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A 2-LOCAL APPROACH TO CONWAY'S SIMPLE GROUP THROUGH THE 2-MODULAR GEOMETRY OF THE LEECH LATTICE

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ABSTRACT

A 2-LOCAL APPROACH TO CONWAY'S SIMPLE GROUP THROUGH THE 2-MODULAR GEOMETRY OF THE LEECH LATTICE

By

P. R. Hewitt

In this dissertation we examine the simple group $\cdot 1$ of J. Conway, and in particular its 2-local geometry which arises from certain of its 2-modular representations.

We proceed from the hypothesis that we have a group $\mathfrak{G}$ with an involution z_0 whose centralizer $\mathcal{C}$ in $\mathfrak{G}$ is an extraspecial 2-group of width 4 extended by the full orthogonal group $\Omega_8^+(\mathbf{F}_2)$. We then examine the fusion of z_0 into $\mathcal{C} \setminus O_2(\mathcal{C})$. Next, we add the hypothesis that z_0 fuses into $O_2(\mathcal{C})$ and construct a flag-transitive, rank-4 simplicial complex Δ for $\mathfrak{G}$. We prove that the normalizer $\mathfrak{G}_0$ of a connected component of Δ contains $\mathcal{C}$ and fuses z_0 into $O_2(\mathcal{C})$.

We then give a nearly complete enumeration of the point suborbits in $\mathfrak{G}_0$. Finally, we use this information to examine representations of $\mathfrak{G}_0$ over $\mathbf{F}_2$ that are given locally by generators and relations for $\mathbf{F}_2 \cdot 1$-modules. In particular, we show that the existence of an adjoint module for $\mathfrak{G}_0$ leads to a module locally isomorphic to the Leech lattice <u>modulo</u> 2.

The techniques we employ throughout most of the dissertation are geometric and combinatorial. In studying the representations of $\mathfrak{G}_0$ we use freely the language of sheaves and homology, but in fact make no essential use of this theory.

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0. INTRODUCTION

The purpose of this work is to apply the geometric representation theory of M. Ronan and S. Smith to the group $\cdot 1$ of J. Conway. This is a finite simple group that lies in the gray area between the sporadic groups and the finite algebraic groups. For example, the simplicial complex determined by its maximal 2-local subgroups is locally the truncation of a building over $\mathbf{F}_2$ [Ronan-Smith 8]. It is natural, then, to try and push the analogy as far as possible. This point of view we adopt in constructing, for example, various candidates for the 'adjoint module' of $\cdot 1$.

The working hypotheses for the thesis are that we are given a finite group that contains an involution whose centralizer has the same shape as that in $\cdot 1$. (*Cf.* (*1.1*) and (*2.1*) below.) The two main results of this thesis are:

(*1*) to show how a complex which is locally isomorphic to that for $\cdot 1$ can be constructed naturally from this class of involutions (*cf.* (*2.7*) and (*2.15*));

(*2*) to examine representations of this complex in projective spaces over $\mathbf{F}_2$: the first of these is closely patterned on the adjoint representations of the algebraic groups over $\mathbf{F}_2$ (*cf.* §*3*, especially (*3.8*)).

The first of these results may be summarized in the following.

THEOREM Let $\mathfrak{G}$ *be a group of type* $\cdot 1$.

(*i*) *At least* 10 *of the suborbits of* $\mathfrak{G}$ *on the conjugates of* z_0 *have the same lengths, parameters, and point-stabilizers as their natural counterparts in* $\cdot 1$.

(*ii*) $\mathfrak{G}$ *admits a flag-transitive* 2-*local complex* Δ *that has the diagram and rank*-2 *residuals of a truncation of the affine building* $\tilde{B}_4(\mathbb{Q}_2)$. *This complex is locally isomorphic to the corresponding* 2-*local complex for* $\cdot 1$.

In §1 we present the general group-theoretic and geometric definitions used throughout the thesis, and we establish some of the basic results that intertwine the group theory and the geometry. Most of these results are well-known.

For background on the foundations of diagram geometries we refer the reader to [Aschbacher 2] or [Tits 15]; for the specific 2-local geometries involved here, [Ronan-Smith 8]; and for the basics of the geometry of groups of $\mathbf{F}_2$-type, [Aschbacher 2] or [Timmesfeld 14]. We use freely the language of sheaves and homology as to be found in [Ronan-Smith 9,10], although the material we present is completely elementary and requires no depth from this theory.

In §2 we begin with the main technical lemma: we examine the fusion, for a group of type $\cdot 1$, of a 2-central involution into its centralizer (cf. (2.4)). We then produce the critical quads - these are elementary abelian subgroups of order 2^4 on which the normalizers induce the symplectic groups $\mathcal{S}_4(\mathbf{F}_2)$ (cf. (2.7)). These help lead quickly to the complex alluded to above.

It is the class of quads which to us demonstrates most clearly the ambiguity of $\cdot 1$'s status. On the one hand the class of quads is sufficiently rich in structure so as to lead to a nearly complete description of the full 2-central involution class. It turns out that the permutation rank of $\cdot 1$ on these involutions is 11: this is more than that encountered in classical groups over $\mathbf{F}_2$ - five, typically - and less than that for, say, Z. Janko's group J_3 - 33 - or the group $\mathfrak{F}\mathfrak{G}$ of B. Fischer and R. Griess - perhaps around 150. On the other hand, the presence of the quads creates problems in defining what should be an analogue of the adjoint module for an algebraic group. Indeed, unlike the case for algebraic groups, the essential defining relations for the 'natural' adjoint module for $\cdot 1$ are not implicit in the 'O_2-geometry' (cf. (3.3)). In particular, we are unable to establish the existence of a 'useful' module for an arbitrary group of type $\cdot 1$.

Thus in §3 we add as hypothesis the existence of one of the choices for 'adjoint module', and then determine the internal structure of this and related modules. The main result of this section is first to produce a small, 'natural' module, and then to dissect this module rather completely (cf. (3.8)).

1. GENERALITIES ON GROUPS OF F_2-TYPE

(1.1) DEFINITIONS AND NOTATION A finite group $\mathfrak{G}$ is said to be of F_2-type in case there is an involution z_0 in $\mathfrak{G}$ whose centralizer $\mathfrak{C}$ satisfies:

(1.2) $$Q := \mathfrak{F}^*(\mathfrak{C}) \text{ is extraspecial.}$$

Note that in such a group $\langle z_0\rangle$ is the centralizer of Q. Thus, the center of any Sylow 2-subgroup of $\mathfrak{G}$ is generated by some conjugate of z_0; equivalently, any Sylow 2-subgroup of $\mathfrak{C}$ is a Sylow 2-subgroup of $\mathfrak{G}$. We will further assume that $Z^*(\mathfrak{G}) = \{1\}$. This rules out merely the case $\mathfrak{G} = \mathcal{O}(\mathfrak{G}).\mathfrak{C}$. Should we have occasion to consider groups under (1.2) without this extra hypothesis, the groups will not be referred to as groups of F_2-type.

We denote by $\mathfrak{L} := \mathfrak{C}/Q$ the Fitting factor of $\mathfrak{C}$, and by $\mathcal{M} := Q/\langle z_0\rangle$ the central factor of Q. Recall that $\mathcal{M}$ is an elementary abelian 2-group of even rank $2n$, say, which affords a faithful, nondegenerate orthogonal $F_2\mathfrak{L}$-module, induced by conjugation. Call n the width of Q, or more generally of $\mathfrak{G}$.

Let $\mathfrak{G}$ be a group of F_2-type, with notation as above. Denote by $P := z_0^{\mathfrak{G}}$ the class of z_0, and call these ~ or, often, the groups $\langle z_0^{g}\rangle$ ~ *points*. For any point z, we will denote its centralizer by $\mathfrak{C}_z$; and we continue this subscripting with $Q_z := \mathfrak{F}^*(\mathfrak{C}_z)$, $\mathfrak{L}_z := \mathfrak{C}_z/Q_z$, and $\mathcal{M}_z := Q_z/\langle z\rangle$. Say that points z and z' are collinear when $z' \in Q_z$. The following fundamental result ~ in a much more general form ~ can be found in [Aschbacher 2, (17.5), pp. 125-126].

(1.3) LEMMA *If $\mathfrak{G}$ is of F_2-type then collinearity is a symmetric relation. Indeed, if $z \neq z' \in P \cap Q_z$ then $\ell := \langle z,z'\rangle$ satisfies*:

(a) $\ell^{\#} \subseteq P$;

(b) $\mathfrak{T}_\ell := \langle Q_x \mid x \in \ell^{\#}\rangle \le N_{\mathfrak{G}}(\ell)$; and

(c) $\mathfrak{T}_\ell/\mathfrak{C}_{\mathfrak{T}_\ell}(\ell) \cong \mathfrak{L}_2(F_2)$.

PROOF If $q \in Q_z$, then $z'^q = z'$ or $z'z$, whence $q \in N_{\mathfrak{G}}(\ell)$. The claims will thus follow from the symmetry of collinearity in that we may choose q (in the above) to be an element of $Q_z \setminus \mathcal{C}_y$.

If the width of Q is 1, then either $\mathcal{C} \cong \mathcal{D}_8$ or else $\mathcal{C} \cong Q_8 \cdot \mathfrak{S}_3$. In either case it is straightforward to check the symmetry.

Assume that symmetry does not hold, so that $z \notin Q_{z'}$, but $z' \in Q_z$, for certain z, z' in P. Thus $N_{\mathfrak{G}}(\ell) \le \mathcal{C}_{\mathfrak{G}}(z)$. We use this to argue that $z \in \mathcal{C}_{\mathfrak{G}}(z')$, a contradiction to the hypothesis that $\mathcal{F}^*(\mathcal{C}_{z'}) = Q_{z'}$.

Consider the groups $\mathfrak{R} := \mathcal{C}_{Q_z}(z')$ and $\mathfrak{R}' := \mathcal{C}_{Q_{z'}}(z)$. Write $\mathfrak{R} \cong \langle z'\rangle \times Q_0$, where $Q_0 \le Q_z$ is extraspecial. The asymmetry yields $[\mathfrak{R} \cap Q_{z'}, \mathfrak{R}] \le \langle z\rangle \cap Q_{z'} = \{1\}$. This implies that $\mathfrak{R} \cap Q_{z'} \le Z(\mathfrak{R}) \cap Q_{z'} = \langle z'\rangle$. Also $[\mathfrak{R}', \mathfrak{R}] \le Q_{z'} \cap \mathfrak{R} = \langle z'\rangle$. Hence $\langle z\rangle \le \Phi(\mathfrak{R}.\mathfrak{R}') \le \ell$. This gives $N_{Q_{z'}}(\mathfrak{R}.\mathfrak{R}') \le \mathfrak{R}'$; that is, $N_{Q_{z'}\mathfrak{R}}(\mathfrak{R}.\mathfrak{R}') \le \mathfrak{R}.\mathfrak{R}'$. In particular $\mathfrak{R}' = Q_{z'}$, or $z \in \mathcal{C}_{\mathfrak{G}}(Q_{z'})$ - a contradiction, as noted. □

(1.4)DEFINITION Let L be the set of foursgroups as in the lemma above. We refer to these as lines. We denote by $\Gamma := (P,L)$ the involution geometry of $\mathfrak{G}$. The distance between two points is their distance in the collinearity graph on P. Let P_d denote the set of points at distance d from z_0, and for general $z \in P$ let $P_z := P \cap Q_z$ be the neighbors of z. Finally, let $P_{d,n} := \{z \in P_d \mid |zz_0| = n\}$, a $\mathcal{C}$-stable set.

Note that $P_1 \cup \{z_0\}$ is a subspace - that is, any line that contains as many as 2 points from the set in fact contains 3. More generally, if $\mathcal{H}$ is any subgroup, then $P(\mathcal{H}) := P \cap \mathcal{H}$ is a subspace. We use also the notation $L(\mathcal{H}) := \{\ell \in L \mid \ell \le \mathcal{H}\}$, and $\Gamma(\mathcal{H}) := (P(\mathcal{H}), L(\mathcal{H}))$.

The demands that $\mathfrak{G}$ be finite and that $Z^*(\mathfrak{G}) = 1$ are requisite to use the following fundamental result. In their stead we might demand merely that L be nonempty.

(1.5) THEOREM (Cf. [Aschbacher 1] and [F. Smith 12].) *If $\mathfrak{G}$ is of F_2-type and L is empty, then either $n = 1$ or $\mathfrak{G}$ is isomorphic to one of $\mathcal{L}_4(\mathsf{F}_3)$, $\cdot 2$, or $\mathcal{U}_{n+2}(\mathsf{F}_2)$.* □

(1.6) LEMMA (*from F. Timmesfeld's* [14, (5.1), pp. 163-164]) *Let $\mathcal{C}$ satisfy hypothesis (1.2), and let $t \in \mathcal{C} \setminus Q$ be an involution. The following hold.*

(i) $[Q,t] = \mathcal{C}_Q(Q_0) = Z(Q_0)$, *where* $Q_0 := \mathcal{N}_Q(\langle t,z_0\rangle)$.

(ii) *If* $\mathcal{C}_Q(t)/\langle z_0\rangle \neq \mathcal{C}_{\mathcal{M}}(t)$, *then* t *is conjugate* via Q *to* tz_0.

(iii) *If* $\mathcal{C}_Q(t)/\langle z_0\rangle = \mathcal{C}_{\mathcal{M}}(t)$, *then* $[Q,t]$ *is elementary abelian, and* $t^{\mathcal{C}} \cap tQ \subseteq t^Q \cup (tz_0)^Q$. *Moreover* $|\mathcal{N}_{\mathcal{C}}(tQ):\mathcal{C}_{\mathcal{C}}(t).Q|$ *divides* 2.

PROOF (i) Now $|[\mathcal{M},t]| = |\mathcal{M}:\mathcal{C}_{\mathcal{M}}(t)| = \frac{1}{2}|\mathcal{C}_Q(Q_0)|$. Also, $[t,q]^{q_0} = [t^{q_0},q^{q_0}] \in [t\langle z_0\rangle,q\langle z_0\rangle] = \langle[t,q]\rangle$ for any $q \in Q$ and $q_0 \in Q_0$. That is, $[Q,t] \leq \mathcal{C}_Q(Q_0)$. Thus, by an order argument, $[Q,t] = \mathcal{C}_Q(Q_0)$. (N.B. Since $[Q,t]$ is normalized by $\langle Q,t\rangle$ it must be that $z_0 \in [Q,t]$, even when z_0 is not itself a commutator of the form $[q,t]$ for $q \in Q$.) However, as $[Q,t]/\langle z_0\rangle = [\mathcal{M},t] \subseteq \mathcal{C}_{\mathcal{M}}(t) = Q_0/\langle z_0\rangle$, necessarily $\mathcal{C}_Q(Q_0) = [Q,t] \leq Z(Q_0)$. That is, $[Q,t] = \mathcal{C}_Q(Q_0) = Z(Q_0)$.

(ii): If $q \in Q$ is such that $q^t = qz_0$, then $t^q = tz_0$.

(iii): Assume now that $\mathcal{C}_Q(t)/\langle z_0\rangle = \mathcal{C}_{\mathcal{M}}(t)$. From (i) we note that $[Q,t] = Z(\mathcal{C}_Q(t)) = \mathcal{C}_Q(\mathcal{C}_Q(t))$ is at least abelian. Assume that $q \in Q$ is of order 4. Choose q' so that $q \equiv [q',t] \bmod \langle z_0\rangle$. Thus $qt \equiv t^{q'} \bmod \langle z_0\rangle$, whence t inverts q (in the dihedral group $\langle t,tq\rangle \cong \mathcal{D}_8$). This gives $q\langle z_0\rangle \in \mathcal{C}_{\mathcal{M}}(t)$ and $q \notin \mathcal{C}_Q(t)$, a contradiction. Hence $[Q,t]$ is of exponent 2.

Consider $tq \in t^{\mathcal{C}} \cap tQ$. The previous paragraph yields $\mathcal{C}_Q(tq)/\langle z_0\rangle = \mathcal{C}_{\mathcal{M}}(tq) = \mathcal{C}_{\mathcal{M}}(t) = \mathcal{C}_Q(t)/\langle z_0\rangle$, so that $tq \in t\mathcal{C}_Q(\mathcal{C}_Q(t)) = t[Q,t] \subseteq t^Q \cup (tz_0)^Q$.

