneous space is well known. The construction of this connection can be found in [K]. Here we describe an easier construction of this connection. We also talk about the Levi-Civita connection of the normal metric as well as briefly describe compact homogeneous spaces of the form G^c/P, where G^c is a complex semi-simple Lie group and P is a parabolic subgroup.

3.1 Notation and Background

Let G be a compact Lie group with identity element e. We identify the Lie algebra $\mathfrak{g}$ of G, with the tangent space T_eG at e.

Each element $a \in G$ defines diffeomorphisms

$$L_a : G \to G \text{ by } L_a(g) = ag \quad \text{(left translation)}$$
$$R_a : G \to G \text{ by } R_a(g) = ga \quad \text{(right translation)}$$
$$C_a : G \to G \text{ by } c_a(g) = aga^{-1} \quad \text{(conjugation)}$$

The adjoint map $Ad_a : \mathfrak{g} \to \mathfrak{g}$ is the differential of the map $C_a : G \to G$ at the identity.

Since $C_{a_1} \circ C_{a_2} = C_{a_1 a_2}$ that gives us a representation of G on the vector space $\mathfrak{g}$. Which we denote by the following map $Ad : G \to GL(\mathfrak{g})$

After taking the derivative, we get the adjoint representation $ad : \mathfrak{g} \to gl(\mathfrak{g})$ of the lie algebra which is given by $ad(X)Y = [X, Y] \;\; \forall \;\; X, Y \in \mathfrak{g}$. We denote the exponential map by $exp : \mathfrak{g} \to G$. So if $t \in \mathbb{R}, X \in \mathfrak{g}$ the map $t \to exp(tX)$ is the unique one parameter subgroup whose tangent vector at $t = 0$ is X. Let $\langle \cdot, \cdot \rangle$ be a bi-invariant metric on G. This implies that

$$\langle X, [Y, Z] \rangle_e = \langle [X, Y], Z \rangle_e \text{ for all } X, Y, Z \in \mathfrak{g} \tag{3.1}$$

Let $K \subseteq G$ be a closed connected subgroup of G and with Lie algebra $\boldsymbol{k}$ then $M = G/K$ is naturally a differentiable manifold. Let $\pi : G \to G/K$ be the canonical projection map. If $g \in G$ we denote its image in G/K by $\bar{g}$ or gK depending on the situation. Let $a \in G$ we define the following map $\overline{L}_a : G/K \to G/K$ by $L_a(gK) = agK$. It is obvious that $\overline{L_a} \circ \pi = \pi \circ L_a$

Definition 1. *A reductive homogeneous space is a homogeneous space with a decomposition* $\mathfrak{g} = \boldsymbol{k} \oplus \mathfrak{m}$ *such that* $Ad_{|K}(\mathfrak{m}) \subseteq \mathfrak{m}$.

Let $\mathfrak{m} = \boldsymbol{k}^{\perp}$, clearly $\mathfrak{g} = \boldsymbol{k} \oplus \mathfrak{m}$. The bi-invariance of the metric implies that the metric is Ad_g-invariant $\forall g \in G$, in particular it is Ad_k invariant $\forall k \in K$, which along with the orthogonality of $\mathfrak{m}$ with $\boldsymbol{k}$ implies that $Ad_{|K}(\mathfrak{m}) \subseteq \mathfrak{m}$. So $\mathfrak{g} = \boldsymbol{k} \oplus \mathfrak{m}$ makes G/K into a reductive homogeneous space.

As $\pi_* \mathfrak{m} = T_{\bar{e}} G/K$, this implies that in a reductive homogeneous space we can identify $\mathfrak{m}$ with $T_{\bar{e}} G/K$. With this identification we can show how left translation is the same as pushing forward the adjoint map.

Proposition 3.1.

$$\pi_* Ad_k(X) = \overline{L}_{k*}X \qquad \forall X \in \mathfrak{m}, \forall k \in K$$

Proof. This follows from

$$\begin{aligned}
\pi_* Ad_k(X) &= \pi_* L_{k*} R_{k^{-1}*} X \\
&= \overline{L}_{k*} \pi_* R_{k^{-1}*} X \\
&= \overline{L}_{k*} \pi_* X \\
&= \overline{L}_{k*} X
\end{aligned}$$

□

The isotropy representation of the group K, $Ad^{G/K} : K \to GL(T_e G/K)$ is defined by

$$Ad^{G/K}(k)(X) = \overline{L}_{k*}(X) \ \forall X \in T_{\bar{e}}(G/K)$$

Let $\omega \in \Lambda^r(TG/K)$ be a covariant vector valued r-tensor. Then ω is G-invariant if

$$\overline{L}_{a*}\omega_{\bar{g}}(X_1, X_2, ...X_r) = \omega_{\bar{a}\bar{g}}(\overline{L}_{g*}X_1, \overline{L}_{g*}X_2.......\overline{L}_{g*}X_r) \ \forall X_i \in T_{\bar{g}}G/K \text{ for } i = 1, 2....r$$

Likewise $\omega \in \Lambda^r T_{\bar{e}} G/K$ is $Ad^{G/K}$ invariant if

$$(Ad^{G/K}(k)\omega_{\bar{e}}(X_1, X_2, ...X_r) = \omega_{\bar{e}}(Ad^{G/K}(k)(X_1), ..., Ad^{G/K}(k)(X_r)) \quad \forall k \in K$$

We have similar definitions for real valued tensors. The next proposition follows from Proposition 3.1

Proposition 3.2. *If $\omega \in \Lambda^r T_{\bar{e}} G/K$ is $Ad^{G/K}$ invariant then ω can be extended to a G invariant r-tensor.*

Restrict $\langle \cdot, \cdot \rangle_e$ to $\mathfrak{m}$. Then the restricted metric is $Ad^{G/K}$ invariant since $\langle \cdot, \cdot \rangle_e$ is Ad^G-invariant. Thus we have a G invariant metric $\langle \cdot, \cdot \rangle_{|G/K}$ on the whole of G/K, which turns out to be nothing but the push forward of the metric $\langle \cdot, \cdot \rangle$ on G.

We give some important examples of invariant tensors. Let $X, Y \in \mathfrak{m}$, lets denote the $\mathfrak{m}$ and $\boldsymbol{k}$ component of the bracket by $[X,Y]_{\mathfrak{m}}$ and $[X,Y]_{\boldsymbol{k}}$ respectively. As $Ad(g)$ commutes with the Lie bracket $\forall g \in G$ and the fact that the decomposition $\mathfrak{m} \bigoplus \boldsymbol{k}$ is reductive, we have the following

$$Ad^G(k)[X,Y]_{\mathfrak{m}} = [Ad^G(k)X, Ad^G(k)Y]_{\mathfrak{m}} \tag{3.2}$$

$$Ad^G(k)[X,Y]_{\boldsymbol{k}} = [Ad^G(k)X, Ad^G(k)Y]_{\boldsymbol{k}} \tag{3.3}$$

From the first equality we have that $[\cdot,\cdot]_{\mathfrak{m}}$ is $Ad^{G/K}$ invariant $(2,1)$ tensor. From the second equality and the fact that the metric $\langle \cdot, \cdot \rangle_{|G/K}$ is invariant under K, we have that that $|[X,Y]_{\boldsymbol{k}}|^2$ is $Ad^{G/K}(k)$ invariant real valued tensor.

Definition 2. $[\cdot,\cdot]_{\mathfrak{m}}$ *and* $|[\cdot,\cdot]_{\boldsymbol{k}}|^2$ *will represent the global tensors obtained by extending the* $Ad^{G/K}$ *invariant tensors* $[\cdot,\cdot]_{\mathfrak{m}}$ *and* $|[\cdot,\cdot]_{\boldsymbol{k}}|^2$ *to* G/K*.*

We now mention a very important property of the bracket tensor $[\cdot,\cdot]_{\mathfrak{m}}$

$$\langle [X,Y]_{\mathfrak{m}}, Z \rangle = \langle X, [Y,Z]_{\mathfrak{m}} \rangle \text{ for all } X, Y, Z \in T_{gK}(G/K) \tag{3.4}$$

This property just follows from (3.1). We will refer to this property as 'associativity of the bracket'.

3.2 The canonical connection

In this section we assume G/K is a reductive homogeneous space, with the decomposition $\mathfrak{g} = \boldsymbol{k} \oplus \mathfrak{m}$. We will define and describe the basic properties of the canonical connection of a reductive homogeneous space.

Consider G as a fiber bundle over the left-coset space G/K with structure group K. The action of K on G is right multiplication. The group G itself acts on the fiber bundle; this action clearly commutes with the projection map and the action of K.

We define a G-invariant connection on the principle bundle G. We set the horizontal space at the identity to be the space $\mathfrak{m}$ and to be $L_{g^*}\mathfrak{m}$ at g. This defines a horizontal distribution. To show that this distribution is a connection we have to show compatibility with the right action:

$$\begin{aligned} R_{k*}L_{g*}m &= L_{g*}R_{k*}m \\ &= L_{g*}L_{k*}L_{k*}^{-1}R_{k*}m \\ &= L_{(gk)*}Ad^{G/K}(k^{-1})_{*}m \\ &= L_{gk*}m \end{aligned}$$

Let $\theta : G \times \mathfrak{m} \to T(G/K)$ is defined by $\theta(g \times X) = \overline{L}_{g*}X \in T_{\overline{g}}(G/K)$. Let $\phi : T(G/K) \to G/K$ denote the projection map and $\pi' : G \times \mathfrak{m} \to (G/K)$ be the natural projection map. Then clearly $\phi \circ \theta = \pi'$.

Define an equivalence relation '$\sim$' on $G \times \mathfrak{m}$ by

$$(g \times X) \sim (gk \times \overline{L}_{k*}^{-1} X) \qquad \forall k \in K$$

Now θ clearly factors through the quotient space $(G \times \mathfrak{m})/\sim$ so we have the quotient map

$$\widetilde{\theta} : (G \times \mathfrak{m})/\sim \to T(G/K)$$

$\widetilde{\theta}$ is clearly one-one, onto and fiber preserving, making $\widetilde{\theta}$ an isomorphisms of vector bundles.

With this connection on the principle bundle G we have a notion of parallel transport along a curve $\gamma : [0,1] \to G/K$ as follows. Let $u_0 \in \pi^{-1}(\gamma(0))$ and let u_t denote the horizontal lift of γ starting at u_0, then the parallel transport of u_0 from $\gamma(0)$ to $\gamma(t)$ is simply u_t.

On the space $(G \times \mathfrak{m})/\sim$ there is a notion of parallel transport (as in section 7 ch2 of [K])that derives from the notion of parallel transport on the principle bundle G. Let $[(u_0, X)]$ represent the equivalence class containing(u_0, X). The parallel transport of $[(u_0, X)]$ along γ, from $\gamma(0)$ to $\gamma(t)$ will just be $[(u_t, X)]$. This is independent of the choice of representative $[(u_0, X)]$.

As we can identify $(G \times \mathfrak{m})/\sim$ with $T(G/K)$ we can talk about the parallel translation of $X \in T_{\gamma(0)}G/K$ along γ. By this identification $[(u_0, L_{u_0*}^{-1}X)]$ can be identified with X, so the parallel translation of X will be $L_{u_t*}L_{u_0*}^{-1}X$.

Let $\tau_0^h : \phi^{-1}(\gamma_0) \to \phi^{-1}(\gamma_h)$ denotes the parallel translation along γ from $\gamma(0)$ to $\gamma(h)$, and let τ_h^0 denote the inverse map. With this notion of parallel transport on $T(G/K)$ we can find a linear connection which has parallel transport coinciding with this (as in sec 1 ch

3 of [K]), as follows.

Define

$$\nabla_{\dot\gamma(0)} X = \lim_{h\to 0} \frac{1}{h}[\tau_h^0(X(\gamma_h)) - X(\gamma_0)]$$

This linear connection will also be called the canonical connection and will be denoted by ∇. The above discussion can be summarized in the following proposition.

Proposition 3.3. *The parallel transport of $X \in T_{\gamma(0)}G/K$ with respect to the canonical connection along a curve γ is given by left translation of some element of G. More precisely $\tau_0^h(X) = L_{u_t *} L_{u_0 *}^{-1} X$ where u_t is any horizontal lift of γ*

Now for a few applications of this proposition. Let $\gamma_X(t)$ be the integral curve to the vector field $\widehat{X}$ in G starting at the identity. Where $\widehat{X}$ is the left-invariant vector field generated by X.

Theorem 3.4. *$\pi(\gamma_X(t))$ is a geodesic with respect to the canonical connection and all geodesics are of this form or a translate of it.*

Proof. To show that $\pi(\gamma_X(t))$ is a geodesic with respect to the canonical connection, it suffices to show that the tangent vector field on the curve is itself parallel. From the above Theorem, the parallel transport of $X \in T_{\bar{e}}G/K$ along $\pi(\gamma_X(t))$ will be $L_{\gamma_X(t_0)*}X \in T_{\gamma_X(t_0)}G/K$. So the vector field $L_{\gamma_X(t)*}X$ is a parallel vector field on $\pi(\gamma_X(t))$. But by the construction of $\pi(\gamma_X(t))$ it is easy to see that this is in fact the tangent vector field on the curve.

