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PRIME IDEALS IN A VECTOR LATTICE
AND ITS DEDEKIND COMPLETION

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ABSTRACT

PRIME IDEALS IN A VECTOR LATTICE AND ITS DEDEKIND COMPLETION

By

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Structure spaces are used to study vector lattices by representing a vector lattice as a class of functions on its structure spaces. Masterson has attempted to study the relationship between the structure spaces of an Archimedean vector lattice E and its Dedekind completion $\hat{E}$. He has proved that, in the presence of $P \cdot P \cdot$, starting with a prime ideal P in E we can get a prime ideal $\hat{P}$ in $\hat{E}$ such that $\hat{P} \cap E = P$. We prove that if E is an arbitrary Archimedean vector lattice, starting with a prime ideal P in E , we can get a prime ideal Q in $\hat{E}$ $\supset$ $Q \cap E = P$. Masterson introduces the property: $\forall x \in \hat{E}^+$, the existence of $y \in E^+$ and scalar α such that $\alpha y \leq x \leq y$, and points out that in the presence of this property, the structure spaces of E and $\hat{E}$ are homeomorphic. We obtain several conditions equivalent to this property and prove that this property is strictly stronger than $P \cdot P \cdot$. It is well known that if $0 < x \leq y_1 + y_2$, where $x, y_1, y_2 \in E$,

then x can be decomposed as $x_1 + x_2$ where $0 \leq x_1 \leq y_1$ and $0 \leq x_2 \leq y_2$. If $x \in E$ and $y_1, y_2 \in \hat{E}$, it is not necessarily true that $x_1 \in E, x_2 \in E$. If every x can be decomposed this way for all $y_1, y_2 \in \hat{E}$, starting with a prime ideal P in E , we can obtain a prime ideal $\tilde{P}$ in $\hat{E}$ such that $\tilde{P} \cap E = P$. This property is strictly stronger than $P \cdot P \cdot$ and weaker than the property introduced by Masterson. Also we obtain a way of characterizing this property using prime ideals. If $g(P) = \tilde{P}$, then $\mathfrak{P}$ and $g(\mathfrak{P})$ are homeomorphic where $\mathfrak{P}$ is the structure space of all prime ideals of E . We obtain two ways of characterizing $P \cdot P \cdot$, one by a property analogous to the one introduced by Masterson, and one by a property analogous to the one mentioned above. Also we mention a property which is strictly stronger than the one introduced by Masterson.

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Table of Contents

	Page
Introduction	iv
Section 0	1
Section 1	6
Section 2	21
Section 3	30
Bibliography	38

Introduction

Several authors have attempted to study vector lattices using prime ideals. Yosida (8), using prime ideals, proved that every Archimedean vector lattice is isomorphic to a vector lattice of extended functions on some locally compact Hausdorff space. Nakano (7) has proved that every Dedekind σ -complete vector lattice E is isomorphic to a vector lattice of extended functions on some totally disconnected Hausdorff space X . The space X is obtained by providing the collection of all maximal dual ideals in the distributive lattice of projectors on E with the dual hull-kernel topology. Johnson and Kist (2) have shown that the representations of Yosida and Nakano can each be obtained by considering a suitable subspace of the space of all prime ideals. They have generalized Nakano's representation to arbitrary Archimedean vector lattices. Using the concept of spectral function, Amemiya (1) has developed a spectral theory for vector lattices, generalizing Nakano's theory for the σ -complete and complete cases. Johnson and Kist (2) have shown that Amemiya's theory can be obtained by ideal-theoretic methods. They do this by showing that the set of all spectral functions defined on a vector lattice E is essentially the same as the set of all prime ideals in E .

The purpose of this thesis is to investigate the relationship between structure spaces of an Archimedean vector lattice E and those of its Dedekind completion $\hat{E}$. Masterson (5) has attempted to answer this question, but his answers are incomplete. He has shown that if E has projection property, for every prime ideal P in E there exists a prime ideal $\hat{P}$ in $\hat{E}$ such that $\hat{P} \cap E = P$. $\hat{P}$ is minimal prime if P is minimal prime. If $\mathfrak{U}$ is a structure space of E consisting entirely of minimal prime ideals, $\mathfrak{U}$ and $f(\mathfrak{U})$ are homeomorphic where $f(P) = \hat{P}$. He also obtains several conditions equivalent to the homeomorphism of $\mathfrak{B}$ and $f(\mathfrak{B})$, where $\mathfrak{B}$ is any structure space of E , in the presence of projection property. He introduces a property in a vector lattice, viz. for every $x \in E^+$ the existence of $y \in E^+$ and positive scalar α $\alpha y \leq x \leq y$, and shows that under this condition $\mathfrak{B}$ and $f(\mathfrak{B})$ are homeomorphic where $\mathfrak{B}$ is any structure space of E .

We show that if P is any prime ideal in an Archimedean vector lattice E , there exists a prime ideal Q in $\hat{E}$ such that $Q \cap E = P$. This prime Q is not unique. We prove two partial converses to Masterson's result: (1) if $\hat{P}$ is prime in $\hat{E}$ for every prime P in E , then E has $P \cdot P \cdot P$. (2) If every prime in $\hat{E}$ is of the form $\hat{P}$ for some prime P in E , then E has $P \cdot P$.

Also the property introduced by Masterson is equivalent to the property that every prime ideal in $\hat{E}$ is of the form $\hat{P}$ for some prime P in E . We obtain other equivalent conditions to this property. We give examples to show that the implications in (1) and (2) above are not reversible. Whether projection property is strictly stronger than the property that $\hat{P}$ is prime in $\hat{E}$ for every prime P in E , is an open question.

We introduce another property: if $0 < x \leq y_1 + y_2$, where $x \in E^+$, $y_1, y_2 \in \hat{E}^+$ the decomposability of x as $x = x_1 + x_2$ where $0 \leq x_1 \leq y_1$ and $0 \leq x_2 \leq y_2$ where $x_1, x_2 \in E$. It is well-known that if x_1, y_1, y_2 are positive elements of a vector lattice E such that $x \leq y_1 + y_2$, then x can be expressed as $x = x_1 + x_2$ where $0 \leq x_1 \leq y_1$ and $0 \leq x_2 \leq y_2$. This property is known as the Riesz dominated decomposition property. But if $x \in E^+$ and $y_1, y_2 \in \hat{E}^+$, it is not necessarily true that $x_1, x_2 \in E$. In the presence of this property, starting with a prime ideal P in E , we can define a prime ideal $\tilde{P}$ in $\hat{E}$; $\tilde{P} \cap E = P$. Also if $\tilde{P}$ is prime in $\hat{E}$ for all prime P in E , then E has this property. If $\mathcal{U}$ is any structure space of E , $\mathcal{U}$ and $f(\mathcal{U})$ are homeomorphic under the mapping $f(P) = \tilde{P}$. We show that the above mentioned property is strictly stronger than $P \cdot P$. The property

introduced by Masterson is stronger than this property, but it is not known whether it is strictly stronger. We also introduce a property strictly stronger than the one introduced by Masterson, viz. for every $x_1, x_2 \in \hat{E}^+$ such that $x_1 < x_2$, the existence of $y \in E^+$ such that $x_1 < y < x_2$.

The whole work is divided into four sections. Section 0 consists of definitions, etc. necessary for the understanding of the text. Sections 1, 2, 3, constitute the main body of the work.

Section 0

Notation 0.1: $E^+ = \{x \in E : x \geq 0\}$.

Definition 0.2: E is Archimedean iff $x, y \in E^+$ such that $nx \leq y$ for all positive integer $n \Rightarrow x = 0$, for all x, y .

Definition 0.3: E is Dedekind Complete iff $\{x_\alpha\} \subset E$, $x_\alpha \leq y \in E \forall \alpha \Rightarrow \sup_\alpha x_\alpha$ exists in E .

Definition 0.4: If E and E' are vector lattices a mapping $\phi : E \rightarrow E'$ is a vector lattice homomorphism iff

- (i) $\phi(\alpha a) = \alpha \phi(a)$ for all $a \in E$, for all scalar α .
- (ii) $\phi(a \vee b) = \phi(a) \vee \phi(b)$ for all $a, b \in E$.

Definition 0.5: A vector lattice isomorphism is a one-one vector lattice homomorphism.

Definition 0.6: The vector lattices E and E' are said to be isomorphic if there exists a vector lattice isomorphism $\phi : E \rightarrow E'$ such that ϕ is onto.

Definition 0.7: $E' \subset E$ is said to be order dense in E if for all $x \in E$, there exists $\{x_\alpha\}_{\alpha \in A} \subset E'$ such that $x = \sup_\alpha x_\alpha$.

Theorem 0.8: (Nakano, 6). If E is a vector lattice, there exists a Dedekind Complete vector lattice $\hat{E}$ such that E can be imbedded as an order dense sub-vector lattice of $\hat{E}$, iff E is Archimedean. If $\hat{E}$ exists, it is unique upto vector lattice isomorphism.