Finally, let $g \in \mathcal{C}$ with $gQ \in \mathcal{C}_{\mathcal{L}}(tQ)$. The above says that $t^g \in t^Q$ or $t^Q z_0$, and either case gives $g \in \mathcal{N}_{\mathcal{C}}(\langle t,z_0\rangle).Q$. □

(1.7) LEMMA *Let $\mathcal{G}$ be a group of F_2-type.*

(*i*) *If z, z', and z'' are pairwise collinear, but do not lie together on a line, then they generate an elementary abelian group of order 8 all of whose involutions are points, with every pair of these collinear. Call these subgroups* planes. *If Π is a plane, then $\mathcal{T}_\Pi := \langle Q_z \mid z \in \Pi^\# \rangle \le N_{\mathcal{G}}(\Pi)$, and $\mathcal{T}_\Pi / \mathcal{C}_{\mathcal{T}_\Pi}(\Pi) \cong \mathcal{L}_3(\mathsf{F}_2)$.*

(*ii*) *If $x \in P_{2,4} = P_2 \setminus \mathcal{C}$, then $|xz_0| = 4$ and $[x,z_0] = (xz_0)^2$ is the unique point collinear with each of x and z_0. Each point of $y^{\mathcal{C}}$ is collinear with precisely $\#(P_1 \setminus \mathcal{C}_Q(y))$ points of $P_{2,4}$.*

(*iii*) *If $x \in P_{i+2}$ and ℓ is a line containing $x' \in P_i$, then x is collinear with at most one point of ℓ.*

PROOF (*i*): Let the three pairwise collinear points be z, y, and x. For $q \in Q_z$ in the centralizer of y but not of x, y is collinear with $x^q = xz \in P$, and q induces the transvection of $\mathcal{A}ut(\langle x,y,z \rangle)$ whose center and axis are $\langle z \rangle$ and $\langle y,z \rangle$. This gives (*i*).

For (*ii*), let y be collinear with each of x and z_0. Since $xz_0 \in Q_y$ is not an involution, it must be that $y = (xz_0)^2 = [x,z_0]$. Thus, y is uniquely determined by x and z_0 in this case. Moreover, the number of such x for a fixed $y \in P_1$ is simply the number of points of Q_y that do not centralize z_0.

Finally, (*iii*) follows from (*i*), since otherwise x and x' would be collinear, and x would be at distance no more than $i + 1$ from z_0. □

(1.8) LEMMA *Let $\mathcal{C}$ be of F_2-type.*

(*i*) *If* $x \in P_{2,4}$, $y = (xz_0)^2$, *and* $w \in \mathcal{C} \cap P_x$ *then* $\langle w,x,y\rangle$ *is a plane, and so* $w \in P_{2,2}$. *If instead* $w \in P_3 \cap P_x$ *then either* $w \in \mathcal{C}_y$ *and* $(wz_0)^2 \in Q_y$, *so that* $|wz_0|$ *is one of* 4 *or* 8; *or else* $x = (wy)^2$ *and* $|wz_0|$ *is* 3 *or* 6, *with* $\langle w,z_0\rangle$ *acting symmetrically on* $\langle x,y\rangle$. *Moreover the case* $|wz_0| = 8$ *can occur only when* $y = (wz_0)^4$, $[\mathcal{M}_y,w]$ *is totally isotropic but not totally singular* (i.e., w *is of* type c *on* $\mathcal{M}$, cf. *proposition* (*2.4*) *below*), *and* y *is on every minimal path joining* w *to* z_0.

(*ii*) *For* $w \in P_3$, *one of the following holds*.

(*a*) $w \in P_x$ *for some* $x \in P_{2,4}$, $\langle w,z_0\rangle$ *has order* 6 *or* 12, *and this dihedral group induces* $\mathcal{L}_2(F_2)$ *on* $\langle x,(xz_0)^2\rangle$; *or*

(*b*) $|wz_0|$ *divides* 8. *If* $|wz_0| = 8$ *then* $w \in P_x$ *such that either* $x \in P_{2,2}$ *and* $[\mathcal{M}_x,z_0]$ *is not totally singular*; *or else* $x \in P_{2,4}$, w *centralizes* $y = (xz_0)^2$, *and* $[\mathcal{M}_y,w]$ *is not totally singular*.

PROOF For (*i*) we argue as in (*1.6.i*). In the first case w is collinear with $x^{z_0} = xy$, and so $\langle w,x,y\rangle = \langle w,x,xy\rangle$ is a plane. In the second case, if $w \in P_3$ then either $w \in \mathcal{C}_y \setminus \mathcal{C}$, from the above; or else w and z_0 induce distinct transpositions in $\langle x,y\rangle^{\#}$, so that $\langle w,z_0\rangle$ induces a copy of $\mathfrak{S}_3$ there. The former possibility says that $[z_0,w] = (z_0w)^2 \in Q_y^{\#}$, so has order either 2 or 4 ~ giving, respectively, a singular or nonsingular point of $\mathcal{M}_y$. The latter possibility gives $(wz_0)^3 = z_0^{\,w}.w^{z_0} \in Q_{xy}$, inverted by both w and z_0. However neither w nor z_0 centralize xy, so that $(wz_0)^6 = 1$.

Moreover, if $x' = (wz_0)^3$ is a point then x' is collinear with $x = (xy)^{z_0}$ and $y = (xy)^w$.

(*ii*): If w is collinear with some $x \in P_{2,4}$, then the statements follow from (*i*). If $w - x - y - z_0$ is a path joining w to z_0, with $x \in \mathcal{C}$, then $[w,z_0] = (wz_0)^2 \in Q_x$, and again the statements follow. □

2. THE ADJOINT GEOMETRY FOR GROUPS OF TYPE $\cdot 1$

(2.1) DEFINITION A group $\mathfrak{G}$ is said to be of *type* $\cdot 1$ in case $\mathfrak{G}$ is of $\mathbb{F}_2$-type (as in (1.2)) and $\mathfrak{C}$ satisfies the following:

(2.2) $Q = \mathfrak{F}^*(\mathfrak{C})$ is an extraspecial 2-group of width and Witt index 4, and $\mathfrak{L} := \mathfrak{C}/Q$ is isomorphic to $\Omega_8^+(\mathbb{F}_2)$.

Naturally, the simple group $\cdot 1$ of J. Conway is of type $\cdot 1$.

(2.3) LEMMA Let $\mathfrak{G}$ be of type $\cdot 1$.

(i) The set L of lines is not empty.

(ii) If ℓ is a line, then $\mathfrak{T}_\ell := \langle Q_z \mid z \in \ell^\# \rangle \leq N_{\mathfrak{G}}(\ell)$, and $\mathfrak{T}_\ell/\mathfrak{C}_{\mathfrak{T}_\ell}(\ell) \cong \mathfrak{L}_2(\mathbb{F}_2)$. Moreover each transvection is induced by a point collinear with its center on the line.

(iii) $\mathfrak{G}$ acts flag-transitively on Γ. Indeed, P_1 is a single $\mathfrak{C}$-orbit, of length 2.135.

PROOF Clearly (i) and (ii) follow from lemma (1.3), in light of theorem (1.5). For (iii) observe that $\mathfrak{G}$ is transitive on points, by design. Now $\mathfrak{C}$ ~ acting as $\mathfrak{L}$ ~ has a single orbit on the singular points of M. Thus, $\mathfrak{C}$ has a single orbit on the foursgroups of Q that contain z_0, and P_1 induces a 2-cycle on each of these. □

(2.4) PROPOSITION Assume $\mathfrak{C}$ satisfies hypothesis (2.2). The lifting of involutions of $\mathfrak{L}$ to involutions of $\mathfrak{C}$ is constrained to the possibilities listed in Table 1 (with the notation detailed in the proof below).

Table 1

The possibilities for involutions of $\mathcal{E}$

$\mathcal{E}$-class			centralizer as $\mathcal{E}_Q(t)\cdot(\mathcal{E}_{\mathcal{E}}(t)/\mathcal{E}_Q(t))$	class length
1	a_2	either	$2^3\circ 2^{1+4}\cdot 2^{1+2.4}[\mathcal{L}_2(F_2)\times\Omega_4^+(F_2)]$	4.1575
			(two classes, equal modulo z_0)	
2		or	$2^3\circ 2^{1+4}\cdot 2^{1+2.4}[3^3.2^2]$	8.1575
			(two Q-classes, fused in $\mathcal{E}$)	
3	a_2		$2^3\circ 2^{2+2}\cdot 2^{1+2.4}[\mathcal{L}_2(F_2)\times 2^2]$	8.9.1575
4	a_4	either	$2^5\cdot 2^6\mathcal{S}_4(F_2)$	16.3780
			(two classes, equal modulo z_0)	
5		or	$2^5\cdot 2^5\mathfrak{U}_6.2$	32.3780
			(two Q-classes, fused in $\mathcal{E}$)	
4′	a_4'		The same possibilities as for 4 above.	
5′	a_4'		The same possibilities as for 5 above.	
6	c_2		$2^2\circ 2^{1+4}\cdot 2^6\mathcal{O}_4^+(F_2)$	8.10.3780
7	c_2		$2^2\circ 2^{1+4}\cdot 2^6\mathcal{O}_4^-(F_2)$	8.6.3780
8	c_4		$2^4\cdot 2^9\mathcal{L}_2(F_2)$	32.56700

PROOF In the notation of [Aschbacher-Seitz 3], the involutions of $\mathcal{L} \cong \Omega_8^+(F_2)$ fall into 5 classes: a_2, the Siegel (long root) elements; c_2, the products of distinct, commuting transvections; a_4 and a'_4, those whose commutator subspace is a maximal totally singular subspace; and c_4, certain involutions that lie in $a_2.c_2$. The 'a' refers to the fact that the commutator subspace is totally singular, and the 'c' to the fact that commutator subspace is totally isotropic but not totally singular. The subscript gives the dimension of the commutator subspace.

The nomenclature serves also to describe the classes in the full symplectic group $\mathcal{S}(\mathcal{V})$. The class of a symplectic involution t can be described as in the following table, meaning that the condition listed for $\mathcal{V}(t) := \{v \in \mathcal{V} \mid \langle v,vt\rangle = 0\}$ suffices - with $d := \dim_{F_2}([\mathcal{V},t])$ - to determine the class of t:

Table 2

The classes of involutions in $\mathcal{S}(\mathcal{V})$ (or $\mathcal{O}(\mathcal{V})$)

class	condition on t
a_d	$\mathcal{V}(t) = \mathcal{V}$
b_d	$\mathcal{V}(t)$ is a hyperplane, d is odd
c_d	$\mathcal{V}(t)$ is a hyperplane, d is even

Of course, d is even for $t \in \Omega(\mathcal{V})$.

The involutions of $\mathcal{L}$ all lift to involutions of $\overline{\mathcal{C}} := \mathcal{C}/\langle z_0\rangle$. More generally, one can show that each involution of $\mathcal{O}^{\epsilon}_{2n}(F_2)$ lifts to an element of the same order in the automorphism group of an extraspecial 2-group Q' of width n and type ϵ using the Frattini argument. Indeed, each respects a decomposition of Q' (or $Q' \circ \mathcal{D}_8$, of the same type but of width $n + 1$) into a central product of subgroups isomorphic to Q_8, whence one obtains that the element normalizes a fixed-point-free, elementary abelian 3-subgroup. (This argument comes from U. Dempwolff's [4, (3), p. 453]; one could in fact use the result given there by embedding Q' in $Q' \circ \mathbb{Z}/4$.)

Now, the involutions of a coset $\bar{t}\mathcal{M}$ are precisely $\bar{t}\mathcal{C}_{\mathcal{M}}(\bar{t})$, where $\bar{t}$ is any involution of $\bar{\mathcal{C}}$. Moreover, $\bar{t}^{\mathcal{M}} = \bar{t}[\mathcal{M},\bar{t}]$. Thus there are $|\mathcal{C}_{\mathcal{M}}(\bar{t}):[\mathcal{M},\bar{t}]|$ $\mathcal{M}$-classes of involutions in $\bar{t}\mathcal{M}$. In particular those of types a_4, a_4', and c_4 - those for which $\mathcal{V}$ is free over $\mathbb{F}_2\langle\bar{t}\rangle$ - lift to a unique $\mathcal{M}$-class of $\bar{\mathcal{C}}$. An order argument now forces $\mathcal{C}_{\bar{\mathcal{C}}}(\bar{t})$ to have shape $[\mathcal{M},\bar{t}]\cdot\mathcal{C}_{\mathcal{L}}(\bar{t})$, for each of these types.

Next we consider the action of $\mathcal{N}^* := \mathcal{N}_{\bar{\mathcal{C}}}(\bar{t}\mathcal{M})$ on the module $\mathcal{M}^* := \langle\bar{t},\mathcal{C}_{\mathcal{M}}(\bar{t})\rangle/[\mathcal{M},\bar{t}]$, for $\bar{t}$ of type a_2 or c_2. Let (†) denote the following:

$$(\dagger) \qquad 0 \longrightarrow \mathcal{C}_{\mathcal{M}}(\bar{t})/[\mathcal{M},\bar{t}] \longrightarrow \mathcal{M}^* \xrightarrow{\pi} \langle\bar{t}\rangle \longrightarrow 0$$

In the case a_2, (†) is split for $\mathcal{O}^2(\mathcal{N}^*)$ since this group induces $\mathcal{D}^1(\Omega_4^+(\mathbb{F}_2)) \cong \mathfrak{U}_3\times\mathfrak{U}_3$. (Here $\Omega_4^+(\mathbb{F}_2) := \langle a_2 \subseteq \mathcal{O}_4^+(\mathbb{F}_2)\rangle$.) However, for the case c_2 $\mathcal{O}^2(\mathcal{N}^*)$ induces $\mathcal{D}^1\mathcal{S}_4(\mathbb{F}_2)) \cong \mathfrak{U}_6$, and it is possible that (†) is nonsplit $(\mathcal{E}xt^1_{\mathfrak{U}_6}(\langle\bar{t}\rangle,\mathcal{C}_{\mathcal{M}}(\bar{t})/[\mathcal{M},\bar{t}]) \neq 0)$. We consider the split and nonsplit cases separately. In either of the split cases we now assume that the $\mathcal{M}$-class of $\bar{t}$ is fixed by $\mathcal{O}^2(\mathcal{N}^*)$. For the nonsplit case we choose any $\bar{t}$ in the coset. In the chart below we list, for either of these t, the orbit lengths under $\mathcal{O}^2(\mathcal{N}^*)$ for those elements of $\mathcal{M}^*$ that map onto $\bar{t}$ under π; the $\mathcal{O}^2(\mathcal{N}^*)$-class lengths of involutions in $\bar{t}\mathcal{M}$; and finally the subgroup $\bar{\mathcal{N}} := \mathcal{O}^2(\mathcal{C}_{\bar{\mathcal{C}}}(\bar{t}))$, expressed as $\mathcal{C}_{\mathcal{M}}(\bar{t})\cdot(\mathcal{O}^2(\mathcal{C}_{\bar{\mathcal{C}}}(\bar{t}))/\mathcal{C}_{\mathcal{M}}(\bar{t}))$.

Table 3
The lifting of involutions from $\mathcal{L}$ to $\bar{\mathcal{C}}$

$\mathcal{L}$-class	$O^2(\mathcal{N}^*)$-orbit-lengths in $\bar{t}\pi^{-1}$	$O^2(\mathcal{N}^*)$-class-lengths in $\mathcal{I}nv(\bar{t}\mathcal{M})$	$\mathcal{N}$
a_2	1	8	$2^{2+4}\cdot 2^{1+2.4}3^3$
	3,3 (nonsingular points)	3.8,3.8	
	9 (singular points)	9.8	
c_2	1	8	$2^{2+4}\cdot 2^5\mathfrak{A}_6$
(split)	15	15.8	
c_2	10 (index-2 quadric)	10.8	$2^{1+4}\cdot 2^4 3^2$
(non-split	6 (index-1 quadric)	6.8	$2^{1+4}\cdot 2^4\Omega^-(F_2)$

In the split cases, $\mathcal{C}_{\bar{\mathcal{C}}}(\bar{t})/\mathcal{C}_{\mathcal{M}}(\bar{t}) = \mathcal{C}_{\mathcal{L}}(\bar{t}\mathcal{M})$. In the nonsplit cases this quotient is $2^6\Omega_4^{\epsilon}(F_2)$. Let $\mathcal{N} \le \mathcal{N}_{\mathcal{C}}(\langle t,z_0\rangle)$ be the preimage of $\bar{\mathcal{N}}$ in $\mathcal{C}$. Note that $\mathcal{N} = O^2(\mathcal{N})$.

We will make repeated use of the fact that, for each class, $[O_2(\mathcal{N}),\mathcal{N}\cap Q]$ covers $[O_2(\bar{\mathcal{N}}),\bar{\mathcal{N}}\cap\mathcal{M}] = sing.rad.([\mathcal{M},\bar{t}])$, and thus equals $\Omega_1([Q,t])$.