The second part of the Theorem follows from G-invariance of the connection. □

Theorem 3.5. *Every G-invariant tensor is parallel with respect to the canonical connection.*

Proof. This follows from Proposition 3.3 and the definition of G-invariant tensor. $\square$

Definition 3. *For a vector field X on G/K define $f_X : G \to \mathfrak{m}$ by*

$$f_X(g) = L_g^{-1}(X(\pi(g)).$$

Proposition 3.6. *Let X be vector field on G/K, let $\gamma(t)$ be a path in G/K, let u_t be a horizontal lift of $\gamma(t)$ and denote $\gamma(0)$ by Y and $u'(0)$ by Y^*. Then $\nabla_Y X = L_{u_0*}(Y^*(f_X))$.*

Proof.

$$\begin{aligned} L_{u_0*}(Y^*(f_X)) &= L_{u_0*}\left\{\lim_{h\to 0}\frac{1}{h}[f_X(u_h) - f_X(u_0)]\right\} \\ &= L_{u_0*}\left\{\lim_{h\to 0}\frac{1}{h}[L_{u_h*}^{-1}X(\gamma_h) - L_{u_0*}^{-1}X(\gamma_0)]\right\} \\ &= \lim_{h\to 0}\frac{1}{h}[L_{u_0*}L_{u_h*}^{-1}X(\gamma_h) - X(\gamma_0)] \\ &= \lim_{h\to 0}\frac{1}{h}[\tau_h^0(X(\gamma_h) - X(\gamma_0)] \\ &= \nabla_Y X \end{aligned}$$

$\square$

Proposition 3.7. *Suppose X, Y, Z are vector fields on G/K and X^*, Y^*, Z^* be the respective horizontal lifts, let $T(X,Y)$ and $R(X,Y)Z$ denote the torsion and curvature on G/K Then*

(a) $T(X,Y)(\bar{g}) = L_{g*}(X^*(f_Y) - Y^*(f_X)) - [X,Y](g)$

(b) $R(X,Y)Z = L_{g*}(X^*(Y^*(f_Z)) - Y^*(X^*(f_Z)) - h([X^*,Y^*])(f_Z)$

$$= L_{g*}v([X^*, Y^*](f_Z))$$

where $h()$ and $v()$ denote the horizontal and vertical components of the vector

Proof. (a) follows from the definition of torsion and the Proposition .

(b) This follows from Proposition 3.6, the definition of curvature, and the fact that $[X, Y]^* = h([X^*, Y^*])$

□

Proposition 3.8. *For the canonical connection*

(a) $T(X, Y) = -[X, Y]_{\mathfrak{m}}$

(b) $R(X, Y)Z = -[[X, Y]_{\boldsymbol{k}}, Z]_{\mathfrak{m}}$

Proof. (a) It suffices to prove it at $\bar{e}$ as $T(\cdot, \cdot)$ and $[\cdot, \cdot]_{\mathfrak{m}}$ are invariant tensors. For $X, Y \in \mathfrak{m}$ let $\widetilde{X}, \widetilde{Y}$ be right-invariant vector fields generated by X, Y in G. The right invariance implies that pushing forward makes sense. So $\pi_*\widetilde{X}, \pi_*\tilde{Y}$ are extensions of X, Y in G/K. We will use these vector fields to show the first part.

Now

$$\begin{aligned} [\pi_*\widetilde{X}, \pi_*\widetilde{Y}] &= \pi_*[\widetilde{X}, \widetilde{Y}] \\ &= \pi_*(-\widetilde{[X, Y]}) \\ &= (-[X, Y])_{\mathfrak{m}} \end{aligned}$$

so at the identity $[\pi_*\widetilde{X}, \pi_*\widetilde{Y}] = -[X, Y]_{\mathfrak{m}}$

Now

$$
\begin{aligned}
f_{\pi_*\widetilde{X}}(g) &= L_g^{-1}(\pi_*\widetilde{X}(\bar{g})) \\
&= L_g^{-1}(\pi_* R_{g*}(X)) \\
&= \pi_*(ad(g^{-1})_* X)
\end{aligned}
$$

Now as we are only calculating the value of the tensors at the identity, we will just note that the horizontal lifts of X, Y at the identity is just X, Y itself. Now

$$
\begin{aligned}
Y^*(e)(f_{\pi_*\widetilde{X}}(g)) &= Y(\pi_*(ad(g^{-1})_* X)) \\
&= \frac{d}{dt}(\pi_*(ad(exp(-tY)_* X)) \\
&= \pi_*[-Y, X] \\
&= [X, Y]_{\mathfrak{m}}
\end{aligned}
$$

Similarly we have $X(f_{\pi_*\widetilde{Y}}(g)) = X(\pi_*(ad(g^{-1})_* Y) = [Y, X]_{\mathfrak{m}}$

So combining the previous equations we get

$$
\begin{aligned}
T(X, Y) &= [Y, X]_{\mathfrak{m}} - [X, Y]_{\mathfrak{m}} - [-X, Y]_{\mathfrak{m}} \\
&= -[X, Y]_{\mathfrak{m}}
\end{aligned}
$$

(b) Firstly let $\widehat{X}, \widehat{Y}$ denote the left invariant vector fields generated by X, Y in G. Notice that $v([h(\cdot), h(\cdot)])$ is a tensor on G, so $v([h(X^*), h(Y^*)]) = v([h(\widehat{X}), h(\widehat{Y})])$, where X^*, Y^* are horizontal lifts of any extensions of X, Y. As $\widehat{X}, \widehat{Y}$ are in fact horizontal vector fields, we have the following calculation.

$$
\begin{aligned}
v([h(\widehat{X}), h(\widehat{Y})] &= v([(\widehat{X}), (\widehat{Y})]) \\
&= v([X, Y]) \\
&= [X, Y]_{\boldsymbol{k}}
\end{aligned}
$$

From the above equations we get.

$$
\begin{aligned}
R(X, Y)Z &= v([h(X^*), h(Y^*)])(f_Z)) \\
&= v([h(\widehat{X}), h(\widehat{Y})] \\
&= [X, Y]_{\boldsymbol{k}}(f_Z) \\
&= [X, Y]_{\boldsymbol{k}}(\pi_*(ad(g^{-1})_* Z) \\
&= \frac{d}{dt}(\pi_*(ad(exp(-t[X, Y]_{\boldsymbol{k}})_* Z)) \\
&= \pi_*[-[X, Y]_{\boldsymbol{k}}, Z] \\
&= [-[X, Y]_{\boldsymbol{k}}, Z]_{\mathfrak{m}}
\end{aligned}
$$

□

Proposition 3.9. *Let J be a G-invariant integrable complex structure on G/K. Then*

a) $\nabla J = 0$, *where* ∇ *is the canonical connection.*

b) $[X, Y]_{\mathfrak{m}} + J[JX, Y]_{\mathfrak{m}} + J[X, JY]_{\mathfrak{m}} - [JX, JY]_{\mathfrak{m}} = 0$

Proof. a) This follows from Proposition 3.5

b) Let X, Y be two vectors at some point, we extend the vector fields in a small neighbourhood and compute the Nijenhuis tensor on the extension.

$$
\begin{aligned}
N(X,Y) &= [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY] \\
&= \nabla_X Y - \nabla_Y X + [X,Y]_{\mathfrak{m}} + J\nabla_{JX} Y - J\nabla_Y JX + J[X,Y]_{\mathfrak{m}} \\
&+ J\nabla_X JY - \nabla_{JY} X + J[X,JY]_{\mathfrak{m}} - \nabla_{JX} JY + \nabla_{JY} JX - J[X,JY]_{\mathfrak{m}} \\
&= [X,Y]_{\mathfrak{m}} + J[JX,Y]_{\mathfrak{m}} + J[X,JY]_{\mathfrak{m}} - [JX,JY]_{\mathfrak{m}} \qquad (3.5)
\end{aligned}
$$

We have used the torsion of the canonical connection (Proposition 3.8)) and part (a) of this proposition to do the calculation. Part (b) follows from this formula since the Nijenhuis tensor is zero if and only if the complex structure is integrable.

□

3.3 The Levi-Civita Connection

We use the metric $\langle \cdot, \cdot \rangle$ to decompose the Lie algebra $\mathfrak{g}$. This makes G/K into a reductive homogeneous space. Whenever we refer to the canonical connection it will refer to the connection from this decomposition hence forth. For the rest of this work we fix the metric $\langle \cdot, \cdot \rangle_{|G/K}$ on G/K and simply denote it by $\langle \cdot, \cdot \rangle$.

Definition 4. *We define a connection* $\widetilde{\nabla}$ *by*

$$\widetilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}[X,Y]_{\mathfrak{m}}$$

Proposition 3.10. *(a)* $\widetilde{\nabla}$ *is the Levi-Civita connection for the metric* $\langle \cdot, \cdot \rangle$*.*

(b) The geodesics of $\widetilde{\nabla}$ *are the same as the geodesics of the canonical connection.*

Proof. (a) Since the torsion of the canonical connection is $-[X,Y]_{\mathfrak{m}}$, it is easy to show that the connection $\widetilde{\nabla}$ has zero torsion. Compatibility with the metric follows from compatibility of the metric with the canonical connection and the associativity of the bracket (3.4).

(b) For a given path γ in G/K we have $\widetilde{\nabla}_{\dot\gamma}\dot\gamma = \nabla_{\dot\gamma}\dot\gamma$. The result now follows.

□

Proposition 3.11. *The curvature of the metric is given by*

$$\langle \bar{R}(X,Y)Y,X\rangle = \frac{1}{4}\langle [X,Y]_{\mathfrak{m}}, [X,Y]_{\mathfrak{m}}\rangle + \langle [X,Y]_{\boldsymbol{k}}, [X,Y]_{\boldsymbol{k}}\rangle$$

Proof. Using Definition 4 and the fact that $[\cdot,\cdot]$ is an invariant tensor we have.

$$\begin{aligned}
\bar{R}(X,Y)Z &= \widetilde{\nabla}_X\widetilde{\nabla}_Y Z - \widetilde{\nabla}_Y\widetilde{\nabla}_X Z - \widetilde{\nabla}_{[X,Y]}Z \\
&= \widetilde{\nabla}_X(\nabla_Y Z + \tfrac{1}{2}[Y,Z]_{\mathfrak{m}}) - \widetilde{\nabla}_Y(\nabla_X Z + \tfrac{1}{2}[X,Z]_{\mathfrak{m}}) - (\nabla_{[X,Y]}Z \\
&+ \tfrac{1}{2}[[X,Y],Z]_{\mathfrak{m}}) \\
&= R(X,Y)Z + \tfrac{1}{2}(\nabla_X[Y,Z]_{\mathfrak{m}} - \nabla_Y[X,Z]_{\mathfrak{m}} - [[X,Y],Z]_{\mathfrak{m}}) \\
&+ \tfrac{1}{2}([X,\nabla_Y Z]_{\mathfrak{m}} - [Y,\nabla_X Z]_{\mathfrak{m}}) + \tfrac{1}{4}([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - [Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}}) \\
&= R(X,Y)Z + \tfrac{1}{2}([\nabla_X Y,Z]_{\mathfrak{m}} - [\nabla_Y X,Z]_{\mathfrak{m}} - [[X,Y],Z]_{\mathfrak{m}}) \\
&+ \tfrac{1}{4}([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - [Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}})
\end{aligned}$$

Using the formulas for the torsion and curvature of the canonical connection (Proposition

3.8), we have

$$\bar{R}(X,Y)Z = R(X,Y)Z + \frac{1}{2}([\nabla_X Y, -\nabla_Y X - [X,Y], Z]_{\mathfrak{m}}) + \frac{1}{4}([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}}$$
$$- [Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}})$$
$$= R(X,Y)Z + \frac{1}{2}([-[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}}) + \frac{1}{4}([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - [Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}})$$
$$= [-[X,Y]_{\boldsymbol{k}}, Z]_{\mathfrak{m}} + \frac{1}{2}([-[X,Y]_{\mathfrak{m}}, Z]_{\mathfrak{m}}) + \frac{1}{4}([X,[Y,Z]_{\mathfrak{m}}]_{\mathfrak{m}} - [Y,[X,Z]_{\mathfrak{m}}]_{\mathfrak{m}})$$

The result now follows from the associativity of the bracket property (3.4). □

3.4 Complex G^c/P

Let G^c be a complex semi-simple Lie group and let $\mathfrak{g}_{\mathbb{C}}$ be the corresponding Lie algebra. Let $\mathfrak{h}$ be a Cartan subalgebra. Let $\Delta \subset \mathfrak{h}^*$ be the set of roots. Let $V_\alpha^{\mathbb{C}} = \{E \in \mathfrak{g}_{\mathbb{C}} | [h,E] = \alpha(h)E\}$ denote the root space corresponding to α. We also have the following decomposition $\mathfrak{g}_{\mathbb{C}} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta} V_\alpha^{\mathbb{C}}$.

