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Einollah Pasha

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ON THE STRUCTURE OF GERM-FIELD MARKOV PROCESSES ON FINITE INTERVALS

By

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ABSTRACT

In a general case of Hilbert space valued Gaussian processes we derived a representation for the processes having Germ Field Markov Property (GFMP) [9] on finite intervals. Aslo we studied the case where the Germ Field is generated by a family of independent Gaussian random variables. In the case where the generating family is finite, these processes is said to be N-ple reciprocal processes and we gave an explicit representation of them in terms of N-ple Markov processes. In a special case these processes conincides with the reciprocal processes introduced by Jamison [5].

To my parents and my family

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INTRODUCTION

This work studies stochastic processes having Markov properties on the family of finite-intervals for Hilbert-space-valued Gaussian processes. In view of the example in [9], these processes need not have Markov property on semi infinite intervals. We show how these processes are related to processes with Markov property on semi-infinite intervals. This allows us to obtain a structural characterization of such processes. This characterization for example allows us to say when the solution of a stochastic differential equation having white noise input with linearly independent boundary conditions is Markov giving the main result of [14]. This is derived from the result on finite-order Markov property introduced here. Under the assumption of existence of $(N-1)$ quadratic mean derivatives, one can show that these are precisely the N-ple Markov processes introduced by Doob [2]. Our representations are motivated from the previous work in [7], [8] and have similar form. This work also constitute an alternative attack on reciprocal processes introduced by Jamison [5]. In fact, our work gives an explicit representation for what one may call "N-ple reciprocal processes" ($N = 1$ being Jamison's case). Thus this work extends the work in [5]. In addition, we also study q-variate case $(q \leq \infty)$. Here the techniques used are from ([7], [8]). This part of the work solves complete generality the question raised by Jamison [5]. In the stationary case, the result of Jamison [5] can be derived. Finally, we study infinite-order Markov processes. Here our work is in some

sense incomplete. However, this part of the study raises some questions about the relationship of these processes to so called T-positive processes. This part will be subject of continuing investigation.
For the convenience of the reader we now describe the results according to chapters.

In Chapter 1, after a brief review of conditional independence, Germ field Markov properly (GFMP) and Markov property [MP] operator valued stochastic processes are studied in detail and a representation for reciprocal processes is given, Theorem (1.5.14). In the special case of differentiable reciprocal processes it is shown that these are the only solution of a linear differential equation of certain type with boundary values, Theorem (1.5.21). Finally in this chapter the form of covariance functions of stationary real valued reciprocal Gaussian processes are obtained.

In Chapter 2, the notion of (Generalized) N-ple Markov processes and N-ple reciprocal processes are introduced and a representation for N-ple Markov processes in the general form of Hilbert-Schmidt operator (HSO)-valued processes is given, Theorems (2.1.4), (2.3.3). The relation between N-ple Markov processes and N-ple reciprocal processes is given in Theorems (2.2.5) and (2.3.7).

The notion of infinite order Markov processes is introduced in Chapter 3. Some properties of this kind of processes have been discussed. A representation for infinite order Markov processes and their T-positivity is of interest.

CHAPTER 1

MARKOV PROPERTY

Let (Ω, F, p) be a probability space and $X = \{X_t, t \in T\}$ be a stochastic process on (Ω, F, p), where T is a topological space. In order to give a definition of Markov property we need the idea of conditional independence and some of its basic consequences.

1.1. Conditional independence [6], [9].

Let F_1, F_2 and G be sub-σ-fields of F. We denote by $F_1 \amalg F_2 | G$ the conditional independence of F_1 and F_2 given G, and it means that $P(A_1 A_2 | G) = P(A_1|G)P(A_2|G)$ for all F_i measurable sets A_i, $i = 1,2$. We have the following basic results of conditional independence:

1.1.1 Lemma [9]. If $F_1 \amalg F_2 | G$ then;

(a) For every A satisfying $G \subseteq A \subseteq G \vee F_2$, we have

$$F_1 \amalg F_2 | A,$$

(b) For every $\mathcal{B}$ satisfying $\mathcal{B} \subseteq G \vee F_2$, we have $F_1 \amalg \mathcal{B} | G$.

1.2. Markov property.

Let A be a subset of T with closure $\overline{A}$ and boundary ∂A. Let :

$$F_A^- = \sigma\{X_t: \ t \in \bar{A}\} \qquad \text{"past"}$$

$$F_A^+ = \sigma\{X_t: \ t \notin \bar{A}\} \qquad \text{"future"}$$

$$\Gamma_A = \sigma\{X_t: \ t \in \partial A\} \qquad \text{"present".}$$

1.2.1. Definition. We say that $X = \{X_t, \ t \in T\}$ has Markov property (M.P.) on A if

$$F_A^- \perp\!\!\!\perp F_A^+ | \Gamma_A.$$

The classical Markov processes are the one with $T = R$ and having Markov property on the sets of the form $A_t = (-\infty, t]$ and the present is given by $\sigma\{X_t\}$, $t \in R$.

In the following we discuss a generalization of this definition.

1.3 Germ field Markov property [9].

As above let (Ω, F, p) be the probability space and $X = \{X_t, \ t \in T\}$ be the stochastic process with T a topological space. Let C be a closed subset of T and define

$$\sum_C = \bigcap_{\substack{c \subseteq 0 \\ 0 \text{ open}}} F_0. \quad \text{For an open set } 0, \ \sum_0 = F_0 = \sigma\{X_t, t \in 0\}.$$

1.3.1. Definition. We say that $X = \{X_t: \ t \in T\}$ has Germ field Markov property (GFMP) on $A \subset T$ if $\sum_{\bar{A}} \perp\!\!\!\perp \sum_{\overline{CA}} | \sum_{\partial A}$.

Germ field Markov property is a weaker condition than Markov property in the sense that if a process has Markov property on a set A then it has GFMP on A, but the converse may not be t e [9].

In this direction, a stochastic process may have GFMP(M.P.) on some particular subsets of T, such as open sets, but not on a larger class of subsets of T. The question is when can we deduce GFMP(M.P.) on some larger class by having GFMP(M.P.) on an smaller one? We have the following answer to this question.