Next we show that the c_2 involutions do not lift to involutions of $\mathcal{C}$, unless (†) is nonsplit. Assume, on the contrary, that (†) is split and $t \in \mathcal{C}$ is an involution that maps to $\bar{t}$. From lemma (1.6) we conclude that $\mathcal{C}_Q(t) \ne \mathcal{N}_Q(\langle t,z_0\rangle)$. The latter group is the central product, over an element of order 4, of $[Q,t] \cong 2\times 4$ with a group $2^{1+5} \cong 4\circ 2^{1+4}$ of symplectic type. However $\mathcal{N}$ as above normalizes no subgroup of $N_Q(<t,z_0>)$ of index 2. Hence $\mathcal{C}_Q(t)$ is *not* properly contained in $N_Q(<t,z_0>)$. The only possible conclusion is that these c_2 involutions cannot lift to involutions of $\mathcal{C}$ when (†) is split.

Rather than determine which of the remaining $\overline{\mathcal{C}}$-involution-classes do in fact lift to involutions of $\mathcal{C}$ we concentrate on the consequences, for each class, of finding an involution in the given coset.

Consider an involution t that maps into c_2 in $\mathcal{L}$. Necessarily π is nonsplit, and so $N_{\mathcal{C}}(<t,z_0>)$ has shape $N_Q(<t,z_0>)\cdot 2^6\Omega_4^{\epsilon}(\mathbb{F}_2)$. Now, $|N_{\mathcal{C}}(<t,z_0>):\mathcal{C}_{\mathcal{C}}(t)|$ is at most 2, but lemma (1.6) forces $|N_Q(<t,z_0>):\mathcal{C}_Q(t)| = 2$. $[N_Q(<t,z_0>),N]$ contains $\Omega_1([Q,t])$ and covers the 4-dimensional orthogonal factor of $N_Q(<t,z_0>)$, and so in fact $|[Q,t]:\mathcal{C}_{[Q,t]}(t)| = 2$.

For the class c_4, $[Q,t]$ is not elementary abelian, so that $|N_Q(<t,z_0>):\mathcal{C}_Q(t)| = 2$. This says that for t of this class $t[Q,t]$ is a single Q-class of involutions, of length 32.

For t of class a_2, $[N_Q(<t,z_0>),N] = N_Q(<t,z_0>)$, a central product, over $<z_0>$, of $[Q,t]$ and an extraspecial group of width 2. Hence $N_Q(<t,z_0>) = \mathcal{C}_Q(t)$ and so there are at most 3 classes of involutions in tQ:

— the two Q-classes of $t[Q,t]$, each of length 4, which are exchanged upon multiplication by z_0, and possibly fused in $\mathcal{C}$;

and

— $t.\{$involutions q of $\mathcal{C}_Q(t) \setminus [Q,t]\}$, a class of length 8.9.

Finally, for t of class a_4 (or a_4'), $[Q,t]$ is either the indecomposable orthogonal module or else a direct sum of $<z_0>$ with the natural symplectic module for $\mathcal{S}_4(\mathbb{F}_2)$, so that $\mathcal{C}_Q(t) = [Q,t]$, and $t[Q,t]$ is exactly 2 Q-classes of involutions, exchanged upon multiplication by z_0, and possibly fused in $\mathcal{C}$.

We have now verified that the entries of Table 1 give all of the classes possible for involutions of $\mathcal{C}$, their lengths and centralizers, and their class in $\mathcal{L}$. $\square$

(2.5)COROLLARY Let $\mathfrak{G}$ be of type $\cdot 1$. If z and z' are commuting points then the action of z on $Q_{z'}$ is isomorphic to that of z' on Q_z. (We are not asserting here that z and z' can be exchanged by an element of $\mathfrak{G}$.)

PROOF If the points are collinear, this is clear. If not, then Table 1 identifies $\mathcal{C}_z \cap \mathcal{C}_{z'}$, and the possibilities are essentially distinct for distinct classes of involutions in $\mathcal{C}_z \cong \mathcal{C}_{z'}$. $\square$

(2.6)COROLLARY Let $\mathfrak{G}$ be of type $\cdot 1$.

(i) If $w \in \mathcal{C}$ is a point then $\mathcal{C}_Q(w)/(Q \cap Q_w) \lesssim \mathcal{C}_{\mathcal{C}}(w)/\mathcal{C}_Q(w)$.

(ii) A point of $\mathcal{C} \setminus (P_2 \cup Q)$ must be as in 4,4′, 5,5′, or 8.

(iii) If $w \in \mathcal{C}$ is a point of $\mathcal{L}$-class a_4 or a'_4, then $\mathcal{C}_Q(w)$ is the indecomposable orthogonal module for $\mathcal{C} \cap \mathcal{C}_w$.

PROOF (i) follows from corollary (2.5) and the appropriate Isomorphism Theorem, since $\mathcal{C}_Q(w).Q_w \le (\mathcal{C} \cap \mathcal{C}_w).Q_w$.

(ii) and (iii) Observe that if $z \in P \setminus \{z_0\}$ then $(Q \cap Q_w)^{\#} = P_1 \cap P_x$. Hence, by (i), $\mathcal{C}_Q(w) \lesssim \mathcal{C}_{\mathcal{C}}(w)/\mathcal{C}_Q(w)$ for $w \in \mathcal{C}$ at distance greater than 2 from z_0. This can happen only for w of class a_4 (or a'_4) or c_4 in $\mathcal{L}$, as one sees by inspection of Table 1. Thus if w is of $\mathcal{L}$-class a_4 or a'_4, then $\mathcal{C}_Q(w)$ must be indecomposable. $\square$

(2.7)PROPOSITION Let $\mathfrak{G}$ be of type $\cdot 1$. The following hold.

(i) Each of P_1, $P_{2,2}$, and $P_{2,4}$ is a $\mathcal{C}$-orbit, of length 2.135, 2.70.90, and $2.2^6 270$, respectively. Each point of P_1 is collinear with precisely 2.70 points of $P_{2,2}$ and 2.2^6 points of $P_{2,4}$.

(ii) There are no planes in Γ. No point of $P_{2,4}$ is collinear with any point of $\mathcal{C} \setminus Q$.

(iii) Each point x of $P_{2,2}$ defines a Siegel (long-root) element of $\mathcal{L}$, and conversely each coset xQ of such a Siegel element contains exactly 8 points of $P_{2,2}$. There are exactly 3 points collinear with both x and z_0.

(iv) If y, y', and y'' are the common neighbors of x and z_0, $x \in P_{2,2}$, then $\mathcal{S} := \langle x,y,y',y'',z_0\rangle$ is an elementary abelian group of order 16 such that the geometry $\Gamma(\mathcal{S})$ induced from Γ on it is isomorpic to the $\mathcal{S}_4(\mathsf{F}_2)$-quadrangle. We call these subgroups quads.

(v) If $\mathcal{S}$ and $\mathcal{S}'$ are distinct quads, then their intersection is either $\{1\}$, a point, or a line. In particular two commuting points at distance 2 from one another lie together on a unique quad.

(vi) The normalizer of a quad induces a copy of either $\mathcal{S}_4(\mathsf{F}_2) \cong \mathfrak{S}_6$ or $\mathfrak{A}_6$, with each of the $\mathfrak{A}_6$-involutions (precisely those of class c_2 on $\mathcal{S}$) induced by a point collinear with its singular center in $\mathcal{S}$ (i.e., the product of the centers in $\mathcal{S}$ of the two commuting transvections of which the given involution is their product). In fact, $\mathcal{T}_{\mathcal{S}} := \langle Q_z \mid z \in \mathcal{S}^{\#}\rangle \le N_{\mathfrak{G}}(\mathcal{S})$, and $\mathcal{T}_{\mathcal{S}}/\mathcal{C}_{\mathcal{T}_{\mathcal{S}}}(\mathcal{S}) \cong \mathfrak{A}_6$.

(vii) If $x \in P_{2,2}$ and $w \in \mathcal{C} \cap P_x$ then $[w,\mathcal{S}_{x,z_0}] = \{1\}$. If also $w \in P_2$ then $w \in \mathcal{S}_{x,z_0}$.

PROOF The transitivity on P_1 follows from lemma (2.3). Lemma (1.6) gives that for each $y \in P_1$, the set $P_{2,4} \cap Q_y$ is conjugate – via $N_{\mathfrak{G}}(\langle y,z_0\rangle)$ – to $P_1 \setminus \mathcal{C}_Q(y)$, which is a single $\mathcal{C} \cap \mathcal{C}_y$-orbit, as $\mathcal{L} \cong \Omega_8^+(\mathsf{F}_2)$ is rank 3 on the singular points of $\mathcal{M}$. Thus, $\mathcal{C}$ is transitive on $P_{2,4}$.

Similarly, since $\mathcal{C} \cap \mathcal{C}_y$ is transitive on the involutions of $Q \cap \mathcal{C}_y$, either all of these lie in Q_y, or none of these do and the set is conjugate *via* $N_{\mathcal{C}}(\langle y,z_0\rangle)$ to $P_{2,2} \cap Q_y$. Now the elementary abelian 2-group $Q \cap Q_y$ has order dividing 2^5, so only the latter case can hold. The flag-transitivity now gives that $P_{2,2}$ is a single $\mathcal{C}$-orbit.

We have shown that there are no triangles of collinearity. A fortiori there are no planes. Thus, by (2.4.i), we conclude (ii).

The first and last orbit lengths are now clear. The length of $P_{2,2}$ is $2.70.\#P_1/\#(P_1 \cap P_x)$, for any fixed $x \in P_{2,2}$. A comparison with Table 1 yields that the only possibility for x is for it to have class a_2 in $\mathcal{L}$ and satisfy one of the situations 1 or 2 of Table 1. The second paragraph of the proof yields that $\langle x,z_0\rangle$ is conjugate into Q, so that both x and xz_0 are points of $P_{2,2}$, and the second case of Table 1 holds.

In particular we have ascertained the following facts. $|\mathcal{C}_{\mathcal{C}}(x)| = |\mathcal{C}|/\#P_{2,2} = 2^{18}3^3$, and the foursgroup $Q/\mathcal{C}_Q(x)$ has two (regular) orbits in $x[Q,x]$. There are exactly 8 points of $P_{2,2}$ in xQ; and $\langle x,z_0\rangle^\# \subseteq P$ but is not a line. Finally, x and z_0 have 3 common neighbors. This is (i) and (iii).

Call these common neighbors y, y', and $y'' = yy'$. The diameter of the geometry $\Gamma(\mathcal{S})$ induced on the group $\mathcal{S} := \langle x,y,y',y'',z_0\rangle$ ~ perforce an elementary abelian group of order 16 ~ is easily checked to be exactly 2, with each pair of noncollinear points joined by at least 2 paths. For example, the distance (in $\Gamma(\mathcal{S})$) between points of $\mathcal{S} \cap Q$ is certainly at most 2, as is the distance between z_0 and any point of $\mathcal{S}$. Moreover, each point of $\mathcal{S} \cap P_{2,2} = x[Q,x]$ is collinear with a unique point of any line in $\mathcal{S}$ that contains z_0. Finally, one uses these observations to show that any two points x', x'' of $x[Q,x]$ are jointly collinear with at least one point z of $[Q,x]$. Indeed, if $P_1 \cap P_{x'} \supseteq z', z''$ and $P_1 \cap P_{x''} \supseteq z'z_0, z''z_0$, then $z = z'z'' = z'z_0z''z_0$.

As a consequence, $\Gamma(\mathcal{S})$ is the <u>geodesic</u> <u>closure</u> of x and z_0 - the smallest subspace that contains x, z_0, and all points lying on all minimal paths joining any pair of points of the subspace. This is because the above gives 3 common neighbors in $\mathcal{S}$ to any pair of noncollinear points of $\mathcal{S}$ - accounting for all of the common neighbors in all of Γ. It is immediate also that $\Gamma(\mathcal{S})$ is isomorphic to the $\mathcal{S}_4(F_2)$-quadrangle. Henceforth we will denote by $\mathcal{S}_{z,z'}$ the unique quad containing two commuting points z and z' at distance 2 from one another.

If $\mathcal{S}$ and $\mathcal{S}'$ are quads with $|\mathcal{S} \cap \mathcal{S}'| \geq 8$, then there is a point z that is the radical, in each of $\mathcal{S}$ and $\mathcal{S}'$, of any eightsgroup in $\mathcal{S} \cap \mathcal{S}'$. This means that $\mathcal{S} = \mathcal{S}_{z,x}$ and $\mathcal{S}' = \mathcal{S}_{z,x'}$ for certain x and x' in $\mathcal{E}_z$, each acting as a Siegel element on $\mathcal{M}_z$. As a result, $x \equiv x'$ <u>mod</u> Q_z, and so $\mathcal{S} = \mathcal{S}'$ (from the definition of a quad above). This finishes (<u>iv</u>) and (<u>v</u>).

Consider now a path $w - x - y - z_0$, where $x \in P_{2,2}$. We assert that exactly one of the following holds for w:

(<u>a</u>) $[w,\mathcal{S}_{x,z_0}] = \{1\}$.

(<u>b</u>) $|wz_0| = 4$, and w acts as an involution of class c_2 on $\mathcal{S} = \mathcal{S}_{x,z_0}$, with singular center x.

By the previous lemma, we know that the order of $|wz_0|$ is 2or 4 (remember that each point x of $P_{2,2}$ defines a Siegel element on $\mathcal{M}$, and z_0 defines a Siegel element on each $\mathcal{M}_x$) so that $\mathcal{E}_{Q_x}(z_0) = \mathcal{E}_{Q_x}([Q_x,z_0])$. Further, $\mathcal{S} = \langle [Q_x,z_0],z_0\rangle = \langle x,[Q,x]\rangle$. Thus w centralizes <u>some</u> point of $\mathcal{S}$ at distance 2 from x if and only if w centralizes <u>each</u> point of $\mathcal{S}$ collinear with x (since if an element centralizes both a hyperplane and a point outside the hyperplane, it centralizes the whole space). These conditions are thus equivalent to $[\mathcal{S},w] = \{1\}$. In particular, w cannot induce a transvection on $\mathcal{S}$. Now consider w collinear with x but not commuting with z_0. First, $\{1\} \neq [\mathcal{S},w] \leq \mathcal{S}.[Q_x,z_0] = \mathcal{S}$.

Second, w normalizes each line on x in the eightsgroup $\mathcal{P} \cap Q_x$, and centralizes a unique one of these, say $\langle x,y'\rangle$, which must also be its commutator on $\mathcal{P}$. If w were to induce an a_2 element on $\mathcal{P}$, then $z_0^w = z_0y'$ (since y' is the unique point of $[\mathcal{P},w]$ collinear with z_0). This contradicts the fact that $w \in \mathcal{C}_{y'}$ satisfies $\mathcal{C}_{Q_{y'}}(w)/\langle y'\rangle = \mathcal{C}_{\mathcal{M}_{y'}}(w)$. Consequently, the class of w in $N_{\mathfrak{C}}(\mathcal{P})/\mathcal{C}_{\mathfrak{C}}(\mathcal{P}) \leq \mathfrak{S}_6$ is as claimed. The singular center of an involution t of type c_2 in any symplectic group is the radical of $\mathcal{V}(t)$. Since w normalizes each line on x, x must in fact be the singular center for w acting on $\mathcal{P}$. Thus (a) and (b) – which are mutually exclusive – exhaust the possibilities for w. Finally, each Q_z is generated by its involutions, so the identification of $\mathcal{T}_{\mathcal{P}}$ is complete.

The above ensures that $[\mathcal{P},x'] = [\mathcal{P}',x] = \{1\}$, where $\mathcal{P} = \mathcal{P}_{x,z_0}$ and $\mathcal{P}' = \mathcal{P}_{x',z_0}$, whenever x and x' are collinear points of $P_{2,2}$. Corollary (2.6) implies that the only Siegel elements of $\mathcal{C}_* := \mathcal{C}_{\mathcal{L}}(xQ)$ that lift to points in $\mathcal{C}_{\mathfrak{C}}(x)$ must normalize some nondegenerate 2^4 in $\mathcal{M}_* := Q_*/\langle xQ\rangle$, where $Q_* := O_2(\mathcal{C}_*)$. Now the only such Siegel elements that also centralize $\mathcal{P}$ lie in Q_*, and thus $\{0\} \neq [\mathcal{M},x] \cap [\mathcal{M},x'] = \langle y\langle z_0\rangle\rangle$, say. In particular, as every quad is geodesically closed, $\langle x,x',z_0\rangle \leq \mathcal{P} \cap \mathcal{P}'$, and so these quads are equal.

