OPTIMAL CONTROL OF LINEAR DISCRETE MACRO-ECONOMIC SYSTEMS

Thesis for the Degree of Ph. D.
MICHIGAN STATE UNIVERSITY
KIOUMARS PARYANI
1972

This is to certify that the

thesis entitled

OPTIMAL CONTROL OF LINEAR
DISCRETE MACRO-ECONOMIC SYSTEMS

presented by

Kioumars Paryani

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Electrical Engineering

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Date April 13, 1972

O-7639

ABSTRACT

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By

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This dissertation presents the results of research directed at the formulation and application of optimal control theory in a dynamic model of the U.S. national economy. This dynamic macro-economic model consists of a set of difference equations. The model incorporates aggregated variables generally considered by economists to be fundamental in determining the dynamic monetary and fiscal characteristics of the U.S. economy. The effects of stochastic environmental influences are provided by adding a noise variable to each behavioral equation.

Analysis and classification of the parameters of the model and its stability properties are considered in this work. The eigenvalues suggest that the natural response of the system is inherently stable. Step perturbations in the control variables are analyzed to determine what weightings of these variables yield the most significant impact on the model output (GNP).

The macro-economic model is reformulated in state-space format as a prerequisite to the application of modern optimal control theory. A quadratic social welfare functional with penalty factors on the system's error and on the activity of the control vector is used. The optimal control policy is derived using dynamic

programming and the principle of optimality. Numerical computations for the minimization of the quadratic performance index are included. Moreover, a sensitivity analysis of the penalty coefficients and weighting factors on the components of the control vector is performed. It is shown that the optimal control policy changes significantly with penalty and weighting values as well as with the length of the planning horizon.

The optimal control variables differ from the actual values of these variables during the period 1954-1963, suggesting the use of more flexible control policies by the decision-makers. During this time, application of optimal economic control policies results in a ten-to one-hundred-fold improvement over the actual performance, with respect to the specific assumptions made in the criterion functional of the study. It also yields a smoothly increasing path for the output (GNP), which implies stable levels of employment and prices.

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Kioumars Paryani

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering
and
Systems Science

1972

To my parents,

Mr. and Mrs. Jállál Paryani

who laid the foundation.

ACKNOWLEDGMENTS

I wish to express my sincere appreciation to Dr. Robert O. Barr and Dr. James B. Ramsey, for their constant assistance and support during the preparation of this thesis. Their time and effort with many helpful suggestions as guidance committee chairmen are especially acknowledged.

I also want to thank the other members of the committee, Dr. Rita Zemach, Dr. Pui K. Wong, and Dr. Thomas J. Manetsch for their concern and interest shown in this work. Special thanks are due to Dr. Jan Kmenta for his generous and kind consultation in regard to the econometric model of this study. I would like to thank my friend Dr. Robert H. Boling for his help and advice.

Finally I owe a great debt of love and gratitude to my wife, Behjat, for her unselfish patience and understanding during my graduate studies and the preparation of this dissertation. It can be fairly said that this thesis is as much hers as mine.

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INTRODUCTION

OPTIMAL CONTROL OF LINEAR DISCRETE MACRO-ECONOMIC SYSTEMS

During the past two decades, optimization models have played an extremely important role in all areas of engineering and science. In this connection, there arises the question of finding the very best (in one sense or another) or, as is said, the optimal control of the process. For example, one can speak about optimality in the sense of rapidity of action, i.e., about achieving the aim of the process in the shortest time; about achieving this aim with a minimum expenditure of energy or cost, etc. Mathematically formulated, these are problems in the calculus of variations, which in fact owes its origin to these problems. However, solution of a whole range of variational problems which are important in contemporary science and technology lies outside the classical calculus of variations. These problems can be approached by the means of Pontryagin's Minimum (Maximum) Principle or Dynamic Programming, the two most important techniques of dynamic optimization procedures.

In economic theory there has been an extensive discussion of aggregate models of capital accumulation, much of it directed toward the determination of investment plans optimal under some specified criterion, starting with the pioneering work of Ramsey (Ref. R-2), see also Refs. C-1, C-2, C-3, C-4, C-5, K-8, and S-5. In these problems classical calculus, calculus of variations, and later, modern control theory, Pontryagin's Minimum Principle and Dynamic Programming, have been of great help. To give some idea of the importance of the application of optimal control theory in economic systems one can refer to Refs. A-2, A-5, B-1, B-2, B-10, B-14, B-15, B-16, C-1, C-2, C-13, D-2, D-3, D-4, F-4, H-6, H-7, I-1, I-2, K-10, L-2, M-1, M-2, P-1, R-1, S-2, S-3, S-6, S-7, S-9, S-10, S-11, S-12, S-13, S-14, and T-10. However, almost all of these studies have been involved with the theoretical aspects of the subject.

In recent years, a few engineers have studied the actual application of optimal control theory to macro-economic systems (or micro-economic systems in some cases) and have obtained numerical results (Refs. A-2, B-14, B-15, H-7, and P-1). Also, the economist Tingergen (Ref. T-7) has treated these problems with little emphasis on quantitative description.

The purpose of this thesis is to apply some of the new developments in the variational calculus, optimal control theory, to models of the U.S. national economy and thus derive new insights for economic decision-making. A discrete linear dynamic macro-economic system is utilized in this study. The importance of applying control theory and optimal growth theory resides not only in investigating whether real world experiences are near-optimal, but also in studying the basic issues in capital theory and economic decision-making. It may be emphasized that the model considered here contains a considerable degree of realism and usefulness.

The dissertation problem was motivated in part by a desire to test empirically some of the propositions given by economists.

For example, there are two propositions which have been closely, but not exclusively, associated with Milton Friedman in recent years. The first is the hypothesis that certain control variables that can be either totally or partially manipulated by a decision-maker are ineffective in stabilizing some target variables, e.g., gross national product. The implications are that policy changes may be detrimental as often as they are beneficial, viewed from the stand point of economic stabilization. Furthermore, the most effective strategy might be either no policy at all or adherence to some simple rule, e.g., a three to four percent increase in a control variable, e.g., money supply, per year (Ref. K-8). The second hypothesis is that the money supply is a more significant and important determinant of consumption and income than are autonomous government expenditures (Ref. K-8).

The outline of this thesis is as follows. In Chapter One some general properties of economic systems, their complexities, and an introduction to the dynamic optimization techniques are presented. Chapter Two contains the application of the minimum principle and dynamic programming in a simple optimal planning model for the illustration of the use of the general theory. Chapter Three describes general systems modeling, the Kmenta/Smith (K/S) model in econometric format, and dynamic analysis and stability of the model. Chapter Four presents a state space formulation of the K/S model and a quadratic social welfare cost functional. Chapter Five illustrates the derivation of the optimizing algorithm using dynamic programming and the principle of optimality. In Chapter Six numerical solutions of the optimal control policy for various penalty and weighting factors are analyzed. Finally, in Chapter Seven a summary of conclusions and some recommendations for further future research are presented.

CHAPTER ONE

SYSTEMS ANALYSIS AND CONTROL THEORY IN SOCIO-ECONOMIC SYSTEMS - BACKGROUND INFORMATION

Systems analysis in general and control theory in particular are relatively new additions to the methodological repertoire of social scientists, especially where research and theory development in new areas are concerned. Whether these techniques are as good as some would have them or as bad as others are eager to assert, there are certain types of research problems for which they are appropriate. This chapter is intended to introduce some basic concepts of economic systems and to demonstrate the most important and well known dynamic optimization techniques.

1.1 Some Basic Properties of Economic Processes

In general, economic processes have three important characteristics. The first and the most obvious characteristic of these systems is that they change over time. Both policy makers and researchers strive to understand these processes, often with the hope of accurately predicting the state of the systems at some future point in time.

Second, economic systems are characterized by complex and often unknown relationships among their constituent components (or variables). Often, the net effect of such complexities and variations among the variables is to inhibit the comprehension and prediction of the effects of changes in key variables.

Finally, feedback plays a control role in many of these systems. Feedback in economic systems may be viewed as the partial return or revision of the effects of a given process to its source, or to some preceeding stage, so as to reinforce or modify that prior definition. In general, the processes and effects associated with feedback are little understood by decision (policy)-makers. Hopefully as the understanding of economic systems increases, a control system can be designed which can monitor the system and guide it to some desired future state.

From the reasoning given above, there is apparently a strong need for research tools which will take into consideration the time-varying nature of these socio-economic systems, the frequently complex relationships among them, and the effect of feedback on the future states of the system. Systems analysis and control offer a methodology which explicitly accounts for these main characteristics of these processes. Such research tools are vital to the social scientist interested in developing, modeling, and testing theories of socio-economic processes. Furthermore, such tools are vital to the policy planner faced with the problems of selecting an optimum strategy from among the options available to him for solution of a particular socio-economic system. It should be mentioned that any specific problem does not necessarily have a unique solution; indeed, it often may have infinite number of solutions. If the various solutions are judged on the basis of a specific payoff function each possible solution is characterized, as a rule, by a different value of this function. A solution associated with the optimum value is referred to as an optimum solution.

1.2 Calculus of Variations

The calculus of variations is concerned primarily with the study of maxima and minima of a real-valued function f of a variable x on a space S. The variable x may represent a point, a curve, or a surface in an Euclidean space. It may represent a point in a Hilbert space or more general spaces.

The general theory for the problem began with the study of path of least time. Methods of variational calculus are in a basic sense extensions of the techniques of point optimization of differentiable functions in differential calculus into the function space, in which the problem is to determine under certain conditions an optimum function rather than an optimum point. Intertemporal optimization problems are usually problems of variational calculus with or without some modifications. The modifications introduced by control theory are the classification of variables into states and controls that may be subject to various types of equality and or inequality constraints (Ref. S-12). Since the overall performance of an optimal system can be visualized as one in which distinct stages may be recognized such that decisions at later stages do not affect performance in the earlier ones, the control theory approach has a very close similarity to the technique of dynamic programming and its associated computational algorithms.

One type of problem that may be approached by variational calculus is:

$$\min_{\underline{x}(t)} I = \int_{t_0}^{t_1} F(\underline{x}(t), \dot{\underline{x}}(t), t)dt \qquad (1.5.1)$$

where $\dot{\underline{x}}(t) = \frac{d}{dt}\underline{x}(t)$ with $\underline{x}(t_0) = \underline{x}_0$ and $\underline{x}(t_1) = \underline{x}_1$ fixed, and $F(\cdot)$, a scalar function of vectors, is convex and differentiable in $\underline{x}$, $\dot{\underline{x}}$ and continuous in t. This problem is the simplest problem in the area since explicit constraints on $\underline{x}$ and $\dot{\underline{x}}$ are not imposed.

An optimal path, $\hat{\underline{x}}(t)$, is found by showing that certain necessary conditions must be satisfied for every $t \in [t_0, t_1]$. This is just the opposite from the dynamic programming approach, where the optimal path is developed in a stagewise manner.

Since the variational method will not be used in the sequel, details are not included here and instead readers are referred to Refs. B-12, G-1, G-2, and H-2.

1.3 Discrete Minimum (Maximum) Principle

The Maximum Principle originally developed in 1956 by Pontryagin and his associates (Ref. P-6), provides an elegant method of obtaining an optimal path solution for very general dynamic systems. An excellent comprehensive treatment of the essential problems in the theory of optimal control, together with which are the use and proof of the maximum principle, is given by Rozonoér (Ref. R-4). The first attempt to extend the principle to the optimization of multistage processes was made by Rozonoér (Ref. R-4, III) in 1959, for systems linear in state variables. In 1960, Chang (Ref. C-6), presented the discrete version of the principle for nonlinear systems, which was further explored in his subsequent papers and a book (Refs. C-6, C-7, C-8, and C-9).

An algorithm essentially identical to Chang's version, but different in notation, was independently developed by Katz (Refs. K-2 and K-3). Following the procedure used by Katz in the derivation of the discrete maximum principle, Fan and Wang (Refs. F-1, F-2, and F-3), found that the same algorithm can be extended with some modifications to solve optimization problems of more complex systems.

Discrete Dynamic System

In the dissertation a system S possessing the following properties is called a "discrete dynamic system":

(i) A set of states $\{\underline{x}\} = \mathcal{X} = \mathcal{R}^n$ called the state space, where $\mathcal{R}^n$ is an n-dimensional Euclidean vector space.

(ii) A set of inputs or controls $\{\underline{u}\} = \mathcal{U} \subseteq \mathcal{R}^m$ called the input space.

(iii) An ordered subset T of the set of positive integers, called the time set, i.e., $T = \{t\} = \{0,1,2,\ldots,N\}$ with N pre-sepcified (time horizon).

(iv) A set of outputs $\{\underline{y}\} = \mathcal{Y} = \mathcal{R}^r$ called the output space, with an algebraic equation $\underline{y}(t) = \underline{h}_t(\underline{x}(t), \underline{u}(t))$, $t = 0,1,\ldots,N$, relating the output vector $\underline{y}(t)$ at t to the state vector $\underline{x}(t)$ and control vector $\underline{u}(t)$ and where $\underline{h}_t(\underline{x}(t), \underline{u}(t))$ is a vector valued function mapping $\mathcal{X} \times \mathcal{U} \to \mathcal{Y}$.

(v) A difference equation describing the evolution of the state of the system in time, i.e.,

$$\underline{x}(t+1) = \underline{f}_t(\underline{x}(t), \underline{u}(t)) , \quad t = 0,\ldots,N-1$$

where $\underline{x}(t)$, $\underline{u}(t)$ are the values of the state vector and the control vector at time t respectively, and $\underline{f}_t(\underline{x}(t), \underline{u}(t))$ is a vector valued function mapping $\mathcal{X} \times \mathcal{U} \to \mathcal{X}$. For every fixed $\underline{u}(t) \in \mathcal{U}$, $\underline{f}_t(\cdot)$ is twice continuously differentiable with respect to $\underline{x}(t)$. Moreover, $\underline{f}_t(\cdot)$ and all its first and second derivatives with respect to $\underline{x}(t)$ are assumed to be bounded over $X \times U$ for any bounded sets $X \subset \mathcal{X}$, $U \subset \mathcal{U}$, and the matrix $\Phi = \dfrac{\partial \underline{f}_t(\cdot)}{\partial \underline{x}(t)}$ is assumed to be non-singular on $\mathcal{X} \times \mathcal{U}$. The difference equation in (v) above, is a rule enabling one to compute the state of the system at time $(t+1)$ from the knowledge of both the state and the control at time t.

Statement of the Optimal Control Problem

Let $\underline{x}(t)$ be the vector of variables describing the state of a discrete dynamic system at time t. The dynamic behavior of $\underline{x}(t)$ is given by:

$$\underline{x}(t + 1) = \underline{f}_t(\underline{x}(t), \underline{u}(t)), \quad t = 0,1,\ldots,N-1 \quad (1.6.1)$$

where $\underline{u}(t)$ is the vector of control (policy) variables.

In addition, the initial state $\underline{x}_0 \in \mathcal{X}$ for $t = 0$, i.e., $\underline{x}(0) = \underline{x}_0$ and a specified terminal (target) set $\mathcal{T} \subseteq \mathcal{X}$ which is a smooth $N-k$ dimensional manifold of the form

$$\mathcal{T} = \{\underline{x} : g_t(\underline{x}(t)) = 0;\ t = 1,2,\ldots,k \le N\} \quad (1.6.2)$$

where the functions $g_1(\underline{x}),\ldots,g_k(\underline{x})$ are given twice continuously differentiable mapping from $\mathcal{X}$ into R^1 such that for every $\underline{x} \in \mathcal{X}$ the vectors $\frac{\partial}{\partial \underline{x}} g_t(\underline{x})$; $t = 1,\ldots,k$ are linearly independent are given.

Then, the optimal control problem is to determine the optimal control sequence

$$\{\hat{\underline{u}}(t);\ t = 0,1,\ldots,N-1\} \qquad (1.6.3)$$

and the corresponding optimal state path (trajectory)

$$\{\hat{\underline{x}}(t);\ t = 0,1,\ldots,N\} \qquad (1.6.4)$$

such that

$$\begin{aligned} &\hat{\underline{x}}(t+1) = \underline{f}_t(\hat{\underline{x}}(t),\ \hat{\underline{u}}(t)) \quad , \ t = 0,1,\ldots,N-1 \\ &\hat{\underline{x}}(0) = \underline{x}_0 \\ &\hat{\underline{u}}(t) \in \mathcal{U} \qquad\qquad\qquad\qquad , \ t = 0,1,\ldots,N-1 \\ &\hat{\underline{x}}(N) \in \mathcal{T} \end{aligned} \qquad (1.6.5)$$

and moreover, among all sequences $\{\underline{u}(t)\}$ and $\{\underline{x}(t)\}$ satisfying the above conditions, the cost functional

$$I_N(\{\underline{u}(t)\}) = \sum_{t=0}^{N-1} L_t(\underline{x}(t),\ \underline{u}(t)) \qquad (1.6.6)$$

attains its minimum value at $\{\underline{u}(t)\} = \{\hat{\underline{u}}(t)\}$, and $\{\underline{x}(t)\} = \{\hat{\underline{x}}(t)\}$.

Necessary Conditions for Optimality - the Mimimum Principle

Given the control problem formulated as above and satisfying the aforementioned assumptions, the following theorem can be proved (Ref. H-1).

Theorem: (Minimum Principle for Discrete Systems)

Let $\{\hat{\underline{x}}(t);\ t = 0,1,\ldots,N\}$, $\hat{\underline{x}} \in \mathcal{X}$, be the state trajectory of system (1.6.1) corresponding to the sequence of controls $\{\hat{\underline{u}}(t)\}$,

$\hat{\underline{u}} \in \mathcal{U}$, originating at $\hat{\underline{x}}(0) = \underline{x}_0$ and terminating at $\hat{\underline{x}}(N) \in \mathcal{T}$, $\mathcal{T}$ defined by (1.6.2). Then, in order that $\{\hat{\underline{u}}(t)\}$ minimize the cost functional (social welfare functional in economics), (1.6.6), it is necessary that there exist a sequence of vectors $\{\hat{\underline{p}}(t);\ t = 0,1,\ldots,N\}$, $\{\underline{p}(t)\} = \mathcal{P} = \mathcal{R}^n$, called the costates (shadow prices in economics), such that:

(i) The scalar function:

$$H(\hat{\underline{x}}(t),\ \hat{\underline{p}}(t+1),\ \underline{u}(t)) = L_t(\hat{\underline{x}}(t),\ \underline{u}(t)) + \langle\hat{\underline{p}}(t+1),\ \underline{f}_t(\hat{\underline{x}}(t),\ \underline{u}(t))\rangle \qquad (1.6.7)^{\#}$$

called the Hamiltonian has an absolute minimum as a function of $\underline{u}(t)$ over $\mathcal{U}$ at $\underline{u}(t) = \hat{\underline{u}}(t)$, $\forall\ t = 0,1,\ldots,N-1$, i.e.,

$$\min_{\underline{u}(t)\in\mathcal{U}} H(\hat{\underline{x}}(t),\hat{\underline{p}}(t+1),\underline{u}(t)) = H(\hat{\underline{x}}(t),\hat{\underline{p}}(t+1),\hat{\underline{u}}(t)) \qquad (1.6.8)^{*}$$

or, equivalently

$$H(\hat{\underline{x}}(t),\hat{\underline{p}}(t+1),\hat{\underline{u}}(t)) \le H(\hat{\underline{x}}(t),\hat{\underline{p}}(t+1),\underline{u}(t))\ \forall\ \underline{u} \in \mathcal{U} \qquad (1.6.8a)$$

(ii) If $\hat{\underline{x}}(t)$ is the corresponding trajectory to $\hat{\underline{u}}(t)$, then the evoluation of $\hat{\underline{x}}(t)$ and $\hat{\underline{p}}(t)$ in time are determined by the canonical system of difference equations:

$$\hat{\underline{x}}(t) = \frac{\partial H}{\partial \underline{p}(t+1)}(\hat{\underline{x}}(t),\hat{\underline{p}}(t+1),\hat{\underline{u}}(t)) = \underline{f}_t(\hat{\underline{x}}(t),\hat{\underline{u}}(t))$$

for all $t = 0,1,\ldots,N$ satisfying the initial condition $\hat{\underline{x}}(t_0) = \underline{x}_0$,

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\# $\langle\underline{x},\underline{y}\rangle$ representing the inner (dot) product $= \sum_{i=1}^{n} x_i \cdot y_i$ for $\underline{x},\underline{y} \in \mathcal{R}^n$.

* The vector $\underline{p}(t)$ in this study and in Ref. A-6 is the negative of the costate vector considered in the maximum principle of Pontryagin. Thus (1.6.8) is expressed as a minimum rather than a maximum and the principle is called "minimum principle".

and

$$\underline{\hat{p}}(t+1) = \frac{\partial H}{\partial \underline{x}(t)} (\underline{\hat{x}}(t), \underline{\hat{p}}(t+1), \underline{\hat{u}}(t))$$

for all $t = 0,1,\ldots,N-1$ satisfying the final condition given below.

(iii) Transversality Condition

There exist real numbers $r_1,\ldots,r_t$ such that

$$\underline{\hat{p}}(N) = \sum_{t=1}^{k} r_t \frac{\partial}{\partial \underline{x}} g_t (\underline{\hat{x}}(N)),$$

that is, $\underline{\hat{p}}(N)$ is normal to $\mathcal{T}$ at $\underline{\hat{x}}(N)$, i.e., $\underline{\hat{p}}'(N)\underline{\hat{x}}(N) = \underline{0}$.

Remarks: (1) If $k = N$ then $\mathcal{T}$ is a point in $\mathcal{R}^n$ and nothing may be said apriori as to the value of $\underline{\hat{p}}(N)$. (2) If $k = 0$ then $\mathcal{T} = \mathcal{R}^n$ and $\underline{\hat{p}}(N) = \underline{0}$.

1.4 Dynamic Programming

The founder and the most important propagator of this method is Bellman (Refs. B-4, B-5, B-6, B-7, B-8, and B-9). Dynamic programming applies primarily to a situation in which many decisions have to be made to optimize the overall performance of a system, but the system is one in which distinct stages may be recognized and decisions at the later stages do not affect the performance of the earlier ones. Dynamic programming works best when the number of decisions at any stage is not too large and above all when the effect of these decisions can be represented in only a few variables. Since a continuous process can always be regarded as the limit of an infinite number of infinitesimal stages, it is not surprising to find that dynamic programming is an excellent way of approaching problems that are in the domain of the calculus of variations.

While the optimal control problem itself is a two-point boundary value problem, the principle of optimality (PO) in dynamic programming reduces it to an initial-value problem and a multi-stage decision process.

Principle of Optimality

An optimal control (sequence of decisions) has the property that whatever the initial state and initial control (decision) are, the remaining control must constitute an optimal one with respect to the state which results from the initial control. For a proof see Meditch (M-5, pp. 331-332). More precisely, a policy which is optimal over the interval $1 \le t \le N$, i.e., for $t \in [1,N]$, is necessarily optimal over any sub-interval $v \le t \le N$, where $1 \le v \le N$.

This principle has proven to be a powerful result for use in the solution of optimal control problems. In discrete-time (sampled-data) systems, it is used to reduce the problem from determining an entire control sequence at once to determining the elements of the sequence singly and recursively. In continuous time problems, its application reduces a calculus of variations problem to one of solving a particular type of partial differential equation.

Statement of the Dynamic Programming Problem and Derivation of the Algorithm

A cascaded (serially connected) multi-stage decision process is one where a number of single-stage processes are connected in series so that the output of one stage is the input to the succeeding stage.

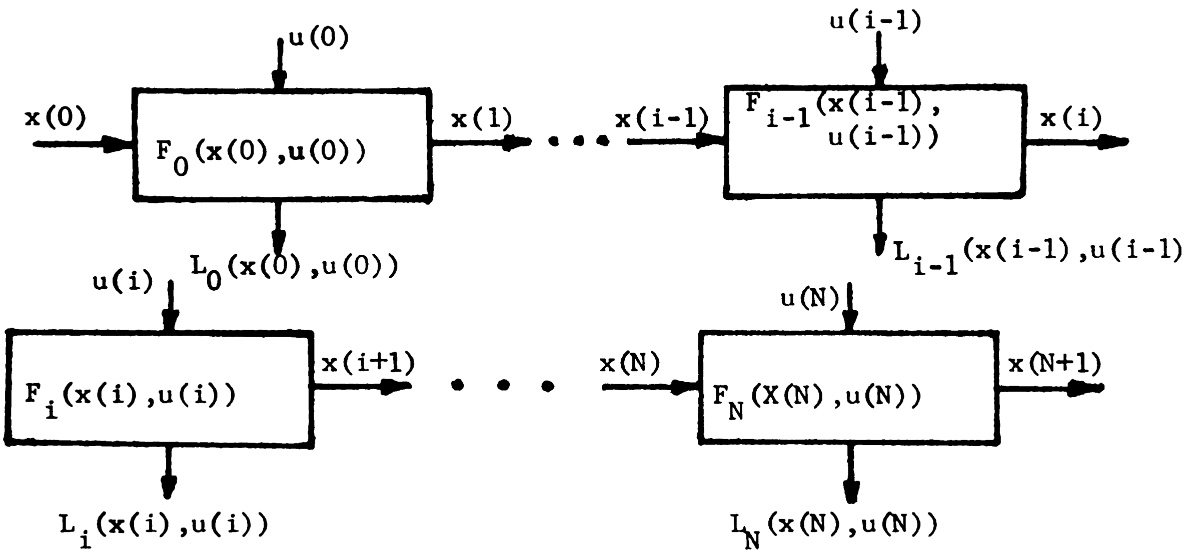

Fig. (1.7.1). A cascaded multi-stage decision process

In Fig. (1.7.1), variables $x(t)$, $t = 0,1,\ldots,N+1$ give all the relevant information about inputs to the boxes and are called state variables; $u(t)$, $t = 0,1,\ldots,N$ are called decision (control) variables. Associated with each control (policy) variable, $u(t)$, and each state variable, $x(t)$, is an output (state variable), $x(t+1)$, which is related to the input and control via a stage transformation function $F_t(x(t),u(t))$, i.e., $x(t+1) = F_t(x(t),u(t))$ for every $t = 0,1,\ldots,N$.

Finally there is a criterion (objective) function, or cost function $L_t(x(t),u(t))$, $t = 0,1,\ldots,N$ that measures the effectiveness of the decisions made and the outputs arising from these decisions. A policy which minimizes the criterion (cost) function $L_N = \sum_{t=0}^{N} L_t(x(t),u(t))$ is said to be an optimal policy.

Considering the implication of the principle of optimality as a multi-stage decision process, it is desired to minimize the N-stage objective function which is given as the sum of the individual stages' costs, i.e.,

$$\min_{\{u(t)\}} I_N = \min_{\{u(t)\}} \sum_{t=0}^{N} L_t(x(t),u(t))$$

$$= \min_{\{u(0),\ldots,u(N)\}} [L_0(x(0),u(0)) + \ldots + L_N(x(N),u(N))]$$

subject to

$$x(t+1) = f_t(x(t),u(t)) \quad , \quad t = 0,1,\ldots,N \ .$$

Let the minimum value of I_N, dependent only upon the initial state, $x(0)$, and the number of stages, N, be denoted by $f_N[x(0)]$, i.e.,

$f_N[x(0)]$ = the total N-stage cost obtained starting in state $x(0)$ using an optimal policy for each stage.

The principle of optimality states that regardless of the initial decision $u(0)$, for $N \geqq 1$, the summation

$$L_0(x(0),u(0)) + [L_1(x(1),u(1)) + \ldots + L_N(x(N),u(N))] =$$
$$L_0(x(0),u(0)) + f_{N-1}[F_0(x(0),u(0))].$$

Note that $x(1) = F_0(x(0),u(0))$. Since this summation holds for all initial decisions $u(0)$, the minimum cost $f_N[x(0)]$ must be minimized over $u(0)$. That is, the basic function equation

$$f_N[x(0)] = \min_{u(0)} \{L_0(x(0),u(0)) + f_{N-1}[x(1)]\}, \ N \geq 1$$

in which

$$f_0[x(0)] = \min_{u(0)} L_0(x(0),u(0))$$

is obtained.

Alternatively, this can be derived as follows:

$$\min_{\{u(0),u(1),\ldots,u(N)\}} I_N = \min_{u(0)} \min_{\{u(1),\ldots,u(N)\}} I_N$$

$$\text{S.T.} \quad x(t+1) = F_t(x(t),u(t)), \; t = 0,1,\ldots,N \; .$$

Hence

$$f_N[x(0)] = \min_{\{u(0),\ldots,u(N)\}} [L_0(x(0),u(0)) + \ldots + L_N(x(N),u(N))]$$

$$= \min_{u(0)} \min_{\{u(1),\ldots,u(N)\}} [L_0(x(0),u(0)) + \ldots + L_N(x(N),u(N))]$$

$$= \min_{u(0)} \{L_0(x(0),u(0)) + \min_{\{u(1),\ldots,u(N)\}} [L_1(x(N),u(N)) + \ldots]\}$$

$$= \min_{u(0)} \{L_0(x(0),u(0)) + f_{N-1}[x(1)]\} \; .$$

CHAPTER TWO

APPLICATION OF MINIMUM PRINCIPLE AND DYNAMIC PROGRAMMING IN A SIMPLE OPTIMAL PLANNING MODEL

In Chapter One two important methods of dynamic optimization techniques, i.e., Pontryagin's Minimum (Maximum) Principle and Dynamic Programming, were discussed and derived. In the present chapter these methods are applied to a simple optimal planning model in economics in order to illustrate the use of the general theory. The optimal controls are determined analytically in order to show the difficulties which arise with Pontryagin's method. One difficulty is that the use of Pontryagin's Principle introduces costate variables, which doubles the dimensionality of the problem to be solved. Furthermore, solving these examples will be helpful in understanding the succeeding chapters, where a more complicated system is studied.

2.1 Example 1

Consider the following elementary model of the dynamics of a macro-economic model of the Samuelson-Hicks type:

$$y(t) = C(t) + I(t) + G(t) \tag{2.1.1}$$

$$C(t) = c_1 y(t-1) + c_2 y(t-2) \tag{2.1.2}$$

$$I(t) = c_0 (y(t-1) - y(t-2)) \tag{2.1.3}$$

where

$C(t)$ = consumer spending

$I(t)$ = gross private investment

$G(t)$ = government purchases of goods and services including federal, state, and local

t = argument, representing time in quarter or year

$c_1 + c_2 = c$, $0 < c < 1$ and $c_0 > 0$

and all variables are real.

Substituting (2.1.2) and (2.1.3) into (2.1.1) and collecting terms yields:

$$y(t) - b_1 y(t-1) - b_2 y(t-2) = G(t) \tag{2.1.4}$$

where

$$b_1 = c_1 + c_0 \quad \text{and} \quad b_2 = c_2 - c_0 \quad .$$

Equation (2.1.4) is a difference equation in $y(t)$ of the second order with the forcing (input) function $G(t)$. This equation can be represented, as shown in Fig. (2.1.1), in the form of a block diagram with input, $G(t)$, and output, $y(t)$.

The solution of the system represented by (2.1.4) consists of two parts, a homogeneous part $(y_h(t))$ and a particular solution $(y_p(t))$, i.e.,

$$y(t) = y_h(t) + y_p(t) \tag{2.1.5}$$

where

$$y_h(t) = \alpha_1 \lambda_1^t + \alpha_2 \lambda_2^t \tag{2.1.6}$$

$$y_p(t) = \frac{G(t)}{1-(c_1+c_2)} = \frac{G(t)}{1-c} \tag{2.1.7}$$

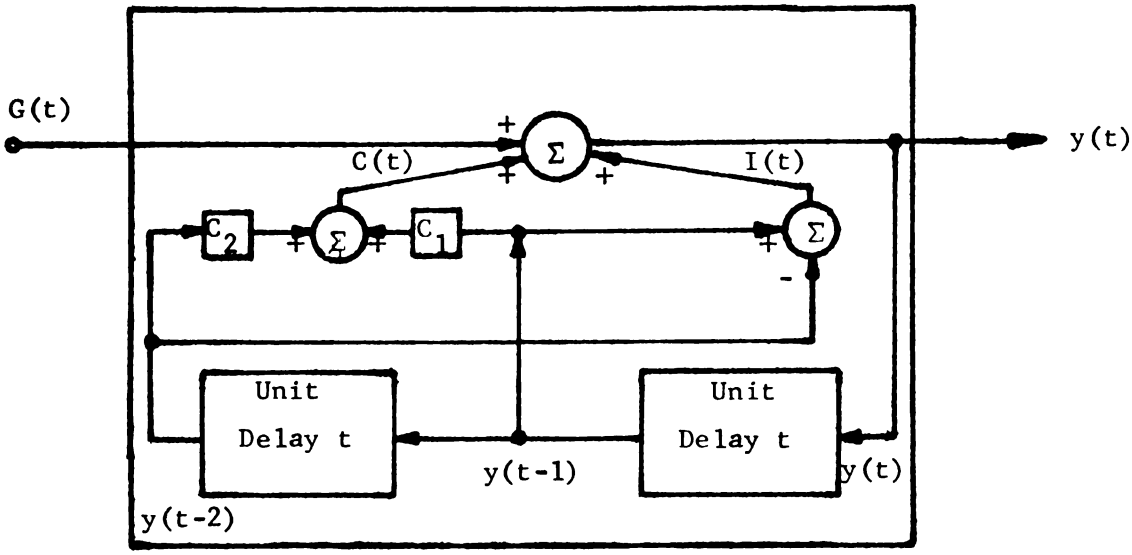

Fig. (2.1.1). Block diagram for Samuelson-Hicks multiplier-accelerator model, Eq. (2.1.4).

and λ_1 and λ_2 are the eigenvalues of the system, with α_1, α_2 depending on the initial conditions $y(0)$, $y(1)$.