For a semi-simple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ the roots and root spaces satisfy the following properties. If α is a root then so is $-\alpha$. Each $V_\alpha^{\mathbb{C}}$ is one-dimensional. The root space satisfies an important property $[V_{\alpha_1}^{\mathbb{C}}, V_{\alpha_2}^{\mathbb{C}}] \subset V_{\alpha_1+\alpha_2}^{\mathbb{C}}$, the bracket is zero if $\alpha_1 + \alpha_2$ is not a root and $[V_{\alpha_1}^{\mathbb{C}}, V_{\alpha_2}^{\mathbb{C}}] \subset \mathfrak{h}$ if $\alpha_1 + \alpha_2 = 0$.

We can choose a base $\Sigma \subset \Delta$ such that any element of Δ can be uniquely written as an integer linear combination of elements of Σ, such that all the co-efficients are either positive or negative. A choice of such a set of roots Σ, are called simple roots. Let Δ^+, Δ^- be the set of elements of Δ that can be written as a positive / negative linear combinations respectively. Δ^+ and Δ^- will be referred to as positive and negative roots.

We have an inner product on the Lie algebra $\mathfrak{g}_{\mathbb{C}}$ namely the Killing form $\kappa(\cdot,\cdot)$. κ is associative in the sense that $\kappa([X,Y],Z) = \kappa([X,[Y,Z]])$ for all $X,Y,Z \in \mathfrak{g}$. It also satisfies the following properties $\kappa(V^{\mathbb{C}}_{\alpha_1}, V^{\mathbb{C}}_{\alpha_2}) = 0$ iff $\alpha_1 + \alpha_2 \neq 0$ and $\kappa(V_{\alpha_i}, \mathfrak{h}) = 0$ for $i = 1,2$, $\alpha_i \in \Delta$. This inner product restricted to $\mathfrak{h}$ is non-degenerate giving us an identification of $\mathfrak{h}$ and $\mathfrak{h}^*$.

For a given root α we will denote its dual by t_α. We also denote by $\mathfrak{h}_{\mathbb{R}}$ the $\mathbb{R}$ linear span of t_α for $\alpha \in \Delta^+$. $\kappa(\cdot,\cdot)$ is positive definite on $\mathfrak{h}_{\mathbb{R}}$. Let $(\cdot,\cdot)$ be the dual of κ. Define the structure constants $c_{\alpha,\beta}$ by $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$. We state a Proposition from [H,sec 25]

Proposition 3.12. *We can choose $E_\alpha \in V^{\mathbb{C}}_\alpha$ such that*

(a) $c_{\alpha,\beta} = -c_{\beta,\alpha}$

(b) $c_{\alpha,\beta} = -c_{-\alpha,-\beta}$

(c) $[E_\alpha, E_{-\alpha}] = \frac{2t_\alpha}{(\alpha,\alpha)}$

E_α are called root vectors, such a choice of root vectors is called a Chevalley basis[H]. As a consequence of this choice $\kappa(E_\alpha, E_{-\alpha}) > 0$.

Let $\Sigma_{\boldsymbol{k}} \subset \Sigma$ and let $\Delta_{\boldsymbol{k}}$ be the set of all roots which can be written down as sums of roots of $\Sigma_{\boldsymbol{k}}$. Let $\Delta^+_{\boldsymbol{k}} = \Delta_{\boldsymbol{k}} \cap \Delta^+$ and $\Delta^-_{\boldsymbol{k}} = \Delta_{\boldsymbol{k}} \cap \Delta^-$. Let $\mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^-_{\boldsymbol{k}} \cup \Delta^+} V^{\mathbb{C}}_\alpha$. Let P be the Lie subgroup corresponding to the subalgebra $\mathfrak{p}$. P is a parabolic subgroup and every parabolic subgroup is of this form, for an appropriate choice of $\mathfrak{h}$, Δ and Σ. [W]

The homogeneous space $V = G^c/P$ is a compact complex homogeneous space and can be written as a quotient G/K where G is a compact subgroup of G^c and $K = G^c \cap P$. We now describe its Lie algebra $\mathfrak{g}$.

Let $X_\alpha = E_\alpha - E_{-\alpha}$ and let $Y_\alpha = iE_\alpha + iE_\alpha$. Then $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = i\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} V_\alpha$$

where $V_\alpha = \operatorname{span}_{\mathbb{R}}\{X_\alpha, Y_\alpha\}$ is the real root space associated to α.

Since $K = G^c \cap P$ is the Lie algebra of K is $\boldsymbol{k} = i\mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+_{\boldsymbol{k}}} V_\alpha$, the restriction of κ to $\mathfrak{g}$ is negative definite, hence the corresponding group G is compact. Denote by $\langle \cdot, \cdot \rangle$ the left-invariant metric on G such that the restriction to $\mathfrak{g}$ is $-\kappa_{|\mathfrak{g}}$. Since κ is associative, that implies that $\langle \cdot, \cdot \rangle$ is a bi-invariant metric.

Let $\Delta_{\mathfrak{m}}$ be the complement of $\Delta_{\boldsymbol{k}}$ in Δ. Let $\mathfrak{m} = \bigoplus_{\alpha \in \Delta^+_{\mathfrak{m}}} V_\alpha$. Using the properties of the Killing form its clear that $\mathfrak{g} = \boldsymbol{k} \oplus \mathfrak{m}$ is an orthogonal decomposition. This makes G/K a reductive homogeneous space. So we can identify $\mathfrak{m}$ with $T_{\bar{e}}G/K$.

Now we define a complex structure on this tangent space $J : T_{\bar{e}}G/K \to T_{\bar{e}}G/K$ by (note that $\alpha \in \Delta^+_{\mathfrak{m}}$)

$$J(X_\alpha) = iY_\alpha$$

$$J(Y_\alpha) = -X_\alpha$$

We note that the complex structure we just defined, implies that $JE_\alpha = iE_\alpha$ and $JE_{-\alpha} = -iE_{-\alpha}$. This can extended to the whole G/K to give us an invariant, integrable and hermitian complex structure.

3.5 The map I

In this section we define a linear operator on a subspace of $\mathfrak{g}$, which along with J gives us quaternionic structure on the subspace. This structure plays an important role in the work below.

We define a bilinear form on $\mathfrak{m}$ by $R_Y X = [Y, X]_{\mathfrak{m}} + J[JY, X]_{\mathfrak{m}}$

Lemma 3.13. *(a) If $X, Y \in \mathfrak{m}$ then $[X^{1,0}, Y^{1,0}]_{\mathfrak{m}} \in \mathfrak{m}^{1,0}$*

(b) $R_Y X \neq 0$ iff $[X^{1,0}, Y^{0,1}]_{\mathfrak{m}} \notin \mathfrak{m}^{0,1}$

(c) If $X, Y \in \mathfrak{m}$ then $[X^{1,0}, Y^{1,0}]_{\boldsymbol{k}} = 0$.

(d) If $[Y, X]_{\boldsymbol{k}} = 0$ and $[Y, JX]_{\boldsymbol{k}} = 0$ iff $[X^{1,0}, Y^{0,1}]_{\boldsymbol{k}} = 0$

Proof. (a) The result follows by decomposing X, Y into their $\{1, 0\}$ and $\{0, 1\}$ components in the integrability condition (Proposition 3.9)

(b) After a brief calculation we find that $R_Y X = -(Z + \overline{Z})$ where $Z = [X^{1,0}, Y^{0,1}]_{\mathfrak{m}} - iJ[X^{1,0}, Y^{0,1}]_{\mathfrak{m}}$ But $Z \in \mathfrak{m}^{1,0}$ so $Z + \overline{Z} = 0$ if and only if $Z = 0$ if and only if $[X^{1,0}, Y^{0,1}]_{\mathfrak{m}} \in \mathfrak{m}^{0,1}$.

(c) It suffices to show that if $\alpha, \beta \in \mathfrak{m}$ the $\alpha + \beta \notin \boldsymbol{k}$. This follows trivially from the construction of $\mathfrak{m}$ and $\boldsymbol{k}$.

(d) Using (c) we arrive at

$$|[X, Y]_{\boldsymbol{k}}|^2 + |[JX, Y]_{\boldsymbol{k}}|^2 = 4\langle [X^{1,0}, Y^{0,1}]_{\boldsymbol{k}}, [X^{0,1}, Y^{1,0}]_{\boldsymbol{k}} \rangle$$

The result now follows from this calculation.

□

Lemma 3.14. *(a) If* $[X^{1,0}, Y^{0,1}]_{\mathfrak{m}} \in \mathfrak{m}^{0,1}$ *then* $J[Y, X]_{\mathfrak{m}} = [JY, X]_{\mathfrak{m}}$

(b) If $[X^{1,0}, Y^{0,1}]_{\mathfrak{m}} \in \mathfrak{m}^{0,1}$ *then* $J[Y, X]_{\mathfrak{m}} = -[Y, JX]_{\mathfrak{m}} + i[Y^{1,0}, X^{1,0}]_{\mathfrak{m}} + i[Y^{0,1}, X^{0,1}]_{\mathfrak{m}}$

Proof. (a) The hypothesis implies $R_Y X = 0$ from Lemma 3.13. The result now follows directly from the definition of $R_Y X$

(b)

$$\begin{aligned} J[Y, X]_{\mathfrak{m}} + [Y, JX]_{\mathfrak{m}} &= J[Y^{1,0} + Y^{0,1}, X^{1,0} + X^{0,1}]_{\mathfrak{m}} + i[Y^{1,0} + Y^{0,1}, X^{1,0} - X^{0,1}]_{\mathfrak{m}} \\ &= i[Y^{1,0}, X^{1,0}]_{\mathfrak{m}} + [Y^{0,1}, X^{0,1}]_{\mathfrak{m}} \end{aligned}$$

The last line follows since $[X^{1,0}, Y^{0,1}]_{\mathfrak{m}} \in \mathfrak{m}^{0,1}$ □

For $\alpha, \beta \in \Delta$ we define the structure constants $c_{\alpha,\beta}$ by $[E_\alpha, E_\beta] = c_{\alpha,\beta} E_{\alpha+\beta}$ whenever $\alpha + \beta \neq 0$

Lemma 3.15. *Let* $\alpha, \beta, \delta \in \Delta^+$ *such that* $\alpha + \beta = \delta$. *Then*

$$c_{\delta,-\alpha} = \frac{-(\beta, \beta)}{(\delta, \delta)} c_{\alpha,\beta}$$
$$c_{\delta,-\beta} = \frac{(\alpha, \alpha)}{(\delta, \delta)} c_{\alpha,\beta}$$

Proof. By the Jacobi identity we have

$$[[E_\alpha, E_\beta], E_{-\delta}] + [[E_\beta, E_{-\delta}], E_\alpha] + [[E_{-\delta}, E_\alpha], E_\beta] = 0 \tag{3.6}$$

Using Proposition 3.12 3), (3.6) becomes

$$\frac{2c_{\alpha,\beta}t_\delta}{(\delta,\delta)} - \frac{2c_{\beta,-\delta}t_\alpha}{(\alpha,\alpha)} - \frac{2c_{-\delta,\alpha}t_\beta}{(\beta,\beta)} = 0 \tag{3.7}$$

Since t_α is the dual of α. $\alpha + \beta = \delta$ implies that

$$t_\delta - t_\alpha - t_\beta = 0 \tag{3.8}$$

Using the linear independence of t_α, t_β and t_δ, (3.7) and (3.8) gives us

$$\frac{2c_{\alpha,\beta}}{(\delta,\delta)} = \frac{2c_{\beta,-\delta}}{(\alpha,\alpha)} = \frac{2c_{-\delta,\alpha}}{(\beta,\beta)}$$

Using the properties of the structure constants in Proposition 3.12, the Lemma follows.