1.3.2. Proposition [9]. (a) If X has GFMP(M.P.) on disjoint open sets O_i, $i = 1,\ldots,$ then it has GFMP(M.P.) on the union $\bigcup_{i=1}^{\infty} O_i$.

(b) If T is locally convex and X has GFMP(M.P.) on convex open sets then it has GFMP(M.P.) on all open sets.

(c) X has GFMP(M.P.) on all sets if it has GFMP(M.P.) on all open sets.

1.3.3. Remark. As a result of this proposition we get that the classical Markov processes have M.P. on all the sets. To see this by (1.3.2) it suffices to show that it has M.P. on all bounded open intervals in addition to the intervals of type $(-\infty,t]$, $t \in R$. Let $S < t$ and $\mathcal{A} = \sigma\{X_u:\ u \leq s\}$, $\mathcal{B} = \sigma\{X_u:\ u \geq t\}$, $G = \sigma\{X_u:\ s < u < t\}$. By the assumption and (1.1.1.) we have:

$$\mathcal{A} \amalg G|\sigma\{X_s,X_t\},\ \mathcal{B} \amalg G|\sigma\{X_s,X_t\} \text{ and } \mathcal{A} \amalg \mathcal{B}|\sigma\{X_s,X_t\}.$$

We want to show that $\mathcal{A} \vee \mathcal{B} \amalg G|\sigma\{X_s,X_t\}$. A typical generating element of $\mathcal{A} \vee \mathcal{B}$ is of the form $A \cap B$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$. So we want to show that:

$$P(A \cap B \cap C|X_s,X_t) = P(A \cap B|X_s,X_t)P(C|X_s,X_t),$$

for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, $C \in G$. But:

$$P(A \cap B \cap C|X_s,X_t) = E(I_{A\cap B}I_C|X_s,X_t)$$

$$= E[I_C E(I_{A\cap B}|X_s,X_t,I_C)|X_s,X_t)]$$

$$= E[I_C E(J_{A\cap B}|X_s,X_t)|X_s,X_t]$$

$$= E(I_{A\cap B}|X_s,X_t)E(I_C|X_s,X_t)$$

$$= P(A \cap B|X_s,X_t)P(C|X_s,X_t).$$

1.3.4. Remark. If we have M.P. on bounded open intervals, then we have M.P. on all bounded open sets and obviously vice-versa. This is the case because any bounded open set on the real line is a countable union of disjoint bounded open intervals.

Having M.P. on bounded open intervals, in general will not imply the Markov property on all open sets (and consequently, having a classical Markov process). But under some condition on P and the triviality of the tail σ-field of the process, NGOC & ROYER [11] proved that the Markov property on all bounded open intervals imply that X is a Markov process. The processes having Markov property on bounded intervals were studied in [5] under the name "Reciprocal processes". In the next section we consider some representation for these processes under very general setting.

1.4. Operator-valued processes.

In [5] Jamison studied reciprocal processes taking values in R and asked whether his result are extendable to the case of processes taking values in R^n at least in Gaussian case. Given a Gaussian process $\{X_t,\ t \in T\}$ taking values in R^n, we can consider the following (finite dimension) operator-valued process:

$$\underline{X}_t(a) = X_t \cdot a \quad a \in R^n,\ t \in T,$$

where $\cdot$ is the inner product in R^n. Here for each $t \in T$, $\underline{X}_t$: $R^n \to L^2(\Omega,F,p)$ with (Ω,F,p) being the probability space on which the original process was defined. In case of a Gaussian process X_t taking values in a Hilbert space, it is well known that $E\|X_t\|_H^2 < \infty$ for each $t \in T$. Thus the operator-valued process $\underline{X}_t$ associated to X_t given by $\underline{X}_t(h) = (X_t,h)_H$, $h \in H$ has the additional property:

$$\sum_1^\infty E|\underline{X}_t(e_i)|^2 = E\sum_1^\infty |(X_t,e_i)|^2 = E\|X_t\|^2 < \infty,$$

where $\{e_i\}$ is an orthonormal basis in H. Therefore $\underline{X}_t$ is a Hilbert-Schmidt operator on H into $L^2(\Omega,F,p)$. As the problems studied here are second-order depending on the Hilbertian properties of H and $L_2(\Omega,F,p)$ we study them as problem involving two Hilbert spaces. Motivated from this we define Hilbert-Schmidt operator-valued processes.

Let H and K be two separable Hilbert spaces with inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_K$ and norms $\|\cdot\|_H$ and $\|\cdot\|_K$; respectively. The set of all linear bounded operator on H into K is denoted by B(H,K), and the dual spaces of H and K is denoted by H^* and K^*; respectively. Before giving a definition of Hilbert-Schmidt operators we need the following lemma:

1.4.1. Lemma ([1], p. 256): Let H and K be two separable Hilbert spaces and A in B(H,K). If the series

$$\sum_{n=1}^\infty \|Ae_n\|_K^2$$

converges for an orthogonal basis $\{e_n\}$ in H, then

$$\sum_{n=1}^{\infty} \|Ae_n\|_K^2 = \sum_{n=1}^{\infty} \|Ae_n'\|^2 = \sum_{n,m=1}^{\infty} |< Ae_n, f_m^* >|^2 = \sum_{n=1}^{\infty} \|A^* f_n^*\|^2$$

$$= \sum_{n,m} |(Ae_n, f_m)|^2,$$

no matter what orthonormal bases $\{e_n'\}$ of H, $\{f_n\}$ in K and $\{f_n^*\}$ of K^* are chosen.

Now we are in a position to give the definition of a Hilbert-Schmidt operator:

1.4.2 Definition. An element A of B(H,K) is called a Hilbert-Schmidt-operator (HSO) if

$$\|A\|^2 = \sum_{n=1}^{\infty} \|Ae_n\|_K^2$$

converges for at least one orthonomal basis $\{e_n\}$ of H. The set of all HSO,s on H into K is denoted by HS(H,K) and it can be considered as a Hilbert space with the inner product given by:

$$(A,B)_{HS} = \text{tr } B^*A = \sum_{i,j} |(B^*Ae_i, e_i)|, \text{ A,B, in HS(H,K)},$$

where $\{e_i\}$ is an orthonormal basis in H and B^* is the conjugate of B.