This is (vii), and the proposition. □

(2.8) REMARKS (i) We have shown that each point of $P_{2,2}$ generates with z_0 a 'fake line' – a foursgroup all of whose involutions are points, but which is not a line. We call the fake lines of the proposition hyperbolic lines; and if the need for emphasis arises we will refer to the lines of the involution geometry as the singular lines. The hyperbolic lines are precisely the foursgroups that lie uniquely in some quad. If $\mathcal{h}$ is a hyperbolic line then one sees that its normalizer induces $\mathcal{L}_2(\mathsf{F}_2) \cong \mathfrak{S}_3$.

(ii) We will see in the proof of (2.15) below that the normalizer of a quad induces all of $\mathcal{S}_4(F_2)$, with a transvection induced by a point at distance 2 from its center in the quad.

(2.9)DEFINITION Call any path $z' - z - z''$, where $\ell' = \langle z,z' \rangle$ and $\ell'' = \langle z,z'' \rangle$ are lines, a corner, of width:

$$\min\{|x'x''| \mid x' \in \ell' \setminus \langle z \rangle,\ x'' \in \ell'' \setminus \langle z \rangle\}.$$

Note that the width of the corner depends only on z and the lines ℓ' and ℓ''.

(2.10)COROLLARY *The involution geometry for a group of type $\cdot 1$ contains no irreducible pentagons. More precisely, if $z_0 - z_1 - z_2 - z_3 - z_4 - z_5 = z_0$ is a cycle of 5 distinct points (subscripts read modulo 5) with no corner of width 1, then for all i it must be that z_i is collinear with $z_{i+2}z_{i+3}$. In particular, if $x \in P_{2,4}$, then x and x^{z_0} are the only points of $P_{2,4}$ collinear with x.*

PROOF If $z_2 \in P_{2,4}$ then so is z_3, by part (ii) of the proposition. However, $z_1z_2 = z_2^{z_0}$ and $z_4z_3 = z_3^{z_0}$ are collinear. Thus z_1 and z_4 are collinear, since now $\langle z_i | i \neq 0 \rangle$ must lie in a quad. This contradicts the fact that there are no planes, since the hypotheses preclude the possibility that $\langle z_1,z_2 \rangle = \langle z_3,z_4 \rangle$.

Thus all z_i lie in $\mathcal{C}$. The proposition now gives that all lie in some quad, in which one checks easily that all pentagons are as described. □

(2.11)COROLLARY *Let $\mathcal{G}$ be of type $\cdot 1$. Let $w \in P_3$ and $x \in P_2$ be collinear. One of the following holds.*

(a) $x \in P_{2,4}$, $(wy)^2 = x$ $(y = (xz_0)^2)$ and $|wz_0| = 3$.

(b) $x \in P_{2,4}$, $(wy)^2 = 1$ $(y = (xz_0)^2)$ and $(wz_0)^2 \in P_{2,2}$.

(c) $x \in P_{2,2}$, and $[w,\mathcal{S}_{x,z_0}] = \{1\}$.

(d) $x \in P_{2,2}$, $(wz_0)^2 = yx$, where $\langle y,x \rangle = [w,\mathcal{S}_{x,z_0}]$, and $y \in Q \cap Q_x$.

PROOF Assume first that $x \in P_{2,4}$, and let $y = (xz_0)^2$. By (1.8.ii) either (b) holds, or else $(wy)^2 = x$ and $|wz_0| = 3$ or 6. The last case gives $(wz_0)^3 = z_0{}^w.w^{z_0} \in P \cap Q_{xy}$, whence $\langle x,y,(wz_0)^3\rangle$ is a plane - a contradiction.

If, on the other hand, $x \in P_{2,2}$, then the remaining cases are lead to (c) and (d) by application of proposition (2.7.vi-vii) and lemma (1.8.i). For if (c) does not hold then $\ell := [\mathcal{P}_{x,z_0},w]$ is a line, and $x \neq (wz_0)^2 \in \ell \setminus Q$. □

(2.12)LEMMA Let $\mathcal{C}$ be of type $\cdot 1$.

(i) $P_{3,2} = P_3 \cap \mathcal{C}$ is a single $\mathcal{C}$-orbit. The product of z_0 and a point w of $P_{3,2}$ is not a point; that is, $P \cap z_0.P_{3,2} = \emptyset$. Such a w acts as an involution of type a_4 on M, and satisfies the conditions in entry 4 of Table 1.

(ii) If $x \in P_{2,2}$ then $\mathcal{C}_{\mathcal{C}}(x)$ has 2 orbits in $P_3 \cap Q_x$, viz. the 2.36 points that centralize z_0 and the 2.96 whose product with z_0 has order 4.

(iii) Each $w \in P_{3,2}$ is collinear with 15 points of $P_{2,2}$. $P_{3,2}$ has length $2^6 3^3 5.7$.

PROOF We prove (ii) first. Corollary (2.5) says that the action of $\mathcal{C}_{\mathcal{C}}(x) = \mathcal{C} \cap \mathcal{C}_x$ in $P_3 \cap Q_x = P_x \setminus [Q_x,z_0]$ is isomorphic to the action of $\mathcal{C}_{\mathcal{C}}(x)$ in $P_1 \setminus [Q,x]$. Here $2^{1+2.4}\cdot 3^3 2^2$ acts with a single orbit in each of the sets of singular points of $\mathcal{C}_M(x) \setminus [M,x]$ - 36 in all - and $M \setminus \mathcal{C}_M(x)$ - 96 in all.

(i) and (iii): The above ensures that $P_{3,2} \neq \emptyset$. From corollary (2.11) any neighbor $x \in P_2$ of a point $w \in P_{3,2}$ must in fact lie in $\mathcal{C}$. Corollary (2.6) implies that the involutions in case 3 of Table 1 are not points. However $w \in \mathcal{C}_{Q_x}(z_0) \setminus [Q_x,z_0]$, whence wz_0 is not a point. Hence, by the corollary again, we have $P_{3,2}$ <u>mod</u> $Q = a_4$ (say), with the conditions of entry <u>4</u> of <u>Table 1</u>. Thus if w is any point of $P_{3,2}$, then $\#P_{3,2} = 16.3780 = 2^6 3^3 5.7 = 8.1575.2.36/\#(P_w \cap P_{2,2})$. This gives the parameters of (<u>iii</u>). Note that a comparison of the size of the sets $P_w \cap P_{2,2}$ and $O_2(\mathcal{C} \cap \mathcal{C}_w) \cap P_{2,2}$ yields containment here. Hence each minimal path joining w and z_0 lies entirely in the elementary abelian group $O_2(\mathcal{C} \cap \mathcal{C}_w)$, of order 2^{11}. □

(<u>2.13</u>)<u>LEMMA</u> <u>Let</u> $\mathfrak{G}$ <u>be of type</u> $\cdot 1$, Γ_0 <u>the connected component of</u> Γ <u>that contains</u> z_0, <u>and</u> $\mathfrak{G}_0$ <u>the normalizer in</u> $\mathfrak{G}$ <u>of</u> Γ_0. <u>The following hold</u>. $\mathfrak{G}_0 = \langle\mathcal{C},\mathcal{D}\rangle$, <u>for any</u> $\mathcal{D} := N_{\mathfrak{G}}(\ell)$, $z_0 \in \ell \in L$. $\mathfrak{G}_0$ <u>is of type</u> $\cdot 1$, $Z^*(\mathfrak{G}_0) = \{1\}$, <u>and</u> Γ_0 <u>is its adjoint geometry</u>.

<u>PROOF</u> Since z_0 is central in a Sylow 2-subgroup of $\mathfrak{G}$, $\Gamma_0 = (P_0, L_0)$ is the unique connected component of $\mathfrak{G}_0$ normalized by z_0. Thus, $\mathcal{C} \le \mathfrak{G}_0$ and so $\mathfrak{G}_0$ is of type $\cdot 1$ - provided that $Z^*(\mathfrak{G}_0) = \{1\}$ (part of the hypothesis (<u>2.1</u>)). In fact $P_{2,4} \subseteq P_0$, so that $z_0^{\mathfrak{G}_0} \supseteq P_1$; whence $P(\mathfrak{G}_0) = z_0^{\mathfrak{G}_0}$, Γ_0 is the adjoint geometry for $\mathfrak{G}_0$, and $Z^*(\mathfrak{G}_0) = \{1\}$. It is now clear that $\mathfrak{G}_0 \ge \langle\mathcal{C},\mathcal{D}\rangle$, whenever $\mathcal{D} = N_{\mathfrak{G}}(\ell)$, ℓ a line on z_0. A Frattini argument gives that $\langle\mathcal{C},\mathcal{D}\rangle = \mathfrak{G}_0$. □

(<u>2.14</u>) <u>Henceforth we assume that</u> Γ <u>is connected, replacing</u> $\mathfrak{G}$ <u>by</u> $\mathfrak{G}_0$ <u>if necessary</u>.

We note that this hypothesis follows from (<u>2.1</u>) and the broad classification [5] of D. Holt of transitive groups in which a 2-central involution fixes a single point. However the connectivity will be used in this section only to establish (<u>2.15.v</u>), whereas Holt's result is immeasurably deeper than this.

(2.15) PROPOSITION Let $\mathfrak{G}$ be of type $\cdot 1$.

(i) The involution geometry Γ extends to a residually connected complex Δ that satisfies the diagram:

D: 

where the vertices of types 1, 2, and 4 are the points, lines and quads, respectively. Those of type 11 are elementary abelian 2-groups of the form $O_2(\mathcal{C}_z \cap \mathcal{C}_{z'})$ where z and z' are commuting points at distance 3 from one another. Call the vertices of type 11 hexes. The normalizer of a hex $\mathcal{K}$ is a split extension of $\mathcal{K}$ by the Mathieu group $\mathfrak{M}_{24}$, and the action it induces on $\mathcal{K}$ is that of $\mathfrak{M}_{24}$ on the (simple factor of the) Golay code. The geometry $\Gamma(\mathcal{K})$ is a near-hexagon on the 759 points of $P(\mathcal{K})$.

(ii) Two commuting points z, z' at distance 3 from one another lie in a unique hex $\mathcal{K}_{z,z'}$. Two distinct hexes intersect in either $\{1\}$, a point, a quad, or else a subgroup that contains no point.

(iii) $\mathfrak{G}$ is flag-transitive on Δ, and the residual geometry for each simplex is isomorphic to the truncation of the classical F_2-geometry that the residual diagram suggests, save for that of a hex, which gives rather a copy of the $\mathfrak{M}_{24}$-geometry over F_2 (cf. [Ronan-Smith 8] for another description of this 2-local geometry). Tacitly we "identify" Δ with the incidence geometry on the vertices of Δ.

(iv) The stabilizers of the simplex σ will be denoted $\mathcal{P}_\sigma$, with $\mathcal{U}_\sigma$ defined to be the kernel of this parabolic on the residual geometry Δ_σ. Fix a flag $\{v_i \mid \mathrm{type}(v_i) = i \in I = \{1,2,4,11\}\}$. The shape of each $\mathcal{P}_i := \mathcal{P}_{\underline{v}_i}$ is given as follows:

$$\mathcal{P}_1 \cong 2^{1+8} \cdot \Omega_8^+(F_2)$$
$$\mathcal{P}_2 \cong 2^{2+2.4} \cdot [\mathcal{L}_2(F_2) \times \mathcal{L}_4(F_2)]$$
$$\mathcal{P}_4 \cong 2^{4+6.2} \cdot [\mathcal{S}_4(F_2) \times \mathcal{L}_2(F_2)]$$
$$\mathcal{P}_{11} \cong 2^{11} \cdot \mathfrak{M}_{24}$$

where $\mathcal{S}_4(\mathbf{F}_2) \leq \Gamma\mathcal{L}_3(\mathbf{F}_4)$ (as the stabilizer of an oval) is a (noncentral) triple cover of $\mathcal{S}_4(\mathbf{F}_2)$. Thus each vertex v may be recovered as $Z(\mathcal{U}_v)$, and $\mathcal{U}_v = \mathcal{S}^*(\mathcal{P}_v)$. The stabilizer of a chamber has a selfnormalizing Sylow 2-subgroup of index 3 which is a Sylow 2-subgroup in all of $\mathfrak{C}$.

(v) I call the sheaf over Δ induced by the points, lines, quads, and hexes (each regarded as a $\mathbf{F}_2$-module for its normalizer) the adjoint sheaf, and denote this by $\mathfrak{J}_{ad}$. I denote $\mathrm{H}_0(\mathfrak{J}_{ad})$ by $\mathfrak{g}$, and refer to this as the adjoint module (cf. [Ronan-Smith 9,10] for details on sheaf homology for geometries). I denote by L_h the set of hyperbolic lines; by $\Gamma_h := (P,L_h)$ the hyperbolic geometry; and by $\Gamma_+ :=$ $(P,L \cup L_h)$, the augmented geometry.

Recall that if $\Gamma' = (P',L')$ is a geometry with lines of size 3, then there is a universal $\mathbf{F}_2\mathcal{A}ut(\Gamma')$-module $H_0(\Gamma') :=$ $\mathbf{F}_2P'/\mathcal{R}'$ generated by the points and subject to the defining relations $\mathcal{R}' := \{R_X := \sum_{\ell\in X} \sum_{z\in\ell} z \mid X \subseteq L'\}$.

Finally, in the following diagram of canonical surjections:

$$\begin{array}{ccccc}
\mathbf{F}_2P & \xrightarrow{\pi} & H_0(\Gamma) & & \\
\downarrow{\scriptstyle \pi_h} & & \downarrow{\scriptstyle \gamma} & & \\
H_0(\Gamma_h) & \xrightarrow{\gamma_h} & H_0(\Gamma_+) & \xrightarrow[-]{\gamma_+} & \mathfrak{g}
\end{array}$$

γ_+ is an isomorphism and $\dim_{\mathbf{F}_2} \ker(\gamma_h) \leq 1$.

PROOF Let $x \in P_{2,2}$ and $w \in P_x \setminus P_{2,2}$ commute with z_0. Such a w is chosen in $\mathfrak{C}_{Q_x}(z_0) \setminus [Q_x,z_0]$, and is forced to be an involution of $\mathfrak{C}$ of type a_4, owing to proposition (2.7) and lemma (2.12). Set $\mathcal{K} := O_2(\mathfrak{C}_{\mathfrak{C}}(w))$. $\mathcal{K}$ has shape 2^{1+4+6} - due to (2.4) - and contains $[Q,w] = \mathfrak{C}_Q(w)$, a maximal elementary abelian subgroup of Q, of order 2^5. $\mathcal{K}/[Q,w]$ is the natural orthogonal module for $N_{\mathfrak{C}}(\mathcal{K})/\mathcal{K} \cong \Omega_6^+(\mathbf{F}_2) \cong \mathcal{L}_4(\mathbf{F}_2)$, and

$\mathcal{K} \setminus [Q,w]$ is composed entirely of involutions: the $2^4.28$ $\mathcal{N}_{\mathcal{C}}(\mathcal{K})$-conjugates of w, the same number of conjugates of wz_0, the $2^3.35$ conjugates of x, and the 24.35 conjugates of xy for any $y \in [Q,w] \setminus [Q,x]$. Thus $\mathcal{K}$ is an elementary abelian subgroup. Set $\mathcal{N} := \mathcal{N}_{\mathcal{C}}(\mathcal{K})$. Observe that since the definition of $\mathcal{K}$ is symmetric in w and z_0, so is the definition of $\mathcal{N}$.

We show next that $z_0^{\mathcal{N}} = \mathcal{P}(\mathcal{K})$. It then follows that $\mathcal{N}$ is irreducible in $\mathcal{K}$; indeed, we show that $z_0^{\mathcal{N}} = z_0^{\mathcal{T}_{\mathcal{K}}}$, and so $\mathcal{T}_{\mathcal{K}} := \langle Q_z | z \in P(\mathcal{K})\rangle$ acts irreducibly in $\mathcal{K}$. To see this note that $P_{3,2}$ is a single $\mathcal{C}$-orbit, so that the orbital $(z_0,w)^{\mathcal{C}}$ is symmetric (or, self-paired). Thus $z_0^{\mathcal{N}} \supseteq w^{\mathcal{N}\cap\mathcal{C}}$. Now $Q \le \mathcal{N}$, so for $x \in P_{2,2} \cap P_w$ as above we have that $x^{Q_w} \subseteq w^{\mathcal{N}\cap\mathcal{C}} \subseteq z_0^{\mathcal{N}}$. This in turn gives the claim, in that that z_0 can be conjugated via $\mathcal{T}_{\mathcal{K}}$ to any point of $\mathcal{K}$ with which it is collinear. It is not difficult to see that $\Gamma(\mathcal{K})$ is connected.