□

Let $\widetilde{c}_{\alpha,\beta} = \sqrt{\frac{(\alpha,\alpha)(\beta,\beta)c^2_{\alpha,\beta}}{(\delta,\delta)^2}}$ and let $[\cdot,\cdot]_{\alpha,\beta}$ denote the projection of the bracket on to the subspace $V_\alpha \oplus V_\beta$. We have following lemma

Lemma 3.16. *Let $X \in V_\alpha \oplus V_\beta$ and $\delta = \alpha + \beta$. For $\widetilde{X}_\delta = aX_\delta + bJX_\delta$*

$$[\widetilde{X}_\delta, [\widetilde{X}_\delta, X]_{\alpha,\beta}]_{\alpha,\beta} = -(a^2 + b^2)c^2_{\alpha,\beta}X$$

Proof. Using the properties of the structure constants (Proposition 3.12) we have

$$\begin{aligned}[X_\delta, X_\alpha] =& [E_\delta - E_{-\delta}, E_\alpha - E_{-\alpha}] \\ =& c_{\delta,\alpha}E_{\alpha+\delta} + c_{-\delta,-\alpha}E_{-\alpha-\delta} - c_{-\delta,\alpha}E_\beta - c_{\delta,-\alpha}E_\beta \\ =& c_{\delta,\alpha}X_{\alpha+\delta} - c_{\delta,-\alpha}X_\beta\end{aligned}$$

So finally we have $[X_\delta, X_\alpha]_{\alpha,\beta} = -c_{\delta,-\alpha}X_\beta$. Similarly we also have $[X_\delta, X_\beta]_{\alpha,\beta} = -c_{\delta,-\beta}X_\alpha$.

Using Lemma 3.15 we have

$$[X_\delta, [X_\delta, X_\alpha]_{\alpha,\beta}]_{\alpha,\beta} = -\tilde{c}^2_{\alpha,\beta}X_\alpha$$
$$[X_\delta, [X_\delta, X_\beta]_{\alpha,\beta}]_{\alpha,\beta} = -\tilde{c}^2_{\alpha,\beta}X_\beta$$

Using these calculations and Lemma 3.14 we also observe that

$$[X_\delta, [X_\delta, JX_\alpha]_{\alpha,\beta}]_{\alpha,\beta} = -\tilde{c}^2_{\alpha,\beta}JX_\alpha$$
$$[JX_\delta, [JX_\delta, X_\alpha]_{\alpha,\beta}]_{\alpha,\beta} = -\tilde{c}^2_{\alpha,\beta}X_\alpha$$
$$[JX_\delta, [JX_\delta, JX_\alpha]_{\alpha,\beta}]_{\alpha,\beta} = -\tilde{c}^2_{\alpha,\beta}JX_\alpha$$

Using these equations and Lemma 3.14 the lemma follows. □

We can define an operator $I_{a,b} : V_\alpha \oplus V_\beta \to V_\alpha \oplus V_\beta$ by

$$I_{a,b}X = \frac{1}{(a^2+b^2)^{\frac{1}{2}}|\tilde{c}_{\alpha,\beta}|}[\tilde{X}_\delta, X]_{\alpha,\beta}$$

We can readily see that $I^2_{a,b} = -Id$ and $I_{a,b}JX = -I_{a,b}JX$ which follows from Lemma 3.14 and 3.16. Let $\mathcal{S}$ be a set of unordered pairs $\{\alpha, \beta\}$ such that $\alpha + \beta = \delta$. We can naturally extend $I_{a,b}$ to be linear operator on the subspace

$$S_0 = \bigoplus_{\{\alpha,\beta\}\in\mathcal{S}} V_\alpha \oplus V_\beta$$

Lemma 3.17. *The map $I_{a,b} : S_0 \to S_0$ defined above satisfies the following properties*

(a) $I_{a,b}^2 = -Id$

(b) $I_{a,b}J = -JI_{a,b}$

(c) For $X \in S_0$ *we have* $|I_{a,b}X| = |X|$

(d) If $X \in S_0$ *then* $\langle [I_{a,b}X, X], \widetilde{X}_\delta \rangle \leqslant -N_0(a^2 + b^2)^{\frac{1}{2}} |X|^2$ *where* N_0 *is a constant only dependant on the Lie algebra* $\mathfrak{g}$

Proof. For the sake of convenience we refer to $I_{a,b}$ as I in the proof. For (a) and (b) this follows from the preceding discussion.

(c)

$$\langle IX, IX \rangle = \sum_{\alpha+\beta=\delta} \frac{1}{(a^2+b^2)|\widetilde{c}_{\alpha,\beta}|^2} \langle [\widetilde{X}_\delta, X]_{\alpha,\beta}, [\widetilde{X}_\delta, X]_{\alpha,\beta} \rangle \tag{3.9}$$

$$= \sum_{\alpha+\beta=\delta} \frac{-1}{(a^2+b^2)|\widetilde{c}_{\alpha,\beta}|^2} \langle X, [\widetilde{X}_\delta, [\widetilde{X}_\delta, X]_{\alpha,\beta}]_{\alpha,\beta} \rangle \tag{3.10}$$

$$= \sum_{\alpha+\beta=\delta} \langle X_{\alpha,\beta}, X_{\alpha,\beta} \rangle \tag{3.11}$$

In the first line we use the fact that the root spaces V_α are orthogonal. In the second line we use associativity of bracket (3.4) and later we use Lemma 3.16.

(d) In the computation we use the associativity of the bracket and part (c) of this lemma.

$$
\begin{aligned}
\langle [IX, X], \widetilde{X}_\delta \rangle &= -\langle IX, [\widetilde{X}_\delta, X] \rangle \\
&= -\sum_{\{\alpha,\beta\}\in\mathcal{S}} \langle IX, [\widetilde{X}_\delta, X]_{\alpha,\beta} \rangle \\
&= -\sum_{\{\alpha,\beta\}\in\mathcal{S}} (a^2 + b^2)^{\frac{1}{2}} |\widetilde{c}_{\alpha,\beta}| |IX_{\alpha,\beta}|^2 \\
&= -\sum_{\{\alpha,\beta\}\in\mathcal{S}} (a^2 + b^2)^{\frac{1}{2}} |\widetilde{c}_{\alpha,\beta}| |X|^2 \\
&< -(a^2 + b^2)^{\frac{1}{2}} N_0 |X_{\alpha,\beta}|^2
\end{aligned}
$$

$\square$

Chapter 4

The complex hat connection

In this Chapter we assume that G/K is equipped with an invariant integrable complex structure J such that the normal metric is hermitian. Let M and N be two complex submanifolds of dimensions m, n respectively. Let $\gamma : [0,1] \to G/K$ be a critical point to the energy functional on the space of paths joining M and N and so it is a geodesic perpendicular to the both manifolds at the endpoints.

If $X(t)$ is any vector field along the geodesic γ such that $X(0)$ and $X(1)$ are in the tangent space of M and N respectively then $X(t)$ is called admissible. For an admissible vector field $X(t)$ we recall the second variation formula

$$E_{**}(X,X) = \langle \widetilde{\nabla}_X X \rangle |_0^1 - \int_0^1 \langle \widetilde{\nabla}_{\dot\gamma} X, \widetilde{\nabla}_{\dot\gamma} X \rangle - \langle R(\dot\gamma, X)X, \dot\gamma \rangle \rangle dt$$

Observe that if $X(t)$ is admissible then $JX(t)$ is admissible too, the following quadratic form can be defined.

Definition 5. *The following quantity is defined as the complex energy hessian.*

$$E_{**}^{\mathbb{C}}(X,X) = \frac{1}{2}E_{**}(X,X) + \frac{1}{2}E_{**}(JX,JX) \tag{4.1}$$

Definition 6. *Define the complex-hat connection* $\widehat{\nabla}$ *by*

$$\widehat{\nabla}_Y X = \nabla_Y X + \frac{1}{2} R_Y(X)$$

where $R_Y(X) = [Y, X]_{\mathfrak{m}} + J[JY, X]_{\mathfrak{m}}$.

4.1 The second variation formula

In this section we rewrite the second variation formula.

Theorem 4.1. *Suppose that* $X(t)$ *is admissible and parallel with respect to the complex hat connection. Then*

$$E^{\mathbb{C}}_{**}(X, X) = -\int_0^1 \frac{1}{2}(|R_{\dot\gamma}(X)|^2) + |[X, \dot\gamma]_{\boldsymbol{k}}|^2 + |[JX, \dot\gamma]_{\boldsymbol{k}}|^2 dt$$

Proof. Using Proposition 2.1 we have

$$E^{\mathbb{C}}_{**}(X, X) = \langle \widetilde{\nabla}_X X + \widetilde{\nabla}_{JX} JX, \dot\gamma\rangle|_0^1 - \int_0^1 \langle \widetilde{\nabla}_{\dot\gamma} X, \widetilde{\nabla}_{\dot\gamma} X\rangle + \langle \widetilde{\nabla}_{\dot\gamma} JX, \widetilde{\nabla}_{\dot\gamma} JX\rangle$$
$$- \langle R(\dot\gamma, X)X, \dot\gamma)\rangle - \langle R(\dot\gamma, JX)JX, \dot\gamma)\rangle dt \qquad (4.2)$$

We begin with the boundary term at $t = 0$.

$$
\begin{aligned}
\langle \widetilde{\nabla}_X X + \widetilde{\nabla}_{JX} JX, \dot{\gamma} \rangle_{t=0} &= \langle \nabla_X X + J \nabla_{JX} X, \dot{\gamma} \rangle_{t=0} \\
&= \langle \nabla_X X + J(\nabla_X JX + [JX, X] - [JX, X]_{\mathfrak{m}}), \dot{\gamma} \rangle_{t=0} \\
&= \langle -J[JX, X]_{\mathfrak{m}}), \dot{\gamma} \rangle_{t=0}
\end{aligned}
$$

In the first line we use the formula for the Levi-Civita connection along with fact that J commutes with ∇. In the next line we use the formula for the torsion of the canonical connection (Proposition 3.8). For the last line we observe that the variations for X, JX at $t = 0$ are tangent to M hence $J[X, JX] \in TM$ is tangent too. As γ is perpendicular to M, so $\langle J([JX, X]), \dot{\gamma}(0) \rangle = 0$.

We can make a similar conclusion for $t = 1$ and we have the following.

$$
\langle \widetilde{\nabla}_X X + \widetilde{\nabla}_{JX} JX, \dot{\gamma} \rangle |_0^1 = \langle -J[JX, X]_{\mathfrak{m}}, \dot{\gamma} \rangle |_0^1 \tag{4.3}
$$

The covariant derivative with respect to the canonical connection vanishes for all G invariant tensors (Theorem 3.5). Applying this to the tensors J, $[\cdot, \cdot]_{\mathfrak{m}}$ and the metric $\langle \cdot, \cdot \rangle$ we have.

$$
\begin{aligned}
\langle -J[JX, X]_{\mathfrak{m}}, \dot{\gamma} \rangle |_0^1 &= \int_0^1 \frac{d}{dt} \langle J[X, JX]_{\mathfrak{m}}, \dot{\gamma} \rangle dt \\
&= \int_0^1 \langle J[\nabla_{\dot{\gamma}} X, JX]_{\mathfrak{m}} + J[X, J\nabla_{\dot{\gamma}} X]_{\mathfrak{m}}, \dot{\gamma} \rangle + \langle J[X, JX]_{\mathfrak{m}}, \nabla_{\dot{\gamma}} \dot{\gamma} \rangle dt \\
&= \int_0^1 \langle \nabla_{\dot{\gamma}} X, [J\dot{\gamma}, JX]_{\mathfrak{m}} + J[J\dot{\gamma}, X]_{\mathfrak{m}} \rangle dt
\end{aligned} \tag{4.4}
$$

In the last line we use the associativity of the bracket (eq 3.4).