1.4.3. Remark. The space HS(H,K) is a module of operators over B(H,H), and in this view a subspace M of HS(H,K) is a subset of HS(H,K) which is a left module of operators over B(H,H); that is,

(i) M is a (sub) Hilbert space,

(ii) for each B in $B(H,H)$ and A in $\mathcal{M}$, AB is in $\mathcal{M}$.

For the subspace $\mathcal{M}$ of $HS(H,K)$ we denote by M the subspace of K generated by the images of the elements of $\mathcal{M}$:

$$M = \overline{sp}\,\{A(h): \; h \in H, \; A \in \mathcal{M}\}.$$

Let M be a subspace of K, and A in $HS(H,K)$, consider the following operator B on H into K given by:

$$B(h) = P_M A(h) \quad h \text{ in } H,$$

where $P_M^{\cdot}$ is the orthogonal projection onto M. For an orthonormal basis $\{e_i\}$ of H and the properties of projections we have:

$$\sum_i \|Be_i\|^2 = \sum_i \|P_M Ae_i\|^2 \leq \sum_i \|P_M^{\cdot}\|^2 \|Ae_i\|^2$$

$$\leq \sum \|Ae_i\|^2 < \infty;$$

that is, B is an HSO on H into M. Thus in this way we have associated to each A in $HS(H,K)$ an element B in $HS(H,M)$, more precisely we have the following map P:

$$P: \; HS(H,K) \longrightarrow HS(H,M)$$

given by

$$P(A)(h) = P_M A(h), \text{ for each } A \in HS(H,K), \; h \in H.$$

We observe that P has the properties of a projection operator: P is linear and $P^2 = P$.

For the linearity of P, let $A,B \in HS(H,K)$ and $u, v \in B(H,H)$, then

$$P(Au + Bv)(h) = P_M(Au + Bv)(h) \qquad (h \in H)$$

$$= P_M\, Au(h) + Bv(h)$$

$$= P_M\, Au(h) + P_M\, Bv(h)$$

$$= P(A)(u(h) + P(B)(v(h))$$

$$= P(A)u(h) + P(B)v(h)$$

so $P(Au + Bv) = P(A)u + P(B)v$.

To see the other property, let $A \in HS(H,K)$, we have:

$$P^2(A)(h) = P(P(S))(h) \qquad (h \in H)$$

$$= P_M{}^{P(A)(h)} = P_M{}^{P_M A(h)} = P_M{}^{A(h)} = P(A)(h).$$

We note that for A in $HS(H,M)$, $P(A)(h) = P_M{}^{A(h)} = A(h)$, for each h in H; thus $P(A) = A$.

Motivated from these properties we have the following definition:

1.4.4. Definition (Payen, [13]). Let $\mathcal{M}$ be a subspace of $HS(H,K)$ and M be the subspace of K generated by the images of the elements of $\mathcal{M}$. For A in $HS(H,K)$ the projection $(A|\mathcal{M})$ of A onto $\mathcal{M}$ is the HSO in $HS(H,M)$ given by:

$$(A|\mathcal{M})(h) = P_M{}^{A(h)} \quad \text{for each } h \text{ in } H.$$

If $N \subset HS(H,K)$, $(N|M) = \{(A|M): A \in N\}$.

The following is a collection of basic properties of the projections, indeed we show that it is an orthogonal projection. Let us first give a definition of orthogonality.

1.4.5. Definition. Let A and B be in HS(H,K). We say A and B are orthogonal $(A \perp B)$ if

$$(A,B)_{HS} = \operatorname{tr} B^*A = 0.$$

We say two subsets M and N of HS(H,K) are orthogonal $(M \perp N)$ if $A \perp B$ for all A in M and B in N. From the definition we note that $A \perp B$ if and only if $B^*A = 0$.

1.4.6. Properties of projections: Let M be a subspace of HS(H,K), then we have:

(a) $(Au|M) = (A|M)u \quad A \in HS(H,K),\ u \in B(H,H)$,

(b) $(A|M) = A \quad A \in M$,

(c) If N is a subspace of HS(H,K) containing M, then $((A|M)|N) = ((A|N)|M) = (A|M),\ A \in HS(H,K)$,

(d) If N and M are two closed orthogonal subspace of HS(H,K) then, $(A|M \oplus N) = (A|M) + (A|N)$

and consequently

$$(A|M \ominus M') = (A|M)-(A|M')$$

for M' a closed subspace of M,

(e) $A- (A|M) \perp M$.

Proof. (a)-(e) are direct consequence of the definition and the properties of the orthogonal projections on the subspaces of K. We give a precise proof for (e).

Let $C \in \mathcal{M}$ and $h \in H$, then:

$$C^*(A-B)(h) = C^*(A(h)-B(h))$$
$$= C^*(A(h)-P_M^{A(h)})$$

where M as usual is the subspace of K generated by the images of the elements of $\mathcal{M}$. In order to show that $C^*(A(h)-P_M^{A(h)})$ as an element of H is 0 we show that it is orthogonal to all the elements x of H:

$$(x,C^*(A(h)-P_M^{A(h)})_H = (Cx,\ A(h)-P_M^{A(h)})_K,$$

but $A(h)-P_M^{A(h)}$ is orthogonal to M, in particular to Cx, thus

$$(x,C^*(A(h)-P_M^{A(h)}))_H = 0 \quad \text{for all } x \text{ in } H,$$

This implies that $C^*(A-B)(h) = 0$, for all h in H; therefore $A-B \perp C$.

