

THESIS



ROOM USE ONLY

ABSTRACT

DERIVATION OF AN ARBITRARY TRIANGULAR PLATE BENDING STIFFNESS MATRIX AND ITS APPLICATION TO LARGE DEFLECTION SHELL PROBLEMS

by George Lasker

This investigation is concerned with a discrete model formulation and solution of certain shell structure problems. The discretization requires a modeling of the shell by flat triangular plate elements which can have any side lengths and thickness. Each element is assigned an independent set of deformed configurations and its elastic properties are described by two stiffness matrices associated with membrane and bending stresses. Displacement and slope compatibility conditions are satisfied at the corner points of elements and in a limited sense these conditions are then satisfied along the common edge of adjacent members.

The main objective of the investigation is to obtain an explicit representation of a bending stiffness matrix for an arbitrary triangular plate element and to examine its applicability to small and large deflection plate and shell problems. Two bending stiffness matrices are obtained. One is associated with a set of deformed configurations that permit compatibility between elements to be completely satisfied for some problems and to be satisfied to a high degree for problems in general. The other is associated with a set of deformed configurations that relaxes slope compatibility but appears to give better numerical results for some problems.

The solution to the geometrically non-linear problem is obtained by a formulation consisting of a sequence of linear solutions which enable equilibrium conditions to be approximately satisfied with respect to the deformed configuration.

A computer program is given for a class of axially loaded shell of revolution problems having a symmetrical or an asymmetrical deformed configuration describable by a half period strip. Most interpretive and computational operations are performed internally from a small amount of input data describing the undeformed geometry, material properties, and boundary conditions.

Numerical results are obtained for several shallow conical shells exhibiting a snap-through type of instability. These results compare favorably with both experimental and numerical results given by several other investigators.

DERIVATION OF AN ARBITRARY TRIANGULAR PLATE
BENDING STIFFNESS MATRIX AND ITS APPLICATION
TO LARGE DEFLECTION SHELL PROBLEMS

By

George Lasker

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of
Metallurgy, Mechanics and Materials Science

1966

03345
7-7-66

ACKNOWLEDGEMENTS

I wish to express my gratitude to Professor William A. Bradley for the guidance and encouragement that he extended to me throughout this investigation.

TABLE OF CONTENTS

	Page
ACKNOWLEDGEMENT	ii
LIST OF TABLES	vi
LIST OF FIGURES.	vii
NOTATIONS	ix
 I. INTRODUCTION	 1
1.1. Preliminary remarks	1
1.2. Previous developments	4
1.3. Present investigation	9
 II. GENERAL FORMULATION	 11
2.1. Some general properties	11
2.2. Discrete method	16
2.3. Simplified model	24
2.4. Matrix relating nodes and members	25
2.5. Interpretation of non-linear problems	26
2.6. Shell stiffness matrix	29
2.7. Member-shell parameter transformation matrix	31
 III. TRIANGULAR MEMBER MATRICES	 38
3.1. Member coordinate systems	38
3.2. Geometric parameters	40
3.3. A symmetric form of the strain energy expression	43
3.4. The member displacement configurations	48
3.5. Member-shell transformation matrix	58
3.6. Member stiffness matrix	72
3.7. Continuity conditions between adjacent members	76

	Page
IV. FORMULATION FOR SHELLS OF REVOLUTION . . .	89
4.1. Interpretation of problem in terms of discrete elements	89
4.2. Coordinate systems	92
4.3. Reduced set of generalized displacement parameters	93
4.4. Node coordinate transformation matrix . . .	97
4.5. Formulation of shell stiffness matrix. . . .	98
4.6. Shell imperfection	102
4.7. Boundary conditions and loading	104
4.8. Computer program	107
V. COMPUTATIONAL RESULTS.	108
5.1. Some general remarks	108
5.2. Linear results.	109
5.3. Non-linear results.	114
5.4. Conclusions	122
BIBLIOGRAPHY	123

	Page
Appendix A. OBLIQUE COORDINATES	126
A.1. Some properties of oblique cartesian coordinates systems	126
A.2. The plane stress strain energy expression referred to oblique cartesian coordinates .	130
A.3. The thin plate bending strain energy expression referred to oblique cartesian coordinates	132
Appendix B. CONSTRUCTION OF TRANSFORMATION MATRIX $D_{(i)}^{(0)}$	134
Appendix C. DERIVATION DETAILS FOR MEMBER STIFFNESS MATRICES	135
C.1. Membrane stiffness matrix for triangular member	135
C.2. Bending stiffness matrix for triangular member	139
Appendix D. SOME ADDITIONAL RESULTS ON THE TRIANGULAR PLATE BENDING STIFFNESS MATRIX	142
D.1. An alternate form of member bending stiffness matrix	142
D.2. Member transformation matrix for bending	144
Appendix E. COMPUTER PROGRAM	147
E.1. Sequence of computer operations	147
E.2. List of Fortran symbols	149
E.3. Fortran program	152
E.4. Binary deck structure	171

LIST OF TABLES

Table		Page
3.1	Functions used to construct displacement modes. .	54
3.2	Membrane stiffness matrix	74
3.3	Bending stiffness matrix	75
4.1	S matrix	99
5.1	Comparison for a uniform stress field	110
5.2	Displacements along a radial line for a concentric plate (Fig. 5.1)	112
C1	Some definite integrals	137
D1	Alternate bending stiffness matrix	146

LIST OF FIGURES

Figure	Page
2.1 Hyperplane formed by load vectors	23
2.2 Triangular member i	33
3.1 Member coordinates	39
3.2 Covariant and contravariant node base vectors . . .	39
3.3 Member parameters and oblique coordinates	41
3.4 Unit normal and tangent vectors	41
3.5 Covariant components of node displacement relative to oblique coordinates	50
3.6 Displacement modes associated with node rotations.	52
3.7 Lines of intersection of planes given in Table 3.1. .	54
3.8 Symbols used to designate subdomains of triangle.	54
3.9 Graphic description of functions Z_{12} and Z_{13}	56
3.10 Force vector subjected to a virtual displacement . .	56
3.11 Relationship of generalized forces to node variables	63
3.12 Relationship between displacement components normal to an edge and node rotation components .	79
3.13 Variation of normal slope for secondary bending modes	81
3.14 Notation for two adjacent triangles	83
3.15 Type of approximation implicit in bending modes . .	87
4.1 Shell geometry	90
4.2 Triangulation	90
4.3 Node coordinates	91

Figure		Page
4.4	Ordering of triangular members	95
4.5	Distribution associated with generalized coordinates v_j	105
5.1	Comparison for a uniform stress field	110
5.2	Vertical displacements along a radial line for a concentric flat plate with fixed boundary conditions	113
5.3	Load-deflection curve for a concentric flat plate	115
5.4	Axial load-deflection curve for a shallow cone. . .	116
5.5	Axial load-deflection curve for a shallow cone. . .	117
5.6	Axial load-deflection curve of the shallow cone shown in Fig. 5.5 for various boundary conditions	118
5.7	Displacements along radial line for the cone of Fig. 5.5 with free boundary conditions	119
5.8	Displacements along radial line for the cone of Fig. 5.5 with hinged boundary conditions	120
5.9	Displacements of node points along radial line for the cone of Fig. 5.5 with the various boundary conditions of Fig. 5.6	121
A1.	Coordinates and base vectors	127
A2.	Base vectors	127

NOTATIONS

A scalar quantity is represented by a lower case Greek or Latin letter except the letter A. A bar above a lower case letter implies the quantity is a vector (not a column matrix). A capital Greek or Latin letter is used to designate a matrix with the exception of Δ . The elements of matrices are scalars with the exception of J , H , H_1 which have elements that are vectors and Ω which has elements that are vector operators. The transpose and inverse matrices of B are respectively designated by B^T and B^{-1} .

Two groups of rectangular coordinate systems, associated with triangular members and node points, and one general coordinate system are used. A bracketed subscript refers a quantity to the appropriate coordinate system and/or node point. Greek, Latin, and the 0 subscripts respectively refer quantities to node, triangular member, and general coordinate systems.

In Chapter II we refer to triangular member i with node points α , β , γ ; in Chapter III we refer to a triangular member with node points 1, 2, 3; and in Chapter IV we refer to a triangular member i, j with node points (α_1, β_1) , (α_2, β_2) , (α_3, β_3) .

Symbol

A	area of triangular member
$A_{a1}, \dots, A_{b3}$	subsets of points of member middle surface
a_1, a_2, a_3	constants
B	matrix relating E to ΔV
b_1, b_2, b_3	constants
C	matrix relating Λ to ΔU
C_R	matrix relating ΔU to ΔV
$C_{R(\beta)}$	submatrices of C_R
C_V	matrix relating $\Delta \Lambda$ to ΔV
$D_{(0)}^{(a)}$	matrix relating $J_{(0)}$ to $J_{(a)}$
d	magnitude of slope discontinuity between adjacent members
D	flexural rigidity
E	matrix of deformation parameters
$\left. \begin{matrix} \bar{e}^1, \bar{e}^2, \bar{e}^3 \\ \bar{e}_1, \bar{e}_2, \bar{e}_3 \end{matrix} \right\}$	contravariant, covariant base vectors of $\zeta^1, \zeta^2, \zeta^3$ oblique coordinates
$\left. \begin{matrix} \bar{e}_{(1)}^1, \bar{e}_{(1)}^2, \bar{e}_{(1)}^3 \\ \bar{e}_{1(1)}, \bar{e}_{2(1)}, \bar{e}_{3(1)} \end{matrix} \right\}$	base vectors defined in Fig. 3.2

$\mathcal{E}$	Young's modulus
F^T	$= [F^T_{(1,1)}, \dots, F^T_{(\gamma, \gamma)}, \dots, F^T_{(p,p)}]$
F_i	generalized forces associated with U_i
$F^T_{(\gamma, e1)}$	$= [f^1_{(\gamma, e1)}, f^2_{(\gamma, e1)}, f^3_{(\gamma, e1)}, m^1_{(\gamma, e1)}, m^2_{(\gamma, e1)}, m^3_{(\gamma, e1)}]$ force components are referred to base vectors $\bar{e}_{1(1)}, \bar{e}_{2(1)}, \bar{e}_{3(1)}$ and moment components are referred to base vectors $\bar{e}^1_{(1)}, \bar{e}^2_{(1)}, \bar{e}^3_{(1)}$
$F^T_{(\gamma i)}$	$= [f^1_{(\gamma i)}, f^2_{(\gamma i)}, f^3_{(\gamma i)}, m^1_{(\gamma i)}, m^2_{(\gamma i)}, m^3_{(\gamma i)}]$ components referred to i-member coordinate of node point γ
$\bar{F}_{(\gamma)}$	force vector of node point γ
G	matrix relating stress resultants to deformation parameters
g_{a_k}, g_{b_k}	$k = 1, 2, 3$; functions used to construct $\bar{h}_{i\ell}$
$g(\beta)$	imperfection function
H	column matrix with elements $\bar{h}_\ell$
H_i	column matrix with elements $\bar{h}_{i\ell}$
$\bar{h}_\ell$	specified displacement configurations of shell
$\bar{h}_{i\ell}$	specified displacement configurations of member i

$$\left. \begin{aligned} J_{(0)}^T &= [J_{(0)}^1, J_{(0)}^2, J_{(0)}^3] \\ J_{(i)}^T &= [J_{(i)}^1, J_{(i)}^2, J_{(i)}^3] \\ J_{(a)}^T &= [J_{(a)}^1, J_{(a)}^2, J_{(a)}^3] \end{aligned} \right\} \begin{array}{l} \text{matrices of base vectors for 0-general,} \\ \text{i-member, a-node coordinate systems} \end{array}$$

K shell stiffness matrix

K_i member stiffness matrix (includes rigid body displacements and rotations)

L_1, L_2 subsets of points along side of member

l_1, l_2, l_3 side lengths of member

M_{A_i} matrix relating Λ_i to components of node displacements and rotations

M_{B_i} matrix relating node displacements and rotations referred to different coordinate systems

M_i matrix relating Λ_i to ΔU_i

M_{2_i} submatrix of M_i

M_{V_i} matrix relating Λ_i to ΔV_i

m_{ij}, m^{ij} covariant, contravariant stress resultant (moment) components

$\bar{m}_{(\gamma)}, \Delta \bar{m}_{(\gamma)}$ moment vectors of node point γ

$m_{(\gamma i)}^1, m_{(\gamma i)}^2, m_{(\gamma i)}^3$ i-coordinate components of $\bar{m}_{(\gamma)}$

$\bar{n}_1, \bar{n}_2, \bar{n}_3$ unit vectors

$P, \Delta P$	column matrices of generalized loads
P_I	generalized implied load matrix
P_R	generalized residual load matrix
$p(\tau), \Delta p$	load intensity, load intensity increment parameters
$\bar{p}(\eta^1, \eta^2, \tau)$	surface load vector
Q	generalized load distribution matrix
$\bar{q}(\eta^1, \eta^2, \tau)$	surface load distribution vector
q_1, q_2, q_3	force components defined in Fig. 3.11b
r, ϕ, z	cylindrical coordinates
$\bar{r}(\eta^1, \eta^2, \tau)$	position vector of shell middle surface
$\bar{r}_{(\gamma)}$	position vector of node point γ
S	matrix relating triangulation node points to member node points; also used to designate middle surface of shell
S_i	same as S but for member i
T	matrix of stress resultants
t	thickness of member
$\bar{t}_1, \bar{t}_2, \bar{t}_3$	unit vectors
ΔU^T	$= [\Delta U_{(1,1)}^T, \Delta U_{(2,2)}^T, \dots, \Delta U_{(\rho,\rho)}^T]$

ΔU_i^T	$= [\Delta U_{(\alpha i)}^T, \Delta U_{(\beta i)}^T, \Delta U_{(\gamma i)}^T]$ of member i with nodes $\alpha \beta \gamma$
$\Delta U_{(\gamma \gamma)}^T$	$= [\Delta u_{(\gamma \gamma)}^1, \Delta u_{(\gamma \gamma)}^2, \Delta u_{(\gamma \gamma)}^3, \Delta \theta_{(\gamma \gamma)}^1, \Delta \theta_{(\gamma \gamma)}^2, \Delta \theta_{(\gamma \gamma)}^3]$
u_a	relative average axial displacement of the two ends of a shell of revolution
$\Delta u_{(\gamma i)}^1, \Delta u_{(\gamma i)}^2, \Delta u_{(\gamma i)}^3$	i -member coordinate components of $\Delta \bar{u}_{(\gamma)}$
$\Delta u_{1(\gamma \gamma)}, \Delta u_{2(\gamma \gamma)}, \Delta u_{3(\gamma \gamma)}$	γ -node coordinate components of $\Delta \bar{u}_{(\gamma)}$
$\bar{u}(\eta^1, \eta^2, \tau), \Delta \bar{u}$	shell middle surface displacement vectors
$\bar{u}_i, \Delta \bar{u}_i$	member i middle surface displacement vectors
$\bar{u}_{L_i}$	displacements relative to S_i'
$\bar{u}_{R_i}$	rigid body displacements of S_i'
$\bar{u}_{(\gamma)}, \Delta \bar{u}_{(\gamma)}$	displacement vectors of node point γ
V	column matrix of generalized coordinates
V_i	a subset of V
v_l	generalized coordinates associated with generalized displacements $\bar{h}_l$
W_e, W_{e_i}	work of surface and boundary loads on shell; on member i
W_I, W_{I_i}	strain energy of shell, of member i
W_{I_1}, W_{I_2}	membrane, bending strain energy

w_1, w_2, w_3	covariant components of $\bar{u}_I$ referred to oblique coordinates $\zeta^1, \zeta^2, \zeta^3$
$w_{1(1)}, w_{2(1)}, w_{3(1)}$	covariant components of $\Delta\bar{u}_{(1)}$ referred to base vectors $\bar{e}_{(1)}^1, \bar{e}_{(1)}^2, \bar{e}_{(1)}^3$
x^1, x^2, x^3	0-general rectangular coordinate system, fixed in space
$x_{(\gamma)}^1, x_{(\gamma)}^2, x_{(\gamma)}^3, r_{(\gamma)}, z_{(\gamma)}$	components of $\bar{r}_{(\gamma)}$
X_1, X_2, X_3, R, Z	column matrices of the above components
y^1, y^2, y^3	i-member rectangular coordinates, redefined after each linear increment
y_R^1, y_R^2, y_R^3	rectangular coordinates of member i, member rigid body rotations (Sec. 3.5) are zero relative to them
z^1, z^2, z^3	α -node rectangular coordinates
α, β	$= \frac{\zeta^1}{l_1}, \frac{\zeta^2}{l_2}$
Γ	diagonal matrix with elements Γ_i
Γ_i	member stiffness matrix
$\Delta(\dots)$	a small finite change in the internal $\Delta\tau_k$
$\delta(\dots)$	a virtual change
δ	displacement increment constant
$\epsilon_1, \dots, \epsilon_6$	middle surface deformation parameters

$\epsilon_{11}, \epsilon_{22}, \epsilon_{12}$	covariant strain components of $\zeta^1, \zeta^2, \zeta^3$ coordinates
$\underline{\epsilon}_{11}, \underline{\epsilon}_{22}, \underline{\epsilon}_{12}$	covariant strain components in rectangular coordinates
η^1, η^2	curvilinear coordinates of a surface
$\Delta\theta^1_{(\gamma\gamma)}, \Delta\theta^2_{(\gamma\gamma)}, \Delta\theta^3_{(\gamma\gamma)}$	γ -node coordinate components of $\Delta\bar{\theta}_{(\gamma)}$
$\Delta\bar{\theta}_{(\gamma)}$	rotation vector of node point γ
κ_{ij}	covariant curvature components
Λ^T	$= [\Lambda_1^T, \dots, \Lambda_m^T]$
Λ_i^T	$= [\lambda_1, \dots, \lambda_{12}]$
λ_l	member i generalized coordinates
μ	$= 2(1 - \nu)$
ν	Poisson's ratio
$\zeta^1, \zeta^2, \zeta^3$	oblique coordinates fixed to member i
Π_i	matrix relating E to Λ_i
Σ^T	$= [\Sigma_1^T, \dots, \Sigma_m^T]$
Σ_i	matrix with elements σ_l
σ_l	generalized forces of member i

σ_{ij}, σ^{ij}	covariant, contravariant stress components
τ	parameter associated with various deformed shell configurations
$\vec{\Phi}(\gamma)$	rotation vector of node point γ measured relative to coordinates y_R^1, y_R^2, y_R^3
Ω	matrix operator
ω	residual load relaxation constant

I. INTRODUCTION

1.1. Preliminary Remarks

In engineering, particularly in the aerospace field where weight minimization is of the utmost importance, thin walled structures, consisting of thin bars, plates and shells, are widely used.

To facilitate the formulation and solution of broad classes of these structural analysis problems and related dynamic and aeroelastic problems, discrete element methods using matrix notation are now widely used. Their increasing use has very closely followed the improvements and increasing availability of digital computers.

These methods are part of the so called "matrix methods of structural analysis" put forward by Langefors¹ and Argyris.²

The components of these structures are characterized by their flexibility, i. e., their relatively small resistance to bending and torsion. When they are loaded the resulting displacements are frequently comparable in magnitude to their linear dimensions. The classical (linear) theory of elasticity and in particular the theory of shells is based on the assumption that the displacements of points in the body are infinitesimal which in turn permits a formulation of equilibrium and compatibility conditions with respect to the undeformed geometry.

If a theory requires the formulation of the conditions for equilibrium and compatibility to be with respect to the deformed geometry, as in reality it is, then it is said to be "geometrically non-linear." If the constitutive equations are non-linear, then the theory is said to be "physically non-linear."

In the following, the Kirchhoff-Love hypothesis is assumed. Some books on shell theory used as source material are given by references 3, 4, and 5.

The discrete model method used requires that the shell geometry be modeled by flat triangular plate members. Each triangle is assigned 12 independent deformed configurations specified so that adjacent member displacement continuity and in a limited sense (see Sec. 3.7) normal slope continuity is maintained between triangular members by maintaining these conditions at corner points. These are used to obtain triangular plate stiffness matrices in the generalized displacement sense which in turn are used to construct the stiffness matrix of shell structures. The modeling is discussed in detail in Chapters II and III.

This investigation is devoted to the formulation of arbitrary flat triangular^{plate} stiffness matrices associated with the membrane and bending stress states, and their applications to geometrically non-linear thin elastic shell problems. The investigation is restricted to shell materials which are isotropic, homogeneous, and linear elastic, and numerical results are presented for axially symmetric geometries. The method is applicable to somewhat general configurations and boundary conditions and can be directly

extended to include physically non-linear materials.

As formulated here, this method may be interpreted as consisting of two parts, i. e., the linear problem and the linear incremented extension of the linear problem into the non-linear range.

The linear problem is closely related to the Rayleigh-Ritz method; however, the concept of a minimizing sequence^{6, 7} cannot be interpreted directly, at least not for the more general problem considered. Each element of such a sequence would be associated with a different triangulation. The inconsistency of the requirement is discussed later.

The method is interpreted as a direct method of the calculus of variations which gives an approximation to the problem, i. e., a set of arbitrary constants (generalized coordinates) associated with a set of admissible shell displacement modes (generalized displacements) are determined so that the integral of potential energy is minimized.

An admissible shell displacement mode is interpreted here as one which forms a compatible field,⁸ i. e., satisfies compatibility conditions and displacement boundary conditions. In this case the potential energy is bounded from above.^{8, 9}

A compatible field is not necessarily an equilibrium field,⁸ i. e., it does not necessarily satisfy the equilibrium equations and stress boundary conditions. If a solution yields a compatible field and also an equilibrium field then it is said to be exact.

In its relationship to the matrix methods of structural analysis the method used is part of the matrix displacement method.² This is also called the direct stiffness method.

The non-linear problem is essentially a step by step procedure based on the linear formulation. The iteration can be interpreted as consisting of two parts; i. e., the load is advanced in increments (since the solution may encounter unstable regions the load can increase or decrease in increments) and it searches for a deformed configuration in equilibrium with the specified load. Equilibrium is here interpreted to mean with respect to the deformed configuration.

1.2. Previous Developments

Much of the impetus for the development of discrete model formulations to shell problems, at least during the past two decades, has come from the aircraft industry and in particular from applications to dynamic and aeroelastic problems.

The various discretization procedures used to approximate the behavior of a structure can be classified according to their properties of either satisfying compatibility but not equilibrium, or satisfying equilibrium but not compatibility, or violating both equilibrium and compatibility. It is desirable that a procedure admit to a refinement which in the limit converges to the exact solution and/or converges monotonically.

Hrennikoff¹⁰ developed and McHenry¹¹ improved on the "frame work analogy method" in which an analogy, consisting of a beam element

lattice, is made to the plane stress problem. It was later generalized to include bending by Parikh and Norris.¹² This method implicitly relaxes both compatibility and equilibrium conditions so that solutions based on it do not in general form either compatible or equilibrium fields. Many procedures similar to this have been and are now being used in both static and dynamic applications.

We note, in this connection, that some finite difference formulations to some differential equations in elasticity implicitly relax both equilibrium conditions and compatibility conditions, and satisfy these conditions in general only in the limit as the mesh size is made smaller.

A plane stress triangular plate element stiffness matrix, in the generalized displacement sense, was put forward in a paper by Turner, Clough, Martin, and Topp.¹³ The stiffness matrix is associated with three independent deformed configurations which have displacements that vary linearly in all directions and strains that are consequently uniform over the entire element. This matrix is now widely used. It was given a different form by Argyris,¹⁴ one which he calls the natural or invariant form.

In the absence of bending, solutions based on this matrix give displacements which form a compatible field for the triangulated model of plane or curved surfaces and for linear or non-linear problems. If bending is present, then compatible fields in general are obtained only for plane surfaces and geometrically linear problems.

In general this matrix does not yield equilibrium fields; however, de Veubeke⁸ gives an alternate approach which yields equilibrium

but not compatible fields and uses it, in what he calls a dual treatment, to obtain upper and lower bounds to static influence coefficients.

In order to satisfy compatibility between a triangular plate and a beam segment de Veubeke¹⁵ generalizes the plane stress stiffness matrix to include parabolic variations in displacements along its sides. The formulation requires nine independent deformed configurations for each triangle and that interelement displacements be satisfied at corner points and at the midpoint of sides.

In their paper Turner et. al¹³ also present a membrane stress rectangular plate stiffness matrix. As pointed out by Melosh¹⁶ this matrix does not in general yield a compatible field. Compatibility conditions are satisfied in the interior of elements and at node points; however, gapping may result, i. e., displacement compatibility between elements is not necessarily maintained.

In the same paper Melosh presents a rectangular stiffness matrix which yields compatible fields. This matrix is associated with five independent deformed configurations. Argyris¹⁴ gives a presentation of the so-called natural forms of these nodes. All the deformed configurations have displacements that vary linearly along the sides of the member, however, the displacements associated with them have terms that are quadratic.

Melosh gives a sufficiency condition for increasing the number of rectangles so that with each refinement the potential energy monotonically approaches a minimum. In essence he requires a refined subdivision, in a sequence of subdivision, to be so constructed that its displacement field can contain any displacement field of a

coarser subdivision. This sufficiency condition can be interpreted with respect to the plane stress triangular plate elements.

Displacement modes for flat surface plane stress problems obtained by a sequence of subdivisions satisfying Melosh's sufficiency condition can be used to obtain a minimizing sequence similar to the Rayleigh-Ritz type; however, the concept of completeness necessary for convergence to the exact solution has not been demonstrated and would, undoubtedly, require additional sufficiency conditions, as noted by de Veubeke.¹⁵

Argyris¹⁴ presents a parallelogram plane stiffness matrix and indicates a method for constructing one for a plane quadrilateral panel which he has obtained. All yield compatible fields.

In a report by Bogner, Mallett, Minich, and Schmit¹⁷ the authors give the displacement modes for constructing the stiffness matrix for a curvilinear rectangle associated with any orthogonal curvilinear coordinate system. For rectangular coordinates it reduces to those of Melosh.

Melosh¹⁶ gives a rectangular bending stiffness matrix. As pointed out by Pian³¹ the displacement modes do not in general maintain slope compatibility between elements and consequently solutions based on this matrix do not yield compatible fields. Melosh¹⁸ previously had given a bending stiffness matrix.

Bogner et. al.¹⁷ present displacement modes which can be used to construct bending stiffness matrices for curvilinear rectangles, which yield compatible fields and can be used with Melosh's sufficiency condition to obtain monotonically converging

sequences. These rectangular member displacement modes and those of Melosh do not form an independent set of deformed configurations in as much as they include rigid body displacements.

Clough,¹⁵ Adini,¹⁹ and Zienkiewicz¹⁵ use a polynomial to numerically calculate the coefficients of a rectangular plate bending stiffness matrix, i. e., the polynomial forms a set of displacement modes which in this case include rigid body displacements.

Zienkiewicz¹⁵ extends this to quadrilateral plates. Clough¹⁵ and Tocher²⁰ use similar polynomials for triangular plates.

Several objections can be raised with this type of procedures as pointed out by the authors. With the exception of the rectangle the use of these polynomials in general results in displacement discontinuities between members and in all cases slope continuity is not maintained between members. A member, and in particular a triangular member, will in general have different stiffness coefficients depending on its orientation with respect to the coordinate system of the polynomial even after they have been properly transformed so that components of node rotations and displacements are with respect to the same coordinate system, i. e., the stiffness coefficients are not uniquely defined by the polynomial.

This points out the desirability in selecting deformed configurations that reflect the geometric properties and symmetries of the member and which are independent of pure rigid body displacements as Argyris did in his plane stress formulations.

The matrix methods of structural analysis were originally developed to facilitate linear formulations of complex problems. These

methods of analysis have been given three classifications: the displacement formulation, the force formulation and the combined formulation. They have been shown to be equivalent to stationary energy principles, i. e., the displacement formulation is equivalent to the principle of stationary potential energy, the force formulation is equivalent to the principle of stationary complementary energy, and the combined formulation is equivalent to Reissner's principle of stationary energy.^{21, 22} A useful survey on linear structural analysis is given by Argyris..²³

During the past few years various investigators have extended these linear procedures to include geometric non-linearities, and both conservative and non-conservative material non-linearities.^{14,17,24-28}

1.3. Present Investigation

This investigation was primarily directed at obtaining an explicit representation for an arbitrary flat triangular^{plate} bending stiffness matrix and to studying its applicability to geometrically linear and non-linear shell problems.

Two bending matrices were obtained. The form in which these matrices are used requires a representation of node variables with respect to rectangular coordinates. A direct representation in this form is, however, so awkward that it has little value. The matrix is represented in the form

$$M_{2_i}^T \Gamma_{2_i} M_{2_i} \quad (1.1)$$

where Γ_{2_i} (Table 3.2 or Table D1) is the bending stiffness matrix with respect to a set of generalized variables and M_{2_i} (D.6) is a transformation matrix that relates the generalized variables to a set of node variables referred to rectangular coordinates. The generalized variables were selected so as to take advantage of the symmetric and geometric properties of an arbitrary triangle and consequently the coefficients of both Γ_{2_i} and M_{2_i} have a particularly simple form.

A derivation obtained by representing all quantities with respect to a rectangular coordinate system was found to be for all practical purposes prohibitive due to the very large volume of algebra required. By using oblique coordinates and representing the bending strain energy expression in terms of a set of deformation parameters that reflect the geometric properties of an arbitrary triangle, the derivation was performed with relatively little algebra.

Since membrane behavior is in general present, the plane stress matrix of Turner et. al.¹³ is used. It is, however, generalized by including three additional inplane displacement modes associated with components of node rotation normal to the plane of the triangle. These were included in order to remove the possibility of obtaining singular shell stiffness matrices.

II. GENERAL FORMULATION

2.1. Some General Properties

Let $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau)$ be the position vector of points on the middle surface of a thin elastic shell possessing a positive-definite strain-energy function, quadratic in the components of strain. The parameters η^1, η^2 are the curvilinear coordinates of the surface and the parameter τ is associated with various deformed configurations. When the shell undergoes a continuous deformation from configuration $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_1)$ to configuration $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_2)$, the parameter τ varies continuously over an interval (τ_1, τ_2) . In particular the configuration of the undeformed middle surface is designated by $\bar{\mathbf{r}}(\eta^1, \eta^2, 0)$.

The displacement vector of a point η^1, η^2 with respect to the interval (τ_1, τ_2) is defined by

$$\bar{\mathbf{u}}(\eta^1, \eta^2, \tau_2 - \tau_1) = \bar{\mathbf{r}}(\eta^1, \eta^2, \tau_2) - \bar{\mathbf{r}}(\eta^1, \eta^2, \tau_1) \quad (2.1)$$

The displacement vector relative to $\bar{\mathbf{r}}(\eta^1, \eta^2, 0)$ is designated by

$$\bar{\mathbf{u}}(\eta^1, \eta^2, \tau) = \bar{\mathbf{r}}(\eta^1, \eta^2, \tau) - \bar{\mathbf{r}}(\eta^1, \eta^2, 0) \quad (2.2)$$

The external force per unit area acting on the shell surface is designated by

$$\bar{\mathbf{p}}(\eta^1, \eta^2, \tau) = p(\tau) \bar{\mathbf{q}}(\eta^1, \eta^2, \tau) \quad (2.3)$$

where $p(\tau)$ is called the load intensity parameter and $\bar{\mathbf{q}}(\eta^1, \eta^2, \tau)$ is called the load distribution vector. The load distribution vector has the property

$$\int_S \bar{\mathbf{q}} \cdot \bar{\mathbf{q}} \, dA = \text{constant} \quad (2.4)$$

where the integration is carried out over the entire middle surface area S .

If the interval $(0, \tau)$ is partitioned into n subintervals

$$\begin{aligned} & (0, \tau_1), (\tau_1, \tau_2), \dots, (\tau_{k-1}, \tau_k), \dots, (\tau_{n-1}, \tau) \\ & = \Delta\tau_1, \Delta\tau_2, \dots, \Delta\tau_k, \dots, \Delta\tau_n \end{aligned} \quad (2.5)$$

then the displacement vector can be expressed in the form

$$\begin{aligned} \bar{\mathbf{u}}(\eta^1, \eta^2, \tau) &= \bar{\mathbf{u}}(\eta^1, \eta^2, \Delta\tau_1) + \dots + \bar{\mathbf{u}}(\eta^1, \eta^2, \Delta\tau_k) + \\ &+ \dots + \bar{\mathbf{u}}(\eta^1, \eta^2, \Delta\tau_n) \end{aligned} \quad (2.6)$$

In general it is assumed that

$$0 < \tau_1 < \tau_2 < \dots < \tau_k < \dots < \tau_{n-1} < \tau. \quad (2.7)$$

This investigation is concerned with shell deflections of sufficient magnitude so as to require a formulation of the equilibrium and compatibility conditions in the deformed state. The resulting non-linearities are dealt with by a formulation consisting of a sequence of linear intervals. Each linear interval has a one to one correspondence with an element of the sequence $\{\Delta\tau_k\}$. The linear problem intrinsic in the non-linear problem is thus one of starting with a deformed configuration $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_k)$ and seeking a deformed configuration $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_{k+1})$.

The following simplified notation is used when convenient:

$$\begin{aligned}
 \Delta\tau &= \Delta\tau_k \\
 \bar{r} &= \bar{r}(\eta^1, \eta^2, \tau_k) \\
 \bar{u} &= \bar{u}(\eta^1, \eta^2, \tau_k) \\
 \Delta\bar{u} &= \bar{u}(\eta^1, \eta^2, \Delta\tau_k) \\
 \bar{p} &= \bar{p}(\eta^1, \eta^2, \tau_k) \\
 \Delta\bar{p} &= \bar{p}(\eta^1, \eta^2, \Delta\tau_k) \\
 \bar{r}' &= \bar{r} + \Delta\bar{u} \\
 \bar{u}' &= \bar{u} + \Delta\bar{u} \\
 \bar{p}' &= \bar{p} + \Delta\bar{p}
 \end{aligned} \tag{2.8}$$

All symbols preceded by Δ are interpreted as finite functional changes in the interval $\Delta\tau_k$.

It is assumed that $\Delta\bar{u}$ together with its first partial derivatives with respect to η^1 and η^2 are sufficiently small, in accordance with linear shell theory, for all partition intervals $\Delta\tau_k$. The assumption of smallness in the interval $\Delta\tau_k$ implies that geometry changes in the interval are small, and that superposition of displacement configurations and corresponding surface loads associated with the interval is admissible.

The middle surface stress resultants and deformation parameters defined in linear shell theory are respectively designated by the six element column matrices

$$\mathbf{T}^T = [t^1(\eta^1, \eta^2, \tau_k), \dots, t^6(\eta^1, \eta^2, \tau_k)] \quad (2.9)$$

$$\mathbf{E}^T = [\epsilon_1(\eta^1, \eta^2, \tau_k), \dots, \epsilon_6(\eta^1, \eta^2, \tau_k)] \quad (2.10)$$

They are related by

$$\mathbf{T} = \mathbf{G} \mathbf{E} \quad (2.11)$$

where $\mathbf{G}$ is a 6×6 symmetric positive definite matrix with elements that are assumed constant throughout the range of τ . We define $\Delta \mathbf{T}$, $\Delta \mathbf{E}$, $\mathbf{T}'$, and $\mathbf{E}'$ in accordance with (2.8). Then

$$\Delta \mathbf{T} = \mathbf{G} \Delta \mathbf{E} \quad (2.12)$$

$$\mathbf{E}' = \mathbf{E} + \Delta \mathbf{E} \quad (2.13)$$

$$\mathbf{T}' = \mathbf{T} + \Delta \mathbf{T} \quad (2.14)$$

The strain energy, associated with the infinitesimal middle surface area dA , and strains $\mathbf{E}$ is given by

$$dW_I = dW_I(\eta^1, \eta^2, \tau_k) = \frac{1}{2} \mathbf{T}^T \mathbf{E} dA \quad (2.15)$$

Substituting from (2.11) into (2.15) we obtain

$$dW_I = \frac{1}{2} \mathbf{E}^T \mathbf{G} \mathbf{E} dA \quad (2.16)$$

The total strain energy is

$$W_I = \frac{1}{2} \int_S \mathbf{E}^T \mathbf{G} \mathbf{E} dA \quad (2.17)$$

The total strain energy after the next increment is similarly related to $\mathbf{E}'$ by

$$W'_I = \frac{1}{2} \int_S \mathbf{E}'^T \mathbf{G} \mathbf{E}' dA \quad (2.18)$$

The change in strain energy W_I due to a virtual change δE is

$$\begin{aligned}\delta W_I &= \frac{1}{2} \int_S \int (E^T + \delta E^T) G(E + \delta E) dA - \frac{1}{2} \int_S \int E^T G E dA \\ &= \frac{1}{2} \int_S \int \{ E^T G \delta E + \delta E^T G E + \delta E^T G \delta E \} dA\end{aligned}\quad (2.19)$$

On dropping higher order terms and noting that G is symmetric

(2.19) reduces to

$$\begin{aligned}\delta W_I &= \int_S \int E^T G \delta E dA \\ &= \int_S \int T^T \delta E dA\end{aligned}\quad (2.20)$$

Similarly the virtual change in the total strain energy after the next increment is related by

$$\begin{aligned}\delta W'_I &= \int_S \int E'^T G \delta E dA \\ &= \int_S \int \{ E^T G \delta E + \Delta E^T G \delta E \} dA \\ &= \int_S \int \{ T^T \delta E + \Delta E^T G \delta E \} dA \\ &= \delta W_I + \int_S \int \Delta E^T G \delta E dA\end{aligned}\quad (2.21)$$

The external work due to the surface load $\bar{p}$ and the virtual displacement $\delta \bar{u}$ is

$$\delta W_e = \int_S \int \bar{p} \cdot \delta \bar{u} dA \quad (2.22)$$

Similarly for $\bar{p}'$ we obtained

$$\begin{aligned}\delta W'_e &= \int_S \int \bar{p}' \cdot \delta \bar{u} dA \\ &= \int_S \int \{ \bar{p} \cdot \delta \bar{u} + \Delta \bar{p} \cdot \delta \bar{u} \} dA\end{aligned}\quad (2.23)$$

$$= \delta W_e + \int_S \Delta \bar{p} \cdot \delta \bar{u} \, dA$$

The principle of virtual work requires

$$0 = \delta W_I - \delta W_e \quad (2.24)$$

and

$$\begin{aligned} 0 &= \delta W'_I - \delta W'_e \\ &= \delta W_I + \int_S \Delta E^T G \, \delta E \, dA - \delta W_e - \int_S \Delta \bar{p} \cdot \delta \bar{u} \, dA \end{aligned} \quad (2.25)$$

From (2.24) and (2.25) we obtain

$$\left\{ \int_S \Delta \bar{p} \cdot \delta \bar{u} \, dA - \int_S \Delta E^T G \, \delta E \, dA \right\} = 0 \quad (2.26)$$

Let Ω designate a six element matrix operator that relates the deformation parameters to the displacements such that

$$\Delta E = \Omega \Delta \bar{u} \quad (2.27)$$

This operator is obtainable from linear shell theory and in the form used is given by (3.27).

If a_1, a_2 are scalar constants and $\bar{u}_1(\eta^1, \eta^2), \bar{u}_2(\eta^1, \eta^2)$ are vector functions then the linear property of this operator requires

$$\Omega(a_1 \bar{u}_1 + a_2 \bar{u}_2) = a_1 \Omega \bar{u}_1 + a_2 \Omega \bar{u}_2 \quad (2.28)$$

2.2. Discrete Method

The term discretization is used here to imply the reduction of the problem from a formulation in terms of unknown functional quantities

whose domain contains all points of the middle surface to a formulation in terms of a finite discrete set of unknown parameters. If the discrete set contains n unknown parameters then the discretization is said to have n degrees of freedom.

The discretization is based on Rayleigh's method, in which the displacement vector is approximated by a linear combination of independent displacement configurations that satisfy compatibility conditions and displacement boundary conditions. Implicit in an n degree of freedom discretization is the existence of an n element set of independent displacement configurations designated in matrix form by

$$H^T = [\bar{h}_1(\eta^1, \eta^2, \bar{r}), \dots, \bar{h}_n(\eta^1, \eta^2, \bar{r})] \quad (2.29)$$

The construction of the vector functions $\bar{h}_i$ depends on a knowledge of $\bar{r}(\eta^1, \eta^2, \tau_k)$ and as used here the $\bar{h}_i$ are functionally dependent on $\bar{r}(\eta^1, \eta^2, \tau_k)$. In accordance with the step-wise linearization discussed above the functions $\bar{h}_i$ are assumed to be independent of $\bar{r}$ in the interval $\Delta\tau$ and corrected after each such interval. Each function $\bar{h}_i$ is specified so as to satisfy displacement boundary conditions and compatibility conditions. The displacements are then approximated by

$$\Delta \bar{u} = H^T \Delta V \quad (2.30)$$

where

$$\Delta V^T = [\Delta v_1, \dots, \Delta v_n] \quad (2.31)$$

are scalar parameters associated with the interval $\Delta\tau$ and called the generalized coordinates. Substituting (2.30) into (2.27) we obtain

$$\Delta E = \Omega H^T \Delta V \quad (2.32)$$

A $6 \times n$ matrix is defined by

$$B = \Omega H^T \quad (2.33)$$

Then (2.32) has the form

$$\Delta E = B \Delta V \quad (2.34)$$

The virtual changes $\delta \bar{u}$ and δE are then related to δV by

$$\delta \bar{u} = H^T \delta V \quad (2.35)$$

$$\delta E = B \delta V \quad (2.36)$$

Let

$$P^T = \int_S \bar{p} \cdot H^T dA \quad (2.37)$$

$$\Delta P^T = \int_S \Delta \bar{p} \cdot H^T dA \quad (2.38)$$

$$Q^T = \int_S \bar{q} \cdot H^T dA \quad (2.39)$$

The dot product is used since the elements of H are vectors. The P , ΔP , and Q are respectively called the generalized load matrix, the generalized load increment matrix, and the generalized load distribution matrix.

On substituting (2.34), (2.35), and (2.36) into (2.26) we obtain

$$\left\{ \int_S \Delta \bar{p} \cdot H^T \delta V \, dA - \int_S \Delta V^T B^T G B \delta V \, dA \right\} = 0 \quad (2.40)$$

Noting that δV and ΔV are not functions of η^1, η^2 and substituting from (2.38) into (2.40) we obtain

$$\{ \Delta P^T - \Delta V^T \int_S B^T G B \, dA \} \delta V = 0 \quad (2.41)$$

Eq. (2.41) can be satisfied for all virtual changes δV only if

$$\Delta P^T - \Delta V^T \int_S B^T G B \, dA = 0 \quad (2.42)$$

The matrix K , called the shell stiffness matrix, is defined by

$$K = \int_S B^T G B \, dA \quad (2.43)$$

Since G is symmetric and positive definitive, K is symmetric and positive semi-definite and if in addition $\det K > 0$ then K is symmetric and positive definite.

In the physical sense the stiffness matrix is positive semi-definite if the elastic system is not tied down, i. e., if an adequate number of constraints have not been imposed so as to prevent rigid body displacements or rotations of the entire structure. Consequently, the displacements associated with a given loading are not unique and the relating coefficient matrix K is not invertable.

Substituting (2.43) into (2.42) and taking the transpose of the entire expression we obtain

$$\Delta P = K \Delta V \quad (2.44)$$

The generalized displacement parameters are chosen so that the displacement boundary conditions may be specified by specifying a subset of these parameters ΔV_2 . The matrices in (2.44) are partitioned as follows:

$$\begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \end{bmatrix} \quad (2.45)$$

where ΔV_1 are unspecified and ΔV_2 are specified generalized coordinates, and ΔP_1 are specified and ΔP_2 are unspecified generalized forces. The matrices ΔP_1 and ΔP_2 are respectively called the generalized load increment and generalized reactions increment.

If in (2.45) the $\det K_{11} > 0$ then K_{11} is invertible and we can solve for the unknown generalized displacements and reactions. We then obtain

$$\Delta V_1 = K_{11}^{-1} \{ \Delta P_1 - K_{12} \Delta V_2 \} \quad (2.46)$$

$$\Delta P_2 = K_{21} K_{11}^{-1} \{ \Delta P_1 - K_{12} \Delta V_2 \} + K_{22} \Delta V_2 \quad (2.47)$$

If a load vector $\bar{p} = p\bar{q}$ is in equilibrium with a distribution of stress resultants T , then from (2.20), (2.22) and (2.24) it follows that

$$\int_S \bar{p} \cdot \delta \bar{u} \, dA = \int_S T^T \delta E \, dA \quad (2.48)$$

Substituting from (2.35) and (2.36) into (2.48), noting that the resulting expression must hold for all δV , and then substituting from (2.37) we obtain

$$P = \int_S \int T^T B \, dA \quad (2.49)$$

The above is a relationship between the externally applied generalized load P and internal distribution of stress resultant T for an elastic system in equilibrium, i. e., it is a form of the equilibrium equation.

In the sequel we use the equation relating the load intensity parameter $p(\tau_k)$ to the generalized load matrix P and generalized load intensity matrix Q .

From (2.3), (2.37), and (2.39) it follows that

$$P = p Q \quad (2.50)$$

$$Q^T P = p Q^T Q \quad (2.51)$$

$$p = \frac{Q^T P}{Q^T Q} \quad (2.52)$$

The matrix H , as already indicated, is functionally dependent on $\bar{r}$, and consequently B and K are also. The distribution of stress resultants T is associated with configuration $\bar{r} = \bar{r}(\eta^1, \eta^2, \tau_k)$, but it is determined from quantities defined with respect to configuration $\bar{r}(\eta^1, \eta^2, \tau_{k-1})$. Consequently the generalized forces implicit in T do not necessarily conform to the specified load distribution of Q . The matrix P_I , associated with $\tau = \tau_k$ and called the implied generalized load matrix, is related to T by (2.49) and is given by

$$P_I = \int_S T^T B \, dA \quad (2.53)$$

The load intensity $p = p(\tau_k)$ is defined by

$$p = \frac{Q^T P_I}{Q^T Q} \quad (2.54)$$

The matrix P_R , associated with τ_k and called the residual generalized load is defined by

$$P_R = P_I - pQ \quad (2.55)$$

The above two definitions (2.54) and (2.55) have a conceptually useful interpretation. We multiply the terms of (2.55) by pQ^T and obtain

$$pQ^T P_R = pQ^T P_I - p^2 Q^T Q \quad (2.56)$$

Solving for $Q^T P_I$ in (2.54), substituting into (2.56) and dividing the resulting expression through by the scalar p we obtain

$$Q^T P_R = 0 \quad (2.57)$$

It then follows that Q is orthogonal to P_R . If we interpret P_I , P_R , and Q geometrically as vectors in a hyperspace, then these three vectors may be interpreted as lying on a hyperplane, pictorially shown in Fig. 2.1, and the vector P_I may be interpreted as being equal to the sum of two orthogonal vectors P_R and pQ .

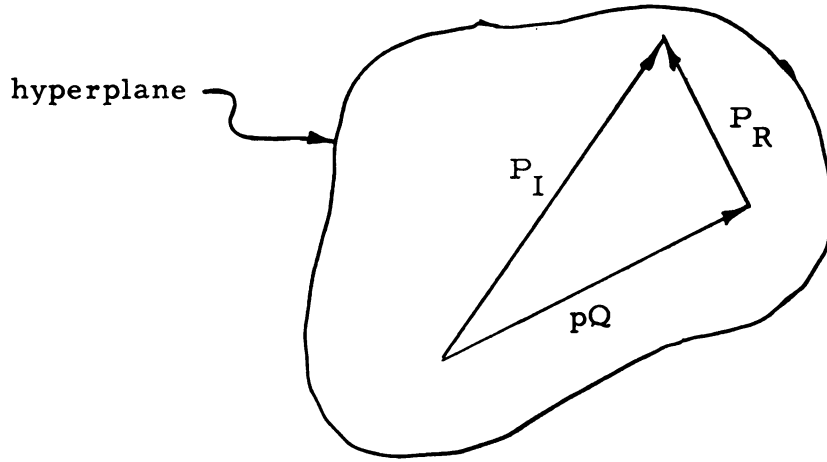


Fig. 2.1. Hyperplane formed by load vectors.

It is desirable to choose a generalized load increment ΔP_1 so as to minimize the next residual load. There does not appear to be any convenient method of making a best choice for ΔP_1 . It is taken in the form

$$\Delta P_1 = \Delta p Q_1 - \omega P_{R_1} \quad (2.58)$$

where Δp is called the load intensity increment and ω is called the residual load relaxation constant. The matrices Q_1 and P_{R_1} in (2.58) are submatrices of Q and P_R obtained by partitioning in accordance with (2.45). Substituting (2.58) into (2.46) we obtain

$$\Delta V_1 = K_{11}^{-1} \{ \Delta p Q_1 - \omega P_{R_1} - K_{12} \Delta V_2 \} \quad (2.59)$$

Note that

$$\begin{aligned}\bar{\mathbf{r}}^i &= \bar{\mathbf{r}} + \Delta \bar{\mathbf{u}} \\ &= \bar{\mathbf{r}} + \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} \Delta V_1 \\ \Delta V_2 \end{bmatrix}\end{aligned}\quad (2.60)$$

The matrices H , B , K are constructed after each linear increment from a knowledge of $\bar{\mathbf{r}}$. This is symbolically represented by $H(\bar{\mathbf{r}})$, $B(\bar{\mathbf{r}})$, $K(\bar{\mathbf{r}})$. The algorithm employed in determining the elements of the sequence $\{\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_k)\}$ is obtained by substituting (2.59) into (2.60) and is given by

$$\bar{\mathbf{r}}^i = \bar{\mathbf{r}} + \begin{bmatrix} H_1(\bar{\mathbf{r}}) & H_2(\bar{\mathbf{r}}) \end{bmatrix} \begin{bmatrix} K_{11}^{-1}(\bar{\mathbf{r}}) \{ \Delta p Q_1 - \omega P_{R_1} - K_{12}(\bar{\mathbf{r}}) \Delta V_2 \} \\ \Delta V_2 \end{bmatrix}\quad (2.61)$$

Associated with every deformed configuration $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_k)$ we have a residual generalized load matrix $P_R(\tau_k)$. If $P_R(\tau_k) = 0$ then $\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_k)$ is a configuration in equilibrium with the specified loading. In general, elements of $\{\bar{\mathbf{r}}(\eta^1, \eta^2, \tau_k)\}$ do not satisfy this condition and consequently their acceptability is determined by interpreting the smallness of the magnitude of $P_R(\tau_k)$.

2.3. Simplified Model

To simplify the analysis of this problem the shell is interpreted as a model consisting of flat triangular plate elements. This process can be interpreted as a ρ node m element triangulation of $\bar{\mathbf{r}}(\eta^1, \eta^2, 0)$ into curvilinear triangles and an isomorphism which deforms the

curvilinear triangles into simple triangles without materially disturbing the relative position of node points.

The assumptions implicit in the simplified model cannot always be justified. This restriction is not, however, very severe for many applications.

In the following, quantities associated with node points have Greek subscripts, quantities associated with triangular elements have Latin subscripts and quantities associated with the shell in general either have no subscripts or the subscript 0.

2.4. Matrix Relating Nodes and Members

The formulation of the shell problem in terms of discrete parameters defined with respect to the simplified model described above deals with configurations of node points and triangular members formed by lines joining node points. This leads to the definition of the mathematical sets N and S . The node set N is defined as the set of all node points of the simplified model and the member set S is defined as the set of members S_i or of ordered node triplets (α, β, γ) that in turn define all triangular members of the simplified model. A correspondence between elements S_i and ordered triplets (α, β, γ) is formed and represented in matrix form by

$$S_i = [\alpha, \beta, \gamma] \quad (2.62)$$

The dependence of S on N is symbolically represented by

$$S = S(N) \quad (2.63)$$

Elements of S and N are ordered respectively by integer numbers

$$\begin{aligned} 1, 2, \dots, i, \dots, m \\ 1, 2, \dots, \alpha, \dots, \rho \end{aligned} \quad (2.64)$$

The functional dependence implicit in (2.63) is explicitly represented by an $m \times 3$ matrix S of integer elements and is

$$S = [s_{ij}] = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ \dots & \dots & \dots \\ s_{i1} & s_{i2} & s_{i3} \\ \dots & \dots & \dots \end{bmatrix} = \begin{bmatrix} \dots & \dots & \dots \\ \dots & \dots & \dots \\ \alpha & \beta & \gamma \\ \dots & \dots & \dots \end{bmatrix} \quad (2.65)$$

The symbol S will serve the dual but analogous role and will also refer to the set of all points on the middle surface of the simplified model, and similarly S_i will refer to the subset of S associated with the i th triangular member. The quantities $\bar{r}$, $\bar{u}$, and $\Delta\bar{u}$ are defined over S and the quantities $\bar{r}_i$, $\bar{u}_i$, and $\Delta\bar{u}_i$ are defined over S_i .

2.5. Interpretation of Non-Linear Problem

The member S_i is displaced and deformed from some initial undeformed position. Part of the rigid body displacements are described by a triangular member S'_i associated with S_i (the node points of S'_i are designated by α' , β' , γ'). The triangle S'_i is identical to the triangle S_i in its undeformed state and is fixed to the deformed member S_i so that point β' , line $\beta'\gamma'$, and plane $\alpha'\beta'\gamma'$ lie respectively on point β , line $\beta\gamma$ and plane $\alpha\beta\gamma$.

The displacements of points on S_i can be represented in the form

$$\Delta \bar{u}_i = \Delta \bar{u}_{R_i} + \Delta \bar{u}_{L_i} \quad (2.66)$$

$$\bar{u}_i = \bar{u}_{R_i} + \bar{u}_{L_i} \quad (2.67)$$

where $\Delta \bar{u}_{R_i}$, $\bar{u}_{R_i}$ are displacements of S_i^1 and $\Delta \bar{u}_{L_i}$, $\bar{u}_{L_i}$ are displacements of S_i relative to S_i^1 .

Since $\Delta \bar{u}_{R_i}$ and $\bar{u}_{R_i}$ are rigid body displacements, the magnitude of the strain deformation parameters corresponding to them are identically zero. From (2.27) we obtain

$$\begin{aligned} \Delta E &= \Omega(\Delta \bar{u}_{R_i} + \Delta \bar{u}_{L_i}) \\ &= \Omega \Delta \bar{u}_{L_i} \end{aligned} \quad (2.68)$$

The displacements u_{L_i} are approximated by

$$\bar{u}_{L_i} = H_i^T \Lambda_i \quad (2.69)$$

and consequently

$$\Delta \bar{u}_{L_i} = H_i^T \Delta \Lambda_i \quad (2.70)$$

where H_i is a column matrix of linearly independent vector valued functions and Λ_i is a column matrix of scalar parameters called the member generalized coordinates. Eq's (2.69) and (2.70) imply that

$$\Lambda_i(\tau_k) = \Delta \Lambda_i(\Delta \tau_k) + \dots + \Delta \Lambda_i(\Delta \tau_k) \quad (2.71)$$

In a manner similar to that used to obtain (2.33), (2.34), (2.38), (2.43), and (2.44) we obtain

$$\Pi_i = \Omega H_i^T \quad (2.72)$$

$$\Delta E = \Pi_i \Delta \Lambda_i \quad (2.73)$$

$$\Delta \Sigma_i = \int_{S_i} \Delta \bar{p} \cdot H_i^T dA \quad (2.74)$$

$$\Gamma_i = \int_{S_i} \Pi_i^T G \Pi_i dA \quad (2.75)$$

$$\Delta \Sigma_i = \Gamma_i \Delta \Lambda_i \quad (2.76)$$

where Σ_i is called the member generalized force matrix and Γ_i is called the member stiffness matrix. The matrix Γ_i is symmetric and positive-definite. From (2.71) it follows that

$$\Sigma_i = \Gamma_i \Lambda_i \quad (2.77)$$

Let

$$\Lambda^T = [\Lambda_1^T, \dots, \Lambda_m^T] \quad (2.78)$$

$$\Sigma^T = [\Sigma_1^T, \dots, \Sigma_m^T] \quad (2.79)$$

$$\Pi = [\Pi_1, \dots, \Pi_m] \quad (2.80)$$

$$\Gamma = \begin{bmatrix} \Gamma_1 & & 0 \\ & \ddots & \\ 0 & & \Gamma_m \end{bmatrix} \quad (2.81)$$

Then

$$\Delta E = \Pi \Lambda \quad (2.82)$$

$$\Sigma = \Gamma \Lambda \quad (2.83)$$

The total strain energy in the shell for $\tau = \tau_k$ is

$$W_I = \frac{1}{2} \Sigma^T \Lambda = \frac{1}{2} \Lambda^T \Gamma \Lambda \quad (2.84)$$

The displacement of a point measured relative to the undeformed shell is interpreted as the sum of two parts. One part is associated with certain rigid body displacements and rotations of the entire triangle and the second part is measured relative to these displacements. The linear relation (2.83) is associated with displacements obtained after the first part has been subtracted off, and is assumed valid through the range of τ .

The non-linearities are, therefore, limited to those resulting from displacements associated with rigid body displacements and rotations of triangles $S_i^!$.

2.6. Shell Stiffness Matrix

The shell generalized coordinates V and generalized forces P are related to Σ and Λ of the same elastic system by the strain energy expression

$$\delta W_I = \delta W_e = P^T \delta V = \Sigma^T \delta \Lambda \quad (2.85)$$

The two sets of generalized coordinates, in accordance with the step-wise linearization used, are related by

$$\Delta \Lambda = C_V \Delta V \quad (2.86)$$

and consequently

$$\delta \Lambda = C_V \delta V \quad (2.87)$$

In general the elements of C_V change with τ so that

$$C_V(\tau_k) \neq C_V(\tau_{k+1}) \quad (2.88)$$

Substituting (2.87) into (2.85) we obtain

$$P^T \delta V = \Sigma^T C_V \delta V \quad (2.89)$$

Eq. (2.89) can hold for all virtual displacements δV only if

$$P = C_V^T \Sigma \quad (2.90)$$

From (2.34), (2.82), and (2.86) we obtain

$$\Delta E = \Pi \Delta \Lambda = \Pi C_V \Delta V = B \Delta V \quad (2.91)$$

and consequently

$$B = \Pi C_V \quad (2.92)$$

Substituting (2.92) into (2.43) and noting that the elements of C_V are not functions of η^1, η^2 we obtain

$$\begin{aligned} K &= \int_S \int B^T G B dA = \int_S \int C_V^T \Pi^T G \Pi C_V dA \\ &= C_V^T \left\{ \int_S \Pi^T G \Pi dA \right\} C_V = C_V^T \left[\int_{S_i} \int_{S_j} \Pi_i^T G \Pi_j dA \right] C_V \end{aligned} \quad (2.93)$$

The integration in the last of the integrals of (2.93) is carried out over points common to both S_i and S_j since Π_i is defined over S_i and Π_j is defined over S_j . Since the triangles do not overlap and from (2.75) it follows that

$$\int_{S_i} \int_{S_j} \Pi_i^T G \Pi_j dA = \begin{cases} 0 & \text{if } i \neq j \\ \Gamma_i & \text{if } i = j \end{cases} \quad (2.94)$$

Then from (2.81), (2.93), and (2.94) we obtain

$$K = C_V^T \Gamma C_V \quad (2.95)$$

2.7. Member - Shell Parameter Transformation Matrix

If the generalized coordinates are interpreted as components of node displacement and rotation vectors then the associated components of generalized forces, being related through the strain energy as in (2.85), can be interpreted as components of force and moment vectors. The shell generalized coordinates and generalized forces having this interpretation are respectively designated by the column matrices U and F .

The formulation in terms of U and F is desirable since most boundary conditions and in a limited sense, load distributions can be interpreted directly in terms of these parameters.

The undeformed simplified model has a continuous middle surface consisting of flat triangular members. Two adjacent undeformed triangles have an intersection line and intersection angle. We have required the displacement vector $\Delta \bar{u}$ to satisfy compatibility conditions on the entire middle surface. This can

be satisfied only if continuity and intersection angles are preserved for all points on the intersection line. These requirements are not strictly satisfied nor essential.

The requirements are useful, however, inasmuch as the approximating surfaces that satisfy them also satisfy compatibility conditions with respect to the simplified model. In this case the elements of the shell stiffness matrix can be interpreted as satisfying certain upperbound requirements.⁹

It is desirable to choose the functions H_i so that whenever compatibility requirements between two adjacent triangles are satisfied at their common node points they are satisfied at all points on their intersection line. The degree to which this condition is satisfied is discussed in Sec. 3.7.

The member displacements obtained by substituting (2.70) into (2.66) are

$$\Delta \bar{u}_i = \Delta \bar{u}_{R_i} + H_i^T \Delta \Lambda_i \quad (2.96)$$

These displacements can be directly related to ΔU . In the formulation used the member displacement vector (2.96) is related only to components of ΔU associated with its nodes. Each node has six elements of ΔU associated with it and consequently each member has 18 degrees of freedom. The rigid body displacements $\Delta \bar{u}_{R_i}$ have six degrees of freedom associated with them, i. e., three translation components and three rotation components. In addition $\Delta \Lambda_i$ has degrees of freedom associated with it equal in number to its dimension. Since the member has 18 degrees of freedom and six can be interpreted as rigid body displacements it follows that the

dimension of $\Delta\Lambda$ must be 12 or reducible to 12. That is, the formulation used requires that all possible displaced configurations Δu_i be linear combinations of six independent rigid body displacements $\Delta \bar{u}_{R_i}$ and 12 independent deformed configurations $H_i^T \Delta \Lambda_i$.

Consider a triangular member i with node points α , β , γ and a reference point 0 as shown in Fig. 2.2. We associate a rectangular cartesian coordinate system

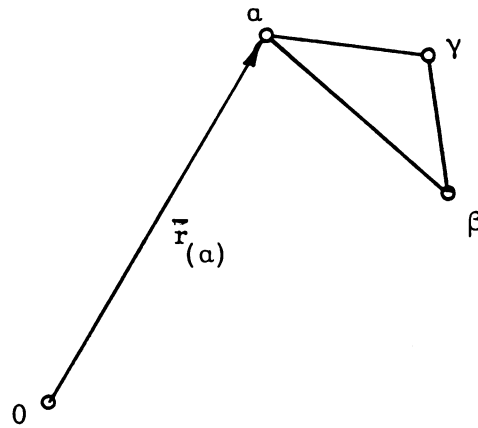


Fig. 2.2. Triangular member i .

with each of the quantities i , 0 , α , β , γ and refer to them as the i -member, 0 -general, α -node, β -node, and γ -node coordinate systems. The base vectors of these coordinate systems have the form $\bar{j}_{(i)}^1, \bar{j}_{(i)}^2, \bar{j}_{(i)}^3$ and the matrix of these base vectors is designated by

$$J_{(i)}^T = [\bar{j}_{(i)}^1, \bar{j}_{(i)}^2, \bar{j}_{(i)}^3] \quad (2.97)$$

The bracketed subscripts on a quantity refer it to a node point and/or a coordinate system. These subscripts are used only when necessary for clarity.

The matrices $J_{(0)}$ and $J_{(i)}$ are related by a 3×3 matrix $D_{1(0)}^{(i)}$ where the indices are interpreted by the relations

$$J_{(0)} = D_{1(0)}^{(i)} J_{(i)} \quad (2.98)$$

$$D_{1(0)}^{(i)} = D_{1(i)}^{(0)-1} = D_{1(i)}^{(0)T} \quad (2.99)$$

The matrix $D_{(0)}^{(i)}$ is defined by

$$D_{(0)}^{(i)} = \begin{bmatrix} D_{1(0)}^{(i)} & 0 \\ 0 & D_{1(i)}^{(0)} \end{bmatrix} \quad (2.100)$$

The position vector of node point a relative to point 0 is $\bar{r}_{(a)}$ and its 0-general coordinate components are $x_{(a)}^1, x_{(a)}^2, x_{(a)}^3$. The matrices of their components are respectively designated by X_1, X_2, X_3 . The elements of these matrices are arranged in accordance with the ordering in (2.64).

The i -member coordinate system is redefined after each interval $\Delta \tau_k$ so that its origin is incident to point a , base vector $\bar{j}_{(i)}^3$ is normal to plane $\alpha \beta \gamma$ and base vector $\bar{j}_{(i)}^1$ lies along line $\alpha \beta$ and directed toward β . The base vectors $J_{(i)}$ can be represented in terms of the position vectors $\bar{r}_\alpha, \bar{r}_\beta, \bar{r}_\gamma$ as follows

$$\begin{aligned} \bar{j}_{(i)}^1 &= \frac{\bar{r}_{(\beta)} - \bar{r}_{(a)}}{|\bar{r}_{(\beta)} - \bar{r}_{(a)}|} \\ \bar{j}_{(i)}^3 &= \frac{\{(\bar{r}_{(\beta)} - \bar{r}_{(a)}) \times (\bar{r}_{(\gamma)} - \bar{r}_{(a)})\}}{|(\bar{r}_{(\beta)} - \bar{r}_{(a)}) \times (\bar{r}_{(\gamma)} - \bar{r}_{(a)})|} \\ \bar{j}_{(i)}^2 &= \bar{j}_{(i)}^3 \times \bar{j}_{(i)}^1 \end{aligned} \quad (2.101)$$

By representing the position vectors in (2.101) in terms of the elements of $J_{(0)}$, the elements of $D_{1(i)}^{(0)}$ can be obtained directly and are given in Appendix B.

The base vectors $J_{(a)}$ of the a -node coordinate system as used here do not change with τ and are specified so that the transformation matrix $D_{1(a)}^{(0)}$ can be constructed directly. Then we obtain

$$D_{1(i)}^{(a)} = D_{1(i)}^{(0)} D_{1(0)}^{(a)} \quad (2.102)$$

Node point a has a displacement vector $\Delta \bar{u}_{(a)}$ and rotation vector $\Delta \bar{\theta}_{(a)}$. The 0-general coordinate components of $\Delta \bar{u}_{(a)}$ and $\Delta \bar{\theta}_{(a)}$ are $\Delta u_{(a0)}^1, \Delta u_{(a0)}^2, \Delta u_{(a0)}^3$ and $\Delta \theta_{(a0)}^1, \Delta \theta_{(a0)}^2, \Delta \theta_{(a0)}^3$. The a -node generalized displacement matrix is

$$\Delta U_{(a0)}^T = [\Delta u_{(a0)}^1, \Delta u_{(a0)}^2, \Delta u_{(a0)}^3, \Delta \theta_{(a0)}^1, \Delta \theta_{(a0)}^2, \Delta \theta_{(a0)}^3] \quad (2.103)$$

The generalized displacement matrix is

$$\Delta U^T = [\Delta U_{(1,1)}^T, \Delta U_{(2,2)}^T, \dots, \Delta U_{(\rho,\rho)}^T] \quad (2.104)$$

The column matrix ΔU_i is defined by

$$\Delta U_i^T = [\Delta U_{(a i)}^T, \Delta U_{(\beta i)}^T, \Delta U_{(\gamma i)}^T] \quad (2.105)$$

We can construct a transformation matrix M_i defined by

$$\Delta \Lambda_i = M_i \Delta U_i \quad (2.106)$$

We partition M_i in the form

$$\Delta\Lambda_i = [M_{i1} \ M_{i2} \ M_{i3}] \begin{bmatrix} \Delta U_{(\alpha i)} \\ \Delta U_{(\beta i)} \\ \Delta U_{(\gamma i)} \end{bmatrix} \quad (2.107)$$

The matrix M_i is explicitly given by (3.74).

The following relationship follows from (2.102)

$$\begin{bmatrix} \Delta U_{(\alpha i)} \\ \Delta U_{(\beta i)} \\ \Delta U_{(\gamma i)} \end{bmatrix} = \begin{bmatrix} D_{(i)}^{(\alpha)} & 0 & 0 \\ 0 & D_{(i)}^{(\beta)} & 0 \\ 0 & 0 & D_{(i)}^{(\gamma)} \end{bmatrix} \begin{bmatrix} \Delta U_{(\alpha \alpha)} \\ \Delta U_{(\beta \beta)} \\ \Delta U_{(\gamma \gamma)} \end{bmatrix} \quad (2.108)$$

Substituting (2.108) into (2.107) we obtain

$$\Delta\Lambda_i = [M_{i1} D_{(i)}^{(\alpha)}, M_{i2} D_{(i)}^{(\beta)}, M_{i3} D_{(i)}^{(\gamma)}] \begin{bmatrix} \Delta U_{(\alpha \alpha)} \\ \Delta U_{(\beta \beta)} \\ \Delta U_{(\gamma \gamma)} \end{bmatrix} \quad (2.109)$$

We define the transformation matrix C , which is similar to C_V , as follows

$$\Delta\Lambda = C \Delta U$$

We partition C into 6×6 submatrices in the form

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{i\rho} \\ \cdots & \cdots & \cdots & \cdots \\ C_{m1} & \cdots & \cdots & C_{m\rho} \end{bmatrix} = [C_{jk}] \quad (2.110)$$

The algorithm for constructing C can be obtained by examining matrix S defined in (2.65) and (2.109) given above. Then for $j = 1, \dots, m$ and $k = 1, \dots, \rho$

$$C_{jk} = 0$$

except for

$$C_{jn} = M_{\ell j} D_{(j)}^{(n)}$$

where

$$n = s_{j\ell}$$

$$\ell = 1, 2, 3$$

III. TRIANGULAR MEMBER MATRICES

3.1. Member Coordinate Systems

Consider a triangular member with middle surface points referred to rectangular coordinates y^1, y^2, y^3 and y_R^1, y_R^2, y_R^3 and oblique coordinates $\zeta^1, \zeta^2, \zeta^3$. The member is displaced from an initial position as shown in Fig. 3.1a to a position as shown in Fig. 3.1b. The coordinates $\zeta^1, \zeta^2, \zeta^3$ are fixed to the member as symbolically shown in Fig. 3.1b. The coordinates y_R^1, y_R^2, y_R^3 are fixed so that the components of rigid body rotation of the triangle (defined explicitly at the end of Sec. 3.5) relative to them vanish. The triangular member rigid body rotations are directly related to node displacements so that if the node displacements relative to the y^1, y^2, y^3 coordinates vanish, the y_R^1, y_R^2, y_R^3 coordinates will lie respectively on them. The $\zeta^1, \zeta^2, \zeta^3$ coordinates have covariant unit base vectors $\bar{e}_1, \bar{e}_2, \bar{e}_3$, and contravariant base vectors $\bar{e}^1, \bar{e}^2, \bar{e}^3$. The coordinates y^1, y^2, y^3 have base vectors $\bar{J}_{(i)}^1, \bar{J}_{(i)}^2, \bar{J}_{(i)}^3$ and are the i -member coordinates of Sec. 2.7.

In addition we introduce three oblique coordinate systems associated with the three node points. Their covariant and contravariant base vectors are as shown in Fig. 3.2. The inplane covariant components of each system are parallel to lines defined by node points and two of the contravariant components are perpendicular to these lines.

The coordinates y^1, y^2, y^3 are redefined after each interval $\Delta\tau_k$ so that the displacements measured relative to all the coordinates

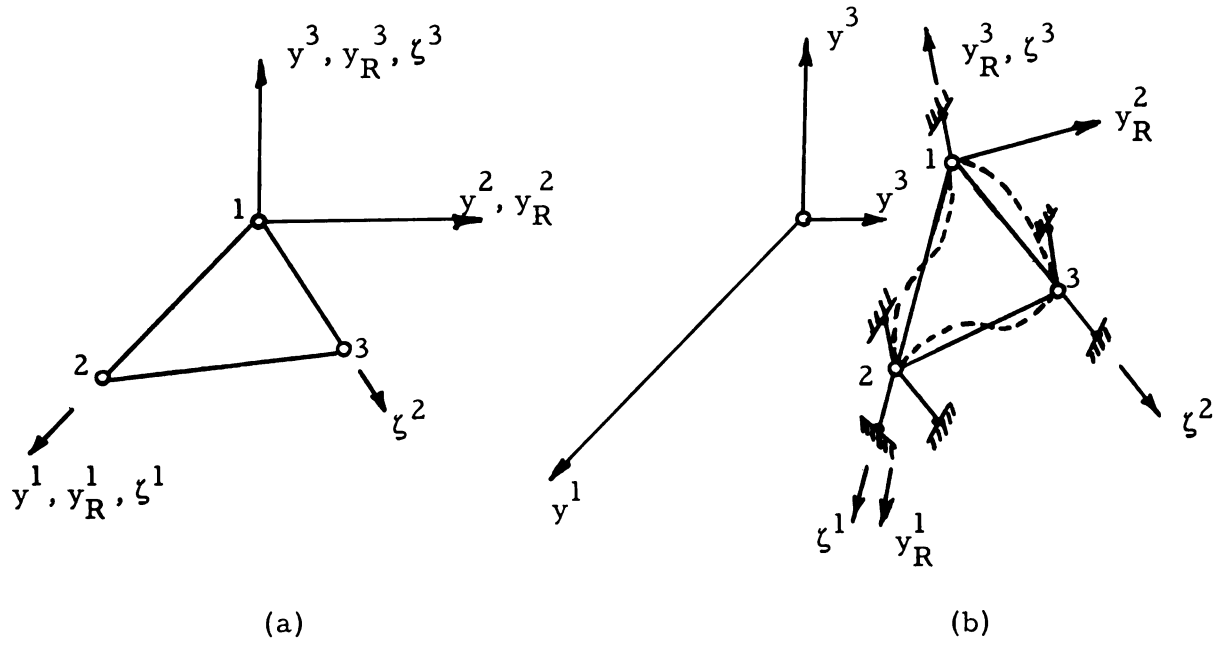


Fig. 3.1. Member coordinates

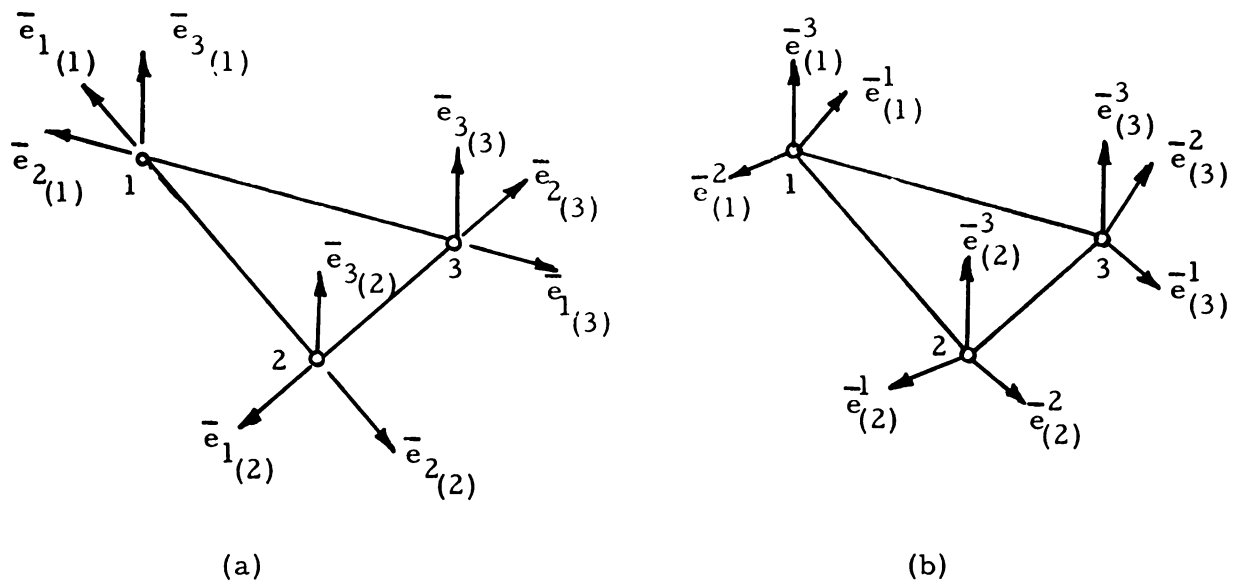


Fig. 3.2. Covariant and contravariant node base vectors

defined in this section are small in accordance with linear theory.

3.2. Geometric Parameters

Consider a triangle of side lengths ℓ_1, ℓ_2, ℓ_3 ; node points 1, 2, 3; included angles ψ_1, ψ_2, ψ_3 ; and referred to oblique cartesian coordinates ζ^1, ζ^2 as shown in Fig. 3.3. The coordinates ζ^1, ζ^2 are components of a position vector

$$\bar{\mathbf{r}} = \zeta^1 \bar{\mathbf{e}}_1 + \zeta^2 \bar{\mathbf{e}}_2 \quad (3.1)$$

where $\bar{\mathbf{e}}_1, \bar{\mathbf{e}}_2$ are the covariant base vectors (see Appendix A). The dimensionless parameters α and β are defined by

$$\begin{aligned} \alpha &= \frac{\zeta^1}{\ell_3} \\ \beta &= \frac{\zeta^2}{\ell_2} \end{aligned} \quad (3.2)$$

We define

$$\begin{aligned} a_1 &= \frac{1}{2} (-\ell_1^2 + \ell_2^2 + \ell_3^2) \\ a_2 &= \frac{1}{2} (\ell_1^2 - \ell_2^2 + \ell_3^2) \\ a_3 &= \frac{1}{2} (\ell_1^2 + \ell_2^2 - \ell_3^2) \end{aligned} \quad (3.3)$$

The area A of the triangle is related by

$$\begin{aligned} A &= \frac{1}{4} (2\ell_1^2\ell_2^2 + 2\ell_2^2\ell_3^2 + 2\ell_3^2\ell_1^2 - \ell_1^4 - \ell_2^4 - \ell_3^4)^{1/2} \\ &= \frac{1}{2} (a_1 a_2 + a_2 a_3 + a_3 a_1)^{1/2} \end{aligned} \quad (3.4)$$

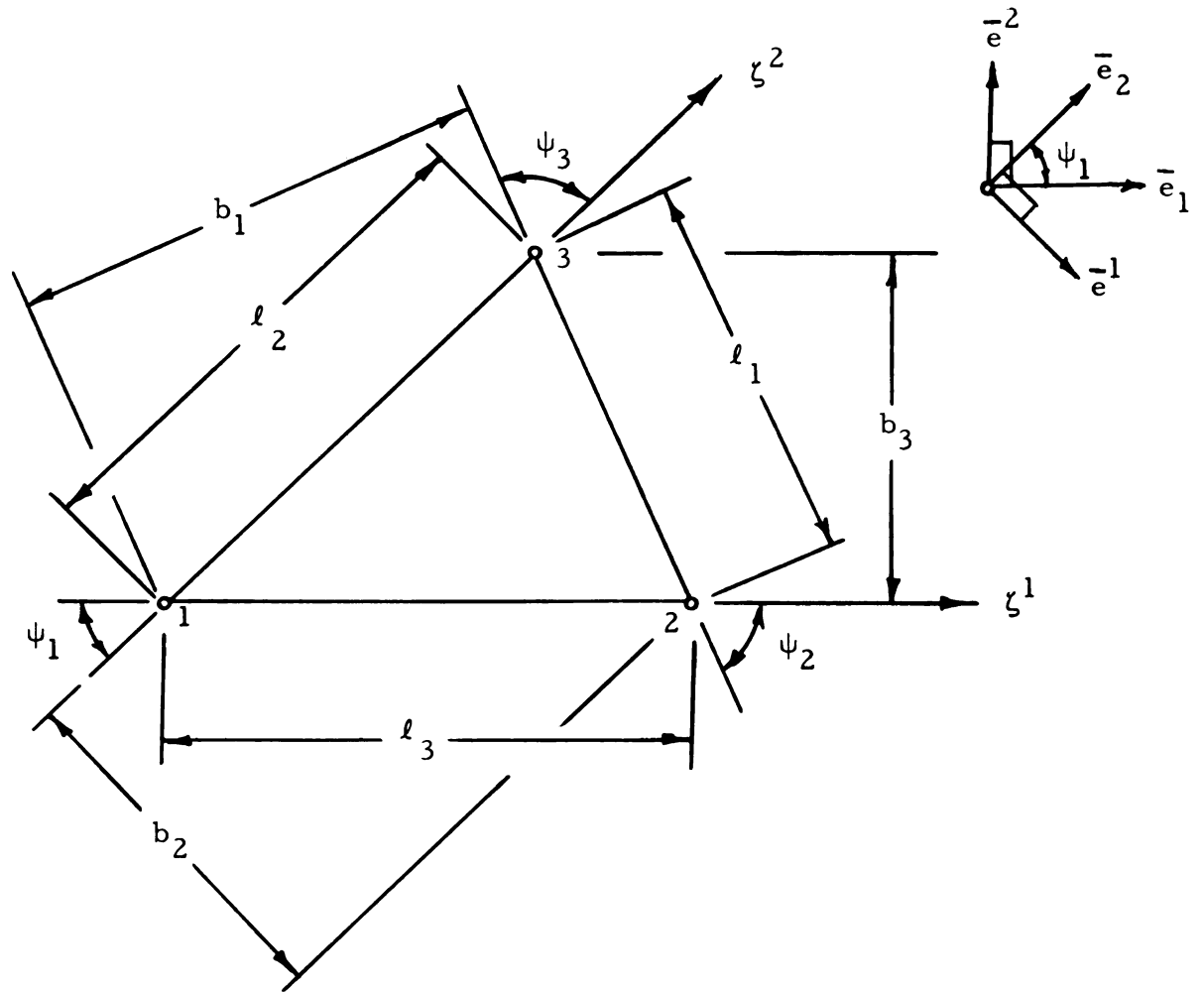


Fig. 3.3. Member parameters and oblique coordinates

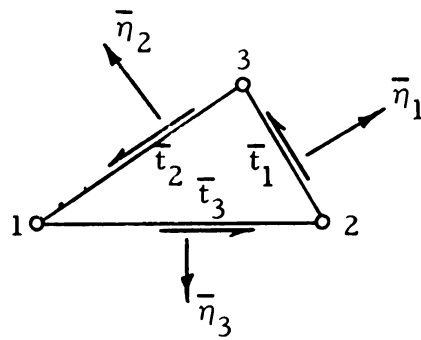


Fig. 3.4. Unit normal and tangent vectors

The trigonometric functions are related by

$$\begin{aligned}\sin \psi_1 &= \frac{2A}{\ell_2 \ell_3}, \quad \sin \psi_2 = \frac{2A}{\ell_3 \ell_1}, \quad \sin \psi_3 = \frac{2A}{\ell_1 \ell_2} \\ \cos \psi_1 &= \frac{a_1}{\ell_2 \ell_3}, \quad \cos \psi_2 = \frac{a_2}{\ell_3 \ell_1}, \quad \cos \psi_3 = \frac{a_3}{\ell_1 \ell_2}\end{aligned}\quad (3.5)$$

The unit normal vectors $\bar{n}_1, \bar{n}_2, \bar{n}_3$ and tangent vectors $\bar{t}_1, \bar{t}_2, \bar{t}_3$ are defined in Fig. 3.4, and are related to the base vectors by

$$\begin{aligned}\bar{n}_1 &= \sin \psi_2 \bar{e}^1 + \sin \psi_3 \bar{e}^2 \\ \bar{n}_2 &= -\sin \psi_1 \bar{e}^1 \\ \bar{n}_3 &= -\sin \psi_1 \bar{e}^2 \\ \bar{t}_1 &= -\cos \psi_2 \bar{e}^1 + \cos \psi_3 \bar{e}^2 = -\frac{\sin \psi_3}{\sin \psi_1} \bar{e}_1 + \frac{\sin \psi_2}{\sin \psi_1} \bar{e}_2 \\ \bar{t}_2 &= -\cos \psi_1 \bar{e}^1 - \bar{e}^2 = -\bar{e}_2 \\ \bar{t}_3 &= \bar{e}^1 + \cos \psi_1 \bar{e}^2 = \bar{e}_1\end{aligned}\quad (3.6)$$

The displacement vector of points on the member in terms of its covariant components is

$$\begin{aligned}\bar{u} &= w_1(\zeta^1, \zeta^2) \bar{e}^1 + w_2(\zeta^1, \zeta^2) \bar{e}^2 + w_3(\zeta^1, \zeta^2) \bar{e}^3 \\ &= w_1(a, \beta) \bar{e}^1 + w_2(a, \beta) \bar{e}^2 + w_3(a, \beta) \bar{e}^3\end{aligned}\quad (3.8)$$

The rotation vector is related to $\bar{u}$ by

$$\bar{\theta}(\zeta^1, \zeta^2) = \frac{1}{\sin \psi_1} \left(\frac{\partial w_3}{\partial \zeta^2} \bar{e}_1 - \frac{\partial w_3}{\partial \zeta^1} \bar{e}_2 \right) + \theta^3 \bar{e}_3 \quad (3.9)$$

Substituting from (3.2) and (3.5) into (3.9) we obtain

$$\bar{\theta}(\alpha\beta) = \frac{1}{2A} (\ell_3 \frac{\partial w_3}{\partial \beta} \bar{e}_1 - \ell_2 \frac{\partial w_3}{\partial \alpha} \bar{e}_2) + \theta^3 \bar{e}_3 \quad (3.10)$$

The rotation component θ^3 is not defined in general. It is, however, given an explicit definition for the three node points at the end of Sec. 3.5.

The normal distances as shown in Fig. 3.3 are designated by b_1, b_2, b_3 and are related by

$$b_1 = \frac{2A}{\ell_1}, \quad b_2 = \frac{2A}{\ell_2}, \quad b_3 = \frac{2A}{\ell_3} \quad (3.11)$$

3.3. A Symmetric Form of the Strain Energy Expression

The plane stress strain energy expression associated with the oblique coordinates ζ^1, ζ^2 (see Appendix A) is

$$W_{I_1} = \frac{\mathcal{E}t}{2(1-\nu^2) \sin \psi_1} \iint_{S_i} \left\{ \frac{1}{\sin^2 \psi_1} (\epsilon_{11} - 2 \cos \psi_1 \epsilon_{12} + \epsilon_{22})^2 - \mu (\epsilon_{11} \epsilon_{22} - \epsilon_{12}^2) \right\} d\zeta^1 d\zeta^2 \quad (3.12)$$

where $\mathcal{E}$ is the Young's modulus, ν is the Poisson's ratio, $\mu = 2(1-\nu)$, and where the strain components are related to the components of displacement by

$$\begin{aligned} \epsilon_{11} &= \frac{\partial w_1}{\partial \zeta^1} \\ \epsilon_{22} &= \frac{\partial w_2}{\partial \zeta^2} \\ \epsilon_{12} &= \frac{1}{2} \left(\frac{\partial w_1}{\partial \zeta^2} + \frac{\partial w_2}{\partial \zeta^1} \right) \end{aligned} \quad (3.13)$$

Introducing the change of variables (3.2) into (3.13) we obtain

$$\begin{aligned}
 \epsilon_{11} &= l_1 \frac{\partial w_1}{\partial \alpha} \\
 \epsilon_{22} &= l_2 \frac{\partial w_2}{\partial \beta} \\
 \epsilon_{12} &= \frac{1}{2} \left(l_3 \frac{\partial w_1}{\partial \beta} + l_2 \frac{\partial w_2}{\partial \alpha} \right)
 \end{aligned} \tag{3.14}$$

For convenience we define the deformation parameters $\epsilon_1, \epsilon_2, \epsilon_3$ in the form

$$\begin{aligned}
 \epsilon_1 &= l_3^2 \epsilon_{11} + l_2^2 \epsilon_{22} - 2 l_2 l_3 \epsilon_{12} \\
 \epsilon_2 &= l_2^2 \epsilon_{22} \\
 \epsilon_3 &= l_3^2 \epsilon_{11}
 \end{aligned} \tag{3.15}$$

The parameters do not have dimensions of strain, but they are, however, related to the elongations in directions defined by the three sides of the triangle. The parameters are related to displacements by

$$\begin{aligned}
 \epsilon_1 &= l_3 \left(\frac{\partial w_1}{\partial \alpha} - \frac{\partial w_1}{\partial \beta} \right) + l_2 \left(\frac{\partial w_2}{\partial \beta} - \frac{\partial w_2}{\partial \alpha} \right) \\
 \epsilon_2 &= l_2 \frac{\partial w_2}{\partial \beta} \\
 \epsilon_3 &= l_3 \frac{\partial w_1}{\partial \alpha}
 \end{aligned} \tag{3.16}$$

Solving for the strain components in terms of the deformation parameters in (3.15) we obtain

$$\begin{aligned}
\epsilon_{11} &= \frac{1}{l_3^2} \epsilon_3 \\
\epsilon_{22} &= \frac{1}{l_2^2} \epsilon_2 \\
\epsilon_{12} &= \frac{1}{2 l_2 l_3} \{ \epsilon_3 + \epsilon_2 - \epsilon_1 \}
\end{aligned} \tag{3.17}$$

Substituting from (3.2), (3.3), (3.5), and (3.17) into (3.12) we obtain after some rearranging of terms

$$\begin{aligned}
W_{I_1} &= \frac{\mathcal{E} t}{4(1-\nu^2)} \iint_{S_i} \left\{ \frac{1}{4A} \left(\frac{a_1^2}{A^2} + \mu \right) \epsilon_1^2 + \frac{1}{4A} \left(\frac{a_2^2}{A^2} + \mu \right) \epsilon_2^2 \right. \\
&+ \frac{1}{4A} \left(\frac{a_3^2}{A^2} + \mu \right) \epsilon_3^2 + \frac{2}{4A} \left(\frac{a_1 a_2}{A^2} - \mu \right) \epsilon_1 \epsilon_2 + \frac{2}{4A} \left(\frac{a_2 a_3}{A^2} - \mu \right) \epsilon_2 \epsilon_3 \\
&+ \left. \frac{2}{4A} \left(\frac{a_3 a_1}{A^2} - \mu \right) \epsilon_3 \epsilon_1 \right\} da \, d\beta = \frac{\mathcal{E} t}{4(1-\nu^2)} \iint_{S_i} \mathbf{E}_1^T \mathbf{G}_1 \mathbf{E}_1 \, da \, d\beta
\end{aligned} \tag{3.18}$$

where

$$\mathbf{E}_1^T = [\epsilon_1 \quad \epsilon_2 \quad \epsilon_3] \tag{3.19}$$

and

$$\mathbf{G}_1 = \begin{bmatrix} \frac{1}{4A} \left(\frac{a_1^2}{A^2} + \mu \right) & \frac{1}{4A} \left(\frac{a_1 a_2}{A^2} - \mu \right) & \frac{1}{4A} \left(\frac{a_1 a_3}{A^2} - \mu \right) \\ \frac{1}{4A} \left(\frac{a_2 a_1}{A^2} - \mu \right) & \frac{1}{4A} \left(\frac{a_2^2}{A^2} + \mu \right) & \frac{1}{4A} \left(\frac{a_2 a_3}{A^2} - \mu \right) \\ \frac{1}{4A} \left(\frac{a_3 a_1}{A^2} - \mu \right) & \frac{1}{4A} \left(\frac{a_3 a_2}{A^2} - \mu \right) & \frac{1}{4A} \left(\frac{a_3^2}{A^2} + \mu \right) \end{bmatrix} \tag{3.20}$$

The plate bending strain energy expression (see Appendix A)

is

$$\begin{aligned}
 W_{I_2} = \frac{\mathcal{D}}{2 \sin \psi_1} \int \int_{S_i} \left\{ \frac{1}{\sin^2 \psi_1} \left(\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^1} - 2 \cos \psi_1 \frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^2} \right. \right. \\
 \left. \left. + \frac{\partial^2 w_3}{\partial \zeta^2 \partial \zeta^2} \right)^2 - \mu \left[\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^1} \frac{\partial^2 w_3}{\partial \zeta^2 \partial \zeta^2} - \left(\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^2} \right)^2 \right] \right\} d\zeta^1 d\zeta^2
 \end{aligned}
 \tag{3.21}$$

where $\mathcal{D}$ is the flexural rigidity. Note that

$$\begin{aligned}
 \frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^1} &= \frac{\partial^2 w_3}{l_3^2 \partial \alpha^2} \\
 \frac{\partial^2 w_3}{\partial \zeta^2 \partial \zeta^2} &= \frac{\partial^2 w_3}{l_2^2 \partial \beta^2} \\
 \frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^2} &= \frac{\partial^2 w_3}{l_2 l_3 \partial \alpha \partial \beta}
 \end{aligned}
 \tag{3.22}$$

We define the deformation parameters $\epsilon_4, \epsilon_5, \epsilon_6$ by

$$\begin{aligned}
 \epsilon_4 &= \frac{\partial^2 w_3}{\partial \alpha^2} + \frac{\partial^2 w_3}{\partial \beta^2} - 2 \frac{\partial^2 w_3}{\partial \alpha \partial \beta} \\
 \epsilon_5 &= \frac{\partial^2 w_3}{\partial \beta^2} \\
 \epsilon_6 &= \frac{\partial^2 w_3}{\partial \alpha^2}
 \end{aligned}
 \tag{3.23}$$

From (3.22) and (3.23) we obtain

$$\begin{aligned}
\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^1} &= \frac{1}{l_3^2} \epsilon_6 \\
\frac{\partial^2 w_3}{\partial \zeta^2 \partial \zeta^2} &= \frac{1}{l_2^2} \epsilon_5 \\
\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^2} &= \frac{1}{2l_2 l_3} (-\epsilon_4 + \epsilon_5 + \epsilon_6)
\end{aligned} \tag{3.24}$$

Substituting from (3.2), (3.3), (3.5), and (3.24) into (3.21) we obtain

$$\begin{aligned}
W_{I2} &= \frac{D}{4} \int \int_{S_i} \left\{ \frac{1}{4A} \left(\frac{a_1^2}{A^2} + \mu \right) \epsilon_4^2 + \frac{1}{4A} \left(\frac{a_2^2}{A^2} + \mu \right) \epsilon_5^2 + \frac{1}{4A} \left(\frac{a_3^2}{A^2} + \mu \right) \epsilon_6^2 \right. \\
&\quad \left. + \frac{2}{4A} \left(\frac{a_1 a_2}{A^2} - \mu \right) \epsilon_4 \epsilon_5 + \frac{2}{4A} \left(\frac{a_2 a_3}{A^2} - \mu \right) \epsilon_5 \epsilon_6 + \frac{2}{4A} \left(\frac{a_3 a_1}{A^2} - \mu \right) \epsilon_6 \epsilon_4 \right\} da d\beta \\
&= \frac{D}{4} \int \int_A E_2^T G_1 E_2 da d\beta
\end{aligned} \tag{3.25}$$

where G_1 is given by (3.20) and

$$E_2^T = [\epsilon_4 \quad \epsilon_5 \quad \epsilon_6] \tag{3.26}$$

If the middle surface deformation parameters (2.10) are taken in the form (3.16) and (3.23) then the matrix operator Ω in Chapter II is given by

$$\Omega = \begin{bmatrix} l_3 \left(\frac{\partial}{\partial \alpha} + \frac{\partial}{\partial \beta} \right) \bar{e}_1 + l_3 \left(\frac{\partial}{\partial \beta} - \frac{\partial}{\partial \alpha} \right) \bar{e}_2 \\ l_2 \frac{\partial}{\partial \beta} \bar{e}_2 \\ l_3 \frac{\partial}{\partial \alpha} \bar{e}_1 \\ \left(\frac{\partial^2}{\partial \alpha^2} - 2 \frac{\partial^2}{\partial \alpha \partial \beta} + \frac{\partial^2}{\partial \beta^2} \right) \bar{e}_3 \\ \frac{\partial^2}{\partial \beta^2} \bar{e}_3 \\ \frac{\partial^2}{\partial \alpha^2} \bar{e}_3 \end{bmatrix} \quad (3.27)$$

Also the matrix G in (2.11) then has the form

$$G = \begin{bmatrix} \frac{t}{4(1-\nu^2)} & G_1 & 0 \\ 0 & \frac{\phi}{4} G_1 \end{bmatrix} \quad (3.28)$$

where G_1 is given by (3.20).

3.4. The Member Displacement Configurations

The vector of displacements measured relative to the $\zeta^1, \zeta^2, \zeta^3$ coordinates in terms of its covariant components is

$$\bar{u}_L(\zeta^1, \zeta^2) = w_1(\zeta^1, \zeta^2) \bar{e}^1 + w_2(\zeta^1, \zeta^2) \bar{e}^2 + w_3(\zeta^1, \zeta^2) \bar{e}^3 \quad (3.29)$$

In terms of the dimensionless parameters (3.2), (3.29) is

$$\bar{u}_L(\alpha, \beta) = w_1(\alpha, \beta) \bar{e}^1 + w_2(\alpha, \beta) \bar{e}^2 + w_3(\alpha, \beta) \bar{e}^3 \quad (3.31)$$

Six of the deformed configurations H_i are associated with membrane stresses and consequently have the component $w_3 = 0$. The other six are associated with bending stresses and have the components $w_1 = w_2 = 0$.

Three of the deformed configurations used have displacements describable in the form

$$\bar{u}_L = (c_0 + c_1\alpha + c_2\beta) \bar{e}^1 + (d_0 + d_1\alpha + d_2\beta) \bar{e}^2 \quad (3.32)$$

where c_i and d_i are arbitrary constants.

This displacement vector is linear in α and β , and consequently has the property of displacing straight lines into straight lines. It is the same one used by Turner et al.¹³

The six arbitrary constants in (3.32) are determined in terms of the three parameters $\lambda_1, \lambda_2, \lambda_3$ called member generalized coordinates. These parameters are respectively defined as the elongations of sides l_1, l_2, l_3 and are the natural forms given by Argyris.¹⁴

The node displacements and constraints, relative to the $\zeta^1, \zeta^2, \zeta^3$ coordinates are shown in Fig. 3.5, and given by

$$\begin{aligned} \bar{u}_L(0, 0) &= w_{1(1)} \bar{e}^1 + w_{2(1)} \bar{e}^2 = c_0 \bar{e}^1 + d_0 \bar{e}^2 \\ \bar{u}_L(1, 0) &= w_{1(3)} \bar{e}^1 = (c_0 + c_1) \bar{e}^1 + (d_0 + d_1) \bar{e}^2 \\ \bar{u}_L(0, 1) &= 0 = (c_0 + c_2) \bar{e}^1 + (d_0 + d_2) \bar{e}^2 \end{aligned} \quad (3.33)$$

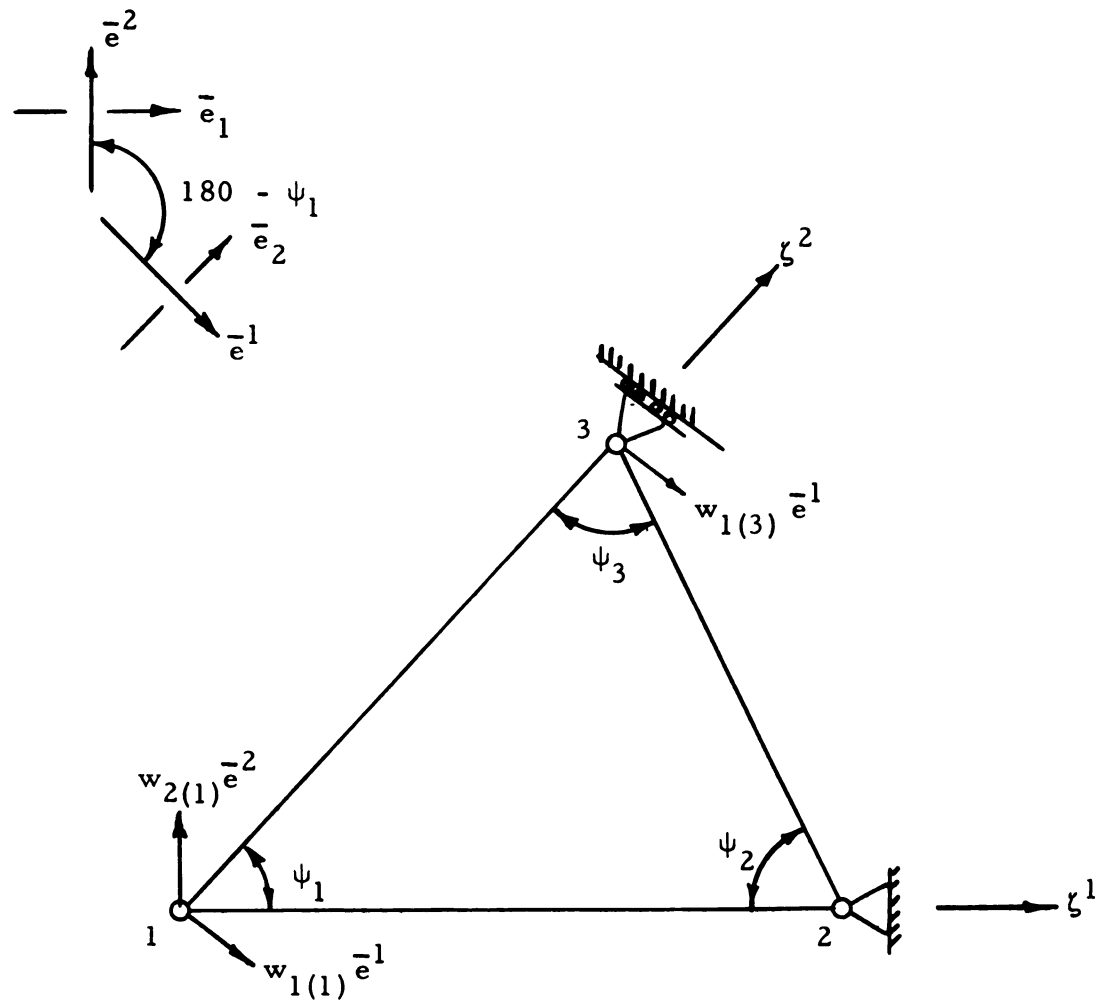


Fig. 3.5. Covariant components of node displacements relative to oblique coordinates.

Solving for the six constants in (3.33) and substituting into (3.32) we obtain

$$\bar{u} = [+\beta w_{1(3)} + (1-\alpha-\beta) w_{1(1)}] \bar{e}^1 + [(1-\alpha-\beta) w_{2(1)}] \bar{e}^2 \quad (3.34)$$

The member generalized displacements are then related to the components of node displacement by

$$\begin{aligned} \lambda_1 &= -\bar{t}_1 \cdot \bar{w}(0,1) = -\frac{\ell_3}{\ell_1} w_{1(3)} \\ \lambda_2 &= -\bar{t}_2 \cdot \bar{w}(0,0) = -w_{2(1)} \\ \lambda_3 &= -\bar{t}_3 \cdot \bar{w}(0,0) = -w_{1(1)} \end{aligned} \quad (3.35)$$

where $\bar{t}_1, \bar{t}_2, \bar{t}_3$ are vectors tangent to the three sides of the triangle and are given in (3.7). Substituting from (3.35) into (3.34) we obtain

$$\begin{aligned} \bar{u}_L &= -\frac{1}{\ell_3} \{ \beta \ell_1 \lambda_1 + (1-\alpha-\beta) \ell_3 \lambda_3 \} \bar{e}^1 \\ &+ \frac{1}{\ell_2} \{ (1-\alpha-\beta) \ell_2 \lambda_2 \} \bar{e}^2 \end{aligned} \quad (3.36)$$

In general when a member is displaced from its initial position (Fig. 3.1a) to its displaced position (Fig. 3.1b) it undergoes a rigid body rotation $\Delta\bar{\theta}_R$ defined below. The coordinate system y_R^1, y_R^2, y_R^3 , see Fig. 3.1b, is defined so that the rigid body member rotation and node 1 displacements relative to it vanish.

Nine deformed configurations are directly related to components of node rotation measured relative to the y_R^1, y_R^2, y_R^3 coordinate system. Three deformed configurations are of a type graphically shown in Fig. 3.6 a, b, c and only have displacement components in

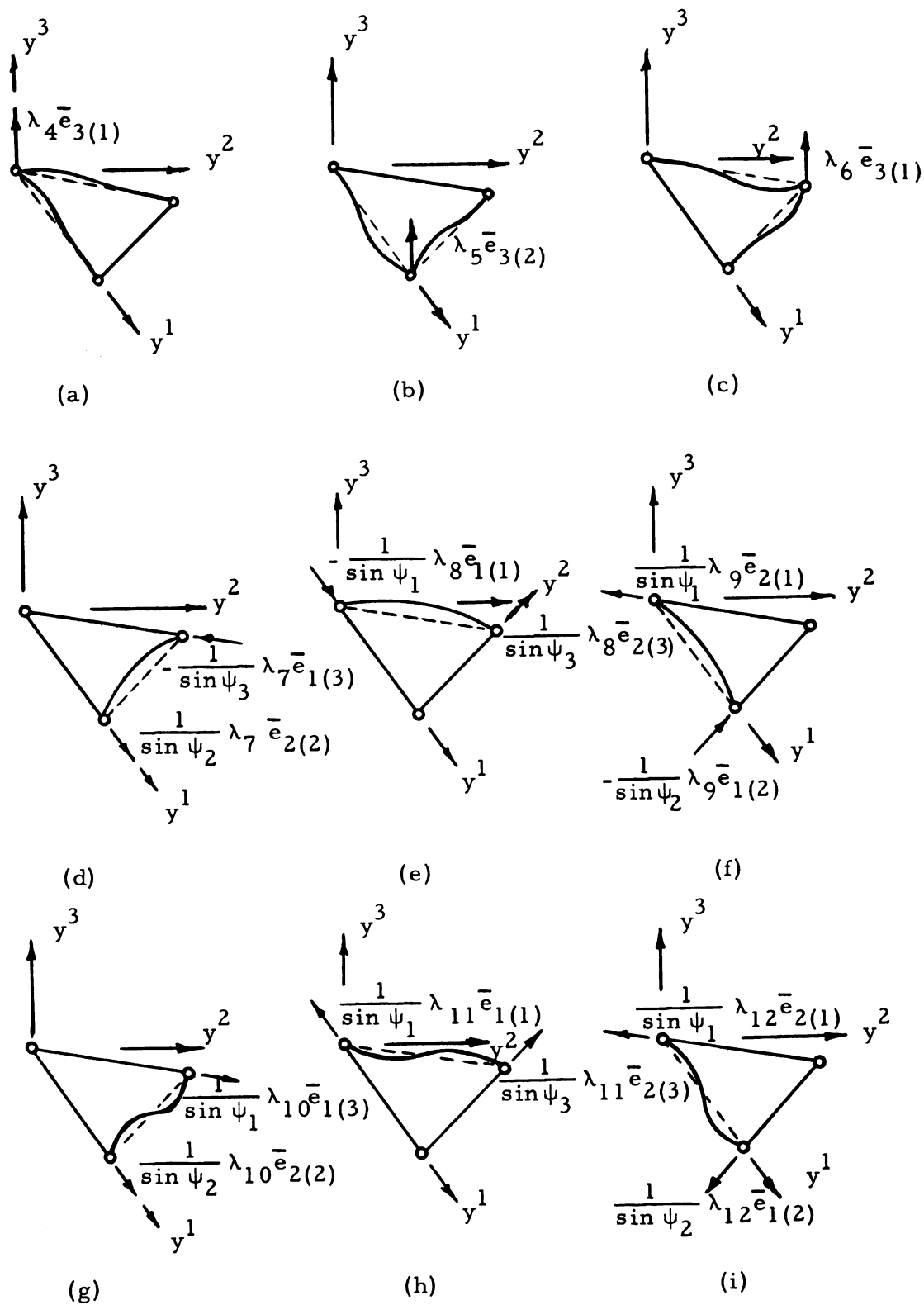


Fig. 3.6. Displacement modes associated with node rotations.

the plane of S_i' (see page 26). The remaining six only have displacements components normal to S_i' and are shown in Fig. 3.6 d-i.

The nine functions of H_i shown in Fig. 3.6 are constructed from the equations of the six planes intersecting the plane of S_i' as shown in Fig. 3.7 and given in Table 3.1. The equation of the planes are normalized so that their principal intersection angle is unity.

The set of points defining the domain of a subtriangle as shown in Fig. 3.8 are designated by $A_{a1}, A_{a2}, A_{a3}, A_{b1}, A_{b2}, A_{b3}$; i.e., A_{a1} corresponds to the set of points lying on triangle 1, 2, 4, etc. The symbol A_{a1} when used with a function g_{a1} in the form $A_{a1} g_{a1}$ implies that g_{a1} is defined over A_{a1} . The form $(A_{a1} g_{a1})(A_{a2} g_{a2}) = A_{a1} A_{a2} g_{a1} g_{a2}$ implies that $g_{a1} g_{a2}$ is defined only for points common to both A_{a1} and A_{a2} .

Consider the two functions taken in the form

$$Z_{12} = \frac{1}{2} \{ g_{a2}(g_{a1} + g_{b3})A_{a3} + g_{a1}(g_{a2} + g_{b3})A_{b3} \} \ell_3 \quad (3.37)$$

$$Z_{13} = \frac{1}{2} \{ g_{a1}(g_{a3} + g_{b2})A_{a2} + g_{a3}(g_{a1} + g_{b2})A_{b2} \} \ell_2 \quad (3.38)$$

The functions Z_{12} and Z_{13} together with their first partial derivatives are continuous for all points of the triangle. The second derivatives of Z_{12} are constant on A_{a3} and A_{b3} , and have a discontinuity along line 3, 6. The second derivatives of Z_{13} are constant on A_{a2} and A_{b2} , and have a discontinuity along line 2, 5. It follows that all third derivative of Z_{12} and Z_{13} vanish for all points of the triangle and consequently strain compatibility is satisfied for inplane displacement configurations constructed from

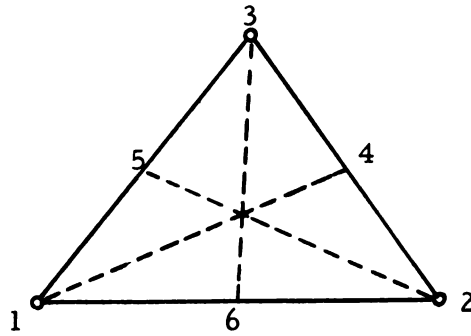


Fig. 3.7. Lines of intersection of planes given in Table 3.1.

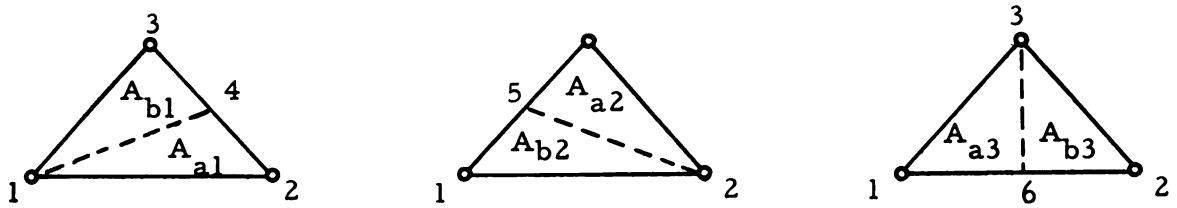


Fig. 3.8. Symbols used to designate subdomains of triangle.

Table 3.1. Functions Used to Construct Displacement Modes.

LINE	FUNCTION	EQUATION
2, 3	$g_{a1} =$	$1 - \alpha - \beta = 0$
3, 1	$g_{a2} =$	$\alpha = 0$
1, 2	$g_{a3} =$	$\beta = 0$
1, 4	$g_{b1} =$	$\alpha - \beta = 0$
2, 5	$g_{b2} =$	$-1 + \alpha + 2\beta = 0$
3, 6	$g_{b3} =$	$1 - 2\alpha - \beta = 0$

these functions. A graphic description of these functions is shown in Fig. 3.9. The function Z_{12} vanishes along lines 1, 3 and 2, 3 as indicated in Fig. 3.9 and its normal derivatives vanishes along 2, 3. It has a slope of unity in the direction of ξ^1 at point 1.

The configuration associated with λ_4 is graphically described in Fig. 3.6a and has the form

$$\bar{u}_L = \lambda_4 (Z_{13} \bar{n}_2 - Z_{12} \bar{n}_3) \quad (3.39)$$

where $\bar{n}_2$ and $\bar{n}_3$ are defined in (3.6) and Fig. 3.4. By two consecutive permutations of the indices 1, 2, 3 into 3, 1, 2 and 2, 3, 1 in (3.36) and (3.37) we construct four additional functions Z_{21} , Z_{23} , Z_{31} , Z_{32} . On substituting from Table 3.1 we obtain

$$\begin{aligned} Z_{12} &= \frac{1}{2} \{ \alpha(2-3\alpha-2\beta)A_{a3} + (1-\alpha-\beta)^2 A_{b3} \} \ell_3 \\ Z_{13} &= \frac{1}{2} \{ -(1-\alpha-\beta)(1-\alpha-3\beta)A_{a2} + \beta(2-2\alpha-3\beta)A_{b2} \} \ell_2 \\ Z_{23} &= \frac{1}{2} \{ \beta(2\alpha-\beta)A_{a1} + \alpha^2 A_{b1} \} \ell_1 \\ Z_{21} &= \frac{1}{2} \{ \alpha^2 A_{a3} - (1-\alpha-\beta)(1-3\alpha-\beta)A_{b3} \} \ell_3 \\ Z_{31} &= \frac{1}{2} \{ -(1-\alpha-\beta)(1-\alpha-3\beta)A_{b2} + \beta^2 A_{b2} \} \ell_2 \\ Z_{32} &= \frac{1}{2} \{ \beta^2 A_{a1} - \alpha(\alpha-2\beta) A_{b1} \} \ell_1 \end{aligned} \quad (3.40)$$

The displacement vector associated with the three additional inplane deformed configurations then has the form

$$\bar{u}_L = \lambda_4 (Z_{13} \bar{n}_2 - Z_{12} \bar{n}_3) + \lambda_5 (Z_{21} \bar{n}_3 - Z_{23} \bar{n}_1) + \lambda_6 (Z_{32} \bar{n}_1 - Z_{31} \bar{n}_2) \quad (3.41)$$

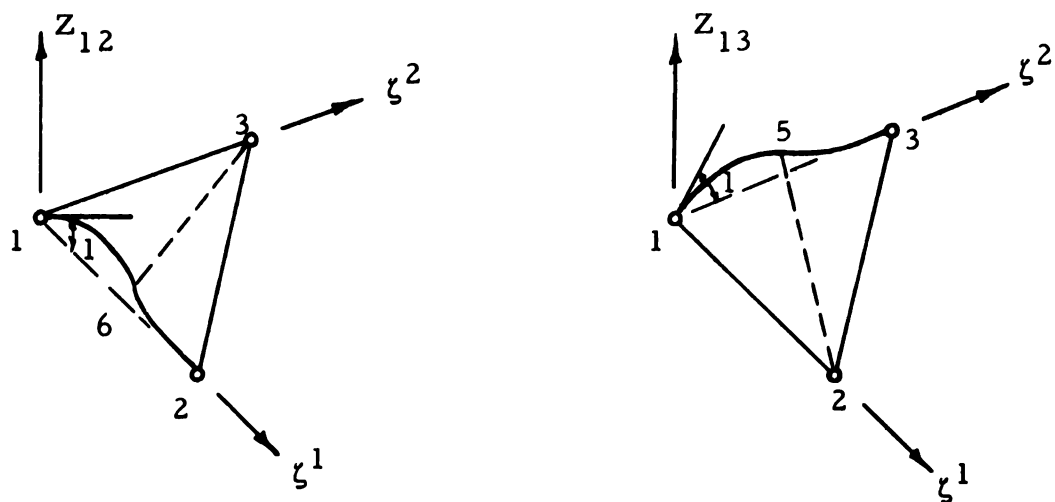


Fig. 3.9. Graphic description of functions Z_{12} and Z_{13} .

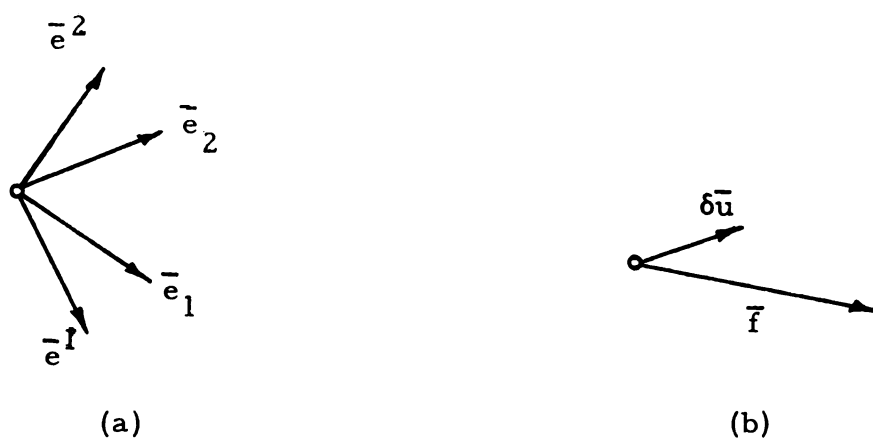


Fig. 3.10. Force vector subjected to a virtual displacement.

On substituting in (3.41) for the functions and unit vectors, rearranging terms and combining with (3.36) we obtain

$$\begin{aligned} \bar{u}_L = \frac{1}{l^3} \{ & -\beta l_1 \lambda_1 - (1-\alpha-\beta) l_3 \lambda_3 - [(1-\alpha-\beta)^2 A_{a2} + \beta(2-2\alpha-3\beta) A_{b2}] A \lambda_4 \\ & - [\beta(2\alpha-\beta) A_{a1} + \alpha^2 A_{b1}] A \lambda_5 + [\beta^2 A_{a1} + \alpha(2\beta-\alpha) A_{b1} \\ & - (1-\alpha-\beta)(1-\alpha-3\beta) A_{a2} + \beta^2 A_{b2}] A \lambda_6 \} \bar{e}^1 + \frac{1}{l^2} \{ & -(1-\alpha-\beta) l_2 \lambda_2 \\ & + [\alpha(2-3\alpha-2\beta) A_{a3} + (1-\alpha-\beta)^2 A_{b3}] A \lambda_4 - [\alpha^2 A_{a3} + (1-\alpha-\beta)(1-3\alpha+\beta) A_{b3} \\ & + \beta(2\alpha-\beta) A_{a1} + \alpha^2 A_{b1}] A \lambda_5 + [\beta^2 A_{a1} + \alpha(2\beta-\alpha) A_{b1}] A \lambda_6 \} \bar{e}^2 \end{aligned} \quad (3.42)$$

Eq. (3.42) contains six of the independent vector functions of H_1 .

The remaining six configurations are taken in the form

$$\begin{aligned} \bar{u}_L = w_3 \bar{e}^3 = \{ & g_{a2} g_{a3} \lambda_7 + g_{a3} g_{a1} \lambda_8 + g_{a1} g_{a2} \lambda_9 \\ & + [g_{b1}(g_{a3} A_{a1} + g_{a2} A_{b1})] \lambda_{10} + [g_{b2}(g_{a1} A_{a2} + g_{a3} A_{b2})] \lambda_{11} \\ & + [g_{b3}(g_{a2} A_{a3} + g_{a1} A_{b3})] \lambda_{12} \} \bar{e}^3 \end{aligned} \quad (3.43)$$

The six functions $g_{a2} g_{a3}, \dots, [g_{b3}(g_{a2} A_{a3} + g_{a1} A_{b3})]$ and their first partial derivatives are continuous for all points of the member. The last three of these functions respectively have second derivative discontinuities along lines 1, 5; 2, 6; 3, 4.

Substituting for the function in (3.43) we obtain

$$\begin{aligned}
\bar{u}_L = & \{ \alpha \beta \lambda_7 + \beta(1-\alpha-\beta)\lambda_8 + \alpha(1-\alpha-\beta)\lambda_9 + [\beta(\alpha-\beta)A_{a1} + \alpha(\alpha-\beta)A_{b1}] \lambda_{10} \\
& + [-(1-\alpha-2\beta)(1-\alpha-\beta)A_{a2} - (1-\alpha-2\beta)\beta A_{b2}] \lambda_{11} \\
& + [\alpha(1-2\alpha-\beta)A_{a3} + (1-\alpha-\beta)(1-2\beta-\beta)A_{b3}] \lambda_{12} \} \bar{e}^3
\end{aligned} \tag{3.44}$$

An alternate form of (3.44) is given in Appendix D.

3.5. Member-Shell Transformation Matrix

The member generalized forces Σ_i and member generalized coordinates Λ_i are related by the strain energy relation

$$\delta W_{I_i} = \Sigma_i^T \delta \Lambda_i \tag{3.45}$$

If F_i and U_i are generalized forces and coordinates so that

$$\delta W_{I_i} = F_i^T \delta U_i \tag{3.46}$$

and if

$$\delta \Lambda_i = M_i \delta U_i \tag{3.47}$$

then

$$\begin{aligned}
F_i^T \delta U_i &= \Sigma_i^T \delta \Lambda_i \\
&= \Sigma_i^T M_i \delta U_i
\end{aligned} \tag{3.48}$$

If (3.48) must hold for all virtual displacements δU_i it follows that

$$F_i^T = \Sigma_i^T M_i \tag{3.49}$$

On taking the transpose of both sides of (3.49) we obtain

$$F_i = M_i^T \Sigma_i \quad (3.50)$$

Here the components of F_i are components of node moment and force vectors, and the components of U_i are components of displacement and rotation vectors.

Consider the displacement vector $\delta \bar{u}$ and force vector $\bar{f}$ shown in Fig. 3.10b. The work due to $\bar{f}$ when it undergoes a virtual displacement $\delta \bar{u}$ is defined by the dot product

$$\delta W = \bar{f} \cdot \delta \bar{u} \quad (3.51)$$

The covariant base vectors (unit vectors) of an oblique Cartesian coordinate system are $\bar{e}_1, \bar{e}_2$ and the contravariant base vectors (not necessarily unit vectors) are $\bar{e}^1, \bar{e}^2$. They are related by

$$\begin{aligned} \bar{e}^1 \cdot \bar{e}_1 &= \bar{e}^2 \cdot \bar{e}_2 = 1 \\ \bar{e}^1 \cdot \bar{e}_2 &= \bar{e}_1 \cdot \bar{e}^2 = 0 \end{aligned} \quad (3.52)$$

The vectors $\bar{f}$ and $\delta \bar{u}$ in terms of their components are

$$\begin{aligned} \delta \bar{u} &= \delta w^1 \bar{e}_1 + \delta w^2 \bar{e}_2 \\ &= \delta w_1 \bar{e}^1 + \delta w_2 \bar{e}^2 \end{aligned} \quad (3.53)$$

$$\begin{aligned} \bar{f} &= f^1 \bar{e}_1 + f^2 \bar{e}_2 \\ &= f_1 \bar{e}^1 + f_2 \bar{e}^2 \end{aligned} \quad (3.54)$$

Substituting from (3.53) and (3.54) into (3.51) we obtain

$$\begin{aligned}
 \delta W &= (f_1 \bar{e}_1 + f_2 \bar{e}_2) \cdot (\delta w_1 \bar{e}_1 + \delta w_2 \bar{e}_2) = f_1 \delta w_1 + f_2 \delta w_2 \\
 &= (f_1 \bar{e}_1 + f_2 \bar{e}_2) \cdot (\delta w_1 \bar{e}_1 + \delta w_2 \bar{e}_2) = f_1 \delta w_1 + f_2 \delta w_2 \\
 &= (f_1 \bar{e}_1 + f_2 \bar{e}_2) \cdot (\delta w_1 \bar{e}_1 + \delta w_2 \bar{e}_2) = \frac{1}{\sin^2 \psi} \\
 &\quad [f_1 \delta w_1 + f_2 \delta w_2 + \cos \psi (f_1 \delta w_2 + f_2 \delta w_1)] \\
 &= (f_1 \bar{e}_1 + f_2 \bar{e}_2) \cdot (\delta w_1 \bar{e}_1 + \delta w_2 \bar{e}_2) = f_1 \delta w_1 + f_2 \delta w_2 \\
 &\quad + \cos \psi (f_1 \delta w_2 + f_2 \delta w_1) \tag{3.55}
 \end{aligned}$$

where ψ is the included angle of the oblique coordinates.

We note that

$$\delta W = [f_1 \quad f_2] \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} = [f^1 \quad f^2] \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} \tag{3.56}$$

but in general

$$\delta W \neq [f^1 \quad f^2] \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} \tag{3.57}$$

$$\delta W \neq [f_1 \quad f_2] \begin{bmatrix} \delta w_1 \\ \delta w_2 \end{bmatrix} \tag{3.58}$$

We then conclude that if the generalized coordinates are contravariant components of a displacement vector then the associated generalized forces are covariant components of a force vector and vice versa.

From (3.35) and Fig. 3.5 we obtain the node displacements relative to the $\zeta^1, \zeta^2, \zeta^3$ coordinates in the form

$$\begin{aligned}\bar{u}_{L(1)} &= \lambda_3 \bar{e}_{(1)}^1 + \lambda_2 \bar{e}_{(1)}^2 \\ \bar{u}_{L(2)} &= 0 \\ \bar{u}_{L(3)} &= \lambda_1 \bar{e}_{(3)}^2\end{aligned}\tag{3.59}$$

where the base vectors are those of Fig. 3.2a.

It then follows from the above discussion that if the components of node displacements are interpreted as components of generalized coordinates then the associated components of generalized force can be interpreted as components of node forces through the energy relation

$$\delta W_I = \bar{f}_{(1)} \cdot \delta \bar{u}_{(1)} + \bar{f}_{(2)} \cdot \delta \bar{u}_{(2)} + \bar{f}_{(3)} \cdot \delta \bar{u}_{(3)}\tag{3.60}$$

To do this we take the variation of (3.59) and then substitute it into (3.60) and obtain

$$\begin{aligned}\delta W_I &= (f_{(1, e1)}^1 \bar{e}_{1(1)} + f_{(1, e1)}^2 \bar{e}_{2(1)}) \cdot (\delta \lambda_3 \bar{e}_{(1)}^1 + \delta \lambda_2 \bar{e}_{(1)}^2) \\ &\quad + (f_{(3, e3)}^1 \bar{e}_{1(3)} + f_{(3, e3)}^2 \bar{e}_{2(3)}) \cdot (\delta \lambda_1 \bar{e}_{(3)}^2) \\ &= f_{(1, e1)}^1 \delta \lambda_3 + f_{(1, e1)}^2 \delta \lambda_2 + f_{(3, e3)}^2 \delta \lambda_1\end{aligned}\tag{3.61}$$

where $f_{(1, e1)}^1, f_{(1, e1)}^2$, etc. are force components referred to the base vectors of Fig. 3.2. If

$$\delta \lambda_4 = \delta \lambda_5 = \dots = \delta \lambda_{12} = 0\tag{3.62}$$

then

$$\delta W_I = \sum_i^T \delta \Lambda_i = \sigma_1 \delta \lambda_1 + \sigma_2 \delta \lambda_2 + \sigma_3 \delta \lambda_3\tag{3.63}$$

Eq.'s (3.61) and (3.63) can hold for all virtual displacements that also satisfy (3.62) only if

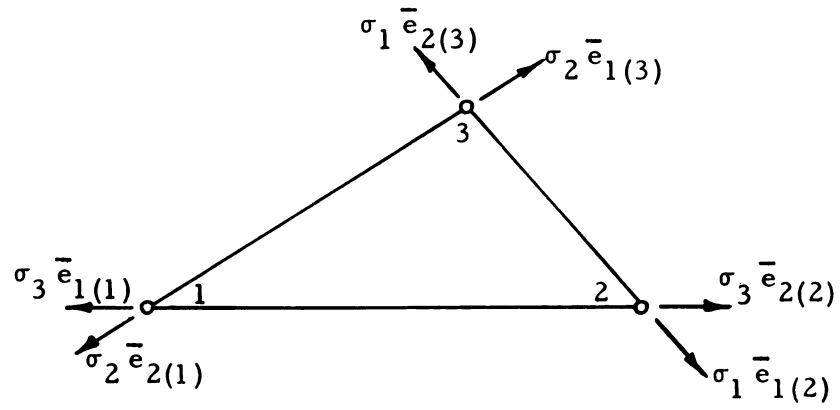
$$\begin{aligned} f_{1(1,e1)} &= \sigma_3 \\ f_{2(1,e1)} &= \sigma_2 \\ f_{2(3,e3)} &= \sigma_1 \end{aligned} \quad (3.64)$$

The components of the node forces $\bar{f}_{(1)}, \bar{f}_{(2)}, \bar{f}_{(3)}$ expressed in terms of $\sigma_1, \sigma_2, \sigma_3$ can be obtained directly from (3.64) and the equations of equilibrium. They are shown in Fig. 3.11a and have the form

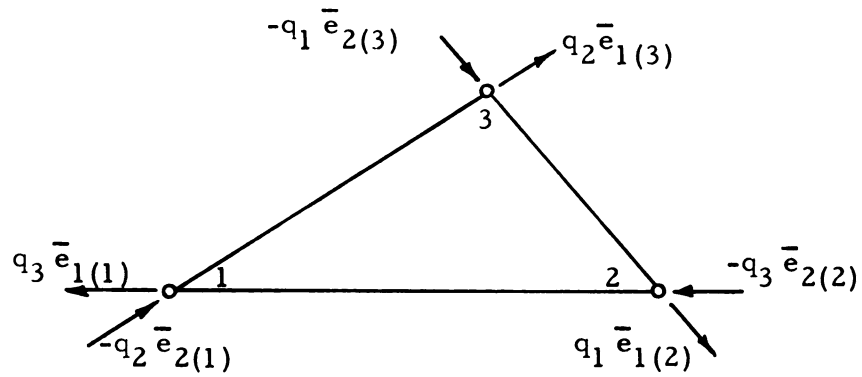
$$\begin{aligned} \bar{f}_{(1)} &= \sigma_3 \bar{e}_{1(1)} + \sigma_2 \bar{e}_{2(1)} \\ \bar{f}_{(2)} &= \sigma_1 \bar{e}_{1(2)} + \sigma_2 \bar{e}_{2(2)} \\ \bar{f}_{(3)} &= \sigma_2 \bar{e}_{1(3)} + \sigma_1 \bar{e}_{2(2)} \end{aligned} \quad (3.65)$$

The member generalized displacements $\lambda_4, \lambda_5, \dots, \lambda_{12}$ are related to the components of the node rotation vector measured relative to the y_R^1, y_R^2, y_R^3 coordinates as shown in Fig. 3.6. The node rotation vectors are

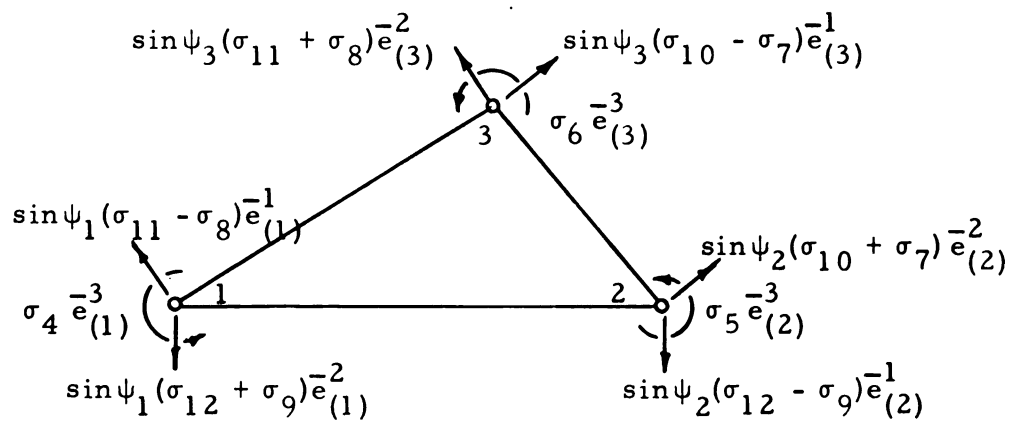
$$\begin{aligned} \bar{\phi}_{(1)} &= \frac{1}{\sin \psi_1} (\lambda_{11} - \lambda_8) \bar{e}_{1(1)} + \frac{1}{\sin \psi_1} (\lambda_{12} + \lambda_9) \bar{e}_{2(1)} + \lambda_4 \bar{e}_{3(1)} \\ \bar{\phi}_{(2)} &= \frac{1}{\sin \psi_2} (\lambda_{12} - \lambda_9) \bar{e}_{1(2)} + \frac{1}{\sin \psi_2} (\lambda_{10} + \lambda_7) \bar{e}_{2(2)} + \lambda_5 \bar{e}_{3(2)} \\ \bar{\phi}_{(3)} &= \frac{1}{\sin \psi_3} (\lambda_{10} - \lambda_7) \bar{e}_{1(3)} + \frac{1}{\sin \psi_3} (\lambda_{11} + \lambda_8) \bar{e}_{2(3)} + \lambda_6 \bar{e}_{3(3)} \end{aligned} \quad (3.66)$$



(a)



(b)



(c)

Fig. 3.11. Relationship of generalized forces to node variables.

If the components of the node moment vectors $\bar{m}_{(1)}, \bar{m}_{(2)}, \bar{m}_{(3)}$ are interpreted as generalized forces and related to the member by means of an energy relation similar to (3.48) then by using arguments similar to the above we can obtain

$$\begin{aligned}\bar{m}_{(1)} &= \sin\psi_1(\sigma_{11} - \sigma_8)\bar{e}_{(1)}^1 + \sin\psi_1(\sigma_{12} + \sigma_9)\bar{e}_{(1)}^2 + \sigma_4\bar{e}_{(1)}^3 \\ \bar{m}_{(2)} &= \sin\psi_2(\sigma_{12} - \sigma_9)\bar{e}_{(2)}^1 + \sin\psi_2(\sigma_{10} + \sigma_7)\bar{e}_{(2)}^2 + \sigma_5\bar{e}_{(2)}^3 \\ \bar{m}_{(3)} &= \sin\psi_3(\sigma_{10} - \sigma_7)\bar{e}_{(3)}^1 + \sin\psi_3(\sigma_{11} + \sigma_8)\bar{e}_{(3)}^2 + \sigma_6\bar{e}_{(3)}^3\end{aligned}\tag{3.67}$$

The above are shown in Fig. 3.11b.

The node forces, necessary for equilibrium when the member is subjected to node moments given by (3.67), can be obtained from the six scalar equations of equilibrium.

The force components normal to the plane of the triangle are associated only with the in-plane moment components and can be obtained directly, by taking moments about the three sides of the triangle.

The in-plane force components are associated only with the moment components normal to the plane of the triangle. Since there are six in-plane force components and only three independent equilibrium equations associated with them, it follows that this system of forces has three redundancies. The redundancies are essentially removed by imposing certain symmetry requirements on the components, i. e., the six components are interpreted in terms of three parameters q_1, q_2, q_3 related as shown in Fig. 3.11b.

The six equilibrium equations are, after some simplification, given by

$$\begin{aligned}
 f_{(1, e1)}^3 &= 2\left(\frac{1}{l_3} \sigma_{12} - \frac{1}{l_2} \sigma_{11}\right) \\
 f_{(2, e2)}^3 &= 2\left(\frac{1}{l_1} \sigma_{10} - \frac{1}{l_3} \sigma_{12}\right) \\
 f_{(3, e3)}^e &= 2\left(\frac{1}{l_2} \sigma_{11} - \frac{1}{l_1} \sigma_{10}\right) \quad (3.68) \\
 q_1 &= \frac{l_1}{4A} (\sigma_4 + \sigma_5 + \sigma_6) \\
 q_2 &= \frac{l_2}{4A} (\sigma_4 + \sigma_5 + \sigma_6) \\
 q_3 &= \frac{l_3}{4A} (\sigma_4 + \sigma_5 + \sigma_6)
 \end{aligned}$$

From (3.65), (3.67), (3.68) we obtain the relationship between node forces and moments referred to the coordinates of Fig. 3.2 a, b, and the member generalized forces. This relationship is given in matrix form by (3.69).

$$\begin{bmatrix} f_{(1, e1)}^1 \\ f_{(1, e1)}^2 \\ f_{(1, e1)}^3 \\ m_{1(1, e1)} \\ m_{2(1, e1)} \\ m_{3(1, e1)} \\ f_{(2, e2)}^1 \\ f_{(2, e2)}^2 \\ f_{(2, e2)}^3 \\ m_{1(2, e2)} \\ m_{2(2, e2)} \\ m_{3(2, e2)} \\ f_{(3, e3)}^1 \\ f_{(3, e3)}^2 \\ f_{(3, e3)}^3 \\ m_{1(3, e3)} \\ m_{2(3, e3)} \\ m_{3(3, e3)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{\ell_3}{4A} & \frac{\ell_3}{4A} & \frac{\ell_3}{4A} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -\frac{\ell_2}{4A} & -\frac{\ell_2}{4A} & -\frac{\ell_2}{4A} & & & & & & & \\ 0 & & & & & & & & \frac{2}{\ell_3} & -\frac{2}{\ell_3} & & \\ 0 & & & & & -\frac{\ell_2 \ell_3}{2A} & & & \frac{\ell_2 \ell_3}{2A} & & & \\ 0 & & & & & & \frac{\ell_2 \ell_3}{2A} & & & \frac{\ell_2 \ell_3}{2A} & & \\ 0 & & 1 & & & & & & & & & \\ 1 & & \frac{\ell_1}{4A} & \frac{\ell_1}{4A} & \frac{\ell_1}{4A} & & & & & & & \\ 0 & 1 & -\frac{\ell_3}{4A} & -\frac{\ell_3}{4A} & -\frac{\ell_3}{4A} & & & & & & & \\ 0 & & & & & & & & \frac{2}{\ell_1} & -\frac{2}{\ell_1} & & \\ 0 & & & & & & -\frac{\ell_1 \ell_3}{2A} & & & \frac{\ell_1 \ell_3}{2A} & & \\ 0 & & & & & \frac{\ell_1 \ell_3}{2A} & & & \frac{\ell_1 \ell_3}{2A} & & & \\ 0 & & & 1 & & & & & & & & \\ 0 & 1 & \frac{\ell_2}{4A} & \frac{\ell_2}{4A} & \frac{\ell_2}{4A} & & & & & & & \\ 0 & 1 & -\frac{\ell_1}{4A} & -\frac{\ell_1}{4A} & -\frac{\ell_1}{4A} & & & & & & & \\ 0 & & & & & & & & -\frac{2}{\ell_1} & \frac{2}{\ell_2} & & \\ 0 & & & & & -\frac{\ell_1 \ell_2}{2A} & & & \frac{\ell_1 \ell_2}{2A} & & & \\ 0 & & & & & & \frac{\ell_1 \ell_2}{2A} & & & \frac{\ell_1 \ell_2}{2A} & & \\ 0 & & & & 1 & & & & & & & \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \\ \sigma_9 \\ \sigma_{10} \\ \sigma_{11} \\ \sigma_{12} \end{bmatrix}$$

The coefficient matrix in (3.69) is designated by $M_{A_i}^T$. The relationship between the components of node forces and moments referred to the i-member coordinates (y^1, y^2, y^3) and those in (3.69) can be obtained directly from (A.22), (A.23), and from the definitions of the quantities. The relationship in matrix form is given by (3.70). The coefficient matrix in (3.70) is designated by $M_{B_i}^T$.

The matrix M_i is related by

$$M_i^T = M_{B_i}^T M_{A_i}^T \quad (3.71)$$

and is given explicitly by (3.72).

$$\begin{bmatrix} f_{(1i)}^1 \\ f_{(1i)}^2 \\ f_{(1i)}^3 \\ f_{(1i)} \\ m_1(1i) \\ m_2(1i) \\ m_3(1i) \\ f_{(2i)}^1 \\ f_{(2i)}^2 \\ f_{(2i)}^3 \\ f_{(2i)} \\ m_1(2i) \\ m_2(2i) \\ m_3(2i) \\ f_{(3i)}^1 \\ f_{(3i)}^2 \\ f_{(3i)}^3 \\ f_{(3i)} \\ m_1(3i) \\ m_2(3i) \\ m_3(3i) \end{bmatrix} = \begin{bmatrix} -1 & -\cos\psi_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin\psi_1 & & & & & & & & & & & & & & & & & & \\ 0 & & 1 & & & & & & & & & & & & & & & & & \\ 0 & & & -1 & 0 & & & & & & & & & & & & & & & \\ 0 & & & \frac{\cos\psi_1}{\sin\psi_1} & -\frac{1}{\sin\psi_1} & & & & & & & & & & & & & & & \\ 0 & & & & & 1 & & & & & & & & & & & & & & \\ 0 & & & & & & \cos\psi_2 & 1 & & & & & & & & & & & & \\ 0 & & & & & & & & -\sin\psi_2 & 0 & & & & & & & & & & \\ 0 & & & & & & & & & & 1 & & & & & & & & & \\ 0 & & & & & & & & & & & 0 & 1 & \frac{\cos\psi_2}{\sin\psi_2} & & & & & & \\ 0 & & & & & & & & & & & & -\frac{1}{\sin\psi_2} & \frac{\cos\psi_2}{\sin\psi_2} & 1 & & & & \\ 0 & & & & & & & & & & & & & & & \cos\psi_1 & -\cos\psi_2 & & & \\ 0 & & & & & & & & & & & & & & & \sin\psi_1 & \sin\psi_2 & & & \\ 0 & & & & & & & & & & & & & & & & 1 & \frac{\sin\psi_2}{\sin\psi_3} & -\frac{\sin\psi_1}{\sin\psi_3} & & \\ 0 & & & & & & & & & & & & & & & & & \frac{\sin\psi_3}{\cos\psi_2} & \frac{\cos\psi_1}{\sin\psi_3} & & \\ 0 & & & & & & & & & & & & & & & & & & \frac{\sin\psi_3}{\sin\psi_3} & 1 & \\ 0 & & & & & & & & & & & & & & & & & & & m_1(3, e3) \\ 0 & & & & & & & & & & & & & & & & & & & m_2(3, e3) \\ 0 & & & & & & & & & & & & & & & & & & & m_3(3, e3) \end{bmatrix} \begin{bmatrix} f_{(1, e1)}^1 \\ f_{(1, e1)}^2 \\ f_{(1, e1)}^3 \\ f_{(1, e1)} \\ m_1(1, e1) \\ m_2(1, e1) \\ m_3(1, e1) \\ f_{(2, e2)}^1 \\ f_{(2, e2)}^2 \\ f_{(2, e2)}^3 \\ f_{(2, e2)} \\ m_1(2, e2) \\ m_2(2, e2) \\ m_3(2, e2) \\ f_{(3, e3)}^1 \\ f_{(3, e3)}^2 \\ f_{(3, e3)}^3 \\ f_{(3, e3)} \\ m_1(3, e3) \\ m_2(3, e3) \\ m_3(3, e3) \end{bmatrix} \quad (3.70)$$

$$\begin{bmatrix} f_{(1i)}^1 \\ f_{(1i)}^2 \\ f_{(1i)}^3 \\ m_{(1i)}^1 \\ m_{(1i)}^2 \\ m_{(1i)}^3 \\ f_{(2i)}^1 \\ f_{(2i)}^2 \\ f_{(2i)}^3 \\ m_{(2i)}^1 \\ m_{(2i)}^2 \\ m_{(2i)}^3 \\ f_{(3i)}^1 \\ f_{(3i)}^2 \\ f_{(3i)}^3 \\ m_{(3i)}^1 \\ m_{(3i)}^2 \\ m_{(3i)}^3 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{a_1}{l_2 l_3} & -1 & -\frac{a_2}{4Al_3} & -\frac{a_2}{4Al_3} & -\frac{a_2}{4Al_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{2A}{l_2 l_3} & 0 & \frac{1}{2l_3} & \frac{1}{2l_3} & \frac{1}{2l_3} & 0 & 0 & 0 & 0 & -\frac{1}{l_2} & \frac{1}{l_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{l_2} & \frac{1}{l_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{A}{l_2 l_3} & 0 & 0 & -\frac{A}{l_2 l_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_1}{2l_2 l_3} & -\frac{1}{2} & 0 & \frac{a_1}{2l_2 l_3} & -\frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{a_1}{l_2 l_3} & 0 & 1 & -\frac{a_1}{4Al_2} & -\frac{a_1}{4Al_3} & -\frac{a_1}{4Al_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{2A}{l_1 l_3} & 0 & 0 & -\frac{1}{2l_3} & -\frac{1}{2l_3} & -\frac{1}{2l_3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{l_1} & 0 & -\frac{1}{l_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{A}{l_1 l_3} & 0 & 0 & \frac{A}{l_1 l_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{a_2}{2l_1 l_3} & 0 & \frac{1}{2} & \frac{a_2}{2l_1 l_2} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{a_2}{l_1 l_3} & \frac{a_1}{l_2 l_3} & 0 & \frac{l_3}{4A} & \frac{l_3}{4A} & \frac{l_3}{4A} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2A}{l_1 l_3} & \frac{2A}{l_2 l_3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{l_1} & \frac{1}{l_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{A}{l_1 l_3} & -\frac{A}{l_2 l_3} & 0 & \frac{A}{l_1 l_3} & -\frac{A}{l_2 l_3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{a_2}{2l_1 l_3} & \frac{a_1}{2l_2 l_3} & 0 & \frac{a_2}{2l_1 l_3} & \frac{a_1}{2l_2 l_3} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \\ \sigma_7 \\ \sigma_8 \\ \sigma_9 \\ \sigma_{10} \\ \sigma_{11} \\ \sigma_{12} \end{bmatrix}$$

Two of the triangle rigid body rotation components have a direct interpretation, i. e., they are the two inplane components of rotation of the plane defined by the three node points of the triangle. The component of rigid body rotation $(\Delta\theta_R^3)$ normal to the plane does not have a direct interpretation and is in fact intrinsically related to the manner in which the redundancies, mentioned above, are removed.

In order to obtain this interpretation we let a triangle in equilibrium undergo a virtual rigid body rotation $\delta\theta_R^3$ so that the magnitude of the forces and internal strain energy do not change. Only inplane components of node forces and displacements need be considered. The virtual displacements of the node points with respect to the oblique coordinates of the node points (Fig. 3.2) can be represented in the form

$$\begin{aligned}\delta\bar{u}_{(1)} &= \delta w_{1(1)} \bar{e}_{(1)}^1 + \delta w_{2(1)} \bar{e}_{(1)}^2 \\ \delta\bar{u}_{(2)} &= \delta w_{1(2)} \bar{e}_{(2)}^1 + \delta w_{2(2)} \bar{e}_{(2)}^2 \\ \delta\bar{u}_{(3)} &= \delta w_{1(3)} \bar{e}_{(3)}^1 + \delta w_{2(3)} \bar{e}_{(3)}^2\end{aligned}\tag{3.73}$$

The node forces can be represented in terms of the six parameters $\sigma_1, \sigma_2, \sigma_3, q_1, q_2, q_3$ (Fig. 3.11a, b). They are

$$\begin{aligned}\bar{f}_{(1)} &= (\sigma_3 + q_3) \bar{e}_{1(1)} + (\sigma_2 - q_2) \bar{e}_{2(1)} \\ \bar{f}_{(2)} &= (\sigma_1 + q_1) \bar{e}_{1(2)} + (\sigma_3 - q_3) \bar{e}_{2(2)} \\ \bar{f}_{(3)} &= (\sigma_2 + q_2) \bar{e}_{1(3)} + (\sigma_1 - q_1) \bar{e}_{2(3)}\end{aligned}\tag{3.74}$$

The components of the node moment vector $(\sigma_4, \sigma_5, \sigma_6)$ normal to the plane are also required.

The work due to the node forces and moments when the triangle is subjected to a rigid body rotation $\delta\theta_R^3$ must vanish. This can be expressed in the form

$$(\sigma_4 + \sigma_5 + \sigma_6) \delta\theta_R^3 + \bar{f}_{(1)} \cdot \delta\bar{u}_{(1)} + \bar{f}_{(2)} \cdot \delta\bar{u}_{(2)} + \bar{f}_{(3)} \cdot \delta\bar{u}_{(3)} = 0 \quad (3.75)$$

Since $\delta\theta_R^3$ does not cause the sides to elongate the work due to the equal in magnitude and oppositely directed force components associated with each of the parameters $\sigma_1, \sigma_2, \sigma_3$ must vanish independently. On removing these parameters from (3.74) and substituting the resulting expression and (3.73) into (3.75) we obtain after some simplification

$$\begin{aligned} (\sigma_4 + \sigma_5 + \sigma_6) \delta\theta_R^3 - q_1 (\delta w_{2(3)} - \delta w_{1(2)}) - q_2 (\delta w_{2(1)} - \delta w_{1(3)}) \\ - q_3 (\delta w_{2(3)} - \delta w_{1(1)}) = 0 \end{aligned} \quad (3.76)$$

Substituting the last three of (3.68) into (3.76) and dividing through by $(\sigma_4 + \sigma_5 + \sigma_6)$ we obtain

$$\begin{aligned} \delta\theta_R^3 = \frac{l_1}{4A} (\delta w_{2(3)} - \delta w_{1(2)}) + \frac{l_2}{4A} (\delta w_{2(1)} - \delta w_{1(3)}) \\ + \frac{l_3}{4A} (\delta w_{2(3)} - \delta w_{1(1)}) \end{aligned} \quad (3.77)$$

From the assumptions of linear theory we have

$$\begin{aligned}\Delta\theta_R^3 &= \frac{\ell_1}{4A} (\Delta w_{2(3)} - \Delta w_{1(2)}) + \frac{\ell_2}{4A} (\Delta w_{2(1)} - \Delta w_{1(3)}) \\ &+ \frac{\ell_3}{4A} (\Delta w_{2(3)} - \Delta w_{1(1)})\end{aligned}\quad (3.78)$$

The rotation of the coordinate system y_R^1, y_R^2, y_R^3 relative to the coordinate system y^1, y^2, y^3 referred to in Sec. 3.1 is $\Delta\bar{\theta}_R$ defined above.

3.6. Member Stiffness Matrix

The triangular member stiffness matrix defined by (2.75) is a 12x12 matrix Γ_i with $\det \Gamma_i > 0$. We partition (2.70) in the form

$$\bar{u}_{L_i} = \begin{bmatrix} H_{1i}^T & H_{2i}^T \end{bmatrix} \begin{bmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{bmatrix} \quad (3.79)$$

where $H_{1i}^T \Lambda_{1i}$ corresponds to the displacement vector given by (3.42) and $H_{2i}^T \Lambda_{2i}$ corresponds to the displacement vector given by (3.44). If the deformation parameters are taken in the form

$$E^T = \begin{bmatrix} E_1^T & E_2^T \end{bmatrix} \quad (3.80)$$

where E_1 and E_2 are respectively given by (3.16) and (3.26) then from (3.27), (2.70), and (2.71) it follows that

$$\begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \begin{bmatrix} \Omega_{1i} \\ \Omega_{2i} \end{bmatrix} \begin{bmatrix} H_{1i}^T & H_{2i}^T \end{bmatrix} \begin{bmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{bmatrix}$$

$$= \begin{bmatrix} \Omega_{1i} H_{1i}^T & \Omega_{1i} H_{2i}^T \\ \Omega_{2i} H_{1i}^T & \Omega_{2i} H_{2i}^T \end{bmatrix} \begin{bmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{bmatrix} \quad (3.81)$$

From (3.27), (3.42), and (3.44) it follows that (3.81) will reduce to

$$\begin{aligned} \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} &= \begin{bmatrix} \Omega_{1i} H_{1i}^T & 0 \\ 0 & \Omega_{2i} H_{2i}^T \end{bmatrix} \begin{bmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{bmatrix} \\ &= \begin{bmatrix} \Pi_{11i} & 0 \\ 0 & \Pi_{22i} \end{bmatrix} \begin{bmatrix} \Lambda_{1i} \\ \Lambda_{2i} \end{bmatrix} \end{aligned} \quad (3.82)$$

On substituting the coefficient matrix of (3.82) into (2.75) we obtain

$$\begin{aligned} \begin{bmatrix} \Gamma_{11i} & \Gamma_{12i} \\ \Gamma_{21i} & \Gamma_{22i} \end{bmatrix} &= \int_{S_i} \int \begin{bmatrix} \Pi_{11i}^T & 0 \\ 0 & \Pi_{22i}^T \end{bmatrix} \cdot \begin{bmatrix} G_1 & 0 \\ 0 & G_1 \end{bmatrix} \begin{bmatrix} \Pi_{11i} & 0 \\ 0 & \Pi_{22i} \end{bmatrix} dA \\ &= \begin{bmatrix} \int_{S_i} \int \Pi_{11i}^T G_i \Pi_{11i} dA & 0 \\ 0 & \int_{S_i} \int \Pi_{22i}^T G_i \Pi_{22i} dA \end{bmatrix} \end{aligned} \quad (3.83)$$

The matrices Γ_{11i} and Γ_{22i} are respectively called the member membrane stiffness matrix and member bending stiffness matrix.

These matrices are given in Tables (3.2) and (3.3), and a detailed derivation is given in Appendix C.

3.7. Continuity Conditions Between Adjacent Members

We now examine the displacement continuity between two adjacent members and the conditions necessary for intersection angles (normal slope continuity) to be preserved. The deformed configurations have been chosen in groups of three, the elements of which are constructed symmetrically with respect to the three sides so that we need only examine one side of a member with arbitrary side lengths l_1, l_2, l_3 .

Since the formulation requires the node displacements of adjacent members to be identical we can examine displacement continuity by examining displacements relative to the line joining the common node point.

Consider the side with node points 1 and 2 lying along the ζ^1 axis as shown in Fig.'s 3.1, 3.5, 3.6. By substituting from (3.40) into (3.41), setting $\beta = 0$ in the resulting expression and in (3.42), and on noting that

$$\begin{aligned}\bar{n}_3 &= -\bar{j}_{(i)}^2 \\ \bar{j}_{(i)}^3 &= \bar{j}_{(i)}^3\end{aligned}\tag{3.84}$$

the desired displacements along the ζ^1 axis are obtained and given by

$$\begin{aligned}\bar{u} &= \left\{ \frac{1}{2} [a(2-3a)A_{a3} + (1-a)^2 A_{b3}] \lambda_4 - \frac{1}{2} [a(2-3a)A_{a3} + (1-a) \right. \\ &\quad \left. (1-3a)A_{b3}] \lambda_5 \right\} \bar{j}_{(i)}^2 + \left\{ a(1-a)\lambda_9 + a^2 A_{b1} \lambda_{10} - (1-a)^2 A_{a2} \lambda_{11} \right. \\ &\quad \left. + [a(1-2a)A_{a3} + (1-a)(1-2a)A_{b3}] \lambda_{12} \right\} \bar{j}_{(i)}^3\end{aligned}\tag{3.85}$$

Let L_1, L_2 correspond respectively to the line segments 1-6, 6-2 shown in Fig. 3.7 so that $f(a)L_1, g(a)(L_1+L_2)$ implies that $f(a)$ is defined over L_1 and $g(a)$ is defined over L_1 and L_2 . Along the y^1 axis the following then holds

$$\begin{aligned} A_{a1} &= L_1 + L_2 \\ A_{b2} &= L_1 + L_2 \\ A_{a3} &= L_1 \\ A_{b3} &= L_2 \end{aligned} \tag{3.86}$$

From the above it then follows that (3.85) can be expressed in the form

$$\begin{aligned} \bar{u} &= \left\{ \frac{1}{2} [a(2-3a)L_1 + (1-a)^2 L_2] \lambda_4 - \frac{1}{2} [a(2-3a)L_1 \right. \\ &\quad \left. + (1-a)(1-3a)L_2] \lambda_5 \right\} \bar{J}_{(i)}^2 + \{ a(1-a)(L_1 + L_2) \lambda_9 \\ &\quad + [a(1-2a)L_1 + (1-a)(1-2a)L_2] \lambda_{12} \} \bar{J}_{(i)}^3 \end{aligned} \tag{3.87}$$

The components of node rotation about the y_R^2 axis $\phi_{2(1,i)}, \phi_{2(2,i)}$ and about the y_R^1 axis $\phi_{3(1,i)}, \phi_{3(2,i)}$ are related to $\lambda_4, \lambda_5, \lambda_9$ and λ_{12} by

$$\begin{bmatrix} \phi_{2(1,i)} \\ \phi_{2(2,i)} \\ \phi_{3(1,i)} \\ \phi_{3(2,i)} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_4 \\ \lambda_5 \\ \lambda_9 \\ \lambda_{12} \end{bmatrix} \tag{3.88}$$

The above can be obtained directly from Fig. 3.6. These rotation components are identical for adjacent members only if the components of rigid body rotation (see Sec. 3.5) about axes normal to the plane of the triangles are identical for the two members. On solving (3.88) for $\lambda_4, \lambda_5, \lambda_9, \lambda_{12}$ and substituting into (3.87) we obtain after some simplification

$$\begin{aligned} \bar{u} = & \frac{1}{2} \{ [a(2-3a)L_1 + (1-a)^2 L_2] \phi_{3(1,i)} \\ & - [a(2-3a)L_1 + (1-a)(1-3a)L_2] \phi_{3(2,i)} \} \bar{j}_{(i)}^2 \\ & + \frac{1}{2} \{ -[a(2-3a)L_1 + (1-a)^2 L_2] \phi_{2(1,i)} \\ & - [-a^2 L_1 + (1-a)(1-3a)L_2] \phi_{2(2,i)} \} \bar{j}_{(i)}^3 \end{aligned} \quad (3.89)$$

A graphic description of the manner in which the displacements along the line are related to components of node rotation, is given in Fig. 3.12.

From (3.89) it can be seen that along the line the inplane displacements are related to the components of node rotation in a manner identical to displacements normal to the plane. Since the components of rotation for common node points of adjacent members are required to be identical it follows that displacement continuity is maintained along the entire line even if the initial intersection angle does not vanish, provided that the normal components of rigid body rotations of the two members are identical.

If in addition the normal slopes of two adjacent members vary continuously and linearly along their intersection line then normal slope continuity is maintained for all points of this line. In the

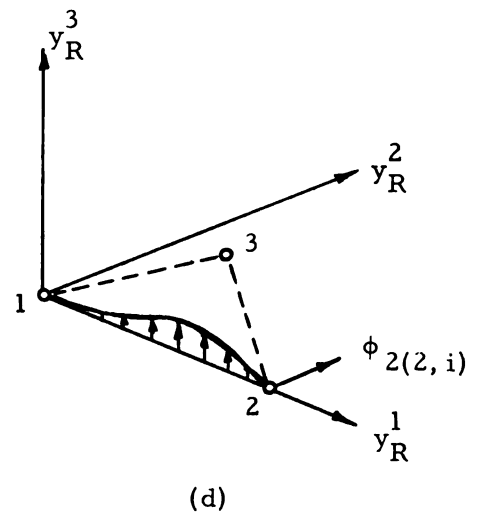
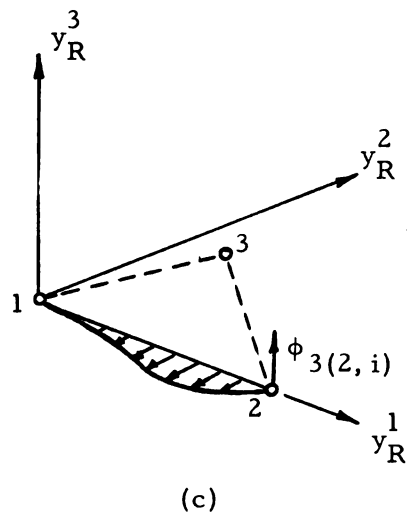
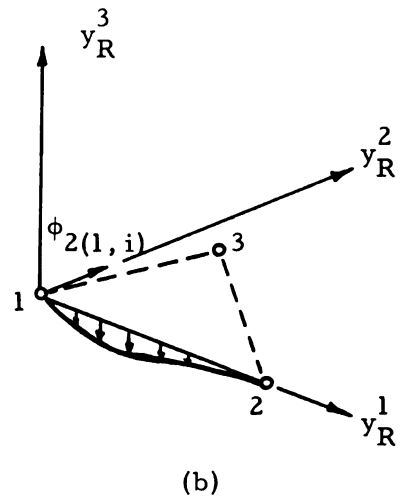
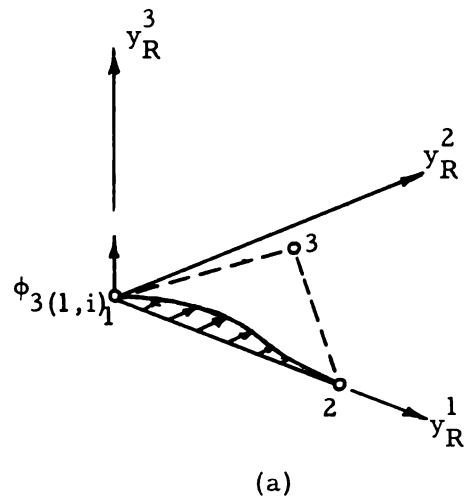


Fig. 3.12. Relationship between displacement components normal to an edge and node rotation components.

formulation used, normal slope continuity is completely satisfied only under a restrictive set of conditions. These conditions are related to the geometric shape of adjacent members and the type of deformation present along the intersection line. The inplane displacements (3.42) do not result in slope change along the edge. The displacements normal to the plane of the triangle are constructed by polynomials no higher than the second degree. Consequently the normal slopes associated with the deformed configurations of $\lambda_7, \dots, \lambda_{12}$ (see Fig. 3.6) vary linearly along line 1-2, but the variation of the configuration associated with λ_{12} can have a finite discontinuity at the midpoint.

The manner in which the normal slope is related to λ_{12} is shown in Fig. 3.13. Also the relationship of the midpoint discontinuity to the geometric parameters is described there. The magnitude of the discontinuity is designated by d and is related to the components of node and midpoint rotations by

$$d = \phi_{1(6,i)} - \frac{1}{2} (\phi_{1(1,i)} + \phi_{1(2,i)}) \quad (3.90)$$

From Fig. 3.13(a) we obtain the relationships between the components in (3.90) and λ_{12} . They are

$$\begin{aligned} \phi_{1(1,i)} &= -\cot \psi_1 \lambda_{12} = -\frac{a_1}{2A} \lambda_{12} \\ \phi_{1(2,i)} &= \cot \psi_2 \lambda_{12} = \frac{a_2}{2A} \lambda_{12} \\ \phi_{1(6,i)} &= \cot \psi_6 \lambda_{12} = \frac{(\ell_2^2 - \ell_1^2)}{4A} \lambda_{12} \end{aligned} \quad (3.91)$$

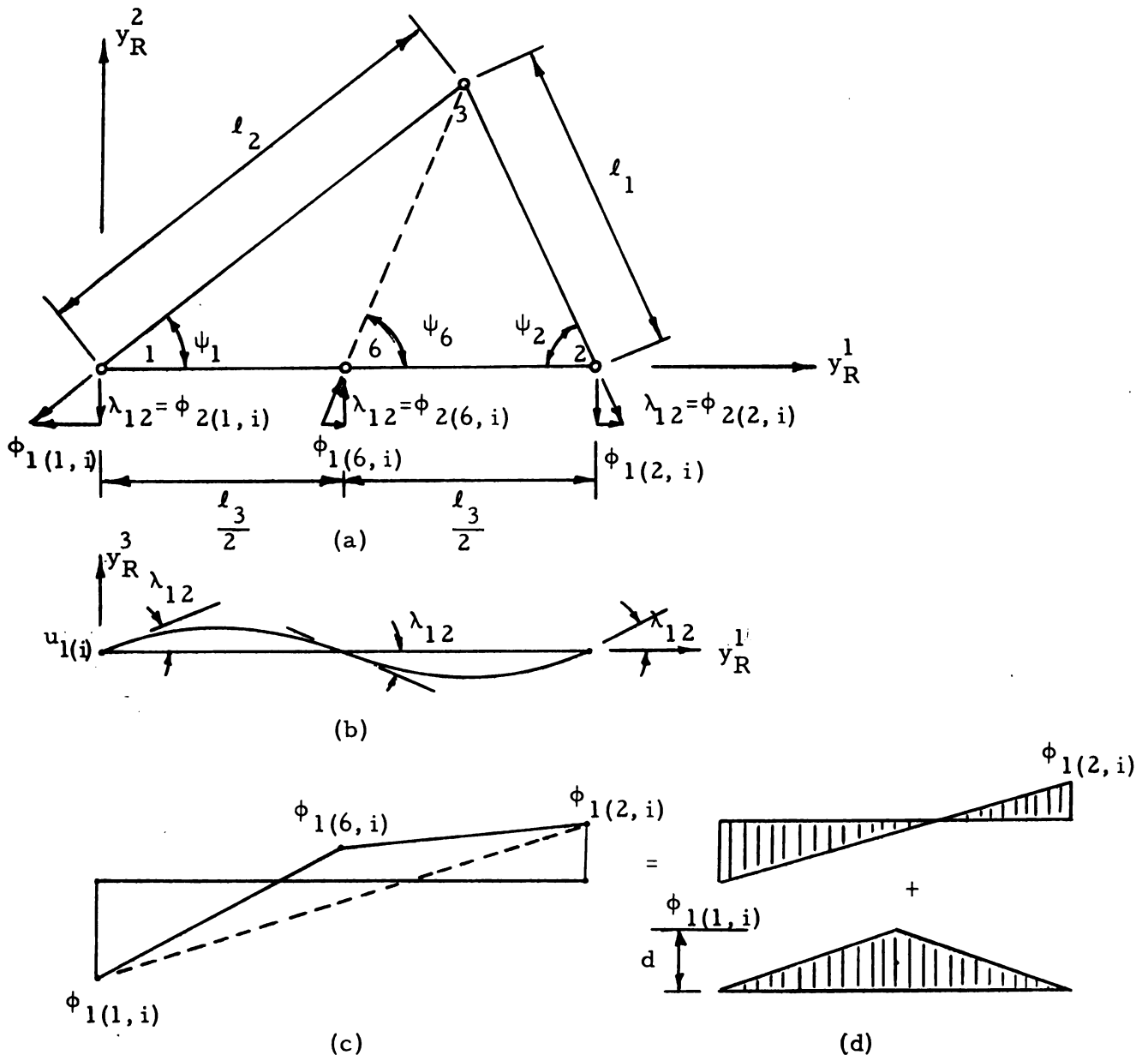


Fig. 3.13. Variation of normal slope for secondary bending modes.

Substituting from (3.91) into (3.90) and simplifying we obtain

$$d = \frac{\ell_2^2 - \ell_1^2}{2A} \lambda_{12} \quad (3.92)$$

In order to interpret the conditions necessary for slope continuity to be satisfied we first consider two adjacent members with their middle surfaces on (near to, in the linear sense) a plane as shown in Fig. 3.14. The various parameters of one member are designated in accordance with earlier notation and its adjacent member has its corresponding parameters distinguished by a prime.

By examining Fig.'s 3.13 and 3.14 it can be seen that

$$\lambda_{12} = -\lambda'_{12} \quad (3.93)$$

Then for normal slope continuity to be preserved

$$d = -d' \quad (3.94)$$

Substituting from (3.92) into (3.94) we obtain

$$\frac{\ell_2^2 - \ell_1^2}{2A} \lambda_{12} = - \frac{(\ell'_2)^2 - (\ell'_1)^2}{2A'} \lambda'_{12} \quad (3.95)$$

Substituting from (3.93) into (3.95) and dividing through by λ_{12} we obtain

$$\frac{\ell_2^2 - \ell_1^2}{2A} = \frac{(\ell'_2)^2 - (\ell'_1)^2}{2A'} \quad (3.96)$$

If the adjacent member does not lie in the same plane then (3.93) is not in general satisfied and normal slope continuity is insured only if

$$d = -d' = 0 \quad (3.97)$$

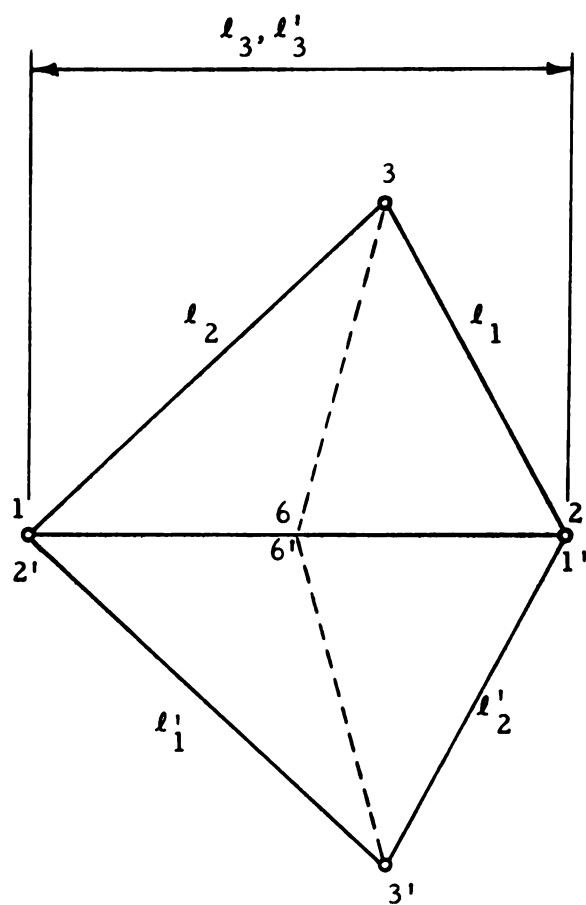


Fig. 3.14. Notation for two adjacent triangles.

This can be satisfied in general only if

$$\frac{l_2^2 - l_1^2}{2A} = 0$$

$$\frac{(l_2')^2 - (l_1')^2}{2A} = 0 \quad (3.98)$$

or simply if

$$l_1 = l_2$$

$$l_1' = l_2' \quad (3.99)$$

If the intersection angle is not very large then $\lambda_{12} \approx -\lambda_{12}'$ and (3.96) will be approximately applicable. If it is large, as for example at a 90° corner, then normal slope continuity can be insured for all points of the intersection line only if (3.99) is satisfied.

As already implied, the normal slope discontinuities along side l_1, l_2, l_3 are respectively related only to the member generalized coordinates $\lambda_{10}, \lambda_{11}, \lambda_{12}$. These can in turn be interpreted as being related to curvature changes along their respective lines. Consequently large discontinuities can occur along a line only if large curvature gradients are present there and then only if geometric properties discussed above admit to it.

For a given triangulation of a shell the slope discontinuities and curvature gradients associated with a solution can be obtained directly. From this information a better triangulation can be obtained by satisfying or more closely satisfying conditions (3.96) and/or (3.99) between members with large curvature gradients.

We classify the bending displacement configurations into those associated with $\lambda_7, \lambda_8, \lambda_9$ which we call the primary bending modes and those associated with $\lambda_{10}, \lambda_{11}, \lambda_{12}$ which we call the secondary bending modes. As already indicated, adjacent member slope discontinuities are associated only with secondary bending modes.

We now examine the normal slope continuity between adjacent members with respect to a limiting process that causes the area of the triangle to vanish without materially disturbing the ratio of its side lengths.

We assume that the deformed intersection line forms part of a regular curve and in the above indicated limit approaches a straight line. There is no ad hoc reason why this should be the case and to prove such an assertion would undoubtedly require several sufficiency conditions on the limiting process together with a proof demonstrating that it is an inherent requirement for minimizing the potential energy. For this the explicit representation of the bending stiffness matrix should prove of value. In any event this assumption leads to a useful speculation.

Consider the displacements along the intersection line of two adjacent members relative to the line joining the common node points. Since the components of displacements tangent to this line do not affect normal slope continuity we need not consider them. The displacements normal to this line can be decomposed into two orthogonal components which, as has already been indicated are identically related to associated node rotation components so that we need only consider one of these components.

Consider a curve described by the function $f(x)$ in the interval $(x, x + \Delta x)$ as shown in Fig. 3.15a. We approximate the curve by summing the two functions associated with the parameters δ_1, δ_2 and graphically shown in Fig. 3.15b, c. The approximating function and its first derivative are required to be identical at the end points of the interval. The functions associated with δ_1 and δ_2 respectively vary relative to line AB (Fig. 3.15a) in a manner identical to the normal displacements along an edge of the primary and secondary bending modes. For the alternate bending modes given in Appendix D this is equivalent to a Hermite interpolation.

From the Taylor series expansion we obtain

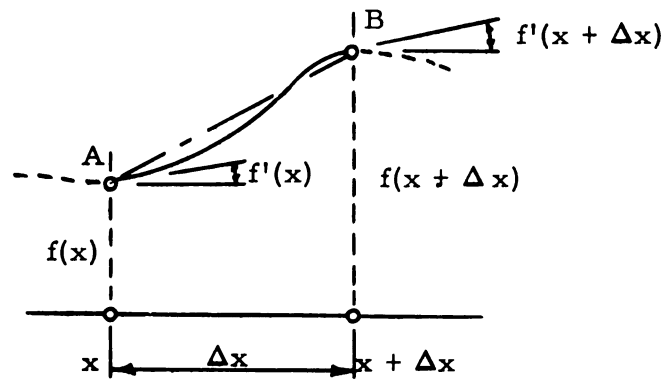
$$\begin{aligned} f(x + \Delta x) &= f(x) + f'(x) \Delta x + f''(x) \frac{\Delta x^2}{2!} + \dots \\ f'(x + \Delta x) &= f'(x) + f''(x) \Delta x + f'''(x) \frac{\Delta x^2}{2!} + \dots \end{aligned} \quad (3.100)$$

where primes denote derivatives. From Fig. 3.15 we obtain

$$\begin{aligned} f'(x) &= a - \delta_1 + \delta_2 \\ f'(x + \Delta x) &= a + \delta_1 + \delta_2 \\ a &= \frac{f(x + \Delta x) - f(x)}{\Delta x} \end{aligned} \quad (3.101)$$

Solving for δ_1 and δ_2 we obtain

$$\begin{aligned} \delta_1 &= \frac{1}{2} [f'(x + \Delta x) - f'(x)] \\ \delta_2 &= \frac{1}{2} [f'(x + \Delta x) + f'(x)] - \frac{f(x + \Delta x) - f(x)}{\Delta x} \end{aligned} \quad (3.102)$$



(a)

≈

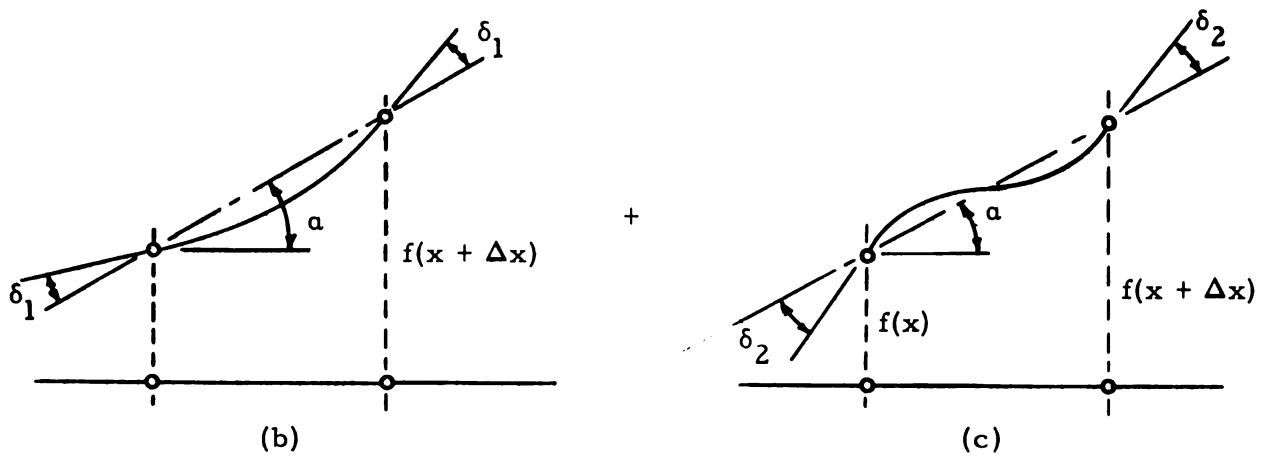


Fig. 3.15. Type of approximation implicit in bending nodes.

Substituting (3.100) into (3.102) and simplifying, we obtain

$$\begin{aligned}\delta_1 &= f''(x)\Delta x \left(\frac{1}{2}\right) + f'''(x)\Delta x^2 \left(\frac{1}{2} \cdot \frac{1}{2!}\right) + f''''(x)\Delta x^3 \left(\frac{1}{2} \cdot \frac{1}{3!}\right) + \dots \\ \delta_2 &= f'''(x)\Delta x^2 \left(\frac{1}{2 \cdot 2!} - \frac{1}{3!}\right) + f''''(x)\Delta x^3 \left(\frac{1}{2} \cdot \frac{1}{3!} - \frac{1}{4!}\right) + \dots\end{aligned}\tag{3.103}$$

As Δx approaches zero

$$\begin{aligned}\delta_1 &\approx f''(x) \frac{\Delta x}{2} \\ \delta_2 &\approx f'''(x) \frac{\Delta x^2}{12}\end{aligned}\tag{3.104}$$

Therefore, when Δx is small, δ_2 is small compared with δ_1 . In its relationship to the limiting process of the triangular element, Δx corresponds to one of the side lengths of the triangle.

From (3.104) it would appear that in the limit the primary bending modes will dominate the behavior, and that the secondary modes and consequently also the slope discontinuities become higher order effects.

We note that for the alternate bending modes given in Appendix D the normal slope discontinuities are associated with cubic terms whereas the highest terms in the primary bending nodes are quadratic.

IV. FORMULATION FOR SHELLS OF REVOLUTION

4.1. Interpretation of Problem in Terms of Discrete Elements

The method presented in Chapters II and III is used to formulate the problem of large deflections of shells of revolution having a small imperfection. The undeformed shell geometry is thus describable by the meridian curve of the middle surface (Fig. 4.1), a function describing the thickness, here assumed to vary only along the meridian, and a function describing the imperfection described later in the Chapter.

We limit the solution to one describable by a half period strip. The strip is in turn described by a simplified model consisting of 48 triangular members (see Fig. 4.2). The pattern used consists of 12 rows of four triangles. In order to accommodate a given geometry the triangles vary in size and shape from one row to the next, but the undeformed triangles of a given row are identical. In the computer program used the location of node points along the meridian can be arbitrarily specified. If one end of the shell is closed the triangles in that end row degenerate into lines and are consequently removed.

In the procedure used it is necessary to invert a matrix of rank equal to the number of unknown generalized coordinates for each linear increment. If the generalized coordinates are interpreted as components of node displacement and rotation vectors, and if the pattern of triangles is as indicated above (Fig. 4.2) then the rank of the matrix is between 121 and 167 depending on the boundary conditions used. If for a given elastic system and loading

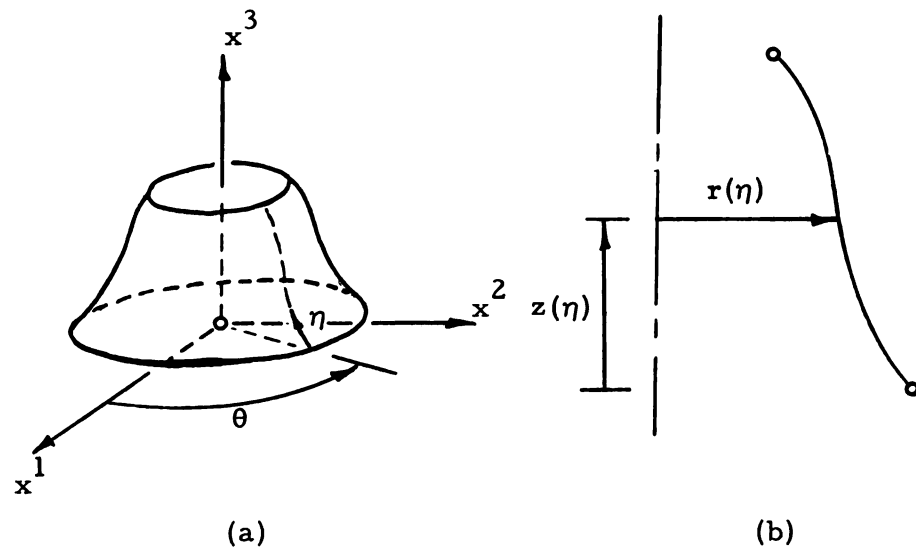


Fig. 4.1. Shell geometry.

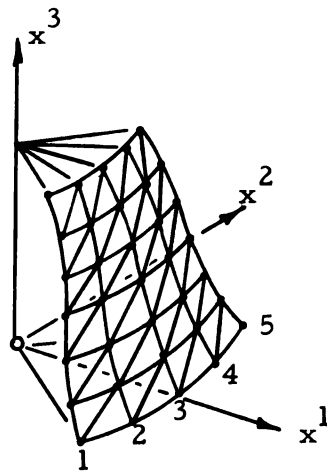


Fig. 4.2. Triangulation.

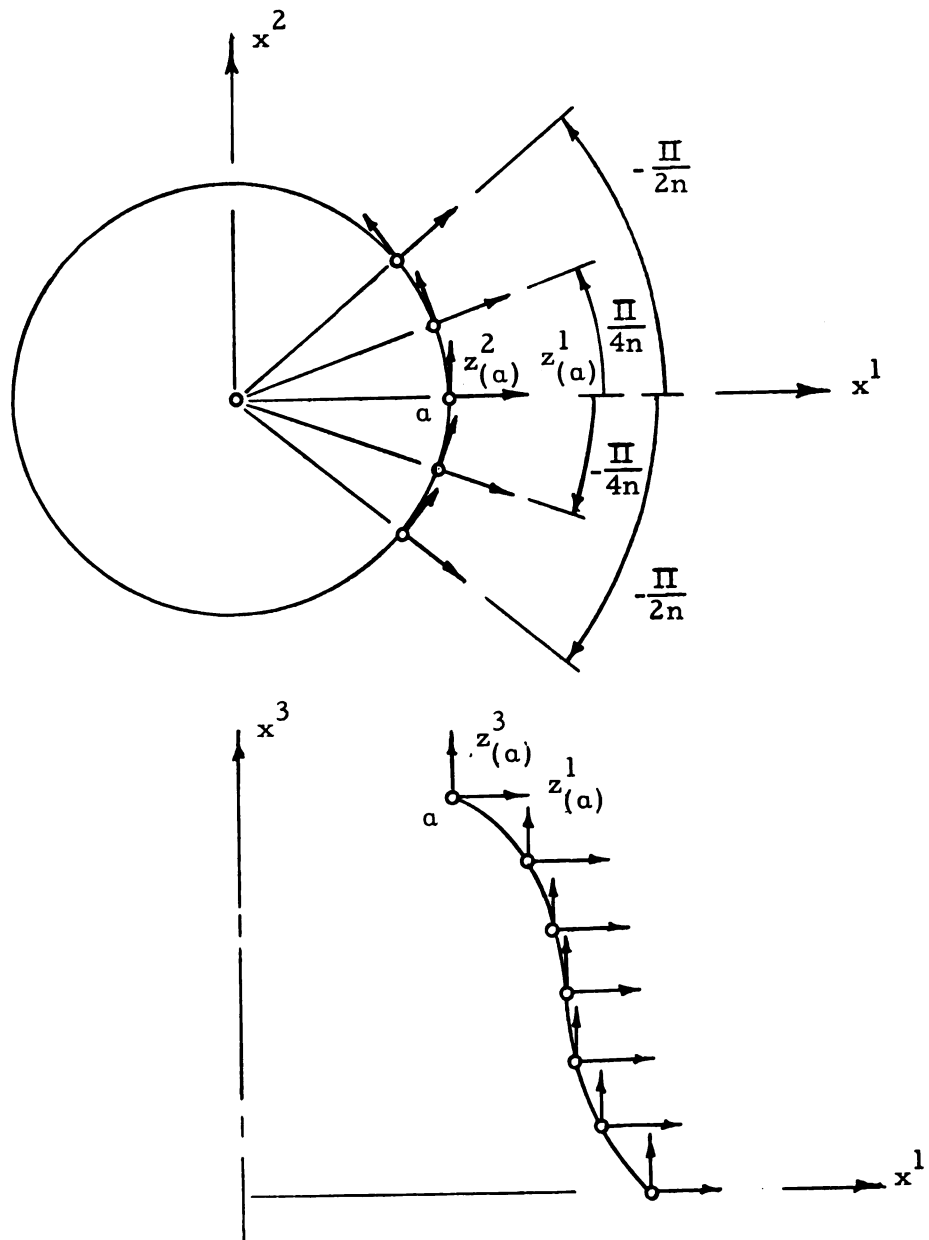


Fig. 4.3. Node coordinates.

the incremented solution does not undergo a region of instability, then it may be possible to obtain acceptable results by using only two to five linear increments; however, if the solution does undergo an unstable region then it is unlikely that acceptable results can be obtained with less than 20 to 30 linear increments. A pattern consisting of a much smaller number of triangles cannot adequately describe the elastic properties of the class of structures being considered.

To deal with problems having a region of instability would require an excessive amount of computer time (approximately 10 times the amount used) especially for the computer program development which required a considerable amount of testing. For this reason, the degree of freedom of the system is reduced by interpreting the node displacements and rotations in terms of a smaller set of unknown parameters. The explicit form of the interpretation (equations of constraint) is presented later in this chapter.

4.2. Coordinate Systems

In accordance with the discussion in Sec. 2.7 three types of right handed orthogonal coordinate systems are used, i. e., the general, node, and member coordinate systems. The general and node coordinate systems are fixed, and the member coordinate are redefined after each linear increment. The 0-general coordinates x^1, x^2, x^3 are defined (see Fig. 4.2) so that the x^3 axis lies on the axis of the shell and is directed upward, the x^1 axis is directed radially

outward and intersects the shell axis and bottom middle node point, and the x^2 axis is perpendicular to the x^1 and x^3 axes. The α -node coordinates $z_{(\alpha)}^1, z_{(\alpha)}^2, z_{(\alpha)}^3$ are defined (see Fig. 4.3) so that the $z_{(\alpha)}^3$ axis is parallel to the shell axis, the $z_{(\alpha)}^1$ axis is directed outward intersecting the shell axis and the α -node point, and the $z_{(\alpha)}^2$ axis is perpendicular to the $z_{(\alpha)}^1$ and $z_{(\alpha)}^3$ axes. Note that the node coordinate systems of node points on a meridian curve have their axes directed in identical directions. The member coordinate systems are defined in accordance with Sec. 2.7.

4.3. Reduced Set of Generalized Displacement Parameters

For convenience each node point is designated by the integers α, β (Fig. 4.4). Since the axes of node coordinates associated with a common meridian curve are parallel, we distinguish them by using only the first integer of the associated node point. The following node displacement and rotation components are with respect to the node coordinate systems.

$$\Delta u_{(\alpha\beta, \alpha)}^1, \Delta u_{(\alpha\beta, \alpha)}^2, \Delta u_{(\alpha\beta, \alpha)}^3, \Delta \theta_{(\alpha\beta, \alpha)}^1, \Delta \theta_{(\alpha\beta, \alpha)}^2, \Delta \theta_{(\alpha\beta, \alpha)}^3 \quad (4.1)$$

These variables are related to a smaller set of generalized coordinates as follows:

$$\begin{aligned} \Delta u_{(\alpha\beta, \alpha)}^1 &= \Delta v_{(\beta)}^1 + \Delta v_{(\beta)}^2 \sin \left[\frac{1}{2} (\alpha - 3) \pi \right] \\ \Delta u_{(\alpha\beta, \alpha)}^2 &= \Delta v_{(\beta)}^3 \cos \left[\frac{1}{2} (\alpha - 3) \pi \right] \\ \Delta u_{(\alpha\beta, \alpha)}^3 &= \Delta v_{(\beta)}^4 + \Delta v_{(\beta)}^5 \sin \left[\frac{1}{2} (\alpha - 3) \pi \right] \end{aligned}$$

$$\begin{aligned}
\Delta \theta_{(\alpha\beta, \alpha)}^1 &= \Delta v_{(\beta)}^6 \cos \left[\frac{1}{2} (\alpha-3) \pi \right] \\
\Delta \theta_{(\alpha\beta, \alpha)}^2 &= \Delta v_{(\beta)}^7 + \Delta v_{(\beta)}^8 \sin \left[\frac{1}{2} (\alpha-3) \pi \right] \\
\Delta \theta_{(\alpha\beta, \alpha)}^3 &= \Delta v_{(\beta)}^9 \cos \left[\frac{1}{2} (\alpha-3) \pi \right]
\end{aligned} \tag{4.2}$$

By using (4.2) we in effect relate the 30 discrete variables of the five node points of a row to nine variables. This in the discrete sense implied by (4.2) constrains the strip to a sinusoidal type of variation. The boundary conditions along the two meridian edges are automatically satisfied when (4.2) is used.

We designate by

$$\Delta U_{(\alpha\beta, \alpha)}^T = [\Delta u_{(\alpha\beta, \alpha)}^1, \Delta u_{(\alpha\beta, \alpha)}^2, \Delta u_{(\alpha\beta, \alpha)}^3, \Delta \theta_{(\alpha\beta, \alpha)}^1, \Delta \theta_{(\alpha\beta, \alpha)}^2, \Delta \theta_{(\alpha\beta, \alpha)}^3] \tag{4.3}$$

$$\Delta v_{(\beta)}^T = [\Delta v_{(\beta)}^1, \Delta v_{(\beta)}^2, \Delta v_{(\beta)}^3, \Delta v_{(\beta)}^4, \dots, \Delta v_{(\beta)}^9] \tag{4.4}$$

$$C_{R(\alpha)} = \begin{bmatrix} 1 & \sin[\frac{1}{2}(\alpha-3)\pi] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cos[\frac{1}{2}(\alpha-3)\pi] & & & & & & & \\ 0 & & 1 & \sin[\frac{1}{2}(\alpha-3)\pi] & & & & & \\ 0 & & & \cos[\frac{1}{2}(\alpha-3)\pi] & & & & & \\ 0 & & & & 1 & \sin[\frac{1}{2}(\alpha-3)\pi] & & & \\ 0 & & & & & \cos[\frac{1}{2}(\alpha-3)\pi] & & & \\ 0 & & & & & & 1 & \sin[\frac{1}{2}(\alpha-3)\pi] & \\ & & & & & & & \cos[\frac{1}{2}(\alpha-3)\pi] \end{bmatrix} \tag{4.5}$$

Then (4.2) can be represented in the form

$$\Delta U_{(\alpha\beta, \alpha)} = C_{R(\alpha)} \Delta v_{(\beta)} \tag{4.6}$$

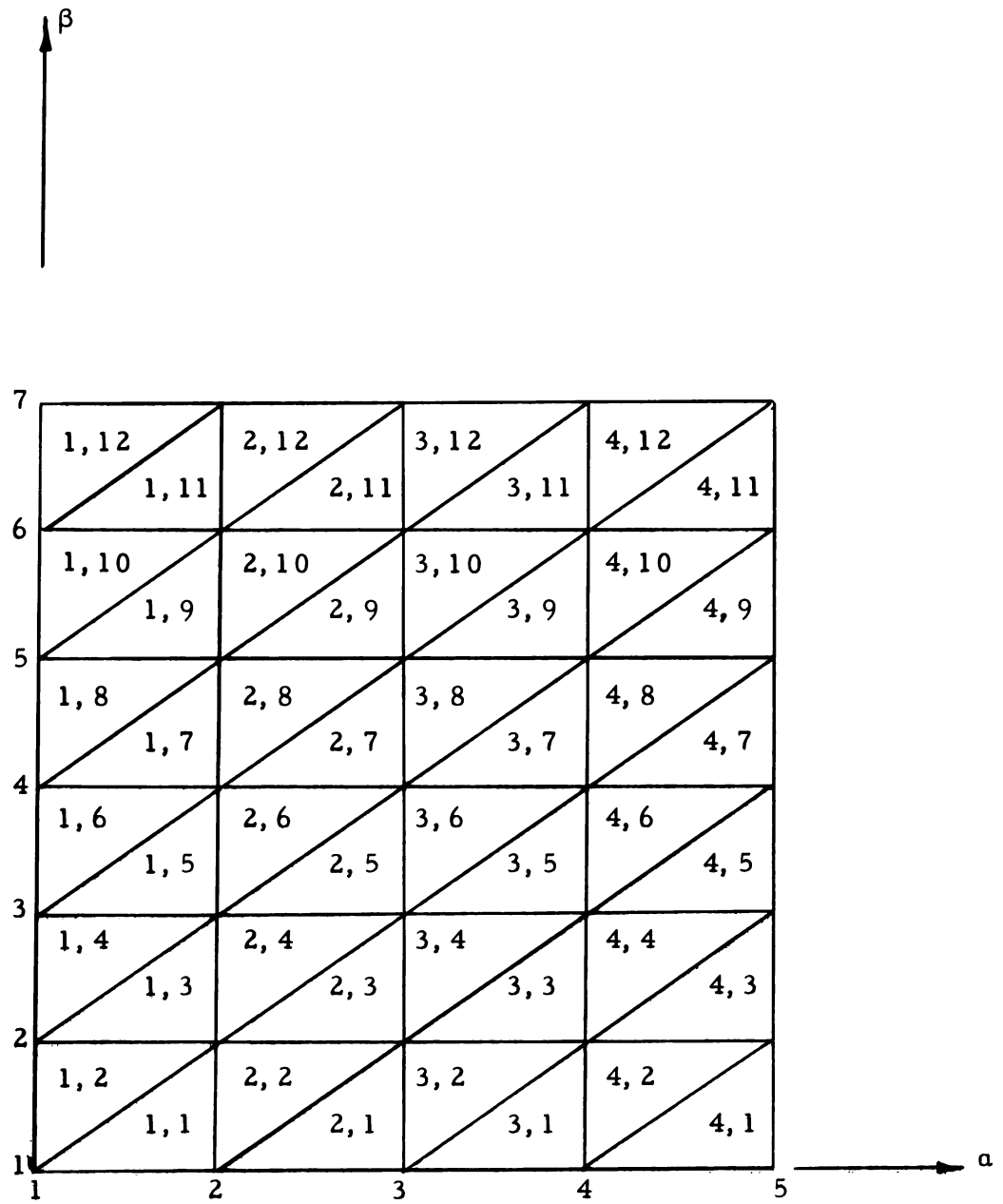


Fig. 4.4. Ordering of triangular members.

Let

$$\Delta U^T = [\Delta U_{(11,1)}^T, \Delta U_{(21,2)}^T, \dots, \Delta U_{(51,5)}^T, \Delta U_{(12,1)}^T, \dots, \Delta U_{(57,5)}^T] \quad (4.7)$$

$$\Delta V^T = [\Delta V_{(1)}^T, \dots, \Delta V_{(7)}^T] \quad (4.8)$$

The generalized forces associated with $\Delta U_{(\alpha\beta, \alpha)}$, $\Delta V_{(\beta)}$, ΔU , and ΔV are respectively designated by $\Delta F_{(\alpha\beta, \alpha)}$, $\Delta P_{(\beta)}$, ΔF , and ΔP . The elements of $\Delta P_{(\beta)}$ are designated by

$$\Delta P_{(\beta)}^T = [\Delta p_{(\beta)}^1, \Delta p_{(\beta)}^2, \dots, \Delta p_{(\beta)}^7] \quad (4.9)$$

Let C_R be defined so that

$$\Delta U = C_R \Delta V \quad (4.10)$$

Then from (4.6) and (4.8) it follows that C_R has the form

$$C_R = \begin{bmatrix} C_{R_{1,1}} & 0 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_{R_{5,1}} & 0 & \dots & 0 \\ 0 & C_{R_{6,2}} & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & C_{R_{12,2}} & \dots & 0 \\ \cdot & \cdot & \dots & 0 \\ 0 & 0 & \dots & C_{R_{31,7}} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & C_{R_{35,7}} \end{bmatrix} = \begin{bmatrix} C_{R_{(1)}} & 0 & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_{R_{(5)}} & 0 & \dots & 0 \\ 0 & C_{R_{(1)}} & \dots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & C_{R_{(5)}} & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & C_{R_{(1)}} \\ \cdot & \cdot & \dots & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \dots & C_{R_{(5)}} \end{bmatrix} \quad (4.12)$$

In a manner similar to that used to derive (2.90) the generalized forces ΔF and ΔP are related by

$$\Delta P = C_R^T \Delta F \quad (4.13)$$

The generalized forces $\Delta P_{(\beta)}$ do not have as simple an interpretation as the node forces and moments. Many boundary conditions can, however, be interpreted directly in terms of the generalized parameters associated with the two end rows. We note that $p_{(7)}^4$ corresponds to the vertical component of the resultant force acting on the top edge of the half period strip and $p_{(1)}^4$ is the corresponding component for the lower edge.

4.4. Node Coordinate Transformation Matrix

If the shell has n periods, the angle enclosed by the half period strip is $\frac{\pi}{n}$. The matrices of base vectors of the general coordinate system are designated by $J_{(0)}$ and those of the α, β -node coordinate system are designated by $J_{(\alpha, \beta)}$ or simply by $J_{(\alpha)}$ since the base vectors of node points on a common meridian are identical. The base vectors are related by

$$J_{(0)} = D_{1(0)}^{(\alpha)} J_{(\alpha)} \quad (4.14)$$

The coefficient matrix in (4.14) can be obtained directly from Fig.'s 4.3 and 4.4 and is given below.

$$D_{1(0)}^{(a)} = \begin{bmatrix} \cos[(a-3)\frac{\pi}{4n}] & -\sin[(a-3)\frac{\pi}{4n}] & 0 \\ \sin[(a-3)\frac{\pi}{4n}] & \cos[(a-3)\frac{\pi}{4n}] & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.15)$$

In a manner similar to (2.100) we defined

$$D_{(0)}^{(a)} = \begin{bmatrix} D_{1(0)}^{(a)} & 0 \\ 0 & D_{1(0)}^{(a)} \end{bmatrix} \quad (4.16)$$

4.5. Formulation of Shell Stiffness Matrix

Since two integers are used to designate a node point, matrix S defined in (2.63) and (2.65) becomes a 48×6 matrix for this formulation and is given in Table 4.1.

We designate the triangular member by integer pairs i, j as indicated in Fig. 4.4. The elements of S are designated by s_{kl} , where the k^{th} row of S defines the node points of triangle k which in turn is ordered by

$$k = i + 4(j-1) \quad (4.17)$$

If (a_1, β_1) , (a_2, β_2) , and (a_3, β_3) are the node points of member i, j then

$$a_1 = s_{k1}$$

$$\beta_1 = s_{k2}$$

$$a_2 = s_{k3}$$

$$\beta_2 = s_{k4}$$

Table 4.1. S Matrix.

2	1	2	2	1	1
3	1	3	2	2	1
4	1	4	2	3	1
5	1	5	2	4	1
1	2	1	1	2	2
2	2	2	1	3	2
3	2	3	1	4	2
4	2	4	1	5	2
2	2	2	3	1	2
3	2	3	3	2	2
4	2	4	3	3	2
5	2	5	3	4	2
1	3	1	2	2	3
2	3	2	2	3	3
3	3	3	2	4	3
4	3	4	2	5	3
2	3	2	4	1	3
3	3	3	4	2	3
4	3	4	4	3	3
5	3	5	4	4	3
1	4	1	3	2	4
2	4	2	3	3	4
2	4	3	3	4	4
4	4	4	3	5	4
2	4	2	5	1	4
3	4	3	5	2	4
4	4	4	5	3	4
5	4	5	5	4	4
1	5	1	4	2	5
2	5	2	4	3	5
3	5	3	4	4	5
4	5	4	4	5	5
2	5	2	6	1	5
3	5	3	6	2	5
4	5	4	6	3	5
5	5	5	6	4	5
1	6	1	5	2	6
2	6	2	5	3	6
3	6	3	5	4	6
4	6	4	5	5	6
2	6	2	7	1	6
3	6	3	7	2	6
4	6	4	7	3	6
5	6	5	7	4	7
1	7	1	6	2	7
2	7	2	6	3	7
3	7	3	6	4	7
4	7	4	6	5	7

$$\alpha_3 = s_{k6}$$

$$\beta_3 = s_{k7}$$

(4.18)

As can be observed from Table 4.1

$$\beta_1 = \beta_3$$

$$\beta_1 > \beta_2 \quad \text{if } i \text{ is even}$$

$$\beta_1 < \beta_2 \quad \text{if } i \text{ is odd}$$

(4.19)

Substituting (2.102) into (2.109) and using the notation of this section we obtain

$$\Delta\Lambda_k = [M_{k_1} D_{(k)}^{(0)} D_{(0)}^{(\alpha_1)} | M_{k_2} D_{(k)}^{(0)} D_{(0)}^{(\alpha_2)} | M_{k_3} D_{(k)}^{(0)} D_{(0)}^{(\alpha_3)}]$$

$$\begin{bmatrix} \Delta U_{(\alpha_1 \beta_1, \alpha_1)} \\ \Delta U_{(\alpha_2 \beta_2, \alpha_2)} \\ \Delta U_{(\alpha_3 \beta_3, \alpha_3)} \end{bmatrix}$$

(4.20)

From (4.6) and (4.19) we obtain

$$\begin{bmatrix} \Delta U_{(\alpha_1 \beta_1, \alpha_1)} \\ \Delta U_{(\alpha_2 \beta_2, \alpha_2)} \\ \Delta U_{(\alpha_3 \beta_3, \alpha_3)} \end{bmatrix} = \begin{bmatrix} C_{R(\alpha_1)} & 0 \\ 0 & C_{R(\alpha_2)} \\ C_{R(\alpha_3)} & 0 \end{bmatrix} \begin{bmatrix} \Delta V_{(\beta_1)} \\ \Delta V_{(\beta_2)} \end{bmatrix}$$

(4.21)

Substituting (4.21) into (4.20) we obtain

$$\Delta \Lambda_k = \left[M_{k_1} D_{(k)}^{(0)} D_{(0)}^{(a_1)} C_{R(a_1)} + M_{k_3} D_{(k)}^{(0)} D_{(0)}^{(a_3)} C_{R(a_3)} \right] \left[\begin{array}{c} M_{k_2} D_{(k)}^{(0)} D_{(0)}^{(a_2)} C_{R(a_2)} \\ \Delta V_{(\beta_1)} \\ \Delta V_{(\beta_2)} \end{array} \right] \quad (4.22)$$

This equation has the form

$$\Delta \Lambda_k = \left[M_{V_{k1}} \quad M_{V_{k2}} \right] \left[\begin{array}{c} \Delta V_{(\beta_1)} \\ \Delta V_{(\beta_2)} \end{array} \right] \quad (4.23)$$

The shell stiffness associated with triangular member k is then given by

$$K_k = \left[\begin{array}{c} M_{V_{k1}}^T \\ M_{V_{k2}}^T \end{array} \right] \Gamma_k \left[\begin{array}{c|c} M_{V_{k1}} & M_{V_{k2}} \end{array} \right] = M_{V_k}^T \Gamma_k M_{V_k} \quad \text{if } j \text{ is odd}$$

$$K_k = \left[\begin{array}{c} M_{V_{k2}}^T \\ M_{V_{k1}}^T \end{array} \right] \Gamma_k \left[\begin{array}{c|c} M_{V_{k2}} & M_{V_{k1}} \end{array} \right] = M_{V_k}^T \Gamma_k M_{V_k} \quad \text{if } j \text{ is even}$$

(4.24)

We partition the stiffness matrix in (4.24) into 9x9 submatrices in the form

$$K_k = \left[\begin{array}{cc} K_{k,1,1} & K_{k,1,2} \\ K_{k,2,1} & K_{k,2,2} \end{array} \right] \quad (4.25)$$

$$r_{(\beta)} = r_{(1)}, r_{(2)}, \dots, r_{(7)} \quad (4.29)$$

$$z_{(\beta)} = z_{(1)}, z_{(2)}, \dots, z_{(7)}$$

The components of node points relative to the 0-general coordinates are then given by

$$\begin{aligned} x_{(a, \beta)}^1 &= r_{(\beta)} \cos \left[(a-3) \frac{\pi}{4n} \right] \\ x_{(a, \beta)}^2 &= r_{(\beta)} \sin \left[(a-3) \frac{\pi}{4n} \right] \\ x_{(a, \beta)}^3 &= z_{(\beta)} \end{aligned} \quad (4.30)$$

The shell surface is assumed to have a slight imperfection with n circumferential periods. The form of the imperfection is taken so that the node components of the position vector of the undeformed surface have the form

$$\begin{aligned} x_{(a, \beta)}^1 &= (r_{(\beta)} + \epsilon g_{(\beta)} \sin \left[(a-3) \frac{\pi}{4} \right]) \cos \left[(a-3) \frac{\pi}{4n} \right] \\ x_{(a, \beta)}^2 &= (r_{(\beta)} + \epsilon g_{(\beta)} \sin \left[(a-3) \frac{\pi}{4} \right]) \sin \left[(a-3) \frac{\pi}{4n} \right] \\ x_{(a, \beta)}^3 &= z_{(\beta)} + \epsilon g_{(\beta)} \sin \left[(a-3) \frac{\pi}{4} \right] \end{aligned} \quad (4.31)$$

where

$$g_{(\beta)} = g_{(1)}, \dots, g_7 \quad (4.32)$$

called the imperfection function and ϵ called the imperfection constant are specified.

4.7. Boundary Conditions and Loading

The generalized displacement parameters of $\Delta V_{(1)}$ and $\Delta V_{(7)}$ describe the deformation respectively along the top and bottom edges of the half period strip. In order to interpret the various types of boundary conditions we give the physical interpretation of these parameters below and a graphic description in Fig. 4.5.

$v_{(1)}^1$	= uniform	radial displacement
$v_{(1)}^2$	= sin	varying radial displacement
$v_{(1)}^3$	= cos	varying tangential displacement
$v_{(1)}^4$	= uniform	vertical displacement
$v_{(1)}^5$	= sin	varying vertical displacement
$v_{(1)}^6$	= cos	varying radial rotation
$v_{(1)}^7$	= uniform	tangential rotation
$v_{(1)}^8$	= sin	varying tangential rotation
$v_{(1)}^9$	= cos	varying vertical rotation

(4.33)

A similar interpretation holds for the elements of ΔV_7 .

In order to tie down the shell we always required

$$v_{(1)}^4 = 0 \quad (4.34)$$

Four types of boundary conditions are considered. These boundary conditions and their requirements on $\Delta V_{(1)}$ along the bottom edge are as follows:

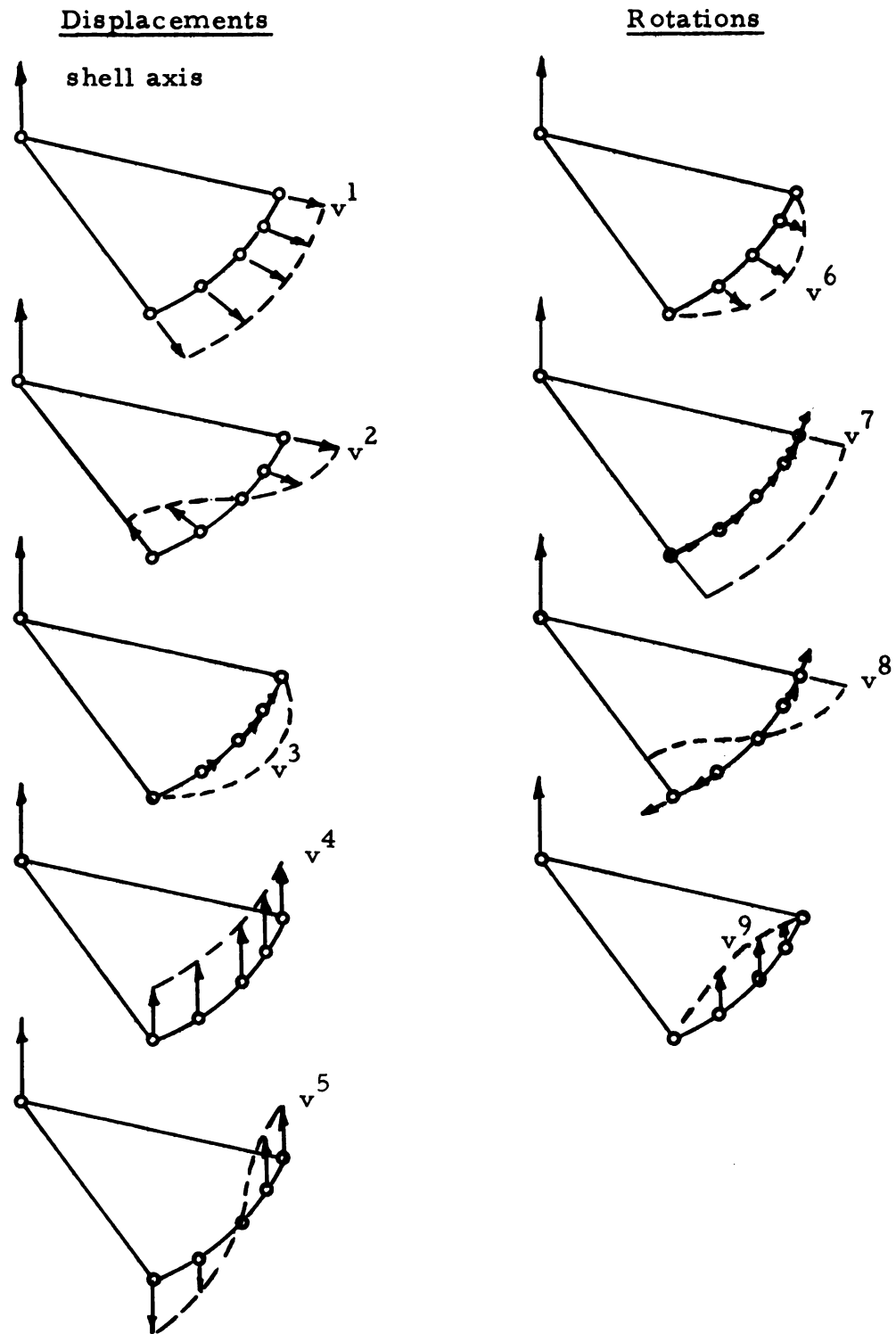


Fig. 4.5. Distributions associated with generalized coordinates v^i .

Free

$$v_{(1)}^4 = 0$$

Hinged

$$v_{(1)}^1 = v_{(1)}^2 = v_{(1)}^3 = v_{(1)}^4 = v_{(1)}^5 = v_{(1)}^6 = v_{(1)}^9 = 0$$

Fixed

$$v_{(1)}^1 = v_{(1)}^2 = v_{(1)}^3 = v_{(1)}^4 = v_{(1)}^5 = v_{(1)}^6 = v_{(1)}^7 = v_{(1)}^8 = v_{(1)}^9 = 0$$

Symmetry

$$v_{(1)}^4 = v_{(1)}^5 = v_{(1)}^6 = v_{(1)}^7 = v_{(1)}^8 = 0$$

(4.35)

By symmetry boundary conditions we imply that the bottom edge lies on a plane of symmetry with respect to the resulting deformations.

The type of load used is limited to an axial load. For convenience we use axial displacement increments so that $v_{(7)}^4$ is always specified. The boundary conditions for the top edge are then as follows:

Free

no zero components

Hinged

$$v_{(7)}^1 = v_{(7)}^2 = v_{(7)}^3 = v_{(7)}^5 = v_{(7)}^6 = v_{(7)}^9 = 0$$

Fixed

$$v_{(7)}^1 = v_{(7)}^2 = v_{(7)}^3 = v_{(7)}^5 = v_{(7)}^6 = v_{(7)}^7 = v_{(7)}^8 = v_{(7)}^9 = 0$$

Symmetry

$$v_{(7)}^5 = v_{(7)}^6 = v_{(7)}^7 = v_{(7)}^8 = 0$$

(4.36)

The axial displacement increment is taken in the form

$$\Delta v_{(7)}^4 = \rho(\tau_k) \delta \quad (4.37)$$

where δ is a specified constant called the displacement increment constant, and $\rho(\tau_k)$ is in general assigned a different value for each linear increment and called the displacement increment function.

4.8. Computer Program

A computer program was written for the class of problems with a geometric configuration, boundary conditions, and triangulation as described in this Chapter. It permits a fixed normal surface load and an incremented average axial displacement of the top edge. It is used to obtain resultant axial force versus average axial displacement curves, and displacement curves.

The program was written in 3600 Fortran source language for the Control Data 3600 computer at Michigan State University. This language contains the features of Fortran-63.

The program was written so that a minimum of input data is needed. This data consists of fifteen computer cards describing the geometry, material properties, boundary conditions, surface load, and magnitude of axial displacement increment.

Computer program details are given in Appendix E.

V. COMPUTATIONAL RESULTS AND CONCLUSIONS

5.1. Some General Remarks

This investigation was primarily directed at the development and explicit representations of arbitrary triangular member bending stiffness matrices, and to determining their applicability for geometrically linear and non-linear plate and shell problems.

Two bending stiffness matrices were obtained. As already indicated, the displacement modes (3.44) for one of them satisfies compatibility to a high degree (see Sec. 3.7) whereas the other (D.2) relaxes slope compatibility along the edge of adjacent members but appears to give better numerical results. The importance of the coupling between membrane and bending behaviors, particularly in geometrically non-linear problems, is of course contained in the shell formulation and reflected in some of the numerical results, but membrane behavior was not a major point of consideration.

Very good comparative results were obtained; however, the numerical results are in no way adequate for the purpose of drawing any general conclusions. For this purpose the solution of problems having regions with various forms of singular behavior and the numerical examination of convergence associated with increased refinement would be very useful.

Most of the numerical results were obtained from the computer program given in Appendix E and based on the formulation given in Chapter IV. The program performs almost all interpretive and

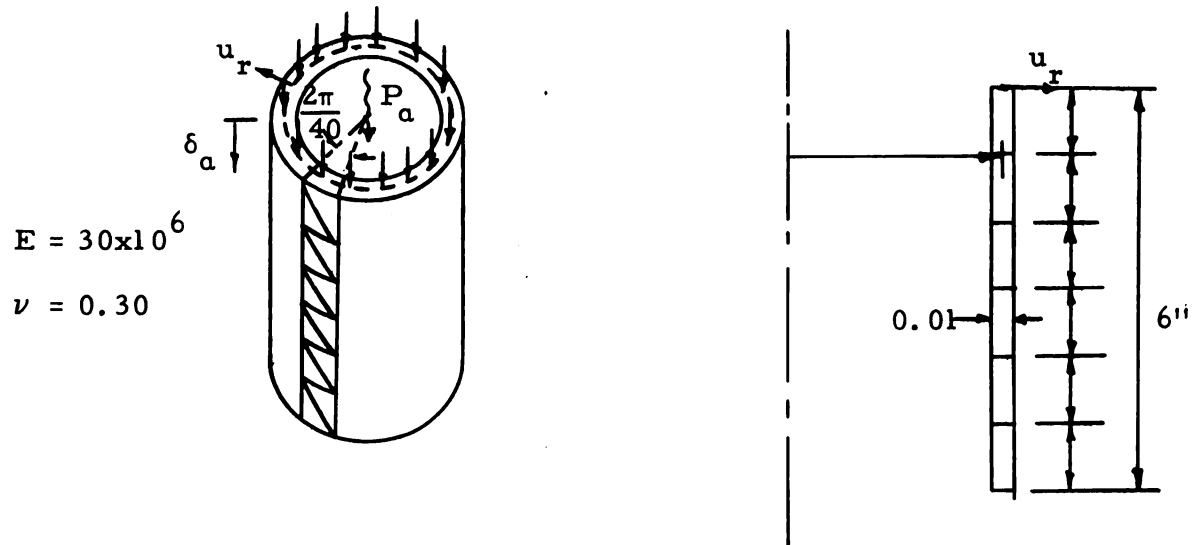
computational operations internally from a relatively small amount of input data necessary for the description of the problem and for both linear and non-linear problems.

5.2. Linear Results

In order to obtain an approximate estimate of the accumulative roundoff error due to all sources in the sequence of computations, an axially loaded perfect cylinder (Fig. 5.1) was solved. The problem results in a uniform membrane stress state which the modeling can describe exactly and yield results that are exact to within computational errors. The results are given in Table 5.1. The accumulative effect in non-linear problems is contained in these results since 25 increments were performed.

Although the cylinder is initially perfect the accumulated round-off error results in a fictitious imperfection. This appears suddenly in the 20th increment and accounts for part of the reduction in the 20th and 25th increments shown in Table 5.1. The imperfection results in both circumferential and longitudinal waves. The longitudinal waves are, however, too long, due to the node distribution, and consequently the sharp reduction in axial load associated with the bifurcation phenomenon is not revealed.

The displacements along a radial line for the linear problem of a concentric plate³ with fixed boundary conditions and axial load are given in Fig. 5.2 and Table 5.2. Results were obtained for both bending stiffness matrices. Both matrices gave good results; however, as already indicated, the matrix given in Appendix D gave particularly



δ_a $\times 10^{-2}$	$P_a (\times 10^3)$		$u_r \times 10^{-2}$	
	Exact	Computed	Exact	Computed
1	9.4248	9.389	.15	.1500
2	18.8495	18.73	.30	.2994
3	28.2743	28.03	.45	.4484
5	47.1238	46.48	.75	.7455
10	94.2477	91.80	1.50	1.482
15	141.3715	136.0	2.25	2.212
20	188.4954	177.4	3.0	--
25	235.6192	212.4	3.75	--

Fig. 5.1 and Table 5.1. Comparison for a uniform stress field.

accurate results, that are in fact to within the roundoff error of the computations.

Some useful results can be obtained by introducing the constants c_1 and c_2 which we respectively call the primary and secondary bending constants. We use these constants to alter the 6x6 member bending stiffness matrix Γ_i as follows

$$\begin{bmatrix} c_1^2 \Gamma_{i_{11}} & c_1 c_2 \Gamma_{i_{12}} \\ c_1 c_2 \Gamma_{i_{21}} & c_2^2 \Gamma_{i_{22}} \end{bmatrix} \quad (5.1)$$

where the $\Gamma_{i_{kl}}$ are the 3x3 submatrices of Γ_i . If $c_1 = c_2 = 1.0$ then the matrix is unaltered. If $0 < c_1 < 1$ then the stiffness associated with the primary bending modes is partially nullified and if $0 < c_2 < 1$ then the stiffness associated with the secondary bending nodes is partially nullified.

The node displacements along a radial line for various values of c_1 and c_2 of the concentric plate shown in Fig. 5.1 are given in Table 5.2. These results clearly show the dominant role that the primary bending modes have on this solution. For $c_1 = 1.0$ and $c_2 = 0.6$ the stiffness associated with the secondary bending modes is essentially reduced by 64% but the resulting increase in the maximum displacement is only 2.2%. From the definition of the constants it follows that for $c_1 = c_2 = 0.6$ the displacement will increase by

$$\left(\frac{1}{(0.6)^2} - 1 \right) 100\% = 177\% \quad (5.2)$$

Table 5.2. Displacements Along a Radial Line for a Concentric Plate (Fig. 5.2) and Percent Error.

Radius of Node			3.0	3.5	4.0	4.5	5.0	5.5	6.0
Exact Displacement			2.852	2.597	2.011	1.307	0.6525	0.1796	0.0
Bending Matrix	c_1	c_2							
Table D.1	1.0	1.0	2.854 0.06	2.606 0.35	2.020 0.44	1.315 0.57	0.6573 0.73	0.1820 1.32	0.0
Table 3.3	1.0	1.0	2.752 3.51	2.516 3.10	1.949 3.06	1.265 3.24	0.6276 3.97	0.1694 5.68	0.0
Table 3.3	1.0	0.8	2.775 2.70	2.536 2.33	1.966 2.75	1.277 2.36	0.6344 2.77	0.1723 4.06	0.0
Table 3.3	1.0	0.6	2.814 1.34	2.569 1.45	1.992 0.97	1.295 0.92	0.6465 0.92	0.1780 0.89	0.0
Table 3.3	1.0	0.5	2.852 0.02	2.601 0.14	2.016 0.25	1.314 0.47	0.6582 0.87	0.1836 2.23	0.0
Table 3.3	1.0	0.62	2.806 1.61	2.563 1.30	1.986 1.25	1.292 1.22	0.6441 1.29	0.1768 1.56	0.0
Table 3.3	0.992	0.62	2.854 0.07	2.607 0.38	2.020 0.44	1.314 0.50	0.6552 0.41	0.1800 0.22	0.0

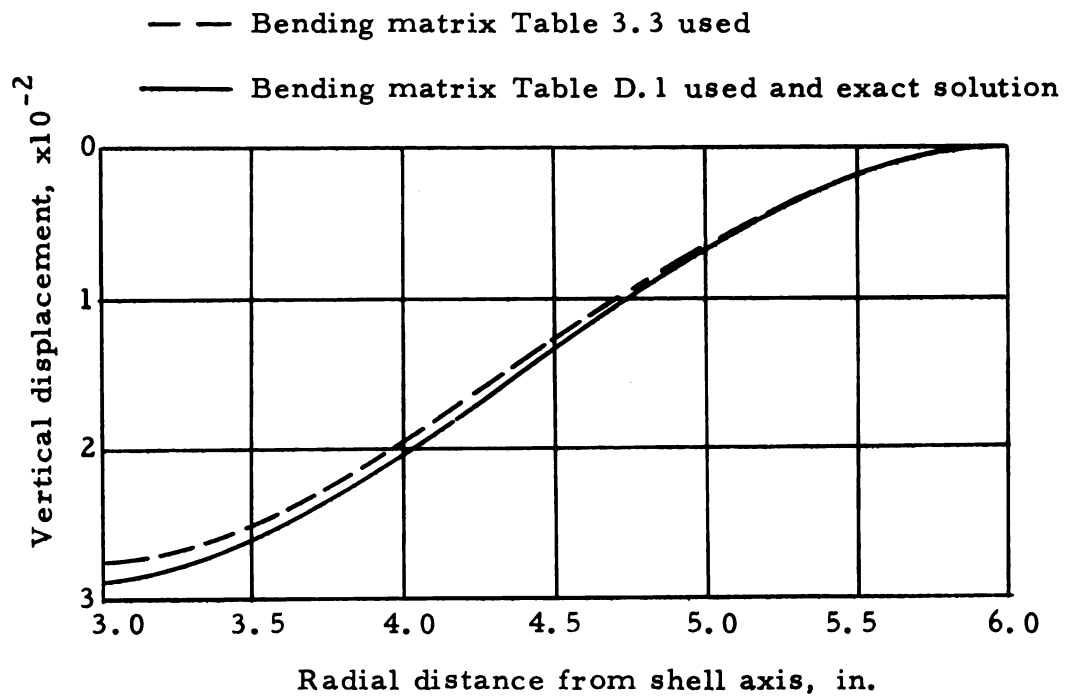
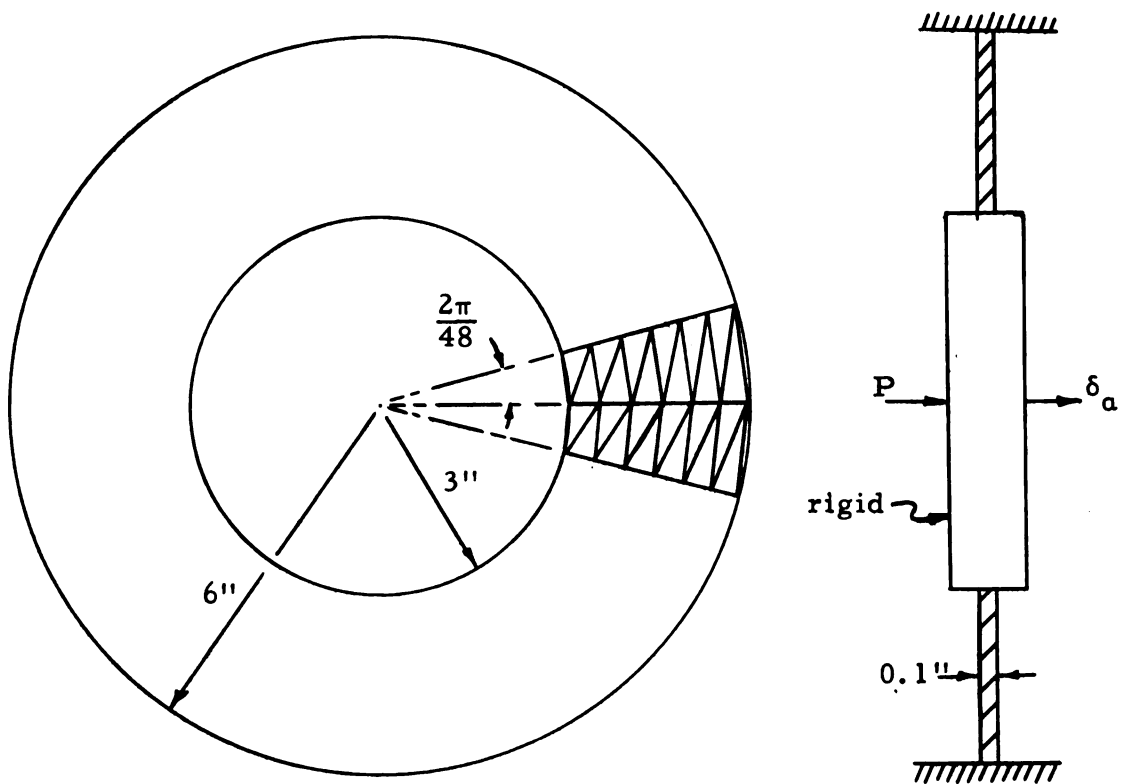


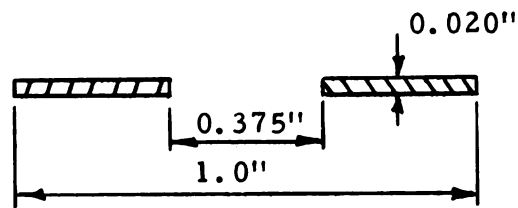
Fig. 5.2. Vertical Displacements Along a Radial Line for a Concentric Flat Plate with Fixed Boundary Conditions.

5.3. Non-Linear Results

Some numerical results for the large deflection problem of a concentric plate (Fig. 5.3) and two shallow conical shells (Fig.'s 5.4, 5.5) were obtained. These results are compared with the numerical results of Newman and Reiss³⁰ and the experimental results of Almen and Lazyle.²⁹ Free boundary conditions were assumed and the bending stiffness matrix given in Table 3.2 was used for all three problems.

The results obtained in general compare favorably. For one of the conical shells, both the numerical results of Newman and Reiss and those obtained have a noticable discrepancy with the experimental results of Almen and Lazyle. This could be due to the presence of a partial constraint (friction) on the boundary when the experiment was performed. To demonstrate this, numerical results were obtained for a variety of boundary conditions (Fig. 5.6). It can be seen from Fig. 5.7 that for hinged boundary conditions the maximum axial load just before instability is approximately 12 times the value obtained for free boundary conditions. A relatively small radial constraint could account for most of the 10% discrepancy present.

Results were obtained for the large displacement problem of the concentric plate with the bending constants $c_1 = 0.992$ and $c_2 = 0.62$. The curve for these results is not shown since there is no discernible difference between it and the curve given by Almen and Lazyle.



—— Almen and Lazyle test

- - - Present solution

- - - Linear solution

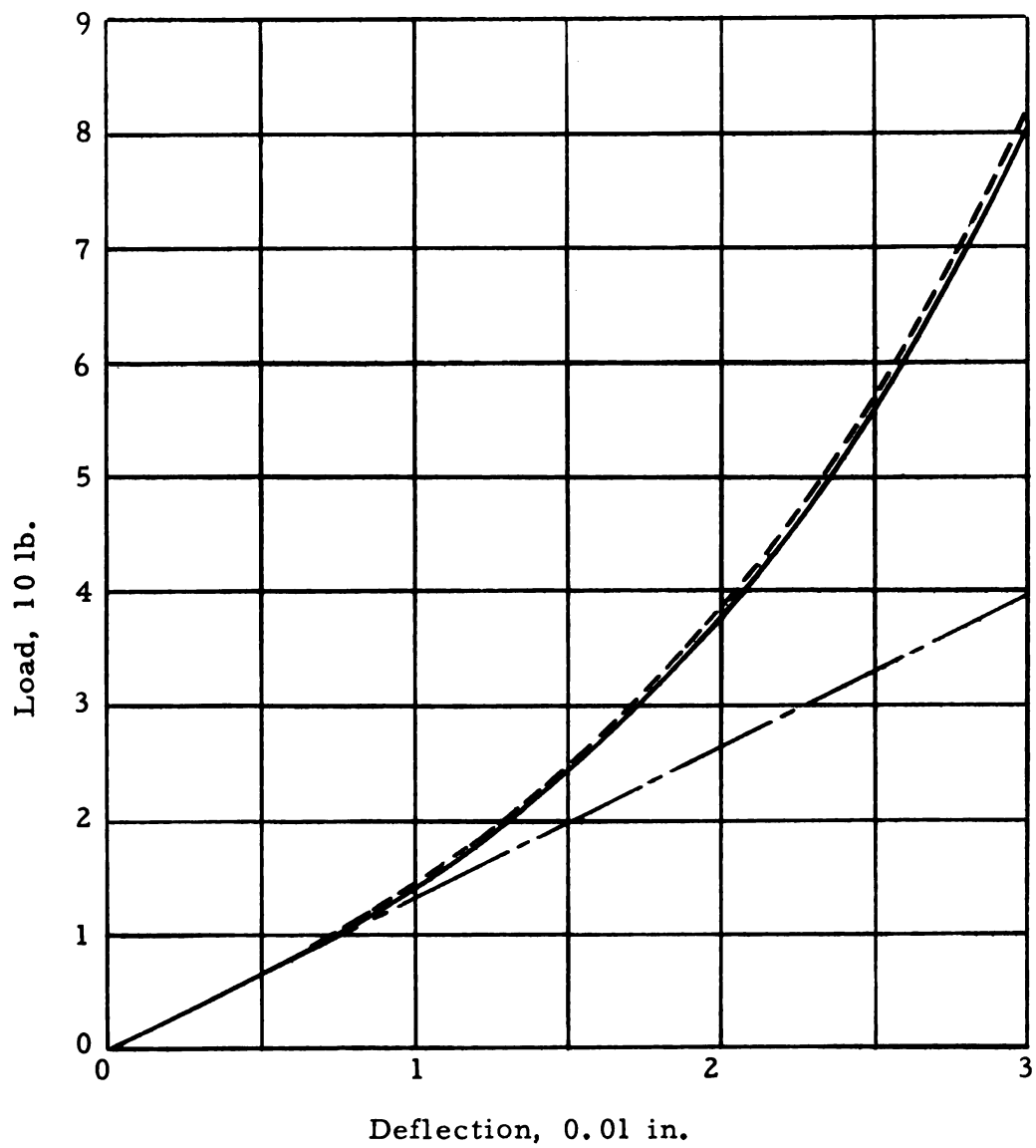
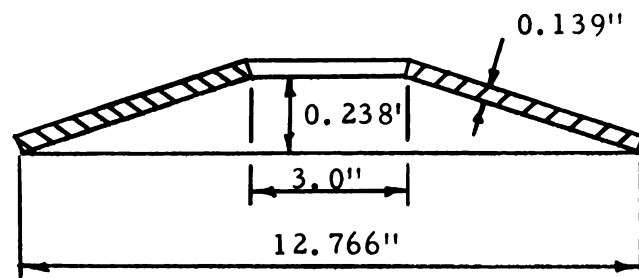


Fig. 5.3. Load-Deflection Curve for a Concentric Flat Plate.



— Almen and Lazyle test
 ○ Newman Reiss solution
 - - - Present solution

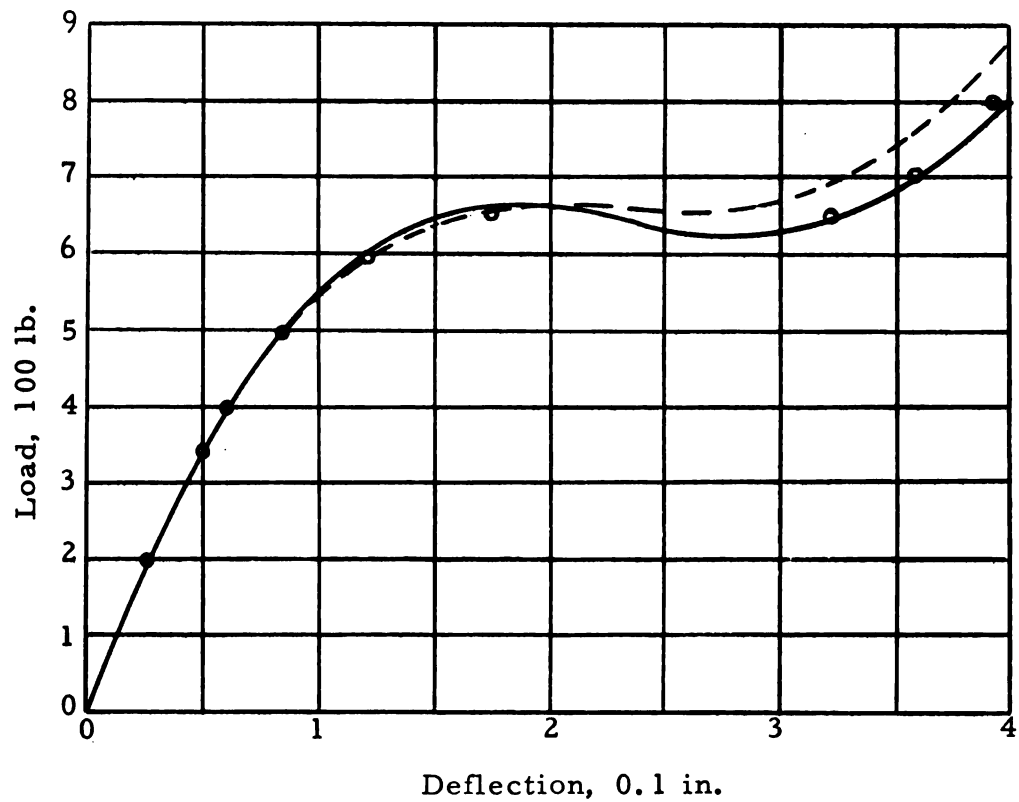
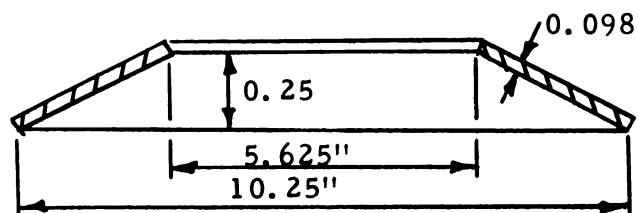


Fig. 5.4. Axial Load-Deflection Curve for a Shallow Cone.



— Almen and Lazyle test
 ○ Newman Reiss solution
 - - - Present solution

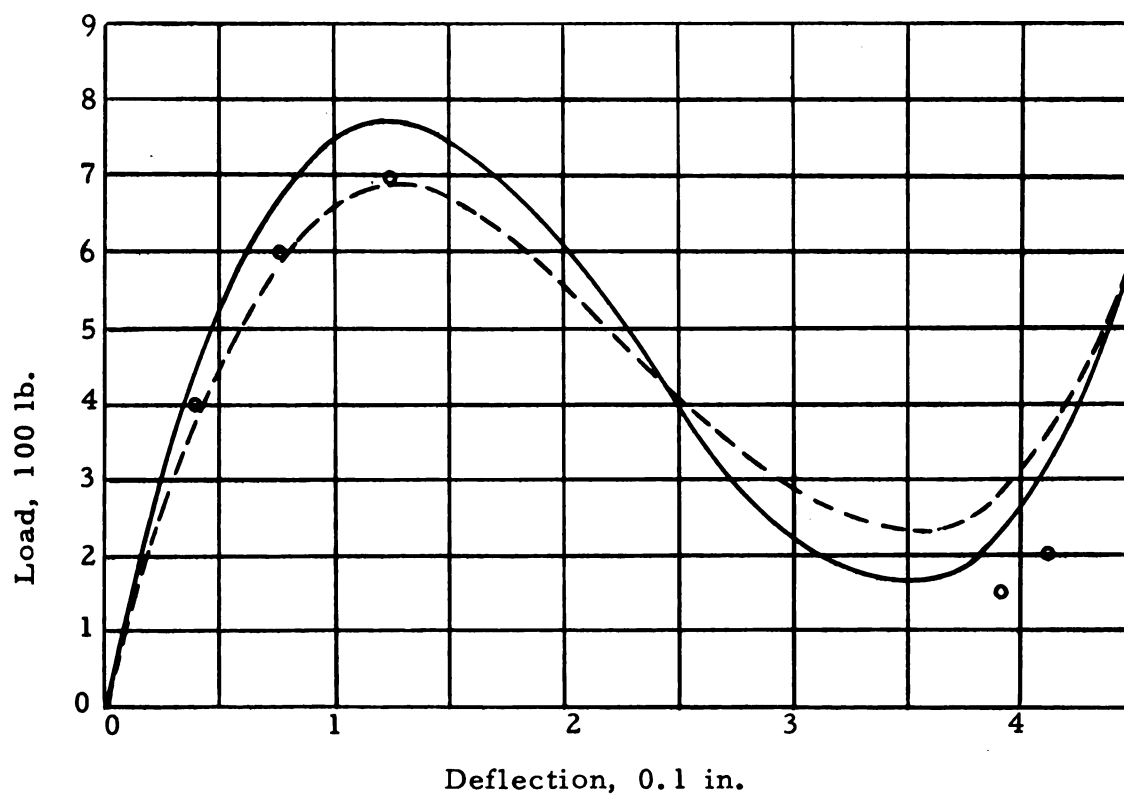


Fig. 5.5. Axial Load-Deflection Curve for a Shallow Cone.

Boundary Conditions

- Ⓘ top and bottom free
- Ⓜ top hinged and bottom free
- Ⓢ top and bottom hinged
- Ⓢ top and bottom symmetry

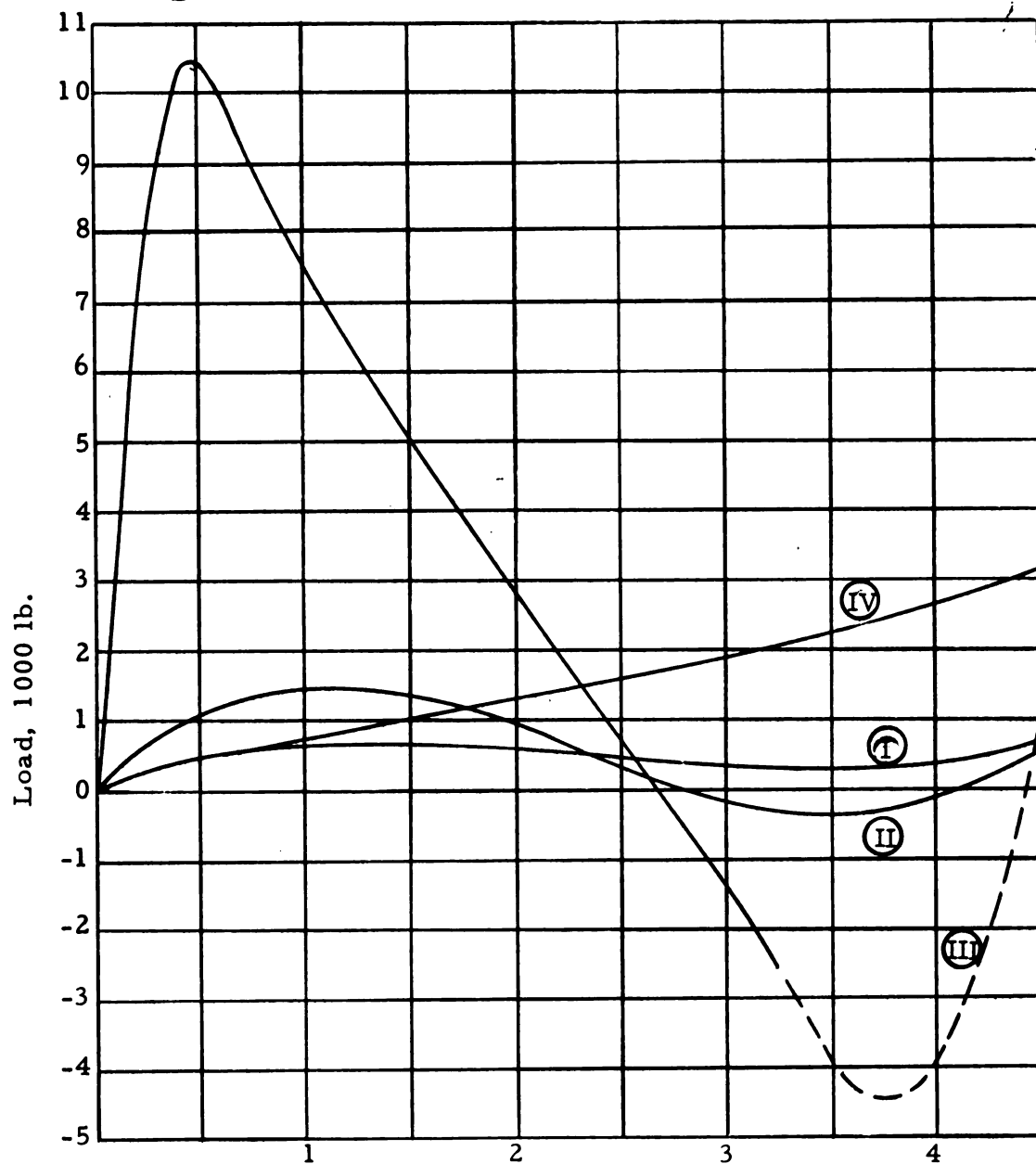


Fig. 5.6. Axial Load-Deflection Curve of the Shallow Cone Shown in Fig. 5.5 for Various Boundary Conditions.

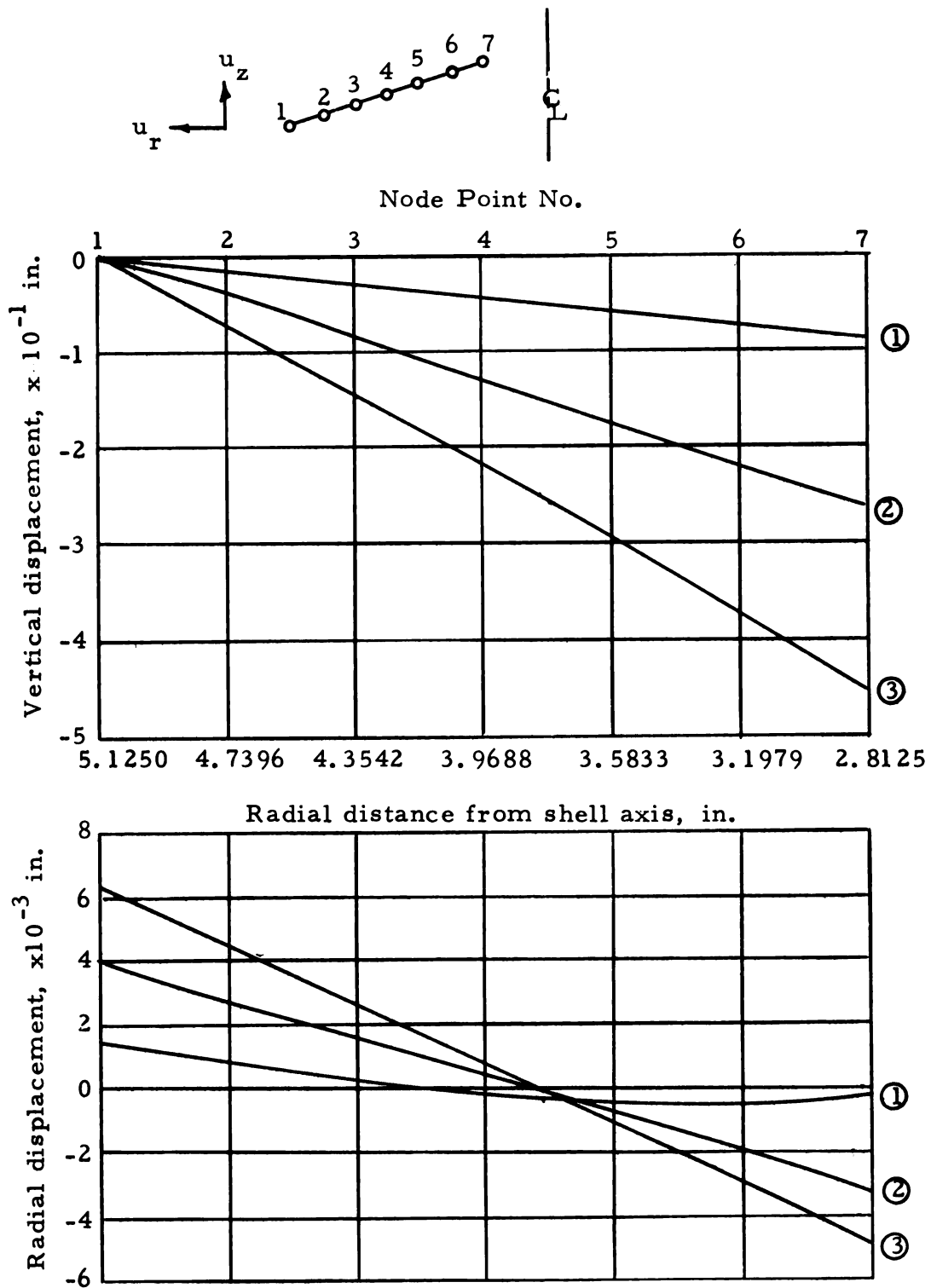


Fig. 5.7. Displacements Along Radial Line for the Cone of Fig. 5.5 with Free Boundary Conditions.

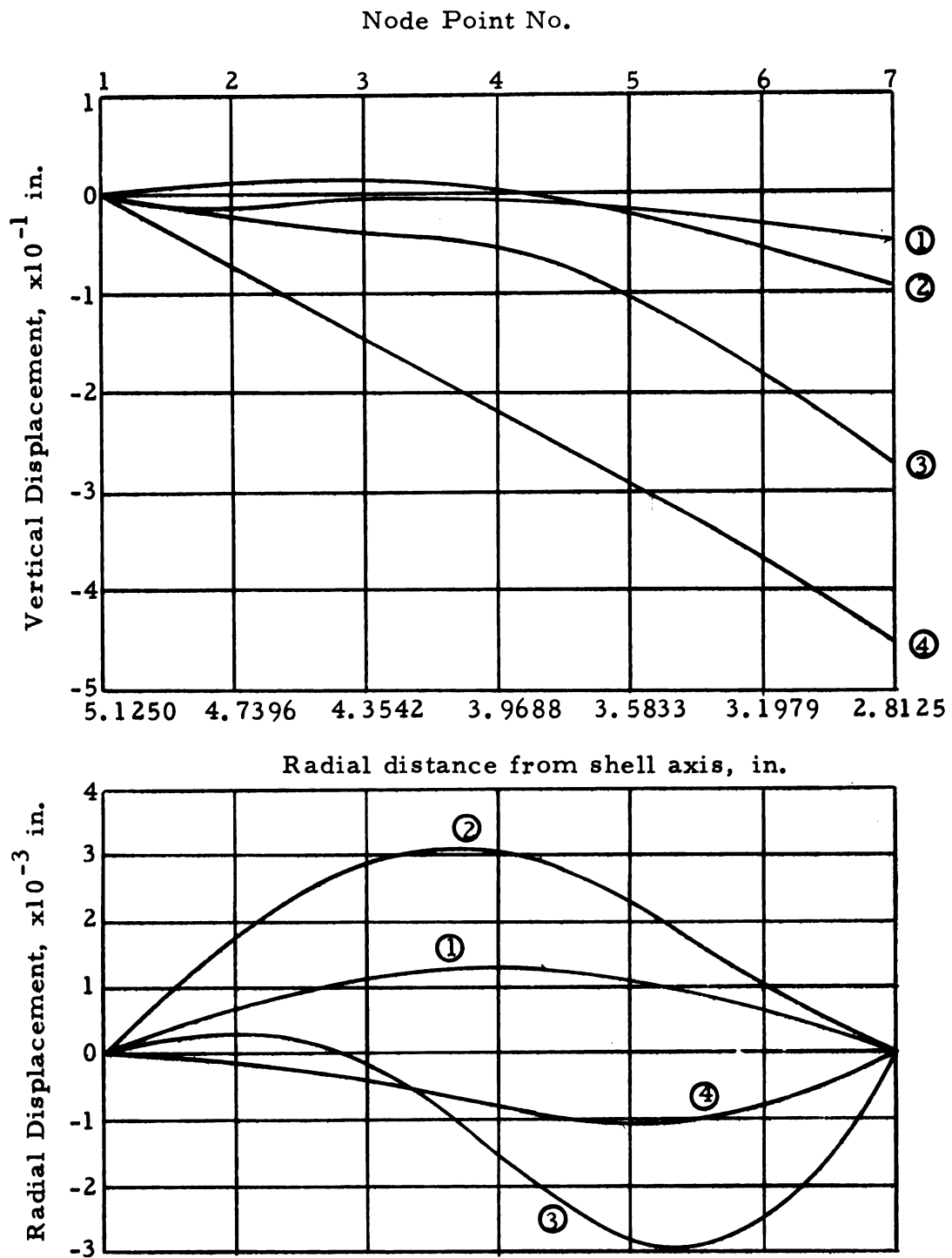


Fig. 5.8. Displacements Along Radial Line for the Cone of Fig. 5.5 with Hinged Boundary Conditions.

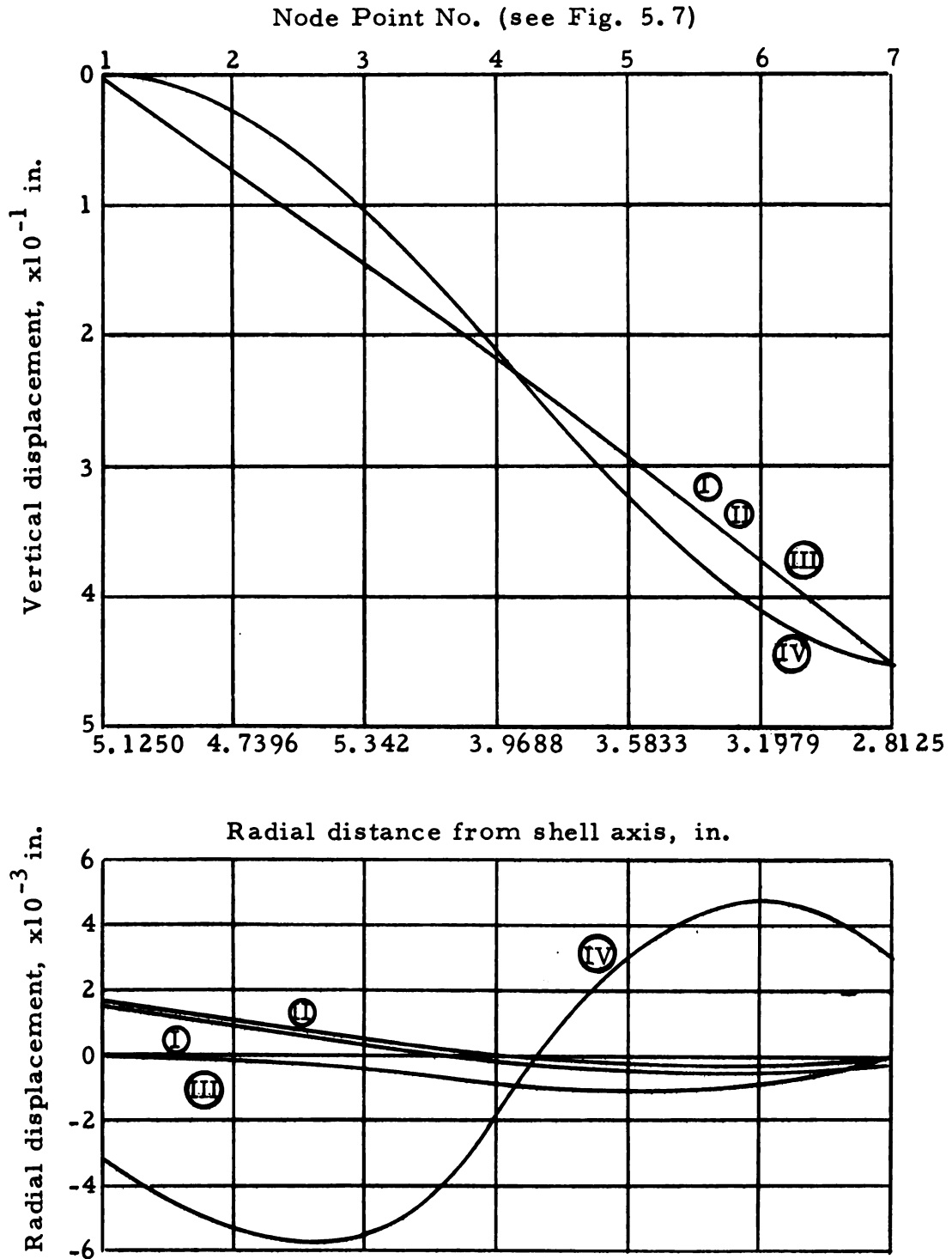


Fig. 5.9. Displacements of Node Points Along Radial Line for the Cone of Fig. 5.5 with the Various Boundary Conditions of Fig. 5.6.

5.4. Conclusions

The usefulness of stiffness matrices associated with arbitrary triangular plate elements is self evident. With the aid of these matrices and modern computing facilities we can formulate and solve many shell problems. The ease with which many geometric configurations can be modeled by triangular plate elements and the inherent ease with which most boundary conditions can be imposed makes this procedure very attractive.

The two bending stiffness matrices obtained for arbitrary triangular plate elements appear to give results of sufficient accuracy for many applications. To ascertain the full flexibility of this procedures, however, will require more numerical results than those obtained, especially for problems with various forms of singular behavior.

This investigator was particularly interested in the bending stiffness matrix given in Table 3.3 because it is associated with displacement modes that have regions of constant curvature and twist, and can consequently be more readily adapted to problems with non-linear stress-strain relations. The modes associated with both bending stiffness matrices can be conveniently used to include the effects associated with thermal strains that vary through the thickness of the shell.

BIBLIOGRAPHY

1. Langefors, B. "Analysis of Elastic Structures by Matrix Transformation with Special Regard to Monocoque Structures." Journal of the Aeronautical Sciences, Vol. 19, No. 7, 1952, pp. 451-458.
2. Argyris, J. H., and Kelsey, S., Energy Theorems and Structural Analysis, Butterworths, London, 1960.
3. Timoshenko, S., and Woinowsky-Krieger, S., Theory of Plates and Shells, McGraw-Hill, New York, 1959.
4. Mushtari, Kh. M., and Galinov, K. Z., Non-Linear Theory of Thin Elastic Shells, Tatknigoizdat, Kazan, 1957. Available in English as NASA-TT-F62, 1961.
5. Novozhilov, V. V., The Theory of Thin Shells, Noordhoff, Groningen, 1959.
6. Langhaar, H. L., Energy Methods in Applied Mechanics, Wiley, New York, 1962.
7. Courant, R., and Hilbert, D., Methods of Mathematical Physics, Interscience, New York, 1953, pp. 175-176.
8. de Veubeke, B. F., "Upper and Lower Bounds in Matrix Structural Analysis," AGARDograph, 72, Pergamon Press, New York, 1964, pp. 115-201.
9. Prager, N., and Synge, J. L., "Approximations in Elasticity Based on the Concept of Function Space," Quart. Appl. Math., Vol. 5, No. 3, 1947, pp. 241-269.
10. Hrennikoff, A. P., "Solution of Problems in Elasticity by the Framework Method," J. Appl. Mech., Vol. 8, No. 4, 1941.
11. McHenry, D., "A Lattice Analogy for the Solution of Stress Problems," J. Inst. Civil Eng., December 1943.
12. Parikh, K. S., and Norris, C. H., "Analysis of Shells Using Framework Analogy," World Conference on Shell Structures, Oct. 1962, pp. 213-222.
13. Turner, M. J., Clough, R. J., Martin, H. C., and Topp, L. J., "Stiffness and Deflection Analysis of Complex Structures," J. Aeron. Sci., Vol. 23, No. 9, 1956, pp. 805-823.

14. Argyris, J. H., "Recent Advances in Matrix Methods of Structural Analysis," Progress in Aeron. Sci., Vol. 4, Macmillan, New York, 1964.
15. Zienkiewicz, O. C., and Holister, G. S., Editors, Stress Analysis; Clough, R. W., "The Finite Element Method in Structural Mechanics," Ch. 7; Zienkiewicz, O. C., "Finite Element Procedures in the Solution of Plate and Shell Problems," Ch. 8; de Veubeke, B. F., "Displacement and Equilibrium Models in the Finite Element Method," Ch. 9; Wiley, New York, 1965.
16. Melosh, R. J., "Basis for Derivation of Matrices for the Direct Stiffness Method," ALAA Journal, Vol. 1, No. 7, 1963, pp. 1631-1637.
17. Bogner, F. K., Mallett, R. H., Minich, M. D., Schmit, L. A., "Development and Evaluation of Energy Search Methods of Nonlinear Structural Analysis," AFFDL-TR-65-113, Wright-Patterson Air Force Base, Ohio, Nov., 1965.
18. Melosh, R. J., "A Stiffness Matrix for the Analysis of Thin Plates in Bending," J. Aerospace Sci., Vol. 28, 1961, pp. 34-43.
19. Adini, A., "Analysis of Shell Structures by the Finite Element Method," Ph.D., Dissertation, California Univ., Berkeley, 1961.
20. Tocher, J. L., "Analysis of Plate Bending Using Triangular Elements," Ph.D., Dissertation, California Univ., Berkeley, 1962.
21. Reissner, E., "On a Variational Theorem in Elasticity," J. Math. Phys., Vol. 29, 1950, pp. 90-95.
22. Reissner, E., "On a Variational Theorem for Finite Elastic Deformations," J. Math. Phys., Vol. 32, 1953, pp. 129-135.
23. Argyris, J. H., "On the Analysis of Complex Elastic Structures," Applied Mechanics Reviews, Vol. 11, No. 7, July 1958, pp. 331-338.
24. Turner, M. J., Dill, E. H., Martin, H. C., and Melosh, R. J., "Large Deflections of Structures Subjected to Heating and External Loads," J. of Aero/Space Sci., Vol. 27, No. 2, Feb., 1960.

25. Gallagher, R. H., and Padlog, J., "Discrete Element Approach to Structural Instability Analysis," AIAA Journal (TN), Vol. 1, No. 6, June 1963, pp. 1437-1439.
26. Denke, P. H., "The Matrix Solution of Certain Nonlinear Problems in Structural Analysis," Journal of the Aeronautical Sciences, Vol. 23, No. 3, March 1956, pp. 231-236.
27. Lansing, W., Jones, I. W., and Ratner, P., "Nonlinear Analysis of Heated, Cambered Wings by the Matrix-Force Method," AIAA Journal, Vol. 1, No. 7, 1963, pp. 1619-1625.
28. Lansing, W., Jones, I. W., and Ratner, P., "Nonlinear Shallow Shell Analysis by the Matrix Force Method," NASA TN D-1510, 1962, pp. 753-761.
29. Almen, J. O., and Laszle, A., "The Uniform-Section Disk Spring," Trans. ASME, Vol. 58, No. 4, 1936, pp. 305-314.
30. Newman, M., and Reiss, E. L., "Axisymmetric Snap Buckling of Conical Shells," NASA TND-1510, Dec. 1962, pp. 451-462.
31. Pian, T. H. H., "Derivation of Element Stiffness Matrices by Assumed Stress Distributions," AIAA Journal, (TN), Vol. II, No. 7, July 1964, pp. 1333-1336.

Appendix A. OBLIQUE COORDINATES

A.1. Some Properties of Oblique Cartesian Coordinate Systems

Consider the oblique Cartesian coordinates designated by ζ^k and the rectangular Cartesian coordinates designated by x^k . These coordinate systems are related as shown in Fig. A1. The coordinates ζ^1, ζ^2 and x^1, x^2 define a common plane, and the coordinates ζ^3 and x^3 are normal to this plane. The angle $\zeta^1 \cap \zeta^2 (= x^1 \cap x^2)$ is designated by ψ . Unit base vectors of the ζ^k and x^k coordinates are respectively designated by $\bar{e}_k$ and $\bar{j}_k$. The associated reciprocal base vectors are $\bar{e}^k$ and $\bar{j}^k (= \bar{j}_k)$, and are defined by the properties

$$\begin{aligned} \bar{e}_k \cdot \bar{e}^\ell &= \begin{cases} 1 & \text{for } k = \ell \\ 0 & \text{for } k \neq \ell \end{cases} \\ \bar{j}_k \cdot \bar{j}^\ell &= \begin{cases} 1 & \text{for } k = \ell \\ 0 & \text{for } k \neq \ell \end{cases} \end{aligned} \quad (\text{A.1})$$

It then follows that the magnitudes of the base vectors are

$$\begin{aligned} |\bar{e}_1| &= |\bar{e}_2| = 1 \\ |\bar{e}^1| &= |\bar{e}^2| = \frac{1}{\sin \psi} \end{aligned} \quad (\text{A.2})$$

$$|\bar{j}^1| = |\bar{j}^2| = |\bar{j}_1| = |\bar{j}_2| = 1 \quad (\text{A.3})$$

A position vector $\bar{r}$ can be represented in any one of the following forms

$$\bar{r} = \zeta^k \bar{e}_k = \zeta_k \bar{e}^k = x^k \bar{j}_k = x_k \bar{j}^k \quad (\text{A.4})$$

where ζ^k are called contravariant components of the vector, ζ_k are called covariant components of the vector, $\bar{e}^k$ are called contravariant

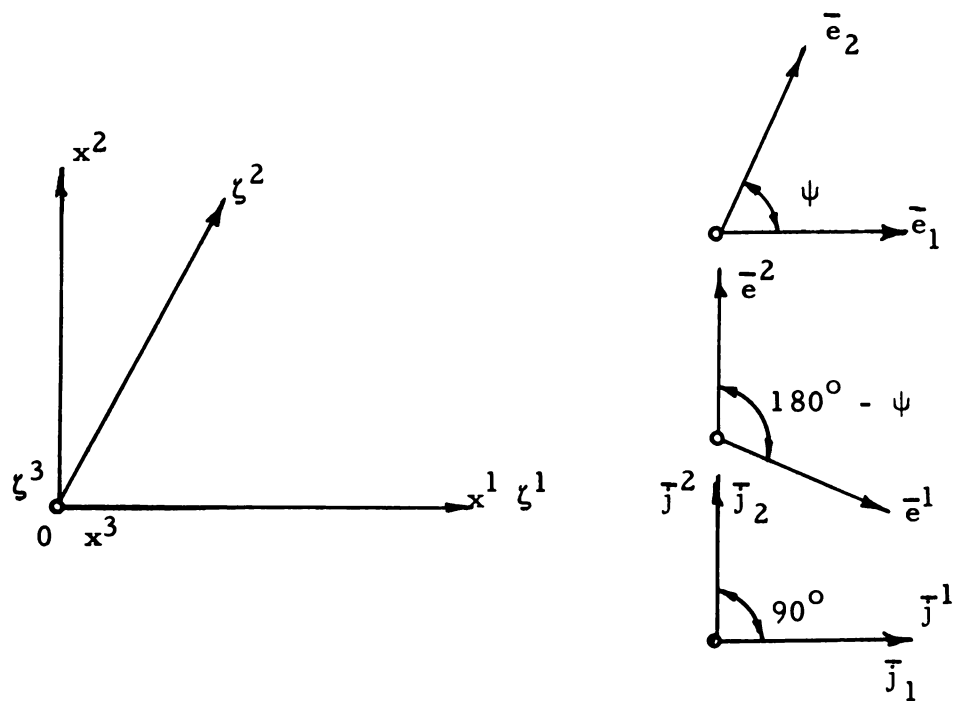


Fig. A1. Coordinates and Base Vectors.

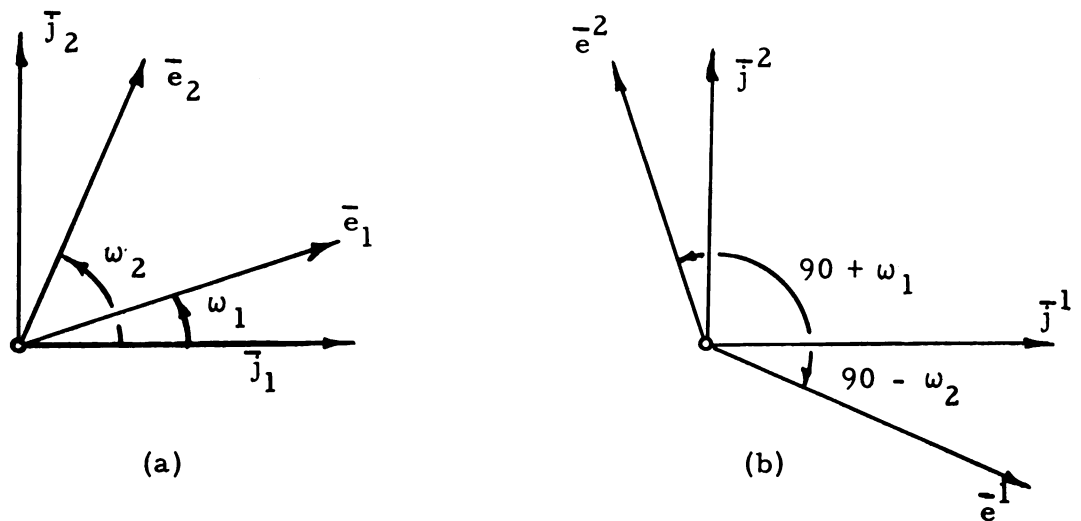


Fig. A2. Base Vectors.

base vectors, and $\bar{e}_k$ are called the covariant base vectors.

The coordinates ζ^k and x^k are related by

$$\begin{aligned} x^1 &= \zeta^1 + \cos \psi \zeta^2 \\ x^2 &= \quad + \sin \psi \zeta^2 \\ x^3 &= \quad + \zeta^3 \end{aligned} \quad (\text{A. 5})$$

In tensor notation (A. 5) can be represented in the form

$$x^k = c_{\ell}^k \zeta^{\ell} \quad (\text{A. 6})$$

The inverse to (A.6) is designated by

$$\zeta^k = d_{\ell}^k x^{\ell} \quad (\text{A. 7})$$

where the matrices $[c_{\ell}^k]$ and $[d_{\ell}^k]$ are related by

$$[d_{\ell}^k] = [c_{\ell}^k]^{-1} \quad (\text{A. 9})$$

The covariant components are related by

$$x_k = d_k^{\ell} \zeta_{\ell} \quad (\text{A. 10})$$

$$\zeta_k = c_k^{\ell} x_{\ell} \quad (\text{A. 11})$$

Substituting (A. 6), (A. 7), (A. 9), (A. 10) into (A. 4) we obtain

$$\zeta^k \bar{e}_k = x^{\ell} \bar{j}_{\ell} = \zeta^k c_k^{\ell} \bar{j}_{\ell} \quad (\text{A. 12})$$

$$\zeta_k \bar{e}^k = x_{\ell} \bar{j}^{\ell} = \zeta_k d_{\ell}^k \bar{j}^{\ell} \quad (\text{A. 13})$$

Eq. 's (A.12) and (A.13) can be satisfied for components of all position vectors $\bar{r}$ only if

$$\bar{e}_k = c_k^\ell \bar{j}_{(x)\ell} \quad (\text{A.14})$$

$$\bar{e}_k = d_\ell^k \bar{j}_{(x)}^\ell \quad (\text{A.15})$$

From (A.5) and (A.9), we obtain

$$\begin{bmatrix} c_1^1 & c_2^1 & c_3^1 \\ c_1^2 & c_2^2 & c_3^2 \\ c_1^3 & c_2^3 & c_3^3 \end{bmatrix} = \begin{bmatrix} 1 & \cos \psi & 0 \\ 0 & \sin \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{A.16})$$

$$\begin{bmatrix} d_1^1 & d_2^1 & d_3^1 \\ d_1^2 & d_2^2 & d_3^2 \\ d_1^3 & d_2^3 & d_3^3 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{\cos \psi}{\sin \psi} & 0 \\ 0 & \frac{1}{\sin \psi} & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{A.17})$$

For the oblique and Cartesian coordinates with covariant base vectors as shown in Fig. A2 the transformation relationships are as follows:

$$\begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix} = \begin{bmatrix} \cos \omega_1 & \sin \omega_1 & 0 \\ \cos \omega_2 & \sin \omega_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{j}_1 \\ \bar{j}_2 \\ \bar{j}_3 \end{bmatrix} \quad (\text{A.18})$$

$$\begin{bmatrix} \bar{j}_1 \\ \bar{j}_2 \\ \bar{j}_3 \end{bmatrix} = \frac{1}{(\cos \omega_1 \sin \omega_2 - \sin \omega_1 \cos \omega_2)} \begin{bmatrix} \sin \omega_2 & -\sin \omega_1 & 0 \\ -\cos \omega_2 & \cos \omega_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{bmatrix}$$

(A.19)

A. 2. The Plane Stress Strain Energy Expression Referred to Oblique Cartesian Coordinates

The covariant components of stress, strain, and displacement when referred to the x^1, x^2 coordinates are respectively designated by $\underline{g}_{ij}$, $\underline{\epsilon}_{ij}$ and u_i ; and when referred to the ζ^1, ζ^2 coordinates are respectively designated by σ_{ij} , ϵ_{ij} , and w_i . The corresponding contravariant components are designated by $\underline{g}^{ij}$, $\underline{\epsilon}^{ij}$, u^i , σ^{ij} , ϵ^{ij} , and w^i . The stress-strain relationships in plane stress are

$$\begin{aligned}\underline{g}^{11} &= \underline{g}_{11} = \frac{E}{1-\nu^2} (\underline{\epsilon}_{11} + \nu \underline{\epsilon}_{22}) \\ \underline{g}^{22} &= \underline{g}_{22} = \frac{E}{1-\nu^2} (\underline{\epsilon}_{22} + \nu \underline{\epsilon}_{11}) \\ \underline{g}^{12} &= \underline{g}_{12} = \frac{E}{1+\nu} \underline{\epsilon}_{12} = \frac{E}{(1-\nu^2)} (1-\nu) \underline{\epsilon}_{12}\end{aligned}\tag{A. 20}$$

The strain displacement relationships are

$$\begin{aligned}\underline{\epsilon}_{ij} &= \frac{1}{2} (u_{i,j} + u_{j,i}) \\ &= \frac{1}{2} \left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i} \right)\end{aligned}\tag{A. 21}$$

and

$$\begin{aligned}\epsilon_{ij} &= \frac{1}{2} (w_{i,j} + w_{j,i}) \\ &= \frac{1}{2} \left(\frac{\partial w_i}{\partial \zeta^j} + \frac{\partial w_j}{\partial \zeta^i} \right)\end{aligned}\tag{A. 22}$$

The components of the two covariant strain tensors are related by

$$\epsilon_{ij} = c_i^k c_j^l \underline{\epsilon}_{kl}\tag{A. 23}$$

$$\underline{\epsilon}_{ij} = d_i^k d_j^l \epsilon_{kl}\tag{A. 24}$$

Eq. (A. 24) is equivalent to the following matrix operations

$$\begin{aligned}
 \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -\frac{\cos \psi}{\sin \psi} & \frac{1}{\sin \psi} \end{bmatrix} \begin{bmatrix} \epsilon_{11} & \epsilon_{12} \\ \epsilon_{21} & \epsilon_{22} \end{bmatrix} \begin{bmatrix} 1 & -\frac{\cos \psi}{\sin \psi} \\ 0 & \frac{1}{\sin \psi} \end{bmatrix} \\
 &= \begin{bmatrix} \epsilon_{11} & -\frac{\cos \psi}{\sin \psi} \epsilon_{11} + \frac{1}{\sin \psi} \epsilon_{12} \\ -\frac{\cos \psi}{\sin \psi} \epsilon_{11} + \frac{1}{\sin \psi} \epsilon_{12} & \frac{\cos^2 \psi}{\sin^2 \psi} \epsilon_{11} + \frac{1}{\sin^2 \psi} \epsilon_{22} - \frac{2 \cos \psi}{\sin^2 \psi} \epsilon_{12} \end{bmatrix}
 \end{aligned}
 \tag{A. 25}$$

The plane stress strain energy for a thin plate of thickness t and middle surface S is given by

$$W_{I_1} = \frac{1}{2} t \int_S \sigma^{ij} \epsilon_{ij} dx^1 dx^2 \tag{A. 26}$$

Expanding (A. 26) and substituting (A. 20) into the resulting expression we obtain

$$W_{I_1} = \frac{Et}{2(1-\nu^2)} \int_S \{(\epsilon_{11} + \epsilon_{22})^2 - 2(1-\nu)(\epsilon_{11} \epsilon_{22} - \epsilon_{12}^2)\} dx^1 dx^2 \tag{A. 27}$$

Substituting the strain relationships given by (A. 25) into (A. 27), putting $\mu = 2(1-\nu)$, and noting that $dx^1 dx^2 = \sin \psi d\zeta^1 d\zeta^2$ we obtain

$$\begin{aligned}
 W_{I_1} &= \frac{Et}{2(1-\nu^2)\sin\psi} \int_S \left\{ \frac{1}{\sin^2 \psi} (\epsilon_{11} - 2 \cos \psi \epsilon_{12} + \epsilon_{22})^2 \right. \\
 &\quad \left. - \mu (\epsilon_{11} \epsilon_{22} - \epsilon_{12}^2) \right\} d\zeta^1 d\zeta^2
 \end{aligned}
 \tag{A. 28}$$

Eq. (A. 28) is the desired relationship.

A. 3. The Thin Plate Bending Strain Energy Expression Referred to Oblique Cartesian Coordinates

The covariant components of bending moments and curvatures when referred to the x^k coordinates are designated by $\underline{m}_{ij}$ and $\underline{\kappa}_{ij}$, and when referred to the ζ^k coordinates are designated by m_{ij} and κ_{ij} . The associated contravariant components are designated identically except that the indices are raised.

The bending moment-curvature relationships in rectangular coordinates are

$$\begin{aligned}\underline{m}^{11} &= \underline{m}_{11} = -\mathcal{D}(\underline{\kappa}_{11} + \nu \underline{\kappa}_{22}) \\ \underline{m}^{22} &= \underline{m}_{22} = -\mathcal{D}(\underline{\kappa}_{22} + \nu \underline{\kappa}_{11}) \\ \underline{m}^{12} &= \underline{m}_{12} = \mathcal{D}(1 - \nu) \underline{\kappa}_{12}\end{aligned}\tag{A. 29}$$

where

$$\mathcal{D} = \frac{\mathcal{E}_t^3}{12(1 - \nu^2)}$$

The curvature displacement relationships are

$$\underline{\kappa}_{ij} = u_{3,ij} = \frac{\partial^2 w_3}{\partial x^i \partial x^j}\tag{A. 30}$$

$$\kappa_{ij} = w_{3,ij} = \frac{\partial w_3}{\partial \zeta^i \partial \zeta^j}\tag{A. 31}$$

The components of the curvature tensors are related by

$$\kappa_{ij} = c_i^{\ell} c_j^k \underline{\kappa}_{kl}\tag{A. 32}$$

$$\underline{\kappa}_{ij} = d_i^k d_j^{\ell} \kappa_{kl}\tag{A. 33}$$

Noting the similarity between (A. 24) and (A. 33) it follows directly from (A. 25) that

$$\begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{12} & \kappa_{22} \end{bmatrix} = \begin{bmatrix} \kappa_{11} & -\frac{\cos \psi}{\sin \psi} \kappa_{11} + \frac{1}{\sin \psi} \kappa_{12} \\ -\frac{\cos \psi}{\sin \psi} \kappa_{11} + \frac{1}{\sin \psi} \kappa_{12} & \frac{\cos^2 \psi}{\sin^2 \psi} \kappa_{11} + \frac{1}{\sin^2 \psi} \kappa_{22} - \frac{2 \cos \psi}{\sin \psi} \kappa_{12} \end{bmatrix} \quad (\text{A. 34})$$

The strain energy W_{I_2} due to bending stresses for a thin plate of thickness t and middle surface S is given by

$$W_{I_2} = \frac{1}{2} t \int_S m^{ij} \kappa_{ij} dx^1 dx^2 \quad (\text{A. 35})$$

Substituting (A. 34) into (A. 29) and then substituting the resulting expression into (A. 35) we obtain

$$W_{I_2} = \frac{t}{2 \sin \psi} \iint_S \left\{ \frac{1}{\sin^2 \psi} (\kappa_{11} - 2 \cos \psi \kappa_{12} + \kappa_{22})^2 - \mu (\kappa_{11} \kappa_{22} - \kappa_{12}^2) \right\} d\zeta^1 d\zeta^2 \quad (\text{A. 36})$$

Substituting (A. 31) into (A. 36) we obtain

$$\begin{aligned} W_{I_2} = \frac{t}{2 \sin \psi} \iint_S \left\{ \frac{1}{\sin^2 \psi} \left(\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^1} - 2 \cos \psi \frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^2} + \frac{\partial^2 w_3}{\partial \zeta^2 \partial \zeta^2} \right)^2 \right. \\ \left. - \mu \left[\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^1} \cdot \frac{\partial^2 w_3}{\partial \zeta^2 \partial \zeta^2} - \left(\frac{\partial^2 w_3}{\partial \zeta^1 \partial \zeta^2} \right)^2 \right] \right\} d\zeta^1 d\zeta^2 \quad (\text{A. 37}) \end{aligned}$$

This is the desired relationship.

Appendix B. CONSTRUCTION OF TRANSFORMATION MATRIX $D_{1(i)}^{(0)}$

This matrix relates the base vectors $J_{(i)}$ of member i with node points 1, 2, 3 to the base vectors $J_{(0)}$ of the general coordinate system and has the form

$$\begin{bmatrix} J_{1(i)} \\ J_{2(i)} \\ J_{3(i)} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} J_{1(0)} \\ J_{2(0)} \\ J_{3(0)} \end{bmatrix} \quad (B.1)$$

where the elements of the coefficient matrix are obtained from (2.101) and are given below.

$$\begin{aligned} b_{11} &= \frac{1}{l_3} [x_{(2)}^1 - x_{(1)}^1] \\ b_{12} &= \frac{1}{l_3} [x_{(2)}^2 - x_{(1)}^2] \\ b_{13} &= \frac{1}{l_3} [x_{(2)}^3 - x_{(1)}^3] \\ b_{21} &= \frac{1}{2Al_3} [-a_2 x_{(1)}^1 - a_1 x_{(2)}^1 + l_3^2 x_{(3)}^1] \\ b_{22} &= \frac{1}{2Al_3} [-a_2 x_{(1)}^2 - a_1 x_{(2)}^2 + l_3^2 x_{(3)}^2] \\ b_{23} &= \frac{1}{2Al_3} [-a_2 x_{(1)}^3 - a_1 x_{(2)}^3 + l_3^2 x_{(3)}^3] \\ b_{31} &= \frac{1}{2A} [x_{(1)}^2 x_{(2)}^3 - x_{(1)}^3 x_{(2)}^2 + x_{(2)}^2 x_{(3)}^3 - x_{(2)}^3 x_{(3)}^2 + x_{(3)}^2 x_{(1)}^3 \\ &\quad - x_{(3)}^3 x_{(1)}^2] \\ b_{32} &= \frac{1}{2A} [x_{(1)}^3 x_{(2)}^1 - x_{(1)}^1 x_{(2)}^3 + x_{(2)}^3 x_{(3)}^1 - x_{(2)}^1 x_{(3)}^3 + x_{(3)}^3 x_{(1)}^1 - x_{(3)}^1 x_{(1)}^3] \\ b_{33} &= \frac{1}{2A} [x_{(1)}^1 x_{(2)}^2 - x_{(1)}^2 x_{(2)}^1 + x_{(2)}^1 x_{(3)}^2 - x_{(2)}^2 x_{(3)}^1 + x_{(3)}^1 x_{(1)}^2 - x_{(3)}^2 x_{(1)}^1] \end{aligned} \quad (B.2)$$

Appendix C. DERIVATION DETAILS FOR MEMBER STIFFNESS MATRICES

C.1. Membrane Stiffness Matrix for Triangular Member

The membrane stresses of a member are associated with the inplane displacements given by (3.42) and have the form

$$\begin{aligned}
 \bar{u}_L &= w_1 \bar{e}^1 + w_2 \bar{e}^2 \\
 &= \frac{1}{l_3} \{ -\beta l_1 \lambda_1 - (1-\alpha-\beta) l_3 \lambda_3 - [(1-\alpha-\beta)^2 A_{a2} + \beta(2-2\alpha-3\beta) A_{b2}] A \lambda_4 \\
 &\quad - [\beta(2\alpha-\beta) A_{a1} + \alpha^2 A_{b1}] A \lambda_5 + [\beta^2 A_{a1} + \alpha(2\beta-\alpha) A_{b1} \\
 &\quad - (1-\alpha-\beta)(1-\alpha-3\beta) A_{a2} + \beta^2 A_{b2}] A \lambda_6 \} \bar{e}^1 \\
 &\quad + \frac{1}{l_2} \{ -(1-\alpha-\beta) l_2 \lambda_2 + [\alpha(2-3\alpha-2\beta) A_{a3} + (1-\alpha-\beta)^2 A_{b3}] A \lambda_4 \\
 &\quad - [\alpha^2 A_{a3} + (1-\alpha-\beta)(1-3\alpha+\beta) A_{b3} + \beta(2\alpha-\beta) A_{a1} + \alpha^2 A_{b1}] A \lambda_5 \\
 &\quad + [\beta^2 A_{a1} + \alpha(2\beta-\alpha) A_{b1}] A \lambda_6 \} \bar{e}^2
 \end{aligned} \tag{C.1}$$

The partial derivatives of the displacement components are

$$\begin{aligned}
 \frac{\partial w_1}{\partial \alpha} &= \frac{A}{l_3} \{ l_3 \lambda_3 + 2[(1-\alpha-\beta) A_{a2} + \alpha \beta A_{b2}] \lambda_4 \\
 &\quad - 2[\beta A_{a1} + \alpha A_{b1}] \lambda_5 + 2[(\beta-\alpha) A_{b1} + (1-\alpha-2\beta) A_{a1}] \lambda_6 \} \\
 \frac{\partial w_2}{\partial \alpha} &= \frac{A}{l_2} \{ l_2 \lambda_2 + 2[(1-3\alpha-\beta) A_{a3} - (1-\alpha-\beta) A_{b3}] \lambda_4 \\
 &\quad - 2[\alpha A_{a3} + (2-3\alpha-\beta) A_{b3} + \beta A_{a1} + \alpha A_{b1}] \lambda_5 \\
 &\quad + 2(\beta-\alpha) A_{b1} \lambda_6 \}
 \end{aligned} \tag{C.2}$$

$$\begin{aligned}
\frac{\partial w_1}{\partial \beta} &= \frac{A}{I_3} \{ -\ell_1 \lambda_1 + \ell_3 \lambda_3 - 2[-(1-\alpha-\beta)A_{a2} + (1-\alpha-3\beta)A_{b2}] \lambda_4 \\
&\quad - 2(\alpha-\beta)A_{a1} \lambda_5 + 2[\beta A_{a1} + \alpha A_{b1} + (2-2\alpha-3\beta)A_{a2} + \beta A_{b2}] \lambda_6 \} \\
\frac{\partial w_2}{\partial \beta} &= \frac{A}{I_2} \{ \ell_2 \lambda_2 - 2[\alpha A_{a3} + (1-\alpha-\beta)A_{b3}] \lambda_4 \\
&\quad - 2[(1-2\alpha-\beta)A_{b3} + (\alpha-\beta)A_{a1}] \lambda_5 + 2[\beta A_{a1} + \alpha A_{b1}] \lambda_6 \}
\end{aligned} \tag{C.2}$$

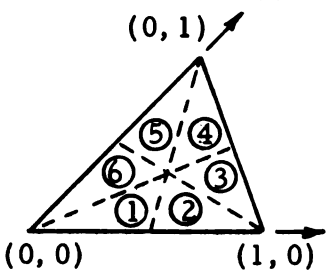
The deformation parameters $\epsilon_1, \epsilon_2, \epsilon_3$ are defined by (3.15) and are related to displacements by (3.16). Substituting (C.2) into (3.16) we obtain

$$\begin{aligned}
\epsilon_1 &= \ell_1 \lambda_1 + 2A \{ [-(1-2\alpha-\beta)A_{a3} + (1-\alpha-2\beta)A_{b2}] \lambda_4 \\
&\quad + [\alpha A_{a3} + (1-\alpha-\beta)A_{b3}] \lambda_5 - [(1-\alpha-\beta)A_{a2} + \beta A_{b2}] \lambda_6 \} \\
\epsilon_2 &= + \ell_2 \lambda_2 + 2A \{ -[\alpha A_{a3} + (1-\alpha-\beta)A_{b3}] \lambda_4 \\
&\quad - [(1-2\alpha-\beta)A_{b3} + (\alpha-\beta)A_{a1}] \lambda_5 + [\beta A_{a1} + \alpha A_{b1}] \lambda_6 \} \\
\epsilon_3 &= \ell_3 \lambda_3 + 2A \{ [(1-\alpha-\beta)A_{a2} + \beta A_{b2}] \lambda_4 - [\beta A_{a1} + \alpha A_{b1}] \lambda_5 \\
&\quad + [-(\alpha-\beta)A_{b1} + (1-\alpha-2\beta)A_{a2}] \lambda_6 \}
\end{aligned} \tag{C.3}$$

The plane stress strain energy expression (3.18) can be expressed in the form

$$\begin{aligned}
W_{I_1} &= \frac{\mathcal{E}t}{4(1-\nu^2)} \{ \frac{1}{4A} (\frac{a_1^2}{A^2} + \mu) \int \int_{S_i} \epsilon_1^2 da d\beta + \frac{1}{4A} (\frac{a_2^2}{A^2} + \mu) \int \int_{S_i} \epsilon_2^2 da d\beta \\
&\quad + \frac{1}{4A} (\frac{a_3^2}{A^2} + \mu) \int \int_{S_i} \epsilon_3^2 da d\beta + \frac{2}{4A} (\frac{a_1 a_2}{A^2} - \mu) \int \int_{S_i} \epsilon_1 \epsilon_2 da d\beta \\
&\quad + \frac{2}{4A} (\frac{a_2 a_3}{A^2} - \mu) \int \int_{S_i} \epsilon_2 \epsilon_3 da d\beta + \frac{2}{4A} (\frac{a_3 a_1}{A^2} - \mu) \int \int_{S_i} \epsilon_3 \epsilon_1 da d\beta \}
\end{aligned} \tag{C.4}$$

Table C1. Some Definite Integrals

	$2592 \times \int \int_{\text{area}} f(\alpha, \beta) d\alpha d\beta$						
	AREA						
$f(\alpha, \beta)$	①	②	③	④	⑤	⑥	Total
1	216	216	216	216	216	216	1296
α	60	132	132	60	24	24	432
β	24	24	60	132	132	60	432
α^2	19	85	85	19	4	4	216
β^2	4	4	19	85	85	19	216
$\alpha\beta$	7	13	34	34	13	7	108
$(1-2\alpha-\beta)^2$	36	36	63	9	9	63	216
$(1-2\alpha-\beta)(1-\alpha-2\beta)$	45	-9	18	18	-9	45	108
$(1-\alpha-2\beta)^2$	63	9	9	63	36	36	216
$\alpha(1-\alpha-\beta)$	34	34	13	7	7	13	108
$(1-\alpha-\beta)^2$	85	19	4	4	19	85	216
$\beta(1-\alpha-\beta)$	13	7	7	13	34	34	108
$\alpha(1-2\alpha-\beta)$	15	-51	-72	-12	3	9	-108
$\alpha(1-\alpha-2\beta)$	27	21	-21	-27	-6	6	0
$(1-\alpha-2\beta)(1-\alpha-\beta)$	72	12	-3	-9	-15	51	108
$(1-2\alpha-\beta)(1-\alpha-\beta)$	51	-15	-9	-3	12	72	72
$\beta(1-2\alpha-\beta)$	6	-6	-27	-21	21	27	0
$\beta(1-\alpha-2\beta)$	9	3	-12	-72	-51	15	-108
$(1-\alpha-\beta)$	132	60	24	24	60	132	432
$(1-2\alpha-\beta)$	72	-72	-108	-36	36	108	0
$(1-\alpha-2\beta)$	108	36	-36	-108	-72	72	0
$(1-\alpha-2\beta)^2$	63	9	9	63	36	36	216
$(\alpha-\beta)^2$	9	63	36	36	63	9	216
$(1-2\alpha-\beta)(\alpha-\beta)$	9	-45	-45	9	-18	-18	-108

From (C.3) and Table C1 we obtain

$$\begin{aligned}
\int_{S_i} \int \epsilon_1^2 \, da \, d\beta &= \frac{1}{2} \ell_1^2 \lambda_1 + \frac{4A\ell_1}{2592} \{ 216 \lambda_1 \lambda_5 - 216 \lambda_1 \lambda_6 \} \\
&+ \frac{4A^2}{2592} \{ 36 \lambda_4^2 + 54 \lambda_5^2 + 54 \lambda_6^2 + 36 \lambda_4 \lambda_5 - 72 \lambda_5 \lambda_6 + 36 \lambda_6 \lambda_1 \} \\
\int_{S_i} \int \epsilon_2^2 \, da \, d\beta &= \frac{1}{2} \ell_2^2 \lambda_2 + \frac{4A\ell_3}{2592} \{ -216 \lambda_4 \lambda_2 + 216 \lambda_2 \lambda_6 \} \\
&+ \frac{4A^2}{2592} \{ 54 \lambda_4^2 + 36 \lambda_5^2 + 54 \lambda_6^2 + 36 \lambda_4 \lambda_5 + 36 \lambda_5 \lambda_6 - 72 \lambda_6 \lambda_4 \} \\
\int_{S_i} \int \epsilon_3^2 \, da \, d\beta &= \frac{1}{2} \ell_3^2 \lambda_3 + \frac{4A\ell_3}{2592} \{ 216 \lambda_3 \lambda_4 - 216 \lambda_3 \lambda_5 \} \\
&+ \frac{4A^2}{2592} \{ 54 \lambda_4^2 + 54 \lambda_5^2 + 36 \lambda_6^2 - 72 \lambda_4 \lambda_5 + 36 \lambda_5 \lambda_6 + 36 \lambda_4 \lambda_5 \} \\
\int_{S_i} \int \epsilon_1 \epsilon_2 \, da \, d\beta &= \frac{1}{2} \ell_1 \ell_2 \lambda_1 \lambda_2 + \frac{2A\ell_2}{2592} \{ 216 \lambda_2 \lambda_5 - 216 \lambda_2 \lambda_6 \} \\
&+ \frac{2A\ell_1}{2592} \{ -216 \lambda_1 \lambda_4 + 216 \lambda_1 \lambda_6 \} + \frac{4A^2}{2592} \{ -18 \lambda_4^2 - 18 \lambda_5^2 - 36 \lambda_6^2 \\
&- 72 \lambda_4 \lambda_5 + 36 \lambda_5 \lambda_6 + 36 \lambda_6 \lambda_4 \} \\
\int_{S_i} \int \epsilon_2 \epsilon_3 \, da \, d\beta &= \frac{1}{2} \ell_2 \ell_3 \lambda_2 \lambda_3 + \frac{2A\ell_2}{2592} \{ 216 \lambda_4 \lambda_2 - 216 \lambda_5 \lambda_2 \} \\
&+ \frac{2A\ell_3}{2592} \{ -216 \lambda_4 \lambda_3 + 216 \lambda_6 \lambda_3 \} \\
&+ \frac{4A^2}{2592} \{ -36 \lambda_4^2 - 18 \lambda_5^2 - 18 \lambda_6^2 + 36 \lambda_4 \lambda_5 - 72 \lambda_2 \lambda_3 + 36 \lambda_4 \lambda_6 \} \\
\int_{S_i} \int \epsilon_3 \epsilon_1 \, da \, d\beta &= + \frac{1}{2} \ell_1 \ell_3 \lambda_1 \lambda_3 + \frac{2A\ell_1}{2592} \{ 216 \lambda_1 \lambda_4 - 216 \lambda_1 \lambda_5 \} \\
&+ \frac{2A\ell_3}{2592} \{ 216 \lambda_5 \lambda_3 - 216 \lambda_6 \lambda_3 \} \\
&+ \frac{4A^2}{2592} \{ -18 \lambda_4^2 - 36 \lambda_5^2 - 18 \lambda_6^2 + 36 \lambda_4 \lambda_5 + 36 \lambda_5 \lambda_6 - 72 \lambda_4 \lambda_6 \}
\end{aligned}$$

Substituting (C. 5) into (C. 4) we obtain the expanded form of

$$W_{I_1} = \frac{1}{2} \Lambda_{1_i}^T \Gamma_{11_i} \Lambda_{1_i} \quad (C.6)$$

where Γ_{11_i} , called the member membrane stiffness matrix, is given in Table 3.2 and

$$\Lambda_{1_i}^T = [\lambda_1 \dots \lambda_6] \quad (C.7)$$

As shown in (3.83) the membrane and bending behaviors of the member are uncoupled so that from the principal of virtual work

$$\Sigma_{1_i} = \frac{\partial W_{I_1}}{\partial \lambda_j} = \Gamma_{11_i} \Lambda_{1_i} \quad j = 1, \dots, 6 \quad (C.8)$$

where Σ_{1_i} are the generalized forces associated with the generalized coordinates Λ_{1_i} .

C.2. Bending Stiffness Matrix for Triangular Member

The bending stresses of a member are associated with the displacements normal to the plane which have the form

$$\begin{aligned} \bar{u}_L &= w_2 \bar{e}_3 \\ &= \alpha \beta \lambda_7 + \beta(1-\alpha-\beta)\lambda_8 + \alpha(1-\alpha-\beta)\lambda_9 \\ &\quad + [\beta(\alpha-\beta)A_{a1} + \alpha(\alpha-\beta)A_{b1}] \lambda_{10} - [(1-\alpha-2\beta)(1-\alpha-\beta)A_{a2} \\ &\quad + \beta(1-\alpha-2\beta)A_{b2}] \lambda_{11} + [\alpha(1-2\alpha-\beta)A_{a3} + (1-\alpha-\beta)(1-2\alpha-\beta)A_{b3}] \lambda_{12} \end{aligned} \quad (C.9)$$

Substituting (C.9) into the expression (3.23) relating the deformation parameters $\epsilon_4, \epsilon_5, \epsilon_6$ to w_3 we obtain

$$\begin{aligned}\epsilon_4 &= -2\lambda_7 + 4(-A_{a1} + A_{b1})\lambda_{10} + 2A_{b2}\lambda_{11} - 2A_{a3}\lambda_{12} \\ \epsilon_5 &= -2\lambda_8 - 2A_{a1}\lambda_{10} + 4(-A_{a2} - A_{b2})\lambda_{11} + 2A_{b3}\lambda_{12} \\ \epsilon_6 &= -2\lambda_9 + 2A_{b1}\lambda_{10} - 2A_{a2}\lambda_{11} + 4(-A_{a3} + A_{b3})\lambda_{12}\end{aligned}\tag{C.10}$$

From (C.10) and Table C1 we obtain

$$\begin{aligned}\int_{S_i} \int \epsilon_4^2 da d\beta &= 2\lambda_7^2 + 8\lambda_{10}^2 + \lambda_{11}^2 + \lambda_{12}^2 - 2\lambda_7\lambda_{11} + 2\lambda_7\lambda_{12} - \frac{4}{3}\lambda_{10}\lambda_{11} \\ &\quad - \frac{4}{3}\lambda_{10}\lambda_{12} - \frac{4}{3}\lambda_{11}\lambda_{12} \\ \int_{S_i} \int \epsilon_5^2 da d\beta &= 2\lambda_8^2 + \lambda_{10}^2 + 8\lambda_{11}^2 + \lambda_{12}^2 + 2\lambda_8\lambda_{10} - 2\lambda_8\lambda_{12} - \frac{4}{3}\lambda_{10}\lambda_{11} \\ &\quad - \frac{4}{3}\lambda_{10}\lambda_{12} - \frac{4}{3}\lambda_{11}\lambda_{12} \\ \int_{S_i} \int \epsilon_6^2 da d\beta &= 2\lambda_9^2 + \lambda_{10}^2 + \lambda_{12}^2 + 8\lambda_{12}^2 - 2\lambda_9\lambda_{10} + 2\lambda_9\lambda_{11} - \frac{4}{3}\lambda_{10}\lambda_{11} \\ &\quad - \frac{4}{3}\lambda_{10}\lambda_{12} - \frac{4}{3}\lambda_{11}\lambda_{12} \\ \int_{S_i} \int \epsilon_4\epsilon_5 da d\beta &= 2\lambda_7\lambda_8 + \lambda_7\lambda_{10} - \lambda_7\lambda_{12} + 2\lambda_{10}^2 + 2\lambda_{11}^2 + 2\lambda_{10}\lambda_{11} \\ &\quad - \lambda_{10}\lambda_{12} - \lambda_8\lambda_{11} + \lambda_8\lambda_{12} - \lambda_{11}\lambda_{12} \\ \int_{S_i} \int \epsilon_5\epsilon_6 da d\beta &= 2\lambda_{11}^2 + 2\lambda_{12}^2 + 2\lambda_8\lambda_9 + \lambda_8\lambda_{11} - \lambda_8\lambda_{10} + 2\lambda_{11}\lambda_{12} \\ &\quad - \lambda_{11}\lambda_{10} - \lambda_9\lambda_{12} + \lambda_9\lambda_{10} - \lambda_{12}\lambda_{10} \\ \int_{S_i} \int \epsilon_6\epsilon_1 da d\beta &= 2\lambda_{12}^2 + \lambda_{10}^2 + 2\lambda_9\lambda_7 + \lambda_9\lambda_{12} - \lambda_9\lambda_{11} + 2\lambda_{12}\lambda_{10} \\ &\quad - \lambda_{12}\lambda_{11} - \lambda_7\lambda_{10} + \lambda_9\lambda_{11} - \lambda_{10}\lambda_{11}\end{aligned}\tag{C.11}$$

Substituting (C.11) into (3.25) we obtain the expanded form of

$$W_{I_2} = \frac{1}{2} \Lambda_{2_i}^T \Gamma_{22_i} \Lambda_{2_i} \quad (C.12)$$

where Γ_{22_i} , called the member bending stiffness matrix, is given in Table 3.3 and

$$\Lambda_{2_i}^T = [\lambda_7, \dots, \lambda_{12}] \quad (C.13)$$

The principle of virtual work requires

$$\Sigma_{2_i} = \frac{\partial W_{I_2}}{\partial \lambda_j} = \Gamma_{22_i} \Lambda_{2_i} \quad j = 7, \dots, 12 \quad (C.14)$$

where Σ_{2_i} are the associated member generalized forces.

Appendix D. SOME ADDITIONAL RESULTS ON THE TRIANGULAR PLATE BENDING STIFFNESS MATRIX

D.1. Alternate Form of Member Bending Stiffness Matrix

The six independent displacement configurations associated with bending and corresponding to those given by (3.43) are taken in the form

$$\begin{aligned} \bar{u}_L = w_3 \bar{e}^3 = \{ & g_{a2} g_{a3} \lambda_7 + g_{a3} g_{a1} \lambda_8 + g_{a1} g_{a2} \lambda_9 \\ & + g_{a2} g_{a3} g_{b1} \lambda_{10} + g_{a3} g_{a1} g_{b2} \lambda_{11} + g_{a1} g_{a2} g_{b3} \lambda_{12} \} e^3 \end{aligned} \quad (D.1)$$

These displacement configurations have the same qualitative form given by (3.43) and shown in Fig. 3.6; however, displacement and slope continuity are not in general preserved for any triangulation. As a result the element of the stiffness matrix cannot be interpreted as satisfying upper bound requirements. This stiffness matrix appears to give better results whenever large curvature gradients are not present (see Ch. V) and is consequently included here.

Substituting from Table 3.1 into (D.1) we obtain

$$\begin{aligned} \bar{u}_L = w_3 \bar{e}^3 = \{ & \alpha \beta \lambda_7 + \beta(1-\alpha-\beta)\lambda_8 + (1-\alpha-\beta)\alpha \lambda_9 \\ & + \alpha \beta(\alpha - \beta)\lambda_{10} + \beta(1-\alpha - \beta)(-1+\alpha+2\beta)\lambda_{11} \\ & + (1-\alpha - \beta)(\alpha)(1-2\alpha - \beta)\lambda_{12} \end{aligned} \quad (D.2)$$

On taking the second derivatives of (D.2) and substituting into the expression for the deformation parameters (3.23) we obtain

$$\epsilon_4 = -2\ell_1 \lambda_7 - 6\ell_1 (\alpha - \beta) \lambda_{10} + 2\ell_2 (1 - \alpha - \beta) \lambda_{11} - 2\ell_3 (1 - \alpha - \beta) \lambda_{12}$$

$$\epsilon_5 = -2\ell_2 \lambda_8 - 2\ell_1 \alpha \lambda_{10} + 6\ell_2 (1 - \alpha - 2\beta) \lambda_{11} + 2\ell_3 \alpha \lambda_{12}$$

$$\epsilon_6 = -2\ell_3 \lambda_9 + 2\ell_1 \beta \lambda_{10} - 2\ell_2 \beta \lambda_{11} - 6\ell_3 (1 - 2\alpha - \beta) \lambda_{12}$$

(D.3)

From (D.3) and Table C1 we obtain

$$\begin{aligned} \int_{S_i} \int \epsilon_4^2 da d\beta &= 2\ell_1^2 \lambda_7^2 + 3\ell_1^2 \lambda_{10}^2 + \frac{1}{3} \ell_2^2 \lambda_{11}^2 + \frac{1}{3} \ell_3^2 \lambda_{12}^2 \\ &\quad - \frac{4}{3} \ell_1 \ell_2 \lambda_7 \lambda_{11} + \frac{4}{3} \ell_1 \ell_2 \lambda_7 \lambda_{12} - \frac{2}{3} \ell_2 \ell_3 \lambda_{11} \lambda_{12} \end{aligned}$$

$$\begin{aligned} \int_{S_i} \int \epsilon_5^2 da d\beta &= 2\ell_2^2 \lambda_8^2 + 3\ell_2^2 \lambda_{11}^2 + \frac{1}{3} \ell_3^2 \lambda_{12}^2 + \frac{1}{3} \ell_1^2 \lambda_{10}^2 \\ &\quad - \frac{4}{3} \ell_2 \ell_3 \lambda_8 \lambda_{12} + \frac{4}{3} \ell_2 \ell_3 \lambda_8 \lambda_{10} - \frac{2}{3} \ell_3 \ell_1 \lambda_{12} \lambda_{10} \end{aligned}$$

$$\begin{aligned} \int_{S_i} \int \epsilon_6^2 da d\beta &= 2\ell_3^2 \lambda_9^2 + 3\ell_3^2 \lambda_{12}^2 + \frac{1}{3} \ell_1^2 \lambda_{10}^2 + \frac{1}{3} \ell_2^2 \lambda_{11}^2 - \frac{4}{3} \ell_3 \ell_1 \lambda_9 \lambda_{10} \\ &\quad + \frac{4}{3} \ell_3 \ell_1 \lambda_9 \lambda_{11} - \frac{2}{3} \ell_1 \ell_2 \lambda_{10} \lambda_{11} \end{aligned}$$

$$\begin{aligned} \int_{S_i} \int \epsilon_4 \epsilon_5 da d\beta &= 2\ell_1 \ell_2 \lambda_7 \lambda_8 + \frac{2}{3} \ell_1^2 \lambda_7 \lambda_{10} - \frac{2}{3} \ell_1 \ell_3 \lambda_7 \lambda_{12} + \frac{1}{2} \ell_1^2 \lambda_{10}^2 \\ &\quad - \frac{5}{3} \ell_1 \ell_2 \lambda_{10} \lambda_{11} - \frac{1}{3} \ell_1 \ell_3 \lambda_{10} \lambda_{12} + \frac{1}{2} \ell_2^2 \lambda_{11}^2 \\ &\quad - \frac{1}{6} \ell_3^2 \lambda_{12}^2 - \frac{2}{3} \ell_2^2 \lambda_8 \lambda_{11} + \frac{2}{3} \ell_2 \ell_3 \lambda_8 \lambda_{12} - \frac{1}{3} \ell_2 \ell_3 \lambda_{11} \lambda_{12} \end{aligned}$$

$$\begin{aligned} \int_{S_i} \int \epsilon_5 \epsilon_6 da d\beta &= 2\ell_2 \ell_3 \lambda_8 \lambda_9 + \frac{2}{3} \ell_2^2 \lambda_8 \lambda_{11} - \frac{2}{3} \ell_2 \ell_1 \lambda_8 \lambda_{10} + \frac{1}{2} \ell_2^2 \lambda_{11}^2 \\ &\quad - \frac{5}{3} \ell_2 \ell_3 \lambda_{11} \lambda_{12} - \frac{1}{3} \ell_2 \ell_1 \lambda_{11} \lambda_{10} + \frac{1}{2} \ell_3^2 \lambda_{12}^2 \\ &\quad - \frac{1}{6} \ell_1^2 \lambda_{10}^2 - \frac{2}{3} \ell_3^2 \lambda_9 \lambda_{12} + \frac{2}{3} \ell_3 \ell_1 \lambda_9 \lambda_{10} - \frac{1}{3} \ell_3 \ell_1 \lambda_{12} \lambda_{10} \end{aligned}$$

$$\begin{aligned}
\int_{S_i} \int \epsilon_6 \epsilon_4 \, da \, d\beta = & 2\ell_3 \ell_1 \lambda_9 \lambda_{10} + \frac{2}{3} \ell_3^2 \lambda_9 \lambda_{12} - \frac{2}{3} \ell_3 \ell_2 \lambda_9 \lambda_{11} + \frac{1}{2} \ell_2^2 \lambda_{12}^2 \\
& - \frac{5}{3} \ell_3 \ell_1 \lambda_{12} \lambda_{10} - \frac{1}{3} \ell_3 \ell_2 \lambda_{12} \lambda_{11} + \frac{1}{2} \ell_1^2 \lambda_{10}^2 \\
& - \frac{1}{6} \ell_2^2 \lambda_{11}^2 - \frac{2}{3} \ell_1^2 \lambda_7 \lambda_{10} + \frac{2}{3} \ell_1 \ell_2 \lambda_7 \lambda_{11} - \frac{1}{3} \ell_1 \ell_2 \lambda_{10} \lambda_{11}
\end{aligned}
\tag{D. 4}$$

Substituting (D. 4) into (3.25) we obtain the expanded form of

$$W_{I_2} = \frac{1}{2} \Lambda_{2_i}^T \Gamma_{22_i} \Lambda_{2_i} \tag{D. 5}$$

where the member bending stiffness matrix Γ_{22_i} is given in Table D1.

D.2. Member Transformation Matrix for Bending

For convenience the member transformation matrix M_{2_i} relating the deformation parameters $\lambda_7, \dots, \lambda_{12}$, associated with bending, to the normal node displacement components and inplane rotation components is given. This can be obtained by taking the transpose of the coefficient matrix in (3.72), deleting the first six rows, and deleting those columns associated with inplane node displacement components and rotation components normal to the plane. The transformation matrix is given by (D. 6).

$$\begin{bmatrix} \lambda_7 \\ \lambda_8 \\ \lambda_9 \\ \lambda_{10} \\ \lambda_{11} \\ \lambda_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1}{l_2} & \frac{1}{l_3} \\ 0 & \frac{A}{l_2 l_3} & -\frac{a_1}{2l_2 l_3} & 0 & -\frac{A}{l_2 l_3} & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & -\frac{1}{l_3} \\ 0 & 0 & 0 & 0 & \frac{1}{l_1} & 0 \\ 0 & 0 & 0 & \frac{1}{l_1} & -\frac{1}{l_1} & \frac{1}{l_2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{a_3}{2l_1 l_3} & -\frac{A}{l_1 l_3} & \frac{a_2}{2l_1 l_3} & 0 & -\frac{A}{l_1 l_3} & -\frac{a_3}{2l_1 l_3} \\ \frac{a_1}{2l_2 l_3} & -\frac{A}{l_2 l_3} & 0 & 0 & -\frac{A}{l_2 l_3} & \frac{a_1}{2l_2 l_3} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{a_3}{2l_1 l_3} & \frac{A}{l_1 l_3} & -\frac{a_2}{2l_1 l_3} & -\frac{1}{l_1} & \frac{A}{l_1 l_3} & \frac{a_3}{2l_1 l_3} \\ \frac{a_1}{2l_2 l_3} & -\frac{A}{l_2 l_3} & 0 & \frac{1}{l_2} & -\frac{A}{l_2 l_3} & \frac{a_1}{2l_2 l_3} \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{(1i)}^3 \\ \theta_{(i,i)}^1 \\ \theta_{(1i)}^2 \\ u_{(2i)}^3 \\ \theta_{(2i)}^1 \\ \theta_{(2i)}^2 \\ u_{(3i)}^3 \\ \theta_{(3i)}^1 \\ \theta_{(3i)}^2 \end{bmatrix}$$

(D.6)

Table D1. Alternate Bending Stiffness Matrix

$\frac{\ell^2}{4A} \left(\frac{a_1^2}{2} + \mu \right)$	$\frac{\ell^2}{4A} \left(\frac{a_1 a_2}{2} - \mu \right)$	$\frac{\ell^2}{4A} \left(\frac{a_1 a_3}{2} - \mu \right)$	$-\frac{\ell^2}{12A} \left\{ \frac{a_1^2}{2} (a_3 - a_2) \right\}$	$-\frac{\ell^2}{12A} \left\{ \frac{a_1 (a_1 - a_3)}{2} + 2\mu \right\}$	$-\frac{\ell^2}{12A} \left\{ \frac{a_1 (a_2 - a_1)}{2} - 2\mu \right\}$
$\frac{\ell^2}{4A} \left(\frac{a_2^2}{2} + \mu \right)$	$\frac{\ell^2}{4A} \left(\frac{a_2 a_3}{2} - \mu \right)$	$\frac{\ell^2}{4A} \left(\frac{a_2 a_3}{2} - \mu \right)$	$-\frac{\ell^2}{12A} \left\{ \frac{a_2^2}{2} (a_3 - a_2) - 2\mu \right\}$	$-\frac{\ell^2}{12A} \left\{ \frac{a_2 (a_1 - a_3)}{2} \right\}$	$-\frac{\ell^2}{12A} \left\{ \frac{a_2 (a_2 - a_1)}{2} + 2\mu \right\}$
$\frac{\ell^2}{4A} \left(\frac{a_3^2}{2} + \mu \right)$	$\frac{\ell^2}{4A} \left(\frac{a_3 a_2}{2} + \mu \right)$	$\frac{\ell^2}{4A} \left(\frac{a_3 a_2}{2} + \mu \right)$	$-\frac{\ell^2}{12A} \left\{ \frac{a_3^2}{2} (a_2 - a_1) + 2\mu \right\}$	$-\frac{\ell^2}{12A} \left\{ \frac{a_3 (a_1 - a_2)}{2} - 2\mu \right\}$	$-\frac{\ell^2}{12A} \left\{ \frac{a_3 (a_2 - a_1)}{2} \right\}$
			$\frac{\ell^2}{4A} \left\{ \frac{1}{6A^2} (9a_1^2 + a_2^2 + a_3^2) \right\}$	$\frac{\ell^2}{4A} \left\{ -\frac{1}{6A^2} (a_3^2 + 5a_1 a_2) \right\}$	$\frac{\ell^2}{4A} \left\{ -\frac{1}{6A^2} (a_2^2 + 5a_1 a_3) \right\}$
			$+ 3a_1 a_2 + 3a_1 a_3$	$+ a_1 a_3 + a_2 a_3 + \mu \}$	$+ a_1 a_2 + a_2 a_3 + \mu \}$
			$- a_2 a_3 + \mu \}$		
				$\frac{\ell^2}{4A} \left\{ \frac{1}{6A^2} (9a_2^2 + a_3^2) \right\}$	$\frac{\ell^2}{4A} \left\{ -\frac{1}{6A^2} (a_1^2 + 5a_2 a_3) \right\}$
				$+ a_1^2 + 3a_2 a_3 + 3a_2 a_1$	$+ a_1 a_2 + a_3 a_1 + \mu \}$
				$- a_3 a_1 + \mu \}$	
					$\frac{\ell^2}{4A} \left\{ \frac{1}{6A^2} (9a_3^2 + a_1^2 + a_2^2) \right\}$
					$+ 3a_3 a_1 + 3a_3 a_2$
					$- a_2 a_1 + \mu \}$

Appendix E. COMPUTER PROGRAM

E.1. Sequence of Computer Operations

- (1) Start.
- (2) Read input data (see Sec. E.4).
- (3) Construct matrix of sin and cos functions associated with the included angles of half period strip.
- (4) Determine the node position vector components for the perfect undeformed shell.
- (5) Complete construction of S matrix.
- (6) Form node coordinate transformation matrices $D_{1(0)}^{(a)}$
- (7) Form the nonzero elements of $C_{R(a)}$
- (8) Interpret boundary conditions and determine the unspecified elements of V .
- (9) Determine the node points associated with member i, j .
- (10) Compute the side lengths of member i, j .
- (11) Compute some constants associated with members i, j and used in (20).
- (12) Form membrane and bending stiffness matrices for member i, j .
- (13) Form member transformation matrix M_i for member ij .
- (14) Repeat (9) to (13) for $j = (1, 12)$ or $(2, 12)$ or $(1, 11)$ or $(2, 11)$ if both ends are open, only bottom is closed, only top is closed, both top and bottom are closed.
- (15) Correct components of node position vectors for small imperfection.

- (16) Start non-linear iteration cycle.
- (17) Clear shell stiffness, generalized implied load, and generalized unit normal load matrices.
- (18) Start construction of stiffness matrix.
- (19) Determine node points of member k .
- (20) Form member coordinate transformation matrix $D_{(k)}^{(0)}$
- (21) Form member transformation matrix M_{V_k} .
- (22) Form $\Gamma_k M_{V_k}$.
- (23) Form shell stiffness of member k $(M_{V_k}^T \Gamma_k M_{V_k})$.
- (24) Calculate generalized forces due to a unit normal load acting on member k .
- (25) Use (22) to determine change in member k generalized forces $\Delta \Sigma_k$ due to previous linear increment, and correct Σ_k .
- (26) Use (21) to obtain generalized implied load associated with member k and add to generalized implied load matrix.
- (27) Add the elements of (23) to appropriate elements of shell stiffness matrix.
- (28) Repeat (19) to (27) for all members and thus form shell stiffness, generalized implied load, and generalized unit normal load matrices.
- (29) Determine axial load.
- (30) Print, iteration number, axial force, axial deflection, generalized displacements, and generalized implied loads.
- (31) If the specified member of linear increment have been performed, stop; otherwise, continue.

- (32) Form load increment ΔP , associated with residual load, displacement increment, and normal surface load. (The bracketed term of 2.59).
- (33) Form shell stiffness submatrix K_{11} (2.45).
- (34) Form K_{11}^{-1} .
- (35) Compute generalized displacement increment ΔV_1 .
- (36) Correct generalized displacements V_1 .
- (37) Correct components of node position vector.
- (38) Return to (17).

E. 2. List of Fortran Symbols

A(12, 9, 2)	M_{V_i}
AREA	A area of triangle
AXDIS	δ_a average axial displacement of top edge
AXLOAD	$P_{(1)}^4, P_{(9)}^4$ resultant axial load of half period strip
B(9, 5)	non zero elements of $C_{R(a)}$
C(12, 9, 12)	M_i for 12 members with zero element submatrices deleted
CMU	$\mu = 2(1-\nu)$
CNP	n number of periods
CNU	ν Poisson's Ratio
COSCON(5)	$= a_1, a_2, a_3, a_4, a_5$ member constants
COSM1(5)	$\cos[(a-3)\frac{\pi}{4n}]$ for $a = 1, \dots, 5$

COSM2(5)	$\cos[(a - 3) \frac{\pi}{4}]$ for $a = 1, \dots, 5$
CSLOAD	constant uniform normal surface load
CT(12, 6)	$M_{i\ell} D_{(i)}^{(0)}$ $\ell = 1$ or 2 or 3 submatrix of $M_i D_{(i)}^{(0)}$
DQ(7, 9)	ΔP generalized load increment
DU(3)	matrix used for intermediate operations
DV(7, 9)	ΔV generalized displacement increment
E	Young's Modulus
EK(1E, 1E)	K_{11} see (4.24) shell stiffness of k member
EP(6, 12)	member constants used to constant $D_{(0)}^{(i)}$
EPS	imperfection constant
FE(4, 12, 12)	the elements of Σ arranged in 3D array
FIMP(7)	imperfection function
GK(63, 63)	complete stiffness matrix K
GRS(63, 63)	submatrix of K designed by K_{11} in (2.45)
I1, I2, I3	integer constant of node points a_1, a_2, a_3
IBC(63)	matrix of integer constants designating non-zero elements of V
IBC2(4), IBC3(4) IBC4(8)	matrix of integer constants describing unspecified elements of $V_{(1)}$ and $V_{(7)}$ for hinged boundary conditions, for symmetry boundary conditions, for fixed boundary conditions
IBCBD, IBCBR, IBCTD, IBCTR	integer constants either 1 or 0 used to specify boundary conditions
ITCYCLE	number of iteration cycles
J1, J2, J3	integer constants of node points $\beta_1, \beta_2, \beta_3$

KE, K9, LE, L9	integer constants
M0(6, 48)	S matrix
M0B(3), M0B(3), M0D(9), M0E(3), M0G(3)	matrices of integers used to locate the position of elements in a matrix
NGKS	integer indicating the rank of K_{11}
NP	number of periods
OPA(3, 3), OPB(3, 3)	matrices used to describe the sign preceding μ in the construction of member stiffness matrices
PINCON(40)	$\rho(\tau_k)$ displacement increment function
Q(63)	ΔV_1 submatrices of ΔV described in (2.45)
QN(7, 9)	generalized forces associated with uniform normal surface load
RR(7), RX(5, 7), RY(5, 7), RZ(5, 7)	components of position vector
SA(3, 3, 5)	$D_{(0)}^{(a)}$ for $a = 1, \dots, 5$ node coordinate trans- formation
SE(3, 3)	$D_{(i)}^{(0)}$
SIDE(8)	$l_1, l_2, l_3, l_1, l_2, \dots$ matrix of member side lengths
SIDESQ(8)	$l_1^2, l_2^2, l_3^2, l_1^2, \dots$
SINM2(5)	$\sin[(a-3)\frac{\pi}{4n}]$ for $a = 1, \dots, 5$
STIFB(6, 6, 12)	bending stiffness matrices for 12 members
STIFM(6, 6, 12)	membrane stiffness matrices for 12 members
TAXLOAD	$2n p_{(7)}^{(4)}$ total axial load
V(7, 9)	matrix of generalized displacements
WB1, WB2, WB3, WM1, WM2, WM3, WS(4)	stiffness coefficients
WCON	residual load relaxation constant

```

JOB,031914,GL,10. LASKER, G
FTN,P,L,X,*.
PROGRAM SASS
COMMON/BK/A(12,9,2),SE(3,3),SA(3,3,5),B(9,5) ,C(12,9,12),RX(5,7),
1  RY(5,7),RZ(5,7),RR(7),SINM1(5),COSM1(5),SINM2(5),COSM2(5),
2  MO(6,48),STIFM(6,6,12),STIFB(6,6,12),T(12),SIDE(8),SIDESQ(8),
3  COSCON(5),EP(6,12),OPA(3,3),MOB(3),MOC(3),FE(4,12,12),Q(63),
4  QN(7,9),DV(7,9),DQ(7,9), DU(3), V(7,9), SK(12,18),
5  EK(18,18),GK(63,63),QI(7,9),GKS(63,63), IBC3(4),IBC4(8),
6  IBC(63) ,IBC2(4),I1,I2,I3,J1,J2,J3,I,J,
7  K,AREA,E,CNU,SLOAD,NP,AXLOAD,MOD(9),MOE(3),MOG(3),CT(12,6)
8  ,QUINNORM, AXDIS,CMU,CNP,OPB(3,3),K8,K9,L8,L9,2T(3,3,2)
9  ,FIMP(7),GUN(7,9),WS(4),WM1,WM2,WM3,WB1,WB2,WB3,PINCON(80)
1  FORMAT(2E10)
2  FORMAT(7F10,4)
4  FORMAT(12F5,3)
5  FORMAT(4I1)
6  FORMAT(E10,3)
8  FORMAT(I2)
9  FORMAT(4OF2,1)
18  FORMAT(1H1)
19  FORMAT(20X,75HAXIAL LOAD DEFLECTION CHARACTERISTICS OF AN AXIALLY
15SYMMETRIC ELASTIC SHELL//20X,44HALL LISTINGS ARE FROM BOTTOM TO TO
2P OF SHELL/20X,34HALL UNITS ARE IN INCHES AND POUNDS/)
20  FORMAT(20X,14HYOUNGS MODULUS,16X,E10,2/20X,14HPOISSONS RATIO,16X,
1F10,3/20X,17HNUMBER OF PERIODS,13X,I10/20X,28HAXIAL DISPLACEMENT I
2NCREMENT,2X,E10,2/20X,26HCONSTANT INTERNAL PRESSURE,4X,F10,3/20X,
3 21HIMPERFECTION CONSTANT,9X,F10,5/20X,19HRELAXATION CONSTANT,11X,
4 F10,3/20X,22HSTIFFNESS COEFFICIENTS,8X,4F8,3)
21  FORMAT(20X,23HDIMENSIONS OF GENERATOR//20X,1HR,10X,7F12,4/20X,1HZ

```

```

1      .10X,7F12.4/20X,1HT,16X,12F6.3)
30 FORMAT(20X,19HBOUNDARY CONDITIONS)
31 FORMAT(25X,11HBOTTOM-FREE)
32 FORMAT(25X,15HBOTTOM-SYMMETRY)
33 FORMAT(25X,13HBOTTOM-HINGED)
34 FORMAT(25X,12HBOTTOM-FIXED)
35 FORMAT(25X,8HTOP-FREE)
36 FORMAT(25X,12HTOP-SYMMETRY)
37 FORMAT(25X,10HTOP-HINGED)
38 FORMAT(25X,9HTOP-FIXED)
40 FORMAT(/ /20X,16HITERATION NUMBER,20X,14)
41 FORMAT(20X,10HAXIAL LOAD,20X,E10.3)
42 FORMAT(20X,16HAXIAL DEFLECTION,14X,E10.3)
43 FORMAT(/20X,35HGENERALIZED DISPLACEMENT COMPONENTS/10X,11X,2HV1,
1 11X,2HV2,11X,2HV3,11X,2HV4,11X,2HV5,11X,2HV6,11X,2+V7,11X,2HV8,
2 11X,2HV9/)
44 FORMAT(10X,9E13.3)
45 FORMAT(25X,24HBOTTOM-CLOSED CONTINUITY)
46 FORMAT(25X,21HTOP-CLOSED CONTINUITY)
47 FORMAT(/20X,21HIMPERFECTION FUNCTION/20X,1HR,10X,7F12.2)
50 FORMAT(8011)
51 FORMAT(9F3,311,311,5F3)
52 FORMAT(411,411,811,511,511,911)
53 FORMAT(9F3)
54 FORMAT(/20X,35HGENERALIZED IMPLIED LOAD COMPONENTS/)
    READ 50,((MO(I,J),I=1,6),J=1,16)
    READ 51,((OPA(I,J),I=1,3),J=1,3),(MOB(K),K=1,3),(MOC(L),L=1,3)
    READ 52,((IBC2(I),I=1,2),(IBC3(J),J=1,4),(IBC4(K),K=1,8)
    READ 53,((OPB(I,J),I=1,3),J=1,3)
    READ 50,((MOD(I),I=1,9),(MOE(J),J=1,3),(MOG(K),K=1,3)
    READ 4,((WS(I),I=1,4)
    READ 1,E,CNU
    READ 2,((RR(I),I=1,7)
    READ 2,((RZ(I,J),J=1,7)
    READ 4,((T(I),I=1,12)
    READ 5,IBCTD,IBCTR,IBCB,IBCB,IBCB
    READ 6,WCON

```

```

READ 6,VINC
READ 6,CSLOAD
READ 9,PINCON
READ8,NP
READ8,ITCYCLE
READ6,EPS
READ 2,(FIMP(I),I=1,7)
PRINT18
PRINT 19
PRINT 20,E,CNU,NP,VINC,CSLOAD,EPS,WCON,WS
PRINT 21,(RR(I),I=1,7),(RZ(1,J),J=1,7),(T(K),K=1,12)
PRINT 47,(FIMP(I),I=1,7)
FORM SINM1,COSM1,SINM2,COSM2
PI=3.14159265
CMU=(1-CNU)*2.0
CNP=NP
DO100I=1,5
CONB1=1
CONA1=((CONB1-3.0)*PI)/(4.0*CNP)
CONA2=((CONB1-3.0)*PI)/4.0
SINM1(I)=SINF(CONA1)
COSM1(I)=COSF(CONA1)
SINM2(I)=SINF(CONA2)
COSM2(I)=COSF(CONA2)
100 FORM NODE POSITION VECTOR FOR UNDEFORMED SHELL
C
DO110I=1,5
DO110J=1,7
RX(I,J)=RR(J)*COSM1(I)
RY(I,J)=RR(J)*SINM1(I)
110 RZ(I,J)=RZ(1,J)
C FORM MO MATRIX
DO120J=17,48
DO120I=1,3
MO(2*I-1,J)=MO(2*I-1,J-16)
120 MO(2*I,J)=MO(2*I,J-16)+2
C FORM NODE COORDINATE TRANSFORMATION MATRIX

```

```

DO140 I=1,5
  SA(1,1,I)=COSM1(I)
  SA(2,1,I)=SINM1(I)
  SA(3,1,I)=0.0
  SA(1,2,I)=-SINM1(I)
  SA(2,2,I)=COSM1(I)
  SA(3,2,I)=0.0
  SA(1,3,I)=0.0
  SA(2,3,I)=0.0
  SA(3,3,I)=1.0
140 C FORM GENERALIZED DISPLACEMENT MATRIX OPERATOR
      DO160 K=1,5
        B(1,K)=1.0
        B(2,K)=SINM2(K)
        B(3,K)=COSM2(K)
        B(4,K)=1.0
        B(5,K)=SINM2(K)
        B(6,K)=COSM2(K)
        B(7,K)=1.0
        B(8,K)=SINM2(K)
        B(9,K)=COSM2(K)
160 C APPLICATION OF BOUNDARY CONDITIONS
      PRINT 30
      IF(RR(1))162,162,163
162 NB=2
      PRINT 45
      L=0
      GO TO 178
163 NB=1
168 IF(IBC8D)169,169,170
169 IF(IBC8R)171,171,172
171 L=0
      PRINT 34
      GO TO 178
172 DO 173 I=1,2
173 IBC(I)=IBC2(I)

```

```

L=2
PRINT 33
GO TO 178
170 IF(IBC(R))174,174,175
174 DO 176 I=1, 4
176 IBC(I)=IBC3(I)
L=4
PRINT 32
GO TO 178
175 DO 177,I=1,8
177 IBC(I)=IBC4(I)
L=8
PRINT 31
178 DO179 I=1,45
179 IBC(I+L)=9+I
M=L+45
IF(RR(7))164,164,165
164 NT=11
NGKS=M
PRINT 46
GO TO 189
165 NT=12
167 IF(IBC(TD))180,180,181
180 IF(IBC(TR))182,182,183
182 NGKS=M
PRINT 38
GO TO 189
183 DO 184 I=1,2
184 IBC(M+I)=IBC2(I)+54
NGKS=M+2
PRINT 37
GO TO 189
181 IF(IBC(TR))185,185,186
185 DO187I=1,4
187 IBC(M+I)=IBC3(I)+54
NGKS=M+4

```

```

PRINT 36
GO TO 189
186 DO 188 I=1,8
188 IBC(M+I)=IBC4(I)+54
NGKS=M+8
PRINT 35
189 CONTINUE
C FORM ELEMENT STIFFNESS AND PARAMETER MATRICES
WM1=WS(1)*WS(1)
WM2=WS(1)*WS(2)
WM3=WS(2)*WS(2)
WB1=WS(3)*WS(3)/WM1
WB2=WS(3)*WS(4)
WB3=WS(4)*WS(4)
DO 190 J=NB,NT
K=4#J-3
I1=MO(1,K)
I2=MO(3,K)
I3=MO(5,K)
J1=MO(2,K)
J2=MO(4,K)
J3=MO(6,K)
CALL EPAR
190 CONTINUE
C CORRECT POSITION VECTOR FOR INITIAL IMPERFECTION
DO 195 I=1,5
DO 195 J=1,7
RZ(I,J)=RZ(I,J)+EPS*FIMP(J)*SINM2(I)
RX(I,J)=RX(I,J)+EPS*FIMP(J)*SINM2(I)*COSM1(I)
195 RY(I,J)=RY(I,J)+EPS*FIMP(J)*SINM2(I)*SINM1(I)
C AXIAL LOAD DEFLECTION ITERATIVE CYCLE
C CLEAR MATRICES
AXDIS=0.0
ANDIS=0.0
DO 200 I=1,7
DO 200 J=1,9

```

```

      QI(I,J)=0.0
      DQ(I,J)=0.0
      DV(I,J)=0.0
      200 V(I,J)=0.0
      DO 210 I=1,4
      DO 210J=1,12
      DO 210 K=1,12
      210 FE(I,J,K)=0.0
      C   START OUTER SHELL ITERATION CYCLE
      DO 355 N=1,ITCYCLE
      DO 205 I=1,63
      DO 205 J=1,63
      205 GK(I,J)=0.0
      DO 215 I=1,7
      DO 215 J=1,9
      QI(I,J)=0.0
      215 QUN(I,J)=0.0
      C   START INNER ELEMENT CYCLE
      DO 280 J=NB,NT
      DO 280 I=1,4
      K=4*(J-1)+I
      I1=M0(1,K)
      J1=M0(2,K)
      I2=M0(3,K)
      J2=M0(4,K)
      I3=M0(5,K)
      J3=M0(6,K)
      IF(J1-J2)220,220,221
      220 K8=1
      K9=2
      L8=J1
      L9=J2
      GO TO 222
      221 K8=2
      K9=1

```

```

L8=J2
L9=J1
222 CONTINUE
      CALL ASKEK
      CORRECT ELEMENT FORCE COMPONENTS
      DO 240 L=1,12
        CONA1=0.0
        CONB1=0.0
      DO 230 M=1,9
        CONA1=CONA1+SK(L,M)*DV(L8,M)
230    CONB1=CONB1+SK(L,M+9)*DV(L9,M)
240    FE(I,J,L)=FE(I,J,L)+CONA1+CONB1
      C      FORM IMPLIED LOAD MATRIX
      DO 260 L=1,9
        CONA1=0.0
        CONB1=0.0
      DO 250 M=1,12
        CONA1=CONA1+FE(I,J,M)*A(M,L,K8)
250    CONB1=CONB1+FE(I,J,M)*A(M,L,K9)
        QI(L8,L)=QI(L8,L)+CONA1
260    QI(L9,L)=QI(L9,L)+CONB1
      C      FORM GENERAL STIFFNESS MATRIX
        JA=9*(L8-1)
      DO 270 L=1,18
      DO 270 M=1,18
270    GK(JA+L,JA+M)=GK(JA+L,JA+M)+EK(L,M)
280 CONTINUE
      C      DETERMINE AXIAL LOAD
        TAXLOAD=(QI(7,4)-CSLOAD*QUN(7,4))*2.0*CNP
        AXLOAD=TAXLOAD-PI*(RX(3,7)**2)*CSLOAD
        PRINT 40,N
        PRINT 41,AXLOAD
        PRINT 42,AXDIS
        PRINT 43
        PRINT 44,((V(L,M),M=1,9),L=1,7)
        PRINT 54

```

```

PRINT 44,((QI(L,M),M=1,9),L=1,7)
IF(ITCYCLE-N)355,355,289
289 CONTINUE
C FORM LOAD INCREMENT
DO 290 L=1,9
DO 290 M=1,7
290 DQ(M,L)=-QI(M,L)*WCON+CSLOAD*QUN(M,L)
C FORM REDUCED STIFFNESS AND FLEXIBILITY MATRICES
DO 300 L=1,NGKS
DO 300 M=1,NGKS
L1=IBC(L)
M1=IBC(M)
300 GKS(L,M)=GK(L1,M1)
DO 310 L=1,NGKS
M=IBC(L)
L1=(IBC(L)-1)/9+1
M1=IBC(L)-9*(L1-1)
310 Q(L)=DQ(L1,M1)-GK(M,58)*VINC*PINCON(N)
CALL MATINV(GKS,NGKS,Q,1)
DO 330 L=1,NGKS
L1=(IBC(L)-1)/9+1
M1=IBC(L)-9*(L1-1)
330 DV(L1,M1)=Q(L)
DV(7,4)=VINC*PINCON(N)
AXDIS=AXDIS+DV(7,4)
C CORRECT GENERALIZED DISPLACEMENT VECTORS
DO 332 L=1,7
DO 332 M=1,9
332 V(L,M)=V(L,M)+DV(L,M)
C CORRECT POSITION VECTORS
DO 350 L=1,5
DO 350 M=1,7
DU(1)=B(1,L)*DV(M,1)+B(2,L)*DV(M,2)
DU(2)=B(3,L)*DV(M,3)
DU(3)=B(4,L)*DV(M,4)+B(5,L)*DV(M,5)
CONA1=0.0

```

```

CONB1=0.0
CONC1=0.0
DO 340 M1=1,3
  CONA1=CONA1+SA(1,M1,L)*DU(M1)
  CONB1=CONB1+SA(2,M1,L)*DU(M1)
340  CONC1=CONC1+SA(3,M1,L)*DU(M1)
      RX(L,M)=RX(L,M)+CONA1
      RY(L,M)=RY(L,M)+CONB1
350  RZ(L,M)=RZ(L,M)+CONC1
355  CONTINUE
END
SUBROUTINE EPAR
COMMON/BK/A(12,9,2),SE(3,3),SA(3,3,5),B(9,5),C(12,9,12),RX(5,7),
1  RY(5,7),RZ(5,7),RR(7),SINM1(5),COSM1(5),SINM2(5),COSM2(5),
2  MO(6,48),STIFM(6,6,12),STIFB(6,6,12),T(12),SIDE(8),SIDESQ(8),
3  COSCON(5),EP(6,12),OPA(3,3),MOB(3),MOC(3),FE(4,12,12),Q(63),
4  QN(7,9),DV(7,9),DQ(7,9),DU(3),V(7,9),SK(12,18),
5  EK(18,18),GK(63,63),QI(7,9),GKS(63,63),IBC3(4),IBC4(8),
6  IBC(63),IBC2(4),I1,I2,I3,J1,J2,J3,I,J,
7  K,AREA,E,CNU,SLOAD,NP,AXLOAD,MOD(9),MOE(3),MOG(3),CT(12,6)
8  ,GUNNORM,AXDIS,CMU,CNP,OPB(3,3),K8,K9,L8,L9,ST(3,3,2)
9  ,FIMP(7),GUN(7,9),WS(4),WM1,WM2,WM3,WB1,WB2,WB3,PINCON(80)
C  FORM ELEMENT SIDE MATRIX
  SIDE(1)=(RX(I3,J3)-RX(I2,J2))**2 +(RY(I3,J3)-RY(I2,J2))**2 +
1  (RZ(I3,J3)-RZ(I2,J2))**2 )**0.5
  SIDE(2)=(RX(I1,J1)-RX(I3,J3))**2 +(RY(I1,J1)-RY(I3,J3))**2 +
1  (RZ(I1,J1)-RZ(I3,J3))**2 )**0.5
  SIDE(3)=(RX(I2,J2)-RX(I1,J1))**2 +(RY(I2,J2)-RY(I1,J1))**2 +
1  (RZ(I2,J2)-RZ(I1,J1))**2 )**0.5
C  FORM SIDE SIDESQ AND COSCON MATRICES
DO 400 I=1,5
400  SIDE(I+3)=SIDE(I)
DO410 I=1,8
410  SIDESQ(I)=SIDE(I)**2
DO 420 I=1,5
420  COSCON(I)=(SIDESQ(I+1)+SIDESQ(I+2)-SIDESQ(I))/2.0

```

```

      AREA=((COSCON(1)*COSCON(2)+COSCON(2)*COSCON(3)+COSCON(3)*COSCON(1)
1      )**0.5)/2.0
      C      FORM ELEMENT PARAMETER MATRIX
      EP(1,J)=1.0/SIDE(3)
      EP(5,J)=1.0/(2.0*AREA)
      EP(2,J)=-COSCON(2)*EP(1,J)*EP(5,J)
      EP(3,J)=-COSCON(1)*EP(1,J)*EP(5,J)
      EP(4,J)=SIDE(3)*EP(5,J)
      EP(6,J)=AREA
      C      FORM STIFFNESS MATRICES
      CONA1=(E*T(J))/(4.0*(1.0-(CNU**2)))
      CONA2=(T(J)**2)/(3.0)
      DO 510 L=1,3
      DO 500 M=1,3
      STIFM(L,M,J)=WM1*((SIDE(L)*SIDE(M))/(4.0*AREA))*((COSCON(L)*
1      COSCON(M))/(AREA**2))-OPB(L,M)*CMU)*CONA1
      STIFB(L,M,J)= CONA2*STIFM(L,M,J)*WB1
      STIFM(L,M+3,J)=WM2*(SIDE(L)/(12.0))*((COSCON(L)*COSCON(M+2))-
1      COSCON(M+1))/(AREA**2)+OPA(L,M)*2.0 *CMU)*CONA1
      STIFB(L,M+3,J) = (CONA2*SIDE(M)*STIFM(L,M+3,J)/AREA)*(-WB2*.75)
      STIFM(M+3,L,J)=STIFM(L,M+3,J)
500  STIFB(M+3,L,J)=STIFB(L,M+3,J)
      STIFM(L+3,L+3,J)=(WM3*CONA1/(72.0*AREA))*((2.0*COSCON(L)**2+3.0*
1      COSCON(L+1)**2+3.0*COSCON(L+2)**2-2.0*COSCON(L)*COSCON(L+1))-
2      4.0*COSCON(L+1)*COSCON(L+2)-2.0*COSCON(L+2)*COSCON(L))+16.0*CMU
3      *AREA**2)
      STIFB(L+3,L+3,J)=WB3*CONA1*CONA2*(SIDE(L)/(4.0*AREA))*((1.0/
1      (2.0*(AREA**2)))+(8.0*COSCON(L)**2+COSCON(L+1)**2+COSCON(L+2)**2
2      +4.0*COSCON(L)*COSCON(L+1)+4.0*COSCON(L)*COSCON(L+2))+CMU)
      N1=MOB(L)
      N2=MOC(L)
      STIFM(N1,N2,J)=(WM3*CONA1*AREA/72.0)*((-2.0*COSCON(L)**2+
1      COSCON(L+1)**2+COSCON(L+2)**2+2.0*COSCON(L)*COSCON(L+1))-4.0*
2      COSCON(L+1)*COSCON(L+2)+2.0*COSCON(L+2)*COSCON(L))
      STIFB(N1,N2,J)= CONA1*CONA2*(SIDE(L+1)*SIDE(L+2)/(4.0*AREA))*
1      ((-1.0/(6.0*AREA**2))*((2.0*COSCON(L)**2+2.0*COSCON(L+1)**2+2.0*

```

```

2. COSCON(L+2)*#2+COSCON(L)*COSCON(L+1)+10.0*COSCON(L+1)*
3. COSCON(L+2)+COSCON(L+2)*COSCON(L)+CMU)*WB3
   STIFM(N2,N1,J)=STIFM(N1,N2,J)
   STIFB(N2,N1,J)=STIFB(N1,N2,J)
510 CONTINUE
C   FORM ELEMENT TRANSFORMATION MATRICES
   DO 600 L=1,12
   DO 600 M=1,9
600 C(L,M,J)=0.0
      CONB1=(2.0*AREA)/(SIDE(2)*SIDE(3))
      CONB2=(2.0*AREA)/(SIDE(1)*SIDE(3))
      CONB3=(2.0*AREA)/(SIDE(1)*SIDE(2))
      CONC1=(COSCON(1))/(SIDE(2)*SIDE(3))
      CONC2=(COSCON(2))/(SIDE(1)*SIDE(3))
      CONC3=(COSCON(3))/(SIDE(1)*SIDE(2))
      COND1=2.0/SIDE(1)
      COND2=2.0/SIDE(2)
      COND3=2.0/SIDE(3)
      C(1,4,J)=CONC2
      C(1,5,J)=-CONB2
      C(1,7,J)=-CONC2
      C(1,8,J)=CONB2
      C(2,1,J)=-CONC1
      C(2,2,J)=-CONB1
      C(2,7,J)=CONC1
      C(2,8,J)=CONB1
      C(3,1,J)=-1.0
      C(3,4,J)=1.0
      C(4,1,J)=-(CONC2*COND3*0.25)/CONB2
      C(4,2,J)=COND3*0.25
      C(4,3,J)=1.0
      C(4,4,J)=-(CONC1*COND3*0.25)/CONB1
      C(4,5,J)=-C(4,2,J)
      C(4,7,J)=(COND2*0.25)/CONB1
      C(5,1,J)=C(4,1,J)

```

C(5.2,J)=C(4.2,J)
C(5.4,J)=C(4.4,J)
C(5.5,J)=C(4.5,J)
C(5.6,J)=1.0
C(5.7,J)=C(4.7,J)
C(6.1,J)=C(4.1,J)
C(6.2,J)=C(4.2,J)
C(6.4,J)=C(4.4,J)
C(6.5,J)=C(4.5,J)
C(6.7,J)=C(4.7,J)
C(6.9,J)=1.0
C(7.5,J)=CONB2*0.5
C(7.6,J)=CONC2*0.5
C(7.8,J)=-CONB2*0.5
C(7.9,J)=-CONC2*0.5
C(8.2,J)=CONB1*0.5
C(8.3,J)=-CONC1*0.5
C(8.8,J)=-CONB1*0.5
C(8.9,J)=CONC1*0.5
C(9.3,J)=-0.5
C(9.6,J)=0.5
C(10.4,J)=COND1*0.5
C(10.5,J)=CONB2*0.5
C(10.6,J)=CONC2*0.5
C(10.7,J)=-COND1*0.5
C(10.8,J)=CONB2*0.5
C(10.9,J)=CONC2*0.5
C(11.1,J)=-COND2*0.5
C(11.2,J)=-CONB1*0.5
C(11.3,J)=CONC1*0.5
C(11.7,J)=COND2*0.5
C(11.8,J)=-CONB1*0.5
C(11.9,J)=CONC1*0.5
C(12.1,J)=COND3*0.5
C(12.3,J)=-0.5
C(12.4,J)=-COND3*0.5
C(12.6,J)=-0.5

```

RETURN
END
SUBROUTINE ASKEK
COMMON/BK/A(12,9,2),SE(3,3),SA(3,3,5),B(9,5),C(12,9,12),RX(5,7),
1  RY(5,7),RZ(5,7),RR(7),SINM1(5),COSM1(5),SINM2(5),COSM2(5),
2  MO(6,48),STIFM(6,6,12),STIFB(6,6,12),T(12),SIDE(8),SIDESQ(8),
3  COSCON(5),EP(6,12),OPA(3,3),MOB(3),MOC(3),FE(4,12,12),Q(63),
4  QN(7,9),DV(7,9),DQ(7,9),DU(3),V(7,9),SK(12,18),
5  EK(18,18),GK(63,63),QI(7,9),GKS(63,63),IBC3(4),IBC4(8),
6  IBC(63),IBC2(4),I1,I2,I3,J1,J2,J3,I,J,
7  K,AREA,E,CNU,SLOAD,NP,AXLOAD,MOD(9),MOE(3),MOG(3),CT(12,6)
8  ,GUNNORM,AXDIS,CMU,CNP,OPB(3,3),K8,K9,L8,L9,ST(3,3,2)
9  ,FIMP(7),QUN(7,9),WS(4),WM1,WM2,WM3,WB1,WB2,WB3,PINCON(80)
C  FORM ELEMENT COORDINATE TRANSFORMATION MATRIX
SE(1,1)=EP(1,J)*RX(12,J2)-RX(11,J1)
SE(1,2)=EP(1,J)*RY(12,J2)-RY(11,J1)
SE(1,3)=EP(1,J)*RZ(12,J2)-RZ(11,J1)
SE(2,1)=EP(2,J)*RX(11,J1)+EP(3,J)*RX(12,J2)+EP(4,J)*RX(13,J3)
SE(2,2)=EP(2,J)*RY(11,J1)+EP(3,J)*RY(12,J2)+EP(4,J)*RY(13,J3)
SE(2,3)=EP(2,J)*RZ(11,J1)+EP(3,J)*RZ(12,J2)+EP(4,J)*RZ(13,J3)
SE(3,1)=EP(5,J)*RY(11,J1)*RZ(12,J2)-RZ(11,J1)*RY(12,J2)
1  +RY(12,J2)*RZ(13,J3)-RZ(12,J2)*RY(13,J3)
2  +RY(13,J3)*RZ(11,J1)-RZ(13,J3)*RY(11,J1)
SE(3,2)=EP(5,J)*RZ(11,J1)*RX(12,J2)-RX(11,J1)*RZ(12,J2)
1  +RZ(12,J2)*RX(13,J3)-RX(12,J2)*RZ(13,J3)
2  +RZ(13,J3)*RX(11,J1)-RX(13,J3)*RZ(11,J1)
SE(3,3)=EP(5,J)*RX(11,J1)*RY(12,J2)-RY(11,J1)*RX(12,J2)
1  +RX(12,J2)*RY(13,J3)-RY(12,J2)*RX(13,J3)
2  +RX(13,J3)*RY(11,J1)-RY(13,J3)*RX(11,J1)
C  FORM MATRIX A FOR TRIANGULAR ELEMENT(I,J)
DO 710 I4=1,12
DO 710 M1=1,9
DO 710 N2=1,2
710 A(I4,M1,N2)=0.0
DO 700 L=1,3
DO 700 M=1,3

```

```

CONA1=0.0
CONB1=0.0
DO 701 N=1,3
  CONA1=CONA1+SE(L,N)*SA(N,M,I1)
  CONB1=CONB1+SE(L,N)*SA(N,M,I3)
701 ST(L,M,1)=CONA1
700 ST(L,M,2)=CONB1
DO 718 N1=1,3
  NA=(N1-1)*3
  IA=MO(2*N1-1,K)
  N2=MOE(N1)
  N3=MOG(N1)
713 DO 716 L=1,6
  DO 716 M=1,3
    CT(L,M)=C(L,1+NA,J)*ST(1,M,N3)+C(L,2+NA,J)*ST(2,M,N3)
    CT(L,M+3)=C(L,3+NA,J)*ST(3,M,N3)
    CT(L+6,M)=C(L+6,1+NA,J)*ST(3,M,N3)
716 CT(L+6,M+3)=C(L+6,2+NA,J)*ST(1,M,N3)+C(L+6,3+NA,J)*ST(2,M,N3)
DO 718 L=1,12
DO 718 M=1,9
  N=MOD(M)
718 A(L,M,N2)=A(L,M,N2)+CT(L,N)*B(M,IA)
  C
  FORM SK
DO 740 L=1,6
DO 740M=1,9
  CONA1=0.0
  CONB1=0.0
  CONC1=0.0
  COND1=0.0
DO 730 N=1,6
  CONA1=CONA1+STIFM(L,N,J)*A(N,M,K8)
  CONB1=CONB1+STIFM(L,N,J)*A(N,M,K9)
  CONC1=CONC1+STIFB(L,N,J)*A(N+6,M,K8)
730 COND1=COND1+STIFB(L,N,J)*A(N+6,M,K9)
  SK(L ,M )=CONA1
  SK(L ,M+9)=CONB1

```

```

      SK(L+6,M )=CONC1
740 SK(L+6,M+9)=COND1
      C
      FORM EK
      DO 770 L=1,9
      DO 755 M=1,L
      CONA1=0.0
      CONB1=0.0
      DO 750 N=1,12
      CONA1=CONA1+A(N,L,K8)*SK(N,M)
750 CONB1=CONB1+A(N,L,K9)*SK(N,M+9)
      EK(L,M)=CONA1
      EK(L+9,M+9)=CONB1
      EK(M,L)=EK(L,M)
755 EK(M+9,L+9)=EK(L+9,M+9)
      DO 765 M=1,9
      CONA1=0.0
      DO 760 N=1,12
760 CONA1=CONA1+A(N,L,K8)*SK(N,M+9)
      EK(L,M+9)=CONA1
765 EK(M+9,L)=CONA1
770 CONTINUE
      C
      FORM UNIT NORMAL LOAD MATRIX
      IF(ST(3,1,1))810,812,812
810 CONA2=-1.0
      GO TO 814
812 CONA2=1.0
814 CONTINUE
      CONA3= EP(6,J)/3.0
      DO 820 L=1,3
      K1=MO(2*L-1,K)
      L1=MO(2*L,K)
      M1=MOG(L)
      DO 820 M=1,5
      M2=MOD(M)
820 QUN(L1,M)=QUN(L1,M)+CONA3*ST(3,M2,M1)*B(M,K1)*CONA2
      RETURN

```

```

END
SUBROUTINE MATINV(A,N,B,M,DETERM)
  DIMENSION IPIVOT(63), A(63,63), B(63,1), INDEX(63,2), PIVOT(63)
  C    INITIALIZATION
    10 DETERM=1.0
    15 DO 20 J=1,N
    20 IPIVOT(J)=0
    30 DO 50 I=1,N
    C    SEARCH FOR PIVOT ELEMENT
      40 AMAX=0.0
      45 DO 105 J=1,N
      50 IF (IPIVOT(J)-1) 60, 105, 60
      60 DO 100 K=1,N
      70 IF (IPIVOT(K)-1) 80, 100, 740
      80 IF (ABSF(AMAX)-ABSF(A(J,K))) 85, 100, 100
      85 IROW=J
      90 ICOLUM=K
      95 AMAX=A(J,K)
    100 CONTINUE
    105 CONTINUE
    110 IPIVOT(ICOLUM)=IPIVOT(ICOLUM)+1
  C    INTERCHANGE ROWS TO PUT PIVOT ELEMENT ON DIAGONAL
    130 IF (IROW-ICOLUM) 140, 260, 140
    140 DETERM=-DETERM
    150 DO 200 L=1,N
    160 SWAP=A(IROW,L)
    170 A(IROW,L)=A(ICOLUM,L)
    200 A(ICOLUM,L)=SWAP
    205 IF(M) 260, 260, 210
    210 DO 250 L=1, M
    220 SWAP=B(IROW,L)
    230 B(IROW,L)=B(ICOLUM,L)
    250 B(ICOLUM,L)=SWAP
    260 INDEX(I,1)=IROW
    270 INDEX(I,2)=ICOLUM
    310 PIVOT(I)=A(ICOLUM,ICOLUM)

```

```

320 DETERM=DETERM*PIVOT(I)
C   DIVIDE PIVOT ROW BY PIVOT ELEMENT
330 A(ICOL,ICOL)=1.0
340 DO 350 L=1,N
350 A(ICOL,L)=A(ICOL,L)/PIVOT(I)
355 IF(M) 380, 380, 360
360 DO 370 L=1,M
370 B(ICOL,L)=B(ICOL,L)/PIVOT(I)
C   REDUCE NON-PIVOT ROWS
380 DO 550 L1=1,N
390 IF(L1-ICOL) 400, 550, 400
400 T=A(L1,ICOL)
420 A(L1,ICOL)=0.0
430 DO 450 L=1,N
450 A(L1,L)=A(L1,L)-A(ICOL,L)*T
455 IF(M) 550, 550, 460
460 DO 500 L=1,M
500 B(L1,L)=B(L1,L)-B(ICOL,L)*T
550 CONTINUE
C   INTERCHANGE COLUMNS
600 DO 710 I=1,N
610 L=N+1-I
620 IF (INDEX(L,1)-INDEX(L,2)) 630, 710, 630
630 JROW=INDEX(L,1)
640 JCOLUMN=INDEX(L,2)
650 DO 705 K=1,N
660 SWAP=A(K,JROW)
670 A(K,JROW)=A(K,JCOLUMN)
700 A(K,JCOLUMN)=SWAP
705 CONTINUE
710 CONTINUE
740 RETURN
750 END
      SCOPE
•LOAD

```


E. 4. Binary Deck Structure

- (a) Job card
- (b) Binary object program
- (c) Run card
- (d) 6 data cards used for all problems
- (e) 15 data cards used to describe problem as follows:

Card (all listing are from bottom to top) Formate

(1) 4 stiffness constants 4F5.2

Primary membrane

Secondary membrane

Primary bending

Secondary bending

(2) Young's modulus, Poisson's ratio E10.2, E10.3

(3) Radial component of node points 7F10

(4) Vertical component of node points 7F10

(5) Thickness of triangles 12F5

(6) Boundary conditions are 4I1

described by 4 integers A, B, C, D.

A, B are for the top edge and

B, C are for the bottom edge.

A, C = 0 or 1 if radial displacement

is 0 or unconstrained

B, D = 0, or 1 if rotation tangent

to the edge is 0 or unconstrained

(7) Relaxation constant E10.3

(8) Axial displacement increment	E10.3
(9) Internal pressure	E10.3
(10) and (11) Axial displacement increment function	80F2.1
(12) Number of periods	I2
(13) Number of axial displacement increments	I2
(14) Imperfection constant	E10.3
(15) Imperfection function	7F10

MICHIGAN STATE UNIVERSITY LIBRARIES



3 1293 03085 6599