Now let us simplify the second term in equation 4.2 and write it in terms of the canonical connection and apply the formula for the curvature (Proposition 3.11)

$$\int_0^1 |\widetilde{\nabla}_{\dot{\gamma}}X|^2 + |\widetilde{\nabla}_{\dot{\gamma}}JX|^2 - \langle R(\dot{\gamma},X)X,\dot{\gamma}\rangle\rangle - \langle R(\dot{\gamma},JX)JX,\dot{\gamma}\rangle\rangle dt \tag{4.5}$$

$$= \int_0^1 |\nabla_{\dot{\gamma}}X + \frac{1}{2}[\dot{\gamma},X]_{\mathfrak{m}}|^2 + |J\nabla_{\dot{\gamma}}X + \frac{1}{2}[\dot{\gamma},JX]_{\mathfrak{m}}|^2 - \langle R(\dot{\gamma},X)X,\dot{\gamma}\rangle\rangle \tag{4.6}$$

$$- \langle R(\dot{\gamma},JX)JX,\dot{\gamma}\rangle\rangle dt \tag{4.7}$$

$$= \int_0^1 2|\nabla_{\dot{\gamma}}X|^2 + \langle\nabla_{\dot{\gamma}}X,[\dot{\gamma},X]_{\mathfrak{m}}\rangle - \langle\nabla_{\dot{\gamma}}X,J[\dot{\gamma},JX]_{\mathfrak{m}}\rangle - |[X,\dot{\gamma}]_{\boldsymbol{k}}|^2 - |[JX,\dot{\gamma}]_{\boldsymbol{k}}|^2 dt \tag{4.8}$$

Now combining eq 4.3 4.4 and 4.8 in equation 4.2 we have

$$\begin{aligned}
E^{\mathbb{C}}_{**}(X,X) &= \int_0^1 2|\nabla_{\dot{\gamma}}X|^2 + \langle\nabla_{\dot{\gamma}}X,[\dot{\gamma},X]_{\mathfrak{m}} - J[\dot{\gamma},JX]_{\mathfrak{m}} + [J\dot{\gamma},JX]_{\mathfrak{m}} + J[J\dot{\gamma},X]_{\mathfrak{m}}\rangle \\
&- \langle[X,\dot{\gamma}]_{\boldsymbol{k}}[X,\dot{\gamma}]_{\boldsymbol{k}}\rangle - \langle[JX,\dot{\gamma}]_{\boldsymbol{k}},[JX,\dot{\gamma}]_{\boldsymbol{k}}\rangle dt \\
&= \int_0^1 2(|\nabla_{\dot{\gamma}}X|^2 + \langle\nabla_{\dot{\gamma}}X,[\dot{\gamma},X]_{\mathfrak{m}} + J[J\dot{\gamma},X]_{\mathfrak{m}}\rangle) - \langle[X,\dot{\gamma}]_{\boldsymbol{k}},[X,\dot{\gamma}]_{\boldsymbol{k}}\rangle \\
&- \langle[JX,\dot{\gamma}]_{\boldsymbol{k}},[JX,\dot{\gamma}]_{\boldsymbol{k}}\rangle dt \\
&= \int_0^1 2(\langle\nabla_{\dot{\gamma}}X,\nabla_{\dot{\gamma}}X + R_{\dot{\gamma}}(X)\rangle - |[X,\dot{\gamma}]_{\boldsymbol{k}}|^2 - |[JX,\dot{\gamma}]_{\boldsymbol{k}}|^2 dt \qquad (4.9)
\end{aligned}$$

Note that we use the integrability condition (Proposition 3.9) in the eq 4.9. As $\nabla_{\dot{\gamma}}X + \frac{1}{2}R_{\dot{\gamma}}(X) = 0$, the result follows. □

4.2 Properties of the complex-hat connection

Proposition 4.2. *(a) The complex-hat connection commutes with the complex structure J*

(b) The parallel transport along a geodesic γ with respect to the complex-hat connection preserves orthogonality between the geodesic and the transported vector.

(c) If $\widehat{\nabla}_{\dot\gamma}X = 0$ along a quadratic γ and $R_{\dot\gamma}(X)$ vanishes at $t = 0$ then $\nabla_{\dot\gamma}X = 0$ along γ.

Proof. (a) Using part (b) of Proposition 3.9 it is easy to show that $R_Y(JX) = JR_Y(JX)$. The result now follows from this observation and part (a) of Proposition 3.9.

(b) Let $X(t)$ be a vector field along a geodesic curve $\dot\gamma$ which is parallel with respect to the complex-hat connection i.e $\nabla_{\dot\gamma}X = -\frac{1}{2}R_{\dot\gamma}(X)$ then

$$\begin{aligned}
\frac{d}{dt}\langle X(t), \dot\gamma(t)\rangle &= \langle \nabla_{\dot\gamma}X, \dot\gamma\rangle + \langle X, \nabla_{\dot\gamma}\dot\gamma\rangle \\
&= \langle -\tfrac{1}{2}([\dot\gamma, X]_{\mathfrak{m}}, \dot\gamma\rangle + \langle -J[J\dot\gamma, X]_{\mathfrak{m}}), \dot\gamma\rangle + 0 \\
&= \tfrac{1}{2}(\langle X, [\dot\gamma, \dot\gamma]_{\mathfrak{m}}\rangle - \langle X, [J\dot\gamma, J\dot\gamma]_{\mathfrak{m}}\rangle) \\
&= 0
\end{aligned}$$

In this computation we have used the associativity of the bracket (3.4) and t

(c) We can use a canonical connection parallel frame. All invariant tensors are parallel with respect to the canonical connection (Theorem 3.5) and so is $\dot\gamma(t)$ (Proposition b). Hence with respect to this frame $R_{\dot\gamma}(X)$ is a linear transformation with respect to X with constant co-efficients, so parallel transport with respect to the complex-hat connection can be thought of as the solution to the linear ODE $\dot X(t) = -\frac{1}{2}R(X)$. From linear ODE Theory it is clear that $\dot X(t) = 0$ iff $R(X(0)) = 0$.

□

Theorem 4.3. *The complex energy hessian of a vector field $X(t)$ which is admissible and parallel with respect to the complex-hat connection is negative iff* $[X^{1,0}(0), \dot{\gamma}^{0,1}(0)] \notin \mathfrak{m}^{0,1}$.

Proof. The sufficient condition follows from Theorem 4.1 and Lemma 3.13. For the necessary condition, now suppose that $[X^{1,0}(0), \dot{\gamma}^{0,1}(0)] \in \mathfrak{m}^{0,1}$. That implies that $R_{\dot{\gamma}}(X(0)) = 0$ and $|[X(0), \dot{\gamma}(0)]_{\boldsymbol{k}}|^2 + |[JX(0), \dot{\gamma}(0)]_{\boldsymbol{k}}|^2 = 0$ by Lemma 3.13. $R_{\dot{\gamma}}(X(0)) = 0$ implies that $X(t)$ is parallel with respect to the canonical connection by part *3*) of Proposition 4.2. But that implies that the value $|[X(t), \dot{\gamma}]_{\boldsymbol{k}}(t)|^2 + |[JX(t), \dot{\gamma}]_{\boldsymbol{k}}(t)|^2$ remains zero along the geodesic and that implies that the complex energy hessian is zero. □

Chapter 5

Index calculations

In this Chapter we work with the complex homogeneous space G^c/P where G^c is a complex simple Lie group and P a parabolic subgroup. This manifold can also be written as G/K where G is a compact Lie group and K a closed subgroup as mentioned in Chapter section 3.4. Equip G/K with the normal metric. Let M and N be two complex submanifolds of dimensions m, n respectively, with the dimension of G/K equal to v. Let $\gamma : [0,1] \to G/K$ be a critical point to the energy functional on the space of paths joining M and N. Thus γ is a geodesic perpendicular to both manifolds at the endpoints. In this Chapter we will focus on giving a lower bound on the index of each geodesic in terms of m, n, v and an invariant ℓ of the Lie algebra $\mathfrak{g}$.

After applying an isometry we can assume that $\gamma(0) = \bar{e}$. So we can identify $T_{\gamma(0)}G/K$ with $\mathfrak{m}$. Recall that Δ and $\Delta_{\boldsymbol{k}}$ denote the roots associated to the root system of $\mathfrak{g}$ and $\boldsymbol{k}$ respectively. $\Delta_{\mathfrak{m}}$ denote the roots complementary to $\Delta_{\boldsymbol{k}}$. $V_\alpha \subset \mathfrak{g}$ is the real root space associated to $\alpha \in \Delta^+$. We define an ordering on Δ^+ by $\alpha < \delta$ if and only if $\delta - \alpha$ is a positive root. This ordering is not partial.

Define $\Gamma = \{\alpha \in \Delta^+ | \dot{\gamma}(0)$ has a non-trivial component in $V_\alpha\}$. Let $\delta \in \Gamma$ be a minimal element such that it also satisfies the following, $\alpha < \beta$ and $\beta < \delta$ implies that $\alpha \notin \Gamma$, we will refer to such δ as superminimal.

$$\mathcal{S}_\delta = \{\alpha \in \Delta^+_{\mathfrak{m}} | \alpha < \delta \text{ and } \delta - \alpha \in \Delta^+_{\mathfrak{m}}\} \tag{5.1}$$

$$\mathcal{T}_\delta = \{\beta \in \Delta^+_{\mathfrak{m}} | \beta \geq \delta\} \cup \{\alpha \in \Delta^+_{\mathfrak{m}} | \delta - \alpha \in \Delta_{\boldsymbol{k}}\} \tag{5.2}$$

To derive an optimal lower bound on the index we impose the following conditions on the sets $\mathcal{S}_\delta$ and $\mathcal{T}_\delta$. These conditions will be shown to be true if $\mathfrak{g}$ is a simply laced Lie algebra i.e Lie algebras of the type A, D and E in Chapter . In the non-simply laced case the conditions are satisfied for most δ. For the remaining choices of δ the arguments given below can be modified.

Condition 1. *If $\beta_0, \beta_1 \in \mathcal{T}_\delta \backslash \{\delta\}$ with $\beta_0 \neq \beta_1$ then $\beta_0 - \delta \neq \beta_1 - \lambda$ for $\lambda \in \Gamma$*

Condition 2. *If $\alpha, \beta \in \mathcal{S}_\delta$ then $\alpha + \beta \in \Gamma$ iff it is equal to δ.*

Observe that if $\alpha < \delta$ then $\alpha, \delta - \alpha$ cannot both lie in $\Delta^+_{\boldsymbol{k}}$ from the construction of $\boldsymbol{k}$, it follows that $|\mathcal{S}_\delta|$ is even. Let $\ell = \frac{1}{2}|\mathcal{S}_\delta| + |\mathcal{T}_\delta|$ and let $h = \frac{1}{2}|\mathcal{S}_\delta|$.

The main goal of this Chapter is to prove the following theorem.

Theorem 5.1. *If Conditions 1 and 2 are satisfied then the index of the geodesic γ is at least $\mathcal{I} = m + n - (v - \ell) - v + 1$*

To prove this theorem we will construct a vector space of dimension 4I. We will use a quaternionic structure on this space and show that there exists a subspace of dimension I such that the hessian E_{**} is negative definite.

Let

$$S_0 = \bigoplus_{\alpha \in \mathcal{S}_\delta} V_\alpha, \quad T_0 = \bigoplus_{\beta \in \mathcal{T}_\delta} V_\beta$$

The spaces S_0, T_0 are invariant under J and are of complex dimension $2h$ and $\ell - h$ respectively. Let $U_0 = S_0 \oplus T_0$ and $\widehat{\tau} : T_{\gamma(0)}(G/K) \to T_{\gamma(1)}(G/K)$ denote the parallel translation with respect to the complex-hat connection. Let

$$U = \{X \in U_0 \cap T_{\gamma(0)}M | \widehat{\tau}(X) \in T_{\gamma(1)}(N)\}$$

.

Proposition 5.2. *The minimum complex dimension of U is $\mathcal{I} + h$.*

Proof. Since the dimension of U_0 is $\ell + h$ the minimum dimension of $U_0 \cap T_{\gamma(0)}M$ is $\ell + h + m - v$. It is clear that the

$$dim_{\mathbb{C}} U = dim_{\mathbb{C}}(\widehat{\tau}(U_0 \cap T_{\gamma(0)}M)) \cap T_{\gamma(0)}M$$

Since both $T_{\gamma(0)}M$, $T_{\gamma(1)}N$ are perpendicular to γ and the complex-hat parallel transport preserves orthogonality with γ (part (b) of Proposition 4.2) we can get an extra dimension in the count. So the dimension of U is $m + \ell + h - v + n - v + 1$.

□

Let $\dot{\gamma}_\delta(0) = aX_\delta + bJX_\delta$ denote the the V_δ component of $\dot{\gamma}(0)$. Lemma 3.17 gives us a linear operator $I_{a,b} : S_0 \to S_0$ satisfying the following properties, $I^2 = -Id$ and $IJ = -JI$ (from now on we omit the subscript).

Let $S_1 = S_0 \cap U$ and let $dim_{\mathbb{C}} S_1 = s_1$. Using the properties of I we get that the subspace $S = S_1 \cap IS_1$ is a closed under I and J. We see that $dim_{\mathbb{C}} S \geq 2(s_1 - h)$ whenever $s_1 > h$ else S is the trivial space. Now let T be the orthogonal complement of S_1 in U and $t = dim_{\mathbb{C}} T$. We observe that as $s_1 + t \geq \mathcal{I} + h$, $t + \frac{1}{2}s \geq \mathcal{I}$.

Let $\widehat{S}$, $\widehat{T}$ and $\widehat{U}$ be the space of vector fields parallel with respect to the complex-hat connection starting from S, T and U respectively. We will use Theorem 4.3 to show that the index of $E_{**|\widehat{T}}$ is at least t. For $\widehat{S}$ Theorem 4.3 does not yield anything, instead one must take suitable linear combinations of vector fields in $\widehat{S}$. The index of E_{**} restricted to these linear combinations is $\frac{s}{2}$.

5.1 The variations $\widehat{T}$

Proposition 5.3. *If $Z(t) \in \widehat{T}$ then $E^{\mathbb{C}}_{**}(Z, Z) < 0$*

Proof. By Theorem 4.3 it suffices to show that $[Z(0)^{1,0}, \dot{\gamma}^{0,1}(0)] \notin \mathfrak{m}^{0,1}$. Since $Z(0) \in T$ we can write $Z(0) = X + Y$ where $X \in S$ and $Y \in T$. As $[X^{1,0}, \dot{\gamma}^{0,1}(0)] \in \mathfrak{m}^{0,1}$ it suffices to prove that $[Y^{1,0}, \dot{\gamma}^{0,1}(0)] \notin \mathfrak{m}^{0,1}$.