Adding Equations (2.1.6) and (2.1.7) yields the equation:

$$y(t) = \alpha_1 \lambda_1^t + \alpha_2 \lambda_2^t + (c-1)^{-1} G(t) \tag{2.1.8}$$

which is the total solution to the system (2.1.4) given $y(0)$ and $y(1)$.

Notice that since λ_i, $i = 1,2$ satisfy the quadratic equation

$$\lambda^2 - b_1 \lambda - b_2 = 0 \tag{2.1.9}$$

then

$$\lambda_1, \lambda_2 = \frac{b_1 \pm \sqrt{b_1^2 + 4b_2}}{2} . \tag{2.1.10}$$

To have real roots, the following condition must hold

$$b_1^2 \geq -4b_2$$

or

$$(c_1 + c_0)^2 \geq 4(c_0 - c_2) \ . \qquad (2.1.11)$$

Now, considering the solution (2.1.8), if $|\lambda_1|$ and $|\lambda_2| < 1$ then

$$\lim_{t\to\infty} \lambda_1^t \ \& \ \lambda_2^t = 0 \quad \text{and} \quad y(t) \to (c-1)^{-1}G(t) \ .$$

If however, either $|\lambda_1|$ or $|\lambda_2|$ or both should be > 1, then $y(t)$, national income, will either exponentially explode, or oscillate violently. If λ_1, λ_2 are complex with unit modulus, national income, $y(t)$, will oscillate steadily about its main value.

A number of optimal control problems now spring to mind. How should $G(t)$ be chosen to control the fluctuations of the economy when in a potentially unstable situation? How should $G(t)$ be chosen in order to obtain a maximum or preassigned rate of growth to the national income at some future time? Doubtless the constants (c_1, c_2, c_0) are affected by $G(t)$, since this must be derived from taxes which in turn affect the propensities to consume, $c = c_1 + c_2$, and to invest, c_0.

To consider the problem of stabilization, the above example is studied with a quadratic cost functional (social disutility functional) in the following example.

2.2 Example 2

Suppose in the above model

$$G(t) = b_0 g(t-1) \tag{2.1.12}$$

where, $g(t)$ is the policy variable, planned government spending at time t.

Suppose further that government planners (decision-makers) wish, by assumption, to set the value of $g(t)$ such that:*

$$\min_{g(t)} I_N = \sum_{t=1}^{N} \{a_1(t)(y_d(t) - y_a(t))^2 + a_2(t)(g(t-1) - g_d(t-1))^2\} \tag{2.1.13}$$

subject to

$$y(t+1) - b_1 y(t) - b_2 y(t-1) = b_0 g(t), \quad t = 1,\ldots,N \tag{2.1.14}$$

$$\begin{aligned} y(0) &= y_0 \\ y(1) &= y_1 \end{aligned} \tag{2.1.15}$$

System (2.1.14) is the same model used in Example 1, logged forward by one t and $G(t)$ replaced by $b_0 g(t-1)$. In the criterion functional, Equation (2.1.13), $a_1(t)$ and $a_2(t)$ are given positive numbers for $t = 2,3,\ldots,N$. Also, $y_d(t)$ and $g_d(t)$ are the desired (target) values for $y(t) = y_a(t)$ (actual $y(t)$) and $g(t)$ respectively. The target values are assumed to be known for all t.

To make the problem formulation compatible with that of optimal control theory, let:

$$y_1(t) = y(t-1)$$

and $$y_2(t) = y_1(t+1) = y(t).$$

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* For a discussion of the reasons for using a quadratic (dis)utility functional in planning models, see Sec. 4.3.

Then

$$y_1(t+1) = y(t)$$

and $$y_2(t+1) = y_1(t+2) = y(t+1) = b_1 y_2(t) + b_2 y_1(t) + b_0 g(t)$$

$$t = 1,\ldots,N-1$$

from Equation (2.1.14).

Hence

$$\begin{bmatrix} y_1(t+1) \\ y_2(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ b_2 & b_1 \end{bmatrix} \begin{bmatrix} y_1(y) \\ y_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} g(t), \quad t = 1,\ldots,N-1 \qquad (2.1.16)$$

is the state model of the system with the output of the system being

$$y(t) = [0 \quad 1] \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} + [0]\, g(t), \quad t = 1,\ldots,N-1 . \qquad (2.1.6a)$$

The variables $y_1(t)$ and $y_2(t)$ are the outputs of the delay elements in Fig. (2.1.1) and they represent the state of the system. The Equations (2.1.16) and (2.1.16a) are in the general form

$$\underline{y}(t+1) = \mathbf{A}\, \underline{y}(t) + \underline{b}\, g(t)$$

$$y(t) = \underline{c}\, \underline{y}(t) + d\, g(t)$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ b_2 & b_1 \end{bmatrix}, \quad \underline{b} = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, \quad \underline{c} = [0 \quad 1], \quad d = 0$$

and $\underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. In Fig. (2.1.2) a simulation diagram for this system is shown.

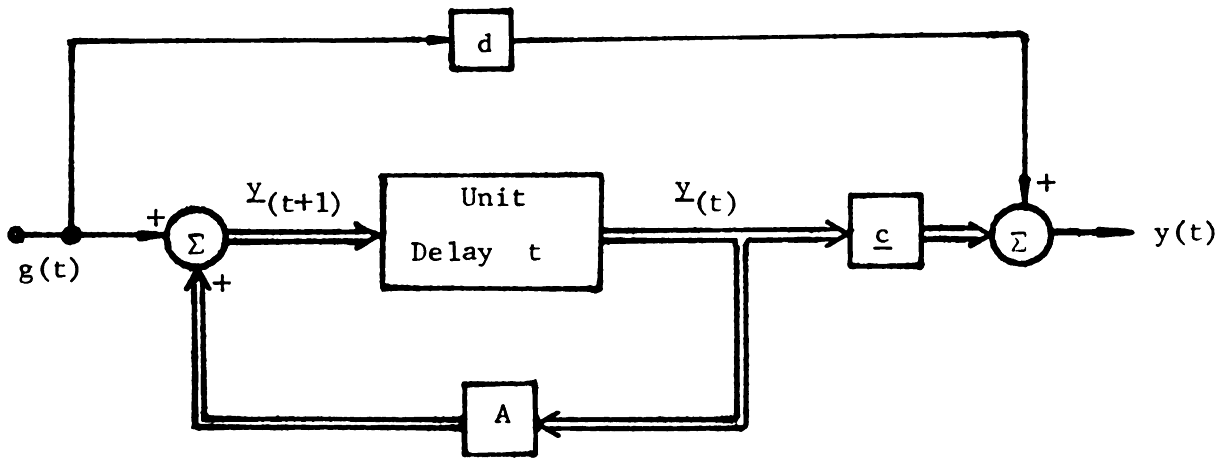

Fig. (2.2.2). Simulation diagram for the above system.

Before continuing with the rest of the problem, it is important that the terms state and state space be clarified. A state of a dynamic system is the smallest collection of numbers which must be specified at time $t = t_0$ in order to be able to predict uniquely the behavior of the system for any time $t \geq t_0$ given any input belonging to the specified input set (space). Such numbers are called state variables. The input space (set) is defined as the space (set) of all possible inputs that can be applied to the system. The state of a system at time t is uniquely determined by the state at time t_0 and the known input for $t \geq t_0$, and is independent of values of the state and input before t_0.

Suppose that at least n state variables $x_1, x_2, \ldots, x_n$ are needed to describe completely the behavior of a given dynamical system. The set of n state variables can be considered as n components of a vector $\underline{x}$. Such a vector $\underline{x}$ is called a state vector. A state space is defined as an n-dimensional (tuple) space in which $x_1, \ldots, x_n$ are coordinates. The state at time t of a system defined by n first-order difference (differential) equations can be represented by a point in an n-dimensional state space. It may be noted that the state space approach of analysis and synthesis of control systems usually deals with a set of n first-order difference (differential) equations rather than a single nth-order difference (differential) equation.

The two Equations (2.1.16) and (2.1.16a) together represent the state space representation of the model with $\underline{y}(t) = (y_1(t)\ y_2(t))'$ being the state vector and $y(t)$ the output of the model.

Since the problem here is concerned with the short-term problems of economic stabilization rather than growth, it can be assumed that $y_d(t)$ and $g_d(t)$ are fixed constants for all t and, since they are purely translation factors, there is no loss of generality in assuming them identically equal to zero (Ref. F-4, pp. 218).

Under these conditions, the problem reduces to:

$$\min_{g(t)} I_N = \sum_{t=1}^{N} \{a_1(t)y_2^2(t) + a_2(t)g^2(t-1)\}$$

$$= \sum_{t=1}^{N} \{\underline{y}'(t)Q(t)\underline{y}(t) + a_2(t)g^2(t-1)\} \qquad (2.1.17)$$

$$= \sum_{t=1}^{N} L(\underline{y}(t), g(t))$$

$$\text{S.T.} \quad \underline{y}(t+1) = A\,\underline{y}(t) + \underline{b}\,g(t) \qquad (2.1.18)$$

where given $\underline{y}(0) = \underline{y}_0$, $g(t)$ unconstrained,

$L(\cdot,\cdot)$ equals $\{\cdot,\cdot\}$ in Equation (2.1.17),

and $$Q(t) = \begin{bmatrix} 0 & 0 \\ 0 & a_1(t) \end{bmatrix}, \; A = \begin{bmatrix} 0 & 1 \\ b_2 & b_1 \end{bmatrix}, \; \underline{b} = \begin{bmatrix} 0 \\ b_0 \end{bmatrix}, \; \underline{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}.$$

Solution

(i) Pontryagin's Principle Approach

Consider the Hamiltonian equation for this problem

$$H(\underline{y}(t), \underline{p}(t), g(t)) = L(\underline{y}(t), g(t)) + \langle \underline{p}(t), \underline{y}(t+1) \rangle$$

or

$$H(\underline{y}(t), \underline{p}(t), g(t)) = a_1(t)y_2^2(t) + a_2(t)g^2(t-1) + p_1(t)y_2(t)$$
$$+ p_2(t)(b_2y_1(t) + b_2y_2(t) + b_0g(t)). \quad (2.1.19)$$

The necessary conditions for optimality imply:

$$\left.\begin{aligned} y_1(t+1) &= \frac{\partial H}{\partial p_1(t)} = y_2(t) \\ y_2(t+1) &= \frac{\partial H}{\partial p_2(t)} = b_2y_1(t) + b_2y_2(t) + b_0g(t) \end{aligned}\right\} \quad t = 1,2,\ldots,N-1 \quad (2.1.20)$$

with the initial conditions $y_1(0)$, $y_2(0)$ given, and

$$\left.\begin{aligned} p_1(t-1) &= \frac{\partial H}{\partial y_1(t)} = b_2p_2(t) \\ p_2(t-1) &= \frac{\partial H}{\partial y_2(t)} = 2a_1(t)y_2(t) + p_1(t) + b_1p_2(t) \end{aligned}\right\} \quad t = 2,\ldots,N-1 \quad (2.1.21)$$

The transversality conditions are:

$$p_1(N-1) \equiv 0 \quad (2.1.22)$$

and $$p_2(N-1) \equiv 2a_1(N)y_2(N). \quad (2.1.23)$$

Since $a_1(t)$ and $a_2(t)$ are given positive numbers for each t, $H(\cdot)$ is a convex function of $g(t)$ for fixed $\underline{y}(t)$, $H(\cdot)$ is differentiable, and the values of $g(t)$, the control variable, are not constrained, it follows that the minimum of $H(\cdot)$ will occur at its stationary point; that is,

$$\frac{\partial H(\cdot)}{\partial g(t)} = 0 = 2a_2(t)g(t) + b_0 p_2(t) \tag{2.1.24}$$

which implies

$$\hat{g}(t) = -\frac{b_0}{2a_2(t)} p_2(t) \quad \text{for} \quad t = 1,\ldots,N-1 \tag{2.1.25}$$

Equation (2.1.25) defines the optimal control for this planning model. The path of $\hat{g}(t)$ is determined after the co-state variable $p_2(t)$ is specified. To find the function $p_2(t)$, the following procedure is used:

Equation (2.1.20) implies:

$$y_2(t+1) = b_1 y_2(t) + b_2 y_2(t-1) - \frac{b_0^2}{2a_2(t)} p_2(t), \tag{2.1.26}$$

after substituting for $y_1(t)$ and $g(t)$ from Equations (2.1.20) and (2.1.25) respectively. Starting from the last time period, $t = N-1$, and using Equations (2.1.21), (2.1.22), (2.1.23) and (2.1.26), successive substitution yields:

$$p_2(t) = k_1(t)y_2(t) + k_2(t)y_2(t-1) + k_3(t) \qquad t = 1,\ldots,N-1 \tag{2.1.27}$$

where the coefficient functions $k_i(t)$, $i = 1,2,3$ are independent of $y(t)$ and are defined recursively by:

$$k_1(t) = h_4(t+1)(b_1 h_1(t+1) + h_2(t+1)),$$

$$k_2(t) = b_2 h_4(t+1) h_1(t+1),$$

$$k_3 = h_4(t+1) h_3(t+1),$$

$$k_1(N) \equiv k_2(N) \equiv k_3(N) \equiv 0; \; k_1(N+1) \equiv k_2(N+1) \equiv k_3(N+1) \equiv 0$$

and

$$h_1(t+1) = 2a_1(t+1) + b_1 k_1(t+1) + b_2 k_2(t+2) + b_2 k_1(t+2)\left[b_1 - \left(\frac{b_0^2}{2a_2(t+1)}\right) k_1(t+1)\right]$$

$$h_2(t+1) = b_1 k_2(t+1) + b_2 k_1(t+2)\left[b_2 - \left(\frac{b_0^2}{2a_2(t+1)}\right) k_2(t+1)\right]$$

$$h_3(t+1) = b_1 k_3(t+1) + b_2 k_3(t+2) - b_2 k_1(t+2)\left(\frac{b_0^2}{2a_2(t+1)}\right) k_3(t+1)$$

$$h_4(t+1) = \left[1 + h_1(t+1)\left(\frac{b_0^2}{2a_2(t)}\right)\right]^{-1}.$$

Given the initial conditions $y(0) = y_0$ and $y(1) = y_1$, the optimal control, $\hat{g}(t)$, may be calculated using Equations (2.1.25) and (2.1.27).

Thus, Pontryagin's Principle increases the dimensionality of the problem to be solved (i.e., because of the costate variables). As the problem gets more complicated, it becomes almost impossible to tackle the problem analytically. For this kind of problem dynamic programming is a more appropriate solution algorithm.

(ii) Dynamic Programming

For relatively simple systems, the generation of the optimum control policy can be accomplished by incorporating a filtering device or an active network. In the case of complicated situations, a digital computer is usually advantageous in generating an optimum control policy. Since problems of optimum digital control may be viewed as a type of multistage decision process (as was discussed in Chapter One), the determination of an optimum control law (policy) is best carried out by means of the dynamic programming technique. Recall the problem:

$$\min_{g(t)} I_N = \sum_{t=1}^{N} (\underline{y}'(t)Q(t)\underline{y}(t) + g'(t-1)a_2(t)g(t-1)) \qquad (2.1.16a)$$

S.T.

$$\underline{y}(t+1) = A\,\underline{y}(t) + \underline{b}\,g(t) \qquad (2.1.17a)$$

$$\underline{y}(0) = \underline{y}_0 \,.$$

Let the minimum value of I_N, which depends only upon the initial stage $\underline{y}(0)$ and the number of stages N over which it is desired to minimize the disutility functional, be denoted by $f_N[\underline{y}(0)]$; that is,

$$f_N[\underline{y}(0)] = \min_{g(t)} I_N = \min_{g(t)} \{ \sum_{t=1}^{N} [\underline{y}'(t)Q(t)\underline{y}(t) + g'(t-1)a(t-1)g(t-1)]\} \qquad (2.1.28)$$

where $f_N[\underline{y}(0)]$ = the total cost of taking the system from initial state $\underline{y}(0)$ through N stages where an optimal policy is pursued at each stage.

More generally, Equation (2.1.28) may be rewritten as:

$$f_{N-k}[\underline{y}(k)] = \min_{g(t)} I_{N-k}$$

$$= \min_{g(t)} \{ \sum_{t=k+1}^{N} [\underline{y}'(t)Q(t)\underline{y}(t) + g'(t-1)a_2(t-1)g(t-1)]\}$$

$$k = 0,1,2,\ldots,N-1 \,. \qquad (2.1.29)$$

When $k = 0$, Equation (2.1.29) reduces to Equation (2.1.28), and it is apparent that $f_0 = 0$.

Now, assuming that the return from the first $(k-1)$ stages is optimum, then the cost of the remaining $(N-k)$ stages is equal to the cost from the k-th stage plus the optimum cost from the remaining $N - (k+1)$ stages, which can be expressed as:

$$\{\underline{y}'(k+1)Q(k+1)\underline{y}(k+1) + g'(k)a_2(k)g(k) + f_{N-(k+1)}[\underline{y}(k+1)]\} .$$

The principle of optimality implies

$$f_{N-k}[\underline{y}(k)] = \min_{g(k)} \{\underline{y}'(k+1)Q(k+1)\underline{y}(k+1) + g'(k)a_2(k)g(k) + f_{N-(k+1)}[\underline{y}(k+1)]\} . \tag{2.1.30}$$

Since the functional f is quadratic in $\underline{y}$, it may be assumed that

$$f_{N-k}[\underline{y}(k)] = \underline{y}'(k)\Omega(N-k)\underline{y}(k) \tag{2.1.31}$$

with $\Omega(N-k)$ being a square matrix to be defined later and

$$f_{N-(k+1)}[\underline{y}(k+1)] = \underline{y}'(k+1)\Omega(N-(k+1))\underline{y}(k+1). \tag{2.1.32}$$

This assumption can readily be justified by mathematical induction. The matrices Ω in Equations (2.1.31) and (2.1.32) are positive definite and symmetric. Substituting Equations (2.1.31) and (2.1.32) into (2.1.30) yields:

$$\begin{aligned} &\underline{y}'(k)\Omega(N-k)\underline{y}(k) \\ &\quad = \min_{g(k)} \{\underline{y}'(k+1)[Q(k+1) + \Omega(N-(k+1))]\underline{y}(k+1) + g'(k)a_2(k)g(k)\} \\ &\quad = \min_{g(k)} \{\underline{y}'(k+1)H(N-(k+1))\underline{y}(k+1) + g'(k)a_2(k)g(k)\} \end{aligned} \tag{2.1.33}$$

where

$$H(N-(k+1)) = Q(k+1) + \Omega(N-(k+1)) \ . \qquad (2.1.34)$$

Let

$$J_{N-k} = \{\underline{y}'(k+1)H(N-(k+1))\underline{y}(k+1) + g'(k)a_2(k)g(k)\} \quad (2.1.35)$$

In view of the system model (constraint equation) Equation (2.1.17a),

$$J_{N-k} = \{[A\ \underline{y}(k) + \underline{b}\ g(k)]'H(N-(k+1))[A\ \underline{y}(k) + \underline{b}\ g(k)]$$

$$+ g'(k)a_2(k)g(k)\}$$

$$= \underline{y}'(k)\Phi_{AA}(N-(k+1))\underline{y}(k) + g'(k)[\varphi_{\underline{b}\underline{b}}(N-(k+1)) + a_2(k)]g(k)$$

$$+ \underline{y}'(k)\underline{\varphi}_{A\underline{b}}(N-(k+1))g(k) + g'(k)\underline{\varphi}_{\underline{b}A}(N-(k+1))\underline{y}(k) \qquad (2.1.36)$$

where

$\Phi_{AA}(N-(k+1)) = A'H(N-(k+1))A$	a 2×2 matrix	(2.1.37)
$\varphi_{\underline{b}\underline{b}}(N-(k+1)) = \underline{b}'H(N-(k+1))\underline{b}$	a scalar	(2.1.38)
$\underline{\varphi}_{A\underline{b}}(N-(k+1)) = A'H(N-(k+1))\underline{b}$	a column vector	(2.1.39)
$\underline{\varphi}_{\underline{b}A}(N-(k+1)) = \underline{b}'H(N-(k+1))A$	a row vector	(2.1.40)

and furthermore,

$$\underline{\varphi}_{\underline{b}A}(N-(k+1)) = \underline{\varphi}'_{A\underline{b}}(N-(k+1)) \ . \qquad (2.1.41)$$

Now the minimization procedure may be readily carried out through ordinary differentiation, since the N-stage decision process has been reduced to a sequence of single-stage decision processes. Differentiating Equation (2.1.36) with respect to $g(k)$

yields:

$$\frac{d\, J_{N-k}}{dg(k)} = 0 = 2\, \underline{\varphi}_{bA}(N-(k+1))\underline{y}(k) + 2[\varphi_{\underline{b}\underline{b}}(N-(k+1)) + a_2(k)]g(k) \tag{2.1.42}$$

Thus, the optimum control policy is:

$$\hat{g}(k) = \underline{q}(N-k)\underline{y}(k) \quad , \quad k = 0,1,\ldots,N-1 \tag{2.1.43}$$

where the <u>feedback matrix</u> (in this simple problem it is a vector) $\underline{q}$ is given by:

$$\underline{q}(N-k) = -[\varphi_{\underline{b}\underline{b}}(N-(k+1)) + a_2(k)]^{-1}\underline{\varphi}_{\underline{b}A}(N-(k+1)) \ . \tag{2.1.44}$$

As Equation (2.1.43) shows, the optimum control policy is a function of the state vector of the system. Since the feedback matrix $\underline{q}$ involves the unknown matrix Ω, the optimum control policy is still undefined. A recursive relationship between matrices Ω and $\underline{q}$ may be found from Equations (2.1.33) and (2.1.43). This recursive relationship together with Equation (2.1.44) provides a computational algorithm for the evaluation of the feedback matrix $\underline{q}(N-k)$ so that $\hat{g}(k)$ is determined for each $k = 0,1,\ldots,N-1$.

Substituting Equation (2.1.43) into Equation (2.1.33) and making use of Equation (2.1.36) yields the minimum value of I_N as:

$$\underline{y}'(k)\Omega(N-k)\underline{y}(k)$$

$$= \underline{y}'(k)\Phi_{AA}(N-(k+1))\underline{y}(k) + [\underline{q}(N-k)\underline{y}(k)]'[\varphi_{\underline{b}\underline{b}}(N-(k+1))$$

$$+ a_2(k)][\underline{q}(N-k)\underline{y}(k)] + \underline{y}'(k)\underline{\varphi}_{A\underline{b}}(N-(k+1))[\underline{q}(N-k)\underline{y}(k)]$$

$$+ [\underline{q}(N-k)\underline{y}(k)]'\underline{\varphi}_{\underline{b}A}(N-(k+1))\underline{y}(k)$$

$$= \underline{y}'(k)\Phi_{AA}(N-(k+1))\underline{y}(k) + \underline{y}'(k)\underline{q}'(N-k)[\varphi_{\underline{b}\underline{b}}(N-(k+1))$$

$$+ a_k(k)]\underline{q}(N-k)\underline{y}(k) + \underline{y}'(k)\underline{\varphi}_{A\underline{b}}(N-(k+1))\underline{q}(N-k)\underline{y}(k)$$

$$+ \underline{y}'(k)\underline{q}'(N-k)\underline{\varphi}_{\underline{b}A}(N-(k+1))\underline{y}(k) \ . \qquad (2.1.45)$$

Using Equation (2.1.44), the above equation reduces to:

$$\underline{y}'(k)\Omega(N-k)\underline{y}(k) = \underline{y}'(k)[\Phi_{AA}(N-(k+1)) + \underline{\varphi}_{A\underline{b}}(N-(k+1))\underline{q}(N-k)]\underline{y}(k) \qquad (2.1.46)$$

Comparing both sides of the above equation yields:

$$\Omega(N-k) = \Phi_{AA}(N-(k+1)) + \underline{\varphi}_{A\underline{b}}(N-(k+1))\underline{q}(N-k) \qquad (2.1.47)$$

Starting with $\Omega(0) = 0$ for $k = N-1$, Equation (2.1.44) gives:

$$\underline{q}(1) = -[\varphi_{\underline{b}\underline{b}}(0) + a_2(N-1)]^{-1}\underline{\varphi}_{\underline{b}A}(0) \qquad (2.1.48)$$

and Equation (2.1.47) gives the value of $\Omega(1)$ as:

$$\Omega(1) = \Phi_{AA}(0) + \underline{\varphi}_{A\underline{b}}(0)\underline{q}(1) \qquad (2.1.49)$$

where

$$\Phi_{AA}(0) = A'Q(N)A \qquad (2.1.50)$$

$$\varphi_{\underline{b}\underline{b}}(0) = \underline{b}'Q(N)\underline{b} \qquad (2.1.51)$$

$$\underline{\varphi}_{A\underline{b}}(0) = A'Q(N)\underline{b} \qquad (2.1.52)$$

For $k = N-2$, Equation (2.1.44) gives $\underline{q}(2)$ and Equation (2.1.47) determines $\Omega(2)$. In like manner, $\underline{q}(3)$, $\Omega(3)$; $\underline{q}(4)$, $\Omega(4)$; ... can be successively evaluated. Hence Equations (2.1.44) and (2.1.47) provide the necessary recursive relationships for the determination of the optimum control policy. These two recurrence relationships illustrate that, even though the process to be controlled is stationary, the over-all control system is a time-varying system since the feedback vector $\underline{q}(N-k)$ varies with time.

Even though the computational procedure showed here may look complicated, it is actually much simpler than that involved using of Pontryagin's Principle. The methods of dynamic programming have proved themselves both powerful and versatile in a number of branches of economics, management, and engineering. It is of interest how naturally they utilize the capabilities of modern high-speed digital computers. Dynamic programming is a simple but very useful concept which finds applications in solving multistage decision problems, as shown in the above example and will be illustrated further in Chapter Five.

CHAPTER THREE

KMENTA/SMITH DYNAMIC MODEL OF THE U.S. NATIONAL ECONOMY IN ECONOMETRIC FORMAT

3.1 Systems Modeling

In dealing with modern control theory it is always assumed that a system (model) is given in the state space formulation. In other words, the analysis of a system (physical or socio-economic) begins with the postulation of a model. This was illustrated in Chapter Two using the simple Samuelson-Hicks optimal planning model. Before proceeding to a more complex model of the U.S. national economy, some general remarks on modeling seem appropriate.

A model is an abstraction of a particular set of properties of a system, knowledge of which suffices for the prediction of the behavior of the system under certain operating conditions. In recent years the word model has been misused to describe almost any attempt at specifying a system under study. There are differences of opinion about what kinds of specification are permitted to constitute a model in the scientific sense. By a model is meant a specification of the inter-relationships of the parts of a system, in verbal or generally in mathematical terms, sufficiently explicit to enable one to study its behavior under a variety of circumstances. In particular, it is desired to control and observe it, and to predict its path over time.

To analyze the system quantitatively, it is necessary to construct a mathematical representation of it. In practice, neither the model nor its mathematical representation is unique and the final choice is usually made on the basis of convenience.

Formal techniques for the modeling of almost all physical components and systems (processes) have been sufficiently developed to permit the construction of reliable predictive mathematical models for them. However, there has not been much work conducted in the modeling of socio-economic systems and generally acceptable component models and modeling procedures for these kinds of systems are now being developed. It is important to realize that in the synthesis of mathematical models, tradeoffs must always be made between the complexity of the model and its ability to represent the characteristics of the system it introduces.

The objective of the present chapter is to study a macroeconomic model which incorporates those variables generally considered fundamental to the dynamics of the U.S. national economy and yet which is mathematically tractable in terms of modern optimal control theory.

Due to recent developments in modern control theory, it is now possible to handle more complex and comprehensive multi-dimensional systems than previously. Therefore, in terms of complexity, the model considered in this work lies somewhere between such analytical macro-economic models as those considered in Refs. H-5, P-3, P-4, T-10, and V-1, and the recent elaborate econometric models used for prediction and simulation, such as Refs. D-5 and K-4.

In modeling a system a number of assumptions are made to reduce the complexity and simplify the subsequent mathematical development. These assumptions may be stated in the following notation. Suppose that to each element of a set A there is assigned a unique element of a set B; the collection, f, of such assignments is called a function (or mapping) from A into B and is written $f: A \rightarrow B$ or $A \xrightarrow{f} B$. In other words, a function is said to be an operator which associates with each of the independent variable a single value for the dependent variable. That is, a function is defined as a set of single-valued ordered pairs given by:

$$f = \{(a_1,b_1),\ (a_2,b_2),\ (a_3,b_3),\ldots\} \tag{3.1.1}$$

where $\{\cdot\}$ denotes a set and $(\cdot,\cdot)$ an ordered pair of elements. Similarly a system S, such as the U.S. national economy, may be characterized as a mathematical operator which maps an input signal in input space X into an output signal in output space Y, i.e.,

$$S : S = \{(\underline{x},\ \underline{y})\} \tag{3.1.2}$$

where $\underline{x} \in X$ is the input vector, and $\underline{y} \in Y$ is the output vector.

In constructing the system S while considering the nature of the system, the following assumptions are made:

(i) Linearity, which states,

if: $(x_1,y_1) \in S$

and: $(x_2,y_2) \in S$

then: $((\alpha x_1 + \beta x_2),\ (\alpha y_1 + \beta y_2)) \in S \quad \forall\ \alpha$ and $\beta \in \mathcal{R}$

where $\mathcal{R}$ is the set of the real numbers, i.e., α and β here are scalar constants.

(ii) Time invariance, that is,

$$\begin{aligned} &\text{if:} && (x_1, y_1) \in S \\ &\text{and:} && x_2(t_1) = x_1(t_1 + T), \\ &\text{and:} && y_2(t_2) = y_1(t_2 + T), \\ &\text{then:} && (x_2, y_2) \in S \end{aligned} \tag{3.1.3}$$

where T is an arbitrary time constant such that, $t_1 + T$ and $t_2 + T$ represent arbitrary shifts in time.

(iii) Deterministic, that is,

$$S = \{(x_k, y_k)\} \quad \text{for all} \quad k = 1,2,\ldots \tag{3.1.4}$$

where the right hand side of (3.1.4) represents a single-valued set of ordered pairs of signals. This says that in a deterministic system, a unique output signal is produced for each input signal as opposed to a stochastic (probabilistic) system for which several output signals may be realized for a given input signal. Note, however, that stochastic (non-deterministic) influences are here considered exogeneous to the basic equations of the model and are treated as external stochastic forcing functions representing environmental noise (disturbances). That is, some of the input signals are allowed to be stochastic, but the system S is assumed to be deterministic.

3.2 The Kmenta/Smith (K/S) Economic Model

The dynamic macro-economic model considered in this thesis is a model of the U.S. national economy developed and estimated by

Kmenta/Smith (Ref. K-6).[#] The model, which represents a dynamic macro-economic system, is expressed in terms of a set of difference equations exhibiting those economic characteristics generally thought to be most important. Yet, it is not too complex to handle. The model is an eight equation system which includes a monetary sector. It is a quarterly model, i.e., the data used in designing the model and estimating its parameters are quarterly, deflated by the implicit price index, seasonally adjusted, and cover the period from 1954 through 1963. The model is characterized by the extensive use of distributed lags and trend factors, both of which enrich its dynamic characteristics. The monetary variables have been deflated by the implicit price index for consumption expenditures. The coefficients of the model were estimated by the two-stage-least-squares (2SLS) method. As a general simplified description of the U.S. economy, the estimated relationships seem quite plausible. Provision for incorporating the effects of stochastic influences from the system's environment is made via noise variables in all behavioral equations.

The model consists of the following set of equations:

Behavioral Equations:

$$C(t) = \alpha_0 + \alpha_1 y(t) + \alpha_2(L(t) - \alpha_3 L(t-1)) + \alpha_3 C(t-1) + u_1(t), \quad (3.2.1)$$

$$I^d(t) = \beta_0 + \beta_1 r(t) + \beta_2(S(t-1) - S(t-2)) + \beta_3 t + \beta_4 I^d(t-1) + u_2(t), \quad (3.2.2)$$

Dr. J. Kmenta is Professor of Economic, Dept. of Economics, Michigan State University, E. Lansing, Michigan, and Dr. P.E. Smith is Professor of Economics, Dept. of Economics, University of Missouri, Columbia, Missouri.

$$I^r(t) = \delta_0 + \delta_1 r(t) + \delta_2(S(t-1) - S(t-2)) + \delta_3 t + \delta_4 I^r(t-1) + u_3(t), \tag{3.2.3}$$

$$I^i(t) = \lambda_0 + \lambda_1 r(t) + \lambda_2(S(t-1) - S(t-2)) + \lambda_3 t + \lambda_4 I^i(t-1) + u_4(t), \tag{3.2.4}$$

$$r(t) = \gamma_0 + \gamma_1 y(t) + \gamma_2 M(t) + \gamma_3 M(t-1) + u_5(t), \tag{3.2.5}$$

Identities:

$$y(t) \equiv C(t) + I^d(t) + I^r(t) + I^i(t) + G(t) , \tag{3.2.6}$$

$$S(t) \equiv y(t) - I^i(t) , \tag{3.2.7}$$

$$L(t) \equiv M(t) + R(t) , \tag{3.2.8}$$

where

C ... consumption expenditures in billions of dollars,

I^d ... investment in producer's outlays on durable plant and equipment in billions of dollars,

I^r ... investment in construction in billions of dollars,

I^i ... investment in inventories in billions of dollars,

r ... yield on all corporate bonds in percentage,

y ... gross national product (GNP) in billions of dollars,

S ... final sales of goods and services in billions of dollars,

L ... money supply plus time deposits in commercial banks in billions of dollars,

R ... time deposits in commercial banks in billions of dollars,

M ... money supply, i.e., demand deposits plus currency outside banks in billions of dollars,

G ... government purchases of goods and services plus net foreign investments in billions of dollars,

t ... time in quarters (first quarter of 1954 is zero).

The variables are dated by the argument t. In a flow variable, such as consumption $C(t)$ measures the total aggregate consumption expenditure during the period $[t, t+1]$, which is viewed, however, as occurring at time t. This notation applies similarly for all other flow variables. In the case of stock variables $(M(t), L(t),$ and $R(t))$, the argument refers to the first day of the quarter. The variables G, L, M, and t are exogeneous and $u_i(t)$ $(i = 1,\ldots,5)$ are independent stochastic disturbances (environmental disturbances) which are assumed to be normally distributed with zero means, constant variances, and non-autocorrelated (Ref. K-6, p.4).

The consumption function is of the form suggested in Ref. Z-1:

$$C(t) = k\, y^P(t) + a(L(t) - L^*(t+1)) + \varepsilon(t) \qquad (3.2.9)$$

$$L^*(t+1) = \eta\, y^P(t) \qquad (3.2.10)$$

$$y^P(t) = y^P(t-1) + b(y(t) - y^P(t-1)) \qquad (3.2.11)$$

where $y^P(t)$ is permanent income, $L^*(t+1)$ denotes desired level of liquid assets for the end of the t-th period, and $\varepsilon(t)$ is a stochastic disturbance. Equation (3.2.11) shows how permanent income is estimated from an adaptive expectations model. Substituting (3.2.10) into (3.2.9) yields:

$$C(t) = (k - a\eta)y^P(t) + a\, L(t) + \varepsilon(t) \qquad (3.2.12)$$

Equation (3.2.11) is recursive, which implies:

$$y^P(t) = b\, y(t) + b(1-b)y(t-1) + b(1-b)^2 y(t-2) + \ldots$$
$$+ b(1-b)^n y(t-n) + \ldots \qquad (3.2.13)$$

Now, letting Equation (3.2.12) lag by one time period, multiplying both sides of this equation by $(1-b)$, substituting the final result from Equation (3.2.12), and substituting for $y^P(t)$ and $y^P(t-1)$ using (3.2.13) the following equation is obtained:

$$C(t) = b(k - a\eta)y(t) + a[L(t) - (1-b)L(t-1)] + (1-b)C(t-1)$$
$$+ (\varepsilon(t) - (1-b)\varepsilon(t-1)) \qquad (3.2.14)$$

By simplifying the notation, allowing for a non-zero constant term; and assuming that $\varepsilon(t)$ follows a first-order autoregressive scheme with the coefficient of autoregression being equal to $(1-b)$, Equation (3.2.1) results.

All three investment equations are based on the proposition that the desired level of investment depends on the rate of interest, $r(t)$, the immediately preceding change in sales, $(S(t-1) - S(t-2))$, and time, t, i.e.,

$$I^*(t) = a_0 + a_1 r(t) + a_2(S(t-1) - S(t-2)) + a_3 t + \varepsilon_1(t) \qquad (3.2.15)$$

The accelerator term involve final sales rather than GNP on the grounds that a large portion of inventory changes may be unintended so that producers are more apt to base the change in their desired stock of capital upon final sales than upon total output. The trend variable, t, is included to take account of automatically induced changes in investment. Further, it is assumed that:

$$I(t) - I(t-1) = \gamma(I^*(t) - I(t-1)) + \varepsilon_2(t) \qquad (3.2.16)$$

Substitution for $I^*(t)$ from (3.2.15) into (3.2.16) and rearranging terms leads to[#]

$$I(t) = a_0\gamma + a_1\gamma\, r(t) + a_2\gamma(S(t-1) - S(t-2)) + a_3\gamma\, t$$
$$+ (1-\gamma)I(t-1) + (\gamma\varepsilon_1(t) + \varepsilon_2(t)) \qquad (3.2.17)$$

which is of the form used in Equations (3.2.2) - (3.2.4).

The money demand equation, Equation (3.2.5), represents a fairly standard formulation and is discussed elsewhere (Refs. C-11 and T-9). It involves the proposition that households and firms are unable to adjust their actual money holdings to the desired level immediately. Since $M(t)$ is considered to be exogenous, $r(t)$ is specified to be the "dependent variable" in the equation as suggested by Chow (Ref. C-11, pp. 10-11). The conventional definition of money is used, i.e., it includes only currency outside of banks and demand deposits. This seems to be the most appropriate monetary policy variable since, as argumed by Laidler (Ref. L-1), the decision maker(s) may be able to control demand deposits, but not the quantities of other financial intermediary liabilities. A different argument for the exclusion of time deposits from the definition of money is presented by Pesek (Ref. P-2).*

[#] Notice that the combined disturbance term, is assumed to be non-autoregressive and normally distributed with zero mean and a constant variance.

* However, it should be emphasized that time deposits are included in the definition of the liquid asset variable in the consumption function, Equation (3.2.1).

The coefficients in the model were estimated using the three-stage-least-squares method (3SLS) except for the adaptive expectations coefficient in the consumption function which was estimated using non-linear two-stage-least-squares (2SLS). For more details on the design and estimation of the model see Ref. K-6. The results together with the estimated standard errors and the coefficients of determination, are given in Ref. K-6 as follows:

Behavioral Equations:

$$\hat{C}(t) = \underset{(.7803)}{-1.7951} + \underset{(.0131)}{.1731}\, y(t) + \underset{(.0277)}{.0421}(L(t) - \underset{(.0665)}{.7275}\, L(t-1)) + \underset{(.0665)}{.7275}\, C(t-1),$$
$$R^2 = .9968;$$

$$\hat{I}^d(t) = \underset{(1.0759)}{2.5624} - \underset{(.1891)}{.4411}\, r(t) + \underset{(.0501)}{.01381}(S(t-1)-S(t-2)) + \underset{(.0110)}{.0237}\, t + \underset{(.0700)}{.8917}\, I^d(t-1),$$
$$R^2 = .8961;$$

$$\hat{I}^r(t) = \underset{(.5779)}{3.6083} - \underset{(.1133)}{.5127}\, r(t) + \underset{(.0335)}{.1267}(S(t-1)-S(t-2)) + \underset{(.0059)}{.0218}\, t + \underset{(.0668)}{.6483}\, I^r(t-1),$$
$$R^2 = .8394;$$

$$\hat{I}^i(t) = \underset{(1.3610)}{3.0782} - \underset{(.4089)}{.8934}\, r(t) + \underset{(.1301)}{.3713}(S(t-1)-S(t-2)) + \underset{(.0208)}{.0450}\, t + \underset{(.1181)}{.3178}\, I^i(t-1),$$
$$R^2 = .5341;$$

$$\hat{r}(t) = \underset{(1.8702)}{13.8928} + \underset{(.0042)}{.0261}\, y(t) - \underset{(.0335)}{.1501}\, M(t) + \underset{(.0338)}{.0588}\, M(t-1),$$
$$R^2 = .8538;$$

Identities:

$$y(t) \equiv C(t) + I^d(t) + I^r(t) + I^i(t) + G(t),$$

$$S(t) \equiv y(t) - I^i(t),$$

$$L(t) \equiv M(t) + R(t).$$

A block diagram of the open-loop system, i.e., Equations (3.2.1) - (3.2.7) is shown in Fig. (3.2.1). This figure shows the

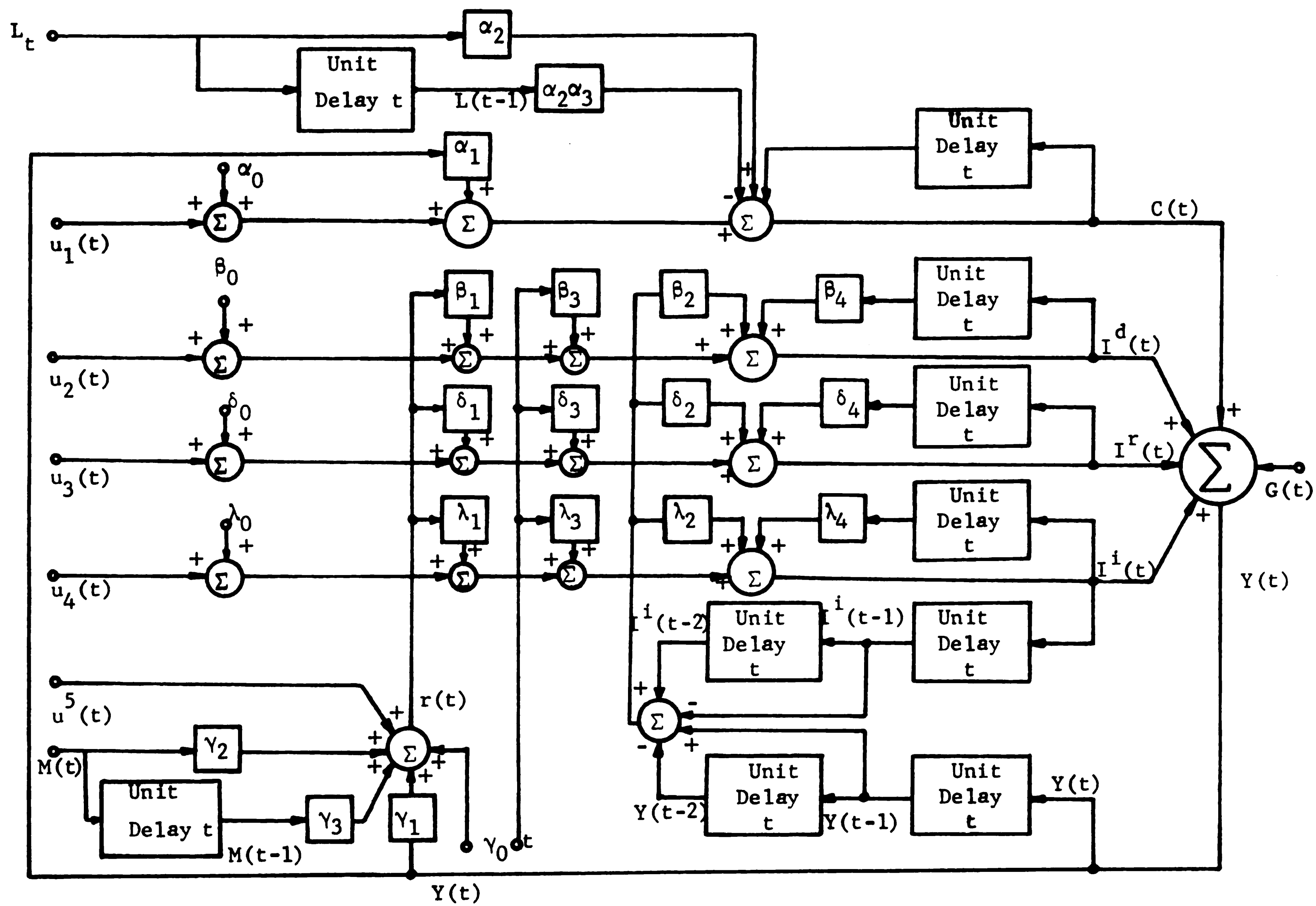

Figure (3.2.1). Block diagram of the open-loop system, Eqs. (3.2.1) - (3.2.7).

channels of the open-loop system is quite involved. The Fig. (3.2.1) shows how the systems' variable are interrelated and also shows the input output relationships.

3.3 Dynamic Analysis and Stability of the Model

Using the estimated structural equations and expressing current endogenous variables in terms of exogenous and lagged endogenous variables, derived reduced form equations are obtained. The coefficients of these equations, which measure the immediate effects of predetermined variables on the current values of the endogenous variables, are called "impact multipliers" (Ref. K-6, p. 10).

The reduced form solution presents a clear picture of the immediate responses of GNP to changes in the predetermined variables and allows to estimate the effects of the exogenous variables given the immediate past history of all endogenous variables.

For an analysis of the past, however, the impact multipliers alone are not very illuminating. Considering the reduced form equations only, it may be found that the main influence on the current values of GNP is its immediate history, and the question of the relative importance of fiscal, monetary and other exogenous variables would remain unresolved. For this problem, the relevant solution is obviously one which determines the time path of GNP in response to autonomous forces alone. Such a solution involves the determination of current GNP in terms of its own lagged values and of current and lagged values of the exogenous variables. The resulting equation may be termed the "fundamental dynamic equation"

(Ref. K-6, pp. 10-11). For the model studied here, this is given by:

$$y(t) + a_4 y(t-1) + a_3 y(t-2) + a_2 y(t-3) + a_1 y(t-4) + a_0 y(t-5)$$
$$= + \underline{a}'_5 \underline{w}(t) + \underline{a}'_4 \underline{w}(t-1) + \underline{a}'_3 \underline{w}(t-2) + \underline{a}'_2 \underline{w}(t-3)$$
$$+ \underline{a}'_1 \underline{w}(t-4) + \underline{a}'_0 \underline{w}(t-5) + \underline{k}'\underline{t} + c_0 + \text{error} \qquad (3.3.1)$$

where

$$a_4 = -3.0716,$$
$$a_3 = 3.6561,$$
$$a_2 = -2.0850,$$
$$a_1 = .5585,$$
$$a_0 = -.0535,$$
$$\underline{w}(t) = (G(t)\ M(t)\ L(t))$$
$$\underline{a}'_5 = (1.1427 \quad .3168 \quad .0481)$$
$$\underline{a}'_4 = (-2.5300 \quad -.7499 \quad -.1065)$$
$$\underline{a}'_3 = (1.3779 \quad .6253 \quad .0580)$$
$$\underline{a}'_2 = (.5853 \quad -.2000 \quad .0246)$$
$$\underline{a}'_1 = (-.7463 \quad .0082 \quad -.0314)$$
$$\underline{a}'_0 = (.1784 \quad .0046 \quad .0075)$$
$$\underline{t}' = (t \quad t-1 \quad t-2 \quad t-3 \quad t-4)$$
$$k' = (.1034 \quad -.2050 \quad .1267 \quad -.0192 \quad -.0032)$$
$$c_0 = -.5113 .$$

The coefficients of Equation (3.3.1) may be classified according to the source of their effects on model dynamics:

1. income effect

$$\partial C(t)/\partial y(t) = \alpha_1$$

$$\partial I^n(t)/\partial y(t) = \frac{\partial I^n(t)}{\partial r(t)} \cdot \frac{\partial r(t)}{\partial y(t)} = m_i \cdot \gamma_1$$

$$\frac{\partial I^n(t)}{\partial y(t-1)} = \frac{\partial I^n(t)}{\partial S(t-1)} \cdot \frac{\partial S(t-1)}{\partial y(t-1)} = m_2 \cdot 1$$

$$\frac{\partial I^n(t)}{\partial y(t-1)} = \frac{\partial I^n(t)}{\partial S(t-2)} \cdot \frac{\partial S(t-2)}{\partial y(t-2)} = -m_2 \cdot 1$$

where n, m_1, and m_2 are replaced by $n = d, r, i$; $m_1 = \beta_1, \delta_1, \lambda_1$; $m_2 = \beta_2, \delta_2, \lambda_2$ respectively,

$$\frac{\partial r(t)}{\partial y(t)} = \gamma_1$$

$$\frac{\partial S(t)}{\partial y(t)} = 1$$

2. interest effect

$$\frac{\partial I^d(t)}{\partial r(t)} = \beta_1 \ , \ \frac{\partial I^r(t)}{\partial r(t)} = \delta_1 \ , \ \frac{\partial I^i(t)}{\partial r(t)} = \lambda_1$$

3. stock effect

$$\frac{\partial C(t)}{\partial L(t)} = \alpha_2 \ , \ \frac{\partial C(t)}{\partial L(t-1)} = -\alpha_2\alpha_3 \ ,$$

$$\frac{\partial I^n(t)}{\partial M(t)} = \frac{\partial I^n(t)}{\partial r(t)} \cdot \frac{\partial r(t)}{\partial M(t)} = m_1 \cdot \gamma_2$$

$$\frac{\partial I^n(t)}{\partial M(t-1)} = \frac{\partial I^n(t)}{\partial r(t)} \cdot \frac{\partial r(t)}{\partial M(t-1)} = m_2 \cdot \gamma_3$$

where n, m_1, and m_2 are as above,

$$\frac{\partial r(t)}{\partial M(t)} = \gamma_2 \ , \ \frac{\partial r(t)}{\partial M(t-1)} = \gamma_3$$

4. trend effect

$$\partial I^d(t)/\partial t = \beta_3 \ , \ \partial I^r(t)/\partial t = \delta_3 \ , \ \partial I^i(t)/\partial t = \lambda_3$$

5. inertia effect

$$\partial C(t)/\partial C(t-1) = \alpha_3,\ \partial I^d(t)/\partial I^d(t-1) = \beta_4,\ \partial I^r(t)/\partial I^r(t-1) = \delta_4,$$

$$\partial I^i(t)/\partial I^i(t-1) = \lambda_4 .$$

Model parameters are influencing coefficients among the model variables, i.e., they measure the effect of variations in one variable upon the value of other variable(s). In other words, these elements correspond to the channels of influence represented in Fig. (3.2.1) and in Equations (3.2.1) - (3.2.5). For example, a rise in $y(t)$ will lead to a positive income effect on the aggregate consumption but this will be partially offset by a negative stock effect in Equation (3.2.1). In Equations (3.2.2) - ((3.2.4) a rise in $y(t)$ will lead to a negative interest effect, because the interest rate has a positive income effect in Equation (3.2.5) whereas the Equations (3.2.2) - (3.2.4) have negative interest effect. Equations (3.2.2) - (3.2.4) have positive trend effects and as the consumption and investments (different kinds) rise, a positive inertia effect will influence them with a delay. In Equation (3.2.5) the stock effect (money supply) has a negative effect in the current period and a positive effect in the preceding period and, therefore, partially offsetting each other's effect. Thus, the values of these parameters which depend upon the design of the model and estimation procedures are fundamental to the dynamic behavior of the GNP model system, which includes its stability, response time constants# and

The time constant of a system response mode is the total elapsed time between application of a step perturbation and the instant the response attains approximately 63% of its steady state value. Time constants are measure of the rapidity of a system's response.

steady state gains.[#]

The fundamental dynamic equation, Equation (3.3.1), determines whether the system is or is not dynamically stable, and it also provides the basis for evaluating the relative importance of individual exogenous variables. The question of dynamic stability can be settled by reference to the auxiliary equation which is obtained from Equation (3.3.1), by transferring all terms involving $y(\cdot)$ to the left-hand side and equating to zero. The eigenvalues (roots of the auxiliary equations) for the system are:

$$\lambda_1 = .2081$$
$$\lambda_2 = .8475 + i\ .0809$$
$$\lambda_3 = .8475 - i\ .0809$$
$$\lambda_4 = .5843 + i\ .1156$$
$$\lambda_5 = .5843 - i\ .1156$$

Fig. (3.3.1) illustrates the plot of these roots in the imaginary λ-plane and their position in the unit circle.

Each eigenvalue characterizes an independent response mode of the open loop system. Eigenvalues falling within the unit circle of the λ-plane represent stable response modes. Fig. (3.3.1) suggests that all response modes of the open-loop economy are stable. Stable eigenvalues closest to the unit circle dominate transient system dynamics. That is, the closer a root is to the unit circle the larger its response time constant. The complex conjugate pair(s) indicates a damped sinusoidal response mode.

[#] Steady state gain is the ratio of output to input at steady state conditions. Steady state gains are analogous to the static multipliers of Keynesian economic theory.

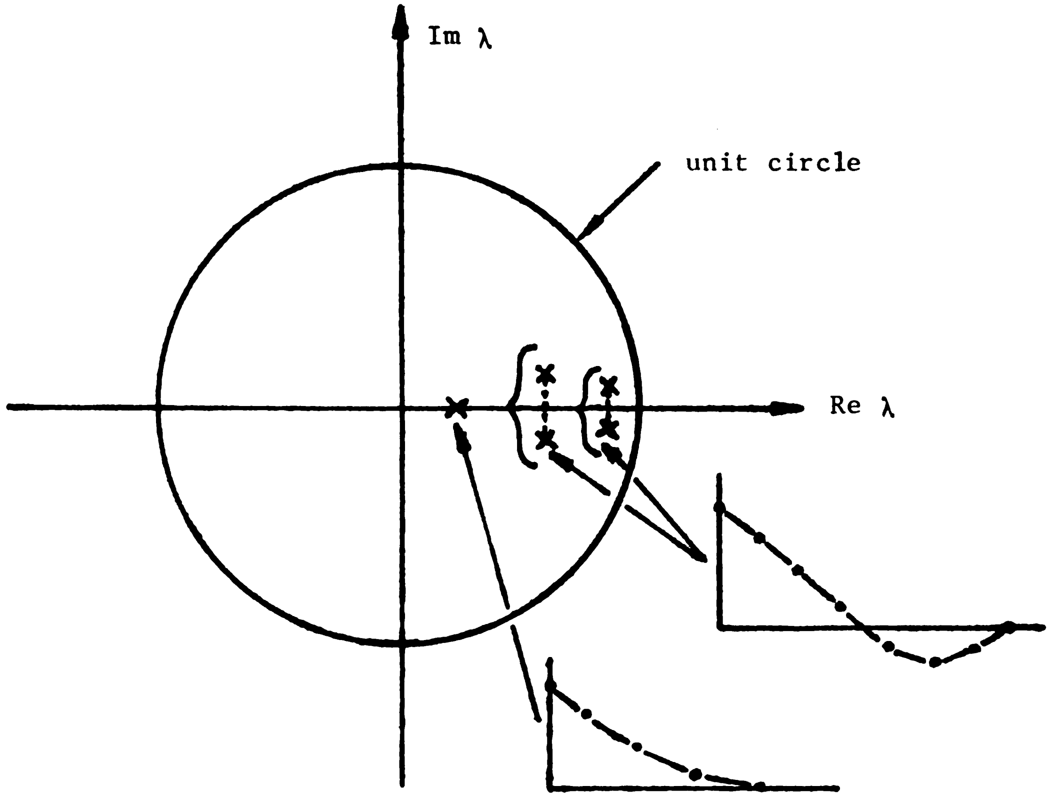

Fig. (3.3.1). λ-plot of the system's eigenvalues with the corresponding responses for the system.

Therefore, the basic homogenous solution for the time path of $y(t)$, apart from the influence of exogenous variables and of initial conditions, is one of damped oscillatory motion, since both the largest real root and the modulus of the conjugate complex roots are less than unity in absolute magnitude (Ref. K-7). It hence appears that the system is inherently stable and that the sources of instability have to be sought in the stimuli from the exogenous variables (including trends) and the random disturbances (Ref. K-6).

CHAPTER FOUR

DERIVATION OF THE OPTIMAL CONTROL MODEL FROM THE KMENTA/SMITH DYNAMIC MODEL

4.1 State Space Equations of Systems Described by Difference Equations

Dynamic systems in which one or more variables can change only at discrete instants of time are called discrete-time systems. The behavior of discrete-time systems can be described in terms of difference equations. By using a set of state variables, higher-order differences may be avoided with a description in the convenient form of vector matrix first order difference equations. In order to solve differential equations by means of a digital computer, differential equations are reduced to difference equations. This class of equations can be treated with algebraic techniques.

Consider the state space representation of the following nth-order difference equation:

$$\begin{aligned} y(t+n) &+ a_{n-1}(t)y(t+(n-1)) + \ldots + a_0(t)y(t) \\ &= b_n(t)u(t+n) + b_{n-1}(t)u(t+(n-1)) + \ldots + b_1(t)u(t+1) \\ &\quad + b_0(t)u(t) \end{aligned} \tag{4.1.1}$$

where $u(t+i)$, $y(t+i)$; $i = 0,1,\ldots,n$ are scalar (single) input and output respectively, with $a_j(t)$, $b_i(t)$; $j = 0,1,\ldots,n-1$; $i = 0,1,\ldots,n$ as the time-varying coefficients of the system. Since $2n$ initial conditions, namely, $y(0), y(1), \ldots, y(n-1)$ and

$u(0), u(1), \ldots, u(n-1)$ must be known in order to obtain the solution of the difference equation (4.1.1), the following approach is used to determine the state vector: Let $\underline{z}(t)$ be an n-vector with components $z_1(t), z_2(t), \ldots, z_n(t)$ which are given by

$$\begin{aligned}
z_1(t) &= y(t) - c_0(t)u(t) \\
z_2(t) &= y(t+1) - c_0(t+1)u(t+1) - c_1(t)u(t) \\
z_3(t) &= y(t+2) - c_0(t+2)u(t+2) - c_1(t+1)u(t+1) - c_0(t)u(t) \qquad (4.1.2) \\
\ldots \quad & \ldots \quad \ldots \quad \ldots \quad \ldots \\
z_n(t) &= y(t+(n-1)) - c_0(t+(n-1))u(t+(n-1)) - c_1(t+(n-2))u(t+(n-2)) - \\
&\qquad \ldots - c_{n-1}(t)u(t)
\end{aligned}$$

where $c_0(t), c_1(t), \ldots, c_n(t)$ are n parameters whose values are still to be determined. Equation (4.1.2) may be written in the form:

$$z_i(t) = y(t+(i-1)) - \sum_{k=0}^{i-1} u(t+k)c_{i-k+1}(t+k), \quad i = 1,2,\ldots,n. \qquad (4.1.3)^{\#}$$

It is time now to determine the difference equations which the $z_i(t)$ satisfy. Observe from Equations (4.1.2) that

$$z_1(t+1) = y(t+1) - c_0(t+1)u(t+1) \qquad (4.1.4)$$

$$= z_2(t) + c_1(t)u(t) \qquad (4.1.5)$$

This deduction is based upon lagging forward the equation defining $z_1(t)$ and upon substitution of the result in the equation defining $z_2(t)$. In a completely analogous way, it may be deduced that, for

Consider that $u(t+0) = u(t)$, $c(t+0) = c(t)$.

$i = 1,2,\ldots,n-1$

$$z_1(t+1) = y(t+i) - \sum_{k=0}^{i-1} u(t+k+1)c_{i-k+1}(t+k+1) \tag{4.1.6}$$

$$= z_{i+1}(t) + c_i(t)u(t) \tag{4.1.7}$$

since $z_{i+1}(t) = y(t+i) - \sum_{k=0}^{i} u(t+k)c_{i-k}(t+k)$

$$= y(t+i) - c_i(t)u(t) = \sum_{k=1}^{i} u(t+k)c_{i-k}(t+k). \tag{4.1.8}$$

As for $z_n(t+1)$, it is noted that lagging Equation (4.1.3) forward once for $i = n$ yields

$$z_n(t+1) = y(t+n) - \sum_{k=1}^{n} u(t+k)c_{n-k}(t+k) \ . \tag{4.1.9}$$

However, the system of difference equation, (4.1.1), gives the relation

$$y(t+n) = -\sum_{i=0}^{n-1} a_i(t)y(t+i) + b_n(t)u(t+n) + \sum_{k=1}^{n-1} b_k(t)u(t+k) + b_0(t)u(t) \tag{4.1.10}$$

It follows from Equation (4.1.8) that

$$\sum_{i=0}^{n-1} a_i(t)y(t+i) = \sum_{i=0}^{n-1} a_i(t)z_{i+1}(t) + \sum_{i=0}^{n-1} a_i(t)\left[\sum_{k=1}^{i} u(t+k)c_{i-k}(t+k)\right] + u(t)\sum_{i=0}^{n-1} a_i(t)c_i(t) \tag{4.1.11}$$

and hence, using Equations (4.1.9), (4.1.10), and (4.1.11) yields

$$z_n(t+1) = -\sum_{i=0}^{n-1} a_i(t)z_{i+1}(t) + b_n(t)u(t+n) + \sum_{k=1}^{n-1} b_k(t)u(t+k) + b_0(t)u(t)$$
$$- \sum_{k=1}^{n} u(t+k)c_{n-k}(t+k) - \sum_{i=0}^{n-1} a_i(t)\left[\sum_{k=1}^{i} u(t+k)c_{i-k}(t+k)\right] - u(t)\sum_{i=0}^{n-1} a_i(t)c_i(t) \ . \tag{4.1.12}$$

However, using direct computation yields:

$$\sum_{i=0}^{n-1} a_i(t)\left[\sum_{k=1}^{i} u(t+k)c_{i-k}(t+k)\right] = \sum_{k=1}^{n-1} u(t+k)\left[\sum_{i=0}^{n-k-1} c_i(t+k)a_{i+k}(t)\right] \tag{4.1.13}$$

where $\sum_{k=1}^{0} u(t+k)c_{i-k}(t+k) \equiv 0$ is assumed.

Upon substitution in Equation (4.1.12) and collecting terms, the following relation is obtained:

$$\begin{aligned} z_n(t+1) = &- \sum_{i=0}^{n-1} a_i(t)z_{i+1}(t) + (b_n(t) - c_0(t+n))u(t+n) \\ &+ \sum_{k=1}^{n-1}\left\{b_k(t) - c_{n-k}(t+k) - \sum_{i=0}^{n-k-1} c_i(t+k)a_{i+k}(t)\right\} \\ &+ u(t)\left\{b_0(t) - \sum_{i=0}^{n-1} a_i(t)c_i(t)\right\} . \end{aligned} \tag{4.1.14}$$

The coefficients in Equation (4.1.14) are selected such that $z_n(t+1)$ is independent of all the lagged values of $u(t)$. This implies

$$c_0(t+n) = b_n(t) \tag{4.1.15}$$

$$c_{n-k}(t+k) = b_k(t) - \sum_{i=0}^{n-k-1} c_i(t+k)a_{i+k}(t) \quad \text{for} \quad k = 1,2,\ldots,n-1.$$

Substituting in Equation (4.1.14) yields

$$z_n(t+1) = - \sum_{i=0}^{n-1} a_i(t)z_{i+1}(t) + c_n(t)u(t) \tag{4.1.16}$$

where $c_n(t) = b_0(t) - \sum_{i=0}^{n-1} a_i(t)c_i(t)$. (4.1.17)

Thus, the difference equations satisfied by the $z_i(t)$ have been determined; they may be written in the vector form as

$$\begin{bmatrix} z_1(t+1) \\ z_2(t+1) \\ \cdots \\ z_{n-1}(t+1) \\ z_n(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ \cdots \\ z_{n-1}(t) \\ z_n(t) \end{bmatrix} + \begin{bmatrix} c_1(t) \\ c_2(t) \\ \cdots \\ c_{n-1}(t) \\ c_n(t) \end{bmatrix} u(t) \tag{4.1.18}$$

or, more succinctly, as

$$\underline{z}(t+1) = A\,\underline{z}(t) + \underline{c}\,u(t) \tag{4.1.19}$$

where the matrix A and the vector $\underline{c}$ are

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & -a_2(t) & \cdots & -a_{n-1}(t) \end{bmatrix}, \quad \underline{c} = \begin{bmatrix} c_1(t) \\ c_2(t) \\ \cdots \\ c_{n-1}(t) \\ c_n(t) \end{bmatrix}. \tag{4.1.20}$$

It is noticed also that the computation of the coefficients $c_i(t)$, $i = 0,1,\ldots,n$, is not particularly difficult since Equations (4.1.15) are of the form

$$\begin{aligned}
c_0(t) &= b_n(t-n) \\
c_1(t) &= b_{n-1}(t-(n-1)) - c_0(t)a_{n-1}(t-(n-1)) \\
c_2(t) &= b_{n-2}(t-(n-2)) - c_0(t)a_{n-2}(t-(n-2)) - c_1(t)a_{n-1}(t-(n-1)) \\
\cdots & \qquad \cdots \qquad\qquad\qquad \cdots \qquad\qquad\qquad \cdots \\
c_n(t) &= b_0(t) - c_0(t)a_0(t) - c_1(t)a_1(t) - \cdots - c_{n-1}(t)a_{n-1}(t)
\end{aligned} \tag{4.1.21}$$

and the $c_i(t)$, $i = 0,1,\ldots,n$, can be found by successive substitution.

It is readily verified that the vector $\underline{z}(t)$ qualifies as a state variable for the system of Equation (4.1.1), since knowledge of $\underline{z}(t_0)$ and $u_{(t_0,t]}$ completely determines the solution of the difference equation (4.1.19) on the interval $(t_0,t]$ (that is, $\underline{z}_{(t_0,t]}$). Moreover, since the output $y(t)$ is given by

$$y(t) = z_1(t) + c_0(t)u(t), \tag{4.1.22}$$

it is clear that $y_{(t_0,t]}$ is indeed determined by $\underline{z}(t_0)$ and $u_{(t_0,t]}$. This equation may be written as

$$y(t) = \underline{e}'\underline{z}(t) + c_0(t)u(t) \tag{4.1.22a}$$

where

$$\underline{e}' = [1 \quad 0 \quad \dots \quad 0] \ .$$

Equations (4.1.19) and (4.1.22a) together constitute the state space formulation of the difference equation (4.1.1).

Figure (4.1.1) illustrates the simulation of the vector difference equation (4.1.18) on a digital computer.

4.2 State Space Representation of the Kmenta/Smith Model

After some rather long and tedious algebriac manipulation the K/S econometric model becomes

$$\begin{aligned} & y(t+5) + a_4y(t+4) + a_3y(t+3) + a_2y(t+2) + a_1y(t+1) + a_0y(t) \\ & = \underline{a}_5'\underline{w}(t+5) + \underline{a}_4'\underline{w}(t+4) + \underline{a}_3'\underline{w}(t+3) + \underline{a}_2'\underline{w}(t+2) + \underline{a}_1'\underline{w}(t+1) + \underline{a}_0'\underline{w}(t) \\ & \quad + \underline{k}'\underline{t} + c_0 + \text{disturbances} \end{aligned} \tag{4.2.1}$$

where

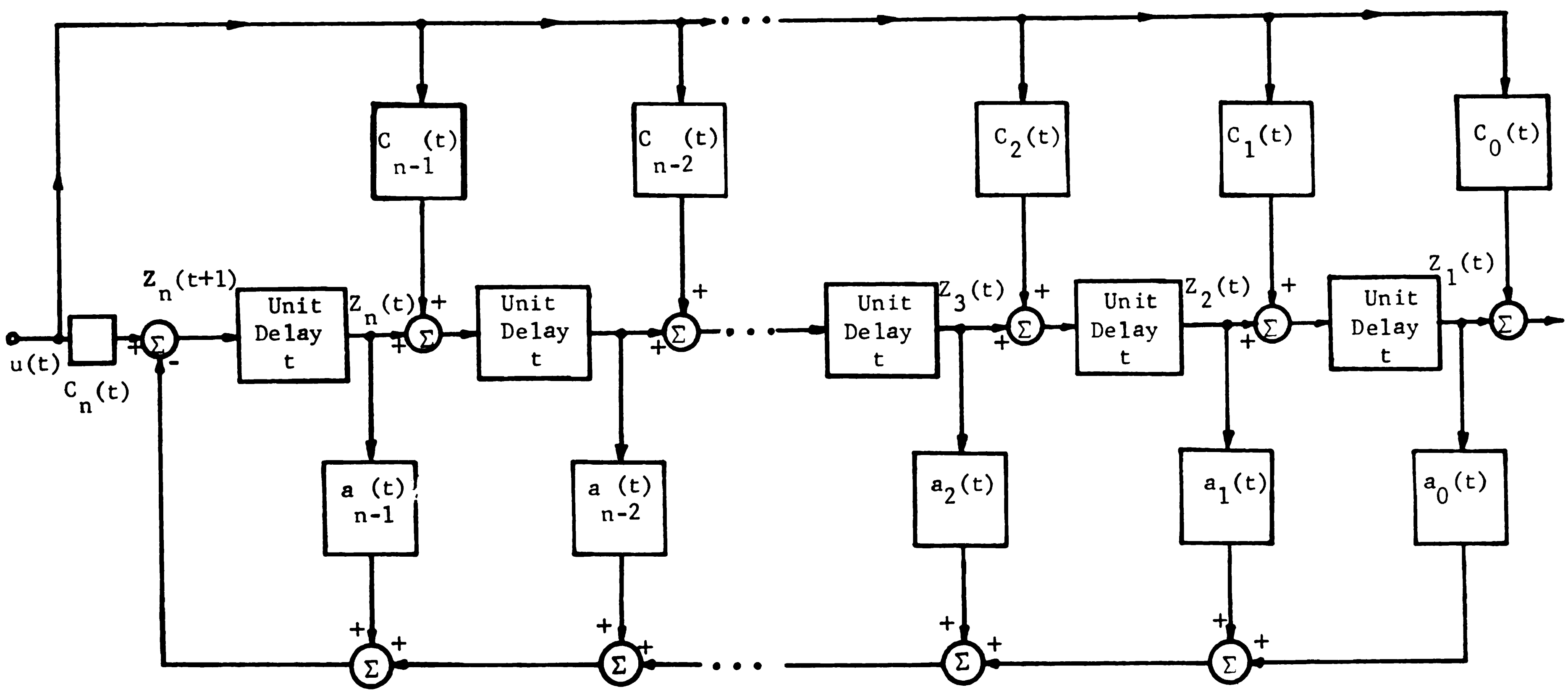

Fig. (4.1.1) The computer simulation representation of the system (4.1.18).

$a_4 = -3.0716;\ a_3 = 3.6561;\ a_2 = -2.0850;\ a_1 = .5585;\ a_0 = -.0535$

$\underline{w}' = (G(t),\ M(t),\ L(t))$

$\underline{a}_5' = (1.1427 \quad .3168 \quad .0481)$

$\underline{a}_4' = (-2.5300 \quad -.7499 \quad -.1065$

$\underline{a}_3' = (1.3779 \quad .6253 \quad .0580)$

$\underline{a}_2' = (.5853 \quad -.2000 \quad .0246)$

$\underline{a}_1' = (-.7463 \quad .0082 \quad -.0314)$

$\underline{a}_0' = (.1784 \quad .0046 \quad .0075)$

$\underline{t}' = (t+5,\ t+4,\ t+3,\ t+2,\ t+1,\ t)$

$\underline{k}' = (.1034 \quad -.2050 \quad .1267 \quad -.0192 \quad -.0032 \quad 0)$

$c_0 = -.5113$

and disturbances consist of environmental noises $u_i(t)$, $u_i(t+1)$, $u_i(t+2)$, $u_i(t+3)$, and $u_i(t+4)$, $i = 1,\ldots,5$.

Applying the results presented in Sec. 4.1 to the 5th-order difference equation (4.2.1) results in the state space representation of the model as follows:

$$\begin{bmatrix} y_1(t+1) \\ y_2(t+1) \\ y_3(t+1) \\ y_4(t+1) \\ y_5(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & -a_4 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \\ y_5(t) \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{bmatrix} \begin{bmatrix} G(t) \\ M(t) \\ L(t) \end{bmatrix}$$

$$+ \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ d_5 \end{bmatrix} t + \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} & d_{15} \\ d_{21} & d_{22} & d_{23} & d_{24} & d_{25} \\ d_{31} & d_{32} & d_{33} & d_{34} & d_{35} \\ d_{41} & d_{42} & d_{43} & d_{44} & d_{45} \\ d_{51} & d_{52} & d_{53} & d_{54} & d_{55} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \\ u_5(t) \end{bmatrix} \qquad (4.2.2)$$

and $y(t) = y_1(t) + \underline{b}_0'\underline{w}(t) + c_0 + d_0 t + \underline{d}_0' \underline{u}(t)$ (4.2.3)

where the state variables y_i, $i = 1,\ldots,5$ are defined by:

$$
\begin{aligned}
y_1(t+1) &= y_2(t) + \underline{b}_1'\underline{w}(t) + c_1 + d_1 t + \underline{d}_1'\underline{u}(t) \\
y_2(t+1) &= y_3(t) + \underline{b}_2'\underline{w}(t) + c_2 + d_2 t + \underline{d}_2'\underline{u}(t) \\
y_3(t+1) &= y_4(t) + \underline{b}_3'\underline{w}(t) + c_3 + d_3 t + \underline{d}_3'\underline{u}(t) \\
y_4(t+1) &= y_5(t) + \underline{b}_4'\underline{w}(t) + C_4 + d_4 t + \underline{d}_4'\underline{u}(t) \qquad (4.2.4) \\
y_5(t+1) &= -a_0 y_1(t) - a_1 y_2(t) - a_2 y_3(t) - a_3 y_4(t) - a_4 y_5(t) \\
&\quad + \underline{b}_5'\underline{w}(t) + c_5 + d_5 t + \underline{d}_5'\underline{u}(t) .
\end{aligned}
$$

The vectors $\underline{b}_i' = (b_{i1} \;\; b_{i2} \;\; b_{i3})$, $i = 1,2,\ldots,5$, and $\underline{d}_i' = (d_{i1} \;\; d_{i2} \;\; d_{i3} \;\; d_{i4} \;\; d_{i5})$, $i = 1,2,\ldots,5$ are the rows of the coefficient matrices of the control vector $\underline{w}'(t) = (G(t) \; M(t) \; L(t))$ and disturbance vector $\underline{u}'(t) = (u_1(t) \;\; u_2(t) \;\; u_3(t) \;\; u_4(t) \;\; u_5(t))$ respectively, in the system (4.2.2).

The state space formulation of the K/S model, Equations (4.2.2) and (4.2.3) are of the form:

$$\underline{y}(t+1) = A_1\underline{y}(t) + A_2\underline{w}(t) + \hat{\underline{a}}_3 + \hat{\underline{a}}_4 t + A_5\underline{u}(t) \qquad (4.2.5)$$

$$y(t) = \underline{e}'\underline{y}(t) + \underline{b}_0' \,\underline{w}(t) + c_0 + d_0 t + \underline{d}_0' \,\underline{u}(t) \qquad (4.2.6)$$

where $\underline{e}' = [1 \;\; 0 \;\; 0 \;\; 0 \;\; 0]$, $\underline{b}_0' = (1.1427 \;\; .3168 \;\; .0481)$. Figure (4.2.1) illustrates the computer simulation (block diagram) of the vector difference equation (4.2.2) and the output equation (4.2.3). Furthermore, applying the results given in Section 4.1, Equations (4.1.21) implies:

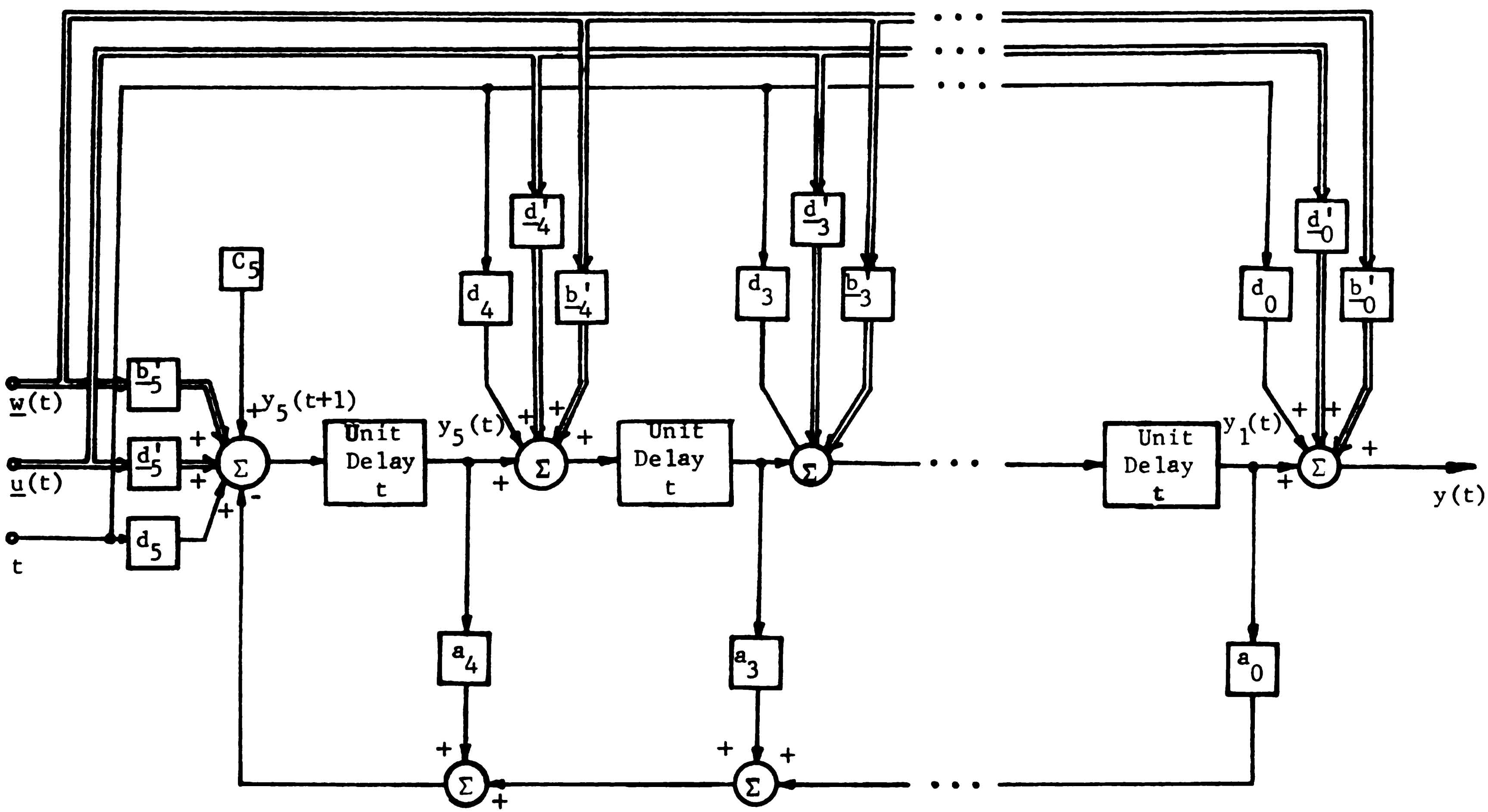

Fig. (4.2.1). The computer simulation illustration of the system represented by Eqs. (4.2.2) and (4.2.3).

$\underline{b}_0' = (1.1427 \quad .3168 \quad .0481)$

$\underline{b}_1' = (.979917 \quad .223183 \quad .041244)$

$\underline{b}_2' = (.209988 \quad .152577 \quad .008827)$

$\underline{b}_3' = (.030154 \quad .113205 \quad .001210)$

$\underline{b}_4' = (-.016487 \quad .086489 \quad -.000825)$

$\underline{b}_5' = (-.030812 \quad .066788 \quad -.001552)$

$d_0 = .1034$; $d_1 = .112603$; $d_2 = .094530$; $d_3 = .075059$;

$d_4 = .058768$; $d_5 = .045527$

and

$c_0 = -.5113$; $c_1 = -1.570509$; $c_2 = -2.954611$;

$c_3 = -4.399506$; $c_4 = -5.700119$; $c_5 = -6.734040$.

The initial conditions are derived by taking the expected value of the equations (4.2.4) with respect to the disturbances as follows:

$$y_1(0) = y(0) - \underline{b}_0'\underline{w}(0) - c_0$$

$$y_2(0) = y(1) - \underline{b}_0'\underline{w}(1) - \underline{b}_1'\underline{w}(0) - c_1 - c_0 - d_0$$

$$y_3(0) = y(2) - \underline{b}_0'\underline{w}(2) - \underline{b}_1'\underline{w}(1) - \underline{b}_2'\underline{w}(0) - c_2 - c_1 - c_0 - d_1 - 2d_0$$

$$y_4(0) = y(3) - \underline{b}_0'\underline{w}(3) - \underline{b}_1'\underline{w}(2) - \underline{b}_2'\underline{w}(1) - \underline{b}_3'\underline{w}(0) - c_3 - c_2 - c_1 - c_0 - d_2 - 2d_1 - 3d_0$$

$$y_5(0) = y(4) - \underline{b}_0'\underline{w}(4) - \underline{b}_1'\underline{w}(3) - \underline{b}_2'\underline{w}(2) - \underline{b}_3'\underline{w}(1) - \underline{b}_4'\underline{w}(0) - c_4 - c_3 - c_2 - c_1 - c_0 - d_3 - 2d_2 - 3d_1 - 4d_0$$

where the c's and the d's are specified in above and

$y(0) = 100.70$ GNP at 1954-1

$y(1) = 100.50$ GNP at 1954-2

$y(2) = 101.80$ GNP at 1954-3

$y(3) = 103.90$ GNP at 1954-4

$y(4) = 107.00$ GNP at 1955-1

$\underline{w}'(0) = (24.10 \quad 144.10 \quad 187.60)$

$\underline{w}'(1) = (23.30 \quad 143.40 \quad 187.80)$

$\underline{w}'(2) = (22.90 \quad 145.10 \quad 191.60)$

$\underline{w}'(3) = (22.60 \quad 146.30 \quad 194.50)$

$\underline{w}'(4) = (22.70 \quad 147.10 \quad 195.60)$

all expressed in billions of dollars. Using these data the initial condition vector becomes:

$$\underline{y}'(0) = (y_1(0) \; y_2(0) \; y_3(0) \; y_4(0) \; y_5(0)) = (20.45 \; -40.45 \; -62.27 \; -82.60 \; -91.31).$$

Substituting for the coefficient matrices in the state model equation (4.2.5), while combining the two column vectors $\underline{\hat{a}}_3$ and $\underline{\hat{a}}_4$ into one vector $\underline{\tilde{a}}_3$ yields:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ .0535 & -.5585 & 2.0850 & -3.6561 & 3.0716 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} .979917 & .223183 & .041244 \\ .209988 & .152577 & .008827 \\ .030154 & .113205 & .001210 \\ -.016487 & .086487 & -.000825 \\ -.030812 & .066788 & -.001552 \end{bmatrix}$$

$$\underline{\tilde{a}}_3 = \underline{\hat{a}}_3 + \underline{\hat{a}}_4 = [-1.570509 + .112603t;\ -2.954611 + .0945430t;$$
$$-4.399506 + .075059t;\ -5.700119 + .058768t;$$
$$-6.734040 + .045527t]\ .$$

That is, the state space formulation of the K/S model, Equations (4.2.5) and (4.2.6) are of the form:

$$\underline{y}(t+1) = A_1\underline{y}(t) + A_2\underline{w}(t) + \underline{\tilde{a}}_3 + A_5\underline{u}(t) \qquad (4.2.5a)$$

$$y(t) = \underline{e}'\underline{y}(t) + \underline{b}_0'\underline{w}(t) + c_0 + d_0t \qquad (4.2.6a)$$

4.3 Criterion Functional

Governmental actions aimed at improving the stability and growth of the economy pose difficult decision problems. Economic policy recommendations for stabilization have usually been discussed in the literature (Refs. A-3 and T-7) on the basis of the multiplier-accelerator type economic models in its different versions. Using the principles of servomechanism and conventional control system theory, Phillips (Refs. P-3 and P-4) first showed that the stability of the time path of the control variable, e.g., government expenditures, differs for different types of economic policy and the certain types of economic policy may themselves give rise to undesired fluctuations or instability. In Phillips' approach the types of stabilization policy operated by the government have the objectives of offsetting a downward shift in demand and of reducing the oscillation in aggregate output. Generally, this involves the addition or subtraction of an official demand to the normal flow of aggregate consumption and investment demands of the economy. However, the recent advances in the theory of control systems have emphasized that the stability alone, which may be a necessary condition of a system design and for which the stability criteria of Hurwitz, Routh and Nyquist have been applied in servo design theory (Ref. S-8), does not necessarily guarantee a suitable and optimum design. In modern control theory, it has been increasingly emphasized that an admissible control must have an optimizing property in some sense, e.g., minimizing the error of the system under control or satisfying certain specifications of accuracy and speed of performance of the system under control (Ref. S-11). Frequently, a criterion function is defined, which is otherwise called a performance measure (index) or social welfare function and the optimal character of a control is defined by minimization (or maximization as the case may be) of the performance index. Thus, in evaluation of control system designs in general, optimality is considered (Ref. B-6) to be a characteristic that is equally if not more important than the property of stability.

The selection of appropriate performance indices, or objective functions is not a trivial step. They may have more than one factor for the decision rule and they may be completely independent of one another. In this study, the performance measure is a functional which involves the expected value (with respect to the environmental disturbances, $\underline{u}(t)$) of a quadratic form in the state and control variables over a fixed interval of time (horizon). The choice of the quadratic performances index is of course not the only choice to be made. This is selected here, because apart from simplicity and the fact that it has been most widely used in adaptive control theory (Ref. S-8), it gives a quadratic criterion function in the integrand and specifies one simple measure of relative fluctuations. The quadratic functional which is otherwise called a quadratic social disutility function in the economic literature on optimum resourse allocation over time has been used quite extensively (Ref. S-11). Such a quadratic social disutility function has also been used as a performance index (Ref. H-5) in considering alternative policies for economic stabilization under a multiplier-type economic model. In the control theory (Refs. A-5, B-9, G-2, K-1, K-5, K-9, L-3, M-5, S-1), information theory (Ref. A-1), production, employment and inventory scheduling (Refs. H-3, H-4, and S-14) and statistical quality theory (Ref. B-3) quadratic performance index which is to be optimized by a given type of control, has been frequently used as a generalized measure of the system performance. The resulting problem, i.e., optimizing a quadratic performance index subject to the linear system model, for either discrete-time or continuous-time systems, is called the stochastic linear regulator problem (Ref. M-5).

Based on the above discussion, the specific criterion functional that is employed in this work is a quadratic social welfare functional of the form:

$$I_N = \sum_{t=1}^{N} \{\mu[y_d(t) - y_a(t)]^2 + \eta\, \underline{w}'(t-1)R\, \underline{w}(t-1)\} \tag{4.3.1}$$

where

$y_a(t)$ ... actual output (actual GNP),

$y_d(t)$ ... potential (desired) output,

μ, η ... weighting factors which may be chosen as functions of t, i.e., in a sense may be viewed as time discounting of the (imputed) cost of the system's error, $\epsilon = y_d(t) - y_a(t)$, and the cost of the activity of the control vector, $\underline{w}(t)$, respectively,

R ... a symmetric positive definite matrix,

$\underline{w}(t)$ ... control vector,

N ... total number of quarters within the optimization interval (planning horizon).

The time horizon t, $t \in \{0,1,\ldots,N\}$ where N is a positive integer, defined as above, need not be fixed or finite. The case of a large N $(N \to \infty)$, however, can be taken into account by certain regularity assumptions regarding the boundedness of the performance index. Since in this study short term stabilization policies are considered, N is assumed to be fixed and finite $(N = 40)$.

In utilizing Equation (4.3.1) as performance index, it is assumed that the decision-maker wants to manipulate fiscal and monetary policies (controls) in such a way as to achieve a performance as close as possible to full employment and price stability (minimum inflation) over the planning horizon. The performance index is therefore, designed to measure the social disutility associated with deviations from these goals. The performance measure is composed of two terms, namely $\mu[y_d(t) - y_a(t)]^2$ and $\eta\, \underline{w}'(t-1)R\, \underline{w}(t-1)$. The first term reflects the social disutility over the planning horizon, resulting from the deviations of actual output from potential output which implies periods of inflation or unemployment. That is, if $y_a(t)$ exceeds $y_d(t)$, inflation of the demand-pull type results, and if $y_a(t)$ stays below $y_d(t)$ it leads to unemployment. Thus the desired output, $y_d(t)$, may be specified as the value of output that the economy can achieve in any period of time at full employment without inflation. Full employment may be defined to include frictional unemployment in the neighborhood of 4% of the labor force, i.e., full employment labor input is equal to the labor force minus frictional unemployment. It is also assumed that there is no autonomous wage-push or profit-push inflation (Ref. B-14). On the other hand, in addition to the primary cost, a realistic performance index must also include a cost associated with applying controls to the system. If there were no penalty on the activity of the control vector, then the optimal problem would have the trivial solution of setting G, M, and L at each quarter such that the error is always zero. The disutility component which arises from the activity of the control function, i.e., $\eta\, \underline{w}'(t-1)R\, \underline{w}(t-1)$ = cost of control (t), has been included in the functional as well (Ref. K-1, K-9, M-5, and T-8). In the case of the U.S. national economy, control costs would be those associated with the synthesis and implementation of optimal economic control policies. Notice that the cost of control at quarter t is a function of the choice of the control policy at quarter $(t-1)$, $t \in \{1,2,\ldots,N\}$.

Klein and Goldberger's aggregate production function (Ref. K-4) is frequently used in dynamic macro models in order to determine potential output (Ref. B-14 and B-15) as:

$$y_d(t) = \alpha_0 + \alpha_1 E(t-1) + \alpha_2 K(t-1) \qquad (4.3.2)$$

$$\alpha_i \geq 0 \ , \ i = 0,1,2$$

where $E(t-1)$ and $K(t-1)$ are the employment and capital input at $(t-1)$ respectively.

However, two points have to be mentioned here. First, the Kmenta/Smith model does not contain employment and capital input variables in it. Secondly, the above formula expresses $y_d(t)$ as a function of the system state variables. Now, when applying feed back control, the system state is a function of the control vector, $\underline{w}(t)$, and therefore, both $y_a(t)$ and $y_d(t)$ are affected by the choice of the control. Thus, an optimal control based on the equations (4.3.1) when $\eta \equiv 0$ and (4.3.2) may be such that the path (trajectory) of $y_d(t)$ is altered along the path of $y_a(t)$ in minimizing that performance measure. Since the objective of this effort is to choose fiscal and monetary policies which when implemented will cause the economy to attain its potential output, control policies which result in $y_d(t)$ to deviate from some desired annual growth rate are unacceptable. That is, by using equation (4.3.2) for potential output, unacceptable or meaningless optimal control policies may result under certain circumstances.

An alternative expression for $y_d(t)$ is suggested by the works of Okun (Ref. O-2). It is a simple function of time (first order difference equation in $y_d(t)$) with a 3.5% annual growth trend. That is,

$$y_d(t) = g\, y_d(t-1) \tag{4.3.3}$$

with g a constant parameter representing the quarterly growth rate equivalent to a 3.5% annual growth rate. With such a definition for $y_d(t)$, only $y_a(t)$ would be a function of $\underline{w}(t)$ so that optimal control policies would have no influence upon the $y_d(t)$ trajectory.

In order to write the objective functional in terms of the state model, equation (4.3.3) is incorporated into the state model, Equation (4.2.5a), to yield

$$\underline{z}(t+1) = A\,\underline{z}(t) + B\,\underline{w}(t) + \underline{c} + \underline{D}\,\underline{u}(t) \tag{4.3.4}$$

$$\underline{z}(0) = \underline{z}_0 \quad \text{given} \tag{4.3.5}$$

and the output equation (4.2.6a) is

$$y(t) = \underline{\hat{e}}'\,\underline{z}(t) + \underline{b}_0'\underline{w}(t) + c_0 + d_0 t \tag{4.3.6}$$

where

$$\underline{z}'(t) = (y_1(t)\ y_2(t)\ y_3(t)\ y_4(t)\ y_5(t)\ y_d(t))$$

$$\underline{w}'(t) = (G(t)\quad M(t)\quad L(t))$$

and $\underline{u}'(t) = (u_1(t)\ u_2(t)\ u_3(t)\ u_4(t)\ u_5(t))$ represents the environmental disturbances.

The matrices are given by:

$$A = \begin{bmatrix} & & A_1 & & & \\ 0 & 0 & 0 & 0 & 0 & 1.00875 \end{bmatrix}, \quad B = \begin{bmatrix} A_2 \\ \underline{0} \end{bmatrix}$$

$$\underline{c} = \begin{bmatrix} \tilde{\underline{a}}_3 \\ 0 \end{bmatrix}, \quad D = \begin{bmatrix} A_5 \\ \underline{0} \end{bmatrix}, \quad \text{and} \quad \underline{\hat{e}}' = \begin{bmatrix} \underline{e}' \\ 0 \end{bmatrix}.$$

Substituting for $y_a(t)$ from Equation (4.2.6a) in the expression $(y_a(t) - y_d(t))^2$ from Equation (4.3.1) yields:

$$(y_d(t) - y_a(t))^2 = m_0 + \underline{m}_1'(t)\underline{z}(t) + \underline{m}_2'(t)\underline{w}(t) + \underline{z}'(t)R_1\underline{z}(t)$$
$$+ \underline{w}'(t)R_2\underline{z}(t) + \underline{w}'(t)R_3\underline{w}(t)$$

where

$$m_0 = c_0^2 + 2c_0d_0t + d_0^2t^2$$

$$\underline{m}_1'(t) = (2(c_0 + d_0t) \quad 0 \quad 0 \quad 0 \quad 0 \quad -2(c_0 + d_0t))$$

$$\underline{m}_2'(t) = 2(1.1427(c_0 + d_0t) \quad .3168(c_0 + d_0t) \quad .0481(c_0 + d_0t))$$

$$R_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 2.2854 & 0 & 0 & 0 & 0 & -2.2854 \\ .6336 & 0 & 0 & 0 & 0 & -.6336 \\ .0962 & 0 & 0 & 0 & 0 & -.0962 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1.305763 & .362007 & .054964 \\ .362007 & .100362 & .015238 \\ .054964 & .015238 & .002314 \end{bmatrix}$$ a symmetric matrix.

The objective functional takes the form:

$$I_N(\underline{z}(0),\underline{w}(t)) = \sum_{t=1}^{N} \{(m_0 + \underline{m}_1'(t)\underline{z}(t) + \underline{m}_2'(t)\underline{w}(t) + \underline{z}'(t)R_1\underline{z}(t)$$
$$+ \underline{w}'(t)R_2\underline{z}(t) + \underline{w}'(t)R_3\underline{w}(t) + \eta\ \underline{w}'(t-1)R\underline{w}(t-1)\} \quad (4.3.7)$$

The output $y_a(t)$ is defined in Equation (4.3.6). The problem now is in a suitable format for optimal control analysis. In short, the problem is to minimize the expected value of (4.3.7) subject to the state model constraint (4.3.4), given the initial condition $\underline{z}(0)$.

Figure (4.3.1) illustrates a computer simulation representation of the state space model, Equations (4.3.4) and (4.3.6), in vector difference equation.

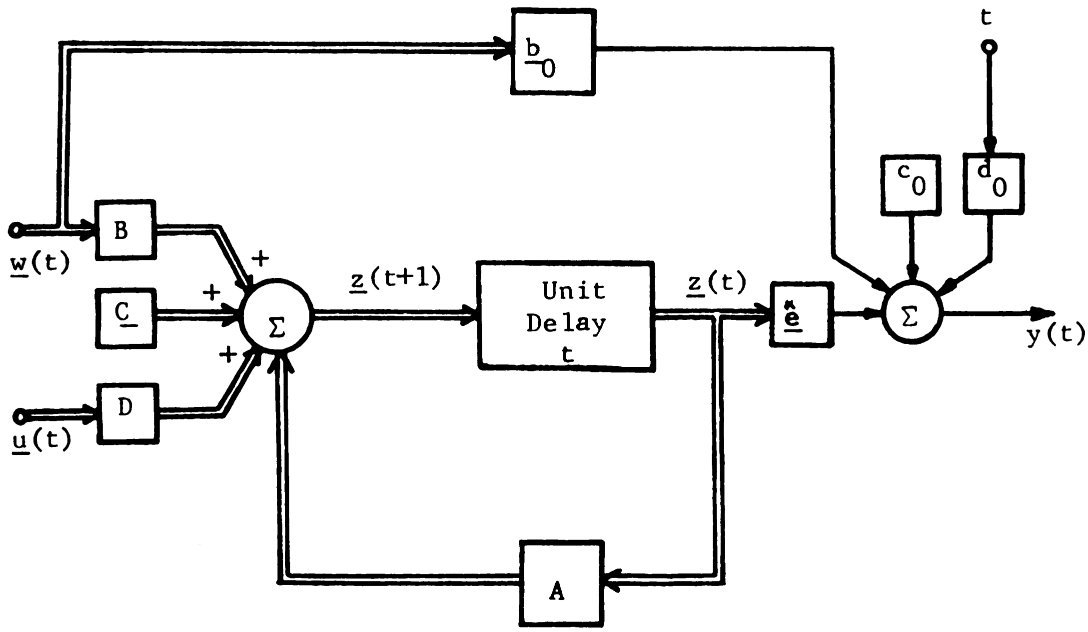

Fig. (4.3.1) Computer simulation (block diagram) representation of the K/S model in state space formulation, Eqs. (4.3.4) and (4.3.6).

CHAPTER FIVE

DERIVATION OF THE OPTIMIZING ALGORITHM

In Chapter Four, the K/S model was used in expressing an optimal control problem. The formulation was

$$\min_{\{\underline{w}(t-1)\}} E\{I_N\} = \min_{\{\underline{w}(t-1)\}} E\left\{ \sum_{i=1}^{N} [m_0 + \underline{m}_1'(t)\underline{z}(t) + \underline{m}_2'(t)\underline{w}(t) + \underline{z}'(t)R_1\underline{z}(t) \right.$$

$$+ \underline{w}'(t)R_2\underline{z}(t) + \underline{w}'(t)R_3\underline{w}(t)$$

$$\left. + \eta\, \underline{w}'(t-1)R\, \underline{w}(t-1)] \right\} \qquad (4.3.7)$$

such that

$$\underline{z}(t+1) = A\, \underline{z}(t) + B\, \underline{w}(t) + \underline{c} + D\, \underline{u}(t) \qquad (4.3.4)$$

$$\underline{z}(0) = \underline{z}_0 \quad \text{given} \qquad (4.3.5)$$

with the output equation

$$y(t) = \underline{e}'\underline{z}(t) + \underline{b}_0'\underline{w}(t) + c_0 + d_0 t\ . \qquad (4.3.6)$$

In Equation (4.3.7), the expected value is taken with respect to the environmental disturbances $\underline{u}(t)$, because I_N is a function of the state variables and control variables where the state vector is a function of the disturbance vector $\underline{u}(t)$ via Equation (4.3.4). An extensive literature exists for the linear constraint-quadratic performance index class of optimal control problems (the so-called linear deterministic and/or stochastic regulator problems). Solution techniques for this class of problems fall into one of the three basic approaches to optimal control problems discussed briefly in

Chapter One:

(1) Classical variational calculus, or calculus of variations (Refs. B-12, G-1, and H-2).

(2) Pontryagin's maximum (minimum) principle, continuous systems (Refs. A-6, G-2, L-3, O-1, and P-5) and discrete (sample-data) systems (Refs. C-6, C-7, F-1, G-2, K-2, K-9, M-5, R-4, S-1, and T-8).

(3) Dynamic programming and Bellman's principle of optimality (Refs. B-5, B-6, B-7, B-8, B-9, G-2, K-1, K-9, M-5, S-1, and T-8).

In this work, dynamic programming and Bellman's principle of optimality is used, for two reasons. One is that since the problem is formulated in terms of discrete time it is easily re-expressed as a multistage decision-process. The second reason is that the use of Pontryagin's maximum principle increases (mostly doubles) the dimensionality of the problem by the addition of the adjoint (costate) variables (see Chapter Two).

To solve the problem by dynamic programming, let the minimum of the expected value of the performance index be defined by:

$$\begin{aligned} f_N[\underline{z}(0)] &= \min_{\{\underline{w}(t-1)\}} E\{I_N\} \\ &= \min_{\{\underline{w}(0)\ \underline{w}(1)\ \dots\ \underline{w}(N-1)\}} E\left\{ \sum_{t=1}^{N} (m_0 + \underline{m}_1'(t)\underline{z}(t) + \underline{m}_2'(t)\underline{w}(t) \right. \\ &\quad \left. + \underline{z}'(t)R_1\underline{z}(t) + \underline{w}'(t)R_2\underline{z}(t) + \underline{w}'(t)R_3\underline{w}(t) + \eta\ \underline{w}'(t-1)R\underline{w}(t-1)) \right\} . \end{aligned} \tag{5.1.1}$$

Function $f_N\{\underline{z}(0)\}$ represents the cost of taking the system from initial state $\underline{z}(0)$ through N stages when an optimal policy is pursued at every stage. The operator $E\{\cdot\}$ represents the expected value of $\{\cdot\}$ with respect to the noise vector $\underline{u}(t)$. Notice that the minimum of the expected value of the performance index depends both on the initial state of the system, $\underline{z}(0)$, and on the length of time, N, over which the optimization process is defined.

To bring the problem into a suitable form for the application of the principle of optimality, the criterion functional may be subdivided as follows:

$$\begin{aligned} I_N = \sum_{t=1}^{N} [m_0 &+ \underline{m}_1'(t)\underline{z}(t) + \underline{m}_2'(t-1)\underline{w}(t-1) + \underline{z}'(t)R_1\underline{z}(t) \\ &+ \underline{w}'(t-1)R_2\underline{z}(t-1) + \underline{w}'(t-1)R_3\underline{w}(t-1) + \eta\underline{w}'(t-1)R\underline{w}(t-1)] \\ &- \underline{m}_2'(0)\underline{w}(0) + \underline{m}_2(N)\underline{w}(N) - \underline{w}'(0)R_2\underline{z}(0) + \underline{w}'(N)R_2\underline{z}(N) \\ &- \underline{w}'(0)R_3\underline{w}(0) + \underline{w}'(N)R_3\underline{w}(N) \ . \end{aligned} \tag{5.1.2}$$

Writing Equation (5.1.1) in a general form (i.e., considering N-k, k = 0,1,...,N-1) and substituting for I_{N-k} from Equation (5.1.2) yields:

$$\begin{aligned} f_{N-k}[\underline{z}(t)] &= \min_{\{\underline{w}(t-1)\}} E\{I_{N-k}\} \\ &= \min_{\underline{w}(k)} E\left\{ \sum_{\substack{t=k+1 \\ k=0,1,\ldots,N-1}}^{N} [m_0 + \underline{m}_1'(t)\underline{z}(t) + \underline{m}_2'(t-1)\underline{w}(t-1) \right. \\ &\left. \quad + \underline{z}'(t)R_1\underline{z}(t) + \underline{w}'(t-1)R_2\underline{z}(t-1) + \underline{w}'(t-1)R_5\underline{w}(t-1)] \right\} \\ &\quad - \min_{\underline{w}(0)} E\{\underline{m}_2'(0)\underline{w}(0) + \underline{w}'(0)R_2\underline{z}(0) + \underline{w}'(0)R_3\underline{w}(0)\} \\ &\quad + \min_{\underline{w}(N)} E\{\underline{m}_2'(N)\underline{w}(N) + \underline{w}'(N)R_2\underline{z}(N) + \underline{w}'(N)R_3\underline{w}(N)\} \end{aligned} \tag{5.1.3}$$

where

$$R_5 = R_3 + R_4$$

and $$R_4 = \eta R$$

with R_5 being a symmetric matrix because R_3 and R are symmetric matrices (see Chapter Four).

The terms minimized at $\underline{w}(0)$ may be included within the summation term by redefining the summation so that at $t = 0$ the terms in $\min\limits_{\underline{w}(0)} E\{\cdot\}$ are subtracted from the other terms within the summation. The terms minimized at $t = N$ give the boundary condition on the process (i.e., on I) at $t = N$. That is, define

$$\min_{\underline{w}(N)} \{\underline{m}_2'(N)\underline{w}(N) + \underline{w}'(N)R_2\underline{z}(N) + \underline{w}'(N)R_3\underline{w}(N)\} = I_1(N) \qquad (5.1.4)$$

Then the first order necessary condition (Ref. A-6) for a minimum applies

$$\frac{\partial I_1(N)}{\partial\, \underline{w}(N)} = 0 = \underline{m}_2(N) + R_2\underline{z}(N) + 2R_3\underline{w}(N)$$

which gives an optimal control at $t = N$ of

$$\hat{\underline{w}}(t) = -\frac{1}{2} R_3^{-1}(\underline{m}_2(N) + R_2\underline{z}(N)) \qquad (5.1.5)$$

Substituting (5.1.5) into (5.1.4) gives the optimum value for the performance index at time $t = N$:

$$\hat{I}_1(N) = -\frac{1}{4} \underline{m}_2'(N)R_3^{-1}\underline{m}_2(N) - \frac{1}{2} \underline{m}_2'(N)R_3^{-1}R_2\underline{z}(N)$$

$$- \frac{1}{4} \underline{z}'(N)R_2'R_3^{-1}R_2\underline{z}(N), \qquad (5.1.6)$$

which is a general quadratic function of the system's state vector. Equation (5.1.6) represents the boundary condition for I_N at $t = N$. Rewriting the summation in Equation (5.1.3) as suggested above and using Equation (5.1.6) as a boundary condition, Equation (5.1.3) then reduces to the terms within the summation. In

Equation (5.1.3) $\underline{z}(k)$ is the system's state vector at any arbitrary k, $k = 0,1,\ldots,N-1$, and when $k = N$ it is apparent that $f_0[\underline{z}(N)] = \hat{I}_1(N)$ of Equation (5.1.6). The function $f_{N-k}[\underline{z}(k)]$ which is quadratic in $\underline{z}(k)$ represents the cost of taking state $\underline{z}$ from $\underline{z}(k)$ through $N-k$ $(k = 0,1,\ldots,N)$ stages using an optimal control (policy) at each stage.

Now, assuming that the return from the first $(k-1)$ stages is optimum, the cost of the remaining $N-k$ stages is equal to the output from the k-th stage plus the optimum cost from the remaining $(N - (k+1))$ stages. That is, the process is now broken up into N-single-stage processes by using the principle of optimality (Ref. T-8):

$$f_{N-k}[\underline{z}(k)] = \min_{\underline{w}(k)} E\{m_0 + \underline{m}_1'(k+1)\underline{z}(k+1) + \underline{m}_2'(k)\underline{w}(k) + \underline{z}'(k+1)R_1\underline{z}(k+1) + \underline{w}'(k)R_2\underline{z}(k) + \underline{w}'(k)R_5\underline{w}(k) + f_{N-(k+1)}[\underline{z}(k+1)]\} \tag{5.1.7}$$

where it has been assumed that an optimal policy has been pursued in bringing the system to state $\underline{z}(k)$ and that an optimal policy will be pursued from $\underline{z}(k)$ through the last $N-(k+1)$ stages.

From Equation (5.1.6), the boundary condition on I_N at $t = N$, it follows that the minimal cost at $t = N$ has the general quadratic form

$$f_0[\underline{z}(N)] = \sigma(0) + \underline{\lambda}'(0)\underline{z}(N) + \underline{z}'(N)\Omega(0)\underline{z}(N) \ . \tag{5.1.6a}$$

By induction from the last state, that is, by working backwards, it may be shown that the cost of any arbitrary state, $\underline{z}(k)$, must have a similar form (Refs. K-9 and M-5); namely:

$$f_{N-k}[\underline{z}(k)] = \sigma(N-k) + \underline{\lambda}'(N-k)\underline{z}(k) + \underline{z}(k)\Omega(N-k)\underline{z}(k) \tag{5.1.8}$$

where $\sigma(N-k)$ a scalar, $\underline{\lambda}(N-k)$ a column vector, and $\Omega(N-k)$ is a square matrix to be determined later in the chapter, $k = 0,1,\ldots,N-1$.

Substituting (5.1.8) into (5.1.7) yields:

$$\sigma(N-k) + \underline{\lambda}'(N-k)\underline{z}(k) + \underline{z}'(k)\Omega(N-k)\underline{z}(k)$$

$$= \min_{\underline{w}(k)} E\{I_{N-k}\}$$

$$= \min_{\underline{w}(k)} E\{m_0 + \sigma(N-(k+1)) + [\underline{m}_1'(k+1) + \underline{\lambda}'(N-(k+1))]\underline{z}(k+1) + \underline{m}_2'(k)\underline{w}(k)$$

$$+ \underline{z}'(k+1)[R_1 + \Omega(N-(k+1))]\underline{z}(k+1) + \underline{w}'(k)R_2\underline{z}(k) + \underline{w}'(k)R_5\underline{w}(k)\}. \tag{5.1.9}$$

Substituting the system constraint equation (4.3.4) into the above yields:

$$\min_{\underline{w}(k)} I_{N-k} = \min_{\underline{w}(k)} E\{m_0 + \sigma(N-(k+1)) + \underline{h}(N-(k+1))[A\ \underline{z}(t) + B\ \underline{w}(k)$$

$$+ \underline{c} + D\ \underline{u}(k)] + \underline{m}_2'(k)\underline{w}(k) + \underline{w}'(k)R_2\underline{z}(k) +$$

$$\underline{w}'(k)R_5\underline{w}(k) + [A\ \underline{z}(k) + B\ \underline{w}(k) + \underline{c} + D\ \underline{u}(k)]'$$

$$H(N-(k+1))[A\ \underline{z}(k) + B\ \underline{w}(k) + \underline{c} + D\ \underline{u}(k)]\} \tag{5.1.10}$$

where

$$\underline{h}(N-(k+1)) = \underline{m}_1'(k+1) + \underline{\lambda}'(N-(k+1)) \quad \text{a row vector} \tag{5.1.11}$$

and

$$H(N-(k+1)) = R_1 + \Omega(N-(k+1)) \qquad \text{a square matrix .} \tag{5.1.12}$$

For the purpose of the optimal control policy the environmental noise, $\underline{u}(k)$, is assumed to be white, as was mentioned earlier.

Carrying out the indicated multiplications, regrouping terms, and performing the expected value operation on Equation (5.1.10) eliminates all terms involving $\underline{u}(k)$ except the last term, since $\underline{u}(k)$ is a zero mean process. Equation (5.1.10) then is reduced to:

$$\begin{aligned}\min_{\underline{w}(k)} I_{N-k} = {} & \varphi_6(N-(k+1)) + \underline{\varphi}_4(N-(k+1))\underline{z}(k) + \underline{z}'(k)\underline{\varphi}_{AC}(N-(k+1)) \\ & + \underline{\varphi}_5(N-(k+1))\underline{w}(k) + \underline{w}'(k)\underline{\varphi}_3(N-(k+1)) \\ & + \underline{z}'(k)\Phi_{AA}(N-(k+1))\underline{z}(k) + \underline{z}'(k)\Phi_{AB}(N-(k+1))\underline{w}(k) \\ & + \underline{w}'(k)\Phi_1(N-(k+1))\underline{z}(k) + \underline{w}'(k)\Phi_2(N-(k+1))\underline{w}(k) \\ & + E\{\underline{u}'(k)\Phi_{DD}(N-(k+1))\underline{u}(k)\} \end{aligned} \qquad (5.1.13)$$

where

$\Phi_1(N-(k+1)) = B'H(N-(k+1))B + R_2$, a rectangular matrix (5.1.14)

$\Phi_2(N-(k+1)) = B'H(N-(k+1))B + R_5$, a square matrix (5.1.15)

$\underline{\varphi}_3(N-(k+1)) = B'H(N-(k+1))\underline{c}$, a column vector (5.1.16)

$\underline{\varphi}_4(N-(k+1)) = \underline{h}(N-(k+1))A + \underline{c}'H(N-(k+1))A$, a row vector (5.1.17)

$\underline{\varphi}_5(N-(k+1)) = \underline{h}(N-(k+1))B + \underline{m}_2'(k) + \underline{c}'H(N-(k+1))B$, a row vector (5.1.18)

$\varphi_6(N-(k+1)) = m_0 + \sigma(N-(k+1)) + \underline{c}'H(N-(k\_1))\underline{c} + \underline{h}'(N-(k+1))\underline{c}$, a scalar (5.1.19)

$\Phi_{AA}(N-(k+1)) = A'H(N-(k+1))A$, a square matrix (5.1.20)

$\Phi_{AB}(N-(k+1)) = A'H(N-(k+1))B$, a rectangular matrix (5.1.21)

$\underline{\varphi}_{AC}(N-(k+1)) = A'H(N-(k+1))\underline{c}$, a column vector (5.1.22)

and

$\Phi_{DD}(N-(k+1)) = D'H(N-(k+1))D$, a square matrix . (5.1.23)

Equation (5.1.13) gives the total cost of the performance index at any arbitrary state $\underline{z}(t)$, in terms of the state at $t = k$,

$\underline{z}(k)$ and the control policy at $t = k$, $\underline{w}(k)$. The minimization of I_{N-k}, Equation (5.1.9), therefore, has been reduced to selecting a control policy, $\underline{w}(k)$, for $N-k$ single-stage processes which will minimize I_{N-k}. The first order necessary condition (Ref. A-6) for $\min_{\underline{w}(k)} I_{N-k}$ is:

$$\frac{\partial I_{N-k}}{\partial \underline{w}(k)} = 0 . \tag{5.1.24}$$

That is:

$$\underline{\varphi}_5'(N-(k+1)) + \underline{\varphi}_3(N-(k+1)) + \Phi_{AB}'(N-(k+1))\underline{z}(k) +$$
$$\Phi_1(N-(k+1))\underline{z}(k) + 2\,\Phi_2(N-(k+1))\underline{w}(k) = 0 .$$

This implies that the optimal control policy is:

$$\hat{\underline{w}}(k) = \underline{q}(N-(k+1)) + Q(N-(k+1))\underline{z}(k) \tag{5.1.25}$$

where

$$\underline{q}(N-(k+1)) = -(\Phi_2(N-(k+1)) + \Phi_2'(N-(k+1)))^{-1}\underline{\varphi}_7(N-(k+1)),$$
a column vector (5.1.26)

$$Q(N-(k+1)) = -(\Phi_2(N-(k+1)) + \Phi_2'(N-(k+1)))^{-1}\Phi_8(N-(k+1)),$$
a rectangular matrix (5.1.27)

and

$$2\Phi_2(N-(k+1)) = \Phi_2(N-(k+1)) + \Phi_2'(N-(k+1)) \quad \text{by symmetry}$$

$$\underline{\varphi}_7(N-(k+1)) = \underline{\varphi}_3(N-(k+1)) + \underline{\varphi}_5'(N-(k+1)), \text{ a column vector} \tag{5.1.28}$$

$$\Phi_8(N-(k+1)) = \Phi_{AB}'(N-(k+1)) + \Phi_1(N-(k+1)), \text{ a rectangular matrix} . \tag{5.1.29}$$

It will be noticed from the optimum control (policy), Equation (5.1.25), that the optimal policy is a function of the state variables of the system. Since the feedback vector $\underline{q}$ and feedback gain matrix Q in Equation (5.1.25) involve the yet unknown variables σ, λ, and Ω, the optimum control law is still undefined until $\underline{q}$ and Q are determined. In order to find known expressions for the variables σ, λ, and Ω and, therefore, to find $\underline{q}$ and Q, Equation (5.1.15) is substituted into Equation (5.1.13) to give the minimal cost of the performance index for an arbitrary state $\underline{z}(k)$:

$$
\begin{aligned}
\hat{I}_{N-k} &\triangleq \sigma(N-k) + \lambda'(N-k)\underline{z}(k) + \underline{z}'(k)\Omega(n-k)\underline{z}(k) \\
&= \varphi_6(N-(k+1)) + \underline{\varphi}_4(N-(k+1))\underline{z}(k) + \underline{z}'(k)\underline{\varphi}_{AC}(N-(k+1)) \\
&\quad + \underline{\varphi}_5(N-(k+1))[\underline{q}(N-(k+1)) + Q(N-(k+1))\underline{z}(k)] + [\underline{q}(N-(k+1)) \\
&\quad + Q(N-(k+1))\underline{z}(k)]'\underline{\varphi}_3(N-(k+1)) + \underline{z}'(k)\Phi_{AA}(N-(k+1))\underline{z}(k) \\
&\quad + \underline{z}'(k)\Phi_{AB}(N-(k+1))[\underline{q}(N-(k+1)) + Q(N-(k+1))\underline{z}(k)] + [\underline{q}(N-(k+1)) \\
&\quad + Q(N-(k+1))\underline{z}(k)]'\Phi_1(N-(k+1))\underline{z}(k) + [\underline{q}(N-(k+1)) + Q(N-(k+1))\underline{z}(k)]' \\
&\qquad \Phi_2(N-(k+1))[\underline{q}(N-(k+1)) + Q(N-(k+1))\underline{z}(k)] \\
&\quad + E\{\underline{u}'(k)\Phi_{DD}(N-(k+1))\underline{u}(k)\} \,.
\end{aligned}
\tag{5.1.30}
$$

Carrying out the indicated transposes and multiplications and re-grouping terms gives:

$$
\begin{aligned}
\hat{I}_{N-k} &\triangleq \sigma(N-(k+1)) + \lambda'(N-(k+1))\underline{z}(k) + \underline{z}(k)\Omega(N-(k+1))\underline{z}(k) \\
&= [\varphi_6(N-(k+1)) + \underline{\varphi}_5(N-(k+1))\underline{q}(N-(k+1)) + \underline{q}'(N-(k+1))\underline{\varphi}_3(N-(k+1)) \\
&\qquad + \underline{q}'(N-(k+1))\Phi_2(N-(k+1))\underline{q}(N-(k+1))] \\
&\quad + [\underline{\varphi}_4(N-(k+1)) + \underline{\varphi}_{AC}'(N-(k+1)) + \underline{\varphi}_5(N-(k+1))Q(N-(k+1)) + \\
&\qquad \underline{\varphi}_3'(N-(k+1))Q(N-(k+1)) + \underline{q}'(N-(k+1))\Phi_{AB}'(N-(k+1)) + \\
&\qquad \underline{q}'(N-(k+1))\Phi_1(N-(k+1)) + 2\underline{q}(N-(k+1))\Phi_2(N-(k+1))Q(N-(k+1))]\underline{z}(k)
\end{aligned}
$$

$$+ \underline{z}'(k)[\Phi_{AA}(N-(k+1)) + \Phi_{AB}(N-(k+1))Q(N-(k+1)) +$$

$$Q(N-(k+1))\Phi_1(N-(k+1)) + Q'(N-(k+1))\Phi_2(N-(k+1))Q(N-(k+1))]\underline{z}(k)$$

$$+ E\{\underline{u}'(k)\Phi_{DD}(N-(k+1))\underline{u}(k)\} \quad . \tag{5.1.31}$$

According to the assumption that $\underline{u}(k)$ (disturbance vector) is white, the separation principle (Refs. M-5, T-8) can be applied to Equation (5.1.31) to eliminate the last term from the optimiza- process. The separation principle allows a problem such as the above to be separated into a deterministic and a stochastic protion with the optimal control policy being based upon the deterministic portion only. Since the problem is restricted to the optimization of a quadratic cost functional subject to the known linear system dynamics with environmental disturbances, it is possible to prove and utilize the separation principle (theorem). This theorem specifically states that, for linear systems with quadratic cost functionals subject to additive white Gaussian noise inputs, the optimum stochastic controller is realized by cascading an optimal estimator with a deterministic optimum controller. This decoupling of the problem into two parts is due to the fact that random noises are white with zero mean and, since they are completely unpredictable, need not be used in the design of the optimal controller. The decoupling is also due to the linearity of the problem. Thus, recursive relationships for σ, λ, and Ω in the quadratic cost functional, Equation (5.1.31) may now be determined from Equation (5.1.31) by simply equating like terms across the equal sign, obtaining:

$$\sigma(N-k) = \varphi_6(N-(k+1)) + \underline{\varphi}_5(N-(k+1))\underline{q}(N-(k+1)) + \underline{q}'(N-(k+1))\underline{\varphi}_3(N-(k+1))$$

$$+ \underline{q}'(N-(k+1))\Phi_2(N-(k+1))\underline{q}(N-(k+1)) \qquad (5.1.32)$$

$$\underline{\lambda}'(N-k) = \underline{\varphi}_4(N-(k+1)) + \underline{\varphi}'_{AC}(N-(k+1)) + \underline{\varphi}_5(N-(k+1))Q(N-(k+1)) +$$

$$\underline{\varphi}'_3(N-(k+1))Q(N-(k+1)) + \underline{q}'(N-(k+1))\Phi'_{AB}(N-(k+1)) +$$

$$\underline{q}'(N-(k+1))\Phi_1(N-(k+1)) + 2\underline{q}'(N-(k+1))\Phi_2(N-(k+1))Q(N-(k+1))$$

$$(5.1.33)$$

$$\Omega(N-k) = \Phi_{AA}(N-(k+1)) + \Phi_{AB}(N-(k+1))Q(N-(k+1)) + Q'(N-(k+1))\Phi_1(N-(k+1))$$

$$+ Q'(N-(k+1))\Phi_2(N-(k+1))Q(N-(k+1)) \quad . \qquad (5.1.34)$$

The recursive relationships, Equations (5.1.32), (5.1.33), and (5.1.34) are used to generate the components of $\sigma(N-k)$, $\underline{\lambda}(N-k)$, and $\Omega(N-k)$, $k = N-1, N-2, \ldots, 1, 0$, i.e., over the time horizon. These in turn are used to generate the optimal feedback gain vector $\underline{q}(N-k)$ and matrix $Q(N-k)$. Hence, the optimal control policy $\underline{\hat{w}}(k)$ in Equation (5.1.25) is obtained. The initial conditions for σ, $\underline{\lambda}$, and Ω, i.e., when $k = N$ are given by Equation (5.1.6) as follows:

$$\sigma(0) = -\frac{1}{4}\,\underline{m}'_2(N)R_3^{-1}\,\underline{m}_2(N) \qquad (5.1.35)$$

$$\underline{\lambda}'(0) = -\frac{1}{2}\,\underline{m}'_2(N)R_3^{-1}\,R_2 \qquad (5.1.36)$$

$$\Omega(0) = -\frac{1}{4}\,R'_2\,R_3^{-1}\,R_2 \quad . \qquad (5.1.37)$$

To summarize, this chapter derives the optimal control policy, Equation (5.1.25), for the control problem stated at the beginning of the chapter, i.e., minimizing (4.3.7) subject to the Equations (4.3.4) and (4.3.5). Dynamic programming and Bellman's principle of optimality were utilized for the derivation of the optimal control algorithm, which is shown in the following flow charts. In the computer flow diagram Fig. (5.1.1) going backwards in time the feedback gain vector $\underline{q}$ and matrix Q are derived and stored in the computer. In Fig. (5.1.2) going forward in time the performance index is computed using the information stored in the computer from the first flow chart in order to compute the control and state vectors given the initial condition $\underline{z}(0)$.

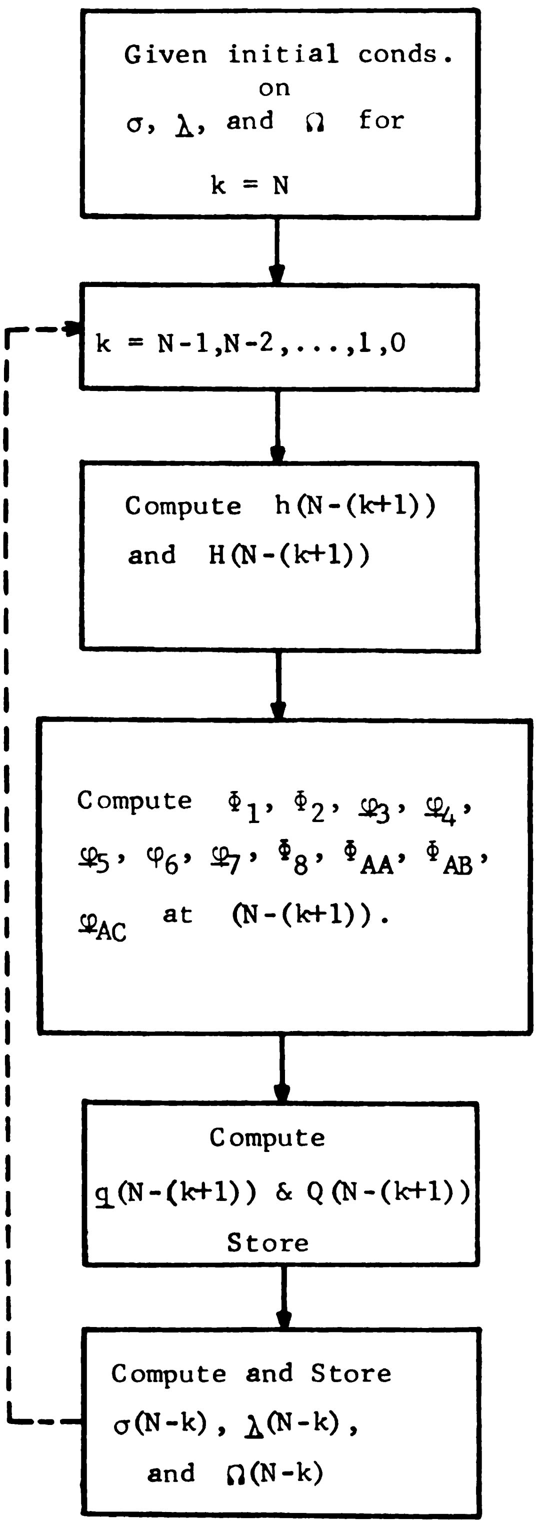

Fig. (5.1.1). Computer flow diagram for computing and storing $\underline{q}$, Q, σ, λ, and Ω.

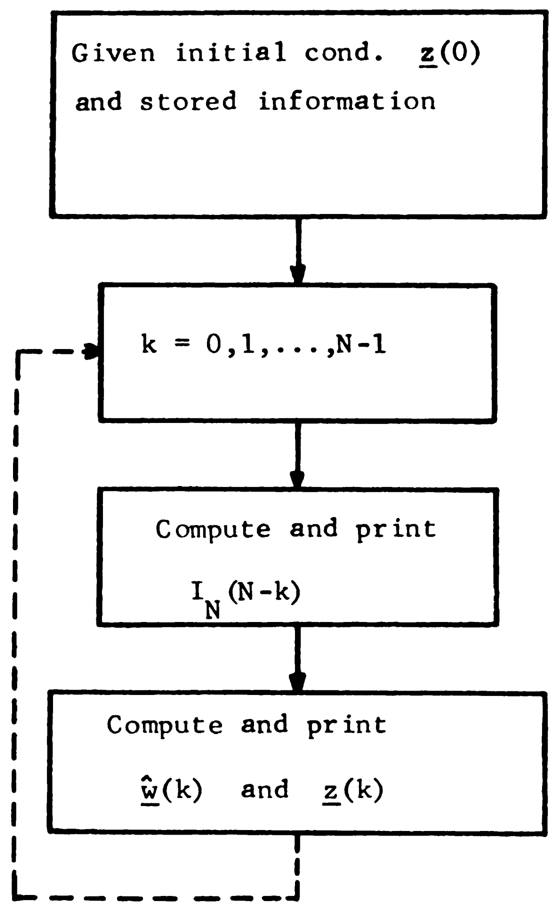

Fig. (5.1.2). Computer flow diagram for computing and printing I_N, $\hat{\underline{w}}(k)$, and $\underline{z}(k)$.

CHAPTER SIX

NUMERICAL SOLUTION OF THE OPTIMAL CONTROL POLICY

6.1 General Discussion

In Chapter Four the optimal control problem for the minimization of the expected value of Equation (4.3.5) subject to the constraints of the system dynamics as given in Equation (4.3.4) was defined. In Chapter Five the optimal control policy for this problem, derived using dynamic programming, is of the form

$$\underline{\hat{w}}(t) = \underline{q}(N-t) + Q(N-t)\underline{z}(t), \quad t = 1,2,\ldots,N-1 \qquad (5.1.25)$$

where the components of the matrix $Q(N-t)$ are the optimal feedback gains from the system state $\underline{z}$ to optimal control $\underline{\hat{w}}$. Since the control vector has three components, G, M, and L, and since the state vector has six components, the optimal gain matrix, $Q(N-t)$, has three rows and six columns. Also note that the optimal gain matrix is a function of the number of quarters in the future over which the problem is to be optimized, N-t, which is called time-to-go in many studies (Ref. B-15). Thus a solution to the optimal control problem is obtained when the vector $\underline{q}(N-t)$ and the optimal gain matrix, $Q(N-t)$, are determined for time-to-go from $N-k = 1$ to $N-k = 40$. This chapter presents the results of the numerical solution of the Matrix-Ricatti equations, Equations (5.1.32), (5.1.33), and (5.1.34), with their initial conditions given by Equations (5.1.35), (5.1.36), and (5.1.37), for σ, λ, and Ω. The results are used to obtain $\underline{q}(N-t)$ and $Q(N-t)$ by Equations (5.1.11), (5.1.12), (5.1.14) through (5.1.22) and (5.1.25). Equation (5.1.25) suggests that the contributions to the components of the control vector from the optimal feedback gain matrix multiplied by the components of the state vector are

$$\partial\hat{G}/\partial y_j = q_{1,j}(N-k) \quad , \quad j = 1,\ldots,6 \tag{6.1.1}$$

$$\partial\hat{M}/\partial y_j = q_{2,j}(N-k) \quad , \quad j = 1,\ldots,6 \tag{6.1.2}$$

$$\partial\hat{L}/\partial y_j = q_{3,j}(N-k) \quad , \quad j = 1,\ldots,6 \tag{6.1.3}$$

where $\underline{z}'(t) = (y_1(t)\ y_2(t)\ \ldots\ y_5(t)\ y_6(t))$ with $y_6(t) = y_d(t)$ and $q_{i,j}(N-k)$, $(i = 1,2,3,\ j = 1,\ldots,6)$, are the elements of the feedback gain matrix, $Q(N-t)$, in the i-th row and j-th column position. Specifically, the optimal control policy vector is

$$\hat{\underline{w}}(t) = \begin{bmatrix} \hat{G}(t) \\ \hat{M}(t) \\ \hat{L}(t) \end{bmatrix} = \begin{bmatrix} q_1(N-t) \\ q_2(N-t) \\ q_3(N-t) \end{bmatrix} + \begin{bmatrix} \frac{\partial\hat{G}}{\partial y_1}(N-k) & \ldots & \frac{\partial\hat{G}}{\partial y_6}(N-t) \\ \frac{\partial\hat{M}}{\partial y_1}(N-t) & \ldots & \frac{\partial\hat{M}}{\partial y_6}(N-t) \\ \frac{\partial\hat{L}}{\partial y_1}(N-t) & \ldots & \frac{\partial\hat{L}}{\partial y_6}(N-t) \end{bmatrix} \underline{z}(t). \tag{6.1.4}$$

Equation (6.1.4) states that the optimum economic policies are functions of the state variables of the entire system including those entering the performance index. The state variables are $y_i(t)$, $i = 1,\ldots,6$, with $y_d(t) = y_6(t)$, as specified in Chapter Four. The matrix equation (5.1.25) yields the appropriate decision policies with respect to the control variables $G(t)$, $M(t)$, and $L(t)$ assuming the objective functional of the system is to be minimized over a time horizon of N-t $(t = 0,1,\ldots,N-1)$ quarters. This information takes into account the dynamics of the system model as well as the stochastic environmental disturbances.

In Chapter Four the objective functional was derived as Equation (4.3.5) with a penalty function for control activity. Since no data could be found for estimating the penalty function for control activity, some sensitivity analysis is performed using the assumed form of the penalty function as

$$P = \eta R = \eta \begin{bmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{bmatrix} \tag{6.1.5}$$

where η is a scalar and R is a symmetric positive definite matrix. The effect of the level of the penalty function is studied numerically in this chapter by considering values of the scalar η (penalty factor) and by altering the weighting ratios r_{11}/r_{22}, r_{11}/r_{33}, and r_{22}/r_{33}.

6.2 Open Loop Results

An open-loop system is one which is unforced. In the present case the deterministic portion of equation (4.3.4), i.e., the state model of the system, without feedback control comprises the open-loop system. A digital simulation of the open-loop system was constructed utilizing the given input data to generate $y_s(t)$, simulated output, shown in Fig. (6.2.1). In this figure the actual path of gross national product (GNP), $y_a(t)$, and the potential (desired) output, $y_d(t)$, during 1954-1963 are also shown. The

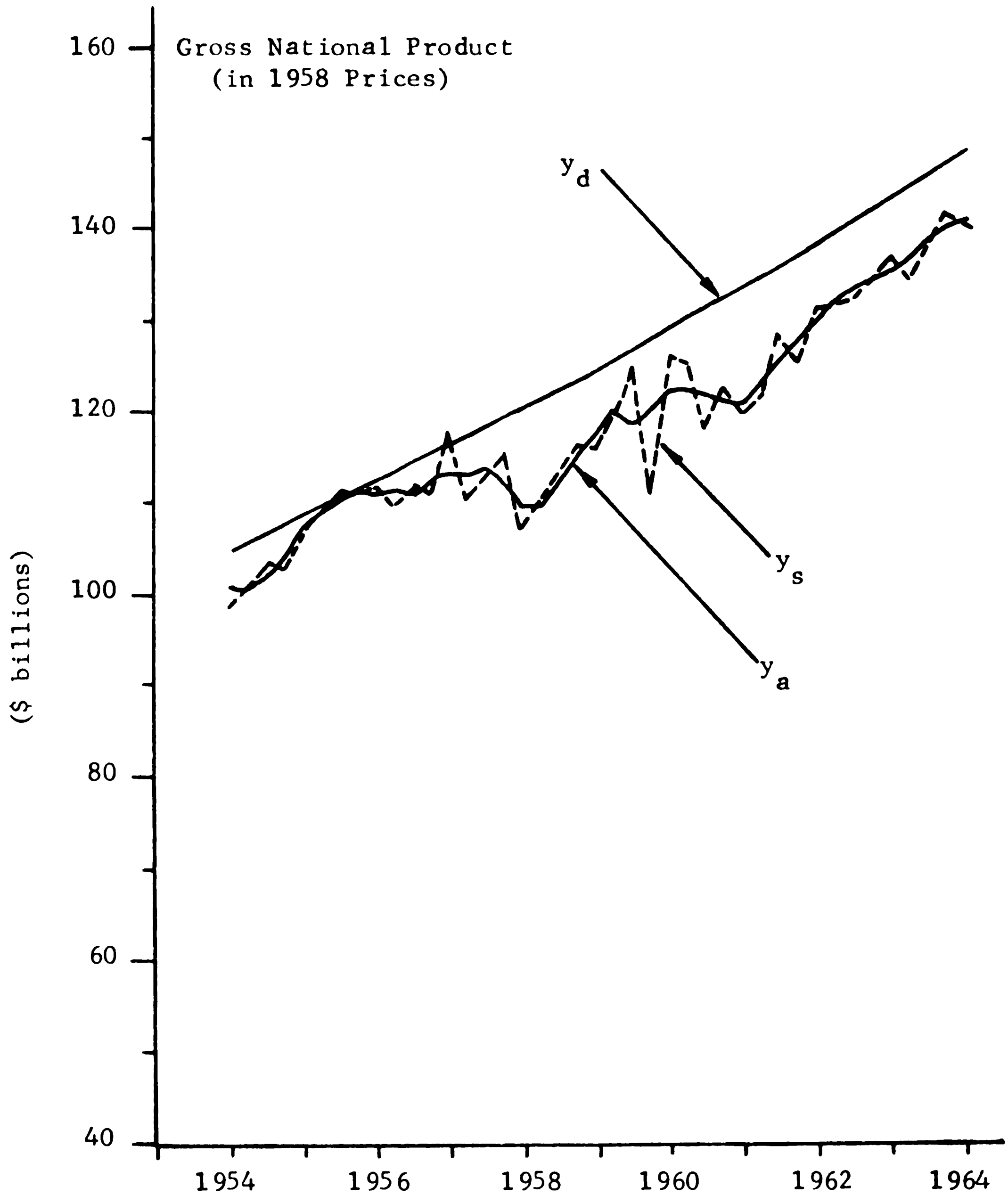

Fig. (6.2.1). Actual Output (GNP), y_a, Compared with Potential Output, y_d, and Simulated Output of the Open-Loop System, y_s.

result of this simulated path of the output $(y_s(t))$ is a curve that varies from time to time about the actual path for the output, $y_a(t)$. This oscillatory behavior may be caused by an imperfect correlation of the unemployment or inflation rate with under or over used potential output. As it will be seen in the coming section (Sec. 6.4) using optimal control policies yields a smooth optimal path for the output.

Potential gross national product, $y_d(t)$, may be defined as that level of output in any quarter (of a year) under conditions of full employment and minimum inflation (price stability), see Sec. 4.3. The work of Phillips (Ref. P-5), among others, suggests that full employment and price stability, as defined in Sec. 4.3, can be simultaneously obtained. Thus, $y_d(t)$ may be considered to lie within a feasible set of $y_a(t)$ values. If the actual national output, $y_a(t)$, exceeds potential output, $y_d(t)$, inflation of the demand-pull type results. When $y_a(t)$ falls below $y_d(t)$, unemployment in excess of four per cent of the total labor force results. Performance index (4.3.1), being quadratic in the variation between $y_d(t)$ and $y_a(t)$, penalizes both of these disutilities.

Okun (Ref. O -2) has considered the problem of estimating potential gross national product in some detail. Based on his study, he suggests the following simple relationship for estimating potential output,

$$y_d(t) = y_a(t)[1 + .032(u(t) - 4)] \tag{6.2.1}$$

where $u(t)$ is the unemployment rate at quarter t measured in percent of total labor force. When the unemployment rate is four percent, potential GNP is estimated as equal to actual; at a five percent rate of unemployment, the estimated "gap" is 3.2 percent of GNP. In the periods from which this relationship has been obtained the unemployment rate varied from about 3 to 7½ percent; the relation is not meant to be extrapolated outside this range. This model and the actual output are both plotted as discrete-time functions in Fig. (6.2.2), using the given data for the system under this study. The dashed line in the figure shows the implied time-series of potential GNP derived by applying the 3.2 coefficient to excess unemployment for the period 1954-1963. The result is a curve that varies from quarter to quarter, even dipping at times. Okun raises the question whether these variations should be taken seriously as indications of irregular or cyclical patterns in the growth of productive capacity or whether they should be attributed to an imperfect correlation of the unemployment rate with unused potential output. In the former case, the irregular path upward shown by the dashed line would be the estimated series of potential GNP. In the latter case, some smoothing of that irregular path would be in order. One way of smoothing which eliminates all the ripples is to substitute a simple exponential curve that corresponds with the trend and level of the varying series. This leads to the second model for $y_d(t)$ suggested by the work of Okun (Ref. O-2), which is a simple function of time with a 3.5% annual growth trend (see Sec. 4.3) that is,

$$y_d(t) = g\ y_d(t-1) \quad . \tag{4.3.3}$$

Such a benchmark trend is obtained by a line that goes through the

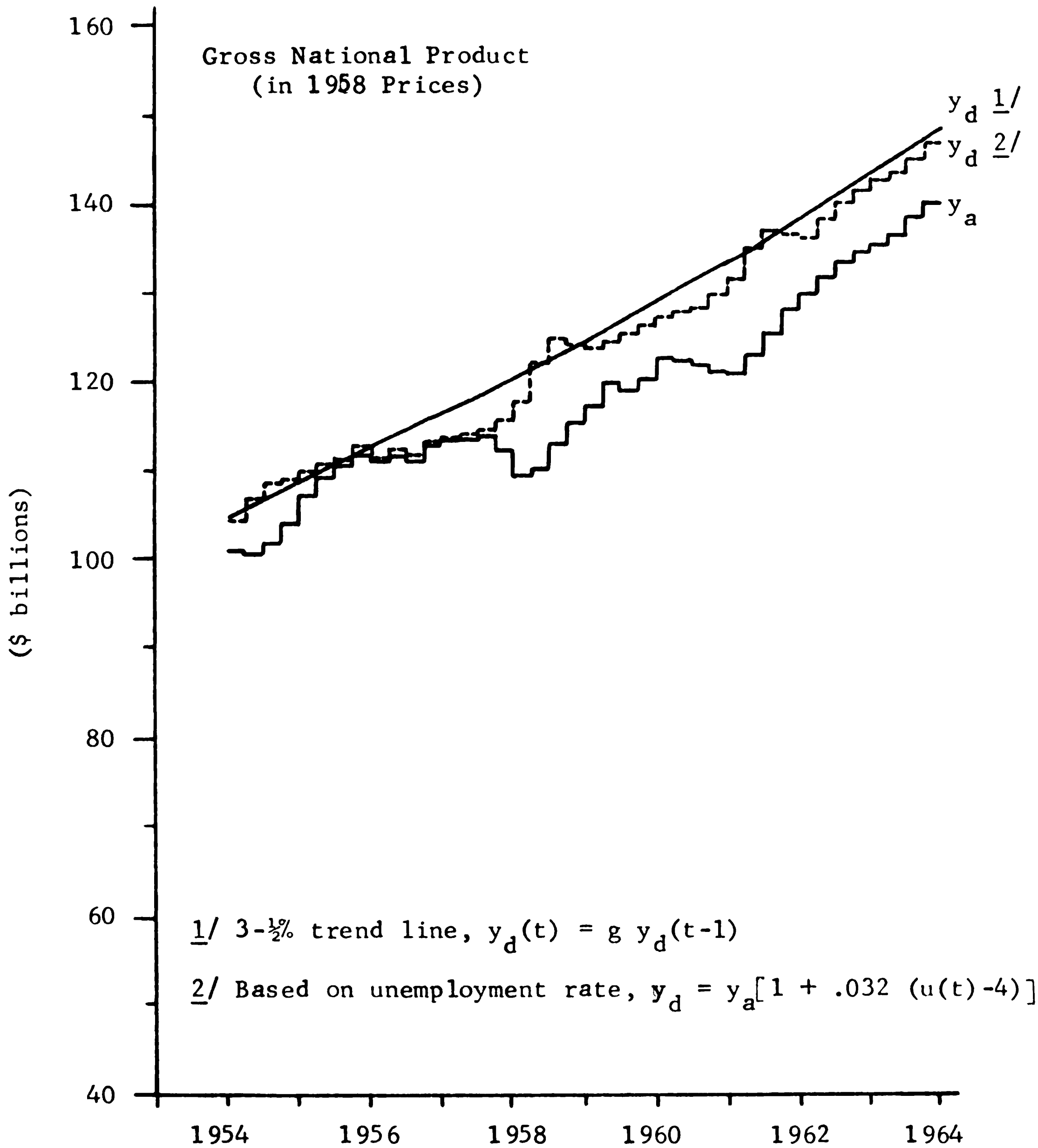

Fig. (6.2.2). Actual Output (GNP), y_a, Compared with Potential Output, y_d.

actual output, $y_a(t)$, in mid-1955 (where $u(t) = 4\%$) and moves upward at a $3\frac{1}{2}$ percent annual growth rate. The trend (time) measure of potential is shown as the solid line in Figs. (6.2.1)-(6.2.3).

A comparison between $y_a(t)$ and both of these $y_d(t)$ [i.e., based on Equation (4.3.3) and Equation (6.2.1)] show that unemployment in excess of 4% exists for almost all optimization periods except for mid 1955. This is consistent with the unemployment figures available for the period 1954-1963, which are shown in Fig. (6.2.3). This figure shows that unemployment rate is between 4 to 7.2 percent; therefore, the estimated potential GNP model, Equation (6.2.1), can be applied for this study. As it was said earlier (Sec. 4.3) and because of the difficulties involved in using $y_d(t)$ based on Equation (6.2.1) which were discussed above, the trend measure of the potential, i.e., Equation (4.3.3) has been used for generating $y_d(t)$.

6.3 Optimal Control Policy Mix

Optimal control policies for various combinations of the components of the control vector, i.e., $G(t)$, $M(t)$, and $L(t)$ may be generated by changing the relative weightings between them in the cost of control term, Equation (4.3.1). Let the cost of control matrix R be given by

$$R = \begin{bmatrix} r_{11} & 0 & 0 \\ 0 & r_{22} & 0 \\ 0 & 0 & r_{33} \end{bmatrix} . \tag{6.3.1}$$

Then by altering the ratios between the r's, the relative magnitudes

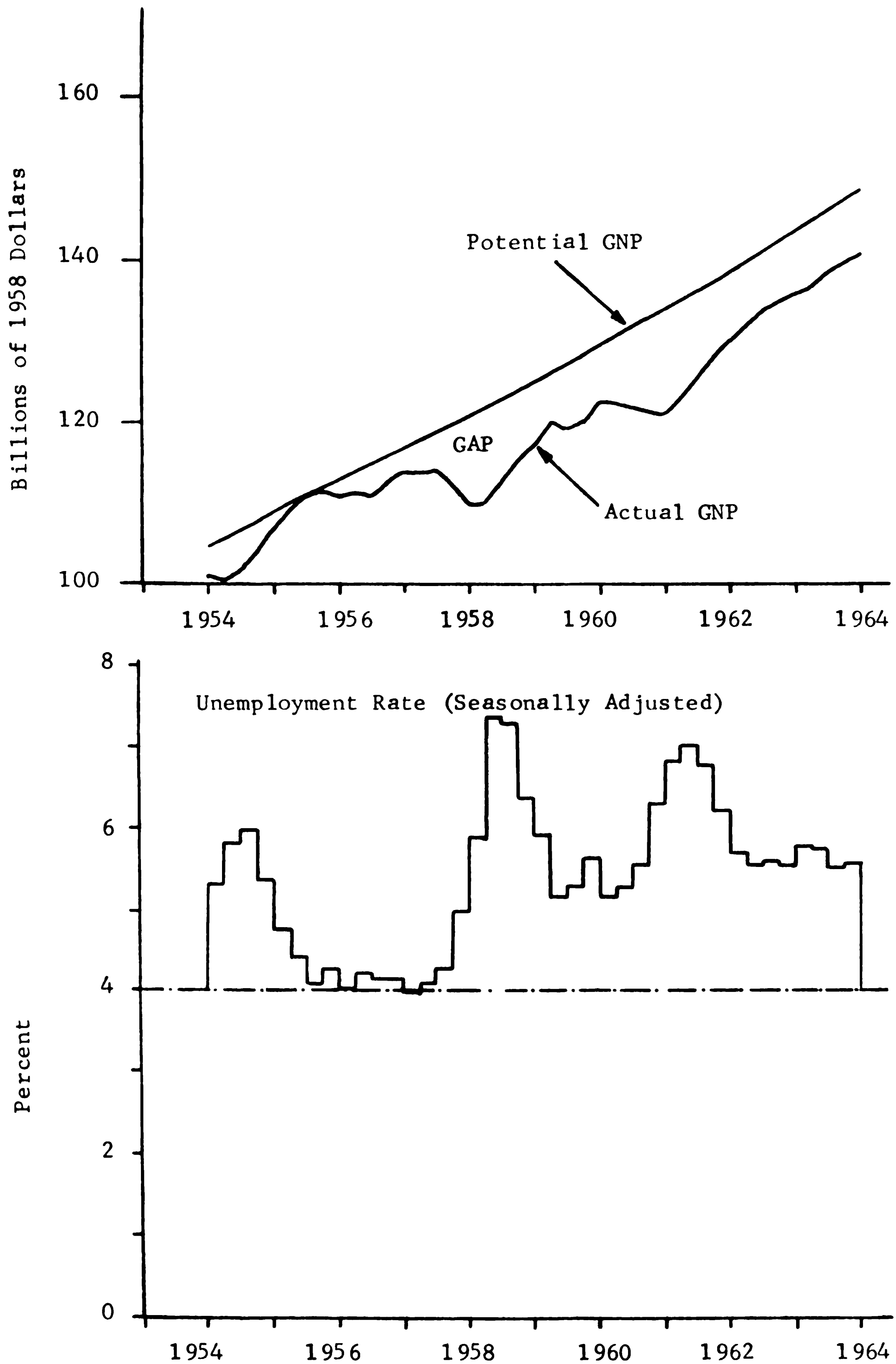

Fig. (6.2.3). Relationship Between the "gap" and the Unemployment Rate.

of G, M, and L may be varied within the optimal control policies. The effect of the level of the penalty function (on the cost of the control vector activity and hence on the cost functional) was studied computationally by considering values of the scalar η (weighting factor). That is, while the matrix R will change the relative magnitudes of the control variables, the penalty factor η will change the overall effect of the control vector activity on the cost functional of the system. Since there does not exist any theoretical or practical information as to how the components of the matrix R and/or scalar η should be chosen, a series of simulation runs with different ratios of r and different values for η were conducted in this study.

When $r_{11}/r_{22} = r_{11}/r_{33} = r_{22}/r_{33} = 1$ optimal control policies over the entire range of η consist of relatively low magnitudes on $\hat{L}$ and $\hat{M}$ with $\hat{G} > \hat{M} > \hat{L}$. The larger the value of η (that is the higher the cost of control), the higher is the value of the performance index. These optimal values are mathematically reasonable but the fact that $\hat{M}$ is greater than $\hat{L}$ indicates negative time deposits $(\hat{R} = \hat{L} - \hat{M})$. These negative values of time deposits specified in the optimal control policies are the result of using an unconstrained control and using equal weighting factors among the different components of the control vector.

In Sec. 3.3 dynamic and stability analyses of the open-loop system were discussed. One important feature of the open-loop system is that using the estimated structural equations (expressed at the end of Sec. 3.2) and expressing current endogenous variables in terms of exogenous and lagged endogenous variables, reduced form equations are obtained. The coefficients of these equations were called impact multipliers in Chapter Three. Focusing attention on the endogenous variable $y_a(t)$, i.e., GNP, the derived reduced form is

$$\begin{aligned} y_a(t) &= -20.8070 + \underline{1.1427}\, G(t) + \underline{.3168}\, M(t) + \underline{.0481}\, L(t) \\ &\quad - .1242\, M(t-1) - .0350\, L(t-1) + .1035\, t + .8313\, C(t-1) \\ &\quad + .7270\, (S(t-1) - S(t-2)) + 1.0190\, I^d(t-1) + .7409\, I^r(t-1) \\ &\quad + .3631\, I^i(t-1) + \text{error}\ . \end{aligned} \tag{6.3.2}$$

Based on the model's estimate, the <u>impact effect</u> (or <u>multiplier effect</u>)# of a \$1 billion increase in $G(t)$ is to increase $y_a(t)$ \$1.1427 billion, while the impact effect of a \$1 billion increase in money supply yield a \$.3649 billion increase. Therefore, it is reasonable that an optimal control policy based on equal cost of control for G, M, and L would heavily favor the use of G over M and L and M over L. In order to obtain control policies with a more balanced policy mix, r_{11}/r_{22}, r_{11}/r_{33}, and r_{22}/r_{33} were varied for different η's, thus, different weighting was given to the activity of each component of the control vector and with penalty on the system's error $\mu = 1$ and 1/2 in Equation (4.3.1). The ratio r_{11}/r_{22} was varied over the range ½,1,5,10,

---

Recalling here that the coefficients attached to the exogenous variables in Equation (3.3.1) are called dynamic multipliers. These multipliers measure the change in $y_a(t)$ corresponding to a unit change in the value of an exogenous variable occurring in any specific present or past period. To note that the dynamic multipliers attached to the current values of exogenous variables are the same as the corresponding impact multipliers, compare the underlined coefficients in Equation (6.3.2) with the coefficient vector $\underline{a}_5'$ in Equation (3.3.1).

in Figs. (6.3.1)-(6.3.3) for initial conditions at the first quarter of 1954. The actual levels of each control variable during 1954-1963 are also shown as discrete-time functions.

Figures (6.3.1)-(6.3.3) clearly show the time relationships between actual and optimal monetary and fiscal policies during 1954-1963. The optimal control variables trajectories represent the paths which minimize the value of the performance index I_N with $\eta = .01$ and $r_{11}/r_{22} = 2,8,10$; $r_{11}/r_{33} = 60,240,500$, $r_{22}/r_{33} = 30,40,50$, respectively. The very high starting values of the control variables are the transient phenomena required to bring $\hat{y}_a(t)$, which initially is 10 to 20 billion dollars below potential output, up to $y_d(t)$. The optimal levels for $\underset{\bullet}{M}$ suggested by this study stay below the actual levels while those for G and L do not. As the ratio r_{11}/r_{22} decreases, the optimal levels for G increase, while those for M and L decrease (see Figs. (6.3.1)-(6.3.3)). As the ratio r_{11}/r_{22} approaches one and continues to decrease, the optimal trajectory for M becomes fairly flat whereas the optimum trajectories for G and L preserve their previous shapes with their magnitudes changing. Figures (6.3.4) and (6.3.5) illustrate the above for the case when $r_{11}/r_{22} = 1$, $r_{11}/r_{33} = r_{22}/r_{33} = 100$ and $\eta = .01$. Figures (6.3.2) and (6.3.5) suggest that the optimal path for the money supply during 1954-1963 in this study is almost constant. Subsequent numerical results in Sec. 6.4 are based on $r_{11}/r_{22} = 8$, $r_{11}/r_{33} = 240$, and $r_{22}/r_{33} = 40$ because these ratios lie in the range of "optimal" policy mix ratios and the resulting optimal control policies are reasonable in terms of relative magnitudes of $\hat{G}$, $\hat{M}$, and $\hat{L}$.

in Figs. (6.3.1)-(6.3.3) for initial conditions at the first quarter of 1954. The actual levels of each control variable during 1954-1963 are also shown as discrete-time functions.

Figures (6.3.1)-(6.3.3) clearly show the time relationships between actual and optimal monetary and fiscal policies during 1954-1963. The optimal control variables trajectories represent the paths which minimize the value of the performance index I_N with $\eta = .01$ and $r_{11}/r_{22} = 2,8,10$; $r_{11}/r_{33} = 60,240,500$, $r_{22}/r_{33} = 30, 40,50$, respectively. The very high starting values of the control variables are the transient phenomena required to bring $\hat{y}_a(t)$, which initially is 10 to 20 billion dollars below potential output, up to $y_d(t)$. The optimal levels for M suggested by this study stay below the actual levels while those for G and L do not. As the ratio r_{11}/r_{22} decreases, the optimal levels for G increase, while those for M and L decrease (see Figs. (6.3.1)-(6.3.3)). As the ratio r_{11}/r_{22} approaches one and continues to decrease, the optimal trajectory for M becomes fairly flat whereas the optimum trajectories for G and L preserve their previous shapes with their magnitudes changing. Figures (6.3.4) and (6.3.5) illustrate the above for the case when $r_{11}/r_{22} = 1$, $r_{11}/r_{33} = r_{22}/r_{33} = 100$ and $\eta = .01$. Figures (6.3.2) and (6.3.5) suggest that the optimal path for the money supply during 1954-1963 in this study is almost constant. Subsequent numerical results in Sec. 6.4 are based on $r_{11}/r_{22} = 8$, $r_{11}/r_{33} = 240$, and $r_{22}/r_{33} = 40$ because these ratios lie in the range of "optimal" policy mix ratios and the resulting optimal control policies are reasonable in terms of relative magnitudes of $\hat{G}$, $\hat{M}$, and $\hat{L}$.

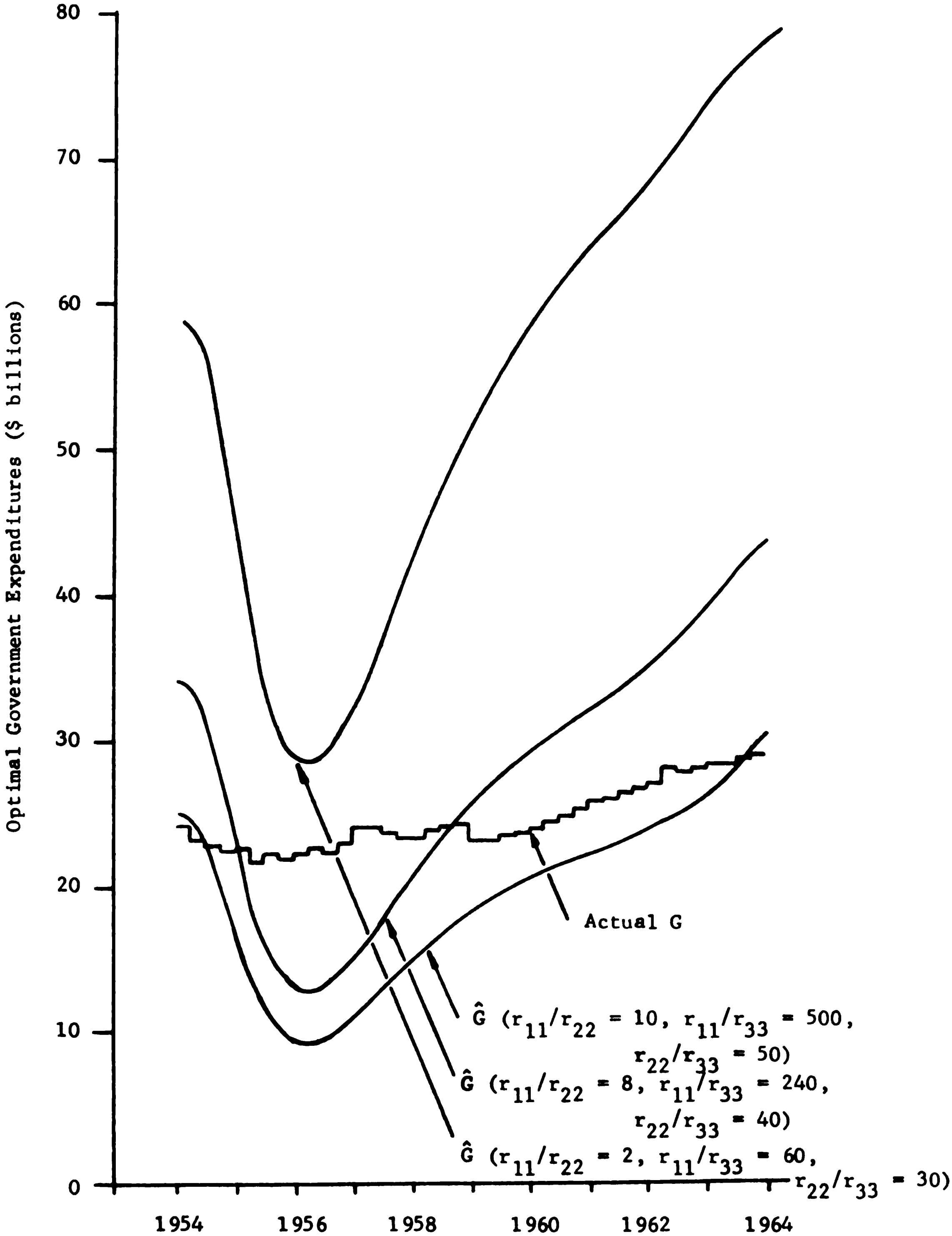

Fig. (6.3.1). Optimal Control Policies for 1954-1963 where $r_{11}/r_{22} = 2,8,10$; $r_{11}/r_{33} = 60,240,500$; $r_{22}/r_{33} = 30,40,50$ ($\eta = .01$).

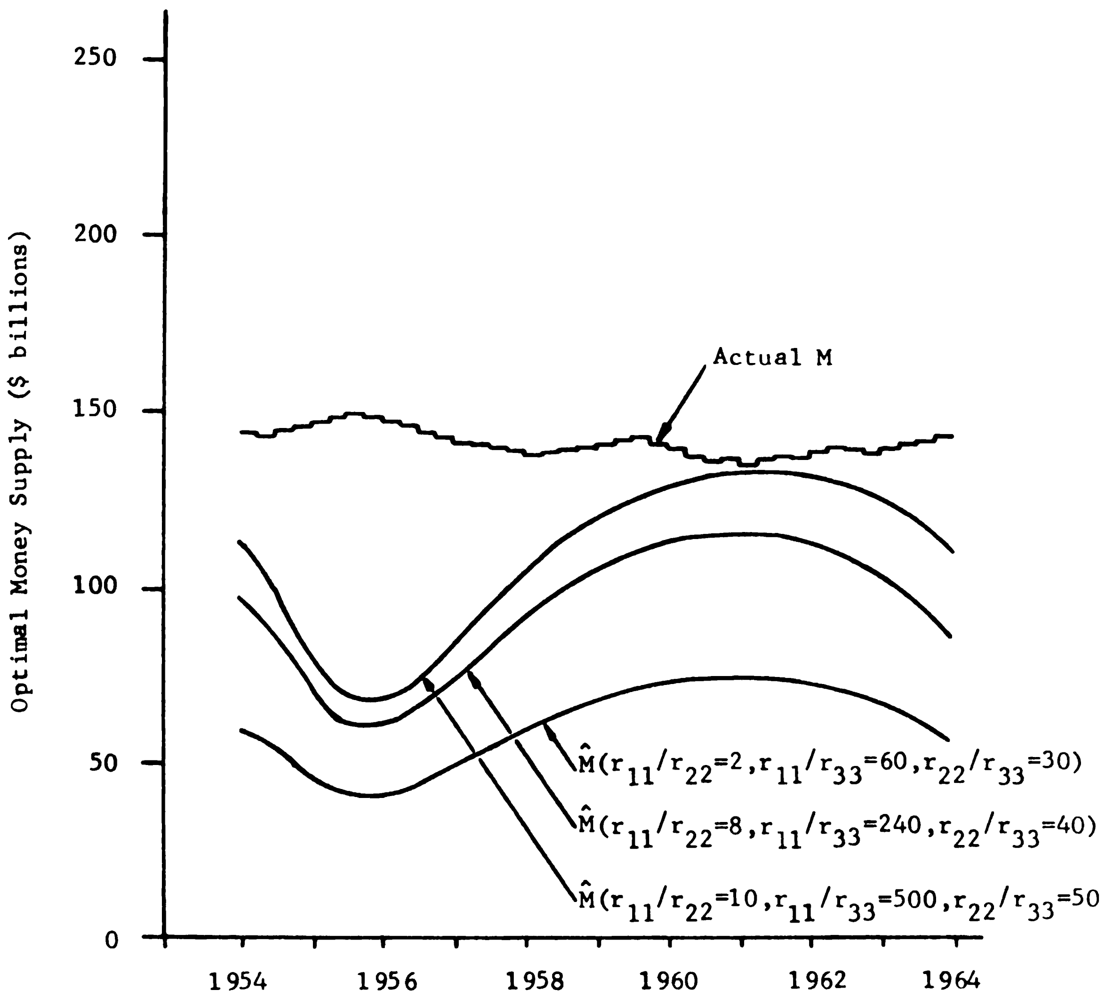

Fig. (6.3.2). Optimal Control Policies for 1954-1963 where r_{11}/r_{22} = 2,8,10; r_{11}/r_{33} = 60,240,500; r_{22}/r_{33} = 30,40,50 (η = .01).

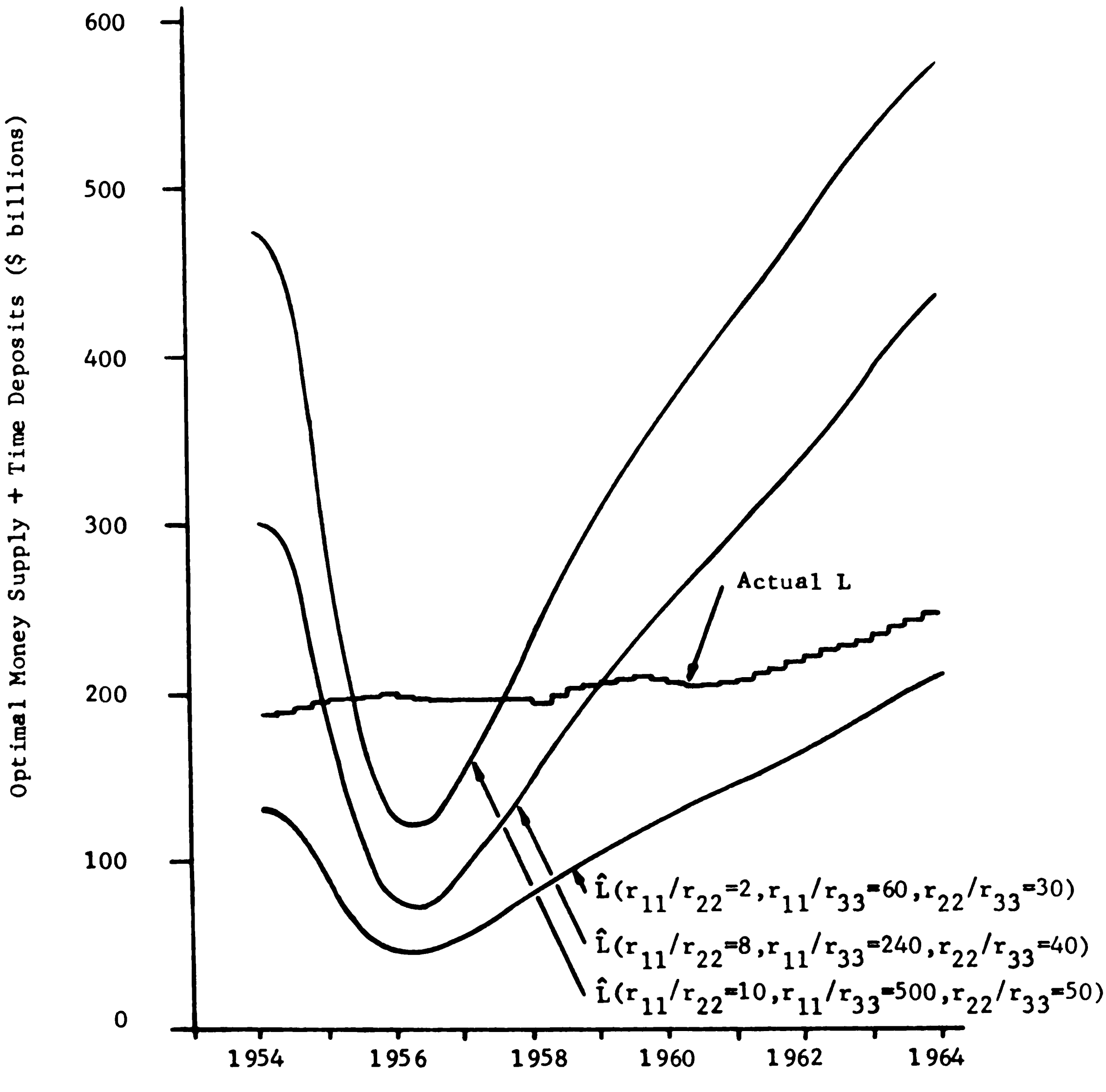

Fig. (6.3.3). Optimal Control Policies for 1954-1963 where $r_{11}/r_{22} = 2,8,10$; $r_{11}/r_{33} = 60,240,500$; $r_{22}/r_{33} = 30,40,50$ $(\eta = .01)$.

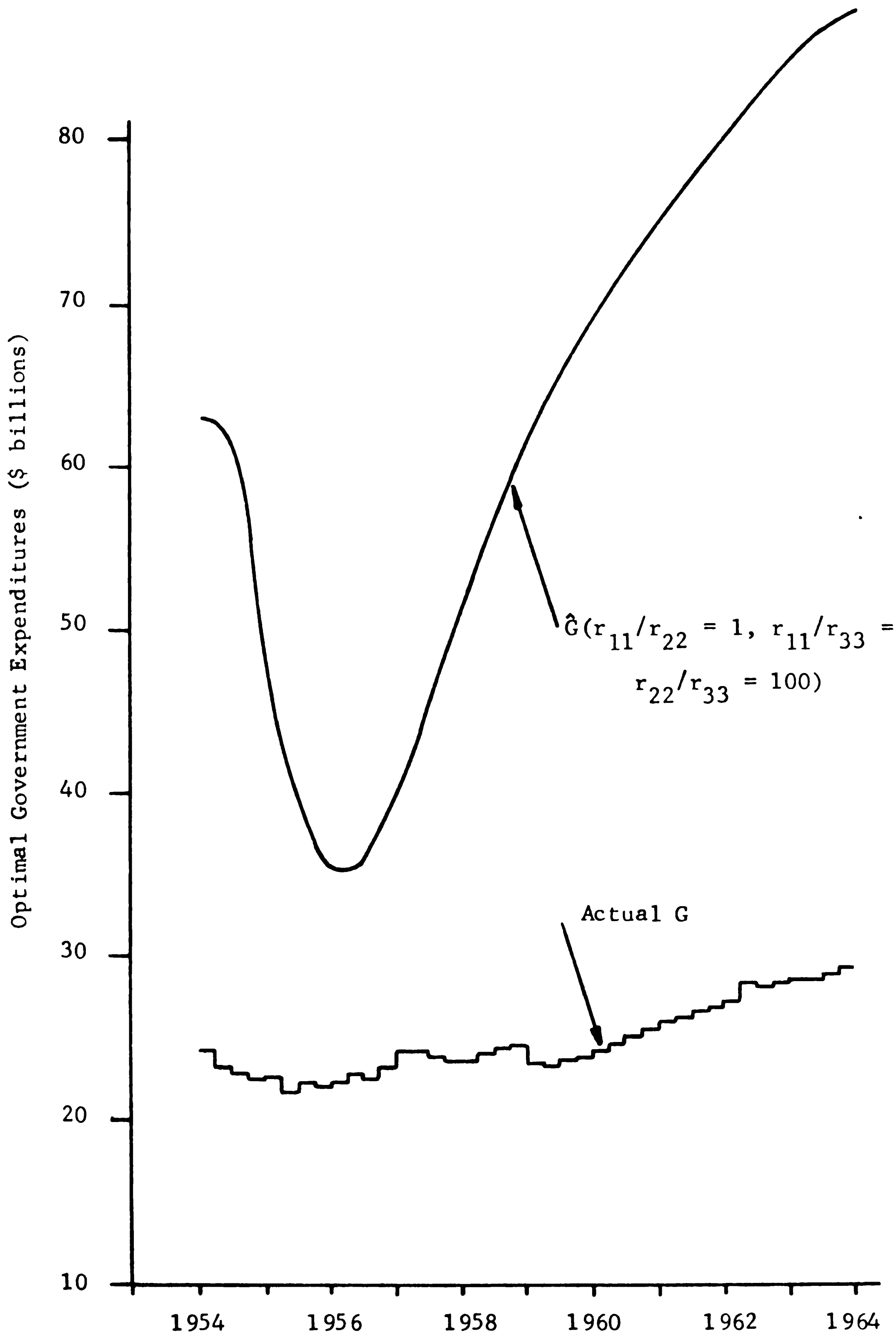

Fig. (6.3.4). Optimal Control Policy for 1954-1963 where $r_{11}/r_{22} = 1$ and $r_{11}/r_{33} = r_{22}/r_{33} = 100$ ($\eta = .01$).

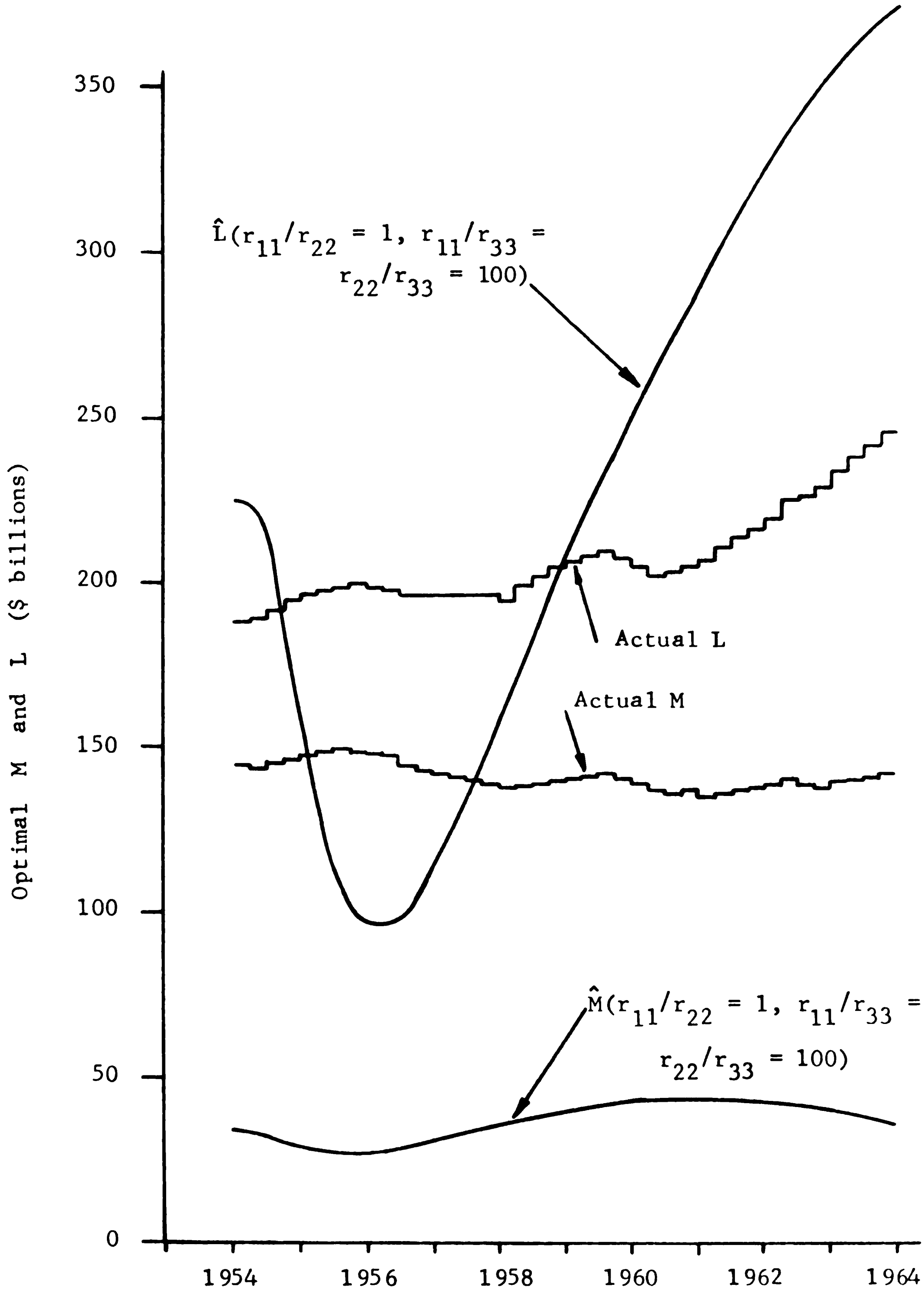

Fig. (6.3.5). Optimal Control Policy for 1954-1963 where $r_{11}/r_{22} = 1$ and $r_{11}/r_{33} = r_{22}/r_{33} = 100$ ($\eta = .01$).

Figures (6.3.6)-(6.3.8) show the optimal control policies when the weighting factor μ on the system's error term is 1/2 for the same weighting factors on the control components shown in Figs. (6.3.1)-(6.3.5). A close look at the Figs. (6.3.6)-(6.3.8) shows that the optimal control paths are flatter than those in Figs. (6.3.1)-(6.3.4) and hence closer to the actual paths for the control variables. On the other hand, the paths are still of the same shape but with different magnitudes. These observations lead to the following results. First, the fact that the optimal paths for the control variables are not the same as the actual paths indicate that decision-maker(s) should consider more flexible control variables in order to achieve a stabilized economy. Secondly, Friedman's hypotheses do not receive support from this work. To recall, the first hypothesis is that policy changes may be detrimental as often as they are beneficial from the standpoint of stabilization policy. Therefore, the most effective strategy might be either no policy at all or adherence to some simple rule, e.g., a three to four percent increase in a control variable, e.g., money supply, per year. The second hypothesis is that the money supply is a more significant and important determinant of consumption and income than are autonomous expenditures (Refs. F-5 and K-8). However, from the analysis of the optimal control policies and the simulation runs made, these hypotheses do not receive a strong support from the model for the period under investigation. With respect to the stabilization question all three control variables are effective. The control variable M (money supply) is not a more significant and important determinant of income than G, simply

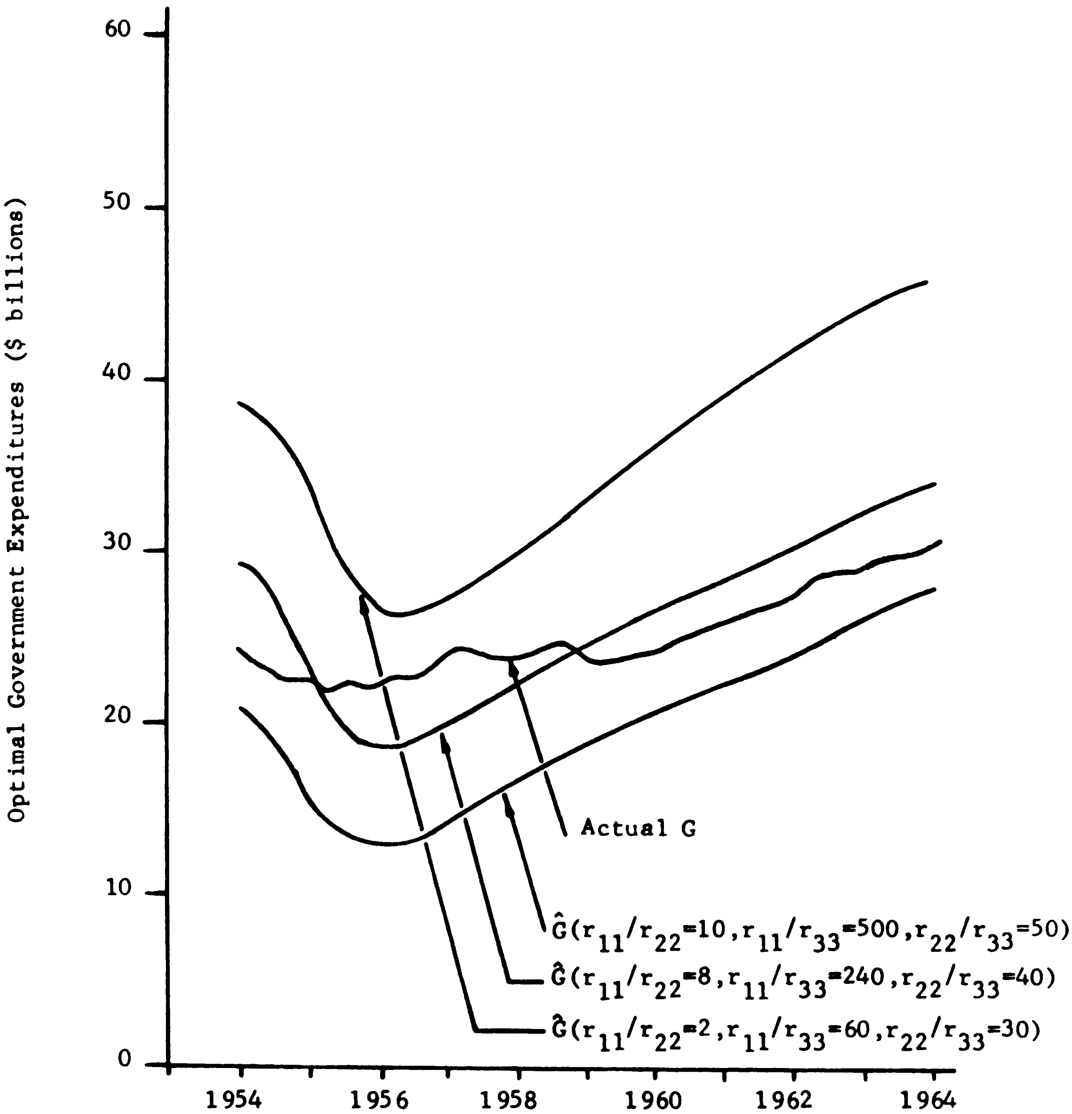

Fig. (6.3.6). Optimal Control Policies for 1954-1963 where r_{11}/r_{22} = 2,8,10; r_{11}/r_{33} = 60,240,500; r_{22}/r_{33} = 30,40,50 (η = .01, μ = 1/2).

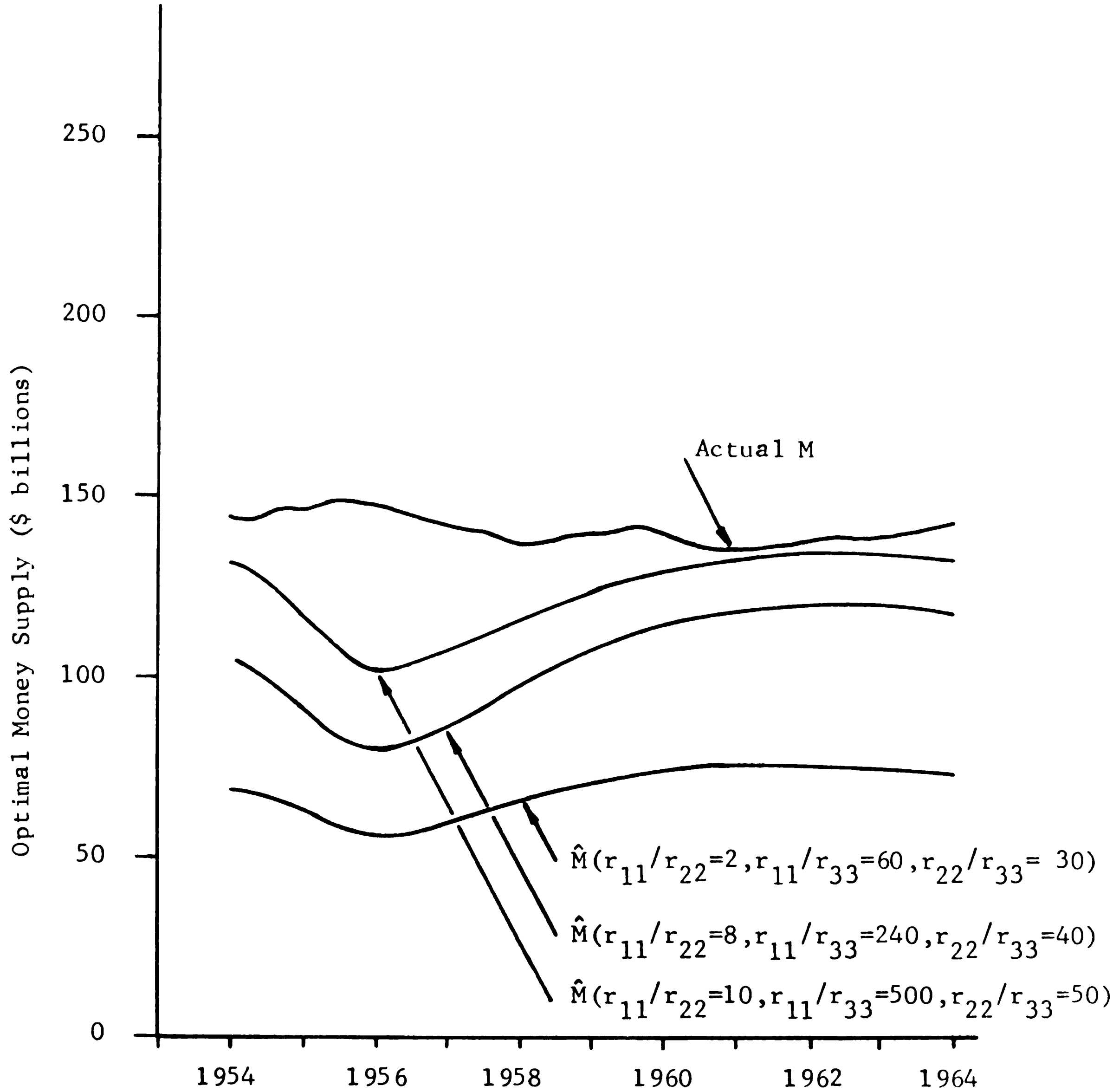

Fig. (6.3.7). Optimal Control Policies for 1954-1963 where $r_{11}/r_{22} = 2,8,10$; $r_{11}/r_{33} = 60,240,500$; $r_{22}/r_{33} = 30,40,50$ ($\eta = .01$, $\mu = 1/2$).

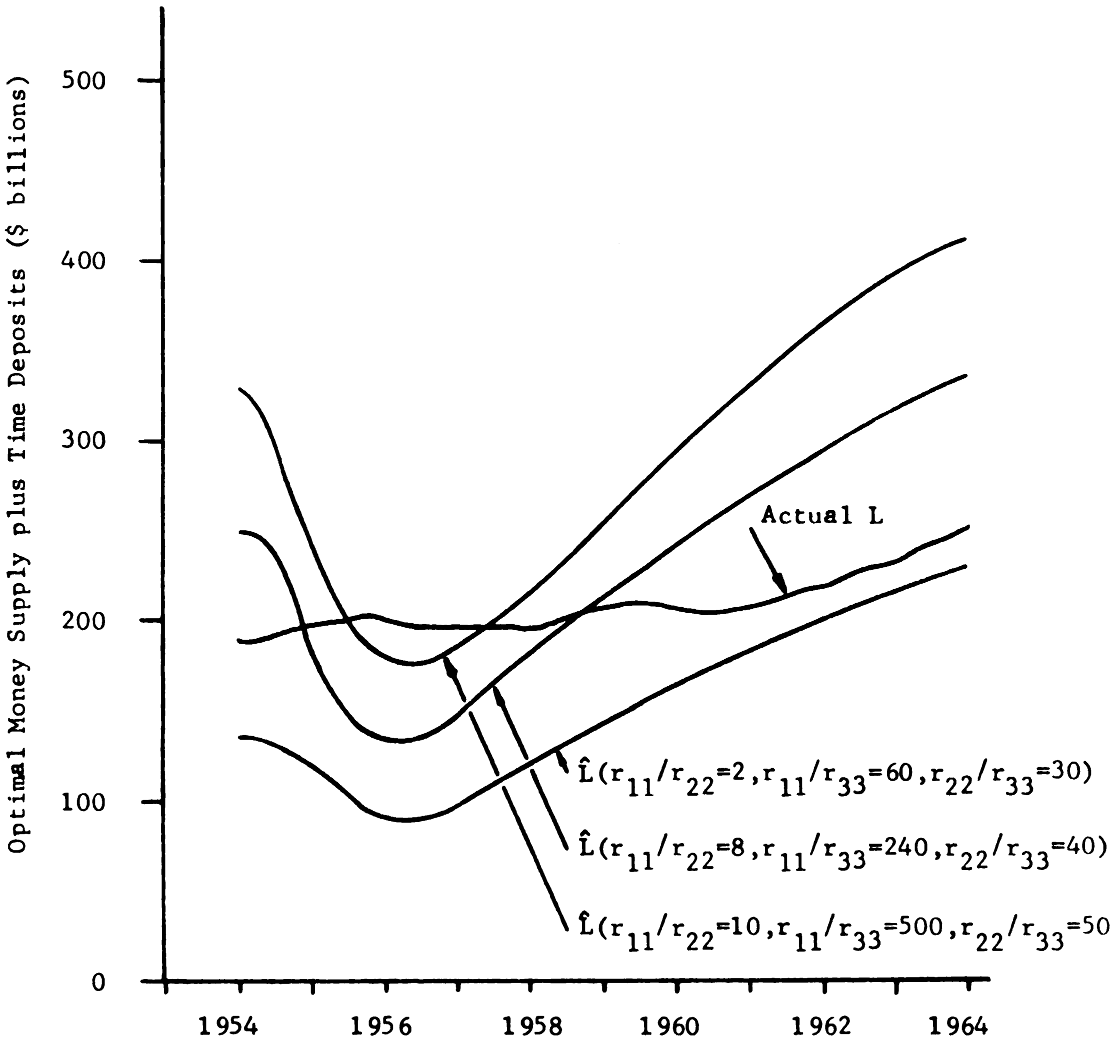

Fig. (6.3.8) Optimal Control Policies for 1954-1963 where $r_{11}/r_{22} = 2,8,10$; $r_{11}/r_{33} = 60,240,500$; $r_{22}/r_{33} = 30,40,50$ ($\eta = .01$, $\mu = 1/2$).

because it is not as sensitive as G and its optimal path always stays below the actual path for M, see Figs. (6.3.2), (6.3.5), and (6.3.7). Finally, it is important to notice that the controls in this study are not constrained. Therefore, the specific high values for the control variables and the large deviations between the optimal and actual paths for control variables are due to the fact that they are free to vary in their space. Obviously further research needs to be done when the controls and/or states are constrained.

6.4 Optimal GNP ($\hat{y}_a(t)$) Trajectory

The optimal trajectories for gross national product from 1954-1963 based on the quadratic disutility functional Equation (4.3.7) with various penalties on the cost of control term are shown in Fig. (6.4.1). Based on this study the figure shows that the optimal output $\hat{y}_a(t)$ reaches its steady state trend line within about one year. Due to the large initial deviation between $y_d(t)$ and $\hat{y}_a(t)$, $\hat{y}_a(t)$ overshoots $y_d(t)$ for the first few quarters after the control has been applied when the penalty on cost of control (η) is relatively low. This underdamped response simply indicates that for the initial start of the output, large initial controls should be applied to rapidly close the gap between $y_d(t)$ and $\hat{y}_a(t)$ followed by a backing off on the controls to limit the resulting overshoots. This point is clearly seen from the optimal paths for the control variables in Figs. (6.3.1)-(6.3.3), since this is indeed the form of the optimal control policy.

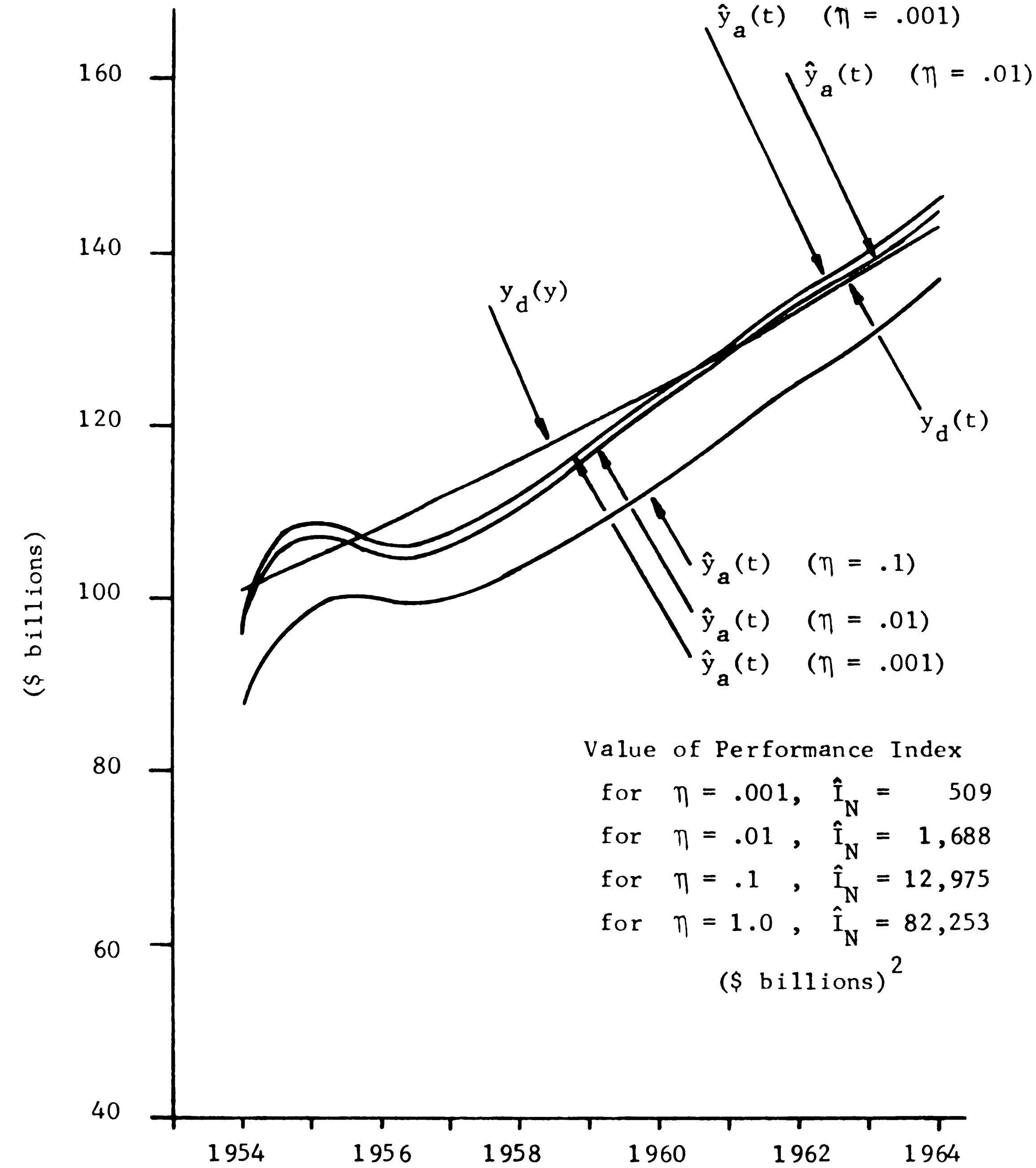

Fig. (6.4.1). Optimal GNP ($\hat{y}_a(t)$) Trajectory Versus Cost of Control $r_{11}/r_{22} = 10$, $r_{11}/r_{33} = 500$, $r_{22}/r_{33} = 50$ and Compared with the Desired GNP ($y_d(t)$).

As the cost of control increases, the initial response of $\hat{y}_a(t)$ slows and the steady state deviation between $y_d(t)$ and $\hat{y}_a(t)$ increases. After the overshoots the optimal path of the output $(\hat{y}_a(t))$ increases steadily and, for the low cost of the control, $\hat{y}_a(t)$ crosses $y_d(t)$ and stays above it for the rest of the optimization period. Figure (6.4.2) shows the same result with a higher level of desired (potential) output $(y_d(t))$. In this case the optimal path for the output $(\hat{y}_a(t))$ stays below $y_d(t)$. The steady state deviation between $y_d(t)$ and $\hat{y}_a(t)$ represents an equilibrium point at which the cost associated with incremental changes in $\hat{\underline{w}}(t)$ (optimal control policy) is just offset by the payoffs resulting from these incremental changes. Perhaps a more realistic form for the cost of control term would be, as was said earlier (Sec. 6.3), one which is a function of the deviation between $y_d(t)$ and $\hat{y}_a(t)$. Thus, as $\epsilon = (y_d(t) - \hat{y}_a(t))$ decreases the cost associated with reducing ϵ also would further decrease. On the other hand, cost of control should never go to zero since there is always a certain level of uncertainty (noise) with respect to the exact state of $y_d(t)$ and $\hat{y}_a(t)$ at any point in time.

Figure (6.4.1) further shows the cost (value) of the performance index for different values of penalties on the cost of control term in the performance index.

6.5 Optimal Feedback Gains

Recall that the optimal control policy, Equation (5.1.25), has the form

$$\hat{\underline{w}}(t) = \underline{q}(N-t) + Q(N-t)\underline{z}(t) \qquad t = 1,\ldots,N-1 \qquad (5.1.25)$$

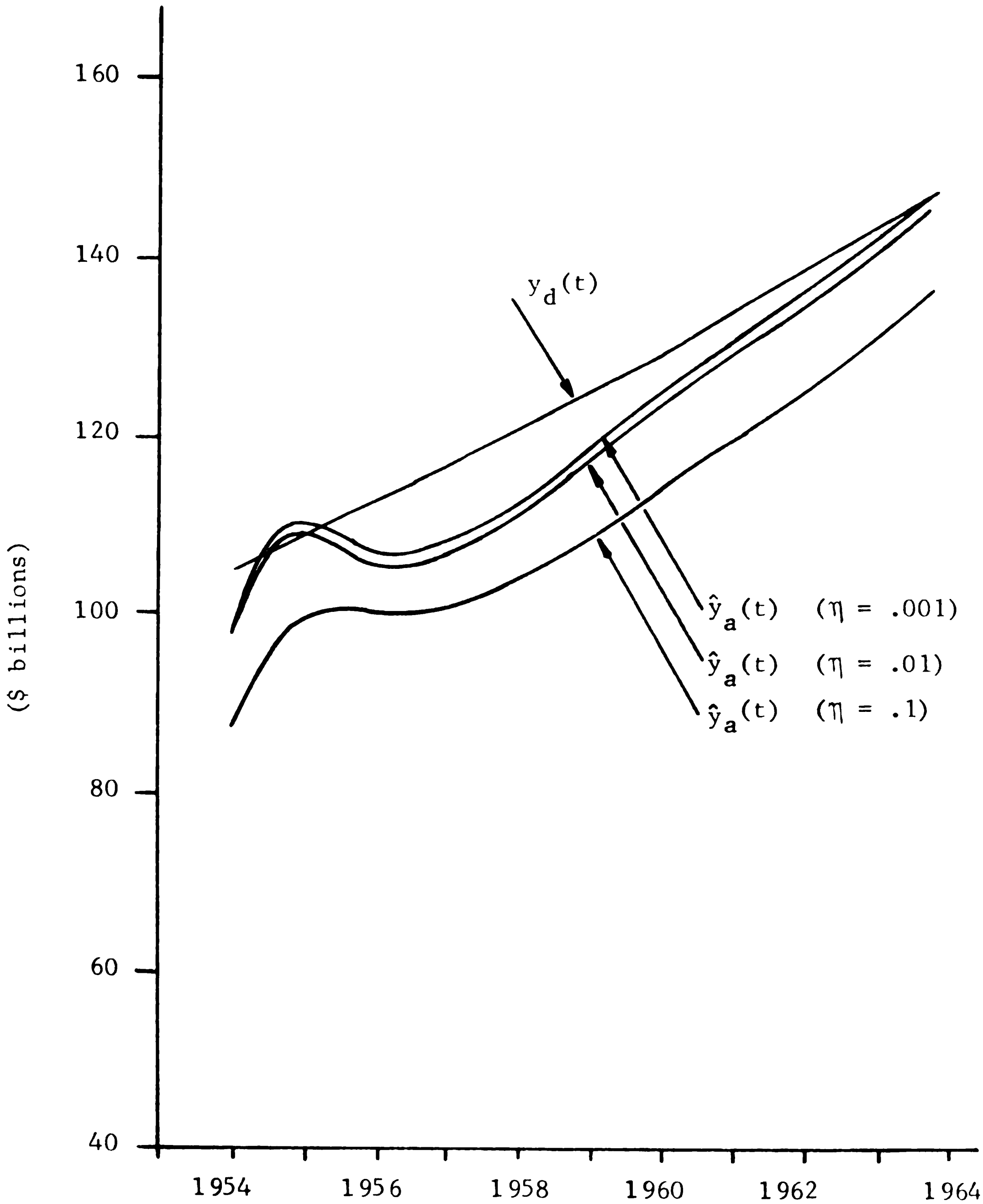

Fig. (6.4.2). Optimal GNP Trajectory Versus Cost of Control $r_{11}/r_{22} = 10$, $r_{11}/r_{33} = 500$, $r_{22}/r_{33} = 50$ and Compared with the Desired GNP.

where the components of the matrix Q are the optimal feedback gains from the system state $\underline{z}$ to the optimal control $\hat{\underline{w}}$. The optimal feedback gains were briefly discussed in Sec. 6.1. In this section the optimal feedback gains for $\eta = .001$ and $\eta = .1$ are compared in Figs. (6.5.1)-(6.5.6) for the optimization interval 1954-1963, where $r_{11}/r_{22} = 2$, $r_{11}/r_{33} = 60$, and $r_{22}/r_{33} = 30$. In general, the shapes of the optimal feedback gains do not appreciably change with η (penalty on the cost of control in the performance index) although their magnitudes do. The feedback gains are plotted on a scale of time-to-go, N-k (N = 40 quarters, k = 0,1,...,39). Thus, feedback gains at stage N-k on the time-to-go scale refer to gains which should be applied at the k-th stage of a N-stage process. The feedback gains given in the Figs. (6.5.1)-(6.5.6) are sufficient to generate the optimal control policies for any length of optimization interval up to 40-quarters and for any arbitrary initial conditions on the state variables. For example, if it is wished to consider the optimal control policies for a three year period, N is set to 12 quarters and feedback gains for N-k, k = 0,1,2,...,11 are used with $\underline{z}(0)$ being chosen arbitrarily.

Some general comments concerning the optimal feedback gains, Figs. (6.5.1)-(6.5.6) are of interest. First, the optimal feedback gains for the control variables G and L are of the same sign whereas those for the control variable M are opposite except for the first and the last ones. Second, the optimal feedback gain matrix Q in Equation (6.1.4) is of the form

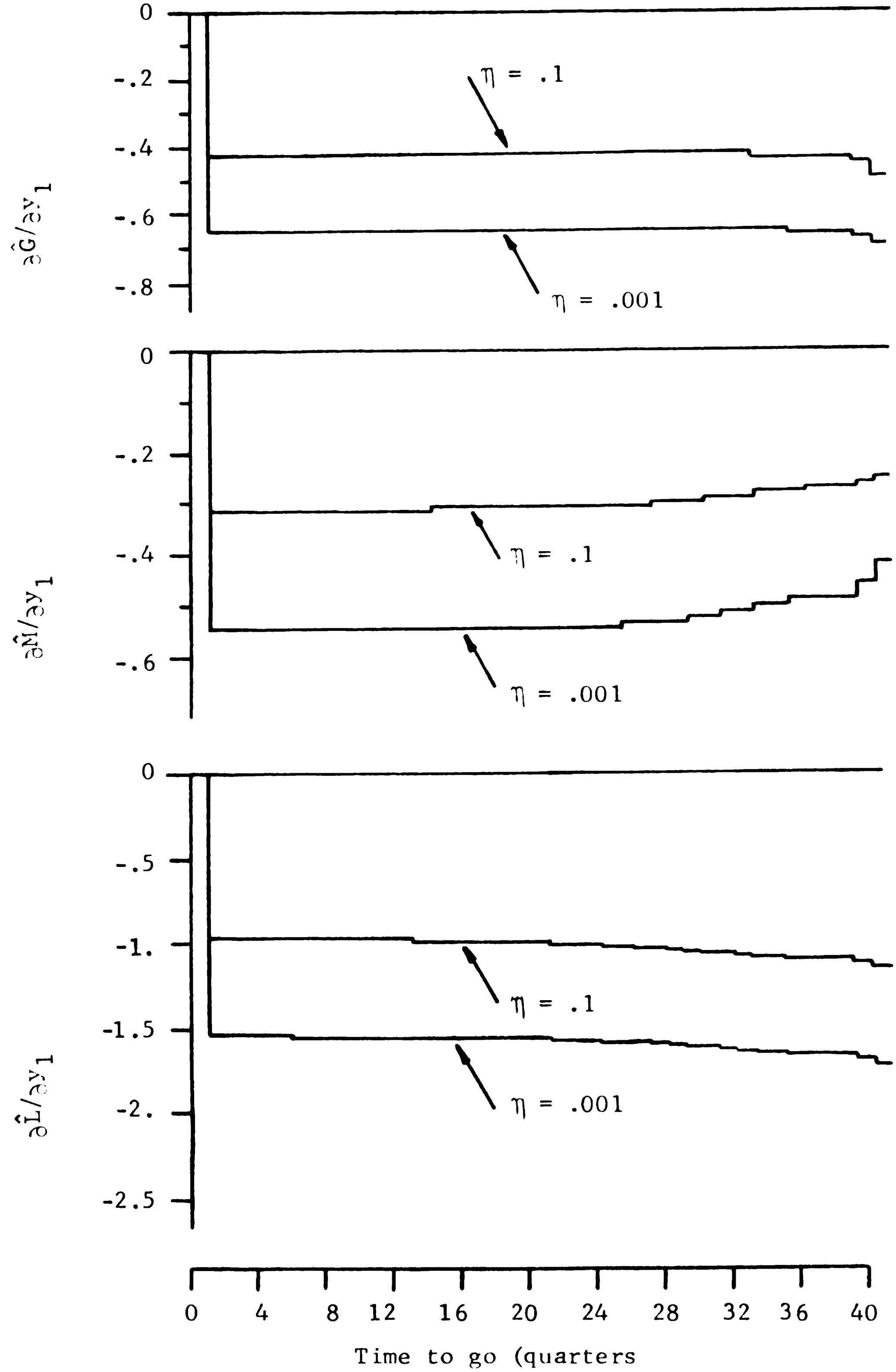

Fig. (6.5.1). Optimal Feedback Gains from State Variable y_1 to Control Variables G, M, and L.

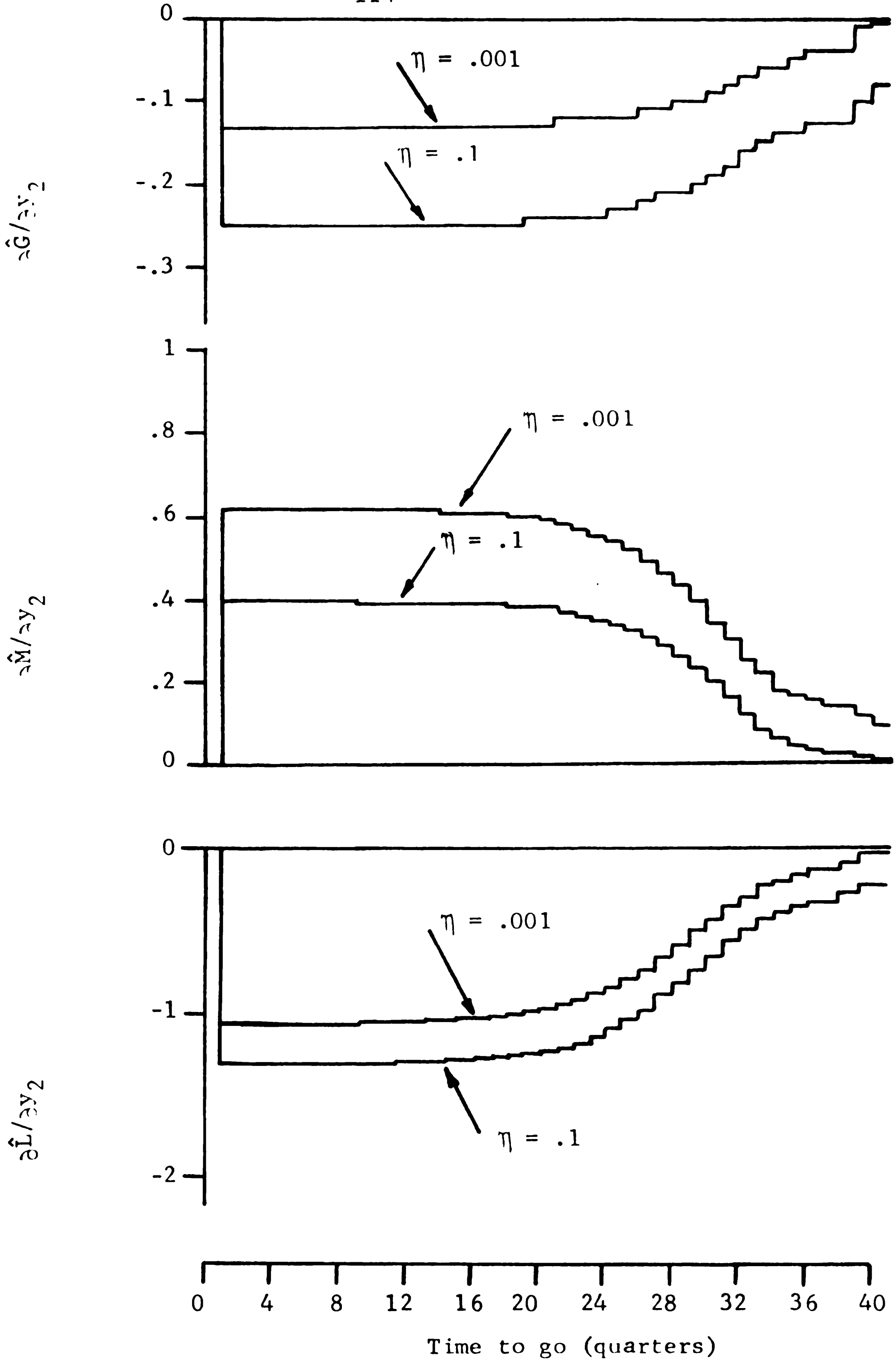

Fig. (6.5.2). Optimal Feedback Gains from State Variable y_2 to Control Variables G, M, and L.

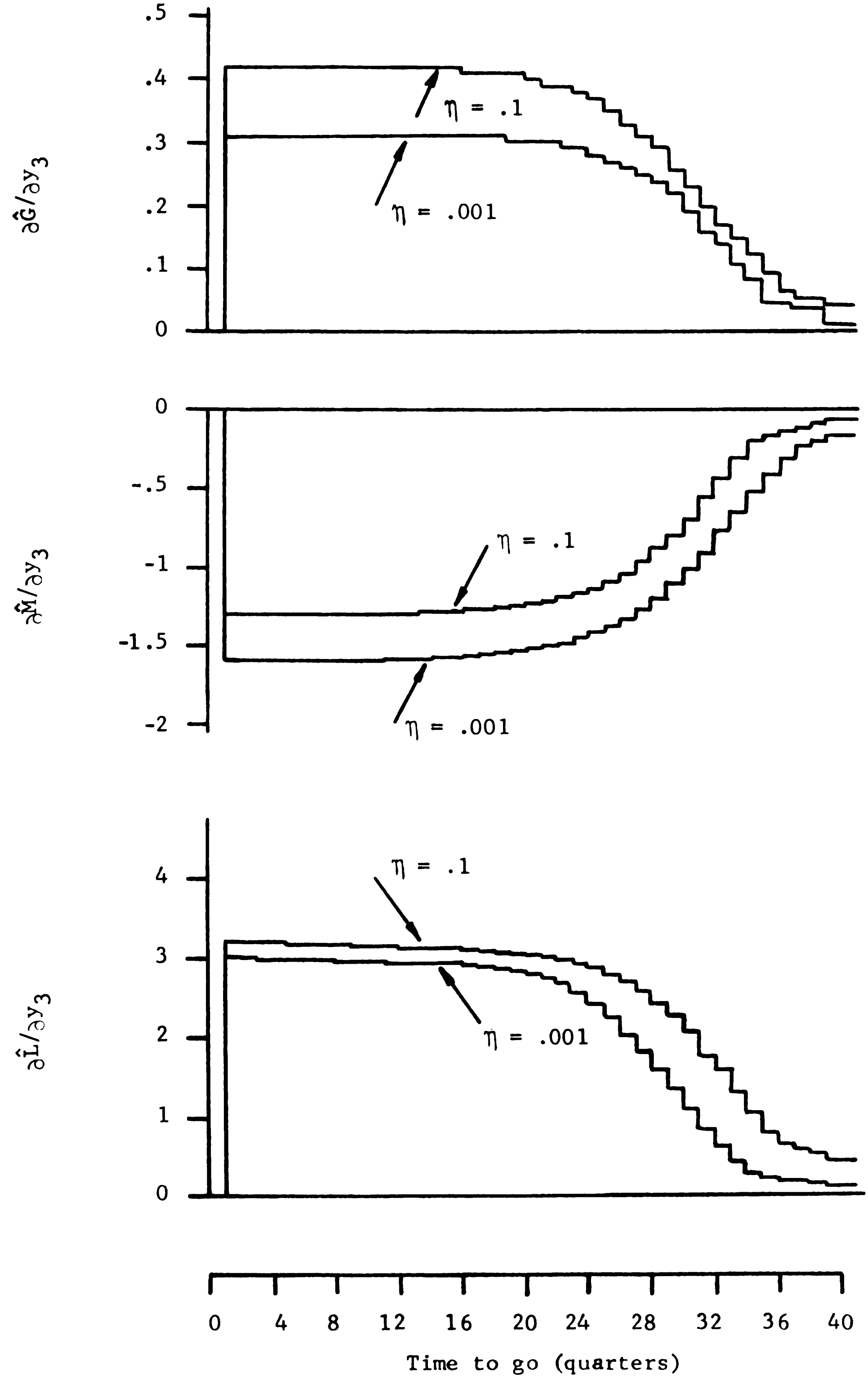

Fig. (6.5.3). Optimal Feedback Gains from State Variable y_3 to Control Variables G, M, and L.

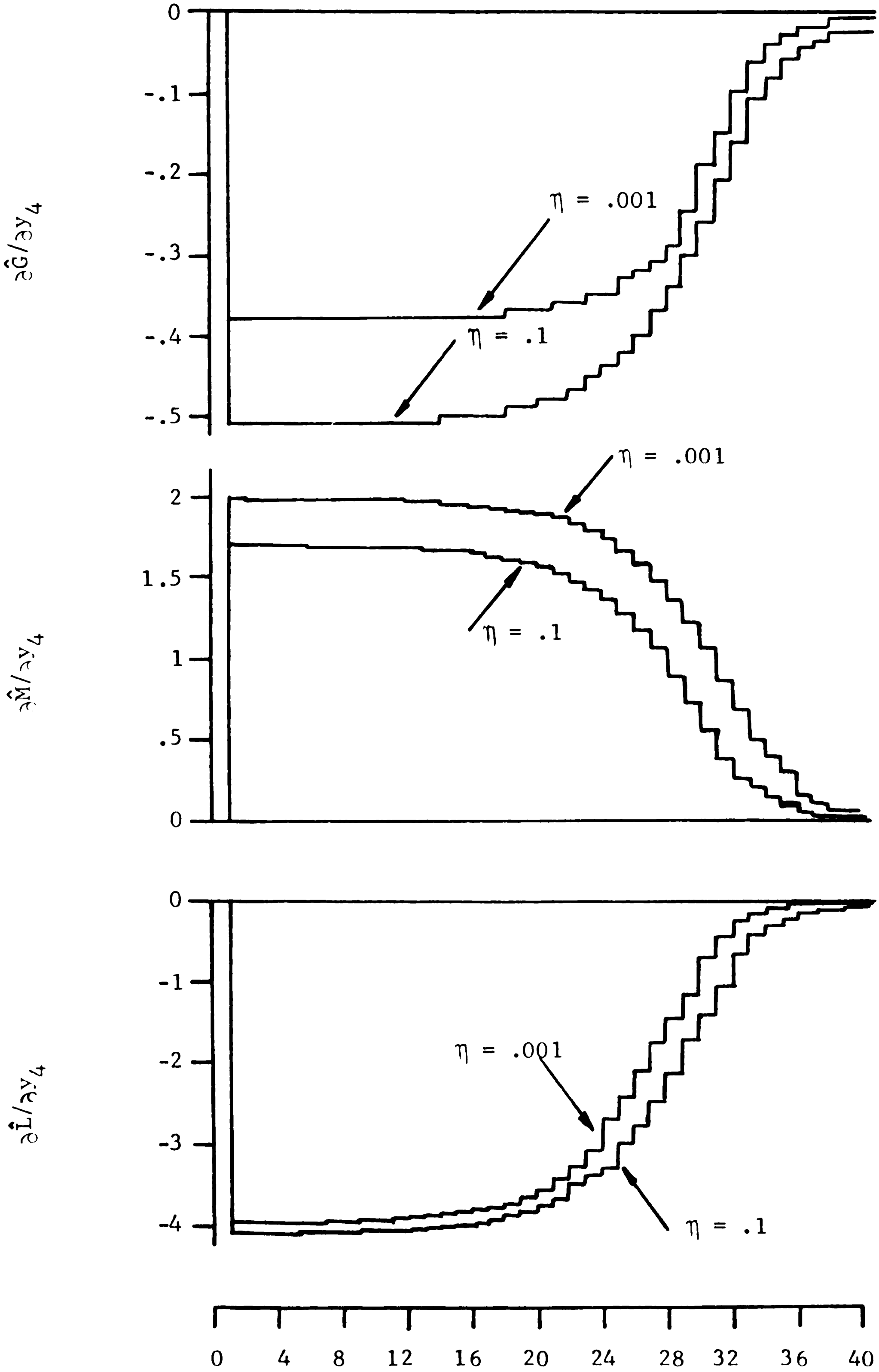

Fig. (6.5.4). Optimal Feedback Gains from State Variable y_4 to Control Variables G, M, and L.

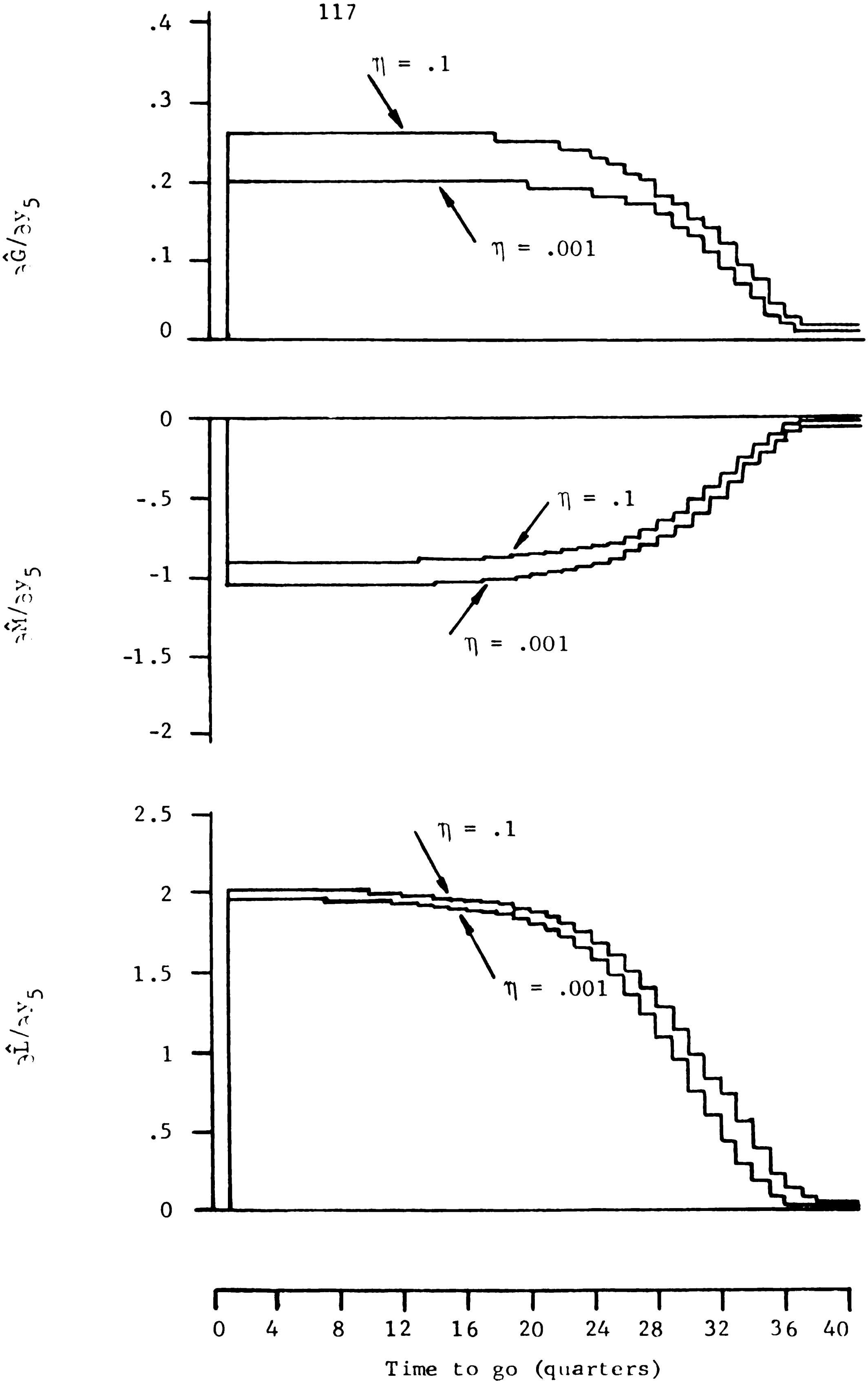

Fig. (6.5.5). Optimal Feedback Gains from State Variables y_5 to Control Variables G, M, and L.

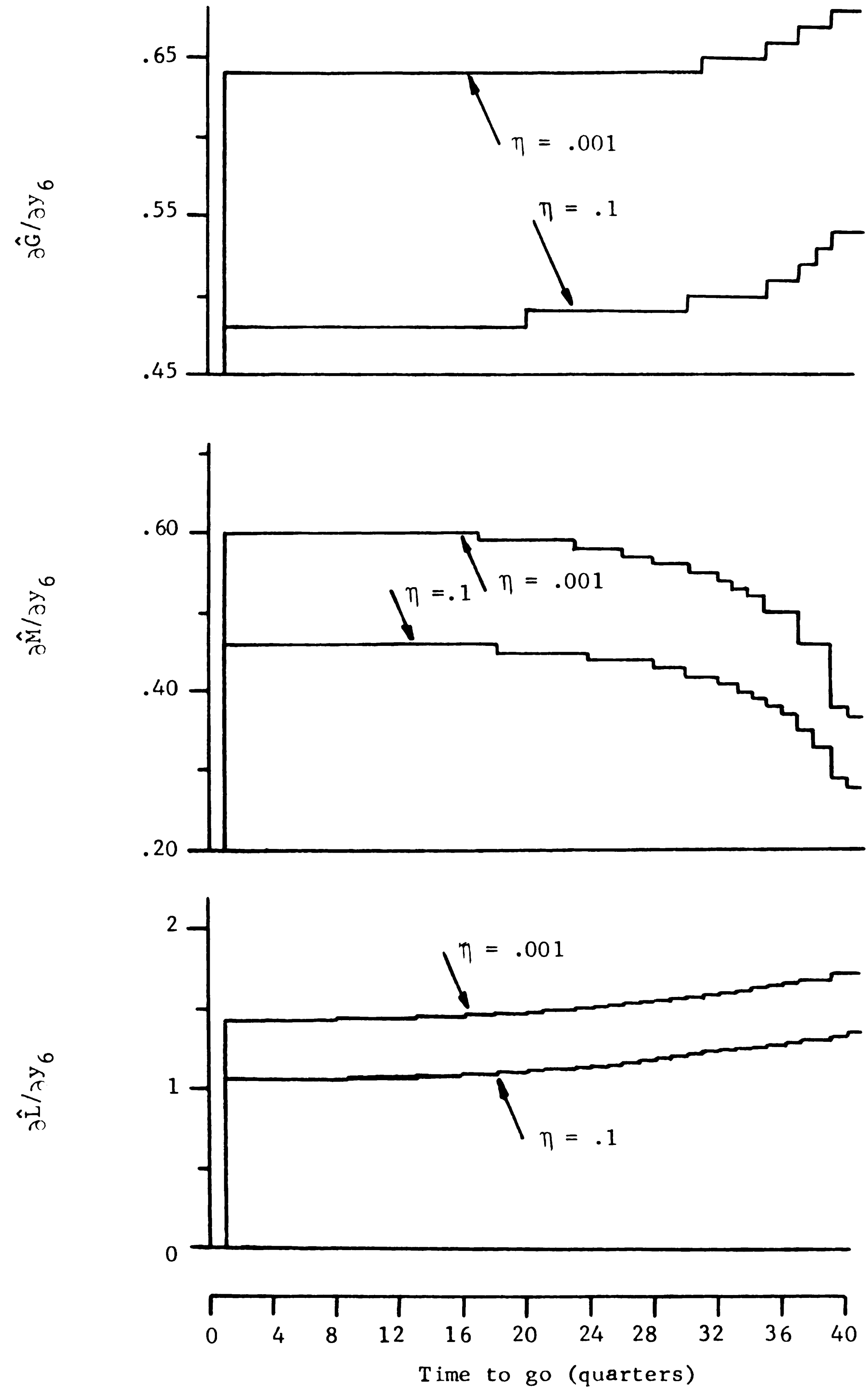

Fig. (6.5.6). Optimal Feedback Gains from State Variable y_6 to Control Variables G, M, and L.

$$Q = \begin{bmatrix} \partial\hat{G}/\partial y_1 < 0 & \partial\hat{G}/\partial y_2 < 0 & \partial\hat{G}/\partial y_3 > 0 & \partial\hat{G}/\partial y_4 < 0 & \partial\hat{G}/\partial y_5 > 0 & \partial\hat{G}/\partial y_6 > 0 \\ \partial\hat{M}/\partial y_1 < 0 & \partial\hat{M}/\partial y_2 > 0 & \partial\hat{M}/\partial y_3 < 0 & \partial\hat{M}/\partial y_4 > 0 & \partial\hat{M}/\partial y_5 < 0 & \partial\hat{M}/\partial y_6 > 0 \\ \partial\hat{L}/\partial y_1 < 0 & \partial\hat{L}/\partial y_2 < 0 & \partial\hat{L}/\partial y_3 > 0 & \partial\hat{L}/\partial y_4 < 0 & \partial\hat{L}/\partial y_5 > 0 & \partial\hat{L}/\partial y_6 > 0 \end{bmatrix}$$

(6.5.1)

The optimal feedback gains from state variables y_1, y_2, and y_4 on both $\hat{G}$ and $\hat{L}$ are negative while those associated with y_3, y_5, and y_6 are positive. Thus an increase in state variables y_1, y_2, and y_3 in one quarter tends to reduce $\hat{G}$ and $\hat{L}$ in the following quarter while the opposite is true of the other state variables. For the other control variable, $\hat{M}$, an increase in y_1, y_3, and y_5 in one quarter tends to reduce $\hat{M}$ in the following quarter. These points can be clearly seen from the optimal path for control variables, Figs. (6.3.1)-(6.3.3). Since the state model of the system is derived in terms of the output and control variables, it is difficult to associate these feedback gains and their effects to the flow and stock variables in the system. Finally, in this study no optimal feedback gains changed its algebraic sign as a function of N-k, i.e., all the gains stayed on either one side or the other of the time axis for the whole optimizing period (1954-1963).

CHAPTER SEVEN

CONCLUSIONS AND RESEARCH RECOMMENDATIONS

7.1 Conclusions

The research of this thesis has demonstrated the primary conclusion that optimal control theory is applicable to the analysis of complex economic systems and very useful in the formulation of optimal fiscal and monetary control policies for a national economy.

Conventional automatic control theory, e.g., stability, is inadequate to dexcribe completely many problems in macro-economic systems. The advancement of high-speed digital computers and modern control theory including optimization results such as dynamic programming and the maximum principle make possible more complete studies of fiscal and monetary decision-making. This dissertation utilizes these advances as well as the progress made by economists in dynamic modeling of macro-economic systems. The optimal economic policies which are derived in the simulations documented in Chapter Six provide several insights: (1) that greater flexibility in the allowed variations of control variables are needed to achieve consistently low levels of unemployment and stable prices; (2) some existing economic hypotheses such as the two stated by Friedman and discussed in the thesis need reconsideration; (3) smoothly increasing paths for the output (GNP) which result from optimal policies also yield up to a one-hundred-fold improvement in the quadratic performance index as compared with the actually observed output.

The dynamic econometric model of this study, consisting of a system of eight difference equations, replicates the time trajectories of the basic aggregated variables of the U.S. economy to within some percentage errors during the period 1954-1963. From the analysis of the optimal control policies and the simulation runs made, the Friedman-Meiselman hypothesis does not receive strong support from the model for the period under investigation. This point was clearly investigated by making some sensitivity simulation analyses of the model with respect to the cost of control and different weightings of G and M. The results suggest that G is more sensitive than M and the optimal path of M is fairly constant, Figs. (6.3.4) and (6.3.5)

The eigenvalues of the open-loop econometric model formulated in state-space suggest that the natural response modes of the model are stable. It hence appears that the system is inherently stable and that the sources of instability have to be sought in the stimuli from the exogenous factors (including trend) and the random disturbances.

An analysis of the model's parameters and their classification based on their effects on the model's variables and output (GNP) are studied. Model parameters influence model variables in that they measure the effect of changes in one variable upon the value of another variable(s). Therefore, their values are fundamental to the dynamic behavior of the GNP model system. That is, they are important for determining stability, response time constants, and steady-state gains. The model's parameters must be designed and estimated properly in order to have a valid dynamic description.

The performance index of this study is based on minimizing the difference between model-predicted GNP $(y_a(t))$ and the theoretical potential output of the economy, $y_d(t)$. A model of $y_d(t)$ studied here is $y_d(t) = f(nt)$. As a standard, $y_d(t)$ should not deviate from an approximate 3½% annual growth trend. This research indicates that $y_d(t) = f(nt)$ preserves the above characteristics and is not influenced by the optimal control policy. The optimal output path, $\hat{y}_a(t)$, stays in a small neighborhood of $y_d(t)$.

When the weighting ratios among G, M, and L in the cost of control term are between the ranges given in Chapter Six, the resulting optimal control policies give lower values of the performance index (less cost of control) than do neighboring optimal control policies with different combinations of G, M, and L.

Application of these optimal economic control policies results in optimal gross national product trajectories which closely track potential output, $y_d(t)$. Optimal control policies for 1954-1963 represent, for low values of cost of control, considerable improvement, in terms of the performance index of this study, over the actual performance of the U.S. national economy during 1954-1963 (see Figs. (6.2.1) and (6.4.1)).

The comparison between the optimal control paths and actual control paths indicate the necessity of a change in the fiscal and monetary decision policies made by the government. That is, this study in agreement with other studies made previously (Refs. B-14 and B-15) urges the consideration of more flexible control policies than currently exist.

7.2 Recommendations for Further Research

Further research should be directed at improving the model of the potential national output, $y_d(t)$, in order to more accurately characterize this variable endogenously. Also, to treat more complex economic models such as the Brookings and Klein/Goldberger models (Refs. D-5 and K-4), refined procedures for obtaining a state space form are needed. This form is essential for effective utilization of dynamic optimization techniques.

Additional research is needed to obtain from economic theory and available economic data, realistic values for the weighting factor on cost of control as well as the relative weights applicable to control variables. Use of some new cost of control term based on the changes in control variables rather than their absolute mangitudes should also be investigated.

The problem considered here had a finite time horizon. This allows one to re-examine the performance index or the decision rule from time to time. The study of the system for different time intervals (N) and the effect of this on the optimal policies also need further research.

The macro-economic model used in this dissertation was chosen to be complex enough to account for the basic performance of an economy like that of the U.S. national economy but simple enough to be mathematically tractable. It appears that future research should be conducted in the direction of optimal control using more complex and comprehensive models. To correlate with the more complex models, multi-level control theory appears to provide a useful framework for research.

Further work should be conducted to improve the estimate of the econometric model parameters since the optimal control policy has been shown to be singificantly dependent on these parameters (see Sec. 3.3). Furthermore, in order to reasonably design any control system a mathematical model of the stochastic characteristics of the parameters is required. Such a noise model does not exist to the author's knowledge at this time.

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APPENDIX

APPENDIX

ECONOMIC DATA FOR ENDOGENOUS VARIABLES

	Quarter (t)	y(t)	c(t)	$I^d(t)$	$I^r(t)$	$I^i(t)$	r(t)
1954	1	100.7	62.7	9.8	4.8	-.7	3.22
	2	100.5	63.3	9.9	4.9	-.9	3.14
	3	101.8	64.2	10.0	5.3	-.6	3.14
	4	103.9	65.5	10.0	5.6	.2	3.13
1955	1	107.0	66.9	10.1	6.0	1.3	3.18
	2	108.9	68.3	10.8	6.4	1.7	3.22
	3	110.5	69.1	11.3	6.4	1.6	3.28
	4	111.6	70.0	12.0	6.3	1.9	3.31
1956	1	110.9	70.0	11.6	6.0	1.6	3.30
	2	111.4	70.1	11.8	5.7	1.2	3.44
	3	111.1	70.2	12.0	5.6	1.0	3.62
	4	112.6	71.2	11.9	5.5	1.1	3.90
1957	1	113.3	71.6	11.9	5.4	.5	4.00
	2	113.3	71.8	11.8	5.2	.6	4.04
	3	113.8	72.3	12.1	5.0	.8	4.36
	4	112.1	72.4	11.6	5.0	-.6	4.42
1958	1	109.4	71.4	10.8	5.0	-1.4	4.04
	2	109.9	71.9	10.4	4.9	-1.3	4.00
	3	112.7	73.0	10.1	5.2	.1	4.19
	4	115.4	73.8	10.3	5.7	1.1	4.40
1959	1	117.2	75.6	10.5	6.2	1.0	4.42
	2	120.0	76.8	11.0	6.4	2.2	4.59
	3	118.8	77.5	11.3	6.2	.1	4.74
	4	120.1	77.5	11.2	6.0	1.6	4.86

Economic Data for Endogenous Variables

	Quarter (t)	y(t)	c(t)	$I^d(t)$	$I^r(t)$	$I^i(t)$	r(t)
1960	1	122.6	78.4	11.6	5.9	2.4	4.87
	2	122.4	79.4	11.9	5.5	1.0	4.78
	3	121.8	79.1	11.8	5.2	.7	4.64
	4	121.0	79.1	11.8	5.1	-.6	4.64
1961	1	120.7	79.1	11.2	5.2	-.8	4.59
	2	123.2	80.1	11.2	5.3	.4	4.59
	3	125.4	81.0	11.5	5.4	1.0	4.72
	4	128.0	82.4	11.7	5.6	1.4	4.72
1962	1	129.9	83.4	11.9	5.8	1.6	4.69
	2	132.0	84.0	12.3	6.0	1.5	4.62
	3	133.4	85.0	12.8	6.0	1.3	4.63
	4	134.6	86.2	12.7	6.0	1.5	4.55
1963	1	135.3	87.1	12.5	6.0	1.1	4.48
	2	136.2	87.5	12.8	6.2	1.2	4.47
	3	138.4	88.8	13.2	6.2	1.4	4.50
	4	140.0	89.1	13.4	6.3	2.0	4.53

ECONOMIC DATA FOR EXOGENOUS VARIABLES

	Quarter (t)	t	G(t)	M(t)	L(t)	[S(t-1) - S(t-2)]
1954	1	0	24.1	144.1	187.6	.2
	2	1	23.3	143.4	187.8	-1.9
	3	2	22.9	145.1	191.6	.0
	4	3	22.6	146.3	194.5	1.0
1955	1	4	22.7	147.1	195.6	1.3
	2	5	21.7	147.7	197.3	2.1
	3	6	22.2	148.1	198.2	1.4
	4	7	22.0	147.8	199.2	1.8
1956	1	8	22.1	146.4	198.2	.7
	2	9	22.7	145.8	197.7	-.3
	3	10	22.4	143.8	196.1	.8
	4	11	23.1	142.7	195.9	-.1
1957	1	12	24.1	142.0	195.8	1.4
	2	13	23.1	140.8	196.0	1.3
	3	14	23.8	139.8	195.9	-.1
	4	15	23.6	138.6	195.9	.3
1958	1	16	23.6	136.6	194.3	-.4
	2	17	24.0	137.3	198.1	-1.8
	3	18	24.3	138.1	201.9	.3
	4	19	24.4	139.1	203.7	1.5
1959	1	20	23.4	139.9	206.1	1.7
	2	21	23.4	140.8	207.3	1.9
	3	22	23.7	141.6	208.1	1.6
	4	23	23.8	140.5	206.6	.8
1960	1	24	24.1	138.1	204.4	.0
	2	25	24.6	136.5	202.3	1.6
	3	26	25.0	135.7	202.7	1.3
	4	27	25.5	135.6	204.6	-.4

Economic Data for Exogenous Variables

	Quarter (t)	t	G(t)	M(t)	L(t)	[S(t-1) - S(t-2)]
1961	1	28	26.0	135.0	206.4	.5
	2	29	26.1	136.2	210.3	-.1
	3	30	26.6	136.8	213.5	1.2
	4	31	26.9	137.2	216.6	1.7
1962	1	32	27.2	138.1	219.4	2.2
	2	33	28.2	138.7	224.7	1.7
	3	34	28.1	138.1	226.2	2.2
	4	35	28.2	138.2	228.6	1.6
1963	1	36	28.6	138.8	233.5	1.0
	2	37	28.6	139.4	237.5	1.1
	3	38	28.9	140.8	241.3	.9
	4	39	29.2	141.2	245.2	1.8