Consider now the complex $\Delta_{\mathcal{K}}$, defined to be the (flag-complex of the) points, lines, and quads that lie in $\mathcal{K}$. The residual geometries in $\Delta_{\mathcal{K}}$ of z_0 and of any line on z_0 are easily checked to be (truncations of) respectively $\mathcal{PG}_4(\mathrm{F}_2)$ and $\mathcal{PG}_3(\mathrm{F}_2)$. (This is done entirely within $\mathcal{C}$ by noting that the lines and quads in $\mathcal{K}$ that lie on z_0 correspond exactly to the point- and line-stabilizers in $(\mathcal{N} \cap \mathcal{C})/\mathcal{K} \cong \mathcal{L}_4(\mathrm{F}_2)$ as this acts in $[Q,w]/\langle z_0\rangle$.) Thus $\Delta_{\mathcal{K}}$ is a flag-transitive complex over F_2 that satisfies the residual diagram D_{11} of D:

D_{11}: 

This complex is plainly embeddable (in $\mathcal{K}$) over F_2, so that the main result of S. Smith [13] yields the identification $\mathcal{N}/\mathcal{K} \cong \mathfrak{M}_{24}$ and $\mathcal{K} \cong_{\mathcal{N}}$ Golay code (i.e., the simple factor of dimension 11 obtained from the span of the octads). We note that this extension is necessarily split, although we could deduce the splitting from that of $\mathcal{N} \cap \mathcal{C}$ over $\mathcal{K}$.

Note that, in $\mathfrak{M}_{24}$, the normalizer of a quad induces $\mathcal{S}_4(F_2)$, with each transvection induced by a point at distance 2 from its center.

The residual connectivity of Δ is a consequence of the connectivity of Γ - cf. (2.14) above.

To prove (ii), we suppose that $\mathcal{H}$ and $\mathcal{H}'$ are hexes, with $P(\mathcal{H} \cap \mathcal{H}') \supseteq z_0$, say. Now $\mathcal{C}$ induces a permutation group on the collection H_{z_0} of hexes containing z_0, with Q acting trivially. $\mathcal{C}$ is transitive on $P_{3,2}$, hence transitive on H_{z_0}. The normalizer of any one of these hexes is, modulo Q, merely the stabilizer of a maximal totally singular subspace of $\mathcal{M}$. Since this action is rank 3, and since there are pairs of distinct hexes in H_{z_0} intersecting variously in $\langle z_0\rangle$ or a quad, we see that these are the only possible intersections for a pair of distinct hexes with a point in common.

The remaining statements of the proposition - save possibly the isomorphism of (v) - now follow straightforwardly from the claims already established, together with the information known on $\Delta_{\mathcal{H}}$, at least once the residual geometry for z_0 in this complex is determined. This can be done by using the correspondence of the lines, quads, and hexes on z_0 with, respectively, the singular-point-, totally-singular-line-, and totally-singular-4-space stabilizers (the 4-spaces corresponding to $w^{\mathcal{C}}$) in $\mathcal{L}$.

For (v) observe first that there are natural surjections:

$$H_0(\Gamma_h) \xrightarrow{\gamma_h} H_0(\Gamma_+) \xrightarrow{\gamma_+} \mathfrak{g},\ z + \mathfrak{R}_h \longmapsto z + \mathfrak{R}_+ \longmapsto im(\mathfrak{J}^{*}_{ad;z}).$$

Thus if $H_0(\Gamma_h) = 0$, then $H_0(\Gamma_+) = \mathfrak{g} = 0$. Assume that $H_0(\Gamma_h) \neq 0$. The relations $\mathfrak{R}_h$ and the flag-transitivity of $\mathfrak{G}$ in Γ_h give that π_h is injective on the set of points in any singular or hyperbolic line, quad, or hex of Γ. Fix a quad $\mathcal{S}$. $R_{\ell} = R_{\ell'}$ mod $\mathfrak{R}_h$, for any singular lines $\ell, \ell' \in L(\mathcal{S})$. The connectivity of Δ now gives that $ker(\gamma_h) = \mathfrak{R}_+\pi_h$ is a trivial module for $\mathfrak{G}$, of dimension at most 1.

To finish (v) consider $\mathscr{S}' := P(\mathscr{S})\pi_h\gamma_h \cup \{0\} \le H_0(\Gamma_+)$. Now $\mathscr{S}'$ is closed under addition: $zz' \in P(\mathscr{S})$ and $zz' = z + z'$ mod $\mathfrak{R}_+$, for distinct z and z' of $P(\mathscr{S})$ (use (2.7)). Thus $\mathscr{S}'$ and $\mathscr{S}$ are isomorphic $N_{\mathfrak{C}}(\mathscr{S})$-modules. Finally, the argument above that provides the hexes of Δ can be repeated to complete a proof that the sheaf $\mathfrak{J}_{ad}$ has a nontrivial image in the constant sheaf $H_0(\Gamma_+)$; use this map to invert γ_+. □

In order to finish the enumeration of the $\mathcal{C}$-orbits in P we need the following.

(2.16) PROPOSITION Consider an $\mathfrak{U}_9 \le \mathcal{C}$ that normalizes a set of 9 mutually nonperpendicular singular points in $\mathcal{M}$. If $\mathfrak{U} \le \mathcal{C}$ is a natural $\mathfrak{U}_5$-subgroup of this $\mathfrak{U}_9$ then $\mathfrak{C}_{\mathfrak{U}} := \mathcal{C}_{\mathfrak{C}}(\mathfrak{U}) \cong HJ$, the simple group of M. Hall and Z. Janko.

We first prove a short lemma.

(2.17) LEMMA Consider a path $z_0 - y - x - w - v$, where $x \in P_{2,4}$, $w \notin \mathcal{C}_y$, $v \in \mathcal{C}_x \setminus \langle w,x\rangle$. We then have $w \in P_{3,3}$, and $v \in P_{4,5}$.

PROOF OF LEMMA We have $w \in P_{3,3}$ by (2.11).

The a_2 involution vQ_x in $\mathfrak{L}_x$ normalizes a unique 2-space in $\mathcal{M}_x$ and containing $y\langle x\rangle$, and vQ_x induces a transvection there. Thus v normalizes a unique quad $\mathscr{S}$ on $\langle x,y\rangle$, and is collinear with a unique point w' of $\mathfrak{m} = \langle x,[y,v]\rangle$. As v is connected to $\mathscr{S}$ it induces the $\mathfrak{U}_6$-involution with singular center w' and axis $\mathfrak{m}$. On the other hand z_0 is connected to $\mathscr{S}$ and induces the $\mathfrak{U}_6$-involution with axis $\ell \neq \langle x,y\rangle$ and singular center y. As ℓ and $\mathfrak{m}$ are disjoint, $e = (vz_0)^5$ centralizes $\mathscr{S}$ (cf. the Appendix).

Now $z = y^{vz_0} = w'^{z_0v}$ is collinear with neither y nor w', and $e = v^{z_0v}.z_0^{vz_0} \in Q_z$ (as in the Appendix). Unless e is 1, z is the unique point z' of $\mathscr{S}$ with $e \in Q_{z'}$. Neither z_0 nor v centralize z, so $e = 1$ and $|vz_0| = 5$.

By (2.11), v cannot be in P_1 or P_3, whence $v \in P_4$. □

<u>PROOF OF PROPOSITION</u> First note that there is exactly one $O_8^+(F_2)$-class of sets of points that are maximal, with respect to inclusion, amongst the sets of mutually nonperpendicular points. The normalizer of any such set is isomorphic to $\mathfrak{A}_9$ (use Witt's Theorem on extending partial isometries).

Next observe that the centralizer $\mathcal{A}$ of a corner $z' - z - z''$ at z of width 4 is the centralizer in $\mathcal{C}_z$ of a subgroup $\mathcal{D}_8$ of Q_z, hence has shape $2^{1+6}\Omega_6^+(F_2)$. $O_2(\mathcal{A})$ acts simply transitively on each of $P_{z'} \setminus \mathcal{C}_z$ and $P_{z''} \setminus \mathcal{C}_z$. Now a point of either set can be viewed to correspond to the subgroup $\Omega_6^+(F_2)$ stabilizing the point; abstractly this is a complement in $\mathcal{A}$ to $O_2(\mathcal{A})$. There are in fact two classes of complements to $O_2(\mathcal{A})$ [Pollatsek 7, (5.2), p. 415]. The respective representatives of the classes act on the 64 lines on z' <u>not</u> perpendicular to $\langle z,z'\rangle$ - as well as the 64 for $\langle z,z''\rangle$ - with orbit decompositions of 1 + 28 + 35 and 8 + 56, respectively. A consequence of this proposition is that (to be proved in the proof of (<u>2.19</u>) below) is that a complement that fixes a point in $P_{z'} \setminus \mathcal{C}_z$ <u>does</u> <u>not</u> fix a point in $P_{z''} \setminus \mathcal{C}_z$.

Now $\mathcal{D} := \mathcal{C}_{\mathfrak{G}_{\mathfrak{U}}}(z_0) = \mathcal{C}_{\mathcal{C}}(\mathfrak{U}) = 2_-^{1+4}\Omega_4^-(F_2)$, with $\mathcal{F}^*(\mathcal{D}) = \mathcal{R} := Q \cap \mathfrak{G}_{\mathfrak{U}} = 2_-^{1+4}$. The copy of $\mathfrak{U}_5$ chosen centralizes a path $z_0 - y - x - w$, where $x \in P_{2,4}$ and $w \in P_{3,3}$. In particular, $z_0^{\mathfrak{G}_{\mathfrak{U}}} \cap \mathcal{R} \neq \langle z_0\rangle$. That is, $\mathfrak{G}_{\mathfrak{U}}$ is of <u>type</u> $\mathcal{HJ}/\mathcal{J}_3$. An old result of Z. Janko [6] asserts that one of the following must hold:

$\mathfrak{G}_{\mathfrak{U}} \cong \mathcal{HJ}$; <u>or</u>

$\mathfrak{G}_{\mathfrak{U}}$ has exactly one class of involutions (and $\mathfrak{G}_{\mathfrak{U}}$ has the character table of $\mathcal{J}_3$).

The second case is impossible since the involutions of $\mathcal{D} \setminus \mathcal{R}$ are of type c_2 on $\mathcal{M}$, and (<u>2.6</u>) rules out these involutions as points on $\mathfrak{G}$. This finishes the proposition. □

For future reference we list the $\mathcal{D}$-orbits in $P_{\mathfrak{U}} := z_0^{\mathfrak{G}_{\mathfrak{U}}}$ and the parameters for the lines between them.

Table 4

The point suborbits for $\mathcal{HJ}$

z_0	2.5 — 1	P_1	2.4 — 1	$P_{2,4}$	2.4 — 4	$P_{3,3}$	2.1 — 5	$P_{4,5}$
1		10		80		160		64

(2.18) PROPOSITION Let $\mathcal{G}$ be of type $\cdot 1$.

(i) Let $w \in P_{3,4}$. There is a unique quad $\mathcal{Q}'$ on w normalized by z_0. $\mathcal{Q}'$ contains a unique point of P_1.

(ii) P_3 breaks up into precisely 3 $\mathcal{C}$-orbits (all self-paired):

$P_{3,2}$, of length $2^6 3^3 5.7$; each of these points is collinear with exactly 15 points of $P_{2,2}$; and each has centralizer in $\mathcal{C}$ of shape $2^5{\cdot}2^6\mathcal{S}_4(\mathbb{F}_2)$;

$P_{3,3}$, of length $34560.2.64/9 = 2^{15}3.5$; each of these points is collinear with exactly 9 points of $P_{2,4}$, with the remaining lines connecting to $P_{4,5}$; and each has centralizer in $\mathcal{C}$ isomorphic to $\mathfrak{A}_9$; this centralizer acts naturally on these 9 neighbors in $P_{2,4}$;

$P_{3,4}$, of length $2^9 3^3 5^2 7$; each of these points (v, say) is collinear with exactly 1 point of $P_{2,2}$ and 2 points of $P_{2,4}$, and lies together with these neighbors in the unique quad on v that is normalized by z_0; and the centralizer in $\mathcal{C}$ of v has shape $2^{2+4.2}\Omega_4^+(\mathbb{F}_2)$.

(iii) Say a point z is connected to a quad $\mathcal{Q}$ in case z is collinear with some point z' of $\mathcal{Q}$. In (2.7.vi) we noted that all the points connected to a given quad normalize the quad. Now each point of $P_3 \setminus P_{3,3}$ lies in a quad to which z_0 is connected. Each point of $P_{3,3}$ or P_4 is connected to a quad to which z_0 is connected.

<u>PROOF</u> This proposition is really a compilation and refinement of the results on P_3 contained in (<u>2.11</u>) and (<u>2.12</u>), and one should have these statements firmly in mind as one reads the following. In fact the bulk of the proof is devoted to proving the statements about $P_{3,3}$. It might be worthwhile to look ahead at this point to <u>Table 5</u> at the end of the section. This gives a global view of the $\mathcal{C}$-orbits in P.

(<u>i</u>): Let $w \in P_{3,4}$. Corollary (<u>2.11</u>) says that w is collinear with $x \in P_2$ where either $x \in \mathcal{C}$ and w induces an $\mathfrak{U}_6$-involution on $\mathcal{S} = \mathcal{S}_{x,z_0}$, or else $y = (xz_0)^2 \in P_1$ and $w \in \mathcal{C}_y$. Consider the first possibility. In this case $w \in Q_x$ acts nontrivially on the unique quad $\mathcal{S}$ on $\langle z_0, z_0^w\rangle \leq Q_y$. Moreover w centralizes a unique line ℓ in $\mathcal{S} \cap Q_x$. Necessarily $\ell = \langle y', x\rangle$ for some y' of $P_x \cap Q$. Thus, z_0 induces an $\mathfrak{U}_6$-involution on $\mathcal{S}_{y',w}$. The first possibility reduces to the one just considered by reversing the roles of w and z_0.

If z is the point of $\ell = \mathcal{S} \cap \mathcal{S}'$ that is collinear with neither w nor z_0, then a point $z' \in P_z \setminus \mathcal{C}$ must normalize $\mathcal{S}$, $\mathcal{S}'$, and ℓ, and conjugate z_0 to one of the 4 points of $\mathcal{S}'$ collinear with yz. If this is not w one can replace z' by another point of P_z or by a product of this point with a point of $\mathcal{C}_z$ that induces a transvection on $\mathcal{S}'$ with center z and obtain thereby a point that conjugates z_0 to w.

We have shown that any point of $P_2 \cap P_w$ lies in the quad $\mathcal{S}_{w,w^{z_0}}$, perforce the unique quad on w that is normalized by z_0. This finishes (<u>i</u>) and yields the parameters for $P_{3,4}$ in (<u>ii</u>).

We will derive the statements about $P_{3,3}$ simultaneously with (<u>iii</u>). This is all that remains to prove. The combination of (<u>2.11</u>), (<u>2.12</u>), and the first part of this proof show that there are but 3 $\mathcal{C}$-orbits in P_3, and provide the statements in (<u>ii</u>) regarding $P_{3,2}$ and $P_{3,4}$.

Consider the set of paths $z_0 - y - x - w$, for a fixed $w \in P_{3,3}$. The stabilizer of any one such path is a copy of $\mathfrak{U}_8$, as was noted in (2.15). In the same proof we saw that the set of lines $\langle x,w \rangle$, for $x \in P_{2,4}$ on one of these paths, determines a set of mutually nonperpendicular points in $\mathcal{M}_w$. Thus there are at most 9 such paths, and in any $\mathfrak{C} \cap \mathfrak{C}_w$-orbit of these paths, a point stabilizer is induced by a copy of $\mathfrak{U}_8$. However, by (2.15), there are at least 4 of these paths. We conclude that there are exactly 9, with $\mathfrak{C} \cap \mathfrak{C}_w \cong \mathfrak{U}_9$ acting naturally.

Finally, any $v \in P_w$ must centralize some $x \in P_w \cap P_{2,4}$. From (2.16) we conclude that $P_w \subseteq P_{2,4} \cup P_{3,3} \cup P_{4,5}$. We have observed that each such point is connected to a quad to which z_0 is connected. This finishes the proof. $\square$

(2.19) PROPOSITION *Let $\mathfrak{G}$ be of type $\cdot 1$.*

(i) *If $v \in P_w$ for some $w \in P_{3,2}$, then either v is contained in the unique hex on w and z_0; or else $v \in P_{4,4}$. In the latter case $x := [v,z_0] = (vz_0)^2 \in P_w \cap P_{2,2}$, whence $w \in \mathcal{S}_{v,x}$, the unique quad on v that is normalized by z_0.*

(ii) *Let $v \in P_w$ for some $w \in P_{3,4}$. Set $x := P_w \cap P_2$ and $\mathcal{S}$ to be the unique quad on w that is normalized by z_0. Assume $v \notin \langle w,x \rangle$. Exactly one of the following three cases can occur.*

v does not centralize x, and $v \in P_{4,5}$;

v does centralize x, but not $\mathcal{S}$, and exactly one of v and vw lies in $P_{4,3}$, while the other lies in $P_{4,6}$;

or

v does centralize $\mathcal{S}$, and $v \in P_{4,4}$.

(<u>iii</u>) P_4 <u>consists of exactly four</u> $\mathcal{C}$-<u>classes</u>:

$P_{4,5}$, <u>of length</u> $2^{16}3^{3}7$; <u>each of these points has</u> 10 <u>neighbors in</u> $P_{3,3}$, 25 <u>in</u> $P_{3,4}$, <u>and</u> 200 <u>in</u> $P_{4,6}$;

$P_{4,3}$, <u>of length dividing</u> $2^{14}5.7.135$;

$P_{4,6}$, <u>of length</u> $2^{14}3^{2}5^{2}7$; <u>each point here has</u> 3 <u>neighbors in each of</u> $P_{4,3}$ <u>and</u> $P_{3,4}$, 96 <u>in</u> $P_{4,5}$, <u>and</u> 36 <u>in</u> $P_{4,4}$;

<u>and</u>

$P_{4,4}$, <u>of length</u> $2^{10}3^{3}5^{2}7$; <u>each points here has</u> 3 <u>neighbors in</u> $P_{3,2}$, 192 <u>in</u> $P_{4,6}$, <u>and</u> 36 <u>in</u> $P_{3,4}$.

(<u>iv</u>) <u>If</u> $v \in P_{4,6}$ <u>then there are</u> $2^{5}3$ <u>lines on</u> v <u>that contain</u> 1 <u>point of</u> $P_{4,5}$ <u>and</u> 2 <u>from</u> $P_{4,6}$; <u>and</u> $2^{2}3^{2}$ <u>that contain</u> 1 <u>point from</u> $P_{4,4}$ <u>and</u> 2 <u>from</u> $P_{4,6}$.

<u>PROOF</u> We show first that if $v \in P_4$, then vz_0 is one of exactly four possibilities, with digressions to finish off (<u>i</u>) and (<u>ii</u>).

From (<u>2.18</u>) we may choose a quad $\mathcal{S}$ to which both v and z_0 are connected. Say $y \in P_1 \cap \mathcal{S}$ and $w \in P_v \cap \mathcal{S}$. Assume first that both v and z_0 act nontrivially on $\mathcal{S}$. If their product induces a 5-cycle in $\mathfrak{U}_6$ then in fact $v \in P_{4,5}$, as noted. Otherwise their product induces a 3-element with fixed points in $\mathfrak{U}_6$ (a 3-cycle, as in the <u>Appendix</u>), and moreover $\mathcal{C}_{\mathcal{S}}(v,z_0)$ consists of a single point x, say. In $\mathcal{C}_x$ it is easy to check that $\langle (vz_0)^3,(vwz_0)^3 \rangle = \langle x \rangle$. This finishes this first case and also gives (<u>ii</u>). Moreover this says that if $v \in P_{4,6}$ then $x = (vz_0)^3$ and v generate a hyperbolic line; and the quad $\mathcal{S}_{v,x}$ contains all of the neighbors of v in $P_{3,4}$. Thus $\#P_{4,6} = 2^{9}3^{3}5^{2}7.2.2^{5}/3$.

If $\mathcal{C}_{\mathcal{G}}(\mathcal{S})$ contains either v or z_0 then we claim that $|vz_0| = 4$. We establish this together with (<u>i</u>). Say z_0 centralizes $\mathcal{S}$, and let $\mathcal{H}$ be the hex on w and z_0. Now $z_0 \in \mathcal{C}_w$ is of type a_4, so that $v' := v^{z_0} \in Q_W$ and v' commutes with v. Since $\mathcal{H} \geq \mathcal{C}_{Q_w}(z_0)$, $v = v'$ would imply $v \in \mathcal{H}$. Rather it must be that $x := vv' = [v,z_0] \in [Q_w,z_0] \le \mathcal{H}$. Thus $\mathcal{S}' := \mathcal{S}_{v,x}$ is the unique quad on v that is normalized by z_0. Moreover, v induces one of the 2-central $\mathfrak{M}_{24}$-involutions in $\mathcal{H}_{w,z_0}$, whence $x = z_0^{\,v}.z_0$ is at distance 2 from z_0, and collinear with w.

As this argument shows, if v is <u>any</u> point collinear with w but not in $P_{2,2} \cup P_{3,2}$, then $v \in P_{4,4}$.

From the above $v \in P_{4,4}$ is connected to precisely 3 points of $P_{3,2}$ - <u>viz</u>. the 3 neighbors of v in $\mathcal{S}'$ - whence $\#P_{4,4} = 2^6 3^3 5.7.240/3$. Assume now that $z \in P_v$, but z is not in $\mathcal{S}'$. Either z centralizes $\mathcal{S}'$ or else z induces an $\mathfrak{U}_6$-involution with v as singular center. Since the center x of z_0 (z_0 as a transvection on $\mathcal{S}$) is not collinear with v, $|zz_0|$ is a multiple of 6 whenever $z \notin \mathcal{C}_{\mathcal{C}}(\mathcal{S})$.

Now count neighbors to see that each point $v \in P_{4,4}$ is connected to $2^2 3^2$ points of $P_{3,4}$ and $2^6 3$ points of $P_{4,6}$; each point $v \in P_{4,6}$ is connected to $2^2 3^2$ points of $P_{4,4}$; while each point $w \in P_{3,4}$ is collinear with $2^3 3^2$ points of $P_{4,4}$, 2^5 from each of $P_{4,3}$ and $P_{4,6}$, and 2^7 from $P_{4,5}$. For example, if $v \in P_{4,4}$, set $x = (vz_0)^2$, and $\mathcal{S} = \mathcal{S}_{v,x}$. Now $\mathcal{C} \cap \mathcal{C}_v = 2^{5+6}3^2$ has just 3 orbits on L_v: the 3 in $\mathcal{S}$, the 36 not in $\mathcal{S}$ that centralize $\mathcal{S}$, and the 96 that do not centralize $\mathcal{S}$. On the other hand if $w \in P_{3,4}$ then any neighbor of wthat centralizes $\mathcal{S}_{v,v^{z_0}}$ must lie in $P_{4,4}$. Hence, the 36 lines of L_v that do not lie in $\mathcal{S}$ but that centralize $\mathcal{S}$ must contain one point each from $P_{3,4}$. We claim that the remaining neighbors of v - those that do not centralize $\mathcal{S}$ - all lie in $P_{4,6}$. Again it is enough to see that w has at least one neighbor in $P_{4,6}$.

For this begin with the observation that v^{z_0} and x^u are collinear points of $\mathcal{S}$, whenever $u \in P_w$ acts nontrivially on $\mathcal{S}$ (<u>cf</u>. the <u>Appendix</u>). Set $w = v^{z_0}.x^u$; this is a point of $P_{3,2}$ that is collinear with each of v and x, and is centralized by u as well as z_0. Hence there is a quad $\mathcal{S}'$ on u and a hex $\mathcal{H}'$ on z_0 whose intersection contains w.

In Δ_w, then, we can find a unique hex $\mathcal{H}''$ containing $\mathcal{S}'$ whose intersection with $\mathcal{H}'$ is a quad $\mathcal{S}''$. Now $z_0 \in \mathcal{H}'$ must be connected to $\mathcal{S}''$, as must be $u \in \mathcal{H}''$. Necessarily, $u \in P_{4,6}$, as we have seen in this case.

With this one calculates that the centralizer $\mathcal{C} \cap \mathcal{C}_v$ of a point v of $P_{4,6}$ - lying in the normalizer of both of the hyperbolic lines $\langle v,x\rangle$ and $\langle x,z_0\rangle$ - has shape $3 \times 2^{1+4}[\mathcal{L}_2(\mathsf{F}_2)\times \mathcal{L}_2(\mathsf{F}_2)]$ and hence has exactly 3 orbits in Γ_v: those of lengths 3 and 36 just mentioned, and the one of length 96 consisting of points that do not centralize $\mathcal{S}_{v,x}$. Each of these lines contains, besides v, a point each of $P_{4,5}$ and $P_{4,6}$.

Now counting the points of $P_{4,5}$ in three ways one sees that $\#P_{4,5} = 2^{16}3^3 5.7/n$, say, where each point $v \in P_{4,5}$ is collinear with $2n$ points of $P_{3,3}$, $5n$ of $P_{3,4}$, and $2^3 5n$ of $P_{4,6}$. If v is collinear with $w \in P_{3,3}$, then $\mathcal{C}_{\mathfrak{C}}(v,w,z_0) \cong \mathfrak{A}_4 \times \mathfrak{A}_5 \leq \mathfrak{A}_9 \leq \Omega_8^+(\mathsf{F}_2)$, stabilizing a (4,5)-partition of the 9-set. Since n is at least 1, 2 divides the index of $\mathcal{C}_{\mathfrak{C}}(v,w,z_0)$ in $\mathcal{C}_{\mathfrak{C}}(v,z_0)$. This leaves only a subgroup $\mathfrak{A}_5 \wr 2$ as a possibility for $\mathcal{C} \cap \mathcal{C}_w \leq \Omega_8^+(\mathsf{F}_2)$, whence $n = 5$.

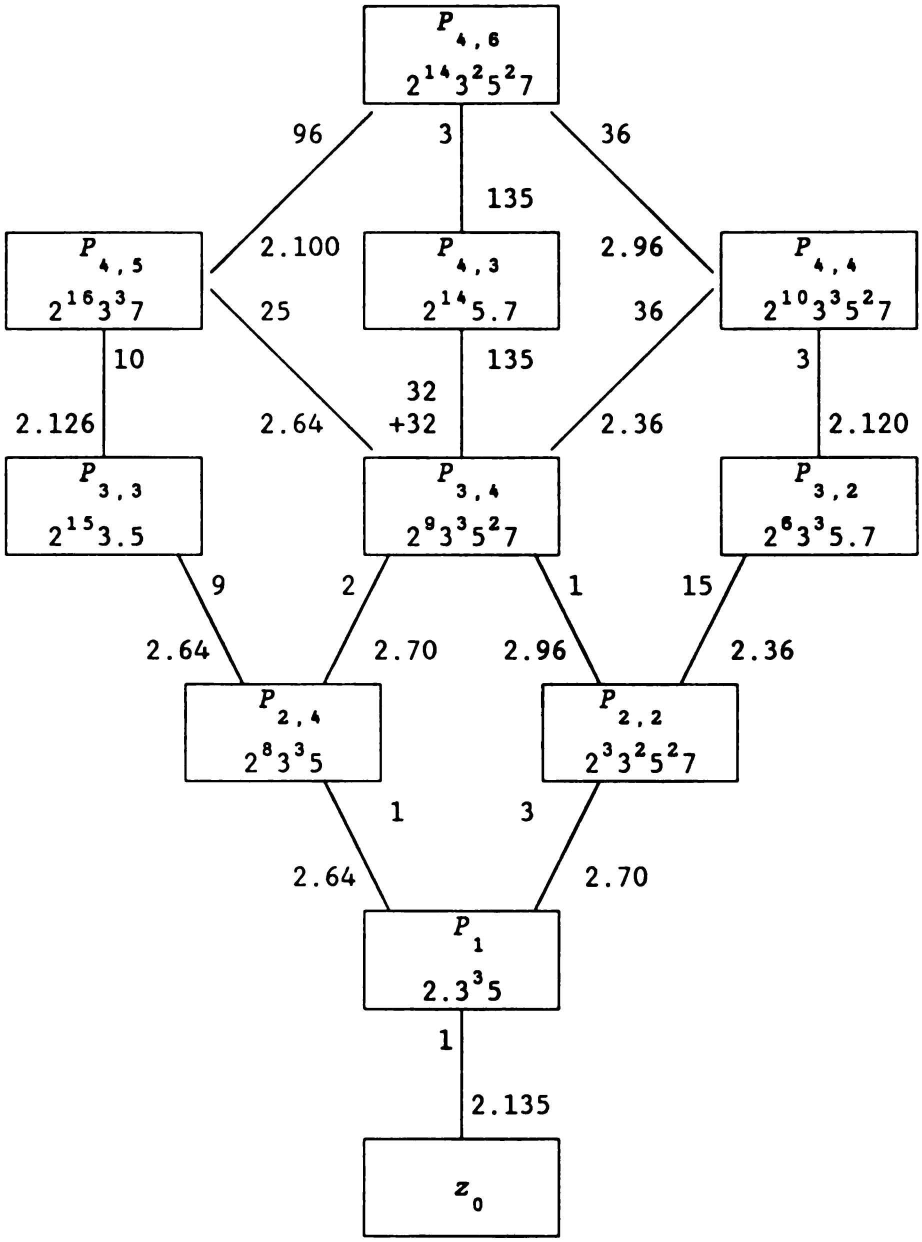
The fact that $n = 5$ implies that if $\langle v,v'\rangle$ is a line, with $v \in P_{4,6}$ and $v' \in P_{4,5}$, then $vv' \in P_{4,6}$. This finishes the last of the $P_{4,6}$ parameters, and thus the proposition. □

(<u>2.20</u>)<u>REMARKS</u>(<u>i</u>) A consequence of the results of this section is that if $\mathcal{S}$ is a fixed quad, $\mathcal{N}$ is its normalizer, and $P_{\mathcal{S}}$ is the set of points connected to $\mathcal{S}$, then the Hecke algebras $\mathcal{E}nd_{\mathcal{N}}(\mathsf{F}_2 P_{\mathcal{S}})$ and $\mathcal{E}nd_{\mathfrak{C}}(\mathsf{F}_2 P)$ are isomorphic (<u>as F_2-spaces</u>).

(<u>ii</u>) It can be shown that in $\cdot 1$ each line on a point $v \in P_{4,3}$ contains 1 point each of $P_{4,6}$ and $P_{3,4}$. This gives the remaining parameters for Γ, and demonstrates that Γ has diameter 4. Moreover if $K_s := P_{4,3} \cup P_{4,6}$, then K_s has the property that each line that contains a point of K_s in fact contains exactly 2. This will be noted again in the next section.

Table 5

The $\mathcal{C}$-orbits in P



3. THE ADJOINT REPRESENTATION FOR GROUPS OF TYPE $\bullet 1$ (AND RELATED REPRESENTATIONS)

Throughout this section $\mathfrak{G}$ remains a group of type $\bullet 1$ (cf. (2.1)), with the assumption (2.14) that its adjoint geometry is connected.

Retain the notation of §§1-2. Much of this is given in (1.1), (1.4), and (2.15). Throughout we will mean by $S^2(\mathcal{A})$ the subspace of $\mathcal{E}nd(\mathcal{A})$ consisting of symmetric matrices ~ as opposed to the appropriate quotient of $\mathcal{E}nd(\mathcal{A})$. Although this involves a choice of basis, $S^2(\mathcal{A})$ will arise only in situations where the action theron is induced from a symplectic representation.

(3.1) DEFINITIONS AND NOTATION We give another description of the homology module $H_0(\Gamma')$ for $\Gamma' = (P', L')$ a geometry where the lines have 3 points each (cf. (2.15.v)). Consider the ($\mathbb{F}_2\mathcal{A}ut\Gamma'$-)permutation modules $\mathbb{F}_2P'$ and $\mathbb{F}_2L'$. Define maps:

$$\mathbb{F}_2L' \xrightarrow{\sigma} \mathbb{F}_2P' , \; l \longmapsto \sum_{p\in l} p \;, \quad \text{and} \quad \mathbb{F}_2P' \xrightarrow{\sigma^*} \mathbb{F}_2L' , \; p \longmapsto \sum_{l\ni p} l \;.$$

Identify each of these modules with its dual through the usual inner product; this identifies σ^* with the dual of σ, as the notation suggests. In this setting, $H_0(\Gamma') = \mathit{coker}(\sigma^*)$. The surjectivity of σ is equivalent to the injectivity of σ^*, in turn equivalent to the nonexistence of a set $K' \neq \emptyset$ of points such that every line meets K' evenly. $P' \setminus K'$ is an example of a hyperplane section, as considered by M. Ronan. We use this in the following.

(3.2) HYPOTHESIS $H_0(\Gamma_h) \neq 0$.

(3.3) REMARK Using (2.15) we conclude from this hypothesis that $\mathfrak{g} = H_0(\mathfrak{J}_{ad}) \neq 0$. At least in $\bullet 1$, the set $P \setminus K_s = P \setminus (P_{4,3} \cup P_{4,6})$ is a hyperplane section for Γ, as was mentioned in (2.20). Thus $H_0(\Gamma) \neq 0$ in this case. What remains to do is establish the existence of a hyperplane section $P \setminus K_h$ for the hyperbolic geometry Γ_h. We could then apply (2.15) to conclude that $\mathfrak{g} \neq 0$.