Suppose $Y^{1,0} = \Sigma_{\beta \in \mathcal{T}_\delta} c_\beta E_\beta$ with $c_{\beta_0} \neq 0$, where $\beta_0 \in \mathcal{T}_\delta \backslash \{\delta\}$ then condition 1 implies that the co-efficient of $E_{\beta_0 - \delta}$ is non-zero in $[Y^{1,0}(0), \dot{\gamma}^{0,1}(0)]$. As either $\beta_0 - \delta > 0$ or $|\beta_0 - \delta| \in \Delta^+_{\boldsymbol{k}}$, implies that $E_{\beta_0 - \delta} \notin \mathfrak{m}^{0,1}$.

If c_δ is non-zero and $c_{\beta_0} = 0$, for all $\beta_0 \in \mathcal{T}_\delta \backslash \{\delta\}$ that implies that $[Y^{1,0}, \dot{\gamma}^{0,1}(0)]_{\mathfrak{m}} \notin \mathfrak{m}^{0,1}$ has a non-trivial component in $\mathfrak{h}$. ◻

Theorem 5.4. *The index of γ when restricted to $\widehat{T}$ is t.*

Proof. $\widehat{T}$ has a natural complex structure J on it, since $\widehat{\nabla}$ commutes with J. To show that the index is t, it suffices to show that every $t+1$ dimensional subspace of the $2t$ dimensional space $\widehat{T}$ has an element X such that $E_{**}(X, X) < 0$. Let W be such a subspace, then $W \cap JW$ is non-empty. Let $X \in W \cap JW$, then from Proposition 5.3 we know that the complex energy hessian $E_{**}(X, X) + E_{**}(JX, JX) < 0$. Since both $X, JX \in W \cap JW$ the result follows. ◻

5.2 The variations $\widehat{S}$

If $\alpha \in \mathcal{S}_\delta$ then $\alpha < \delta$ and by the choice of δ that implies that either $\alpha < \lambda$ or α is not comparable with λ for $\lambda \in \Gamma$. If $X(t) \in \widehat{S}$ then $X(0) \in S$ which implies $[X^{1,0}, \dot{\gamma}^{0,1}(0)]_{\mathfrak{m}} \in \mathfrak{m}^{0,1}$. Hence $R_{\dot{\gamma}}(X(0)) = 0$ so Theorem 4.2 gives us that X is parallel to the canonical connection.

Let $\tau_0^t : T_{\gamma(0)}G/K \to T_{\gamma(t)}G/K$ denote parallel translation with respect to the canonical connection. We have $I : S \to S$ where $S \subset T_{\gamma(0)}M$, using parallel translation we can also define $I_t : \tau_0^t(S) \to \tau_0^t(S)$. Since the invariant tensors J and $[\cdot,\cdot]_{\mathfrak{m}}$, the vector fields $\dot{\gamma}(t)$ and $X(t)$ are all parallel the properties of Lemma 3.17 carry over to I_t. Using this, we can now define a natural operation I on the vector fields $\widehat{S}$.

We define

$$\widehat{S}_k = \{Z | \nabla_{\dot{\gamma}} Z = -kIZ \text{ and } Z(0) \in S\}$$

On $\widehat{S}_k$ we can define operations $\bar{I}$ and $\bar{J}$. For $Z \in \widehat{S}_k$ we have $(\bar{I}Z)(0) = I(Z(0)), (\bar{J}Z)(0) = J(Z(0))$. We observe that $\bar{I}^2 = -Id, \bar{J}^2 = -Id$ and $\bar{I}\bar{J} = -\bar{J}\bar{I}$.

Proposition 5.5.

$$E_{**}(Z,Z) + E_{**}(\bar{I}Z,\bar{I}Z) + E_{**}(\bar{J}Z,\bar{J}Z) + E_{**}(\bar{I}\bar{J}Z,\bar{I}\bar{J}Z) < 0 \textit{ for sufficiently small } k \tag{5.3}$$

Proof. Let $X \in \widehat{S}$ such that $X(0) = Z(0)$. It is easy to verify that

$$Z = \cos(kt)X - \sin(kt)IX \tag{5.4}$$

$$\bar{I}Z = \sin(kt)X + \cos(kt)IX \tag{5.5}$$

$$\bar{J}Z = \cos(kt)JX + \sin(kt)JX \tag{5.6}$$

$$\bar{I}\bar{J}Z = -\cos(kt)JIX + \sin(kt)JX \tag{5.7}$$

We see that all these vector fields are admissible since X and IX are. We begin by analysing $E_{**}(Z, Z)$

$$\begin{aligned} E_{**}(Z, Z) &= \langle \nabla_Z Z, \dot{\gamma}\rangle|_0^1 + \int_0^1 |\widetilde{\nabla}_{\dot{\gamma}} Z|^2 - \langle R(\dot{\gamma}, Z)Z, \dot{\gamma}\rangle dt \\ &= \langle \nabla_Z Z, \dot{\gamma}\rangle|_0^1 + \int_0^1 |\nabla_{\dot{\gamma}} Z + \frac{1}{2}[\dot{\gamma}, Z]_{\mathfrak{m}}|^2 - \frac{1}{4}|[\dot{\gamma}, Z]_{\mathfrak{m}}|^2 - |[\dot{\gamma}, Z]_{\boldsymbol{k}}|^2 dt \\ &= \langle \nabla_Z Z, \dot{\gamma}\rangle|_0^1 + \int_0^1 k^2|IZ|^2 - k\langle IZ, [\dot{\gamma}, Z]_{\mathfrak{m}}\rangle - |[\dot{\gamma}, Z]_{\boldsymbol{k}}|^2 dt \end{aligned}$$

If $\alpha, \beta \in \mathcal{S}_\delta$ then from condition 2 we have that $\alpha+\beta \notin \Gamma\backslash\{\delta\}$ and since δ is superminimal, we have that $\eta < \alpha$ implies $\eta \notin \Gamma$. So we have $\alpha - \beta \notin \Gamma$. We can now conclude that $\langle [Z, IZ]_{\mathfrak{m}}, \dot{\gamma}(0) - \dot{\gamma}_\delta(0)\rangle = 0$. So for sufficiently small k we have

$$
\begin{aligned}
E_{**}(Z,Z) &= \langle \nabla_Z Z, \dot\gamma\rangle|_0^1 + \int_0^1 k^2|IZ|^2 - k\langle IZ, [\dot\gamma_\delta, Z]_{\mathfrak{m}}\rangle - |[\dot\gamma, Z]_{\boldsymbol{k}}|^2 \\
&< \langle \nabla_Z Z, \dot\gamma\rangle|_0^1 + \int_0^1 k^2|IZ|^2 - kN_0(a^2+b^2)^{\frac{1}{2}} \sum_{\alpha+\beta=\delta} |X_{\alpha,\beta}|^2 - |[\dot\gamma, Z]_{\boldsymbol{k}}|^2 \\
&< \langle \nabla_Z Z, \dot\gamma\rangle|_0^1 \qquad (5.8)
\end{aligned}
$$

To get the inequality we use part 3 of Lemma 3.17. We can get a similar calculation for $E_{**}(\bar I Z, \bar I Z), E_{**}(\bar J Z, \bar J Z)$ and $E_{**}(\bar I \bar J Z, \bar I \bar J Z)$. We now calculate the corresponding boundary terms.

$$
\begin{aligned}
&\langle \nabla_Z Z, \dot\gamma\rangle + \langle \nabla_{\bar J Z} \bar J Z, \dot\gamma\rangle + \langle \nabla_{\bar I Z} \bar I Z, \dot\gamma\rangle + \langle \nabla_{\bar I \bar J Z} \bar I \bar J Z, \dot\gamma\rangle|_0^1 \\
&\qquad = \langle \nabla_X X, \dot\gamma\rangle + \langle \nabla_{JX} JX, \dot\gamma\rangle + \langle \nabla_{IX} IX, \dot\gamma\rangle + \langle \nabla_{IJX} IJX, \dot\gamma\rangle|_0^1 \\
&\qquad = \langle [-JX, X]_{\mathfrak{m}}, \dot\gamma\rangle|_0^1 + \langle [-JIX, IX]_{\mathfrak{m}}, \dot\gamma\rangle|_0^1 \\
&\qquad = 0 \qquad (5.9)
\end{aligned}
$$

For the last line we observe that X, $[\cdot,\cdot]_{\mathfrak{m}}$ and J are parallel to the canonical connection, which implies that $\langle [-JX, X]_{\mathfrak{m}}, \dot\gamma\rangle$ and $\langle [-JIX, IX]_{\mathfrak{m}}, \dot\gamma\rangle$ are independent of t.

Using the inequality 5.8 and eqn 5.9 the proposition is proved.

□

Theorem 5.6. *For sufficiently small k the index of γ when restricted to $\widehat{S}_k$ is $\frac{s}{2}$.*

Proof. The real dimension of $\widehat{S}_k$ is $2s$, so its suffices to show that for any $3\frac{s}{2}+1$ real dimensional subspace W, there exists $Z \in \widehat{S}_k$ such that $E_{**}(Z,Z) < 0$. By the properties of $\bar I$ and $\bar J$ we know that there exist $Z \in W$ such that Z, $\bar I Z$, $\bar J Z$, $\bar I \bar J Z$ belong to W. From

Proposition 5.5 the Theorem follows . $\square$

5.3 Reconciling $\widehat{S}_k$ and $\widehat{T}$

In the Theorems 5.4 and 5.6 we have shown the existence of two subspaces of dimension t and $\frac{s}{2}$ belonging to $\widehat{T}$ and $\widehat{S}_k$ such that E_{**} is negative definite. But that is not sufficient to show that the index is $\mathcal{I} = t + \frac{s}{2}$. We will tackle this problem by twisting the space $\widehat{T}$.

Let $\widehat{T}_k = \{Z|Z = \cos(kt)X - \sin(kt)Y \text{ with } X, Y \in \widehat{T}\}$. From equations 5.4 to 5.7 we observe that $\widehat{S}_k = \{Z|Z = \cos(kt)W - \sin(kt)IW \text{ with } W \in \widehat{S}\}$. We now define $\widehat{U}_k = \widehat{S}_k \bigoplus \widehat{T}_k$. So if $Z \in \widehat{U}_k$ then $Z = cos(kt)(X) - sin(kt)(Y)$ for some $X, Y \in \widehat{U}$.

We can define two operations on $\widehat{U}_k$, $\bar{I}Z = \sin(kt)(X + W) + \cos(kt)(Y + IW)$ and $\bar{J}Z = \cos(kt)J(X + W) + \sin(kt)J(Y + IW)$. Observe that $\widehat{U}_k$ is closed under the two operations $\bar{I}$ and $\bar{J}$. It can be easily verified that $\bar{I}^2 = -Id$ and $\bar{J}^2 = -Id$ and that $\bar{I}\bar{J} = -\bar{J}\bar{I}$. From eqns 5.4 to 5.7, we see that the definitions of $\bar{I}$ and $\bar{J}$ coincide.

Recall that the complex energy hessian $E^{\mathbb{C}}_{**}(X, X) = E_{**}(X, X) + E_{**}(JX, JX)$. For $Z \in \widehat{U}_k$ we define $Q(Z) = E_{**}(Z, Z) + E_{**}(\bar{I}Z, \bar{I}Z) + E_{**}(\bar{J}Z, \bar{J}Z) + E_{**}(\bar{J}\bar{I}Z, \bar{J}\bar{I}Z)$

Now we are ready to prove an important proposition.