Property 0.9: (Riesz) If $0 < x < y_1 + y_2$, $x, y_1, y_2 \in E$, there exists a decomposition of x such that $x = x_1 + x_2$, where $0 < x_1 < y_1$ and $0 < x_2 < y_2$. This property is known as Riesz dominated decomposition property.

Definition 0.10: A subset $E' \subset E$ is order closed iff $\{x_\alpha\}_{\alpha \in A} \subset E'$, $\sup_\alpha x_\alpha$ exists in $E \Rightarrow \sup_\alpha x_\alpha \in E'$.

Definition 0.11: A linear subspace $I \subset E$ is an ideal iff I is solid, i.e. $x \in I$, $|y| \leq |x| \Rightarrow y \in I$.

Definition 0.12: A principal ideal is an ideal generated by a single element. Equivalently, an ideal I is principal iff there exists $a \in E \ni I = \{x: |x| \leq n|a|, \text{ for some integer } n\}$.

Definition 0.13: A band is an order closed ideal.

Definition O.14: A principal band is the order closure of a principal ideal.

Notation O.15:

For all $x \in E$, $x^+ = x \vee 0$

$$x^- = -x \vee 0$$

$$|x| = x^+ \vee x^- = x^+ + x^-$$

Definition O.16: $x \perp y$ iff $|x| \wedge |y| = 0$.

Notation O.17: If $A \subset E$, $A^\perp = \{x \in E: x \perp a \text{ for all } a \in A\}$.

Theorem O.18: $A^\perp$ is a band for any subset A of E .

Theorem O.19: E is Archimedean iff $A^{\perp\perp} = A$, for every band A in E .

Definition O.20: E has Projection Property (P.P.) iff $E = B \oplus B^\perp$, for any band B of E .

Definition O.21: E has Principal Projection Property (P.P.P.) iff $E = B \oplus B^\perp$, for any principal band B of E .

Definition O.22: E is Dedekind σ -Complete iff $\{x_n\}_{n \in \mathbb{N}}, x_n \leq x \in E \Rightarrow \sup_n \{x_n\}$ exists in E .

Theorem 0.23:

$$\begin{array}{ccccc}
 E \text{ Ded. Complete} & \Rightarrow & E \text{ has P.P.} & \Rightarrow & E \text{ has P.P.P.} \Rightarrow E \\
 & & & & \text{is Archimedean} \\
 & \Rightarrow & E \text{ is Ded. } \sigma\text{-Complete} & \Rightarrow &
 \end{array}$$

None of the above implications is reversible. Projection Property and Dedekind σ -Completeness are independent.

If E is Dedekind σ -Complete and has P.P. it is Dedekind Complete.

Definition 0.24: An ideal P is prime $\Leftrightarrow x \wedge y \in P$
 $x, y \in E \Rightarrow x \in P$ or $y \in P$.

Theorem 0.25: (Johnson & Kist, 2): The following are equivalent:

- (1) P is a prime ideal.
- (2) If $x \wedge y = 0$, then $x \in P$ or $y \in P$.
- (3) The quotient vector lattice $\frac{E}{P}$ is linearly ordered.
- (4) If $P \supset A \cap B$, where A and B are ideals in E , then either $P \supset A$ or $P \supset B$.

Definition 0.26: Let $\mathfrak{P}$ denote the collection of all prime ideals in E . For $\mathfrak{P}' \subset \mathfrak{P}$, the kernel of $\mathfrak{P}' = \bigcap \{P : P \in \mathfrak{P}'\}$. The kernel is an ideal, not necessarily prime. The hull of an ideal is the collection of all prime ideals containing that ideal.

Hence if $\mathfrak{P}' \subset \mathfrak{P}$, note that $h(k(\mathfrak{P}')) \supset \mathfrak{P}'$, where h denotes the hull and k , the kernel. Taking $h(k(\mathfrak{P}'))$ as the closure of $\mathfrak{P}'$, the closure of any subset of $\mathfrak{P}$ is uniquely defined. And this closure operation defines a topology on $\mathfrak{P}$. This topology is known as the hull-kernel topology. $\mathfrak{P}$, with this topology is known as the structure space of E . The class $\{\mathfrak{P}_a\}_{a \in E}$ is a base for this topology, where $\mathfrak{P}_a = \{P \in \mathfrak{P} : a \notin P\}$.

More generally, let $\mathfrak{P}$ denote any collection of prime ideals in E such that $\bigcap \{P : P \in \mathfrak{P}\} = 0$. Define the topology on $\mathfrak{P}$ as above. $\mathfrak{P}$, with this topology is a structure space of E .

Section 1

It is well known that in a vector lattice, maximal ideals are prime. Yosida (8) has generalized this result. He has shown that relatively maximal ideals are prime. An ideal is relatively maximal if it is maximal with respect to not containing a fixed element.

In this section we give an alternative proof of Yosida's result (Theorem 1.1). We also give some generalizations of his result (Theorem 1.5, Theorem 1.9). We obtain many beautiful results on prime ideals, earlier obtained by Johnson and Kist (Johnson & Kist 2) as easy corollaries to these theorems (Corollaries 1.3, 1.4, 1.7, 1.8).

Masterson (5) has shown that starting with a prime ideal P in an Archimedean vector lattice E , an ideal $\hat{P}$ can be defined in $\hat{E}$, the Dedekind Completion of E , such that $\hat{P} \cap E = P$. He shows that if E has P.P., $\hat{P}$ is prime in $\hat{E}$. We show that for a prime ideal P in an Archimedean vector lattice E , there exists a prime ideal Q in $\hat{E}$ such that $Q \cap E = P$ (Theorem 1.10). This prime ideal Q is not unique (Example 1.11). We show that if $\hat{P}$ is prime in $\hat{E}$ for every prime P in E , E has P.P.P. (Theorem 1.12). Also this property is strictly stronger than P.P.P. (Example 1.13). We give

an example to show that Dedekind σ -Completeness, a property stronger than P.P.P., is not strong enough to imply this property (Example 1.14). Recall Masterson's result that P.P. implies this property. Whether P.P. is strictly stronger than this property is not known. Hence we have the chain

$$\begin{array}{ccccc}
 \text{P.P.} & \Rightarrow & \bigwedge P & \text{prime in } E \text{ for every prime } P & \Rightarrow & \text{P.P.P.} \\
 & & * & P \text{ in } E & & \nleftrightarrow \\
 \text{Ded. } \sigma & & & & & \\
 \text{Completeness} & & & & &
 \end{array}$$

Theorem 1.1: Let E be a vector lattice and $a \in E$. Let I be an ideal in E , maximal with respect to not containing a . Then I is prime.

Proof: Existence by Zorn's Lemma:

Let $\mathfrak{I}_a = \{I : I \text{ an ideal in } E, a \notin I\}$. We can take a positive for, $a \in I \Rightarrow |a| \in I$. $\mathfrak{I}_a$ is non-empty for, $\mathfrak{I}_a$ empty $\Rightarrow$ every ideal contains $a \Rightarrow E$ has only one non-trivial ideal $\Rightarrow E$ is linearly ordered. $\mathfrak{I}_a$ is partially ordered by inclusion. If $I_1 \subset I_2 \subset \dots$ is a chain in $\mathfrak{I}_a$, $\bigcup I_i$ is an upper bound for this chain. $\therefore \mathfrak{I}_a$ has a maximal element, by Zorn's lemma.

Let M be a maximal element. We show that M is prime. Suppose not. Then $\exists b, c \in E, b \wedge c = 0$, and $b \notin M, c \notin M. \Rightarrow a \in (M, b)$ and $a \in (M, c)$ where (M, b) denotes the smallest ideal containing all the

elements of M and b etc. But $a \leq Am + Bb$ and $a \leq A'm' + B'c$ where A, B, A', B' are positive scalars and $m, m' \in M$.

By Riesz dominated decomposition property (0.9),

$$a = a_1 + a_2 \quad \text{where} \quad 0 \leq a_1 \leq Am$$

$$0 \leq a_2 \leq Bb$$

$$a = a'_1 + a'_2 \quad \text{where} \quad 0 \leq a'_1 \leq A'm'$$

$$0 \leq a'_2 \leq B'c$$

$$a_2 \leq a = a'_1 + a'_2$$

$$\therefore a_2 = a_{21} + a_{22}, \quad \text{where} \quad 0 \leq a_{21} \leq a'_1$$

$$0 \leq a_{22} \leq a'_2$$

$$b \wedge c = 0 \Rightarrow a_2 \wedge a'_2 = 0 \Rightarrow a_{22} = 0$$

$$\therefore a_2 = a_{21} \leq a'_1$$

$$\therefore a = a_1 + a_2 \leq a_1 + a'_1 \leq Am + A'm'$$

$$\Rightarrow a \in M, \text{ contradiction.}$$

Corollary 1.2: The intersection of all relatively maximal ideals in a vector lattice is $\{0\}$.

The following two results were obtained by Johnson and Kist(2). They are immediate from the above theorem.

Corollary 1.3: The intersection of all prime ideals in a vector lattice is $\{0\}$.

Corollary 1.4: The intersection of all minimal prime ideals in a vector lattice is $\{0\}$.

Theorem 1.5: Let I be an ideal in E and $0 < a \notin I$. Let M be maximal in the class of ideals containing I and not containing a . Then M is prime.

Proof: Existence by Zorn's Lemma:

Let $J_{I,a} = \{J: J \text{ an ideal in } E, J \supset I, a \notin J\}$
 $J_{I,a}$ is non-empty, for $I \in J_{I,a}$. $J_{I,a}$ is partially ordered by inclusion. Let $J_1 \subset J_2 \subset \dots$ be a chain in $J_{I,a}$. $\bigcup J_i$ is an upper bound for this chain. $J_{I,a}$ has a maximal element, by Zorn's Lemma.

Let M be a maximal element. We show that M is prime. Suppose not.

$$\Rightarrow \exists b, c \in E, \quad b \wedge c = 0, \quad b \notin M, \quad c \notin M$$

$$\Rightarrow a \in (M, b) \quad \text{and} \quad a \in (M, c)$$

$$\Rightarrow a \leq Am + Bb, \quad a \leq A'm' + B'c$$

where A, B, A', B' are positive scalars and $m, m' \in M$.

By Riesz dominated decomposition property

$$a = a_1 + a_2 \quad \text{where} \quad 0 \leq a_1 \leq Am$$

$$0 \leq a_2 \leq Bb$$

$$a = a'_1 + a'_2 \quad \text{where} \quad 0 \leq a'_1 \leq A'm'$$

$$0 \leq a'_2 \leq B'c$$

$$a_2 \leq a = a'_1 + a'_2$$

$$\therefore a_2 = a_{21} + a_{22} \quad \text{where} \quad 0 \leq a_{21} \leq a'_1$$

$$0 \leq a_{22} \leq a'_2$$

$$\begin{aligned}
b \wedge c = 0 &\Rightarrow a_2 \wedge a'_2 = 0 \Rightarrow a_{22} = 0 \\
&\Rightarrow a_2 = a_{21} \leq a'_1 \\
\therefore a = a_1 + a_2 &\leq a_1 + a'_1 \leq A_m + A'_m \\
&\Rightarrow a \in M. \quad \text{Contradiction.}
\end{aligned}$$

Definition 1.6 (Johnson and Kist): A prime ideal belonging to an ideal I is a prime ideal containing the ideal I . A minimal prime ideal belonging to an ideal I is a minimal element in the class of prime ideals belonging to I .

The following two results were obtained by Johnson and Kist. (Johnson and Kist, 2). They follow immediately from the above theorem.

Corollary 1.7: The intersection of all prime ideals belonging to an ideal I is I .

Corollary 1.8: The intersection of all minimal prime ideals belonging to an ideal I is I .

Theorem 1.9: Let $\{x_\alpha\}_{\alpha \in A} \subset E^+$ such that $x_{\alpha_1} \wedge x_{\alpha_2} \wedge \dots \wedge x_{\alpha_k} > 0$, for all integers k and for all choices of indices $\alpha_1, \alpha_2, \dots, \alpha_k \in A$. Let M be a maximal element in the class of ideals not containing any of these finite infima. Then M is prime.

Proof: Suppose not. Then $\exists a, b \in E$, $a \wedge b = 0$ and $a \notin M$, $b \notin M$. Hence $\exists$ indices $\alpha_1, \alpha_2, \dots, \alpha_n$ and

$$\begin{aligned}
& \beta_1, \beta_2, \dots, \beta_m \quad ; \\
& x_{\alpha_1} \wedge x_{\alpha_2} \wedge \dots \wedge x_{\alpha_n} \in (M, a) \\
& x_{\beta_1} \wedge x_{\beta_2} \wedge \dots \wedge x_{\beta_m} \in (M, b) \\
& \Rightarrow x_{\alpha_1} \wedge x_{\alpha_2} \wedge \dots \wedge x_{\alpha_n} \leq Am + Ba \\
& x_{\beta_1} \wedge x_{\beta_2} \wedge \dots \wedge x_{\beta_m} \leq Cm' + Db
\end{aligned}$$

where A, B, C, D are positive scalars and $m, m' \in M$.

$$\begin{aligned}
\text{So, } & x_{\alpha_1} \wedge x_{\alpha_2} \wedge \dots \wedge x_{\alpha_n} \wedge x_{\beta_1} \wedge \dots \wedge x_{\beta_m} \\
& \leq (Am + Ba) \wedge (Cm' + Db) \\
& \leq (Am + Ba) \wedge Cm' + (Am + Ba) \wedge Db \\
& \leq Am \wedge Cm' + Ba \wedge Cm' + Am \wedge Db + Ba \wedge Db \\
& \leq Am \wedge Cm' + Cm' + Am, \quad \because a \wedge b = 0 \\
\Rightarrow & 0 < x_{\alpha_1} \wedge x_{\alpha_2} \wedge \dots \wedge x_{\alpha_n} \wedge x_{\beta_1} \wedge \dots \wedge x_{\beta_m} \in M,
\end{aligned}$$

a contradiction.

Q.E.D.

On the other hand if P is a prime ideal in a vector lattice E and $\{x_\alpha\}_{\alpha \in A}$ is the class of all positive elements not belonging to P , P is the unique maximal element in the class of all ideals not containing any of the finite intersections of the x_α 's. P does not contain any finite intersection, for, if it does, at least one x_α will be in P . P is maximal in the class, for we have excluded all positive elements not contained in P . P is

unique, for, if P and Q are distinct prime ideals, the classes of positive elements not contained in P, Q will be distinct.

As mentioned earlier, starting with a prime ideal P in E , Masterson (5) defines an ideal $\hat{P}$ in $\hat{E}$, where $\hat{P} = \{y \in \hat{E} : |y| \leq |x|, \text{ for some } x \text{ in } P\}$. This ideal, in general, is not prime. This has the property that $\hat{P} \cap E = P$. If E has $P \cdot P \cdot$, $\hat{P}$ is prime. The following theorem shows that given a prime ideal P in an Archimedean vector lattice E , there exists a prime ideal Q in $\hat{E}$ $\ni Q \cap E = P$.

Theorem 1.10 Let E be an Archimedean vector lattice, $\hat{E}$ its Dedekind completion and P a prime ideal in E . Then there exists a prime ideal Q in $\hat{E}$ such that $Q \cap E = P$.

Proof: Let $\{x_\alpha\}_{\alpha \in A}$ be the class of positive elements of E , which are not in P . This class has the property that

$$x_{\alpha_1} \wedge x_{\alpha_2} \wedge \cdots \wedge x_{\alpha_k} > 0, \text{ for all integers } k \text{ and all}$$

choices of indices $\alpha_1, \alpha_2, \dots, \alpha_k$, for if at least one such infimum is 0 at least one x_α is in P . For the same reason, this class is closed under finite infima.

Consider the class

$\mathfrak{J} = \{J \text{ an ideal in } \hat{E}: x_\alpha \notin P \text{ for all } \alpha, J \supset P\}$. The class $\mathfrak{J}$ is non-empty for $\hat{P} \in \mathfrak{J}$ where $\hat{P} = \{y \in \hat{E}: |y| \leq |x|, \text{ for some } x \text{ in } P\}$ and $\hat{P}$ is an ideal in $\hat{E}$.

$\mathfrak{J}$ is partially ordered by inclusion. If $J_1 \subset J_2 \subset \dots$ is a chain in $\mathfrak{J}$, $\cup J_i$ an upper bound for this chain. $\therefore$ the class $\mathfrak{J}$ has a maximal element, by Zorn's lemma.

Let Q be a maximal element in $\mathfrak{J}$. We show that Q is prime in $\hat{E}$. Suppose not. Then $\exists a, b \in \hat{E} \setminus Q$ such that $a \wedge b = 0$ and $a \notin Q, b \notin Q$

$$\Rightarrow \exists \alpha \in A \setminus Q \text{ such that } x_\alpha \leq Aq + Ba$$

$$\text{and } \exists \beta \in A \setminus Q \text{ such that } x_\beta \leq Cq' + Db$$

where A, B, C, D are positive scalars, and

$$0 \leq q, q' \in Q$$

$$\Rightarrow 0 < x_\alpha \wedge x_\beta \leq (Aq + Ba) \wedge (Cq' + Db)$$

$$\leq Aq \wedge Cq' + Aq \wedge Db + Ba \wedge Cq' + Ba \wedge Db$$

$$\leq Aq \wedge Cq' + Aq + Cq'. \quad \because a \wedge b = 0$$

$\Rightarrow 0 < x_\alpha \wedge x_\beta \in Q$, contradiction. $Q \cap E = P$, by construction.

The following example shows that the prime ideal Q obtained in the above theorem is not unique, for a given P in E .

Example 1.11: Let $E = C$, the space of real convergent sequences, $\hat{E} = m$, the space of real bounded sequences. C_0 , the space of zero convergent sequences is a prime ideal in C , but $\hat{C}_0$ is not prime in $\hat{E}$,

where $\hat{C}_0 = \{x \in m: |x| \leq y, \text{ for some } y \text{ in } C_0\}$. $\hat{C}_0$ is an ideal in m . By Corollary 1.7 there is more than one prime ideal in m containing $\hat{C}_0$, for, if there is only one, it has to be $\hat{C}_0$. Let q be any one of them. Then $q \cap C \supset C_0$. But $q \cap C = C_0$, for, if not, a sequence converging to a non-zero number is in $q \Rightarrow 1 \in q \Rightarrow q = m$. In fact we prove that there are an infinite number of choices for q .

$$\text{Let } a = (0 \ 0 \ \cdots \underset{\alpha_1}{0 \ 1 \ 0} \ \cdots \underset{\alpha_2}{0 \ 1 \ 0} \ \cdots \underset{\alpha_3}{0 \ 1 \ 0} \ \cdots)$$

be an element of m , which has 1's in the positions $\alpha_1, \alpha_2, \dots$ and 0's at $N - \{A\}$ where N denotes the positive integers and $A = \{\alpha_i\}_1^\infty$. Let $\mathfrak{J}_{\hat{C}_0, a} = \{I:$

$I \text{ an ideal in } m, I \supset \hat{C}_0, a \notin I\}$. This class is non-empty for $\hat{C}_0$ is in it. By Zorn's lemma, this class has a maximal element. Let q_a be one such. By methods similar to the ones employed in the proof of Theorem 1.10, we can show that q_a is prime in m . Also $a' \in q_a$ where a' is the element in m with zeros at A and 1's at $N - A$, since $a' \wedge a = 0 \Rightarrow$ there are at least two prime ideals viz. q_a and q'_a ; $q_a \cap C = C_0$; $q'_a \cap C = C_0$.

In fact there are an infinite number of choices for q_a , by lemma 2.4. For we can have a sequence $a_1 < a_2 < \dots$, where the inequalities are strict

and each a_i is an element of the type a . By lemma 2.4, $R_a = R_b \Leftrightarrow I_a = I_b$ where R_a, R_b denote the class of relatively maximal ideals not containing a, b and I_a, I_b denote the principal ideals generated by a, b . $\Rightarrow R_{a_1} \subset R_{a_2} \subset R_{a_3} \dots$
 $a_1 \neq a_2 \neq a_3$

Masterson (5) has proved that if E has $P \cdot P \cdot$, and if P is a prime ideal in E , $\hat{P}$ is a prime ideal in $\hat{E}$, where $\hat{P} = \{y \in \hat{E} : |y| \leq |x|, \text{ for some } x \text{ in } P\}$. We prove below that if $\hat{P}$ is prime in $\hat{E}$ for all prime P in E , then E has $P \cdot P \cdot P$. The example given after the Theorem shows that the converse implication is not true.

Theorem 1.12: Let E be an Archimedean vector lattice and P a prime ideal in E . Let $\hat{P} = \{y \in \hat{E} : |y| \leq |x|, \text{ for some } x \text{ in } P\}$. If $\hat{P}$ is prime in $\hat{E}$, for all prime P in E , then E has $P \cdot P \cdot P$.

Proof: Suppose E does not have $P \cdot P \cdot P$. Then $\exists a, b \in E^+$ such that $\sup_n (b \wedge na)$ does not exist in E .

By theorem 0.23, $\sup_n (b \wedge na)$ exists in $\hat{E}$.

Let this sup be denoted by $a(b)$.

$$\therefore b = a(b) + b - a(b) \text{ in } \hat{E}.$$

Consider the class $\mathfrak{J} = \{I \text{ an ideal in } \hat{E} :$

$x \in E, |x| < a(b) \Rightarrow x \in I; a(b) \notin I\}$. This class is

non-empty, for, the ideal generated by all elements of E less than $a(b)$ is an element of the class. The definition of I is consistent, for if $0 < x \in E$ and $x < a(b)$, $\nexists$ scalar α $\} a(b) < \alpha x$; for suppose such an α exists. $x < a(b) \Rightarrow x \wedge (b - a(b)) = 0$
 $\Rightarrow \alpha x \wedge b = \alpha x \wedge (a(b) + b - a(b))$, in $\hat{E}$
 $= \alpha x \wedge a(b) + \alpha x \wedge (b - a(b))$, in $\hat{E}$
 $= \alpha x \wedge a(b) = a(b)$

$\therefore \alpha x \wedge b = a(b)$.

But $\alpha x \wedge b \in E \Rightarrow a(b)$ exists in E , which is not the case.

Partially order the class $\mathfrak{J}$ by inclusion. Each chain has an upper bound. By Zorn's lemma the class $\mathfrak{J}$ has a maximal element M . By Theorem 1.5, M is prime in $\hat{E}$. $a(b) \notin M \Rightarrow b - a(b) \in M$. Let $P = M \cap E$. P is prime in E . We will show that $\nexists y \in E$, $y > b - a(b)$ such that $y \in M$. This proves that $\hat{P}$ is not prime in $\hat{E}$. Let $y \in E$, $y > (b - a(b))$

$$b \geq y \wedge b \neq b - a(b)$$

$y \wedge b \neq b - a(b)$, since $y \wedge b \in E$ and $b - a(b) \in \hat{E} - E$

This says: $b - (b - a(b)) > b - y \wedge b$

i.e., $a(b) > b - y \wedge b \in E$.

By construction, $b - y \wedge b \in M$

if $y \in M$, then $y \wedge b \in M$

$$\Rightarrow (b - y \wedge b) + y \wedge b = b \in M$$

which cannot be.

So, $\nexists y > (b - a(b))$, $y \in E$ such that $y \in M$.

Q.E.D.

Example 1.13: Let L be the set of all real bounded functions f on the point set $X = \{1, 2, \dots, \infty\}$ such that $f(x) \neq f(\infty)$, for at most finitely many x . This is a Riesz space with the usual point wise operations. This has $P \cdot P \cdot P \cdot$, but does not have $P \cdot P \cdot$ and is not Dedekind σ -complete. Let A be an arbitrary principal band in L , generated by $v \in L^+$, and denote the set $\{x: f(x) \neq 0 \text{ for at least one } f \in A\} = \{x: v(x) > 0\}$ by X_1 . If $v(\infty) = 0$, then X_1 consists only of a finite number of points and $X_2 = X - X_1$ contains the point ∞ . If $v(\infty) > 0$, then X_1 consists of all but a finite number of points, and X_1 contains the point ∞ . Hence given $f \in L$, we have in both cases

$$f = f \cdot \chi_{X_1} + f \cdot \chi_{X_2} \text{ with } f \cdot \chi_{X_1} \in A; f \cdot \chi_{X_2} \in A^\perp$$

$$\therefore L = A \oplus A^\perp, \quad \therefore L \text{ has } P \cdot P \cdot P \cdot$$

L is not Dedekind complete by Theorem 0.23. To show that L does not have $P \cdot P \cdot$, consider the band A in L , defined by

$$A = \{f \in L : f(x) = 0 \text{ for } x = 1, 3, 5, \dots\}$$

$$\text{then } A^\perp = \{f \in L : f(x) = 0 \text{ for } x = 2, 4, 6, \dots\}$$

$$A \oplus A^\perp = \{f \in L : f(\infty) = 0\} \neq L.$$

In this space, a prime ideal is the class of all functions vanishing at a particular point. The class of functions vanishing at a given point x_0 is clearly an ideal. It is a prime ideal, since if $f \wedge g = 0$, $f(x_0) = 0$ or $g(x_0) = 0$. These are all the prime ideals, for suppose an ideal I contains only functions which vanish on at least two points. Let the two points be x_1 and x_2 , $x_1 < x_2$, where x_2 can be $= \infty$. Consider the functions

$$\begin{aligned} h_1 &= 1 & \text{at } x \leq x_1 \\ &= 0 & \text{at } x > x_1 \\ h_2 &= 0 & \text{at } x \leq x_1 \\ &= 1 & \text{at } x > x_1 \end{aligned}$$

$h_1 \wedge h_2 = 0$, but $h_1 \notin I$, $h_2 \notin I$.

Suppose an ideal I contains a function k which does not vanish at any point. Let $k(\infty) = a$. $\Rightarrow k(x) = a$ for all but finitely many x . Let the points y with $k(y) \neq a$ be $y_1, y_2, \dots, y_m$. Let

$$\min (|k| (y_1), |k| (y_2), \dots, |k| (y_m)) = b > 0. \text{ Let } |a| \wedge b = c > 0$$

$\Rightarrow k' \in I$ where $k'(x) = c$ for $x = 1, 2, \dots, \infty$

$\Rightarrow I = L \quad \therefore I$ is not proper.

Let the prime ideals be $P_1, P_2, \dots, P_\infty$. P_∞ has the property that its elements are functions vanishing at all but a finite number of points. $\hat{L}$ is the class of all bounded functions on the point set X . Clearly $\hat{P}_\infty$ is

not prime in $\hat{L}$, for $\hat{P}_\infty$ contains only functions which vanish at ∞ and all but finitely many points.

Example 1.14: We have noted (Theorem 0.23) that Dedekind σ -completeness is a property stronger than $P \cdot P \cdot P$. The example below shows that even this property is not strong enough to imply the primeness of $\hat{P}$, for every prime P in E .

Let L be the collection of all real bounded functions f on $[0, 1]$ such that $f(x) \neq f(0)$ for at most countably many x . This is a vector lattice with the usual pointwise operations. Given that $0 \leq u_n < v$, $\sup u_n$ exists in L and is the pointwise limit. Hence L is Dedekind σ -complete. To show that L does not have the $P \cdot P$, let A be the band in L consisting of all $f \in L$ vanishing on $[0, \frac{1}{2}]$. Then $A^\perp$ is the band of all $f \in L$ vanishing on $(\frac{1}{2}, 1]$. Any $f \in A$ satisfies $f(x) \neq 0$ for at most countably many x . The same is true for any $f \in A^\perp$ and hence for any f in $A \oplus A^\perp$. $\therefore L \neq A \oplus A^\perp$. $\therefore L$ does not have $P \cdot P$. I_0 is a prime ideal in L where I_0 is the collection of all functions vanishing at 0. The Dedekind completion $\hat{L}$ of L is the class of all bounded functions on $[0, 1]$. $\hat{I}_0$ is not prime in $\hat{L}$, for let $A \subset [0, 1]$, such that A and A^c are uncountable. Let $f, g, \in \hat{L}$ such that

$$\begin{aligned}
 f &= 1 & \text{at} & A ; & g &= 0 & \text{at} & A \\
 &= 0 & \text{at} & A^c ; & &= 1 & \text{at} & A^c
 \end{aligned}$$

$f \wedge g = 0$, but $f \notin \hat{I}_0$, $g \notin \hat{I}_0$, since $\hat{I}_0$ is the collection of functions vanishing at 0 and all but countably many x in $[0, 1]$. Hence, the example.

Section 2

We had noted in Section 1 that Masterson has proved that if E has $P \cdot P \cdot \hat{P}$ is prime in $\hat{E}$, if P is prime in E . (Theorem 2.1, Masterson 5) We obtain a partial converse to this result (Theorem 2.6) Masterson has mentioned the property (which we will call the property *), viz. $\forall x \in \hat{E}^+$, the existence of $y \in E^+$ $\rightarrow \alpha y \leq x \leq y$ for some scalar α . He notes that if E has this property, the structure spaces of E and $\hat{E}$ are homeomorphic. He also obtains a characterization of this property (Masterson 5, Theorem 2.6). We obtain other characterizations of this property (Theorem 2.5). We show that this property is strictly stronger than $P \cdot P \cdot$ (Theorem 2.6, Theorem 2.7, Example 3.5). We also obtain a characterization of $P \cdot P \cdot$ in terms of a property analogous to property * (Theorem 2.7). We also mention a property which is strictly stronger than property * (Theorem 2.9, Example 2.10).

Lemma 2.1: Let E be an Archimedean vector lattice and A a band in E . Then any positive element x of E can be written as $x = x_1 + x_2$, where $x_1 = \sup_{\alpha} (x_{\alpha})$ $x_{\alpha} \in A$ and $x_2 = \sup_{\beta} (x_{\beta})$, $x_{\beta} \in A^{\perp}$, the sups taken in $\hat{E}$.

Proof: If $x \in A \oplus A^\perp$, there is nothing to prove. $\therefore$ Assume $x \in E \setminus A \oplus A^\perp$
 $\hat{E} = A^{\perp\perp}(\text{in } \hat{E}) \oplus A^\perp(\text{in } \hat{E})$ (since $\hat{E}$ has P.P. by Theorem 0.22) $\therefore x = x_1 + x_2$, where $x_1 \in A^{\perp\perp}(\text{in } \hat{E})$
 $x_2 \in A^\perp(\text{in } \hat{E})$
 $x_2 = \sup_\beta (x_\beta)$, $x_\beta \in E \quad \forall \beta$. $x_\beta \in A^\perp(\text{in } E) \quad \forall \beta$
for if $x_\beta \in A$ for some β , $x_2 \notin A^\perp(\text{in } \hat{E})$. Also if $x_\beta \in E - A \oplus A^\perp(\text{in } E)$ then $x_\beta \wedge y > 0$ for some $y \in A$
 $\Rightarrow x_2 \notin A^\perp(\text{in } \hat{E})$. Also $x_1 = \sup_\alpha (x_\alpha)$, $x_\alpha \in E \quad \forall \alpha$.
 $x_\alpha \in A \quad \forall \alpha$, for if $x_{\alpha_0} \in A^\perp(\text{in } E)$ for some α then
 $x_1 \notin A^{\perp\perp}(\text{in } \hat{E})$. $x_\alpha \notin E - A \oplus A^\perp(\text{in } E) \quad \forall \alpha$ for if
 $x_{\alpha_0} \in E - A \oplus A^\perp(\text{in } E)$, then $x_{\alpha_0} \wedge z > 0$ for some
 $z \in A^\perp(\text{in } E)$ (for if $x_{\alpha_0} \wedge z = 0 \quad \forall z \in A^\perp(\text{in } E)$,
then $x_{\alpha_0} \in A^{\perp\perp}(\text{in } E) = A$, since E is Archimedean)
 $\Rightarrow x_1 \notin A^{\perp\perp}(\text{in } \hat{E})$. Q.E.D.

Lemma 2.2: If $E = A \oplus A^\perp$, where A is an ideal in E , then $\hat{E} = \hat{A} \oplus \hat{A}^\perp$, where $\hat{E}$, $\hat{A}$, $\hat{A}^\perp$ are the Dedekind completions of E , A , $A^\perp$ respectively, as vector lattices.

Proof: Let $x \in (\hat{E})^+$ and let $x = \sup_\alpha x_\alpha$, where
 $x_\alpha \in E^+ \quad \forall \alpha$.

W.L.O.G. let, $x \leq y \in E^+$, for any element of $\hat{E}$ is majorized by an element of E .

Let $y = y_1 + y_2$ where $y_1 \in A$, $y_2 \in A^\perp$.

Let $x_\alpha = x_{\alpha_1} + x_{\alpha_2}$ where $x_{\alpha_1} \in A$,

$x_{\alpha_2} \in A^\perp \quad \forall \alpha$

$$x = \sup_{\alpha} x_{\alpha} = \sup_{\alpha} (x_{\alpha_1} + x_{\alpha_2})$$

$$= \sup_{\alpha} (x_{\alpha_1}) + \sup_{\alpha} (x_{\alpha_2})$$

Since $x_{\alpha_1} \perp x_{\beta_2} \quad \forall \alpha, \beta$.

(all the sups taken in $\hat{E}$). Also $x_{\alpha_1} \leq y_1, \quad \forall \alpha$,

$$x_{\alpha_2} \leq y_2 \quad \forall \alpha$$

$$\therefore \sup_{\alpha} (x_{\alpha_1}) \leq y_1; \quad \sup_{\alpha} (x_{\alpha_2}) \leq y_2$$

$$\Rightarrow \sup_{\alpha} (x_{\alpha_1}) \in \hat{A}; \quad \sup_{\alpha} (x_{\alpha_2}) \in \hat{A}^\perp$$

Corollary 2.3: If E has P.P., $\hat{E} = \hat{B} \oplus \hat{B}^\perp$ for any band B of E .

Lemma 2.4 Let E be any vector lattice. Let $a \in E^+$. Let $B = \{b \in E^+ : \exists \text{ scalar } \alpha \text{ } a < \alpha (b_1 \wedge b_2 \wedge \dots \wedge b_k) \quad \forall \text{ integers } k \text{ and all choices of } b_1, b_2, \dots, b_k, \exists \text{ scalar } \beta \text{ } (b_1 \wedge b_2 \wedge \dots \wedge b_n) < \beta a, \quad \forall \text{ integers } n \text{ and all choices of } b_1, b_2, \dots, b_n.\}$

Then there exists a prime ideal P in E $\} \quad a \in P,$
 $P \cap B = \emptyset.$

Note: B is the class of all elements in E^+ such that if b_0 is any finite infimum of elements of B , the principal ideal generated by a is a proper subset of the principal ideal generated by b_0 . Also B is closed under finite infima.

Proof: Consider the class

$$\mathfrak{J} = \{I : I \text{ an ideal in } E, a \in I, I \cap B = \emptyset\}$$

$\mathfrak{J} \neq \emptyset$, for $I_a \in \mathfrak{J}$, where I_a is the principal ideal generated by a . Partially order $\mathfrak{J}$ by inclusion. Each chain has an upper bound. $\therefore \mathfrak{J}$ has a maximal element M by Zorn's lemma. We shall prove that M is prime.

Suppose not. Then $\exists x, y \in E, x \wedge y = 0, x \notin M, y \notin M$.

But $\exists m_1 \in M, b_1 \in B$ and positive scalars C_1, C'_1 $\} \quad$

$b_1 \leq C_1 x + C'_1 m_1$; also $\exists m_2 \in M, b_2 \in B$ and positive

scalars C_2, C'_2 $\} \quad b_2 \leq C_2 y + C'_2 m_2$

$$\therefore b_1 \wedge b_2 \leq (C_1 x + C'_1 m_1) \wedge (C_2 y + C'_2 m_2)$$

$$\leq (C_1 x \wedge C'_2 m_2) + (C'_1 m_1 \wedge C_2 y) + (C'_1 m_1 \wedge C'_2 m_2)$$

(Since $x \wedge y = 0$)

$$\leq C'_2 m_2 + C'_1 m_1 + C'_1 m_1 \in M. \text{ This is a}$$

Contradiction, since B is closed under finite infima and

M is an ideal.

Theorem 2.5: The following are equivalent:

- (1) Every ideal in $\hat{E}$ is of the form $\hat{I}$,
for some ideal I in E .
- (2) For each $u \in \hat{E}^+$ $\exists v \in E$ and scalar
 $\alpha > 0$ $\alpha v \leq u \leq v$.
- (3) Every prime ideal in $\hat{E}$ is of the form $\hat{P}$
for some prime ideal P in E .

Proof: Masterson (5, Theorem 2.6) has proved the equivalence of statements (1) and (2). $\therefore$ We have only to prove the equivalence of (2) and (3).

(2) $\Rightarrow$ (3): Let Q be prime in $\hat{E}$. Let $Q \cap E = P$ to prove $\hat{P} = Q$. $\hat{P} \subset Q$, by definition.
 $\therefore$ We have only to prove that $Q \subset \hat{P}$. Let $u \in Q \cap \hat{E}^+$
by (2) $\exists v \in E$ and positive scalar $\alpha > 0$ $\alpha v \leq u \leq v$
 $\Rightarrow \alpha v \in P \Rightarrow v \in P \Rightarrow u \in \hat{P}$

(3) $\Rightarrow$ (2): Assume (3) and suppose (2) is not true. Then, $\exists u \in \hat{E}^+$ $\nexists$ for no $x \in E^+$ is $x > u$, $\exists$ positive scalar $\alpha > 0$ $\alpha x \leq u \leq x$. Let $X = \{x \in E^+ : u \leq x\}$. The hypothesis of Lemma 2.4 are satisfied
 $\therefore \exists$ prime Q in $\hat{E}$ $\ni u \in Q$ and $Q \cap X = \emptyset$; then Q is not of the form $\hat{P}$ for any prime P in E , for
 $\nexists v \in E \cap Q \ni v > u$. This contradicts the hypothesis.

Q.E.D.

Theorem 2.6: Let E be an Archimedean vector lattice and $\hat{E}$ its Dedekind completion. Let P be any prime ideal in E . Let $\hat{P} = \{y \in \hat{E} : |y| \leq |x|, \text{ for some } x \text{ in } P\}$. If $\hat{P}$ is prime in $\hat{E}$ for all prime P in E , and every prime in $\hat{E}$ is of the form $\hat{P}$ for some prime P in E , then E has $P \cdot P$.

Proof: Suppose E does not have $P \cdot P$. Then E band $B \rightarrow E \setminus B \oplus B^\perp \neq \emptyset$. Let $E \setminus B \oplus B^\perp = \{x_\alpha\}_{\alpha \in A}$. Let $x_{\alpha_0} \in \{x_\alpha\}_{\alpha \in A}$. $\hat{E} = B^{\perp\perp} \text{ (in } \hat{E}) \oplus B^\perp \text{ (in } \hat{E})$ (Theorem 0.23). Let $x_{\alpha_0} = x_{01} + x_{02}$ where $x_{01} \in B^{\perp\perp} \text{ (in } \hat{E})$, $x_{02} \in B^\perp \text{ (in } \hat{E})$. By Lemma 2.1 x_{01} is the sup of a subset of elements of B . By theorem 2.5 $\exists y_1 \in E \rightarrow \alpha y_1 \leq x_{01} \leq y_1$. $y_1 \in B$, for, if not $y_1 = x_\alpha$ for some α or $y_1 = y_{11} + y_{12}$ where $y_{11} \in B$, $y_{12} \in B^\perp \text{ (in } E)$. $y_1 = x_\alpha \in E \setminus B \oplus B^\perp \text{ (in } E)$
 $\Rightarrow \exists x_{\alpha'} \in B^\perp \text{ (in } E) \rightarrow x_\alpha \wedge x_{\alpha'} > 0$. $x_\alpha \wedge x_{\alpha'} \in I_{y_1}$ and $x_\alpha \wedge x_{\alpha'} \in B^\perp \text{ (in } E)$ which cannot be since $I_{y_1} \subset B^{\perp\perp} \text{ (in } \hat{E})$. Similarly y_1 cannot be of the form $y_{11} + y_{12}$ where $y_{11} \in B$, $y_{12} \in B^\perp \text{ (in } E)$. $\therefore$ We have $x_{01} < y_1$ where $y_1 \in B$. Similarly $\exists y_2 \in B^\perp \text{ (in } E) \rightarrow x_{02} < y_2$.
 $\therefore x_{\alpha_0} = x_{01} + x_{02} < y_1 + y_2 \in B \oplus B^\perp \text{ (in } E) \Rightarrow x_{\alpha_0} \in B \oplus B^\perp \text{ (in } E)$ since $B \oplus B^\perp$ is an ideal. Contradiction!

Q.E.D.

Theorem 2.7: Let E be an Archimedean vector lattice and $\hat{E}$ its Dedekind completion. Then E has P.P. $\Leftrightarrow \forall a \in \hat{E}^+ \exists d_1 \in E^+ \} a < d_1$ and the principal bands generated by a and d_1 in $\hat{E}$ are identical

Proof: ($\Rightarrow$) Let $C = \{c \in E^+ : c < a\}$. Let B be the band generated by C in E . Let $d \in E \} d > c \forall c \in C$. Such a d exists, for, if $x \in \hat{E}^+ \exists y \in E \} y > x$. $E = B \oplus B^\perp$.

Let $d = d_1 + d_2$, $d_1 \in B$, $d_2 \in B^\perp$. $a < d \Rightarrow a < d_1$
 $\therefore B_a(\text{in } \hat{E}) \subset B_{d_1}(\text{in } \hat{E})$; also $d_1 \in B \subset B_a(\text{in } \hat{E})$

$\therefore B_{d_1}(\text{in } \hat{E}) \subset B_a(\text{in } \hat{E}) \therefore B_a(\text{in } \hat{E}) = B_{d_1}(\text{in } \hat{E})$

($\Leftarrow$) Suppose E does not have P.P. Then $\exists$ band B in $E \} E \setminus B \oplus B^\perp \neq \emptyset$. Let $0 < x_0 \in E \setminus B \oplus B^\perp$
 $\hat{E} = B^{\perp\perp}(\text{in } \hat{E}) \oplus B^\perp(\text{in } \hat{E})$. $x_0 = x_{01} + x_{02}$ where $x_{01} \in B^{\perp\perp}(\text{in } \hat{E})$, $x_{02} \in B^\perp(\text{in } \hat{E})$.

By hypothesis $\exists y_1 \in E \} x_{01} < y_1$ and $B_{x_{01}}(\text{in } \hat{E}) = B_{y_1}(\text{in } \hat{E}) \therefore y_1 \in B_{x_{01}}(\text{in } \hat{E}) \subset B^{\perp\perp}(\text{in } \hat{E})$

$\Rightarrow y_1 \in B$. Similarly $\exists y_2 \in E \} x_{02} < y_2$ and $B_{x_{02}}(\text{in } \hat{E}) = B_{y_2}(\text{in } \hat{E}) \Rightarrow y_2 \in B^\perp(\text{in } E)$

$\therefore x_0 = x_{01} + x_{02} < y_1 + y_2 \in B \oplus B^\perp(\text{in } E) \Rightarrow x_0 \in B \oplus B^\perp$,
 since $B \oplus B^\perp$ is an ideal.

2.8 A property strictly stronger than Property* .

Let E be an Archimedean vector lattice and $\hat{E}$ its Dedekind completion. Consider the property:
for $x, y \in \hat{E}^+$ such that $x < y$, $\exists z \in E^+$ such that $x < z < y$. We will show that this property implies property*. The converse implication is not true. There are vector lattices which have this property, but which are not Dedekind complete.

Proof: Let $x \in \hat{E}^+$; $2x > x \therefore \exists z \in E$ }
 $x < z < 2x \Rightarrow I_x(\text{in } \hat{E}) = I_z(\text{in } \hat{E})$. Hence property*.
The rationals is a vector lattice which has the above-mentioned property. The following example shows that there are spaces without this property, but which have property*.

Example 2.9: Let E be the set of bounded sequences (a_n) of real numbers such that the set of points $\{a_n : n = 1, 2, \dots\}$ has a finite or countable closure. E is an Archimedean vector lattice with the usual pointwise operations. E is not Dedekind complete since if $\{r_n ; n = 1, 2, \dots\}$ is an ordering of rationals in the closed interval $[0, 1]$ and (a_n) is the sequence which takes the value r_n at n and 0 elsewhere then the set $\{(a_n) : n = 1, 2, \dots\}$ is bounded in E , but does not have a sup in E . $\hat{E} = m$, the bounded sequences of real numbers.

Masterson (5) has proved that E has property * . Consider $x = (r_1, r_2, \dots, r_n, \dots)$ where $\{r_n\}$ is the ordinary of rationals in $[0, 1]$, as mentioned above. Let $y = (r_1, r_2, \dots, r_{n-1}, r_n + 1, r_{n+1}, \dots)$. $x < y$, $x, y \in \hat{E}^+$. But there exists no $z \in E$ $\} x < z < y$, for any $z = (z_1, z_2, \dots)$ has the property that the closure of $\{z_1, z_2, \dots\}$ is uncountable.

Theorem 2.10: If E has P.P. every prime ideal Q in $\hat{E}$ is either

- (a) of the form $\hat{P}$ for some prime ideal P in E , or
- (b) order dense.

Proof: Let Q be prime in $\hat{E}$. Let $P = Q \cap E$. Suppose $Q \neq \hat{P}$. Then, $\exists a \in Q$ $\} \nexists b \in P$ $\} a < b$. Let $A = \{a_\alpha \in E^+ : a_\alpha < a\}$. Let B_A be the band generated by elements of A . $E = B_A \oplus B_A^\perp$. Let $c \in E$ $\} a < c$. Let $c = c_1 + c_2$ where $c_1 \in B_A$; $c_2 \in B_A^\perp$ $\therefore a < c_1$ by assumption ($\nexists b \in P$ $\} a < b$), $c_1 \notin P$. $c_1 \notin P \Rightarrow B_A^\perp$ (in $\hat{E}$) $\subset Q$, since B_{c_1} (in $\hat{E}$) $= B_A$ (in $\hat{E}$) also $a \in Q \Rightarrow I_a \subset Q \Rightarrow Q$ is order dense.

Section 3

It is well-known that if x, y_1, y_2 are positive elements of a vector lattice E such that $x \leq y_1 + y_2$, there exists a decomposition of $x = x_1 + x_2$ where $0 \leq x_1 \leq y_1$ and $0 \leq x_2 \leq y_2$. (property 0.9). If E is Archimedean, $x \in E$ and $y_1, y_2 \in \hat{E}$, it is not in general true that $x_1, x_2 \in E$. If every $x \in E^+$ can be expressed as, $x = x_1 + x_2$ for all $y_1, y_2 \in \hat{E}^+$, we say that E has property^{**}. In the presence of property^{**}, starting with a prime ideal P in E , we can obtain a prime ideal $\tilde{P}$ in $\hat{E}$. Under the same conditions, if $\mathcal{P}$ is the structure space of all prime ideals in E , $\mathcal{P}$ and $f(\mathcal{P})$ are homeomorphic, where $f(P) = \tilde{P}$ (Theorem 3.2). Also there is a way of characterizing property^{**} in terms of prime ideals (Theorem 3.1). We show that Property^{*} $\Rightarrow$ Property^{**} $\Rightarrow P \cdot P \cdot$ (Theorem 3.3, Theorem 3.4). Property^{**} is strictly stronger than $P \cdot P \cdot$ (Example 3.5), but it is not known whether property^{*} is strictly stronger than property^{**}. We also show that there is a way of characterizing $P \cdot P \cdot$ by a property analogous to property^{**} (Theorem 3.4)

Theorem 3.1: Let E be an Archimedean vector lattice, P a prime ideal in E , and $\hat{E}$ the Dedekind completion of E . Let $\tilde{P} = \{a \in \hat{E} : A_a \subset P\}$ where

$A_a = \{b \in E : |b| \leq |a|\}$. If E has property^{**}, $\tilde{P}$ is prime in $\hat{E}$. On the other hand if $\tilde{P}$ is prime in $\hat{E}$ for all prime P in E , then E has property^{**}

Proof: Let E have property^{**}

- (1) Let $a \in \tilde{P}$ and $|c| \leq |a|$. If $|d| \leq |c|$ and $d \in E$ then $d \in P$, since $|d| \leq |a|$.

So, $c \in \tilde{P} \Rightarrow \tilde{P}$ is solid

- (2) If $a \in \tilde{P}$, we show $\alpha a \in \tilde{P}$, where α is a scalar: Let $|b| \leq |\alpha a| = |\alpha| |a|$, $b \in E$.

$$\therefore \frac{|b|}{|\alpha|} \leq |a| \Rightarrow \frac{b}{\alpha} \in P \Rightarrow b \in P \Rightarrow \alpha a \in \tilde{P}$$

- (3) To show that $\tilde{P}$ is closed under addition:

let $a_1, a_2 \in \tilde{P}$. Let $b \leq a_1 + a_2$, where $a_1, a_2 \in \hat{E}^+$, $b \in E^+$. By property^{**}, $b = b_1 + b_2$ where $0 \leq b_1 \leq a_1$, $0 \leq b_2 \leq a_2$, $b_1, b_2 \in E$, $a_1 \in \tilde{P} \Rightarrow b_1 \in P$; $a_2 \in \tilde{P} \Rightarrow b_2 \in P$
 $\therefore b = b_1 + b_2 \in P \Rightarrow a_1 + a_2 \in \tilde{P}$

- (4) To show that $\tilde{P}$ is prime: let $a \wedge b = 0$, $a, b \in \hat{E}$. Suppose $a \notin \tilde{P}$, $b \notin \tilde{P}$. Then, by definition, $\exists a_1 \in E$, $a_1 < a \Rightarrow a_1 \notin P$; $\exists b_1 \in E$, $b_1 < b \Rightarrow b_1 \notin P$. This cannot be since $a \wedge b = 0 \Rightarrow a_1 \wedge b_1 = 0$. Contradiction.

On the other hand, assume $\tilde{P}$ prime in $\hat{E}$ for all prime P in E . Suppose E does not have property^{**}. Then $\exists a \in E^+$, $b_1, b_2 \in \hat{E}^+ \Rightarrow a < b_1 + b_2$ and a cannot be written

as = $a_1 + a_2$, $a_1, a_2 \in E$, $a_1 < b_1$, $a_2 < b_2$.

So, at last one of b_1, b_2 is in $E^\wedge \setminus E$.

Consider the classes $B_1 = \{b'_1 \in E^+ : b'_1 \leq b_1\}$

$B_2 = \{b'_2 \in E^+ : b'_2 \leq b_2\}$

$\mathcal{U} = \{A : A \text{ an ideal in } E, a \notin A, B_1 \subset A, B_2 \subset A\}$

$\nexists c \in E : a < c < b_1 + b_2$ where c can be written

as = $c_1 + c_2$, $c_1 < b_1$, $c_2 < b_2$, $c_1, c_2 \in E$; for if

such a c exists, a can be decomposed in the required

way. Hence the class $\mathcal{U}$ is non-empty, for the ideal

generated by $B_1 \cup B_2$ is an element of $\mathcal{U}$.

Partially order $\mathcal{U}$ by inclusion. Any chain has an upper bound. $\therefore \mathcal{U}$ has a maximal element M . We will show that M is prime in E . Suppose not. Then $\exists x, y \in E$, $x \wedge y = 0$, $x \notin M$, $y \notin M$.

So, $\exists m_1, m_2 \in M$ and scalars s_1, s_2, s_3, s_4

$$\} \quad a \leq s_1 m_1 + s_3 x$$

$$a \leq s_2 m_2 + s_4 y$$

$$\therefore a \leq (s_1 m_1 + s_3 x) \wedge (s_2 m_2 + s_4 y)$$

$$\leq s_1 m_1 \wedge s_2 m_2 + s_3 x \wedge s_2 m_2 + s_1 m_1 \wedge s_4 y$$

$$\leq s_1 m_1 \wedge s_2 m_2 + s_2 m_2 + s_1 m_1$$

$\in M$. This is a contradiction.

Hence M is prime.

By hypothesis, $\tilde{M}$ is prime in $E^\wedge$. $b_1, b_2 \in \tilde{M}$, since $B_1 \subset M$ and $B_2 \subset M$. But $b_1 + b_2 \notin \tilde{M}$ since $a \notin M$, a contradiction. Q.E.D.

Masterson (5) has attempted to compare the topological properties of structure spaces of E and $\hat{E}$. He considers the homeomorphism under the mapping $f(P) = \hat{P}$, where P is a prime ideal in E and $\hat{P} = \{y \in \hat{E} : |y| \leq x\}$, when E has $P \cdot P$. He shows that if $\mathfrak{P}$ is a structure space of E consisting entirely of minimal prime ideals, $\mathfrak{P}$ is homeomorphic to $f(\mathfrak{P})$, where $f(\mathfrak{P})$ has the relative topology from the structure space of $\hat{E}$. He also has several conditions which are necessary and sufficient for the homeomorphism of pairs $\mathfrak{P}$ and $f(\mathfrak{P})$ where $\mathfrak{P}$ is any structure space of E , and $f(\mathfrak{P})$ has the relative topology from the structure space of $\hat{E}$, when E has the $P \cdot P$. We show below that if E has property** and $\mathfrak{M}$ is the structure space of E , $\mathfrak{M}$ and $g(\mathfrak{M})$ are homeomorphic where $g(P) = \tilde{P}$, defined above.

Theorem 3.2 Let E satisfy property** . Let $\mathfrak{M}$ be the structure space of all prime ideals of E and $\mathfrak{N}$, that of $\hat{E}$. The mapping $f : \mathfrak{M} \rightarrow \mathfrak{N}$ where $f(P) = \tilde{P}$ is one-one, into. Under this mapping $\mathfrak{M}$ and $f(\mathfrak{M})$ are homeomorphic where $f(\mathfrak{M})$ has the relative topology from $\mathfrak{N}$.

Proof: Let $\mathfrak{M}_a = \{P \in \mathfrak{M} : a \notin P\}$, $a \in E^+$, be a basic open set in $\mathfrak{M}$.

$$a \notin P \Rightarrow a \notin \tilde{P} \Rightarrow f(\mathfrak{M}_a) = \mathfrak{N}_a \cap f(\mathfrak{M}) .$$

Hence $f(\mathfrak{M}_a)$ is open in the relative topology. On the other hand, if $b \in \hat{E}$

$$f^{-1}(\mathfrak{M}_b \cap f(\mathfrak{M})) = \cup \{\mathfrak{M}_a : a \in E, |a| < |b|\},$$

since $b \notin \tilde{P} \Leftrightarrow \exists |a| \leq |b|, a \in E, a \notin P$.

$\therefore f$ is one-one onto from $\mathfrak{M}$ to $f(\mathfrak{M})$ and bicontinuous. So, f is a homeomorphism.

Q.E.D.

Theorem 3.3 The following are equivalent:

- (1) For $x \in \hat{E}^+, \exists y \in E^+ : \alpha y \leq x \leq y$.
- (2) Every ideal in $\hat{E}$ is of the form $\hat{I}$ for some ideal I in E .
- (3) Every prime ideal in $\hat{E}$ is of the form $\hat{P}$ for some prime ideal P in E .
- (4) Every prime ideal in $\hat{E}$ is of the form $\tilde{P}$ for some prime ideal P in E .
- (5) Every ideal in $\hat{E}$ is of the form $\tilde{I}$ for some ideal I in E .

Proof: We have already proved the equivalence of statements (1), (2), (3) (Theorem 2.5). So it is enough to prove the equivalence of (1), (4) and (5).

(1) $\Rightarrow$ (4): Let Q be a prime ideal in $\hat{E}$. Let $P = Q \cap E$. Let $a \in Q \cap (\hat{E}^+ \setminus E)$. By hypothesis $\exists b \in E^+$ and scalar α such that $\alpha b \leq a \leq b$

$$a \in Q \Rightarrow \alpha b \in P \Rightarrow b \in P \Rightarrow A \subset P$$

where $A = \{a' \in E^+ : a' < a\} \Rightarrow Q = \tilde{P}$.

(4) $\Rightarrow$ (1): Suppose $\exists a \in \hat{E}^+$ such that
 $\nexists b \in E^+$ and scalar α such that $\alpha b \leq a \leq b$
 Then $\nexists b_1, b_2, \dots, b_k \rightarrow$

$\alpha(b_1 \wedge b_2 \wedge \dots \wedge b_k) \leq a \leq (b_1 \wedge b_2 \wedge \dots \wedge b_k)$
 for any finite collection of elements of E . By Theorem
 1.5, there exists a prime Q in $\hat{E} \rightarrow a \notin Q$ and
 $B \subset Q$ where $B = \{b \in E^+ : b < a\}$.

Clearly Q is not of the form $\tilde{P}$ for any prime
 P in E , since $B \subset Q$ and $a \notin Q$.

(1) $\Rightarrow$ (5): Let J be an ideal in $\hat{E}$.
 $J \cap E = I$ is an ideal in E . Let $x \in J \cap \hat{E}^+$. By
 hypothesis $\exists y \in E^+ \rightarrow \alpha y \leq x \leq y$. $\alpha y \leq x \Rightarrow y \in I$
 $\Rightarrow B_x \subset I$ where $B_x = \{z \in E : |z| \leq x\} \Rightarrow J = \tilde{I}$

(5) $\Rightarrow$ (1): Assume (5). Suppose (1) is not
 true. $\therefore \exists x \in \hat{E}^+$ such that there does not
 exist $y \in E^+$ such that $\alpha y \leq x \leq y$. Let
 $A = \{y \in E : |y| \leq x\}$. Consider the ideal generated by A
 in E . Let it be I_A . Consider $\hat{I}_A$: this is an ideal in
 $\hat{E}$; $x \notin \hat{I}_A \Rightarrow \hat{I}_A$ is not of the form $\tilde{I}$, for any I
 in E .

Theorem 3.4: E has $P \cdot P = V \quad x \in E^+$,
 $y_1, y_2 \in \hat{E}^+ \rightarrow y_1 + y_2$ and $x \leq y_1 + y_2$, x can be

expressed as $x = x_1 + x_2$ where $x_1 < y_1$, $x_2 < y_2$,
 $x_1, x_2 \in E$.

Proof: ($\Leftarrow$) Suppose E does not have P.P. $\Rightarrow$

$\exists$ band B in E and $a \in E^+$

$\supset a \notin B \oplus B^\perp$

$\hat{E} = B^{\perp\perp}(\text{in } \hat{E}) \oplus B^\perp(\text{in } \hat{E})$

$\therefore a = a_1 + a_2$, where $a_1 \in B^{\perp\perp}(\text{in } \hat{E})$, $a_2 \in B^\perp(\text{in } \hat{E})$.

By hypothesis, $a = a'_1 + a'_2$ where $a'_1 \leq a_1$; $a'_2 \leq a_2$,

$a'_1 \in E$, $a'_2 \in E$. But $a'_1 = a_1$ and $a'_2 = a_2$ for if

$a'_1 < a_1$ or $a'_2 < a_2$, $a = a'_1 + a'_2 < a_1 + a_2 = a$.

Contradiction.

$\therefore a_1 \in B$ and $a_2 \in B^\perp$

$\therefore E$ has P.P.

($\Rightarrow$) Let $a \in E^+$, $b_1, b_2 \in \hat{E}^+$, $a \leq b_1 + b_2$, $b_1 \perp b_2$

Let $B = \{b'_1 \in E^+ : b'_1 < b_1\}$. Let $\mathfrak{B}$ be the band generated

by B in E . $E = \mathfrak{B} \oplus \mathfrak{B}^\perp$, $a = a_1 + a_2$, $a_1 \in \mathfrak{B}$, $a_2 \in \mathfrak{B}^\perp$.

Also $\hat{E} = \hat{\mathfrak{B}} \oplus \hat{\mathfrak{B}}^\perp$ and $b_1 \in \hat{\mathfrak{B}}$, $b_2 \in \hat{\mathfrak{B}}^\perp$

$a \leq b_1 + b_2 \Rightarrow a_1 \leq b_1$ for $a_1 \perp b_2$. Similarly
 $a_2 \leq b_2$.

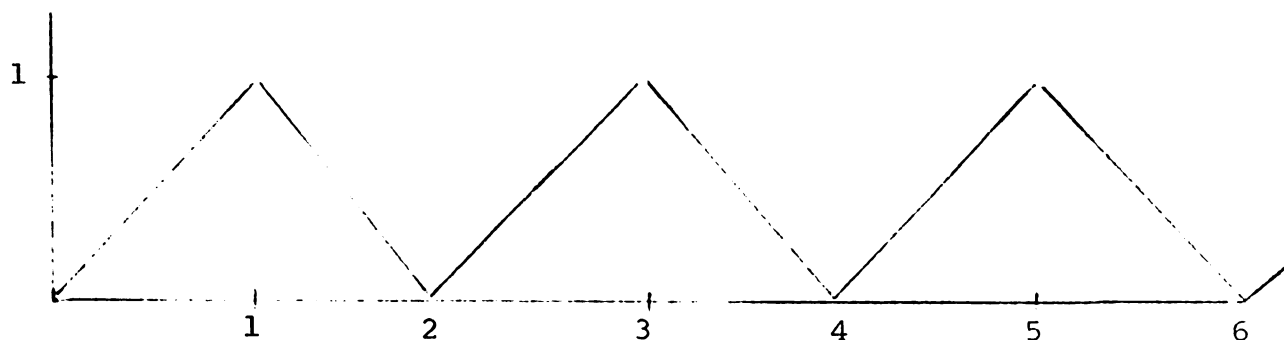
Q.E.D.

Example 3.5: Let E be the collection of all bounded real-valued functions f on the positive real axis such that there is a t , depending upon f , such that f is finite valued on $[t, \infty)$. Under the usual pointwise

operations E is a vector lattice.

Masterson (5) has shown that E has P.P. The Dedekind completion $\hat{E}$ of E is the set of all bounded functions on $[0, \infty)$.

Now consider a to be the identity function on $[0, \infty)$ and b_1 , the function shown in the figure.



$b_2 = 1 - b_1$. $a \leq b_1 + b_2$, but a cannot be expressed as $a = a_1 + a_2$, $a_1 \leq b_1$, $a_2 \leq b_2$, $a_1 \in E$, $a_2 \in E$, for, if $a = a_1 + a_2 \leq b_1 + b_2$ then $a_1 = b_1$; $a_2 = b_2$. Hence the example.

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