Whatever K_h might be, it is <u>not</u> K_s. In fact, there is <u>no</u> $\mathcal{C}$-invariant hyperplane section for the hyperbolic geometry, as we show in (<u>3.11</u>) below.

The anomoly of having two distinct hyperplane sections is not unusual in geometries where there are both 'singular' and 'hyperbolic' lines. For example the symplectic spaces over F_2 (or, analogously, any field of even characteristic) have as hyperplane sections, in addition to the linear hyperplanes of subspaces perpendicular to a given point, the set of points of an orthogonal quadric. These are 'linear' only in the universal (homology) module for the hyperplane section ~ <u>i</u>.<u>e</u>., the orthogonal modules of 1 dimension greater.

The near-hexagon on the octads for $\mathfrak{M}_{24}$ has a hyperplane section consisting of the set of all octads at distance no greater than 2 from a given octad. However, the representation of the near-hexagon in the 11-dimensional factor of the Golay code does not realize these hyperplane sections in linear hyperplanes.

(<u>3.4</u>)<u>DEFINITIONS</u> <u>AND</u> <u>NOTATION</u> Let S be the set of quads and H the set of hexes. The shadow geometry $\Delta(H) := (H, \{H_{\mathcal{S}} | \mathcal{S} \in S\})$ over H is a partial linear space, where we regard as *H-lines* the sets $H_{\mathcal{S}} := H \cap \Delta_{\mathcal{S}}$ for $\mathcal{S} \in S$. The H-lines contain 3 points each.

Fix a hex $\mathcal{H}_*$ and define K_* as the set of all hexes $\mathcal{H}$ that lie on paths of the form $\mathcal{H}_* - z - \mathcal{H}' - \mathcal{S}' - \mathcal{H}$, where $\langle z\rangle = \mathcal{H}_* \cap \mathcal{H}'$ is a point, $\mathcal{S}' = \mathcal{H}' \cap \mathcal{H}$ is a quad, but $\mathcal{S}' \cap Q_z = \{1\}$. K_* can be shown to be the complement to the unique $N_* := N_{\mathfrak{G}}(\mathcal{H}_*)$-invariant hyperplane section of $\Delta(H)$. For $\mathfrak{G} = \cdot 1$, K_* corresponds to the set of coördinate frames of the Leech lattice Λ, taken <u>modulo</u> 2, that are <u>not</u> perpendicular (<u>modulo</u> 2) to the frame (that corresponds to) $\mathcal{H}_*$. Verifying that $H \setminus K_*$ is indeed a hyperplane section is tantamount to sorting the hexes into N_*-orbits, of which there are 6: one at distance 1 from $\mathcal{H}_*$, and two each at distances 2 and 3.

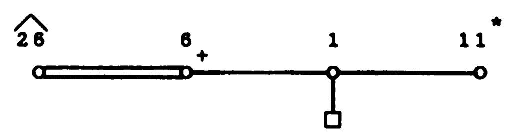
Rather than establish the nonvanishing of the module $H_0(\Delta(H))$ as above, we show below how (3.2) implies that this module is nonzero. This approach involves only some elementary local calculations in various homology modules.

Y. Segev has studied the geometry $\Delta(H)$ extensively. He uses elegant geometric arguments to obtain delicate information about the geometry. He then constructs a concrete isomorphism of $\Delta(H)$ with the $\cdot 1$-geometry on the coördinate frames of Λ. A corollary to his work is the existence of hyperplane sections in $\Delta(H)$. There is some obvious overlap of our projects, although our aims are somewhat different.

I thank Dr. Segev for useful conversations that we had at the Noordwijkerhout meeting 'Groups and Geometries', and for providing me with a draft of his paper while I prepared this manuscript.

(3.5) LEMMA (i) Δ supports a sheaf $\mathfrak{F}_S$ for $\mathfrak{G}$ as follows:

$\mathfrak{F}_S$:



where $\widehat{26}$ denotes the homology of the sheaf of fixed-points associated to the simple factor of $\Lambda^2(\mathcal{M})$; 6_+ denotes an orthogonal module for the factor $\Omega_6^+(F_2) \cong \mathcal{L}_4(F_2)$ in $\mathcal{P}_2$, embedded in $\Lambda^2(\mathcal{M})$ as the exterior square of the $\mathcal{P}_2$-invariant totally-singular subspace of $\mathcal{M}$; 11^* denotes the dual of the (simple) Golay code 11 for $\mathfrak{M}_{24}$; and 11^* intersects 6_+ in a totally-singular 3-dimensional sub-space, and intersects $\widehat{26}$ in a 6-dimensional orthogonal subspace.

(ii) If $\mathcal{E} := \Lambda^2(\mathfrak{g})$ then there is a nonzero $\mathfrak{G}$-map $\mathfrak{F}_S \xrightarrow{\psi} \mathfrak{K}_{\mathcal{E}}$, the constant sheaf for $\mathcal{E}$. In particular, $H_0(\mathfrak{F}_S) \neq 0$.

PROOF We use the term S-point to mean an element of S; an S-line is a shadow $S_{\ell,\mathcal{H}} := S \cap \Delta_{\ell,\mathcal{H}}$, where $\ell \in L(\mathcal{H})$ and $\mathcal{H} \in H$. Notice that the partial linear space $\Delta(S)$ has 'lines' in correspondence with the 7 points of $\mathcal{PG}_3(\mathbb{F}_2)$, and so a universal $\mathbb{F}_2\mathfrak{G}$-representation of $\Delta(S)$ is not exactly as in (2.15.v) or (3.4).

Given the definition of $\mathfrak{F}_S$ above for the faces of a fixed chamber of Δ it is not hard to check that the definition extends equivariantly to all of Δ (cf. [Ronan-Smith 10, (4.2), p. 142]).

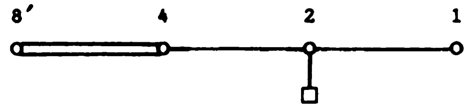
We finish by showing that $\mathfrak{F}_S$ has a nontrivial image in $\mathfrak{X}_{\mathcal{E}}$. The point is that $H_0(\mathfrak{F}_S)$ is the limit of the system $\mathfrak{F}_S$, so that $\mathfrak{F}_S \longrightarrow \mathcal{E} \cong H_0(\mathfrak{X}_{\mathcal{E}})$ must factor through $H_0(\mathfrak{F}_S)$. See the Reciprocity of [Ronan-Smith 10, (1.2), p. 139]: $\mathcal{H}om(\mathfrak{F},\mathfrak{X}_{\mathcal{X}}) \cong \mathcal{H}om_{\mathfrak{G}}(H_0(\mathfrak{F}),\mathcal{X})$ for $\mathfrak{F}$ any sheaf over Δ and $\mathcal{X}$ any $\mathbb{F}_2\mathfrak{G}$-module.

Recall that (2.15.v) allows us to regard $\mathfrak{g}$ as generated by P, subject to the defining relations determined by L and L_h. Fix $\mathcal{S} \in S$. The element $z_1 \wedge z_1' + z_2 \wedge z_2'$ is the central element of $\Lambda^2(\mathcal{S})$ whenever the two pairs $\{z_i, z_i'\}$ are orthogonal hyperbolic pairs. Denote this by $\zeta_{\mathcal{S}}$. Now if $\mathcal{H}$ is a hex and $\mathcal{S} \le \mathcal{H}$ is a quad, then $N_{\mathfrak{G}}(\mathcal{S},\mathcal{H})$ induces a group $\mathfrak{G} \cong 2^6\mathcal{S}_4(\mathbb{F}_2) \le \mathfrak{M}_{24}$ such that $\mathcal{H}$ is uniserial for $\mathfrak{G}$: $\mathcal{S} \le \mathcal{H}_{\sim} \le \mathcal{H}$, where $\mathcal{H}_{\sim}$ is the linear hyperplane containing the points of $\mathcal{H}$ that are connected to $\mathcal{S}$. Thus $\mathfrak{G}$ has a single fixed point in $\Lambda^2(\mathcal{H})$. Perforce this is $\zeta_{\mathcal{S}}$, which then lies in a copy of $\mathcal{H}^* \le \Lambda^2(\mathcal{H})$ (the dual of the kernel of the cocycle for the extension of $\mathcal{H}^*$ over $\mathcal{H}$ that lives in the permutation module of dimension 24).

The remaining terms of the subsheaf of $\mathfrak{X}_{\mathcal{E}}$ can be constructed straightforwardly from the terms for the quads and hexes. See the proof of (3.8) below. Note that it is unimportant what actually appears in $\mathcal{E}$ at the vertices P of Δ: $\mathfrak{F}_S$ will map to $\mathfrak{X}_{\mathcal{E}}$ in any case. $\square$

(3.6) LEMMA (i) $H_0(\Delta(H)) \neq 0$.

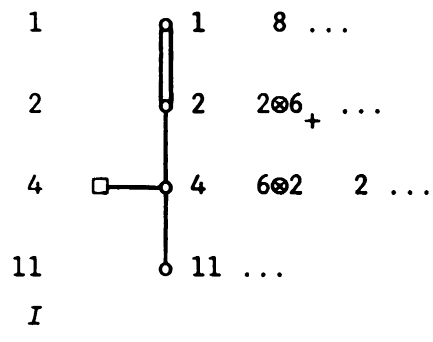
(ii) *In each* $H_0(\Delta(H))^{\mathfrak{U}}\sigma$, $\sigma \in \Delta$ *there is the term* $\mathfrak{J}_{\Lambda;\sigma}$ *of the following sheaf over* Δ:

$\mathfrak{J}_\Lambda$: 

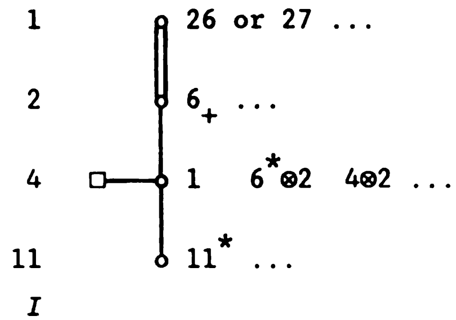
Here 1 *denotes a trivial module; while* 2 *and* 4 *denote natural modules of these dimensions for the respective parabolic factors isomorphic to* $\mathcal{L}_2(\mathbf{F}_2)$ *and* $\mathcal{L}_4(\mathbf{F}_2)$; 8′ *indicates an orthogonal module that is twisted by a triality automorphism from the orthogonal module* $\mathcal{M}$; *and* 4 *is contained in* 8′ *as a totally singular subspace*.

PROOF (i) We show that $\mathcal{E}\otimes\mathfrak{g}$ represents the geometry $\Delta(H)$. That is, we show that for each $K \in H$ there is a fixed point $\epsilon_K \in \mathcal{E}\otimes\mathfrak{g}$ for $N_{\mathfrak{G}}(K)$ such that $\epsilon_K + \epsilon_{K'} + \epsilon_{K''} = 0$ whenever $\{K, K', K''\}$ is an H-line. Indeed for each $K \in H$, the image of $\mathfrak{J}_{S;K} \otimes \mathfrak{J}_{\Lambda;K} \cong \mathcal{E}nd(11)$ in $\mathcal{E}\otimes\mathfrak{g}$ has a 1-dimensional fixed-point space for $N_{\mathfrak{G}}(K)$; say this is $\mathbf{F}_2\epsilon_K$.

It is straightforward to verify, exactly as in the proof of (3.8) below, that $\mathfrak{g}\downarrow_{\mathcal{P}_i}$, $i \in I$, contains the following composition factors:



Thus, for example, there is a uniserial $\mathcal{P}_4$-submodule $\mathfrak{g}_4$ of $\mathfrak{g}$ that contains $\mathfrak{J}_{ad;11}$ and $\mathfrak{J}_{ad;4}$ and has composition factors $\mathfrak{J}_{ad;4} \cong 4$, $6\otimes2$, and 2 (6 is the $\mathbf{F}_4$-semilinear module for $\mathcal{P}_4(\mathbf{F}_2) \cong \mathfrak{S}_6$, as before). Similarly, in $\mathcal{E}$ we find the uniserial $\mathcal{P}_4$-submodule $\mathcal{E}_4$ containing 11^* of shape:



In particular $\mathcal{H}om_{\mathcal{P}_4}(\mathrm{F}_2,\mathcal{E}_4\otimes\mathcal{S}_4) \cong \mathcal{H}om_{\mathcal{P}_4}(\mathcal{S}_4{}^*,\mathcal{E}_4) = 0$.

Now consider the H-line $\{\mathcal{X},\mathcal{X}',\mathcal{X}''\}$ on the vertex $v_4 \in S$. Since $\epsilon_{\mathcal{X}} + \epsilon_{\mathcal{X}'} + \epsilon_{\mathcal{X}''} \in \mathcal{E}_4\otimes\mathcal{S}_4$ is centralized by $\mathcal{P}_4$, the remark above forces $\epsilon_{\mathcal{X}} + \epsilon_{\mathcal{X}'} + \epsilon_{\mathcal{X}''} = 0$. This concludes (i).

(ii): Let $\hat{\Lambda}$ be $H_0(\Delta(H))$ ~ nonzero, from (i) ~ and let $\mathcal{V}_{11}$ and $\mathcal{V}_4$ be, respectively, the images in $\hat{\Lambda}$ of the subspaces $\mathrm{F}_2\mathcal{X}$ and $\mathrm{F}_2 H_{\mathcal{P}}$ of the permutation module $\mathrm{F}_2 H$. Define $\mathcal{V}_2 = \sum \mathcal{V}_{11}\mathcal{P}_2$. $\mathcal{V}_2$ cannot vanish, lest $\hat{\Lambda} = \sum \mathcal{V}_{11}\mathfrak{G}$ vanish. Thus $\mathcal{V}_2$ is generated by the 15 points $\mathcal{V}_{11}^{\#}.\mathcal{P}_2$ subject to the relations determined by the lines of the natural module for $\mathcal{L}_2 \cong \mathcal{L}_4(\mathrm{F}_2)$ in which $\mathcal{P}_{2,11}/\mathcal{U}_2$ fixes a point ~ a presentation for this 4-dimensional natural module.

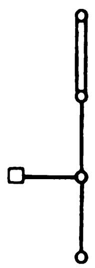
Similarly the sum $\mathcal{V}_1$ of the images of $\mathcal{V}_{11}$ under elements of $\mathcal{C} = \mathcal{P}_1$ must be the natural module for $\mathcal{C}$ (as this acts through its quotient $\Omega_8^+(\mathrm{F}_2)$) in which $\mathcal{P}_{1,11}$ fixes a point (One could either repeat the argument of (2.15.v), or else invoke the general theorem of [Ronan-Smith 9, (4.1), pp. 338-339]). □

(3.7) REMARKS (i) The notation $\mathfrak{I}_\Lambda$ is meant to suggest a connection with the Leech lattice Λ. Indeed for each $\sigma \in \Delta$, $\mathfrak{I}_{\Lambda;\sigma}$ is just the subspace $\mathfrak{I}_{\bar{\Lambda};\sigma}$ of $\bar{\Lambda} := \Lambda$ mod 2 that consists of the fixed points for $\mathcal{U}_\sigma \leq \cdot 1$. We regard this as saying that the pair $\mathfrak{G},\mathfrak{I}_\Lambda$ is 'locally isomorphic to' the pair $\cdot 1,\mathfrak{I}_{\bar{\Lambda}}$. In proposition (3.8) we strengthen this to say that $\hat{\Lambda} := H_0(\mathfrak{I}_\Lambda)$ 'is locally isomorphic to' $\bar{\Lambda}$ (cf. the precise statement in Table 6 below).

(ii) The assumption that $H_0(\mathfrak{J}_\Lambda) \neq 0$ would, similarly, lead to the nonvanishing of $H_0(\Gamma)$ through a nontrivial image of Γ in $S^2(H_0(\mathfrak{J}_\Lambda))$. This is noted also in corollary (3.9).

(3.8) PROPOSITION Let $\mathfrak{G}$ be of type $\cdot 1$, and $\mathfrak{J}_\Lambda$ the sheaf as defined in lemma (3.6). For each maximal parabolic $\mathcal{P}_i$ of $\mathfrak{G}$ we list a composition series for $H_0(\mathfrak{J}_\Lambda)\big\downarrow_{\mathcal{P}_i}$ in the following table.

Table 6
The local composition series' of $H_0(\mathfrak{J}_\Lambda)$



I	$\mathcal{V}$	$\mathcal{W}/\mathcal{V}$	$\mathcal{X}/\mathcal{W}$	$\mathcal{Y}/\mathcal{X}$	$\mathcal{Z}/\mathcal{Y}$
1	8′		8″		8′
2	4	$2\otimes 4^*$		$2\otimes 4$	4^*
4	2	6^*	$4\otimes 2$	6	2
11	1	11^*		11	1

REMARK Since Conway's group $\cdot 1$ is a group of type $\cdot 1$ this result is a 'local uniqueness' theorem for such groups.

PROOF Set $\hat{\Lambda} = H_0(\mathfrak{J}_\Lambda)$. Lemma (3.6) (that $\hat{\Lambda} \neq 0$) provides us with identifications $\mathfrak{J}_{\Lambda;v_i} \xrightarrow{\sim} \mathcal{V}_i \leq \hat{\Lambda}$, $i \in I$. Starting with these the chart gives a $\mathcal{P}_i$-series $\mathcal{V}_i \leq \mathcal{W}_i \leq \mathcal{X}_i \leq \mathcal{Y}_i \leq \mathcal{Z}_i$, $i \in I$. The entries in the chart identify the composition factors by their dimensions - a blank indicates 0 - possibly in conjunction with other notational information which will be explained as the proof develops. The proof is carried out in twelve steps, one for each nonzero factor not given as a sheaf term.

<u>Step 1</u> Define $\mathcal{W}_{11} := \sum \mathcal{V}_4\mathcal{P}_{11}$. This is a $\mathcal{P}_{11}$-submodule. In $\mathcal{W}_{11}/\mathcal{V}_{11}$ there is a sheaf over $\Delta_{v_{11}}$ as follows. First, $\mathcal{V}_{11} \leq \mathcal{V}_4 \leq \mathcal{W}_{11}$, and $\mathcal{V}_4/\mathcal{V}_{11}$ is a 1-dimensional $\mathcal{P}_{4,11}$-submodule of $\mathcal{W}_{11}/\mathcal{V}_{11}$. Next, $\mathcal{V}_4 \leq \mathcal{V}_2$, and $\sum \mathcal{V}_4\mathcal{P}_{2,11} = \mathcal{V}_2$, so that $\mathcal{V}_2/\mathcal{V}_{11} \leq \mathcal{W}_{11}/\mathcal{V}_{11}$ is a natural 3-dimensional module for $\mathcal{P}_{2,11}$. Finally, the $\mathcal{P}_{1,11}$-conjugates of $\mathcal{V}_4$ generate the subspace of the orthogonal module $\mathcal{V}_1$ that is perpendicular to $\mathcal{V}_{11}$. We summarize these calculations in the following table:

$$6_+ = (\textstyle\sum \mathcal{V}_4\mathcal{P}_{1,11})/\mathcal{V}_{11} = \mathcal{V}_{11}^{\perp}/\mathcal{V}_{11} \leq \mathcal{V}_1/\mathcal{V}_{11}$$

$$3 = (\textstyle\sum \mathcal{V}_4\mathcal{P}_{2,11})/\mathcal{V}_{11} = \mathcal{V}_2/\mathcal{V}_{11}$$

$$1 = \mathcal{V}_4/\mathcal{V}_{11}$$

6_+ is the 6-dimensional orthogonal module for $\mathcal{L}_4(F_2) \cong \Omega_6^+(F_2)$.

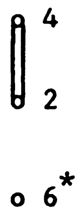
The homology of this sheaf is the simple $\mathfrak{M}_{24}$-module denoted by 11^* - the <u>Todd</u> <u>module</u> (<u>cf</u>. [S. Smith 13]). The fact that $\hat{\Lambda} \neq 0$ forces the image, in $\mathcal{W}_{11}/\mathcal{V}_{11}$, of the residual homology module to be nonzero. Hence $\mathcal{W}_{11}/\mathcal{V}_{11}$ contains a copy of the Todd module. Since $\mathcal{W}_{11}/\mathcal{V}_{11}$ is generated by the subspaces $(\mathcal{V}_4/\mathcal{V}_{11})\mathcal{P}_{11}$ in this copy of the Todd module, these are perforce equal, and we have verified the first of the twelve entries: <u>the</u> <u>composition</u> <u>factor</u> $\mathcal{W}/\mathcal{V}$ <u>over</u> $i = 1$ <u>is</u> <u>the</u> <u>Todd</u> <u>module</u> <u>for</u> $\mathcal{P}_1$.

Observe that $\mathcal{W}_{11} \geq \mathcal{V}_{11}$, and that $\mathcal{W}_{11} \cap \mathcal{V}_1$ has a $\mathcal{P}_{1,11}$-series $1{:}1{:}6_+$. (We abbreviate a composition series by such a sequence.)

All of the other steps follow this same line, and so we finish the proof with but a sketch of the calculations.

<u>Step</u> <u>2</u> $\mathcal{W}_4 := \sum \mathcal{V}_2\mathcal{P}_4$ leads, in exactly the same way as <u>step</u> <u>1</u>, to the sheaf:

$2 \cap 6^* = 2$

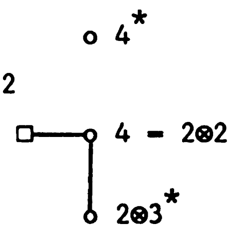


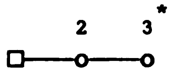
6 denotes the $\mathbb{F}_4$-semilinear module for $\mathcal{S}_4(\mathbb{F}_2)$, as before. (The convention will be that the symbol for the dual of a module will be the symbol for that module with a superscript '*'.) Note here that only the subgroup $O_2(\mathfrak{U}_4) = O^3(\mathfrak{U}_4)$ of $\mathfrak{U}_4$ acts trivially. Also, 6^* contains the $\mathcal{P}_{2,4}$- and $\mathcal{P}_{1,4}$-subspaces 2 and 4, respectively. As $H_0($$) \cong 6^*$, the second entry of the chart is verified.

$\mathcal{V}_2 \le \mathcal{W}_4 \le \mathcal{W}_{11}$, and $\mathcal{W}_4 \cap \mathcal{V}_1 = 2{:}4$.

<u>Step</u> <u>3</u> $\mathcal{W}_2 := \sum \mathcal{V}_1\mathcal{P}_2$ leads to

$4^* \cap 2\otimes 2 = 2$

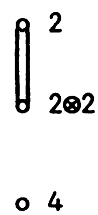


Since $4^* \cap 4 = 2$, the $\mathcal{P}_{1,2}$-subsheaf $\mathfrak{J}_{4^*}$:  generates, in the homology of the above, a factor of $2\otimes H_0(\mathfrak{J}_{4^*}) \cong 2\otimes 4^*$. Again, the simplicity of the residual homology module, together with the nonvanishing of $\hat{\Lambda}$, gives $\mathcal{W}_2/\mathcal{V}_2 \cong 2\otimes 4^*$.

$\mathcal{W}_2 \ge \mathcal{V}_1, \mathcal{V}_4$; and $\mathcal{W}_2 \cap \mathcal{W}_{11} = 1{:}3{:}2\otimes 3^*$.

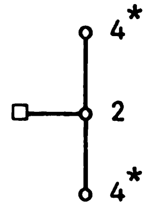
<u>Step</u> <u>4</u> $\mathcal{X}_4 := \sum \mathcal{W}_{11}\mathcal{P}_4$ leads to:

$2 \cap 4 = 1$



$H_0(\overset{1}{\circ}\!=\!=\!\overset{2}{\circ})$ is a copy of the 5-dimensional orthogonal module for $\mathcal{S}_4(F_2) \cong \Omega_5(F_2)$. This and the fact that $2 \cap 4 = 1$ imply that the homology of the above is a factor of $5\otimes 2$. However, there does not exist a copy of the symplectic 4 in $5\otimes 2$ (for the parabolic $\mathcal{P}_{4,11}$), so that this homology module is $4\otimes 2 \cong \mathcal{X}_4/\mathcal{W}_4$.

Step 5 $\mathcal{X}_1 := \sum\,[\mathcal{W}_{11},\mathcal{P}_{1,11}]\mathcal{P}_1 = \sum \mathcal{W}_2\mathcal{P}_1$ leads to:



whence to $\mathcal{X}_1/\mathcal{V}_1 \cong 8''$, a triality-twisting of $\mathcal{V}_1$.

$\mathcal{X}_1 \geq \mathcal{W}_2,\ \mathcal{W}_4$; and $\mathcal{X}_1 \cap \mathcal{W}_{11} = 1{:}6_+{:}4$.

All of the major steps of the proof are now complete.

Steps 6-8 "Dualize" the arguments in steps 3, 2, and 1, in that order, to obtain:

$$\mathcal{Y}_2/\mathcal{W}_2 \cong 2\otimes 4;\ \mathcal{Y}_4/\mathcal{W}_4 \cong 6;\ \text{and}\ \mathcal{Y}_{11}/\mathcal{W}_{11} \cong 11.$$

Steps 9-12 First define $\mathcal{Z}_1 := \sum \mathcal{Y}_2\mathcal{P}_1$, and verify that $\mathcal{Z}_1/\mathcal{X}_1 \cong 8'$. $\mathcal{Z}_1$ is then stable for $\mathcal{P}_2$, $\mathcal{P}_4$, and $\mathcal{P}_{11}$ ($\sum \mathcal{Z}_1\mathcal{P}_i$ generates a constant sheaf for each).

Now $\mathcal{Z}_1 \leq \hat{\Lambda}$ is stable for each of the parabolics and contains the generators for $\hat{\Lambda}$, hence $\hat{\Lambda} = \mathcal{Z}_{11} = \mathcal{Z}_4 = \mathcal{Z}_2 = \mathcal{Z}_1$. □

(3.9) COROLLARY (i) *$\hat{\Lambda}$ is self-dual for $\mathfrak{G}$.*

(ii) *The singular lines of $\mathfrak{G}$ act quadratically on $\hat{\Lambda}$: $[\hat{\Lambda},\ell,\ell] = 0$ whenever $\ell \in L$.*

(iii) *$S^2(\hat{\Lambda})$ represents the adjoint geometry for $\mathfrak{G}$.*

PROOF (i) The Table 6 shows that there is a nonzero map $\mathfrak{F}_\Lambda \longrightarrow \hat{\Lambda}^*$, and so the reciprocity applies.

In particular $\mathfrak{G}$ is represented in $\mathcal{S}p(\hat{\Lambda})$, so that (without loss) $\mathfrak{G}$ leaves $S^2(\hat{\Lambda})$ invariant.

(ii) and (iii) Let $\ell = \langle z,z'\rangle$ be a singular line. From the proof of (3.8) observe that $[\hat{\Lambda},z_0] = \mathcal{V}_z \leq \mathcal{X}_z = \mathcal{C}_{\hat{\Lambda}}(z)$. Now $\mathcal{V}_{z'} \leq \mathcal{V}_z.\mathcal{N}_{\mathfrak{C}}(\ell) \leq \mathcal{W}_z \leq \mathcal{X}_z$. The $\mathfrak{C}$-map $\mathbb{F}_2P \longrightarrow S^2(\hat{\Lambda}) \leq \mathcal{E}nd(\hat{\Lambda})$, $z \longmapsto 1 + z$ (of trace 0) factors through $H_0(\Gamma)$, since $1 + z + 1 + z' + 1 + zz' = (1 + z)(1 + z') = 0$. □

(3.10)REMARK Counts of the vertices of Δ and of the image of these vertices in $\hat{\Lambda}$ shows that the map is indeed an embedding.

As a final result we indicate how a proof of the nonvanishing of the adjoint module that includes a description of a hyperplane section will not be so easy.

(3.11)LEMMA There is no $\mathcal{C}$-invariant hyperplane section for the hyperbolic geometry of $\cdot 1$.

PROOF Any $\mathcal{C}$-invariant subset of P must be a union of various orbits $P_{d,n}$. Now there are hyperbolic lines that lie entirely in P_1, and hyperbolic lines entirely in $P_{4,5}$; and for any point v not in $P_{4,5}$, there is a hyperbolic line that intersects $v^{\mathcal{C}}$ in precisely v. This gives the result.

To see these claims, first note that if y,y' are in P_1, $\langle y,z_0\rangle \neq \langle y',z_0\rangle$, then $\langle y,y'\rangle \subseteq P_1$ is a hyperbolic line. Consider now $w \in P_{3,3}$ and $x \in P_w \cap P_{2,4}$. Choose $v,v' \in P_w \cap \mathcal{C}_x$, $[v,v'] = 1$, and such that $\langle v,w\rangle$, $\langle v',w\rangle$, $\langle vv',w\rangle$, and $\langle x,w\rangle$ are pairwise distinct. We have then that $\langle v,v'\rangle \subseteq P_{4,5}$ is a hyperbolic line.

For $x \in P_{2,2}$, there is a $v \in P_{4,4}$ such that $\langle v,x\rangle$ is a hyperbolic line with $vx \in P_{4,4}$. Similarly if $w \in P_{3,2}$ resp. $P_{3,4}$ there is a point $v \in P_{4,6}$ such that $\langle w,v\rangle$ is a hyperbolic line with $vw \in P_{4,3} \cup P_{4,6}$. And finally, if $v \in P_{4,4}$ resp. $P_{4,3} \cup P_{4,6}$ there is a point $v' \in P_{4,5}$ such that $\langle v,v'\rangle$ is a hyperbolic line with $vv' \in P_{4,5}$. □

4. CONCLUDING REMARKS

We became interested in $\cdot 1$ while trying to characterize the Friendly Giant $\mathfrak{FG}$ of B. Fischer and R. Griess. $\mathfrak{FG}$ is of $\mathbf{F}_2$-type, with $\mathcal{C}$ of shape $2^{1+24}\cdot 1$. We have been investigating an inductive approach to the adjoint module $H_0(\Gamma)$ for $\mathfrak{FG}$, trying in particular to establish that this does not vanish, based on the existence of the adjoint module for $\cdot 1$. As yet we have not succeeded in this, although we can demonstrate that if $H_0(\Gamma)$ is nontrivial then $\mathfrak{FG}$ admits a 'nice' 2-modular representation. This representation is given by a sheaf locally isomorphic to the sheaf of O_2-fixed-points in the Griess-module modulo 2.

The approach of the present paper does produce for a group of type $\mathfrak{FG}$ a 2-local geometry as described in [Ronan-Smith 8]. However it seems completely hopeless to enumerate sufficiently the $\mathcal{C}$-orbits in P to give the desired nonvanishing as was done here. Indeed, the number of these suborbits is at least:

$$|\mathfrak{FG}|/|\mathcal{C}|^2 \approx 41.514\ldots$$

S. Norton has suggested that there may be around 150 suborbits. We have enumerated most of those out to distance 4, and there seems to be no end in sight!

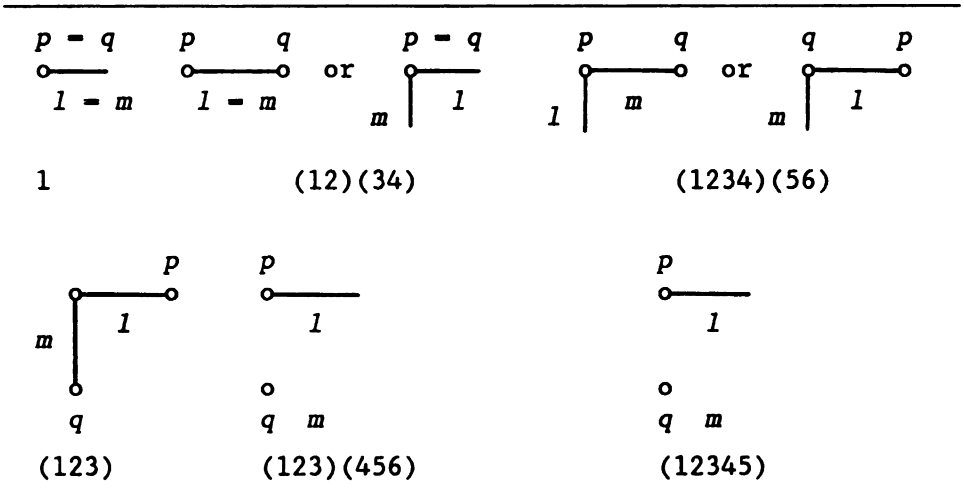
APPENDIX

APPENDIX

CALCULATIONS IN THE 15-POINT QUADRANGLE

It is perhaps easiest to calculate in $\mathfrak{A}_6$ as it acts on the 15-point quadrangle by viewing the quadrangle as the transposi-tions of $\mathfrak{S}_6$. Thus the singular center of (12)(34) is (56); its axis is {(12),(34),(56)}. One sees that the class of st - for involutions s and t with axes and singular centers l, m, p, q - is determined by the incidence structure on $\{p,q,l,m\}$:

Table 7
The suborbits for the $\mathfrak{A}_6$ involutions



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