Proposition 5.7. *There exists a $k > 0$ such that for any $Z \in \widehat{U}_k$, $Q(Z) < 0$*

We first state and prove a few lemmas

Lemma 5.8. *If $X \in T$ then there exists $M > 0$ such that $\frac{1}{2}||R_{\dot{\gamma}}(X(t)|^2 + |[X(t), \dot{\gamma}]_{\boldsymbol{k}}|^2 + |[JX(t), \dot{\gamma}]_{\boldsymbol{k}}|^2 > M|X(t)|^2$ for any $t \in [0, 1]$ where $X(t)$ is a vector field which is parallel with respect to the complex-hat connection with initial value X. As a consequence $E_{**}(X, X) + E_{**}(JX, JX) > M\int_0^1 |X(t)|^2 dt$*

Proof. For $X \in T$ we know from the proof of the above Propsition 5.4 that either $\frac{1}{2}|R_{\dot{\gamma}}(X(0)| \neq 0$ or $|[X(0), \dot{\gamma}]_{\boldsymbol{k}}|^2 + |[JX(0), \dot{\gamma}]_{\boldsymbol{k}}|^2) \neq 0$. If the first case is true then we know from Proposition 4.2 part (c) that $R_{\dot{\gamma}}(X(t)) \neq 0$ for all t. If the first case were not true then the vector field X is parallel with respect to the canonical connection and so $|[X(t), \dot{\gamma}]_{\boldsymbol{k}}|^2 + |[JX(t), \dot{\gamma}]_{\boldsymbol{k}}|^2$ is the same for all t because $|[\cdot, \cdot]_{\boldsymbol{k}}|$ is an invariant tensor. But at $t = 0$ we know that this quantity is non-zero and hence it is non-zero for all t. As the interval $[0, 1]$ is compact the lemma follows. □

Lemma 5.9. *If* $Z = \cos(kt)X - \sin(kt)Y$ *with* $X, Y \in \widehat{U}$ *then*

$$Q(Z) = E^{\mathbb{C}}_{**}(X, X) + E^{\mathbb{C}}_{**}(Y, Y) + \int_0^1 2k^2|X|^2 + 2k^2|Y|^2 + 2k\langle [Y, X]_{\mathfrak{m}} - [JY, JX]_{\mathfrak{m}}, \dot{\gamma}\rangle dt$$

Proof. Let us begin by rewriting the second variational form as the sum of two quadratic forms for ease of computation. Now

$$\begin{aligned} E_{**}(X, X) &= \langle \widetilde{\nabla}_X X, \dot{\gamma}\rangle|_0^1 + \int_0^1 \langle \widetilde{\nabla}_{\dot{\gamma}} X, \widetilde{\nabla}_{\dot{\gamma}} X\rangle - \langle R(\dot{\gamma}, X)X, \dot{\gamma})\rangle dt \\ &= \langle \widetilde{\nabla}_X X, \dot{\gamma}\rangle|_0^1 + \int_0^1 |\nabla_{\dot{\gamma}} X + \frac{1}{2}[\dot{\gamma}, X]_{\mathfrak{m}}|^2 - \frac{1}{4}|[X, \dot{\gamma}]_{\mathfrak{m}}|^2 - |[X, \dot{\gamma}]_{\boldsymbol{k}}|^2 dt \\ &= \langle \widetilde{\nabla}_X X, \dot{\gamma}\rangle|_0^1 + \int_0^1 |\nabla_{\dot{\gamma}} X|^2 + \langle \nabla_{\dot{\gamma}} X, [\dot{\gamma}, X]_{\mathfrak{m}}\rangle - |[X, \dot{\gamma}]_{\boldsymbol{k}}|^2 dt \\ &= H(X, X) + G(X, X) \end{aligned}$$

Where $H(X, X) = \langle \widetilde{\nabla}_X X, \dot{\gamma}\rangle|_0^1 - \int_0^1 [X, \dot{\gamma}]_{\boldsymbol{k}}|^2$ &
$G(X, X) = \int_0^1 |\nabla_{\dot{\gamma}} X|^2 + \langle \nabla_{\dot{\gamma}} X, [\dot{\gamma}, X]_{\mathfrak{m}}\rangle dt$ We observe that $H(\cdot, \cdot)$ is linear with respect to

differentiable functions and a simple calculation gives us

$$H(Z,Z) + H(\bar{I}Z,\bar{I}Z) = H(X,X) + H(Y,Y) \tag{5.10}$$

So we focus on $G(Z,Z) + G(\bar{I}Z,\bar{I}Z)$. We start by simplifying

$$\begin{aligned}|\nabla_{\dot\gamma}Z|^2 + |\nabla_{\dot\gamma}\bar{I}Z|^2 &= |\nabla_{\dot\gamma}X|^2 + |\nabla_{\dot\gamma}Y|^2 + k^2|X|^2 + k^2|Y|^2 - 2k\langle\cos(kt)\nabla_{\dot\gamma}X \\ &- \sin(kt)\nabla_{\dot\gamma}Y, \bar{I}Z\rangle + 2k\langle\sin(kt)\nabla_{\dot\gamma}X + \cos(kt)\nabla_{\dot\gamma}Y, Z\rangle \\ &= |\nabla_{\dot\gamma}X|^2 + |\nabla_{\dot\gamma}Y|^2 + k^2|X|^2 + k^2|Y|^2 - 2k\langle\cos^2(kt)\nabla_{\dot\gamma}X, Y\rangle \\ &- 2k\langle\sin^2(kt)\nabla_{\dot\gamma}X, Y\rangle + 2k\langle\cos^2(kt)\nabla_{\dot\gamma}Y, X\rangle + 2k\langle\sin^2(kt)\nabla_{\dot\gamma}Y, X\rangle \\ &= |\nabla_{\dot\gamma}X|^2 + |\nabla_{\dot\gamma}Y|^2 + k^2|X|^2 + k^2|Y|^2 - 2k\langle\nabla_{\dot\gamma}X, Y\rangle + 2k\langle\nabla_{\dot\gamma}Y, X\rangle\end{aligned} \tag{5.11}$$

and similar computation leads us to

$$\langle\nabla_{\dot\gamma}Z, [\dot\gamma, Z]_{\mathfrak{m}}\rangle + \langle\nabla_{\dot\gamma}\bar{I}Z, [\dot\gamma, \bar{I}Z]_{\mathfrak{m}\rangle} = \langle\nabla_{\dot\gamma}X, [\dot\gamma, X]_{\mathfrak{m}}\rangle + \langle\nabla_{\dot\gamma}Y, [\dot\gamma, Y]_{\mathfrak{m}}\rangle + 2k\langle[Y,X]_{\mathfrak{m}}, \dot\gamma\rangle \tag{5.12}$$

From equation 5.11 and 5.12 we arrive at

$$\begin{aligned}G(Z,Z) + G(\bar{I}Z,\bar{I}Z) &= k\int_0^1 k|X|^2 + k|Y|^2 - 2\langle\nabla_{\dot\gamma}X, Y\rangle + 2\langle\nabla_{\dot\gamma}Y, X\rangle + 2\langle[Y,X]_{\mathfrak{m}}, \dot\gamma\rangle dt \\ &+ G(X,X) + G(Y,Y)\end{aligned} \tag{5.13}$$

Equations 5.10 and 5.13 give us

$$E_{**}(Z,Z) + E_{**}(\bar{I}Z, \bar{I}Z)$$
$$= E_{**}(X, X) + E_{**}(Y, Y) + \int_0^1 k^2|X|^2 + k^2|Y|^2 - 2k\langle \nabla_{\dot\gamma} X, Y\rangle + 2k\langle \nabla_{\dot\gamma} Y, X\rangle$$
$$+ 2k\langle [Y, X]_{\mathfrak{m}}, \dot\gamma\rangle \tag{5.14}$$

Keeping in mind that $\bar{J}Z = \cos(kt)JX + \sin(kt)JY$ and $\bar{J}\bar{I}Z = \cos(kt)JY - \sin(kt)JX$ we can use equation 5.14 to obtain

$$E_{**}(\bar{J}Z, \bar{J}Z) + E_{**}(\bar{I}\bar{J}Z, \bar{I}\bar{J}Z)$$
$$= E_{**}(JX, JX) + E_{**}(JY, JY) + \int_0^1 k^2|JX|^2 + k^2|JY|^2 - 2k\langle \nabla_{\dot\gamma} JX, J(-Y)\rangle$$
$$+ 2k\langle \nabla_{\dot\gamma} J(-Y), JX\rangle + 2k\langle [J(-Y), JX]_{\mathfrak{m}}, \dot\gamma\rangle dt$$
$$= E_{**}(JX, JX) + E_{**}(JY, JY) + \int_0^1 k^2|JX|^2 + k^2|JY|^2 + 2k\langle \nabla_{\dot\gamma} X, Y\rangle$$
$$- 2k\langle \nabla_{\dot\gamma} Y, X\rangle - 2k\langle [JY, JX]_{\mathfrak{m}}, \dot\gamma\rangle dt$$

The result follows from adding the formulas for $E_{**}(Z, Z)+E_{**}(\bar{I}Z, \bar{I}Z)$ and $E_{**}(\bar{J}Z, \bar{J}Z)+$ $E_{**}(\bar{J}\bar{I}Z, \bar{J}\bar{I}Z)$ □

Lemma 5.10. *Let* $P(X(t), Y(t)) = \langle [Y(t), X(t)]_{\mathfrak{m}} - [JY(t), JX(t)]_{\mathfrak{m}}, \dot\gamma\rangle$

(a) There exists a N *such that* $P(X(t), Y(t)) < N|X(t)||Y(t)|$ *for* $X, Y \in \widehat{U}_k$ $\forall\, t \in [0, 1]$

(b) $P(W(t), IW(t)) < -2kN_0(a^2 + b^2)^{\frac{1}{2}})|W(t)|^2$

Proof. (a) Clear.

(b) Since all the objects we deal with are parallel with the canonical connection, it suffices to prove it in the case $t = 0$. It is easy to see that $[W(0), IW(0)]_{\mathfrak{m}} - [JW(0), JIW(0)]_{\mathfrak{m}} = 4Re[W^{1,0}, IW^{1,0}]_{\mathfrak{m}}$. Now $IW, W \in S_0$ implies that $[W^{1,0}, IW^{1,0}]_{\mathfrak{m}} \in V_\delta$ via condition 2. As a consequence $\langle [W(0), IW(0)]_{\mathfrak{m}} - [JW(0), JIW(0)]_{\mathfrak{m}}, \dot{\gamma}(0) - \dot{\gamma}_\delta(0)\rangle$. The result now follows form (c) of Lemma 3.17.

□

Proof. of Proposition 5.7 If $Z \in \widehat{U}_k$ there exists $X, Y \in \widehat{T}$ and $W \in \widehat{S}$ so that $Z = cos(kt)(X+W) - sin(kt)(Y+IW)$. Note that $R_{\dot{\gamma}}W(0) = R_{\dot{\gamma}}(IW(0)) = 0$, so from eqn 4.9 of Theorem 4.1 we get that $E^{\mathbb{C}}_{**}(X, X) = E^{\mathbb{C}}_{**}(X+W)$ and $E^{\mathbb{C}}_{**}(Y, Y) = E^{\mathbb{C}}_{**}(Y+IW, Y+IW))$. So now we can rewrite Lemma 5.9.

$$Q(Z) = E^{\mathbb{C}}_{**}(X, X) + E^{\mathbb{C}}_{**}(Y, Y) + \int_0^1 2k^2(|X + W|^2 + |Y + IW|^2) + 2k(P(X, Y) + P(X, IW) + P(W, Y) + P(W, IW))dt$$

Using part (a) and part (b) of Lemma 5.10 we have

$$Q(Z) < E^{\mathbb{C}}_{**}(X, X) + E^{\mathbb{C}}_{**}(Y, Y) + \int_0^1 4k^2(|X|^2 + |Y|^2) + 2kN(|X||Y| + |X||W| + |W||Y|) + (8k^2 - 4k(a^2 + b^2)^{\frac{1}{2}})|W|^2 dt$$

We choose $\epsilon^2 N = 2(a^2+b^2)^{\frac{1}{2}}$ and apply the AM-GM inequality.

$$\begin{aligned} Q(Z,Z) &< E^{\mathbb{C}}_{**}(X,X) + E^{\mathbb{C}}_{**}(Y,Y) + \int_0^1 (4k^2+kN)(|X|^2+|Y|^2) + kN(\frac{|X|^2}{\epsilon^2} + \epsilon^2|W|^2 \\ &+ \frac{|Y|^2}{\epsilon^2} + \epsilon^2|W|^2) + (8k^2 - 4k(a^2+b^2)^{\frac{1}{2}})|W|^2 dt \\ &< \int_0^1 (4k^2 + kN + \frac{kN}{\epsilon^2})(|X|^2+|Y|^2) + k(8k - 2(a^2+b^2)^{\frac{1}{2}})|W|^2 dt \\ &+ E^{\mathbb{C}}_{**}(X,X) + E^{\mathbb{C}}_{**}(Y,Y) \end{aligned}$$

Finally applying Lemma 5.8 we conclude that.

$$Q(Z) = \int_0^1 (4k^2 + 2kN + \frac{kN}{\epsilon^2} - M)(|X|^2+|Y|^2) + k(8k - 2(a^2+b^2)^{\frac{1}{2}})|W|^2 dt$$

It is clear that k and can be chosen small enough so that $Q(Z)$ is negative for all $\bar{Z} \in \widehat{U}_k$. $\square$

Proof. of Theorem 5.1 To prove this we demonstrate the existence of an $\mathcal{I}$ dimensional subspace of the variation vector fields $\widehat{U}_k$ such that E_{**} is negative definite. We recall that the real dimension of $\widehat{U}_k$ is $4\mathcal{I}$. To show that the index is $\mathcal{I}$ all we have to do is show that for every $3\mathcal{I}+1$ real subspace W there exists a vector field of negative index. As $\bar{J}^2 = -Id$ that implies that in every $3\mathcal{I}+1$ dimensional real space we can find a $2\mathcal{I}+2$ dimensional space which is closed under $\bar{J}$. Now $\bar{I}^2 = -Id$ and $\bar{I}\bar{J} = -\bar{J}\bar{I}$ forces the existence of a 4 dimensional space which is closed under both $\bar{J}$ and $\bar{I}$. So we can find a vector field Z such that $\bar{J}Z$ $\bar{I}Z$ and $\bar{J}\bar{I}Z$ belong to W. By Theorem 5.7 we know that $Q(Z) < 0$ for uniform k so the hessian is negative for one the following vector fields Z, $\bar{J}Z$, $\bar{I}Z$ and $\bar{J}\bar{I}Z$ $\square$

Chapter 6

Lie algebra calculations

From Theorem 5.1 have shown that the index of a geodesic is $\mathcal{I} = m + n - (v - \ell) - v + 1$ where n, m are the dimensions of the submanifolds M, N and $\ell = \frac{1}{2}|\mathcal{S}_\delta| + |\mathcal{T}_\delta|$, if the roots satisfy certain conditions. For the simply laced Lie algebras A_r,D_r and E_r we will show that these conditions are satisfied. For simple Lie algebra B_r these conditions are not satisfied when δ is a short root. In this case we explicitly work with the roots and get over this handicap.