The interesting subspaces are the one generated by a family of the operators in HS(H,K). Let $\{X_t\}_{t\in I}$, (I an index set), be a family of HSO's. Denote by $\mathcal{M}_X$ the closure of the set $\{\sum_{t\in J} X_tB_t$, J a finite subset of I, $B_t \in B(H,H)\}$ under the norm $\|\cdot\|_{HS}$. Let M_X be the subspace of K generated by the images of the elements of the family $\{X_t\}_{t\in I}$. Now we have the following:

1.4.7. Theorem [11]: $HS(H,M_X) = \mathcal{M}_X$, where M_X and $\mathcal{M}_X$ are as above.

Proof [11]. Let $Z \in HS(H,M_X)$ and $\{e_i\}$ be a complete orthonormal basis in H. Since $\sum\|Ze_i\|^2 < \infty$, for a given $\epsilon > 0$ there exists an integer N such that:

$$\sum_{N+1}^{\infty} \|Ze_i\|^2 < \frac{\epsilon}{2} . \tag{1.4.8}$$

Let $Z_N = Z P_N$, where P_N is the projection onto the subspace of H generated by $e_1,\dots,e_N$. Clearly $Z_N \in HS(H,M_X)$, therefore [by (2) page 335 [13]] there are $A_i \in B(H,H), i = 1,\dots,k$, such that

$$\sum_{i=1}^{N} \|(\sum_{j=1}^{k} X_jA_j)e_i - Z_N e_i\|^2 < \frac{\epsilon}{2} . \tag{1.4.9}$$

Let $B_j = A_j P_N$, then from (1.4.9) we get:

$$\sum_{i=1}^{N} \|(\sum_{j=1}^{k} X_jB_j) e_i - Z_N e_i\|^2 < \frac{\epsilon}{2} , \tag{1.4.10}$$

Thus by (1.4.8) and (1.4.10) we get:

$$\|\sum_{j=1}^{k} X_jB_j - Z\|^2 = \sum_{i=1}^{\infty} \|(\sum_{j=1}^{k} X_jB_j)e_i - Ze_i\|^2$$

$$= \sum_{i=1}^{N} \|(\sum_{j=1}^{k} X_jB_j)e_i - Ze_i\|^2 + \sum_{i=N+1}^{\infty} \|(\sum_{j=1}^{k} X_jB_j)e_i - Ze_i\|^2$$

$$= \sum_{i=1}^{N} \|(\sum_{j=1}^{k} X_jB_j)e_i - Z_Ne_i\|^2 + \sum_{i=N+1}^{\infty} \|Ze_i\|^2$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore $HS(H,M_X) \subseteq M_X$. It is clear that $M_X \subset HS(H,M_X)$, thus $HS(H,M_X) = M_X$, and the proof is complete.

1.4.11. Remark. In particular case of a finite family $\{X_1,\dots X_n\}$ of the elements of $HS(H,K)$ we may wish to have:

$$(Y|M_X) = \sum_{i=1}^{n} X_i B_i \quad \text{for some } B_i \text{ in } B(H,H), i = 1,\dots,n.$$

But, in general, this is not true. All we can have is the following:

$$(Y|M_X) = \sum_{i=1}^{n} \mathrm{ext}(X_i B_i), \ B_i \text{ some linear, not necessarily bounded,}$$

mapping in H.

where $\mathrm{ext}(A) = A^{**}$ is the extention of A. For example let M be generated by $\{X_1, X_2\}$, where $X_1 \perp X_2$, then by definition of the projections onto M we have

$$(Y|M) = P^Y_{\overline{R}\{X_1,X_2\}} = P^Y_{\overline{R}\{X_1\} \oplus \overline{R}\{Y_1-(X_1|M_{X_2})\}}$$

$$= P^Y_{\overline{R}\{X_1\}} + P^Y_{\overline{R}(X_2)}$$

where $\overline{R}(X)$ is the closure of the range of X in K. By Lemma 1.4 [11] we have:

$$(Y|M) = \mathrm{ext}(X_1C_1) + \mathrm{ext}(X_2C_2),$$

for some linear map C_1, C_2 in H.

In the sequel we make an assumption similar to [2.9 [11]] in order to have bounded B_i's.

1.4.12. Definition. Let T be a Borel subset of R (usually $T = R$ or $T = [a,b]$ for some real numbers $a < b$) and for each t in T let X_t be in HS(H,K), then $\{X_t, t \in T\}$ is called an HSO-valued stochastic process. In the special case of $K = L^2(\Omega,F,P)$ and $H = R^n$, $\{X_t, t \in T\}$ is a second order multivariate stochastic process.

Associated with the process $\{X_t, t \in T\}$ are defined:

$\Gamma(s,t) = X_t^* X_s$, the covarianse of the process,

$M_{X_t} = G\{X_t\}$ over $B(H,H)$, where G denotes the closed subspace generated by X_t.

$M_t^-(x) = G\{X_\tau: \tau \leq t\}$ over $B(H,H)$ t in T,

$M_{u,v}^+(x) = G\{X_\tau: \tau \notin (u,v)\}$ over $B(H,H)$, $u < v$ in T,

$M_\infty^X = \bigcap_{u<v} M_{u,v}^+(X)$, $M_{-\infty}^X = \bigcap_t M_t^-(X)$,

$M_t^+(X) = G\{X_\tau: \tau \geq t\}$ over $B(H,H)$ and $M_{+\infty}^X = \bigcap_t M_t^+(X)$.

For simplicity we will write "G{...}" instead of "G{...} over B(H,H)", and M_t^-, $M_{u,\nu}^+, \ldots$, for $M_t^-(X)$, $M_{u,\nu}^+(X), \ldots$, unless otherwise stated.

Having the remark (1.4.11) in mind, in the sequel we make the following assumption:

1.4.13. Assumption. $R(\Gamma(s,t)) \subseteq R(\Gamma(t,t))$, where $\Gamma(s,t)$ is the covariance of the process given by $\Gamma(s,t) = X_t^* X_s$.

Under this assumption we will have:

$$(X_s | G\{X_{t_1}, \ldots X_{t_n}\}) = \sum_{i=1}^{n} X_{t_i} B_i, B_i \in B(H,H).$$

To see this let us first prove it for $n = 2$:

$$(X_s | G\{X_{t_1}, X_{t_2}\}) = P_{\overline{R}\{X_{t_1}, X_{t_2}\}}^{X_s}$$

$$= P_{\overline{R}\{X_{t_1}\} \oplus \overline{R}\{X_2 - (X_2 | G\{X_1\})\}}^{X_s}$$

$$= P^{X_s}_{\overline{R\{X_1\}}} + P^{X_s}_{\overline{R\{X_2-(X_2|G\{X_1\})}}$$

$$= (X_s|G\{X_1\}) + (X_s|G\{X_2-(X_2|G\{X_1\})\}).$$

By [1.4. [11]] and assumption (1.4.13) we get:

$$(X_s|G\{X_1\}) = X_1A_1$$

$$(X_s|G\{X_2-(X_2|G\{X_1\})\}) = (X_2-(X_2|G\{X_1\}))A_2$$

$$(X_2|G\{X_1\}) = X_1A_3,$$

for some A_1, A_2, A_3 in $B(H,H)$. Therefore:

$$(X_s|G\{X_t, X_{t_2}\}) = X_1A_1 + (X_2-X_1A_3)A_2$$

$$= X_1(A_1-A_3A_2) + X_2A_2.$$

For $n > 2$ we note that:

$$(X_s|G\{X_{t_1},\ldots,X_{t_n}\}) = (X_s|G\{X_{t_1},\ldots,X_{t_{n-1}}\}) +$$

$$(X_s|G\{X_n-(X_n|G\{X_{t_1},\ldots,X_{t_{n-1}}\})\}),$$

now by induction we get the result.

1.5. Reciprocal processes.

As stated before, Jamison [5] introduced the notion of reciprocal processes which were called Markov-like processes by Slepian [15]. In the following we give a representation of an HSO-valued reciprocal processes in terms of HSO-valued Markov processes and under further conditions in terms of HSO-valued martingales. The notations are the same as in (1.4.4) and 1.4.12).

1.5.1. Definition. ([11]) An HSO-valued process $\{X_t, t \in T\}$ is called a

(i) Martingale, if $(X_t|M_s^-) = X_s$ for all $t \geq s$ in T,

(ii) Markov process, if $(X_t|M_s^-) = (X_t|M_{X_s})$ for all $t \geq s$ in T,

(iii) reciprocal process, if $(X_t|M_{u,v}^+) = (X_t|G\{X_u, X_v\})$, $u \leq t \leq \nu$.

It is clear that (i) implies (ii). In fact, under some conditions there is a very close tie between the martingales and markov processes. In [11] it is shown that if $\Gamma^-(s,s)\Gamma(s,t)$ is one-to-one for all $s \leq t$, then

$$X_t = U_t \, \Phi(t)$$

where U_t is a martingale and $\Phi(t)$ is in $B(H,H)$, moreover this representation is unique and it is a necessary and sufficient condition for $\{X_t, t \in T\}$ to be a markov process.

Before discussing the relations between (ii) and (iii) of definition (1.5.1) let us give some expected elementary properties of HSO-valued Markov processes.

1.5.2. Theorem: Let $X = \{X_t, t \in T\}$ be an HSO-valued stochastic process, then:

(a) X is Markov if and only if for each $N \subset M_t^+$, $t \geq s$

$$(N|M_s^-) = (N|M_{X_s})$$

(b) If X is Markov, then $(X_t|M_v^+) = (X_t|M_{X_v})$, $t \leq v$.

Proof. (a) is obvious from the definition of a Markov process.

(b) By definition of the projection for $t \leq v$ we have:

$$(X_t|M_v^+) = P^{X_t}_{\overline{R}\{X_u:\ u \geq v\}} = P^{X_t}_{\overline{R}\{X_v\} \oplus [\overline{R}(X_u:\ u \geq v) \ominus \overline{R}(X_v)]}$$

$$= P^{X_t}_{\overline{R}\{X_v\}} + P^{X_t}_{\overline{R}\{X_u:\ u \geq v\} \ominus \overline{R}\{X_v\}}.$$

Let $X_s \perp X_v$ for some $s > v$, then by Markov property of X we have

(1.5.3) $$(X_s|M_v^-) = (X_s|M_{X_v}) = P^{X_s}_{\overline{R}\{X_v\}} = 0.$$

Since $\overline{R}\{X_t\} \subset \overline{R}\{X_u:\ u \leq v\}$ and (1.5.3) we get $P^{X_s}_{\overline{R}\{X_t\}} = 0$; that is X_s is orthogonal to X_t. So X_s is orthogonal to all the generator elements of $\overline{R}\{X_u:\ u \geq v\} \ominus \overline{R}\{X_v\}$, thus $P^{X_t}_{\overline{R}\{X_u:\ u \geq v\} \ominus \overline{R}\{X_v\}} = 0$, and we get

$$(X_t|M_v^+) = (X_t|M_{X_v}).$$

Now we return to the definition (1.5.1) and prove (ii) implies (iii).

1.5.4 Theorem: If $\{X_t,\ t \in T\}$ is a Markov process, then it has reciprocal property.

Proof: Let $u < t < v$ and $M_v^+ = G\{X_t:\ t \geq v\}$, then we have:

$$M_{u,v}^+ = M_u^- \oplus (M_{u,v}^+ \ominus M_u^-)$$

$$= M_v^+ \oplus (M_{u,v}^+ \ominus M_v^+)$$

so we have:

$$(X_t|M^+_{u,v}) = (X_t|M^-_u \oplus (M^+_{u,v} \ominus M^-_u))$$

(1.5.5)
$$= (X_t|M^-_u) + (X_t|M^+_{u,v} \ominus M^-_u)$$

$$= (X_t|M_{X_u}) + (X_t|M^+_{u,v} \ominus M^-_u).$$

On the other hand we have:

$$(X_t|M^+_{u,v}) = (X_t|M^+_v) + (X_t|M^+_{u,v} \ominus M^+_v)$$

$$= (X_t|M_{X_v}) + (X_t|M^+_{u,v} \ominus M^+_v).$$

we note that $M^+_{u,v} \ominus M^+_v \subset M^-_u$ and $M^+_{u,v} \ominus M^-_u \subset M^+_v$. Also we have (comparing the two values of $(X_t|M^+_{u,}$ $))(X_t|M_{X_u})-(X_t|M_X) =$ $(X_t|M^+_{u,v} \ominus M^+_v)-(X_t|M^+_{u,v} \ominus M^-_u)$. But by lemma 1.4 of [11] there exists $A,B \in B(H,H)$ such that

(1.5.6)
$$(X_t|M_{X_u}) = X_u A$$

$$(X_t|M_{X_v}) = X_v B.$$

so we get

$$X_uA - X_vB = (X_t|M^+_{u,v} \ominus M^+_v)-(X_t|M^+_{u,v} \ominus M^-_u).$$

Now by projecting the above equality on M^+_v we get:

$$(X_u|M^+_v)A-X_vB = -(X_t|M^+_{u,v} \ominus M^-_u)$$

[the projection of the first term on M_v^+ is 0 and since $M_{u,v}^+ \ominus M_u^- \subset M_v^+$ the second term will remain the same].
Again using Lemma (1.4) of [11] we get:

(1.5.7) $$X_v(CA-B) = -[X_t|M_{u,v}^+ \ominus M_u^-]$$

Now, by (1.5.5), (1.5.6) and (1.5.7) we get

(1.5.8) $$(X_t|M_{u,v}^+) = X_uA - X_v(CA-B).$$

Therefore by (1.4.4)(c) and (1.5.6) we get

$$(X_t|G(X_u,X_v)) = ((X_t|M_{u,v}^+|G\{X_u,X_v\})$$

$$= (X_uA-X_v((A-B)|G\{X_u,X_v\})$$

$$= X_uA-X_v(CA-B),$$

and by (1.5.8) this is equal to $(X_t|M_{u,v}^+)$. Therefore $(X_t|M_{u,v}^+) = (X_t|G\{X_u,X_v\})$, $u < t < v$. This completes the proof.

In general (iii) of (1.5.1) does not imply (ii):

1.5.9. EXAMPLE: Let $T = R$ and

$$X_t = \begin{cases} X & \text{if } t = 0 \\ Y & \text{if } t \neq 0 \end{cases}$$

where X and Y are two HSO's in $HS(H,K)$ such that $X(H) \perp Y(H)$ and none of them are constant, i.e. $X(H) \neq 0$ and $Y(H) \neq 0$. Then X_t is reciprocal but not Markov.

Now the question is under what conditions reciprocal property will imply Markov property. As it is stated in Remark [1.3.4] ROYER and NGOC [12] studied this question and gave the following answer:

1.5.10 Theorem (ROYER and NGOC [12]). Let $T = R$ and $X = \{X_t, t \in T\}$ be an E-valued (E any state space) stochastic process such that:

(i) X has Markov property on each open bounded intervals (a,b),

(ii) either $\bigcap_t \sigma\{X_u, u \geq t\} = \{\emptyset,\Omega\}$ or $\bigcap_t \sigma\{X_u: u \leq t\} = \{\emptyset,\Omega\}$,

(iii) for each $t' < t < t''$ in T there are three finite measures $\nu_{t'},\nu_t,\nu_{t''}$ such that the joint distribution $\mu_{t',t,t''}$ of $X_{t'},X_t,X_{t''}$ is absolutely continous with respect to the direct product $\nu_{t'} \otimes \nu_t \otimes \nu_{t''}$ of $\nu_{t'},\nu_t$ and $\nu_{t''}$, then X is a Markov process.

1.5.11. Remark (i). In the case of non degenerated Gaussian processes condition (iii) of (1.5.10) is valid automatically, i suffices to take $\nu_{t'},\nu_t$ and $\nu_{t''}$ to be the distribution of $X_{t'},X_t$ and $X_{t''}$; respectively. (ii) In case of $T = [a,b]$, (1.5.10)(ii) implies that either X_a or X_b is constant, in this case we have the result of (1.5.10) even without the condition (1.5.10)(iii) [6].

A theorem similar to (1.5.10) for the general case of HSO-valued processes is of interest. But in the following we consider non degenerated Hilbert-space valued Gaussian processes.

1.5.12 Theorem: Let $X = \{X_t, t \in R\}$ be a non degenerated Hilbert space valued reciprocal Gaussian processes and either $M_{-\infty}$ or $M_{+\infty} = \{0\}$, then X is a Markov process.

Proof. Assume $M_{+\infty} = \{0\}$. A similar proof can be given if $M_{-\infty} = \{0\}$. Let $s < t < n$, where s,t in T and n is an integer.

Now,

$$M_s^- \vee G\{X_n, X_{n+1}, \ldots\} \subset M_{s,n}^+ \quad \text{implies}$$

$$(X_t | M_s^- \vee G\{X_n, X_{n+1}, \ldots\}) = ((X_t | M_{s,n}^+) | M_s^- \vee G\{X_n, X_{n+1}, \ldots).$$

By reciprocal property this equals $((X_t | G\{X_s, X_n\}) | M_s^- \vee G\{X_n, X_{n+1}, \ldots\})$, and by (1.4.6)(b) this is equal to $(X_t | G\{X_s, X_n\})$. Again using reciprocal property we get:

$$(X_t | G\{X_s, X_n\}) = (X_t | G\{X_s\} \vee G\{X_n, X_{n+1}, \ldots\}).$$

Therefore we have the following equality:

(1.5.13) $\quad (X_t | M_s^- \vee G\{X_n, X_{n+1}, \ldots\}) = (X_t | M_{X_s} \vee G\{X_n, X_{n+1}, \ldots\}).$

Now by the assumption on the process and [12] we get

$$G\{X_s\} \vee G\{X_n, X_{n+1}, \ldots\} \to G\{X_s\}$$

as $n \to \infty$. Therefore by the properties of the projections we get:

$$(X_t | M_s^- \vee G\{X_n, X_{n+1}, \ldots\}) \to (X_t | X_s) \quad \text{as} \quad n \to \infty$$

Now by projecting both side of the above equality on M_s^- we get

$$(X_t | M_s^-) = (X_t | X_s),$$

and this completes the proof.

1.5.14. Remark. In the case of finite interval $T = [a,b], M_{+\infty} = \{0\}$ is equivalent to $X_b = 0$, and in this case the Theorem can be stated even for the general case of HSO-valued processes and proved very easily:

$$(X_t|M_s^-) = (X_t|M_s^- \vee G\{X_b\}),$$

by reciprocal property this equals to $(X_t|G\{X_s\} \vee G\{X_b\})$ which is equal to $(X_t|G\{X_s\})$.

What follows is the main theorem of this chapter, it gives a representation of reciprocal processes. We recall that we assume (1.4.13).

1.5.14. Representation theorem. A non degenerate Gaussian Hilbert-space-valued process $\{X_t, t \in T\}$ is a reciprocal process if and only if it has the following representation:

$$X_t = Y_t + Z_t$$

where

(i) $\{Y_t, t \in T\}$ is a Markov process and orthogonal to $\{Z_t\}$ with $M_\infty^{(Y)} = \{0\}$,

(ii) Z_t is in M_∞ for all t in T.

Moreover this representation is unique in the sense that if $X_t = Y_t^{(1)} + Z_t^{(1)}$, where $Y_t^{(1)}$ and $Z_t^{(1)}$ satisfying (i) and (ii) instead of Y_t and Z_t; respectively, then $Y_t^{(1)} = Y_t$ and $Z_t^{(1)} = Z_t$.

Proof: Let $Z_t = (X_t|M_\infty)$ and $Y_t = X_t - Z_t$. It is clear that Z_t is in M_∞ and Y_t is orthogonal to Z_t. All we have to show is the Markov property of Y_t. To show this we prove that Y_t has reciprocal property and $M_\infty^Y = \{0\}$, then by Theorem (1.5.12) we get the Markov property of Y_t. We note that:

$$M^+_{u,v}(Y) = G\{Y_t: \ t \notin (u,v)\} = G\{X_t - Z_t: \ t \notin (u,v)\}$$

$$= G\{X_t: \ t \notin (u,v)\} \ominus G\{Z_t: \ t \notin (u,v)\}$$

$$= M^+_{u,v} \ominus M^+_{u,v}(Z)$$

This equality gives that

$$M^+_{u,v} = M^+_{u,v}(Y) \oplus M^+_{u,v}(Z)$$

which implies that

$$M_\infty = M^Y_\infty \oplus M_\infty,$$

therefore $M^Y_\infty = \{0\}$.

To see the reciprocal property of $\{Y_t\}$, let $u < v$ and $t \notin (u,v)$, then

$$(Y_t | M^+_{u,v}(Y)) = (Y_t | M^+_{u,v})$$

$$= (X_t | M^+_{u,v}) - Z_t,$$

and by reciprocal property of $\{X_t\}$ we get:

$$(Y_t | M^+_{u,v}(Y)) = (X_t | G\{X_u, X_v\}) - Z_t$$

$$= X_u A + X_v B - Z_t$$

for some A,B in $B(H,H)$ (A,B depend on u,v,t).

Now by substituting for X_u and X_v in terms of Y and Z we get:

$$X_uA + X_vB = Y_uA + Y_vB + Z_uA + Z_vB$$

$$= Y_uA + Y_vB + (X_uA + X_vB|M_\infty)$$

$$= Y_uA + Y_vB + ((X_t|M^+_{u,v})|M_\infty)$$

$$= Y_uA + Y_vB + (X_t|M_\infty).$$

so we get

$$(Y_t|M^+_{u,v}(Y)) = Y_uA + Y_vB + (X_t|M_\infty) - (X_t|M_\infty)$$

$$= Y_uA + Y_vB.$$

Therefore $\{Y_t\}$ is a reciprocal process with trivial tails, so it is a Markov process.

Conversely: Let X_t be represented in the form given in the Theorem and let $t \in (u,v)$, for $u < v$ in T, then we have:

$$(X_t|M^+_{u,v}) = (Y_t + (X_t|M_\infty)|M^+_{u,v})$$

$$= (Y_t|M^+_{u,v}) + (X_t|M_\infty)$$

$$= (Y_t|M^+_{u,v}(Y) \oplus M^+_{u,v}(Z)\ ) + (X_t|M_\infty).$$

Since $Y_t \perp M^+_{u,v}(Z)$ we get

$$(X_t|M^+_{u,v}) = (Y_t|M^+_{u,v}(Y)) + (X_t|M_\infty).$$

By reciprocal property of $\{Y_t\}$ we get:

$$(X_t|M^+_{u,v}) = (Y_t|G\{Y_u,Y_v\}) + (X_t|M_\infty)$$

$$(*) \qquad \begin{aligned} &= Y_u A + Y_v B + (X_t | M_\infty) \\ &= X_u A + X_v B - (X_u A + X_v B | M_\infty) + (X_t | M_\infty). \end{aligned}$$

(A,B are in B(H,H) and depend on t,u,v).

On the other hand we have:

$$\begin{aligned} (X_t | M^+_{u,v}) &= (X_t | (M^+_{u,v} \ominus M_\infty) \oplus M_\infty) \\ \text{(A)} \qquad &= (X_t | M^+_{u,v} \ominus M_\infty) + (X_t | M_\infty). \end{aligned}$$

Comparing the two values of $(X_t | M^+_{u,v})$ and noting that $X_u A + X_v B - (X_u A + X_v B | M_\infty)$ is orthogonal to M_∞ and the uniqueness of the representation of the form (*) we get

$$(X_t | M^+_{u,v} \ominus M_\infty) = X_u A + X_v B - (X_u A + X_v B | M_\infty).$$

This implies that

$$(X_t | M^+_{u,v}) = X_u A + X_v B,$$

i.e. $\{X_t\}$ is reciprocal.

Uniqueness: Since $M^Y_\infty = \{0\}$ and $Y \perp M^Z_\infty$ we get

$$M_\infty = M^Z_\infty = M^{Z(1)}_\infty$$

so

$$(X_t | M_\infty) = (Y_t + Z_t | M_\infty) = Z_t$$

$$(X_t | M_\infty) = (Y^{(1)}_t + Z^{(1)}_t | M_\infty) = Z^{(1)}_t$$

therefore $Z_t = Z^{(1)}_t$ and $Y_t = Y^{(1)}_t$ for all t in T.

In the case of a finite interval $T = [a,b]$ we observe that in the above argument:

$$M_\infty = G\{X_a, X_b\},$$

so

$$Z_t = (X_t | M_\infty)$$

$$= A_a A(t) t X_b B(t) \quad A(t), B(t) \text{ in } B(H,H)$$

Thus we get

$$X_t = Y_t + X_a A(t) + X_b B(t), \; t \in T.$$

In the following we consider a special choice of H and derive a representation for the vector valued stochastic processes.

1.5.16. Special case. Let H be the set of real or complex numbers, and A be a linear, bounded operator on H into a Hilbert space K. For each r in H we have $A(r) = rA(1)$, this will lead us to the fact that we can identify K with $HS(H,K)$ in the sense that there is a one-to-one norm preserving correspondence φ on K onto $HS(H,K)$. For each k in K, φ at k is given by φ_k where

$$\varphi_k(r) = rk \quad r \text{ in } H.$$

We note that if $\varphi(k) = \varphi(k')$, then for each r in H we have $rk = rk'$, which gives $k = k'$, so φ is one-to-one. The linearity of φ is obvious by its definition. Also φ is onto and its inverse is given by

$$\varphi^{-1}(A) = A(1) \quad \text{for } A \text{ in } HS(H,K)$$

Finally φ is norm preserving

$$\|\varphi_u\| = \sup_{|r|\leq 1} \|\varphi_u(r)\| = \sup_{|r|\leq 1} \|rk\| = \|k\|.$$

Now let K be a q-dimensional Gaussian space, and $X = \{X_t;\ t \in T\}$ be a q-variate Gassian stochastic process, then by (1.5.15) and (1.5.16) we have the following:

1.5.17. Corollary. Let $\{X_t,\ t \in T\}$ be a q-variate Gaussian stochastic process, then it has reciprocal property if and only if it has the following representation:

$$X_t = Y_t + Z_t \qquad t \in T$$

where Y_t is a q-variate Markov process with trivial tails and Z_t is independent of Y_t and measurable with respect to the tail of X_t. In the case of $T = [a,b]$ we have:

$$X_t = Y_t + A(t)X_a + B(t)X_b$$

where Y_t is the same as before and $A(t)$, $B(t)$ are some $q \times q$ matrix.

In this representation if we know that Y_t is continous in quadratic mean and $R(t,s) = E\, Y_t Y_s^*$ is nonsingular for all s,t in T, then by [11] we have the following representation:

$$Y_t = \Phi(t)\underline{U}(t)$$

where $\Phi(t)$ is a nonsingular $q \times q$ matrix and $\underline{U}(t)$ is a q-variate martingale. The two conditions on $\{Y_t,\ t \in T\}$ will be satisfied if we assume that $\{X_t,\ t \in T\}$ is continous in quadratic mean, and $Y_t \neq 0$ for all t in T. By Corrollary (1.5.17) the cotinuity of X_t implies the continuity of $\{Y_t\}$ and $\{Z_t\}$. It remains to show that $R(t,s)$ is nonsingular. By Markov property of $\{Y_t\}$ and for each $s \leq t' \leq t$ we have

$$R(t,s) = R(t,t')R^{-1}(t',t')R(t',s). \tag{1.5.18}$$

Let $s = s_0 \quad s_1 < \ldots < s_k = t$ be such that $|s_i - s_{i-1}| < \epsilon$, $i = 1,\ldots,k$, for a given $\epsilon > 0$, then by (1.5.18) we get

$$R(t,s)R^{-1}(s,s) = \prod_{j=1}^{k} R(s_j,s_{j-1})R^{-1}(s_{j-1},s_{j-1}),$$

therefore

$$\det R(t,s) \det R^{-1}(s,s) = \prod_{j=1}^{k} \det R(s_j,s_{j-1}) \det R^{-1}(s_{j-1},s_{j-1})$$

(det A is the determinant of matrix A). Now if we have $\det R(s,t) = 0$, we get

$$\det R(s_i,s_{i-1}) = 0 \quad \text{for some } i \tag{1.5.19}$$

Let $\epsilon \to 0$ and u be an accumulation point of collection $\{s_i\}$ satisfying (1.5.19), then by continuity of the covariance and its determinant we get

$$0 = \lim \det R(s_i,s_{i-1}) = \det R(u,u),$$

but by assumption $\det R(u,u) \neq 0$, hence $\det R(s,t) \neq 0$ for all s and t in T. Thus we have the following:

1.5.20. Corollary. Let X_t be a centered continuous in quadratic mean and Gaussian reciprocal process such that in the representation

(1.5.17) $Y_t \neq 0$ for all t, then it has the following representation:

$$\dot{X}_t = \Phi(t)\underline{U}(t) + Z(t)$$

where $Z(t)$ is as in (1.5.17) and $\Phi(t)$ is a nonsingular $q \times q$ matrix and $\underline{U}(t)$ is a q-variate martingale.

Under the assumption of Corollary (1.5.20) we have the following result concerning differentiable reciprocal process which extends a result of ([7]):

1.5.21. Theorem. Let $\{X_t, t \in [0,T]\}$ be a centered differentiable Gaussian process, then it is reciprocal if and only if it is the solution of stochastic differentical equation of the following form with boundary values X_a, X_b:

$$d\left(\frac{X_t}{a(t)}\right) = du_t + Y\left(\frac{b(t)}{a(t)}\right)' dt + Z\left(\frac{c(t)}{a(t)}\right)' dt \tag{1.5.22}$$

where U_t is a martingale independent of Y and Z, and $U_0 = U_T = 0$, $a(t)$, $b(t)$, and $c(t)$ are some real functions, and $X_0 = Y$, $X_T = Z$.

Proof. Let X_t be reciprocal, then by Corollary (1.5.20) it has the following representation:

$$X_t = a(t)U_t + b(t)X_0 + c(t)X_T \quad \text{with} \quad a(t) \neq 0$$

therefore

$$\frac{X_t}{a(t)} = u_t + X_0 \left(\frac{b(t)}{a(t)}\right) + X_T \left(\frac{c(t)}{a(t)}\right)$$

and by differentiating we get (1.5.22).

Conversely, if X_t satisfies (1.5.22), then by integrating both sides from 0 upto t we get:

$$\frac{X_t}{a(t)} = u(t) - u(0) + Y(B(t)) + