SYMMETRIC GROUP ANALYSIS OF MULTISPINOR LAGRANGIANS

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ABSTRACT

SYMMETRIC GROUP ANALYSIS OF MULTISPINOR LAGRANGIANS

By

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If the requirement of symmetric group invariance is imposed on the Lagrangian for multispinor fields, then valuable information about the structure of the couplings can be obtained. The number and symmetries of the auxiliary fields which must be introduced in order to obtain the Bargmann-Wigner equations are related to the irreducible representations of S_n. The form of the non-vanishing kinetic couplings may be predicted from the direct product series, while the operators themselves may be constructed directly on the basis of symmetry.

The present work reviews the second- and third-rank multispinor Lagrangians and identifies the respective wave functions as objects which transform under the appropriate symmetric group. Each multispinor is transformed into tensor components, and field equations are found in that formulation.

The fourth-rank multispinor wave functions are presented, and the spin-0 Lagrangian for this rank is constructed on the basis of symmetric group considerations. The transformation from multispinor to tensor form is carried out. The field equations on the tensor components which result upon variation assure the Bargmann-Wigner

equations on the spin-0 field and the vanishing of the auxiliary fields.

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Not chaos-like together crush'd and bruis'd,
But, as the world, harmoniously confus'd:
Where order in variety we see,
And where, though all things differ, all agree.

-Alexander Pope

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INTRODUCTION

The pioneering work of Dirac in describing the spin-½ field led to various suggestions for the representation of fields of higher spin. Rarita and Schwinger [1] proposed a spinor-tensor method, Bargmann and Wigner a multispinor form [2], and Fierz and Pauli [3] a hybrid of the former two. The Bargmann-Wigner representation is the most natural generalization of a Dirac spinor since the indices on the wave functions all correspond to spinor components, and each spin degree of freedom obeys the Dirac algebra separately. Moreover, the Bargmann-Wigner formulation facilitates the introduction of interactions. For all spins, both integral and half-odd integral, the wave functions possess indices that are homogeneous in the sense that they are all spinors and hence may be treated alike. This offers an advantage not present in other systems where one needs to perform vector operations on some indices and spinor operations on others. The present work investigates the general multispinor Lagrangian formalism with examples given for spinors of the second, third, and fourth rank. It also illustrates how a fourth-rank spin-0 multispinor can be transformed into tensor notation.

The familiar Dirac equation:

$$(\gamma\cdot\partial)_{\alpha\alpha'}\,\Psi_{\alpha'} = -m\Psi_{\alpha} \tag{I-1}$$

is generalized into the Bargmann-Wigner equations for multispinor fields [4]:

$$(\gamma\cdot\partial)_{\alpha_1\alpha_1'}\,\Psi_{\alpha_1'\alpha_2\alpha_3\ldots\alpha_{2s}} = -m\Psi_{\alpha_1\alpha_2\alpha_3\ldots\alpha_{2s}} \tag{I-2}$$

$$(\gamma\cdot\partial)_{\alpha_2\alpha_2'}\,\Psi_{\alpha_1\alpha_2'\alpha_3\ldots\alpha_{2s}} = -m\Psi_{\alpha_1\alpha_2\alpha_3\ldots\alpha_{2s}}$$

$$\vdots$$

$$(\gamma\cdot\partial)_{\alpha_{2s}\alpha_{2s}'}\,\Psi_{\alpha_1\alpha_2\alpha_3\ldots\alpha_{2s}'} = -m\Psi_{\alpha_1\alpha_2\alpha_3\ldots\alpha_{2s}} .$$

Here $\underline{s}$ is the maximum value of the spin. The $(\gamma\cdot\partial)$ are 4×4 matrices that act on the spinor components as indicated. We use the metric defined by Minkowski; namely that

$$a\cdot b = a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

$$= a_1b_1 + a_2b_2 + a_3b_3 - a_4b_4 , \tag{I-3}$$

and take the γ_μ to be hermitian matrices satisfying

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\,\delta_{\mu\nu} . \tag{I-4}$$

The field totally symmetric in its spinor indices:

$$\Psi_{\alpha_1\alpha_2\alpha_3\ldots\alpha_{2s}}$$

must be a faithful representation of particles of spin-s. That this is indeed the case is seen by counting the number of independent, non-vanishing components of Ψ when the Bargmann-Wigner equations are applied and observing that this is the same as the number of spin components. This is done by Bargmann and Wigner for the general spin s [5]. As an illustrative example let us look at the fourth-rank multispinor: $\Psi_{\alpha\beta\gamma\delta}$. Each index has four values so that if order were significant there would be $4^4 = 256$ components. However, the symmetry of the indices under exchange of any pair reduces the number to

$$\frac{[4 + (4-1)]!}{4!\ (4-1)!} = 35 \text{ components [6].}$$

This number is further restricted by the Bargmann-Wigner equations. They can also be written:

$$i(\gamma_k p_k)_{\alpha\alpha'}\ \Psi_{\alpha'\beta\gamma\delta} = -m\Psi_{\alpha\beta\gamma\delta}\ , \text{ etc.} \tag{I-5}$$

In the rest frame p may be taken to be (0,0,0,mi). In standard notation, γ_4 is diagonal and has elements 1,1,-1, -1. From Eq. (I-5) it is evident that the spinor components of the Ψ must vanish if they correspond to the third or fourth rows of γ_4. Also, the first two components are restricted to representing either the spin-up or the spin-down cases and can be chosen to be either

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} ,$$

respectively. Since the Ψ's must also be eigenstates of the spin matrix $-i\gamma_1\gamma_2$, with diagonal elements $\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}$, the surviving positive energy spin states for the four spinors are:

$$
\begin{array}{ccccl}
+ & + & + & + & m = 2 \\
+ & + & + & - & m = 1 \\
+ & + & - & - & m = 0 \\
+ & - & - & - & m = -1 \\
- & - & - & - & m = -2
\end{array}
$$

Thus there are five independent components remaining, corresponding to the five spin states allowed for a particle of spin-2.

At this point, one notices that the symmetry of the wave function can be generalized so that functions antisymmetric and symmetric in various combinations of indices can be introduced. Since the number of totally symmetric combinations will always exceed those of lesser symmetry, once the Bargmann-Wigner equations are applied to the wave function, the totally symmetric function will always represent the highest spin possible for a particular rank while other combinations will represent lower spins.

The postulation that third rank multispinors of mixed symmetry might represent quark combinations in baryons was noted by Salam [7] and Sakita [8] and cited in work by Repko [9]. Such a generalization is reasonable since the quarks that are believed to comprise baryons are not identical so that symmetric combinations of spinors are not to be expected.

If construction of multispinors of mixed symmetry is suggested by a logical extension from the totally symmetric ones and shown desirable by the quark model, their existence is demanded by the construction of Lagrangians for fields of spin greater than one. The Lagrangian density:

$$L = \overline{\Psi}(\gamma\cdot\partial + m)\Psi \tag{I-6}$$

leads, under variation with respect to $\overline{\Psi}$, to the usual Dirac equation. A similar Lagrangian for second-rank spinors:

$$L = \overline{\Psi}\left(\frac{(\gamma\cdot\partial)_1 + (\gamma\cdot\partial)_2}{2} + m\right)\Psi \tag{I-7}$$

leads to the Bargmann-Wigner equations for spin-1. (Here the subscripts 1 and 2 refer to the index upon which $\gamma\cdot\partial$ is acting.) If one attempts to write a Lagrangian for a third-rank multispinor in analagous fashion, however, one does not arrive at the equations of motion. It is only by the introduction of auxiliary fields of definite symmetry [10] in two indices that suitable terms can be added to the Lagrangian so that variation successively with respect to each field leads to the Bargmann-Wigner equations on the totally symmetric field and to the vanishing of all fields of lower symmetry.

Any further attempt to generalize this technique to spinors of rank higher than three requires a careful analysis of the constraints placed on the couplings in the Lagrangian. The nature of the auxiliary fields will be specified by the requirement that they transform as irreducible representations of the symmetric group, and the couplings possible will be only those that are invariant under the transformations of the symmetric group. This will be illustrated for the cases $s = 1$ and $s = 3/2$, using second- and third-rank multispinors, respectively. For the fourth-rank spinor, the simplest non-trivial representation (that is, the field of mixed symmetry of spin-0) will be considered. The information gained from symmetric group considerations will establish what kinds of couplings can be used to write the most general Lagrangian. The Lagrangian can then be varied to find the strengths of the various couplings with the requirements that the equations of motion result for the spin-0 field and the auxiliary fields vanish. Transformations for each of the fourth-rank multispinors to tensor form will be presented in order to demonstrate the Bargmann-Wigner equations in tensor form.

The symmetric group constraints mentioned above may prove too severe for the formation of some higher spin Lagrangians. An alternative to be explored is the use of the alternating group. If the less stringent restraints of the alternating group prove to be sufficient for multispinors of even rank (that is to say, integral spin) then there will be a significant mathematical difference in the treatment of bosons and fermions. This difference is not apparent in Bargmann-Wigner theory as it now stands with multispinors of even rank on the same footing as those of odd rank.

Whatever the results of the above speculation, the value of the symmetric group as a guide in constructing Lagrangians for fields of higher spin is established. Only the symmetric group representations (or those of its subgroup, the alternating group) lead to fields of definite spin. The tensor formulation contains no such result. Thus the group theoretical approach studied provides a valuable tool for constructing Lagrangians for specific fields of higher spin.

Chapter I

The Second Rank Multispinor Fields

For purposes of orientation, consider the second rank multispinor fields: $\Psi_{\alpha\beta}$, which is symmetric in α, β; and $\phi_{\alpha\beta}$, which is anti-symmetric. The implications of the Bargmann-Wigner equations for these fields will be analyzed in some detail in order to provide a guide to the higher-rank multispinor equations which will be treated in subsequent chapters.

The symmetric second rank multispinor $\Psi_{\alpha\beta}$ satisfies the Bargmann-Wigner equations

$$(\gamma\cdot\partial)_{\alpha\alpha'}\Psi_{\alpha'\beta} = -m\,\Psi_{\alpha\beta} \qquad (1\text{-}1a)$$

$$(\gamma\cdot\partial)_{\beta\beta'}\Psi_{\alpha\beta'} = -m\,\Psi_{\alpha\beta}. \qquad (1\text{-}1b)$$

These equations can be expressed in the more conventional tensor form by using the expansion

$$\Psi_{\alpha\beta} = (\gamma_\mu C)_{\alpha\beta}V_\mu + \tfrac{1}{2}i(\sigma_{\mu\nu}C)_{\alpha\beta}T_{\mu\nu}, \qquad (1\text{-}2)$$

where $C_{\alpha\beta}$ is the antisymmetrical matrix which satisfies

$$C^{-1}\gamma_\mu C = -\gamma_\mu{}^T, \qquad (1\text{-}3)$$

and $\sigma_{\mu\nu}$ denotes the (six) matrices

$$\sigma_{\mu\nu} = \frac{1}{2i}\,(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu). \qquad (1\text{-}4)$$

The superscript T denotes the transpose. With the aid of equation (1-3) it is easy to check that the expansion (1-2) is symmetric in α, β.

The Bargmann-Wigner equations (1-1) then take the form

$$(\gamma\cdot\partial\gamma_\lambda C)_{\alpha\beta}V_\lambda+\tfrac{1}{2}i(\gamma\cdot\partial\sigma_{\lambda\rho}C)_{\alpha\beta}T_{\lambda\rho} = -m\Psi_{\alpha\beta} \qquad (1\text{-}5a)$$

$$(\gamma_\lambda C\gamma^T\cdot\partial)_{\alpha\beta}V_\lambda+\tfrac{1}{2}i(\sigma_{\lambda\rho}C\gamma^T\cdot\partial)_{\alpha\beta}T_{\lambda\rho} = -m\Psi_{\alpha\beta}, \qquad (1\text{-}5b)$$

and it is possible to obtain equations for V_μ and $T_{\mu\nu}$ using the relations

$$(C^{-1}\gamma_\mu)_{\beta\alpha}\Psi_{\alpha\beta} = 4V_\mu,\ (C^{-1}\sigma_{\mu\nu})_{\beta\alpha}\Psi_{\alpha\beta} = 4iT_{\mu\nu}. \qquad (1\text{-}6)$$

Multiplying equation (1-5a) by $(C^{-1}\gamma_\mu)_{\beta\alpha}$ and $(C^{-1}\sigma_{\mu\nu})_{\beta\alpha}$ and performing the traces gives

$$\partial_\nu T_{\mu\nu} = mV_\mu \qquad (1\text{-}7a)$$

$$T_{\mu\nu} = -\frac{1}{m}(\partial_\mu V_\nu - \partial_\nu V_\mu). \qquad (1\text{-}7b)$$

Since $T_{\mu\nu}$ is antisymmetric, equation (1-7a) implies

$$\partial_\mu V_\mu = 0, \qquad (1\text{-}8)$$

and (1-7b) reduces to

$$(\Box^2 - m^2)V_\mu = 0. \qquad (1\text{-}9)$$

Hence V_μ represents a spin 1 field.

The antisymmetric second rank multispinor $\phi_{\alpha\beta}$ satisfies the Bargmann-Wigner equations (1-1), and it too can be expanded in tensor form. Specifically, the antisymmetric matrices C, $\gamma_\mu\gamma_5 C$ and $\gamma_5 C$ can be used to write

$$\phi_{\alpha\beta} = C_{\alpha\beta}S + (\gamma_\lambda\gamma_5 C)_{\alpha\beta}A_\lambda + (\gamma_5 C)_{\alpha\beta}P \ , \tag{1-10}$$

so that the Bargmann-Wigner equation becomes

$$(\gamma\cdot\partial C)_{\alpha\beta}S + (\gamma\cdot\partial\gamma_\lambda\gamma_5 C)_{\alpha\beta}A_\lambda$$
$$+ (\gamma\cdot\partial\gamma_5 C)_{\alpha\beta}P = -m\phi_{\alpha\beta}. \tag{1-11}$$

To obtain equations for S, A and P, note that

$$(C^{-1})_{\beta\alpha}\phi_{\alpha\beta} = 4S,\ (C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}\phi_{\alpha\beta} = 4A,$$
$$(C^{-1}\gamma_5)_{\beta\alpha}\phi_{\alpha\beta} = 4P, \tag{1-12}$$

which allows equation (1-11) to be written

$$0 = -mS \tag{1-13a}$$

$$\partial_\mu P = -mA_\mu \tag{1-13b}$$

$$\partial_\mu A_\mu = -mP. \tag{1-13c}$$

From (1-13), it follows that P is the only independent field and it satisfies

$$\Box^2 P = m^2 P. \tag{1-14}$$

Hence the antisymmetric field $\phi_{\alpha\beta}$ has spin o.

We now turn to the Lagrangian density for the second rank multispinor:

$$L = -\overline{\Psi}[\frac{(\gamma\cdot\partial)_1 + (\gamma\cdot\partial)_2}{2} + m]\Psi. \tag{1-15}$$

In this expression $\overline{\Psi}$ is defined in terms of Ψ as:

$$\overline{\Psi}_{\alpha\beta} = \Psi^*_{\alpha'\beta'}(\gamma_4)_{\alpha'\alpha}(\gamma_4)_{\beta'\beta} \qquad (1\text{-}16)$$

where * denotes the Hermitian conjugate. Also, since $(\gamma\cdot\partial)_1$ and $(\gamma\cdot\partial)_2$ operate in different spaces, they are seen to commute.

Variation with respect to $\overline{\Psi}$ gives:

$$\left[\frac{(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2}{2} + m\right]\Psi = 0. \qquad (1\text{-}17)$$

If this is multiplied by $[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]$:

$$\frac{\Box^2 - \Box^2}{2}\,\Psi + m[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2] = 0$$

or

$$(\gamma\cdot\partial)_1\Psi = (\gamma\cdot\partial)_2\Psi. \qquad (1\text{-}18)$$

Equations (1-17) and (1-18) together constitute the Bargmann-Wigner equations on Ψ. The same development follows for ϕ.

It is to be emphasized at this point that the simplicity of the Lagrangian (1-15) is a consequence of the simple nature of the derivative coupling. This alone makes possible the single multiplication by $[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]$, which leads to (1-18). It is clear that for higher rank multispinors more effort in writing a Lagrangian that leads to the Bargmann-Wigner equations is to be expected.

With a view to generalization, consider the most general Lagrangian which is linear in the derivatives

$(\gamma\cdot\partial)$ and contains second-rank multispinors of definite symmetry:

$$L = -\bar{\Psi}[\frac{(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2}{2} +m]\Psi+a\bar{\Psi}[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]\phi$$

$$+ a\bar{\phi}[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]\Psi+b\bar{\phi}[\frac{(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2}{2} +m]\phi. \tag{1-19}$$

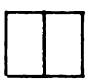
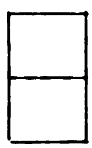
Variation with respect to $\bar{\Psi}$ and $\bar{\phi}$ respectively gives:

$$[\frac{(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2}{2} +m]\Psi + a[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]\phi = 0, \tag{1-20a}$$

$$a[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]\Psi+b[\frac{(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2}{2} +m]\phi = 0. \tag{1-20b}$$

To obtain the Bargmann-Wigner equations on the symmetric field, equation (1-20a) can be multiplied by $[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2]$. This will result in (1-18) if and only if the constant a=0. Putting b=0, the Lagrangian (1-15) is recovered.

As a final note, the two possible symmetries of a second rank multispinor may be thought of as the two irreducible representations of the symmetric group, S_2. They may be represented in Young's tableaux as follows:

$\Psi_{\alpha\beta}$ (2)  and $\phi_{\alpha\beta}$ (1^2) 

This point is significant for generalizations to be introduced in the work to follow.

Chapter II

The Symmetric Group and Multispinor Lagrangians

The simple Dirac Lagrangian for spin-½ fields and that for second-rank spinors lead one to speculate that the appropriate Lagrangian for third-rank spinors might be:

$$L = -\overline{\Psi}[\frac{(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2+(\gamma\cdot\partial)_3}{3} +m]\Psi. \qquad (2\text{-}1)$$

However, if this is varied with respect to $\overline{\Psi}$ there is no operation that can be performed on the resulting equation that leads to

$$(\gamma\cdot\partial)_1\Psi = (\gamma\cdot\partial)_2\Psi = (\gamma\cdot\partial)_3\Psi, \qquad (2\text{-}2)$$

which is necessary for the Bargmann-Wigner equations to result. There is, therefore, a need for additional terms in the Lagrangian. As suggested in the introduction, multispinors with two or three antisymmetric indices are available for other couplings that might be devised. Let us denote the additional fields by

$$\chi_{[\alpha\beta]\gamma},\ \Omega_{[\alpha\beta\gamma]}, \qquad (2\text{-}3)$$

where the square brackets enclose indices that are antisymmetric under exchange of adjacent elements. Rounded brackets will designate symmetric indices. Clearly, mass couplings of fields of different symmetry vanish, for example

$$m\overline{\Psi}_{(\alpha\beta\gamma)}\chi_{[\alpha\beta]\gamma} = 0. \qquad (2\text{-}4)$$

This is shown by

$$\overline{\Psi}_{(\alpha\beta\gamma)}\chi_{[\alpha\beta]\gamma} = \overline{\Psi}_{(\beta\alpha\gamma)}\chi_{[\beta\alpha]\gamma}$$

$$= \overline{\Psi}_{(\alpha\beta\gamma)}[-\chi_{[\alpha\beta]\gamma}]$$

$$= 0.$$

The first step merely exchanges the roles of summation indices while the second makes use of the symmetry properties of the functions.

Derivative couplings in general do not vanish, for example

$$\overline{\Psi}_{\alpha\beta\gamma}(\gamma\cdot\partial)_{\alpha\alpha'}\Psi_{\alpha'\beta\gamma} \neq 0.$$

A Lagrangian incorporating auxiliary fields of mixed symmetry has been given by Guralnik and Kibble [11] and reformulated by Gupta and Repko [12]:

$$L = -\overline{\Psi}[(\gamma\cdot\partial)_1+m]\Psi + \frac{2}{3}\overline{\chi}[(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_3+3m]\chi$$
$$+\overline{\Omega}[(\gamma\cdot\partial)_3-m]\Omega - \frac{1}{\sqrt{3}}[\overline{\chi}(\gamma\cdot\partial)_1\Psi + \overline{\Psi}(\gamma\cdot\partial)_1\chi]$$
$$+ \frac{1}{2}[\overline{\chi}(\gamma\cdot\partial)_3\Omega + \overline{\Omega}(\gamma\cdot\partial)_3\chi]. \qquad (2\text{-}5)$$

The variation of this function with respect to the various fields and the solution of the resulting equations have been worked out by Repko and Gupta. It is found that with the particular choice of constants weighting each coupling in (2-5), the Bargmann-Wigner equations result on the totally symmetric spinor $\Psi_{(\alpha\beta\gamma)}$, and $\chi_{[\alpha\beta]\gamma}$ and $\Omega_{[\alpha\beta\gamma]}$ vanish. $\Psi_{\alpha\beta\gamma}$ thus represents the spin-3/2 field. If one constructs a Lagrangian from χ and Ω only, there is a choice of constants such that the Bargmann-Wigner equations result on χ, and Ω vanishes. χ is then found to represent a field of spin-½.

The choice of auxiliary fields in the third-rank case is not particularly difficult because the options for various symmetries are limited. For higher spin cases the auxiliary fields needed are not so obvious and the derivative couplings tedious to develop. At this point the question may be asked: Is there some underlying unity among the fields of a particular rank?

The fact that in the process of variation all the permutations of the indices must be taken into account is suggestive of the symmetric group, the group of permutations of n objects. Furthermore the process of constructing auxiliary fields requires a knowledge of the number of distinct symmetric and antisymmetric arrangements of indices, information which comes from the representations of S_n. For example, the number of antisymmetric combinations is the same as the number of partitions of n, which in turn is the number of irreducible representations in S_n. The dimensions of each irreducible representation can also be found from symmetric group theory. Thus the number and nature of the auxiliary fields can be deduced.

The physics of many-particle systems makes use of the fact that the symmetry group of the Lagrangian contains the group S_n [13]. Since the spinor indices take up the role of the spins of individual particles for the fields studied here, the terms in the Lagrangian must be invariant under symmetric group transformations. This makes it possible for the nature of the derivative couplings to be found.

There are evidently n linearly-independent combinations of the operators $(\gamma\cdot\partial)_i$. If the projection operators described in Appendix A are used, there is one totally symmetric sum:

$$\alpha_n = \frac{[(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2+\ldots+(\gamma\cdot\partial)_n]}{n}, \qquad (2\text{-}6)$$

and n-1 others, designated for the present purposes β_k. If these β_k are arranged as a vector, this vector is found to transform under the (n-1,1) representation of S_n. The general form of the β_k is found to be:

$$\beta_k = \frac{1}{\sqrt{6(k^2+k)}}\left[\sum_{i=1}^{k}(\gamma\cdot\partial)_i - k(\gamma\cdot\partial)_{k+1}\right] \qquad (2\text{-}7)$$

$$k = 1,2,\ldots,n-1.$$

(Other properties and notation for the symmetric group are also listed in Appendix A.)

The terms in the Lagrangian are all of the form:

$$\overline{F}\theta G = \overline{F}_{\alpha_1\alpha_2\ldots\alpha_n}\theta_{\alpha_1\alpha_2\ldots\alpha_n,\ \alpha'_1\alpha'_2\ldots\alpha_n}G_{\alpha'_1\alpha'_2\ldots\alpha'_n}, \qquad (2\text{-}8)$$

where F and G are any of the fields transforming under a definite representation of S_n. Invariance under the symmetric group requires that:

$$\overline{F}\hat{\theta}G = \overline{F}'\hat{\theta}'G' = \overline{F}\Gamma^{(F)^T}(S)\hat{\theta}'\Gamma^{(G)}(S)G, \tag{2-9}$$

or

$$\hat{\theta} = \Gamma^{(F)^T}(S)\hat{\theta}'\Gamma^{(G)}(S)G. \tag{2-10}$$

Thus $\hat{\theta}$ must transform under the symmetric group:

$$\hat{\theta}' = \Gamma^{(i)}(S)\hat{\theta}\Gamma^{(j)^T}(S). \tag{2-11}$$

Here, the matrices $\Gamma^{(i)}(S)$ for a particular permutation S may be from different representations and therefore of different dimension.

The coupling (2-8) represents a Kronecker product of three representations which itself must transform as the identity since it is a scalar. Now α_n belongs to the identity representation and the β_k transform under (n-1,1) as we have already seen. Since these operators exhaust the combinations of the derivative couplings, only they or linear combinations of them are candidates for components of $\hat{\theta}$. If (λ) is the representation to which F belongs and (μ) is the representation to which G belongs, then it follows from a general result of group theory that the Clebsch-Gordon series

$$(\lambda)\times(\mu) = \sum_i a_j(j) \tag{2-12}$$

must contain either the representation (n) or (n-1,1) for a non-vanishing coupling like (2-8) with the invariance property (2-9) to occur. If the series (2-12) contains (n), then an α-type coupling is possible:

$$\bar{F}\alpha G; \tag{2-13}$$

if (n-1,1), then the coupling is of the form (2-8) with combinations of the β_k occurring as components of $\hat{\theta}$. This severely limits the number of couplings that can be made.

The components of $\hat{\theta}$ may be constructed as follows. For a coupling of the form (2-8) with F belonging to the (λ) representation and G belonging to the (μ) representation, the js-component of $\hat{\theta}$ may be computed as follows:

$$[\hat{\theta}(\lambda,\mu)]_{js} \equiv \beta_k \langle \lambda j,(n-1)k|\mu s\rangle$$

where for brevity n-1 denotes the (n-1,1) representation. The $\langle \lambda j,(n-1)k|\mu s\rangle$ is the Clebsch-Gordan coefficient connecting the (λ) and (n-1,1) representations to the (μ) representation. The Clebsch-Gordan coefficients may be computed according to the method outlined by Hamermesh [14] (pp. 260-62). The labor in so doing is considerably shortened due to relationships that exist between some of these coefficients from different representations.

This method of selecting appropriate auxiliary fields and finding the corresponding couplings will be illustrated for fields of second, third and fourth rank.

A general result from group theory states that if the Clebsch-Gordan series for $(\lambda)\times(\mu)$ contains (ν), then the product $(\nu)\times(\lambda)$ contains (μ). From this it is evident that the inner product of a representation with itself contains the identity since $(n)\times(\lambda) = (\lambda)$. Also the series for the product of the identity representation with itself contains only the identity.

With these principles it is possible to deduce some of the couplings immediately.

For the second-rank multispinor there are two possible symmetric arrangements as we have seen in Chapter I. The number 2 has just two possible partitions:

$$2 \text{ and } 1 + 1,$$

and these correspond to the irreducible representations of S_2: (2) and (1^2) under which the fields $\Psi_{(\alpha\beta)}$ and $\phi_{[\alpha\beta]}$ transform respectively. That each of these representations is one-dimensional can be illustrated graphically using the Young's tableaux method. The tableaux at the end of Chapter I correspond to each of the representations. It is clear that if one starts with the block to the left or at the top, there is, for each diagram, only one way of assigning the numerals 1,2 to the blocks. Thus each representation is one-dimensional.

What couplings exist? The Clebsch-Gordan series for the symmetric field taken with itself is just:

$$(2) \times (2) = (2). \tag{2-15}$$

Thus there is only one coupling of this type and the operator is α_2:

$$\alpha_2 = \frac{(\gamma\cdot\partial)_1 + (\gamma\cdot\partial)_2}{2} .$$

Similarly, the antisymmetric spinor has just one coupling:

$$(1^2) \times (1^2) = (2), \tag{2-16}$$

and it is again with α_2.

If we look at the coupling between Ψ and ϕ, the series is just:

$$(2) \times (1^2) = (1^2). \tag{2-17}$$

There are just two elements of S_2 and the values of Γ for the (1^2) representation are:

$$\Gamma(e) = 1; \; \Gamma(1,2) = -1. \tag{2-18}$$

The projection operator is, for this case,

$$P^{(1^2)}(\gamma\cdot\partial)_1 = \tfrac{1}{2}[P(E) - P(1,2)], \tag{2-19}$$

which leads (up to a normalization factor) to

$$\beta_1 = \tfrac{1}{2}[(\gamma\cdot\partial)_1 - (\gamma\cdot\partial)_2]. \tag{2-20}$$

The character table is simple to construct:

	(2)	(1^2)
Γ(e)	1	1
Γ(1,2)	1	-1

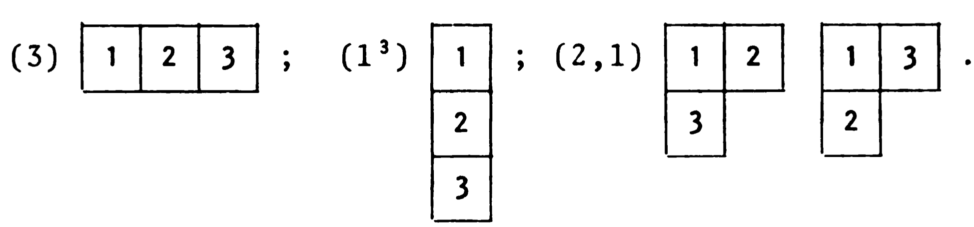
If the couplings involving the operators α_2 and β_1 are assembled into a Lagrangian, they produce that written down in (1-15).

The Third Rank Case

There are three partitions of 3:

3, 2+1, 1+1+1,

so there are three irreducible representations of S_3. Moreover, using Young's tableaux, one sees that there is only one way of assigning 1,2,3 to the blocks corresponding to the (3) and the (1^3) representations. However, for the (2,1) representation there are two ways, indicating a multiplicity of two for the components of this representation. The diagrams are:



Thus there are three distinct fields, one of which can be represented as a two component object:

$$\Psi_{(\alpha\beta\gamma)}\,;\ \Xi_{\alpha\beta\gamma} = \begin{pmatrix} \chi'_{(\alpha\beta)\gamma} \\ \chi_{[\alpha\beta]\gamma} \end{pmatrix}\ ;\ \Omega_{[\alpha\beta\gamma]}\ .$$

Here $\Psi_{(\alpha\beta\gamma)}$ is the totally symmetric spinor, $\Omega_{[\alpha\beta\gamma]}$ transforms according to the alternating representation, and the components of $\Xi_{\alpha\beta\gamma}$ have the following properties:

$$\chi'_{(\alpha\beta)\gamma} = \chi'_{(\beta\alpha)\gamma} \qquad \text{(a)} \qquad\qquad (2\text{-}21)$$

$$\chi'_{\alpha\beta\gamma} + \chi'_{\gamma\alpha\beta} + \chi'_{\beta\gamma\alpha} = 0 \qquad \text{(b)}$$

$$\chi_{[\alpha\beta]\gamma} = -\chi_{[\beta\alpha]\gamma} \qquad \text{(c)}$$

$$\chi_{\alpha\beta\gamma} + \chi_{\gamma\alpha\beta} + \chi_{\beta\gamma\alpha} = 0 \qquad \text{(d)}$$

These components are related to one another through:

$$\chi'_{\alpha\beta\gamma} = \frac{1}{\sqrt{3}}\,(\chi_{\beta\gamma\alpha} - \chi_{\gamma\alpha\beta}) \qquad\qquad (2\text{-}22)$$

which, together with (2-21) leads to:

$$\chi_{\beta\gamma\alpha} = -\tfrac{1}{2}\,\chi_{\alpha\beta\gamma} + \frac{\sqrt{3}}{2}\,\chi'_{\alpha\beta\gamma} \qquad \text{(a)} \qquad\qquad (2\text{-}23)$$

$$\chi_{\gamma\alpha\beta} = -\tfrac{1}{2}\,\chi_{\alpha\beta\gamma} - \frac{\sqrt{3}}{2}\,\chi'_{\alpha\beta\gamma} \qquad \text{(b)}$$

From the last two equations it is evident that $\chi'_{\alpha\beta\gamma}$ has the symmetry claimed in (2-21) provided (2-22) is true. If we solve (2-22) for $\chi_{\alpha\beta\gamma}$ and use that result to express $\chi'_{\alpha\beta\gamma}$ with permuted indices:

$$\chi'_{\beta\gamma\alpha} = - \quad \chi'_{\alpha\beta\gamma} - \frac{\sqrt{3}}{2}\chi_{\alpha\beta\gamma}, \tag{2-24a}$$

$$\chi'_{\gamma\alpha\beta} = - \quad \chi'_{\alpha\beta\gamma} + \frac{\sqrt{3}}{2}\chi_{\alpha\beta\gamma}. \tag{2-24b}$$

Combining (2-23) and (2-24) in matrix form:

$$\begin{pmatrix} \chi'_{\beta\gamma\alpha} \\ \chi_{\beta\gamma\alpha} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \chi_{\alpha\beta\gamma} \end{pmatrix} \tag{2-25a}$$

$$\begin{pmatrix} \chi'_{\gamma\alpha\beta} \\ \chi_{\gamma\alpha\beta} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \chi_{\alpha\beta\gamma} \end{pmatrix} \tag{2-25b}$$

These operations are clearly P(123) $\Xi_{\alpha\beta\gamma}$ and P(132) $\Xi_{\alpha\beta\gamma}$ respectively. Also:

$$P(12) \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \chi_{\alpha\beta\gamma} \end{pmatrix} = \begin{pmatrix} \chi'_{\beta\alpha\gamma} \\ \chi_{\beta\alpha\gamma} \end{pmatrix} = \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \chi_{\alpha\beta\gamma} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \chi_{\alpha\beta\gamma} \end{pmatrix} \tag{2-26}$$

We may generate P(23) from the first equation of each pair in (2-23) and (2-24) by interchanging the role of α and β.

$$\begin{aligned} \chi_{\alpha\beta\gamma} &= -\tfrac{1}{2}\ \chi_{\beta\alpha\gamma} + \frac{\sqrt{3}}{2}\ \chi'_{\beta\alpha\gamma} \\ &= +\tfrac{1}{2}\ \chi_{\alpha\beta\gamma} + \frac{\sqrt{3}}{2}\ \chi'_{\alpha\beta\gamma} \end{aligned} \tag{2-27a}$$

and

$$\begin{aligned} \chi'_{\alpha\gamma\beta} &= -\tfrac{1}{2}\ \chi'_{\beta\alpha\gamma} - \frac{\sqrt{3}}{2}\ \chi_{\beta\alpha\gamma} \\ &= -\tfrac{1}{2}\ \chi'_{\alpha\beta\gamma} + \frac{\sqrt{3}}{2}\ \chi_{\alpha\beta\gamma} \end{aligned} \tag{2-27b}$$

In matrix form:

$$\begin{pmatrix} \chi'_{\alpha\gamma\beta} \\ \\ \chi_{\alpha\gamma\beta} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & +\frac{\sqrt{3}}{2} \\ \\ \frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix} \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \\ \chi_{\alpha\beta\gamma} \end{pmatrix} \qquad (2\text{-}28)$$

Similarly, P(13) can be generated by exchanging the role of α and β in the second equation of each pair in (2-23) and (2-24).

$$\begin{pmatrix} \chi'_{\gamma\beta\alpha} \\ \\ \chi_{\gamma\beta\alpha} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \\ -\frac{\sqrt{3}}{2} & +\frac{1}{2} \end{pmatrix} \begin{pmatrix} \chi'_{\alpha\beta\gamma} \\ \\ \chi_{\alpha\beta\gamma} \end{pmatrix} \qquad (2\text{-}29)$$

Thus we have generated a matrix representation of S_3 corresponding to the irreducible representation (2,1):

$$\Gamma^{(2,1)}(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Gamma(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \Gamma(23) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma(123) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ +\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad \Gamma(132) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix} \qquad (2\text{-}30)$$

From the cycle structure of S_3 it can be seen that there are three classes:

$$S_3 = \{e;\ (12),\ (13),\ (23),\ (123),\ (132)\},$$

from which the character table can be constructed.

	C_e	C_1	C_2
(3)	1	1	1
(2,1)	2	0	-1
(1^3)	1	-1	1

To find out what couplings are possible it is necessary to look at the Clebsch-Gordan series for all possible inner product pairs. The character table can be used to facilitate this process. An example will be used to explain how this may be done. If we seek the decomposition of

$$(2,1) \times (1^3),$$

we multiply corresponding elements of the rows of the character table that belong to the representations in the Kronecker product, we have

$$2\cdot 1 \qquad 0(-1) \qquad (-1)\cdot 1$$

or:

$$2 \qquad 0 \qquad -1 \;.$$

Since this is the row that belongs to the (2,1) representation, the decomposition contains only (2,1). The row that is the product of two representations will be the sum of the rows that correspond to representations in the Clebsch-Gordan series. This leads to

$$(3)\times(3) = (3) \qquad (2,1)\times(2,1) = (3)+(2,1)+(1^3)$$

$$(3)\times(2,1) = (2,1) \qquad (2,1)\times(1^3) = (2,1)$$

$$(3)\times(1^3) = (1^3) \qquad (1^3)\times(1^3) = (3) \tag{2-32}$$

The only coupling that contains neither (3) nor (2,1) is $(3)\times(1^3)$ so there is no term like $\bar{\Psi}\hat{\theta}\Omega$. However, the other five couplings are allowed.

Next it is necessary to construct the derivative combinations. Using the projection operator given in Appendix A, we may construct the operator transforming under the identity representation.

$$P^{(3)}(\gamma\partial)_1 = \tfrac{1}{6}[P(e)+P(12)+P(13)+P(23)+P(123)+P(132)] \tag{2-33}$$

so that:

$$P^{(3)}(\gamma\partial)_1 = \tfrac{1}{3}[(\gamma\partial)_1+(\gamma\partial)_2+(\gamma\partial)_3] \equiv \alpha_3. \tag{2-34}$$

Thus we have the totally symmetrical operator. It is not difficult to see that the projection operator operating on $(\gamma\partial)_1$ gives 0 for the (1^3) representation:

$$\begin{aligned} P^{(1^3)}(\gamma\partial)_1 &= \tfrac{1}{6}[P(e)-P(12)-P(13)-P(23)+P(123)+P(132)] \\ &= \tfrac{1}{6}[(\gamma\partial)_1-(\gamma\partial)_2-(\gamma\partial)_3-(\gamma\partial)_1+(\gamma\partial)_2+(\gamma\partial)_3] \\ &= 0. \end{aligned}$$

For the (2,1) representation:

$$P^{(2,1)}_{11}(\gamma\partial)_1 = \tfrac{2}{6}[P(e)+P(12)-\tfrac{1}{2}P(13)-\tfrac{1}{2}P(23)-\tfrac{1}{2}P(123)-\tfrac{1}{2}P(132)]$$

$$P_{11}^{(2,1)}(\gamma\partial)_1 = \frac{2}{6}[(\gamma\partial)_1+(\gamma\partial)_2-\tfrac{1}{2}(\gamma\partial)_3-\tfrac{1}{2}(\gamma\partial)_1-\tfrac{1}{2}(\gamma\partial)_2-\tfrac{1}{2}(\gamma\partial)_3]$$

$$= \frac{1}{6}[(\gamma\partial)_1+(\gamma\partial)_2-2(\gamma\partial)_3] \equiv \beta_1 \qquad (2\text{-}35a)$$

$$P_{12}^{(2,1)}(\gamma\partial)_1 = \frac{2}{6}[0\cdot P(e) + 0\cdot P(12) - \frac{\sqrt{3}}{2}P(13) + \frac{\sqrt{3}}{2}P(23) - \frac{\sqrt{3}}{2}P(123) + \frac{\sqrt{3}}{2}P(132)]$$

$$P_{12}^{(2,1)}(\gamma\partial)_1 = \frac{1}{2\sqrt{3}}[(\gamma\partial)_1 - (\gamma\partial)_2] \equiv \beta_2 \qquad (2\text{-}35b)$$

Thus α, β_1, and β_2 are three linearly independent operators.

From the series in (2-32) it can be seen that the couplings of $\Psi_{\alpha\beta\gamma}$ are as follows:

i) Ψ couples with itself through the operator α which transforms as the identity representation.

ii) Ψ couples with Ξ through a (2,1)-type operator.

iii) Ψ does not couple with Ω.

The couplings of Ξ are:

i) Ξ couples through two operators to itself.

ii) Ξ couples to Ω through the operator which transforms under (2,1).

Finally, Ω couples with itself only one way, through the operator which transforms under the identity.

$\overline{\Psi}$-Couplings

The term in the Lagrangian in which Ψ couples with itself must contain α since this is the operator which transforms under the (3) representation. It can be seen that α satisfies (2-11) when it is remembered that for this representation all the Γ's are 1. The coupling of Ψ with Ξ must involve β_1 and β_2 in an operator which transforms under the (2,1) representation. Such an operator is

$$\hat{\theta} = (\beta_1, \beta_2) \qquad (2\text{-}36)$$

since:

$$\begin{aligned} \hat{\theta}' &= \Gamma^{(3)}(S)\ (\beta_1, \beta_2)\ \Gamma^{(2,1)}(S) \\ &= 1(\beta_1, \beta_2)\ \Gamma^{(2,1)}(S) \\ &= (\beta_1', \beta_2') \ ; \end{aligned}$$

that is, it fulfills the requirement of (2-11) transforming as required. Thus the most general derivative coupling of Ψ is:

$$a_1 \overline{\Psi} \alpha \Psi + a_2 \overline{\Psi} (\beta_1 \beta_2) \Xi \ . \qquad (2\text{-}37)$$

$\overline{\Xi}$-Couplings

Ξ couples to Ψ, Ξ, and Ω. The coupling of $\overline{\Xi}$ to Ψ is just the transpose of that of $\overline{\Psi}$ with Ξ. The coupling of Ξ with itself contains (3) and (2,1). Ξ is a two dimensional vector so these operators must be 2×2 matrices.

The operator which transforms under the identity representation must be:

$$\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} , \qquad (2\text{-}38)$$

which satisfies (2-11) since the matrices Γ are orthogonal. The (2,1) operator may be constructed using the definition (2-14) for the elements of $\hat{\theta}$ and the method given in Hamermesh [15] for finding the Clebsch-Gordan coefficients. We proceed with the second task first. The detail is carried out in this relatively simple case to illustrate the method which will be merely quoted in the discussion of the operators for higher rank multispinors.

The basic relation needed for determining the Clebsch-Gordan coefficients is

$$D^{(\lambda)}_{ts}(R)\ D^{(\mu)}_{ij}(R)\ D^{(\nu)}_{kl}(R)\ S^{\lambda\tau_\lambda\mu\nu}_{s\ \ jl} = S^{\lambda\tau_\lambda\mu\nu}_{t\ \ ik} . \qquad (2\text{-}39)$$

Here the $S^{\lambda\tau_\lambda\mu\nu}_{s\ \ jl}$ are related to the coefficient in (2-14) by:

$$S^{\lambda\tau_\lambda\mu\nu}_{s\ \ jl} = \langle \mu j, \nu l | \lambda, \tau_\lambda ; s\rangle . \qquad (2\text{-}40)$$

In the present case τ_λ, which denotes the multiplicity of the occurance of the representation (λ), is 1. The representations (λ), (μ), and (ν) are all (2,1), so these superscripts will be suppressed. The $D_{ts}(R)$ are the matrices of the representation, in this case the Γ's, (2-30). The summation indices run over 1,2. There is a symmetry relation which saves computation of all eight of the S_{ijk}. That is, if

$$(\lambda) = (\mu) = (\nu),$$

then

$$S^{\lambda\,\lambda\,\lambda}_{i\,j\,k} = S^{\lambda\,\lambda\,\lambda}_{i\,k\,j} = S^{\lambda\,\lambda\,\lambda}_{k\,i\,j}, \text{ etc.} \tag{2-41}$$

The relation limits the need to evaluate (2-39) for S_{111}, S_{112}, S_{122}, S_{222}. In the following calculation, matrix elements from $\Gamma_{(1,2)}$ and $\Gamma_{(2,3)}$ are used in the expressions for S_{122}, S_{112}, and S_{222}.

$$\begin{aligned} S_{112} = {} & D_{11}D_{11}D_{21}S_{111}+D_{11}D_{11}D_{22}S_{112}+D_{11}D_{12}D_{21}S_{121} \\ & +D_{11}D_{12}D_{22}S_{122}+D_{12}D_{12}D_{22}S_{222}+D_{12}D_{11}D_{21}S_{211} \\ & +D_{12}D_{11}D_{12}S_{212}+D_{12}D_{12}D_{21}S_{221} \ . \end{aligned}$$

In $\Gamma(12)$, $D_{12} = D_{21} = 0$, $D_{11} = 1$, $D_{22} = -1$. Hence, $S_{112} = -S_{112}$. Thus, making use of (2-41):

$$S_{112} = S_{121} = S_{211} = 0. \tag{2-42}$$

$$\begin{aligned} S_{122} = {} & D_{11}D_{21}D_{21}S_{111}+D_{11}D_{22}D_{22}S_{122}+D_{12}D_{21}D_{22}S_{212} \\ & +D_{12}D_{22}D_{21}S_{221}+D_{12}D_{22}D_{22}S_{222} \ . \end{aligned}$$

Using $\Gamma(23) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$,

$$S_{122} = -\frac{3}{8} S_{111} + \frac{5}{8} S_{122} + \frac{\sqrt{3}}{8} S_{222} ,$$

or,

$$\frac{3}{8} S_{222} = \frac{3\sqrt{3}}{8} S_{122} + \frac{3\sqrt{3}}{8} S_{111} . \qquad (2\text{-}43)$$

The series for S_{222} is:

$$S_{222} = D_{21}D_{21}D_{21}S_{111}+3D_{21}D_{22}D_{22}S_{122}+D_{22}D_{22}D_{22}S_{222}.$$

Applying $\Gamma(12)$ to this:

$$S_{222} = -S_{222} = 0. \qquad (2\text{-}44)$$

This combined with (2-43) gives:

$$S_{122} = S_{212} = S_{221} = -S_{111}. \qquad (2\text{-}45)$$

We may take $S_{122} = 1$ and $S_{111} = -1$. The components of $\hat{\theta}(2,1;2,1)$ are:

$$\hat{\theta}_{11} = \beta_1 S_{111}+\beta_2 S_{112} = -\beta_1$$

$$\hat{\theta}_{12} = \beta_1 S_{211}+\beta_2 S_{212} = \beta_2$$

$$\hat{\theta}_{21} = \beta_1 S_{121}+\beta_2 S_{122} = \beta_2$$

$$\hat{\theta}_{22} = \beta_1 S_{221}+\beta_2 S_{222} = \beta_1 .$$

Or:

$$\hat{\theta}(2,1;2,1) = \begin{pmatrix} -\beta_1 & \beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} . \qquad (2\text{-}46)$$

To find the derivative coupling of Ξ with Ω, a less tedious method than that outlined in the previous pages suggests itself. We seek a two component column operator which transforms as (2-11). If $\Gamma^{(i)}(12)$ from the appropriate representations are applied to

$$\begin{pmatrix} \beta_i \\ \beta_j \end{pmatrix}$$

we have:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \beta_i \\ \beta_j \end{pmatrix} (-1) = \begin{pmatrix} -\beta_i \\ \beta_j \end{pmatrix} = P(12) \begin{pmatrix} \beta_i \\ \beta_j \end{pmatrix} .$$

The operator which has this symmetry under the exchange of indices 1 and 2 is:

$$\hat{\theta}(2,1;1^3) = \begin{pmatrix} +\beta_2 \\ -\beta_1 \end{pmatrix} , \qquad (2\text{-}47)$$

orthogonality with (β_1, β_2) requiring the minus sign.

Hence the couplings are:

$$b_1 \bar{\Xi} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \Psi + \bar{\Xi} \left[b_2 \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} + b_3 \begin{pmatrix} -\beta_1 & \beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} \right] \Xi$$

$$+ b_4 \bar{\Xi} \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix} \Omega \qquad (2\text{-}48)$$

$\bar{\Omega}$ Couplings

Using the considerations of the preceding sections, the Ω couplings can be written down:

$$c_1 \bar{\Omega} \alpha \Omega + c_2 \bar{\Omega} (\beta_2, -\beta_1) \Xi . \qquad (2\text{-}49)$$

Fourth-Rank Multispinor

The number of partitions of 4 is five, corresponding to the number of irreducible representations, and hence the number of independent auxiliary fields for the fourth-rank multispinor. The partitions are listed below together with the Young's tableaux showing the dimensionality of each representation. Our designation for the auxiliary field corresponding to each representation is the last item in the line. (Table 1)

Partitions	Young's Tableaux	
(4)	1 2 3 4	Ψ
(3,1)	1 2 3 / 4 ; 1 2 4 / 3 ; 1 3 4 / 2	χ
(2^2)	1 2 / 3 4 ; 1 3 / 2 4	ϕ
$(2,1^2)$	1 2 / 3 / 4 ; 1 3 / 2 / 4 ; 1 4 / 2 / 3	ζ
(1^4)	1 / 2 / 3 / 4	Ω

TABLE 1

In the representation for S_4 carried by the matrices Γ as given in Hamermesh [16], Table 7-3, it can be seen that the requirement that the auxiliary fields transform under these representatives leads to the following symmetries for the components of the irreducible representation.

$$(4) \qquad \Psi_{(\alpha\beta\gamma\delta)}$$

$$(3,1) \qquad \chi_{\alpha\beta\gamma\delta} = \begin{pmatrix} \chi'_{(\alpha\beta\gamma)\delta} \\ \chi''_{(\alpha\beta)\gamma\delta} \\ \chi'''_{[\alpha\beta](\gamma\delta)} \end{pmatrix}$$

$$(2^2) \qquad \phi_{\alpha\beta\gamma\delta} = \begin{pmatrix} \phi'_{(\alpha\beta)(\gamma\delta)} \\ \phi''_{[\alpha\beta](\gamma\delta)} \end{pmatrix}$$

$$(2,1^2) \qquad \zeta_{\alpha\beta\gamma\delta} = \begin{pmatrix} \zeta'_{(\alpha\beta)[\gamma\delta]} \\ \zeta''_{[\alpha\beta]\gamma\delta} \\ \zeta'''_{[\alpha\beta\gamma]\delta} \end{pmatrix}$$

$$(1^4) \qquad \Omega_{[\alpha\beta\gamma\delta]} \tag{2-50}$$

Transformation under the orthogonal matrices Γ assures orthogonality of the components of each vector above and is sufficient for specifying the wave functions in their symmetric group properties.

The functions above fulfill the following cyclic relations over all four indices.

$$\chi_{\alpha\beta\gamma\delta}+\chi_{\delta\alpha\beta\gamma}+\chi_{\gamma\delta\alpha\beta}+\chi_{\beta\gamma\delta\alpha} = 0 \tag{2-51a}$$

$$\zeta_{\alpha\beta\gamma\delta}-\zeta_{\delta\alpha\beta\gamma}+\zeta_{\gamma\delta\alpha\beta}-\zeta_{\beta\gamma\delta\alpha} = 0 \tag{2-51b}$$

$$\phi_{\alpha\beta\gamma\delta}-\phi_{\delta\alpha\beta\gamma}-\phi_{\gamma\delta\alpha\beta}+\phi_{\beta\gamma\delta\alpha} = 0 \tag{2-51c}$$

$\Psi_{(\alpha\beta\gamma\delta)}$ transforms under the identity representation, and $\Omega_{[\alpha\beta\gamma\delta]}$ transforms under the alternating representation. The relation (2-51c) above is weak; a study of the symmetry of the $\phi_{\alpha\beta\gamma\delta}$ gives:

$$\phi_{\alpha\beta\gamma\delta} = \phi_{\gamma\delta\alpha\beta} , \tag{2-52}$$

that is, it is symmetric under exchange of the first pair of indices with the second pair. Moreover, scrutiny of the Young's tableaux for this representation shows that the only choice for assigning the numbers 1, 2, 3, and 4 to the diagram falls on the numerals 2 and 3. Thus there is a structural similarity between this function and $\chi_{\alpha\beta\gamma}$ from the (2,1) representation of the third-rank spinor. This suggests a cyclic relation over three indices which is verified.

$$\phi_{\alpha\beta\gamma\delta} + \phi_{\gamma\alpha\beta\delta} + \phi_{\beta\gamma\alpha\delta} = 0. \tag{2-53}$$

The investigation of what couplings are possible proceeds along the same lines as that for the third-rank case. We first observe that S_4 is of order 24 and that the elements fall into five classes. The character table can be constructed merely by looking at the matrices in Appendix B and taking traces.

	C_e	$C_{1^2 2}$	$C_{1,3}$	C_{2^2}	C_4
$\Gamma(4)$	1	1	1	1	1
$\Gamma(3,1)$	3	1	0	-1	-1
$\Gamma(2^2)$	2	0	-1	2	0
$\Gamma(2,1^2)$	3	-1	0	-1	1
$\Gamma(1^4)$	1	-1	1	1	-1

(2-54)

The Clebsch-Gordan series for the inner products can be found from the character table. They are:

$(4)\times(4) = (4)$

$(4)\times(3,1) = (3,1)$

$(4)\times(2^2) = (2^2)$

$(4)\times(2,1^2) = (2,1^2)$

$(4)\times(1^4) = (1^4)$

$(3,1)\times(3,1) = (4) + (3,1) + (2^2) + (2,1^2)$

$(3,1)\times(2^2) = (3,1) + (2,1^2)$

$(3,1)\times(2,1^2) = (3,1) + (2^2) + (2,1^2) + (1^4)$

$(3,1)\times(1^4) = (2,1^2)$

$(2^2)\times(2^2) = (4) + (1^4) + (2,1^2)$

$(2^2)\times(2,1^2) = (3,1) + (2,1^2)$

$(2^2)\times(1^4) = (2^2)$

$(2,1^2)\times(2,1^2) = (4) + (3,1) + (2^2) + (2,1^2)$

$(2,1^2)\times(1^4) = (3,1)$

$(1^4)\times(1^4) = (4)$

If a series contains (4) we will designate the coupling by α where the operator is to be understood as α times the identity for the representation. If a series contains (3,1) we designate the coupling by $\hat{\theta}$, where $\hat{\theta}$ is a matrix of the desired dimension connecting the two representations, and which has components which transform under the (3,1) representation. If a series contains neither (4) nor (3,1) then that coupling is not allowed.

The β_k which transform under (3,1) are found by the projection operator method. They are for the present case:

$$\beta_1 = \frac{1}{6\sqrt{2}} [(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2+(\gamma\cdot\partial)_3-3(\gamma\cdot\partial)_4]$$

$$\beta_2 = \frac{1}{6} [(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2-2(\gamma\cdot\partial)_3]$$

$$\beta_3 = \frac{1}{2\sqrt{3}} [(\gamma\cdot\partial)_1-(\gamma\cdot\partial)_2] \qquad (2\text{-}55)$$

The scalar α is just:

$$\alpha_4 = \frac{1}{4} [(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2+(\gamma\cdot\partial)_3+(\gamma\cdot\partial)_4] \qquad (2\text{-}56)$$

The couplings which occur are those listed below and their respective transposes:

$\bar{\Psi}\alpha\Psi$	$\bar{\phi}\alpha\phi$
$\bar{\Psi}\hat{\theta}\chi$	$\bar{\phi}\hat{\theta}\zeta$
$\bar{\chi}\alpha\chi$	$\bar{\zeta}\alpha\zeta$
$\bar{\chi}\hat{\theta}\chi$	$\bar{\zeta}\hat{\theta}\zeta$
$\bar{\chi}\hat{\theta}\phi$	$\bar{\zeta}\hat{\theta}\Omega$
$\bar{\chi}\hat{\theta}\zeta$	$\bar{\Omega}\alpha\Omega$ (2-57)

The matrices may be found from the definition (2-14) after the needed Clebsch-Gordan coefficients have been found. These matrices are listed on the following page with the abbreviations for the representations:

$$(4) \rightarrow 4$$

$$(3,1) \rightarrow 3$$

$$(2^2) \rightarrow 2$$

$$(2,1^2) \rightarrow \overline{3}$$

$$(1^4) \rightarrow \overline{4}$$

$$\hat{\theta}(3,3) = \frac{1}{\sqrt{6}} \begin{pmatrix} -2\beta_1 & \beta_2 & \beta_3 \\ \beta_2 & \beta_1 - \sqrt{2}\beta_2 & \sqrt{2}\beta_3 \\ \beta_3 & \sqrt{2}\beta_3 & \beta_1 + \sqrt{2}\beta_2 \end{pmatrix}$$

$$\hat{\theta}(\overline{3},\overline{1}) = \frac{1}{\sqrt{3}} \begin{pmatrix} -\beta_3 \\ \beta_2 \\ -\beta_1 \end{pmatrix}; \quad \hat{\theta}(3,2) = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{2}\beta_2 & -\sqrt{2}\beta_3 \\ -\sqrt{2}\beta_1 - \beta_2 & \beta_3 \\ \beta_3 & -\sqrt{2}\beta_1 + \beta_2 \end{pmatrix}$$

$$\hat{\theta}(3,1) = \frac{1}{\sqrt{3}} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}; \quad \hat{\theta}(3,\overline{3}) = \frac{1}{\sqrt{2}} \begin{pmatrix} \beta_2 & \beta_3 & 0 \\ -\beta_1 & 0 & \beta_3 \\ 0 & -\beta_1 & -\beta_2 \end{pmatrix}$$

$$\hat{\theta}(\overline{3},\overline{3}) = \frac{1}{\sqrt{6}} \begin{pmatrix} \beta_1 + \sqrt{2}\beta_2 & -\sqrt{2}\beta_3 & \beta_3 \\ -\sqrt{2}\beta_3 & \beta_1 - \sqrt{2}\beta_2 & -\beta_2 \\ \beta_3 & -\beta_2 & -2\beta_1 \end{pmatrix}$$

$$\hat{\theta}(\overline{3},2) = \frac{1}{\sqrt{6}} \begin{pmatrix} -\sqrt{2}\beta_1 + \beta_2 & -\beta_3 \\ -\beta_3 & -\sqrt{2}\beta_1 - \beta_2 \\ -\sqrt{2}\beta_3 & \sqrt{2}\beta_2 \end{pmatrix}$$

Thus the symmetric group method applied to multi-spinor fields provides a systematic method for finding the auxiliary fields and the possible couplings among them. This process has been shown in detail for n=2, n=3, and n=4.

Chapter III

The Third-Rank Multispinor

In the previous chapter the couplings of the various auxiliary fields with one another were developed in order to facilitate the writing of the most general Lagrangian. If the couplings in (2-37), (2-48) and (2-49) are combined with the mass terms, the Lagrangian may be written:

$$
\begin{aligned}
L = & -\bar{\Psi}\alpha\Psi + b_1\bar{\Xi}\alpha\Xi + b_2\bar{\Xi}\begin{pmatrix} -\beta_1 & \beta_2 \\ \beta_2 & \beta_1 \end{pmatrix}\Xi + c_1\bar{\Omega}\alpha\Omega \\
& + a_2\left[\bar{\Psi}(\beta_1,\beta_2)\Xi + \bar{\Xi}\begin{pmatrix}\beta_1 \\ \beta_2\end{pmatrix}\Psi\right] \\
& + a_3\left[\bar{\Xi}\begin{pmatrix}\beta_2 \\ -\beta_1\end{pmatrix}\Omega + \bar{\Omega}(\beta_2,-\beta_1)\Xi\right] \\
& -m\bar{\Psi}\Psi + d_2 m\bar{\Xi}\Xi + d_3 m\bar{\Omega}\Omega.
\end{aligned} \tag{3-1}
$$

Here the coefficients of $\bar{\Psi}\alpha\Psi$ and $m\bar{\Psi}\Psi$ have been arbitrarily chosen to be -1 since no loss of generality is involved. Hermiticity requires that the terms that appear with their transposes have the same constant multiplier as the transpose terms. Variation with respect to each of the barred fields in turn gives the field equations:

$$-(\alpha+m)\Psi + a_2(\beta_1,\beta_2)\Xi = 0 \tag{3-2a}$$

$$b_2 \begin{pmatrix} -\beta_1 & \beta_2 \\ \beta_2 & \beta_1 \end{pmatrix} \Xi + b_1 \alpha \Xi + d_2 m \Xi + a_2 \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} \Psi + a_3 \begin{pmatrix} \beta_2 \\ -\beta_1 \end{pmatrix} \Omega = 0 \qquad (3\text{-}2b)$$

$$c_1 \alpha \Omega + d_3 m \Omega + a_3 (\beta_2, -\beta_1) \Xi = 0 \qquad (3\text{-}2c)$$

We first make note of the following identities which can be verified by direct computation:

$$\alpha \beta_2 = -\beta_1 \beta_2 \qquad (3\text{-}3a)$$

$$\beta_1{}^2 - \beta_2{}^2 = 2\alpha\beta_1 \qquad (3\text{-}3b)$$

$$\beta_1{}^2 + \beta_2{}^2 = -\frac{1}{2}(\alpha^2 - \Box^2) \qquad (3\text{-}3c)$$

If (3-2b) is multiplied by (β_1, β_2) and the identities above are used, we have

$$b_2[-2\alpha\beta_1, -2\alpha\beta_2]\Xi + b_1 \alpha (\beta_1, \beta_2) \Xi + d_2 m (\beta_1, \beta_2) \Xi + (-\frac{1}{2} a_2)(\alpha^2 - \Box^2)\Psi = 0, \qquad (3\text{-}4a)$$

where Ω has been eliminated by the multiplication. Rewriting:

$$[(b_1 - 2b_2)\alpha + d_2 m](\beta_1, \beta_2)\Xi - \frac{1}{2} a_2 (\alpha^2 - \Box^2)\Psi = 0. \qquad (3\text{-}4b)$$

If (3-4b) is multiplied by a_2 and (3-2a) by $[(b_1 - 2b_2)\alpha + d_2 m]$, the results may be combined to yield a single equation on Ψ:

$$\{[(b_1-2b_2) - \frac{1}{2} a_2{}^2]\alpha^2 + \frac{a_2{}^2}{2} \Box^2\}\Psi$$

$$+ [d_2+(b_1-2b_2)]m\alpha\Psi = -d_2m^2\Psi. \qquad (3\text{-}5)$$

To obtain an equation for Ψ, note that the operator [17]

$$\frac{9\alpha^2-\Box^2}{8} \qquad (3\text{-}6)$$

has the following properties:

$$\frac{9\alpha^2-\Box^2}{8}\,\beta_k = 0. \quad [k=1,2] \qquad (3\text{-}7a)$$

Hence, if it is shown that

$$\frac{9\alpha^2-\Box^2}{8}\,\Psi = m^2\Psi \;, \qquad (3\text{-}7b)$$

the Bargmann-Wigner equations result on Ψ. Operating on (3-2a) with this operator leads to:

$$(\alpha+m)\Psi = 0. \qquad (3\text{-}8)$$

From (3-7a) and (3-7b) we have:

$$\frac{9\alpha^2-\Box^2}{8}\,\beta_k\Psi = \beta_k\,\frac{9\alpha^2-\Box^2}{8}\,\Psi = m^2\beta_k\Psi = 0.$$

$$\beta_k\Psi = 0. \qquad (3\text{-}9)$$

(3-8) and (3-9) are equivalent to the Bargmann-Wigner equations on Ψ. Thus making (3-5) equivalent to (3-7b) is sufficient.

Equation (3-5) reduces to (3-7b) if:

$$d_2 = 1 ,$$

$$b_1 - 2b_2 = -1,$$

$$a_2{}^2 = \frac{1}{4}. \qquad (3\text{-}10)$$

Returning to (3-2b), we operate on both sides this time with $(\beta_2, -\beta_1)$ to eliminate Ψ. Again exploiting the identities (3-3) we have

$$[(b_1+2b_2)\alpha+m](\beta_2,-\beta_1)\Xi - \frac{1}{2} a_3(\alpha^2-\Box^2)\Omega = 0. \qquad (3\text{-}11)$$

If this is multiplied by a_3 and (3-2c) by $[(b_1+2b_2)\alpha+m]$, an equation is obtained for Ω:

$$\{(b_1+2b_2)c_1\alpha^2 + [c_1+d_3(b_1+2b_2)]m\alpha + d_3m^2$$

$$+ \frac{a_3{}^3}{2}(\alpha^2-\Box^2)\}\Omega = 0. \qquad (3\text{-}12)$$

The requirement that Ω satisfy (3-7b) implies the following choice of constants:

$$d_3 = -1 \qquad c_1{}^2 = 1$$

$$a_3{}^2 = \frac{1}{4} \qquad b_1+2b_2 = c_1 \qquad (3\text{-}13)$$

Ω therefore satisfies (3-7a) and (3-7b) in a way analagous to Ψ.

The equations now reduce to:

$$(\beta_1,\beta_2)\Xi = 0$$

$$b_1\alpha\Xi + b_2\begin{pmatrix}-\beta_1 & \beta_2\\ \beta_2 & \beta_1\end{pmatrix}\Xi + m\Xi = 0$$

$$(\beta_2, -\beta_1)\Xi = 0 \tag{3-14}$$

The relations among the constants (3-10) and (3-13) have two sets of values. If $c_1 = -1$,

$$b_1 = -1,\ b_2 = 0; \tag{3-15a}$$

but if $c_1 = +1$,

$$b_1 = 0,\ b_2 = \frac{1}{2}\ . \tag{3-15b}$$

In the first case, the equations (3-14) become:

$$\begin{aligned} \beta_1\chi' + \beta_2\chi &= 0 \\ (\alpha - m)\chi' &= 0 \\ (\alpha - m)\chi &= 0 \\ \beta_2\chi' - \beta_1\chi &= 0 \end{aligned} \tag{3-16}$$

If the first and last of these are combined:

$$\begin{aligned} (\beta_1{}^2 + \beta_2{}^2)\chi' = (\beta_1{}^2 + \beta_2{}^2)\chi &= -\tfrac{1}{2}(\alpha^2 - \Box^2)\chi' \\ &= -\tfrac{1}{2}(\alpha^2 - \Box^2)\chi = 0. \end{aligned} \tag{3-17}$$

Multiplying the middle two equations of (3-16) by $(\alpha + m)$ and combining with (3-17) leads to:

$$(\Box^2 - m^2)\chi = (\Box^2 - m^2)\chi' = 0. \qquad \text{(3-18}$$

Then Ξ is found to satisfy (3-7b), its components each satisfying the Bargmann-Wigner equations with m replaced by -m.

In the second case specified by conditions (3-15b), the equations (3-14) are:

$$\beta_1\chi'+\beta_2\chi = 0$$

$$-\beta_1\chi'+\beta_2\chi+2m\chi' = 0$$

$$\beta_2\chi'+\beta_1\chi+2m\chi = 0$$

$$\beta_2\chi'-\beta_1\chi = 0 \qquad (3\text{-}19)$$

As in the previous case, the first and last equations imply (3-17). The middle two equations may be combined to give:

$$(\beta_1{}^2+\beta_2{}^2)\chi' = -2m(-\beta_1\chi'+\beta_2\chi) = 0$$

$$(\beta_1{}^2+\beta_2{}^2)\chi = -2m(\beta_1\chi+\beta_2\chi) = 0 \qquad (3\text{-}20)$$

This result follows from (3-17) and, when this result is compared with the middle two equations of (3-19), it is seen immediately that χ and χ' vanish.

Transformation to Spinor-Tensor Form

The transformation of the spin-3/2 Lagrangian density to spinor-tensor form has been done by Repko [18]. The method and results are reproduced here for completeness.

The totally symmetric field can be expanded in terms of operators symmetric in the first two indices:

$$\Psi_{\alpha\beta\gamma} = \frac{1}{2}\,[(\gamma_\mu C)_{\alpha\beta}(\Psi_\mu)_\gamma + \frac{1}{2}(\sigma_{\mu\nu}C)_{\alpha\beta}(\Psi_{\mu\nu})_\gamma]. \qquad (3\text{-}21)$$

Symmetrization of the second two indices is accomplished by multiplying $\Psi_{\alpha\beta\gamma}$ by each of the antisymmetric operators:

$$C^{-1}_{\beta\gamma},\ i(C^{-1}\gamma_5\gamma_\lambda)_{\beta\gamma},\ (C^{-1}\gamma_5)_{\beta\gamma},$$

and setting the result equal to zero. When this is done, the following conditions result:

$$\Psi_\mu = -i\gamma_\nu\Psi_{\mu\nu},\ \sigma_{\mu\nu}\Psi_{\mu\nu} = 0$$

$$\gamma_\mu\Psi_\mu = 0 \qquad (3\text{-}22)$$

The most general expansions for $\chi_{\alpha\beta\gamma}$ and Ω are:

$$\chi_{[\alpha\beta]\gamma} = \frac{1}{4}\sqrt{3}\,[(C)_{\alpha\beta}(\eta)_\gamma + i(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\chi_\mu)_\gamma + (\gamma_5 C)_{\alpha\beta}(\chi)_\gamma]$$

$$\Omega_{[\alpha\beta\gamma]} = \frac{\sqrt{3}}{4}\,[(C)_{\alpha\beta}(\omega)_\gamma + i(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\Omega_\mu)_\gamma + (\gamma_5 C)_{\alpha\beta}(\Omega)_\gamma] \qquad (3\text{-}23)$$

which, when their symmetry requirements are applied, give:

$$\eta = \gamma_5(\chi - i\gamma_\nu\chi_\nu),$$

$$\omega = -\gamma_5\Omega,\ \Omega_\mu = i\gamma_\mu\Omega. \qquad (3\text{-}24)$$

The Lagrangian being written down and varied, the field equations lead to:

$$\Omega = \chi = \partial_\mu \chi_\mu = 0$$

$$\gamma_\lambda \chi_\lambda = \partial_\lambda \Psi_\lambda = 0$$

$$\chi_\mu = 0 \tag{3-25}$$

so that:

$$(\gamma\partial + m)\Psi_\mu = 0$$

$$\gamma_\mu \Psi_\mu = 0$$

$$\Psi_{\mu\nu} = \frac{i}{m}(\partial_\mu \Psi_\nu - \partial_\nu \Psi_\mu) \tag{3-26}$$

Chapter IV
The Fourth-Rank Multispinor

The symmetric group methods outlined in previous chapters will be applied here to the fourth-rank multispinor. We choose to develop for illustrative purposes the spin-0 field which represents the smallest dimension, non-trivial case. We transform the multispinor Lagrangian density to tensor form and show that the Bargmann-Wigner equations in tensor form are equivalent to the multispinor counterpart.

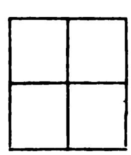
We first establish that if ϕ satisfies

$$(\alpha+m)\phi = 0, \tag{4-1}$$

then ϕ is a valid representative of a spin-0 field. Here α is the symmetric operator

$$\alpha = \tfrac{1}{4}[(\gamma\cdot\partial)_1+(\gamma\cdot\partial)_2+(\gamma\cdot\partial)_3+(\gamma\cdot\partial)_4]. \tag{4-2}$$

ϕ belongs to the irreducible representation given by (2^2) and shown by the Young's tableaux:



We choose the representative component of ϕ that is antisymmetric in each pair of indices:

$$\phi \equiv \phi_{[\alpha\beta][\gamma\delta]}. \tag{4-3}$$

Since the Lorentz boost cannot change the number of spin components, we may look at equation (4-1) in the rest frame. Then, assuming the Klein-Gordon equation is satisfied, the derivatives with respect to the space coordinates vanish and the time derivative gives a factor of -m. Equation (4-1) is then:

$$-m[\gamma_1^{(4)} + \gamma_2^{(4)} + \gamma_3^{(4)} + \gamma_4^{(4)}]_{\alpha\beta\gamma\delta,\alpha'\beta'\gamma'\delta'}$$

$$\times \phi_{\alpha'\beta'\gamma'\delta'} = -m\, \phi_{\alpha\beta\gamma\delta} \; . \qquad (4\text{-}4)$$

The ϕ's on the left hand side vanish unless the components of the spinor corresponding to the third and fourth rows of $\gamma^{(4)}$ are zero. Thus only the ϕ's with subscripts 1 and 2 survive, but the antisymmetry leaves only one possibility:

$$\phi_{[12][12]}.$$

For the negative energy solution, only the component

$$\phi_{[34][34]}$$

survives. Thus there is only one component each for the positive and negative solutions, as is to be expected for a spin-0 object.

The couplings that are allowed are those among ϕ, ζ, and Ω listed for the fourth-rank multispinor in Chapter II. When mass terms and Hermitian conjugates are included, the Lagrangian may be written:

$$L = -\bar{\phi}(\alpha+m)\phi + a_1[\bar{\phi}\hat{\theta}(2,\bar{3})\zeta+\bar{\zeta}\hat{\theta}(\bar{3},2)\phi]$$

$$+\bar{\zeta}[b_1\alpha+b_2\hat{\theta}(\bar{3},\bar{3})+c_1m]\zeta + \bar{\Omega}[b_3\alpha+c_2m]\Omega$$

$$+ a_2[\bar{\zeta}\hat{\theta}(\bar{3},\bar{1})\Omega + \bar{\Omega}\hat{\theta}(\bar{1},\bar{3})\zeta]. \qquad (4\text{-}5)$$

This is the most general Lagrangian for the spin-0 part of the fourth-rank multispinor field.

We next seek to investigate the Bargmann-Wigner equations in tensor form. To do this, we first expand the fields ϕ, ζ, and Ω in tensor representations.

ϕ Expansion

We choose the component ϕ, (4-3) above. Then the most general expansion which carries this symmetry is:

$$\phi_{[\alpha\beta][\gamma\delta]} = FC_{\alpha\beta}C_{\gamma\delta}+[C_{\alpha\beta}(\gamma_\mu\gamma_5C)_{\gamma\delta}+(\gamma_\mu\gamma_5C)_{\alpha\beta}C_{\gamma\delta}]F_\mu$$

$$+ [C_{\alpha\beta}(\gamma_5C)_{\gamma\delta}+(\gamma_5C)_{\alpha\beta}C_{\gamma\delta}]F^5$$

$$+ (\gamma_\mu\gamma_5C)_{\alpha\beta}(\gamma_\nu\gamma_5C)_{\gamma\delta}F_{\mu\nu}$$

$$+ [(\gamma_\mu\gamma_5C)_{\alpha\beta}(\gamma_5C)_{\gamma\delta}+(\gamma_5C)_{\alpha\beta}(\gamma_\mu\gamma_5C)_{\gamma\delta}]G_\mu$$

$$+ (\gamma_5C)_{\alpha\beta}(\gamma_5C)_{\gamma\delta}G. \qquad (4\text{-}6)$$

$\phi_{[\alpha\beta][\gamma\delta]}$ of itself has twenty components. This follows from the fact that each antisymmetric pair of indices may be composed of any of six combinations of the values 1, 2, 3, and 4. Symmetry under exchange of the pairs of indices reduces the number of independent components to twenty-one. Then the cyclic relation, (2-53), imposes an added condition which leaves twenty components.

The quantities on the right hand side of (4-6), however, comprise twenty-one independent values. Thus, we seek an additional relationship among the amplitudes in the expansion. This is to be found by imposing the cyclic relation (2-53). Then operating on the resulting equation with the inverse of any of the antisymmetric matrices leads to:

$$G = F + F_{\mu\mu}, \qquad (4\text{-}7)$$

the desired relation.

Next, we require that ϕ satisfy the Bargmann-Wigner equations:

$$[(\gamma\cdot\partial)_{\alpha\alpha'} + \delta_{\alpha\alpha'} m]\phi_{\alpha'\beta\gamma\delta} = 0. \qquad (4\text{-}8)$$

$$\vdots$$

If we use the tensor expansion for $\phi_{\alpha'\beta\gamma\delta}$ in (4-8) and multiply by $(C^{-1})_{\delta\gamma}(C^{-1})_{\beta\alpha}$, we find that:

$$F = 0. \qquad (4\text{-}9)$$

Multiplication by $[(\gamma_\mu\gamma_5 C)^{-1}]_{\delta\gamma}(C^{-1})_{\beta\alpha}$ yields:

$$F_\mu = 0. \qquad (4\text{-}10)$$

Operation with $(C^{-1}\gamma_5)_{\delta\gamma}(C^{-1})_{\beta\alpha}$ gives:

$$F^5 = 0. \qquad (4\text{-}11)$$

It is to be noted that (4-9) and (4-7) give:

$$F_{\mu\mu} = G. \tag{4-12}$$

Multiplication by $(C^{-1}\gamma_5\gamma_\rho)_{\delta\gamma}(C^{-1}\gamma_5\gamma_\lambda)_{\beta\alpha}$ gives the relation:

$$mF_{\mu\nu} = \tfrac{1}{2}[\partial_\mu G_\nu + \partial_\nu G_\mu], \tag{4-13}$$

whereas, operation with $(C^{-1}\gamma_5)_{\delta\gamma}(C^{-1}\gamma_5\gamma_\lambda)_{\beta\alpha}$ yields:

$$\partial_\lambda G = mG_\lambda. \tag{4-14}$$

Combining the last two equations:

$$m^2F_{\mu\nu} = \partial_\mu\partial_\nu G \ , \tag{4-15}$$

but, contracting on μ and ν and using (4-12), we obtain the Klein-Gordon equation on G:

$$(\Box^2 - m^2)G = 0. \tag{4-16}$$

Moreover, equations (4-13) and (4-14) show that there is only one independent parameter (namely G), since G_λ and $F_{\mu\nu}$ may be derived from it by taking appropriate derivatives.

ζ Expansion

A general expansion for the fourth-rank multispinor from the representation $(2,1^2)$ is:

$$\zeta_{[\alpha\beta\gamma]\delta} = C_{\alpha\beta}C_{\gamma\delta}S + C_{\alpha\beta}(\gamma_\mu C)_{\gamma\delta}V_\mu + C_{\alpha\beta}(\sigma_{\mu\nu}C)_{\gamma\delta}iT_{\mu\nu}$$
$$+ C_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}A_\mu + C_{\alpha\beta}(\gamma_5 C)_{\gamma\delta}P + (\gamma_5 C)_{\alpha\beta}C_{\gamma\delta}S'$$

$$+(\gamma_5 C)_{\alpha\beta}(\gamma_\mu C)_{\gamma\delta}V'_\mu+(\gamma_5 C)_{\alpha\beta}(\sigma_{\mu\nu}C)_{\gamma\delta}iT'_{\mu\nu}$$

$$+(\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}A'_\mu+(\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta}P'$$

$$+(\gamma_\mu\gamma_5 C)_{\alpha\beta}C_{\gamma\delta}X_\mu+(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\nu C)_{\gamma\delta}X_{\mu\nu}$$

$$+(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\sigma_{\rho\sigma}C)iX_{\mu\rho\sigma}+(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta}Y_\mu$$

$$+(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\nu\gamma_5 C)_{\gamma\delta}Y_{\mu\nu}, \tag{4-17}$$

where that component is chosen which is totally antisymmetric in the first three indices and corresponds to the Young's tableau:

1	4
2	
3	

Relation (4-17) is manifestly antisymmetric in the first two indices. The antisymmetrization of the second and third indices can be achieved with multiplication by the symmetric operator $(\gamma_\lambda C)^{-1}{}_{\beta\gamma}$ and requiring the result to vanish. When this is done, multiplication by $(C)^{-1}{}_{\delta\alpha}$ produces the following relation:

$$V_\mu+A'_\mu-Y_\mu = -\varepsilon_{\mu\nu\rho\sigma}X_{\nu\rho\sigma}. \tag{4-18}$$

However, if the antisymmetrized equation is multiplied by $(\gamma_5 C)^{-1}{}_{\delta\alpha}$:

$$V'_\mu + A_\mu - X_\mu = 2X_{\sigma\mu\sigma}. \tag{4-19}$$

If multiplied by $(\gamma_\pi C)^{-1}{}_{\delta\alpha}$ and the symmetric and antisymmetric parts separated, the antisymmetrized equation leads to two relations:

$$S\text{-}P' = \tfrac{1}{2}Y_{\mu\mu}, \tag{4-20}$$

$$T_{\mu\nu} + \tilde{T}'_{\mu\nu} + \tilde{X}_{\mu\nu} = 0. \tag{4-21}$$

Here the notation for the dual is used:

$$\tilde{T}_{\mu\nu} \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}T_{\rho\sigma}. \tag{4-22}$$

Multiplication by $(\gamma_\pi\gamma_5 C)^{-1}{}_{\delta\alpha}$ also yields a symmetric and an antisymmetric part:

$$P\text{-}S' = \tfrac{1}{2}X_{\mu\mu}, \tag{4-23}$$

$$\tilde{T}_{\mu\nu} + T'_{\mu\nu} - \tilde{Y}_{\mu\nu} = 0. \tag{4-24}$$

From the last two operations it is also found that for the off-diagonal elements of $X_{\mu\nu}$ and $Y_{\mu\nu}$:

$$X_{[\mu\nu]} = -X_{[\nu\mu]}, \quad \mu\neq\nu, \tag{4-25a}$$

$$Y_{[\mu\nu]} = -Y_{[\nu\mu]}, \quad \mu\neq\nu. \tag{4-25b}$$

Combining (4-21) and (4-24) yields:

$$X_{\mu\nu} = -Y_{\mu\nu} \; . \tag{4-26}$$

Returning to the relation (4-17) we may also antisymmetrize by multiplication with $(\sigma_{\pi\lambda}C)^{-1}{}_{\beta\gamma}$. When this is done, and the resulting equation multiplied by $(C)^{-1}{}_{\delta\alpha}$, one finds:

$$T_{\mu\nu} = \tilde{T}'_{\mu\nu}. \qquad (4\text{-}27)$$

$(\sigma_{\pi\lambda}C)^{-1}{}_{\delta\alpha}$ acting on the antisymmetrized expression yields:

$$S + P' = 0, \qquad (4\text{-}28)$$

which, when combined with (4-20) gives:

$$S = -P' = \tfrac{1}{4}Y_{\mu\mu}. \qquad (4\text{-}29a)$$

Similarly, the result of operation with $(\sigma_{\pi\lambda}\gamma_5 C)_{\delta\alpha}$ combined with (4-23) gives:

$$S' = -P = -\tfrac{1}{4}X_{\mu\mu}. \qquad (4\text{-}29b)$$

Combining the relations among the antisymmetric tensors:

$$T_{\mu\nu} = \tilde{T}'_{\mu\nu} = -\tfrac{1}{2}\tilde{X}_{\mu\nu} = \tfrac{1}{2}Y_{\mu\nu}. \qquad (4\text{-}30)$$

If the antisymmetrized relation is multiplied by $(\sigma_{\pi\lambda}\gamma_\tau C)^{-1}{}_{\delta\alpha}$, the following results:

$$V_\mu = A'_\mu, \qquad (4\text{-}31)$$

whereas multiplication by $(\gamma_\tau C)^{-1}{}_{\delta\alpha}$ alone gives:

$$3V_\mu - 3A'_\mu - 3Y_\mu = -\varepsilon_{\mu\nu\rho\sigma}Y_{\nu\rho\sigma}. \qquad (4\text{-}32)$$

The last two relations together with (4-18) give:

$$V_\mu = A'_\mu = -Y_\mu = -\frac{1}{3}\varepsilon_{\mu\nu\rho\sigma}Y_{\nu\rho\sigma}. \qquad (4\text{-}33)$$

Operation with $(\gamma_\lambda\gamma_5 C)^{-1}{}_{\delta\alpha}$ gives:

$$V'_\mu = A_\mu, \qquad (4\text{-}34)$$

while the multiplication by $(\gamma_\pi \gamma_5 C)^{-1}{}_{\delta\alpha}$ gives:

$$X_\mu = -\frac{2}{3} X_{\sigma\mu\sigma}. \tag{4-35}$$

Combining the last two results with (4-19), we have:

$$V'_\mu = A_\mu = -X_\mu = \frac{2}{3} X_{\sigma\mu\sigma}. \tag{4-36}$$

These relations restrict the number of tensor components to sixteen, so that all of the amplitudes in (4-17) may be represented by the set: $\{S, P, V_\mu, A_\mu, T_{\mu\nu}\}$. However $\zeta_{[\alpha\beta\gamma]\delta}$ is a fifteen-component object. This is clear from the fact that the antisymmetry in the first three indices provides for four independent combinations while there are four values for the fourth index. The symmetry relation (2-51b) restricts the number to fifteen. Thus we seek an additional relation among the sixteen tensor components. The relation (4-17) may now be written:

$$\begin{aligned}
\zeta_{[\alpha\beta\gamma]\delta} = {} & [C_{\alpha\beta}C_{\gamma\delta} - (\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta} + (\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}]S \\
& + [C_{\alpha\beta}(\gamma_5 C)_{\gamma\delta} - (\gamma_5 C)_{\alpha\beta}C_{\gamma\delta} + (\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\mu C)_{\gamma\delta}]P \\
& + [C_{\alpha\beta}(\gamma_\mu C)_{\gamma\delta} + (\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta} - (\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta} \\
& - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}(\gamma_\nu\gamma_5 C)_{\alpha\beta}(\sigma_{\rho\sigma}C)_{\gamma\delta}]V_\mu \\
& + [C_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta} + (\gamma_5 C)_{\alpha\beta}(\gamma_\mu C)_{\gamma\delta} - (\gamma_\mu\gamma_5 C)_{\alpha\beta}C_{\gamma\delta} \\
& + i(\gamma_\nu\gamma_5 C)_{\alpha\beta}(\sigma_{\mu\nu}C)_{\gamma\delta}]A_\mu \\
& + [iC_{\alpha\beta}(\sigma_{\mu\nu}C)_{\gamma\delta} - i(\gamma_5 C)_{\alpha\beta}(\sigma_{\mu\nu}\gamma_5 C)_{\gamma\delta} \\
& - \epsilon_{\mu\nu\rho\sigma}(\gamma_\rho\gamma_5 C)_{\alpha\beta}(\gamma_\sigma C)_{\gamma\delta}
\end{aligned}$$

$$+\{(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\nu\gamma_5 C)_{\gamma\delta}-(\gamma_\nu\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)\}]T_{\mu\nu} \tag{4-37}$$

Imposing the cyclic relation (2-51b) on this expression and multiplying by any of the antisymmetric operators, for example, $(C)^{-1}{}_{\beta\alpha}$, yields

$$S = 0. \tag{4-38}$$

Thus we arrive at the tensor decomposition of $\zeta_{[\alpha\beta\gamma]\delta}$:

$$\zeta_{[\alpha\beta\gamma]\delta} = (\Gamma_5)_{\alpha\beta\gamma\delta}P + (\Gamma_\mu)_{\alpha\beta\gamma\delta}\, V_\mu$$

$$+ (\Gamma_{5\mu})_{\alpha\beta\gamma\delta}\, A_\mu + (\Gamma_{\mu\nu})_{\alpha\beta\gamma\delta}\, T_{\mu\nu}, \tag{4-39}$$

where the various $(\Gamma)_{\alpha\beta\gamma\delta}$ are the coefficients of P, V_μ, A_μ and $T_{\mu\nu}$ in (4-37).

Ω-Expansion

The most general expansion for the totally antisymmetric fourth-rank spinor is

$$\Omega_{[\alpha\beta\gamma\delta]} = C_{\alpha\beta}C_{\gamma\delta}\omega+\tfrac{1}{2}[C_{\alpha\beta}(\gamma_5 C)_{\gamma\delta}+(\gamma_5 C)_{\alpha\beta}C_{\gamma\delta}]\omega'$$

$$+ \tfrac{1}{2}[C_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}+(\gamma_\mu\gamma_5 C)_{\alpha\beta}C_{\gamma\delta}]\omega_\mu$$

$$+(\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta}\omega^5 + \tfrac{1}{2}[(\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}$$

$$+(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta}]\omega_\mu' + \tfrac{1}{2}[(\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\nu\gamma_5 C)_{\gamma\delta}$$

$$+ (\gamma_\nu\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}]\omega_{\mu\nu}. \tag{4-40}$$

To antisymmetrize in the indices β and γ we multiply by $(\gamma_\lambda C)^{-1}{}_{\lambda\gamma}$ and set the result equal to zero. When the definition of $\sigma_{\lambda\mu}$ is taken into account, the result can be written:

$$(\gamma_\lambda C)_{\alpha\delta}\omega + i(\sigma_{\lambda\mu}\gamma_5 C)_{\alpha\delta}\omega_\mu + i(\sigma_{\lambda\mu}C)_{\alpha\delta}\omega'_\mu - (\gamma_\lambda C)_{\alpha\delta}\omega^5 + \tfrac{1}{2}[\gamma_\mu\gamma_\lambda\gamma_\nu C + \gamma_\nu\gamma_\lambda\gamma_\mu C]_{\alpha\delta}\omega_{\mu\nu} = 0. \quad (4\text{-}41)$$

If this is multiplied by $(C^{-1}\gamma_5\sigma_{\lambda\rho})_{\delta\alpha}$, we find:

$$\omega_\mu = 0. \quad (4\text{-}42)$$

If multiplied by $(C^{-1}\sigma_{\lambda\rho})_{\delta\alpha}$, we have:

$$\omega'_\mu = 0. \quad (4\text{-}43)$$

Multiplication by $(C^{-1}\gamma_\rho)_{\delta\alpha}$ yields, for $\rho \neq \lambda$,

$$\omega_{\lambda\rho} = -\omega_{\rho\lambda}\ , \quad (4\text{-}44)$$

but from the original expansion, $\omega_{\mu\nu}$ is manifestly symmetric in μ and ν. Therefore, the off-diagonal elements of $\omega_{\mu\nu}$ vanish. If $\rho=\lambda$,

$$\omega - \omega^5 - 2\omega_{\mu\mu} = 0. \quad (4\text{-}45)$$

Now we antisymmetrize the original expansion by multiplication with $(\sigma_{\lambda\rho}C)^{-1}{}_{\beta\gamma}$, again using the definition of $\sigma_{\lambda\rho}$.

$$(\sigma_{\lambda\rho}C)_{\alpha\delta}\omega + (\sigma_{\lambda\rho}\gamma_5 C)_{\alpha\delta}\omega' + (\sigma_{\lambda\rho}C)_{\alpha\delta}\omega^5 + \tfrac{1}{2}[-\gamma_\mu\sigma_{\lambda\rho}\gamma_\nu C - \gamma_\nu\sigma_{\lambda\rho}\gamma_\mu C]_{\alpha\delta}\omega_{\mu\nu} = 0. \quad (4\text{-}46)$$

Operation on this expression with $(C^{-1}\sigma_{\lambda\rho}\gamma_5)_{\delta\alpha}$ gives:

$$\omega' = 0. \qquad (4\text{-}47)$$

If we simultaneously multiply by $(C^{-1}\sigma_{\lambda\rho})_{\delta\alpha}$ and use the observation above that $\omega_{\mu\nu}$ is diagonal, we obtain

$$\omega + \omega^5 = 0. \qquad (4\text{-}48)$$

If the last result is combined with (4-45):

$$\omega = -\omega^5 = \omega_{\mu\mu} . \qquad (4\text{-}49)$$

The original expansion for Ω can be written:

$$\Omega_{\alpha\beta\gamma\delta} = [C_{\alpha\beta}C_{\gamma\delta} - (\gamma_5 C)_{\alpha\beta}(\gamma_5 C)_{\gamma\delta} + (\gamma_\mu\gamma_5 C)_{\alpha\beta}(\gamma_\mu\gamma_5 C)_{\gamma\delta}]\omega. \qquad (4\text{-}50)$$

The total antisymmetry of Ω restricts the number of its components to one, and clearly the expansion (4-40) has just one independent component.

Couplings in Tensor Form

Having thus constructed the tensor expansions for ϕ, ζ, and Ω, we now proceed to investigate all the independent couplings of the form $\bar{F}\hat{O}G$. The number and the nature of the non-vanishing independent combinations may then be compared with the predictions made on the basis of symmetric group considerations as a test of the group theoretical analysis.

The expansions for $\bar{\phi}$, $\bar{\zeta}$, and $\bar{\Omega}$ are obtained by taking the complex conjugate transpose of the Dirac matrices in the expansions for the respective fields. They are:

$$\overline{\Phi} = C^{-1}_{\beta\alpha}\ C^{-1}_{\delta\gamma}\ F+[C^{-1}_{\beta\alpha}(C^{-1}\gamma_5\gamma_\mu)_{\delta\gamma}+(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}C^{-1}_{\delta\gamma}]F_\mu$$
$$-[C^{-1}_{\beta\alpha}(C^{-1}\gamma_5)_{\delta\gamma}+(C^{-1}\gamma_5)_{\beta\alpha}C^{-1}_{\delta\gamma}]F^5+(C^{-1}\gamma_5)_{\beta\alpha}(C^{-1}\gamma_5)_{\delta\gamma}G$$
$$-[(C^{-1}\gamma_5)_{\beta\alpha}(C^{-1}\gamma_5\gamma_\mu)_{\delta\gamma}+(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}(C^{-1}\gamma_5)_{\delta\gamma}]G_\mu$$
$$+(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}(C^{-1}\gamma_5\gamma_\nu)_{\delta\gamma}F_{\mu\nu}. \qquad (4\text{-}51)$$

$$\overline{\Omega} = [C^{-1}_{\beta\alpha}\ C^{-1}_{\delta\gamma}\ -(C^{-1}\gamma_5)_{\beta\alpha}(C^{-1}\gamma_5)_{\delta\gamma}$$
$$+\ (C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}(C^{-1}\gamma_5\gamma_\mu)_{\delta\gamma}]\omega \qquad (4\text{-}52)$$

$$\overline{\zeta} = [-C^{-1}_{\beta\alpha}\ (C^{-1}\gamma_5)_{\delta\gamma}+(C^{-1}\gamma_5)_{\beta\alpha}C^{-1}_{\delta\gamma}-(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}(C^{-1}\gamma_\mu)_{\delta\gamma}]P$$
$$-[C^{-1}_{\beta\alpha}\ (C^{-1}\gamma_\mu)_{\delta\gamma}+(C^{-1}\gamma_5)_{\beta\alpha}(C^{-1}\gamma_5\gamma_\mu)_{\delta\gamma}$$
$$-(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}(C^{-1}\gamma_5)_{\delta\gamma}\ +\ \frac{i}{2}\ \varepsilon_{\mu\nu\rho\sigma}(C^{-1}\gamma_5\gamma_\nu)_{\beta\alpha}$$
$$\times\ (C^{-1}\sigma_{\rho\sigma})_{\delta\gamma}]V_\mu$$
$$+[C^{-1}_{\beta\alpha}\ (C^{-1}\gamma_5\gamma_\mu)_{\delta\gamma}+(C^{-1}\gamma_5)_{\beta\alpha}(C^{-1}\gamma_\mu)_{\delta\gamma}$$
$$-(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}C^{-1}_{\delta\gamma}\ -\ i(C^{-1}\gamma_5\gamma_\nu)_{\beta\alpha}(C^{-1}\sigma_{\mu\nu})_{\delta\gamma}]A_\mu$$
$$+\{-iC^{-1}_{\beta\alpha}\ (C^{-1}\sigma_{\mu\nu})_{\delta\gamma}+i(C^{-1}\gamma_5)_{\beta\alpha}(C^{-1}\gamma_5\sigma_{\mu\nu})_{\delta\gamma}$$
$$+[(C^{-1}\gamma_5\gamma_\mu)_{\beta\alpha}(C^{-1}\gamma_5\gamma_\nu)_{\delta\gamma}-(C^{-1}\gamma_5\gamma_\nu)_{\beta\alpha}(C^{-1}\gamma_5\gamma_\mu)_{\delta\gamma}]$$
$$-\ \varepsilon_{\mu\nu\rho\sigma}(C^{-1}\gamma_5\gamma_\rho)_{\beta\alpha}(C^{-1}\gamma_\sigma)_{\delta\gamma}\}T_{\mu\nu}. \qquad (4\text{-}53)$$

The symmetry properties of ϕ and Ω make sufficient the calculation of just one kinetic coupling, which may be taken to be $(\gamma\cdot\partial)_1$, with each pair of these fields. When the tensor expansions are inserted and traces are taken, these couplings are:

$$\bar{\Omega}(\gamma\cdot\partial)_1\Omega = 0 \tag{4-54a}$$

$$\bar{\Omega}(\gamma\cdot\partial)_1\phi = \bar{\phi}(\gamma\cdot\partial)_1\Omega = 0 \tag{4-54b}$$

$$\bar{\phi}(\gamma\cdot\partial)_1\phi = -16[F^5(\partial_\mu F_\mu) - F_\mu(\partial_\mu F^5) + G_\mu(\partial_\mu G) - G(\partial_\mu G_\mu) + G_\nu(\partial_\mu F_{\mu\nu}) - F_{\mu\nu}(\partial_\mu G_\nu)] \tag{4-54c}$$

The symmetry of ζ requires the calculation of both $(\gamma\cdot\partial)_1$- and $(\gamma\cdot\partial)_4$-type couplings in each term where ζ appears:

$$\bar{\Omega}(\gamma\cdot\partial)_1\zeta = 32\omega(\partial_\mu V_\mu)$$

$$\bar{\zeta}(\gamma\cdot\partial)_1\Omega = -32V_\mu(\partial_\mu\omega)$$

$$\bar{\Omega}(\gamma\cdot\partial)_4\zeta = -96\omega(\partial_\mu V_\mu)$$

$$\bar{\zeta}(\gamma\cdot\partial)_4\Omega = 96V_\mu(\partial_\mu\omega)$$

$$\bar{\phi}(\gamma\cdot\partial)_1\zeta = 16[-F_\mu(\partial_\mu P)+F^5(\partial_\mu A_\mu)+F_{\mu\nu}(\partial_\mu V_\nu) - 2G_\nu(\partial_\mu T_{\mu\nu}) - G(\partial_\mu V_\mu)]$$

$$\bar{\zeta}(\gamma\cdot\partial)_1\phi = 16[P(\partial_\mu F_\mu)-V_\nu(\partial_\mu F_{\mu\nu})+V_\mu(\partial_\mu G) - A_\mu(\partial_\mu F^5) - 2T_{\mu\nu}(\partial_\nu G_\mu)]$$

$$\bar{\phi}(\gamma\cdot\partial)_4\zeta = \bar{\zeta}(\gamma\cdot\partial)_4\phi = 0$$

$$\begin{aligned}\bar{\zeta}(\gamma\cdot\partial)_1\zeta = -32[&P(\partial_\mu A_\mu) + 2V_\nu(\partial_\mu T_{\mu\nu}) - A_\mu(\partial_\mu P)\\ &+ 2T_{\mu\nu}(\partial_\nu V_\mu) - \varepsilon_{\lambda\tau\mu\nu}A_\lambda(\partial_\tau T_{\mu\nu})\\ &+ \varepsilon_{\mu\nu\tau\lambda}T_{\mu\nu}(\partial_\tau A_\lambda)]\end{aligned}$$

$$\begin{aligned}\bar{\zeta}(\gamma\cdot\partial)_4\zeta = -96[&P(\partial_\mu A_\mu)+2V_\nu(\partial_\mu T_{\mu\nu})-A_\mu(\partial_\mu P)\\ &+2T_{\mu\nu}(\partial_\nu V_\mu) + \varepsilon_{\lambda\tau\mu\nu}A_\lambda(\partial_\tau T_{\mu\nu})\\ &- \varepsilon_{\mu\nu\tau\lambda}T_{\mu\nu}(\partial_\tau A_\lambda)]\end{aligned} \qquad (4\text{-}55)$$

There is therefore just one independent coupling between ζ and Ω, and one between ζ and ϕ. The vanishing of the kinetic term involving Ω and ϕ is consistent with symmetric group predictions. The mass terms are:

$$m\bar{\phi}\phi = 16[F^2+2F_\mu F_\mu-2(F^5)^2+F_{\mu\nu}F_{\mu\nu}-2G_\mu G_\mu+G^2]m$$

$$m\bar{\Omega}\Omega = 96m\omega^2$$

$$m\bar{\zeta}\zeta = 96[P^2+V_\mu V_\mu-A_\mu A_\mu-2T_{\mu\nu}T_{\mu\nu}]m \qquad (4\text{-}56)$$

The Lagrangian and the Field Equations

Before writing down the Lagrangian using the couplings found above, it is to be noted that F, G and $F_{\mu\mu}$ are related by:

$$G = F + F_{\mu\mu}\ , \qquad (4\text{-}7)$$

and thus cannot be varied independently. We therefore write:

$$F_{\mu\nu} = F_{\mu\mu} + F'_{\mu\nu} , \qquad (4\text{-}57)$$

where $F'_{\mu\nu}$ is traceless. The traceless condition may be represented explicitly:

$$F'_{\mu\nu} = \overline{F}_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}\,\overline{F}_{\lambda\lambda}. \qquad (4\text{-}58)$$

In order to avoid cross terms in the mass coupling, it is convenient to introduce the following transformation of F and G:

$$G = 1/\sqrt{2}\,(G'-F')$$

$$F = 1/\sqrt{2}\,(G'+F') \qquad (4\text{-}59)$$

Thus:

$$F_{\mu\nu} = (G-F) + \overline{F}_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}\,\overline{F}_{\lambda\lambda}$$

or,

$$F_{\mu\nu} = -(1/2\sqrt{2})\,F'\delta_{\mu\nu} + \overline{F}_{\mu\nu} - \tfrac{1}{4}\delta_{\mu\nu}\,\overline{F}_{\lambda\lambda}. \qquad (4\text{-}60)$$

The Lagrangian is shown in Table 2 with the various tensor terms written opposite the multispinor couplings from which they arose. The primed coefficients are introduced to absorb certain multiples of sixteen which occur in the traces. They are related to the script constants as follows:

$$32\mathcal{a}_2 \equiv a'_2 \qquad 96\mathcal{b}_2 \equiv b'_2$$

$$16\mathcal{a}_0 \equiv 1 \qquad 96\mathcal{c}_2 \equiv c'_2$$

$$16\mathcal{a}_1 \equiv a'_1 \qquad 96\mathcal{c}_1 \equiv c'_1$$

$$32\mathcal{b}_1 \equiv b'_1 \qquad (4\text{-}61)$$

Spinor Terms	Lagrangian in Tensor Form
$a_2\bar{\Omega}(\gamma\cdot\partial)_1\zeta$	$a_2'\ \omega\ (\partial_\mu V_\mu)$
$a_2\bar{\zeta}(\gamma\cdot\partial)_1\Omega$	$-a_2'\ V_\mu(\partial_\mu\omega)$
$-a_0\bar{\phi}(\gamma\cdot\partial)_1\phi$	$+[F^5(\partial_\mu F_\mu)-F_\mu(\partial_\mu F^5)+\frac{3}{2\sqrt{2}}\{F'(\partial_\mu G_\mu)-G_\mu(\partial_\mu F')\}$
	$+\frac{1}{\sqrt{2}}\{G_\mu(\partial_\mu G')-G'(\partial_\mu G_\mu)\}+\{G_\nu\partial_\mu\bar{F}_{\mu\nu}-\bar{F}_{\mu\nu}\partial_\mu G_\nu\}$
	$+\frac{1}{4}\{\bar{F}_{\lambda\lambda}\delta_{\mu\nu}(\partial_\mu G_\nu)-G_\nu(\partial_\mu\bar{F}_{\lambda\lambda}\delta_{\mu\nu}\}]$
$a_1\bar{\phi}(\gamma\cdot\partial)_1\zeta$	$+a_1'[\frac{1}{2\sqrt{2}}F'(\partial_\mu V_\mu)-\frac{1}{\sqrt{2}}G'(\partial_\mu V_\mu)-\frac{1}{4}\bar{F}_{\lambda\lambda}\delta_{\mu\nu}(\partial_\mu V_\nu)$
	$+\bar{F}_{\mu\nu}(\partial_\mu V_\nu)+F^5(\partial_\mu A_\mu)-F_\mu(\partial_\mu P)-2G_\nu(\partial_\mu T_{\mu\nu})]$
$a_1\bar{\zeta}(\gamma\cdot\partial)_1\phi$	$+a_1'[-\frac{1}{2\sqrt{2}}V_\mu(\partial_\mu F')+\frac{1}{\sqrt{2}}V_\mu(\partial_\mu G')+\frac{1}{4}V_\mu(\partial_\mu\bar{F}_{\lambda\lambda})-V_\nu(\partial_\mu\bar{F}_{\mu\nu})$
	$-A_\mu(\partial_\mu F^5)+P(\partial_\mu F_\mu)-2T_{\mu\nu}(\partial_\nu G_\mu)]$
$b_1\bar{\zeta}(\gamma\cdot\partial)_1\zeta$	$-b_1'[P(\partial_\mu A_\mu)+2V_\nu(\partial_\mu T_{\mu\nu})-A_\mu(\partial_\mu P)+2T_{\mu\nu}(\partial_\nu V_\mu)$
	$-\varepsilon_{\lambda\tau\mu\nu}A_\lambda(\partial_\tau T_{\mu\nu})+\varepsilon_{\mu\nu\tau\lambda}T_{\mu\nu}(\partial_\tau A_\lambda)]$
$b_2\bar{\zeta}(\gamma\cdot\partial)_4\zeta$	$-b_2'[P(\partial_\mu A_\mu)+2V_\nu(\partial_\mu T_{\mu\nu})-A_\mu(\partial_\mu P)+2T_{\mu\nu}(\partial_\nu V_\mu)$
	$+\varepsilon_{\lambda\tau\mu\nu}A_\lambda(\partial_\tau T_{\mu\nu})-\varepsilon_{\mu\nu\tau\lambda}T_{\mu\nu}(\partial_\tau A_\lambda)]$
$-a_0 m\bar{\phi}\phi$	$-m[-2(F^5)^2+2F_\mu F_\mu-2G_\mu G_\mu+\frac{3}{2}(F')^2+(G')^2$
	$-\frac{1}{4}(\bar{F}_{\lambda\lambda})^2+\bar{F}_{\mu\nu}\bar{F}_{\mu\nu}]$
$c_2 m\bar{\Omega}\Omega$	$+c_2'\ \omega^2\ m$
$c_1 m\bar{\zeta}\zeta$	$-c_1'\ m[P^2+V_\mu V_\mu-A_\mu A_\mu-2T_{\mu\nu}T_{\mu\nu}]$

TABLE 2

Variation with respect to F^5, F_μ, P, and A_μ, respectively, leads to the following equations:

$$\partial_\mu F_\mu + a_1' \partial_\mu A_\mu + 2mF^5 = 0 \tag{4-62}$$

$$2mF_\mu + \partial_\mu F^5 + a_1' \partial_\mu P = 0 \tag{4-63}$$

$$a_1' \partial_\mu F_\mu - b' \partial_\mu A_\mu - c_1' m P = 0 \tag{4-64}$$

$$c_1' m A_\mu - a_1' \partial_\mu F^5 + b' \partial_\mu P = 0 \tag{4-65}$$

where $b' = (b_1' + b_2')$. Eliminating $\partial_\mu F_\mu$ between (4-62) and (4-64), and letting $(a_1')^2 = -b'$, we obtain:

$$a_1' F^5 = -\frac{c_1'}{2} P . \tag{4-66}$$

Eliminating $\partial_\mu F^5$ between (4-63) and (4-65) gives:

$$2a_1' F_\mu = -c_1' A_\mu . \tag{4-67}$$

Using (4-66) in (4-65), it is easily seen that A_μ vanishes if

$$c_1' = -2b'. \tag{4-68}$$

From (4-67) it is seen that if $A_\mu = 0$, then $F_\mu = 0$; then from (4-62) and (4-64), $F^5 = P = 0$. Thus we have the following restraints on the constants:

$$a_1'^2 = -b' = \frac{c_1'}{2} . \tag{4-69}$$

Variation with respect to V_μ, G_μ, and $T_{\mu\nu}$, respectively, gives:

$$-2a_2' \partial_\mu \omega - \frac{2a_1'}{2\sqrt{2}} \partial_\mu F' + \frac{2a_1'}{\sqrt{2}} \partial_\mu G' + \frac{a_1'}{2} \partial_\mu \overline{F}_{\lambda\lambda} - 2a_1' \partial_\nu \overline{F}_{\mu\nu}$$

$$+ 4 b' \partial_\nu T_{\mu\nu} - 2c_1 m V_\mu = 0. \quad (4\text{-}70)$$

$$-\frac{6a_1'}{2\sqrt{2}} \partial_\mu F' + \frac{4a_1'}{2\sqrt{2}} \partial_\mu G' - \tfrac{1}{2} \partial_\mu \overline{F}_{\lambda\lambda} + 2 \partial_\nu \overline{F}_{\mu\nu}$$

$$+ 4a_1' \partial_\nu T_{\mu\nu} + 4 m G_\mu = 0. \quad (4\text{-}71)$$

$$[\partial_\nu G_\mu - \partial_\mu G_\nu] + b'[\partial_\nu V_\mu - \partial_\mu V_\nu] - 2 c_1 m T_{\mu\nu} = 0. \quad (4\text{-}72)$$

Taking ∂_τ through (4-70) and (4-71) and separating the antisymmetric part of each:

$$-2a_1' A_{\tau\mu}(\partial_\tau \partial_\nu \overline{F}_{\mu\nu}) + 4b' A_{\tau\mu}(\partial_\tau \partial_\nu T_{\mu\nu})$$

$$- 2c_1' mA_{\tau\mu}(\partial_\tau V_\mu) = 0. \quad (4\text{-}73a)$$

$$+2 A_{\tau\mu}(\partial_\tau \partial_\nu \overline{F}_{\mu\nu}) + 4a_1' A_{\tau\mu}(\partial_\tau \partial_\nu T_{\mu\nu})$$

$$+ 4 m A_{\tau\mu}(\partial_\tau G_\mu) = 0. \quad (4\text{-}73b)$$

$A_{\tau\mu}$ represents the part of each function antisymmetric under exchange of τ and μ. If we now set $a_1' = 1$, $b' = -1$, and $c_1' = 2$, consistent with (4-69), and add the two parts of (4-73), we find:

$$(\partial_\tau V_\mu - \partial_\mu V_\tau) = (\partial_\tau G_\mu - \partial_\mu G_\tau). \quad (4\text{-}74)$$

Using this result in (4-72),

$$T_{\mu\nu} = 0. \tag{4-75}$$

Variation with respect to F' and G' gives:

$$\frac{3}{\sqrt{2}} \partial_\mu G_\mu + \frac{1}{\sqrt{2}} \partial_\mu V_\mu - 3\, m\, F' = 0, \tag{4-76}$$

$$- \frac{1}{\sqrt{2}} \partial_\mu G_\mu - \frac{1}{\sqrt{2}} \partial_\mu V_\mu - m\, G' = 0. \tag{4-77}$$

Combining these last two equations and transforming back to the unprimed variables:

$$\partial_\mu V_\mu = - 3\, m\, F\ , \tag{4-78}$$

$$\partial_\mu G_\mu = m(2F-G). \tag{4-79}$$

Variation with respect to ω leads to:

$$\partial_\mu V_\mu = - \frac{c_2'}{a_2'}\, m\, \omega\ . \tag{4-80}$$

Variation with respect to $\overline{F}_{\mu\nu}$ gives:

$$(\partial_\nu G_\mu + \partial_\mu G_\nu) - (\partial_\nu V_\mu + \partial_\mu V_\nu) - \tfrac{1}{2}\, \partial_\lambda G_\lambda \delta_{\mu\nu}$$

$$+ \tfrac{1}{2}\, \partial_\lambda V_\lambda \delta_{\mu\nu} - \tfrac{1}{2} m \overline{F}_{\lambda\lambda} \delta_{\mu\nu} + 2\overline{F}_{\mu\nu} m = 0. \tag{4-81}$$

Combining this with (4-74), we have:

$$\partial_\nu G_\mu - \partial_\nu V_\mu - \tfrac{1}{4} \partial_\lambda G_\lambda \delta_{\mu\nu} + \tfrac{1}{4} \partial_\lambda V_\lambda \delta_{\mu\nu} + m(\overline{F}_{\mu\nu} - \tfrac{1}{4} \overline{F}_{\lambda\lambda} \delta_{\mu\nu}) = 0, \tag{4-82a}$$

or

$$m(\overline{F}_{\mu\nu} - \tfrac{1}{4} \overline{F}_{\lambda\lambda} \delta_{\mu\nu}) = -\partial_\nu G_\mu + \partial_\nu V_\mu + \tfrac{1}{4} \delta_{\mu\nu} \partial_\lambda G_\lambda - \tfrac{1}{4} \delta_{\mu\nu} \partial_\lambda V_\lambda . \tag{4-82b}$$

The V_μ equation, (4-70), can now be written in terms of the unprimed variables:

$$a_2' \partial_\mu \omega - \frac{3}{4} \partial_\mu G - \frac{1}{4} \partial_\mu F + \partial_\nu (F_{\mu\nu} - \frac{1}{4} \delta_{\mu\nu} F_{\lambda\lambda})$$

$$+ 2 m V_\mu = 0. \tag{4-83}$$

Taking ∂_μ through (4-82b) gives:

$$m \partial_\mu (\overline{F}_{\mu\nu} - \frac{1}{4} \overline{F}_{\lambda\lambda} \delta_{\mu\nu}) = - \frac{3}{4} \partial_\nu (\partial_\lambda G_\lambda) + \frac{3}{4} \partial_\nu (\partial_\lambda V_\lambda) \tag{4-84}$$

But from (4-78) and (4-79):

$$\partial_\nu \partial_\mu G_\mu = m \partial_\nu (2F-G), \tag{4-85a}$$

$$\partial_\nu \partial_\mu V_\mu = - 3 m \partial_\nu F. \tag{4-85b}$$

Using these results in (4-84):

$$\partial_\mu (\overline{F}_{\mu\nu} - \frac{1}{4} \overline{F}_{\lambda\lambda} \delta_{\mu\nu}) = \frac{3}{4} \partial_\nu G - \frac{15}{4} \partial_\nu F. \tag{4-86}$$

Exchanging the role of μ and ν in (4-86) and substituting in (4-83):

$$a_2' \partial_\mu \omega - 4 \partial_\mu F + 2 m V_\mu = 0. \tag{4-87}$$

From (4-78) and (4-80)

$$\partial_\mu F = \frac{c_2'}{3a_2'} \partial_\mu \omega . \tag{4-88}$$

Using (4-88) in (4-87), we obtain:

$$(a_2' - \frac{4c_2'}{3a_2'}) \partial_\mu \omega = -2 m V_\mu . \tag{4-89}$$

Thus for V_μ to vanish, it is necessary for the relation to be satisfied:

$$3a_2'^2 - 4c_2' = 0. \tag{4-90}$$

The simplest non-trivial values satisfying (4-90) are:

$$a_2' = 2; \quad c_2' = 3, \tag{4-91}$$

which, together with the values previously found:

$$a_1' = 1; \; b_1' + b_2' = -1; \; c_1' = 2, \tag{4-92a}$$

determine the Lagrangian completely. Since only the sum $b_1' + b_2'$ is fixed, there is no contradiction and some benefit in the choice $b_1' = b_2'$. Thus we have:

$$b_1' = b_2' = -\tfrac{1}{2}. \tag{4-92b}$$

The vanishing of V_μ is sufficient to show that F and ω also vanish, as may be seen from equations (4-78) and (4-80). From (4-79) we obtain the relation:

$$\partial_\mu G_\mu = - m\, G. \tag{4-93}$$

The G_μ equation (4-71) may now be written:

$$\frac{5}{4}\, \partial_\mu G + \partial_\nu(\overline{F}_{\mu\nu} - \frac{1}{4}\, \overline{F}_{\lambda\lambda}\delta_{\mu\nu}) + 2\, m\, G_\mu = 0. \tag{4-94}$$

but, using (4-86),

$$\partial_\mu G = -m\, G_\mu. \tag{4-95}$$

Using the definitions of $F_{\mu\nu}$, $F'_{\mu\nu}$, and $\overline{F}_{\mu\nu}$, (4-57) and (4-58), and incorporating (4-93) with the observation that now

$$G = F_{\mu\mu}, \tag{4-96}$$

we have:

$$\partial_\nu G_\mu + mF_{\mu\mu} + mF'_{\mu\nu} = 0. \tag{4-97}$$

This can be written more symmetrically:

$$\tfrac{1}{2}(\partial_\nu G_\mu + \partial_\mu G_\nu) = -m\ F_{\mu\nu}. \tag{4-98}$$

The relations (4-93), (4-95) and (4-98) constitute the field equations fulfilled by the tensor components of the multispinor ϕ which represents the spin-0 field of the fourth rank multispinor.

The fourth-rank, spin-0 Lagrangian in Table 2 may therefore be written:

$$\begin{aligned} L = \frac{1}{16}\{ & -\bar{\phi}(\gamma\cdot\partial)_1\phi + [\bar{\phi}(\gamma\cdot\partial)_1\zeta + \bar{\zeta}(\gamma\cdot\partial)_1\phi] \\ & - \frac{1}{12}\ \bar{\zeta}[3(\gamma\cdot\partial)_1 + (\gamma\cdot\partial)_4]\zeta \\ & + [\bar{\Omega}(\gamma\cdot\partial)_1\zeta + \bar{\zeta}(\gamma\cdot\partial)_1\Omega] \\ & + \frac{1}{6}\ m[2\bar{\zeta}\zeta + 3\bar{\Omega}\Omega - 6\bar{\phi}\phi]\}. \end{aligned} \tag{4-99}$$

CONCLUSION

It has been shown that the symmetric group provides a useful tool in the construction of multispinor Lagrangians of second, third, and fourth rank. The irreducible representations of each group correspond to the wave functions which must be introduced, while the direct product series provides information about the nature and number of the couplings which survive. Moreover the operators in the kinetic terms may be constructed directly using general group theoretical methods and the transformation matrices of S_n.

Symmetric group analysis has been applied successfully to the construction of the spin-0 fourth rank multispinor Lagrangian. The fact that the number of independent, non-vanishing couplings that arise from the nature of the fields is the same as that from the symmetric group predictions lends cogent support to the validity of symmetric group invariance as a requirement for multispinor Lagrangians in general.

The existence of a set of coupling constants that lead to the equations of motion on the spin-0 component and to the vanishing of all other components is by no means assured in advance since imposing these requirements on the field equations overdetermines these constants. The consistency of these equations leading to a unique set of values for the constants thus offers further evidence that the symmetric group method leads to the desired results.

Another observation to be made is that a single element from each irreducible representation is sufficient in each coupling. This fact is of great consequence in the determination of the coupling constants. This can be seen from the fact that a single constant appears as the coefficient of each coupling term. If that term involves all the elements of each representation and the Lagrangian is varied as though each element were an independent field, then there are more field equations implying more constraints on the constants. The likelihood of the inconsistencies mentioned above is thus greatly enhanced.

There are several lines of inquiry that invite further investigation. The spin-2 Lagrangian may be written down, the fields transformed into tensor components, and the field equations solved in a manner similar to the example worked here for spin-0.

The general theory of higher rank multispinors deserves some exploration with the application of the observations presented here. Because the spinor indices only possess four components, any function totally antisymmetric in more than four indices must vanish. Moreover, the sufficiency of one element from an irreducible representation in any coupling greatly reduces the number of auxiliary fields that must be introduced. Thus higher spin Lagrangians may prove more tractible than might have been suspected.

APPENDICES

APPENDIX A

The Symmetric Group

A brief summary of symmetric group properties is included primarily for the purpose of clarifying notation used.

The symmetric group S_n is the group of permutations on n objects and is of order n!. The number of classes of S_n, equal to the number of irreducible representations, is the same as the number of partitions of n. The number of elements in each class, as well as the number of classes, can be found from the method of Young's tableaux.

The tableaux are constructed according to the following rules:

1. There are n squares in each tableau arranged in rows and columns such that no row may exceed in length any row above it. The total number of distinct arrangements is equal to the number of classes of S_n.

2. Numerals 1, 2, 3, ..., n are assigned to each square beginning with the upper left hand corner and proceeding to the right or down, each row being filled in in order to the right and each column from the top down. For each class the number of distinct configurations of numerals is the order of the class.

The character tables for the symmetric groups considered in this thesis appear in the appropriate sections. Methods for constructing character tables for arbitrary n exist and are described in standard references.

The matrices effecting a particular transformation s and belonging to the irreducible representation i are designated:

$$\Gamma^{(i)}(s).$$

The general form of the projection operator [13] is:

$$P_{ik}^{(\ell)} = \frac{d_\ell}{g} \sum_s (\Gamma^{(\ell)}(s))_{ik} P(s).$$

Here d_ℓ is the order of the representation, g is the order of the group and P is the particular transformation corresponding to s. If this operator acts on an arbitrary function F, the result is $F^{(\ell)}(s)$, a function belonging to the irreducible representation $\Gamma^{(\ell)}(s)$, provided the result is not zero.

APPENDIX B

Transformation Matrices for S_4

The transformation matrices as they are used in this work appear below. Only the six matrices corresponding to the transposition of two objects (ij) appear here since any element of S_n can be resolved into a product of such two-cycles.

Representation

(4) [1] for all s.

(1^4) [1] for all permutations s that are even.

[-1] for all permutations s that are odd.

(2^2)

$$\Gamma(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \Gamma(12) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\Gamma(23) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \Gamma(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$\Gamma(34) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \Gamma(14) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

(3,1)

$$\Gamma(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Gamma(13) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \qquad \Gamma(34) = \begin{pmatrix} -\frac{1}{3} & \frac{\sqrt{8}}{3} & 0 \\ \frac{\sqrt{8}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Gamma(14) = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{5}{6} & -\frac{\sqrt{3}}{6} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{3}}{6} & \frac{1}{2} \end{pmatrix} \quad \Gamma(24) = \begin{pmatrix} -\frac{1}{3} & -\frac{\sqrt{2}}{3} & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{3} & \frac{5}{6} & \frac{\sqrt{3}}{6} \\ \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{6} & \frac{1}{2} \end{pmatrix}$$

$(2,1^2)$

$$\Gamma(e) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \Gamma(12) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Gamma(13) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \Gamma(34) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

$$\Gamma(14) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{6} & \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{3}}{6} & -\frac{5}{6} & -\frac{\sqrt{2}}{3} \\ \frac{\sqrt{6}}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix} \quad \Gamma(24) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{6} & -\frac{\sqrt{6}}{3} \\ \frac{\sqrt{3}}{6} & -\frac{5}{6} & -\frac{\sqrt{2}}{3} \\ -\frac{\sqrt{6}}{3} & -\frac{\sqrt{2}}{3} & \frac{1}{3} \end{pmatrix}$$

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