Recall that Δ is the set of roots for a simple Lie algebra $\mathfrak{g}$. Here $\mathfrak{g}$ is not the algebra G_2. Define $W_\delta = \{\alpha \in \Delta | \delta - \alpha \text{ is a root}\}$. Let $(\cdot, \cdot)$ be the dual of the Killing form on the Lie algebra $\mathfrak{g}$ restricted to $\mathfrak{h}^*_{\mathbb{R}}$. Normalize $(\cdot, \cdot)$ so that the inner product corresponding to the root system Δ of $\mathfrak{g}$ is normalized so that the length of each long root is 2.

In this chapter we refer the reader to [H] for all standard results on root systems and Lie algebras.

Lemma 6.1. *Suppose δ and α are roots such that δ is long then $(\alpha, \delta) > 0$ is $(\alpha, \delta) = 1$*

Proof. Since $(\alpha, \delta) > 0$ the value $2\frac{(\alpha,\delta)}{(\alpha,\alpha)}$ is either 1 or 2. Using table 1 from [H,sec 9.4] it is clear that $2\frac{(\alpha,\delta)}{(\alpha,\alpha)}$ is 1 if α is long and 2 if α is short. In either case it is easy to verify that $(\alpha, \delta) = 1$.

□

Lemma 6.2. *If $\delta \in \Delta$ is a long root then, for $\eta_1, \eta_2 \in W_\delta$, $\eta_1 + \eta_2$ is a root if and only if it is equal to δ.*

Proof. Since the Weyl group is transitive on the set of long roots it suffices to prove this in the case where δ is the highest root vector.

Let the α-chain of roots through δ be

$$-p\alpha + \delta, \ldots, \delta, \ldots q\alpha + \delta \quad \text{where } p, q \geq 0$$

Then it is well known that $p - q = 2\frac{(\alpha,\delta)}{(\alpha,\alpha)}$. Since δ is the highest root we have $q = 0$. Since $\alpha \in W_\delta$ we have that $p > 0$ making $(\alpha, \delta) > 0$. From Lemma 6.1 $(\alpha, \delta) = 1$. So if $\eta_1, \eta_2 \in W_\delta$ then $(\eta_1 + \eta_2, \delta) = 2$. So if $\eta_1 + \eta_2$ is a root then it has to be equal to δ.

□

Recall that

$$\mathcal{S}_\delta = \{\alpha \in \Delta^+_{\mathfrak{m}} | \alpha < \delta \text{ and } \delta - \alpha \in \Delta^+_{\mathfrak{m}}\}$$

$$\mathcal{T}_\delta = \{\beta \in \Delta^+_{\mathfrak{m}} | \beta \geq \delta\} \cup \{\alpha \in \Delta^+_{\mathfrak{m}} | \delta - \alpha \in \Delta_{\boldsymbol{k}}\}$$

Proposition 6.3. *If δ is a long root then condition 1 and condition 2 are satisfied.*

Proof. We first recall condition 1 and 2

Condition 1. *If $\beta_0, \beta_1 \in \mathcal{T}_\delta \backslash \{\delta\}$ with $\beta_0 \neq \beta_1$ then $\beta_0 - \delta \neq \beta_1 - \lambda$ for $\lambda \in \Gamma$*

Condition 2. *If $\alpha, \beta \in \mathcal{S}_\delta$ then $\alpha + \beta \in \Gamma$ iff it is equal to δ.*

For condition 1, begin by assuming that $\beta_0 - \delta = \beta_1 - \lambda$. So $\lambda = \beta_1 + \delta - \beta_0$ is a root. Observe that β_1 and $\delta - \beta_0$ belong to W_δ. So by the Lemma 6.2 the only way that

$\beta_1 + \delta - \beta_0$ is a root is if it is equal to δ, that means $\beta_1 = \beta_0$. A contradiction and hence the lemma readily follows. Condition 2 follows trivially. □

Proposition 6.4. *If the root system associated to $\mathfrak{g}^{\mathbb{C}}$ is such that all roots are long, the value $\ell = \frac{1}{2}|\mathcal{S}_\delta| + |\mathcal{T}_\delta|$ is independent of δ and only dependent on the lie algebra $\mathfrak{g}^{\mathbb{C}}$. The values of ℓ are given below*

(a) If $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_{r+1}(\mathbb{C})$ then $\ell = r$

(b) If $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}_{2r}(\mathbb{C})$ then $\ell = 2r - 3$

(c) If $\mathfrak{g}^{\mathbb{C}} = E_6$,E_7 ,E_8 then $\ell = 11, 17$ and 29 respectively

Proof. We begin by showing that ℓ is independent of δ and later calculate 'ℓ'by choosing δ appropriately.

Define $\widetilde{W}_\delta = \{\{\alpha, \beta\} | \alpha + \beta = \delta, \alpha, \beta \in \Delta\}$. It is clear that $|\widetilde{W}_\delta| + 1 = \frac{1}{2}|\mathcal{S}_\delta| + |\mathcal{T}_\delta|$. Since the Weyl group acts linearly $|\widetilde{W}_\delta|$ is preserved by the Weyl group. As these Lie algebras are simply-laced, it is well known that the Weyl group acts transitively on the roots [H] therefore $|\widetilde{W}_\delta|$ is independent of δ. We observe that $|W_\delta| = 2|\widetilde{W}_\delta|$ and calculate $|W_\delta|$ for a suitable root δ in each case. We refer to [H] for a description of the respective root systems that we will use.

$\mathbf{A}_r$: The roots are given by $\Delta = \{e_i - e_j | 1 \le i, j \le r + 1\}$ where $e_i \in \mathbb{R}^{r+1}$ are the standard basis. Let us choose $\delta = e_1 - e_2$. Then it is clear that $W_\delta = \{e_1 - e_j | 2 < j\} \cup \{e_k - e_2 | k \le r + 1\}$. Thus $|W_\delta| = 2r - 2$,so $\ell = r$

$\mathbf{D}_r$: The roots are given by $\Delta = \{\pm e_i \pm e_j | i \ne j, 1 \le i, j \le r + 1\}$ where $e_i \in \mathbb{R}^{r+1}$ are the standard basis. Choose $\delta = e_1 - e_2$. we then have that $W_\delta = \{e_1 \pm e_j, e_k \pm e_2 | 2 < j, k \le r\}$. So $|W_\delta| = 4r - 8$

$\mathbf{E_8}$:The roots are given by

$$\Delta = \{\pm e_i \pm e_j | i \neq j, \text{and } 1 \leq i,j \leq 8\} \cup \left\{ \frac{1}{2}\sum_{i=1}^{8} t_i e_i | t_i = \pm 1, \prod_{i=1}^{8} t_i = 1 \right\}$$

where $e_i \in \mathbb{R}^8$. Choose $\delta = e_1 - e_2$. A little computation gives us that

$$W_\delta = \{e_1 \pm e_j | 2 < j\} \cup \{e_k \pm e_2 | k \leq 8\} \cup \left\{ \frac{1}{2}(e_1 - e_2) + \frac{1}{2}\sum_{i=3}^{8} t_i e_i | t_i = \pm 1, \prod_{i=3}^{8} t_i = 1 \right\}$$

So we have $|W_\delta| = 56$ and so $\ell = 29$. A similar calculation for E_6 and E_7 can be made.

$\square$

Proof of Theorem 1.1

Parts i,ii and v

In Lie algebras of type A, D and E all roots are long, so parts *i*, *ii* and *iv* follow from Theorem 5.1, Proposition 6.3 and Propsition 6.4.

Part iii

Referring back to [H] we see that the roots of B_r are given by

$$\Delta = \{\pm e_i \pm e_j | 1 \leq i,j \leq r\} \cup \{\pm e_i | 1 \leq i \leq r\}$$

where e_i are the standard basis of $\mathbb{R}^r$. The positive roots are given by

$$\Delta^+ = \{e_i + e_j | 1 \leq i,j \leq r\} \cup \{e_i - e_j | 1 \leq i < j \leq r\} \cup \{e_i | 1 \leq i \leq r\}.$$

Recall that given a $\Gamma \subset \Delta^+$ we chose $\delta \in \Gamma$ to be a superminimal element i.e a minimal element such that if $\alpha < \beta$ and $\beta < \delta$ then $\alpha \notin \Gamma$.

Suppose we can choose a superminimal element δ such that it is a long root. In this case we can repeat the arguments of the the simply laced case, using Proposition 6.3 and Theorem 5.1. We can make similar calculations and conclude that $\ell = 2r - 2$.

Suppose no superminimal elements are long. In this case $\delta = e_i$ where i is the largest index such that $e_i \in \Gamma$. Observe that $e_l + e_j \notin \Gamma$ for $i < l < j$ and that $e_l - e_j \notin \Gamma$ for $l < j$. For if any of these statements were not true it would imply that there exists a superminimal long root.

We now claim that if $e_k \notin \Delta_{\boldsymbol{k}}$ for $i < k \leq r$ then $\delta = e_i$ satisfies conditions 1 and 2. To verify the claim we argue as follows. It is clear that $\mathcal{S}_\delta$ is a subset of the following $\{e_i - e_l | i < l\} \cup \{e_l | \; i < l\}$. Thus condition 2 can be easily verified. Let $U_\delta = \{\beta \geq \delta | \beta \in \Delta^+_{\mathfrak{m}}\}$ and let $V_\delta = \{\alpha \in \Delta^+_{\mathfrak{m}} | \delta - \alpha \in \Delta^+_{\boldsymbol{k}}\}$. Then $\mathcal{T}_\delta = U_\delta \cup V_\delta$. We can clearly see that $U_\delta = \{e_a | a < i\} \cup \{e_i + e_a | \; a < i\}$. Due to the assumption that $e_k \notin \Delta_{\boldsymbol{k}}$, we have $V_\delta = \{e_b | \; i < b, e_i - e_b \in \Delta^+_{\boldsymbol{k}}\}$. Thus

$$\mathcal{T}_\delta = U_\delta \cup V_\delta = \{e_a | a < i\} \cup \{e_i + e_a | \; a < i\} \cup \{e_b | \; i < b, e_i - e_b \in \Delta^+_{\boldsymbol{k}}\}$$

Note that if $\beta_1, \beta_2 \in \mathcal{T}_\delta$ then $\beta_1 + \delta - \beta_2$ is a positive root if and only if its equal to δ or of the form $e_s - e_t$ where $s < t$. To see this note that

$$\{\delta - \beta | \beta \in \mathcal{T}_\delta\} = \{-(e_a - e_i) | a < i\} \cup \{-e_a | a < i\} \cup \{e_i - e_b | \; i < b, e_i - e_b \in \Delta^+_{\boldsymbol{k}}\}.$$

We have already observed that roots of the form $e_s - e_t$ does not belong to Γ with $s < t$, we now see that condition 1 is satisfied thereby verifying the claim.

To finish the proof we assume that $e_k \in \delta_{\boldsymbol{k}}$ for k such that $i < k$. Let $g_t = exp(tX_{e_k})$ where X_{e_k} is a root vector for the root e_k. Since $X_{e_k} \in K$ that implies that $g_t \in K$. Then the left translation L_{g_t} is an non-trivial isometry as well as a biholomorphism on the space G/K, that fixes the base point $\bar{e}$. So it suffices to study the index of the geodesic $L_{g_{t*}}\dot{\gamma}$. For small values of t the set Γ associated to the geodesic will contain the long root $e_i - e_k$ by the Baker-Campbell-Hausdorff formula. So δ can be chosen in the form $e_p - e_q$ where $p < q$. We can then use the arguments above to prove the Theorem in the main case.

Parts iv,vi and vii The proofs of the remaining cases are similar and details will appear in a forthcoming paper. $\square$

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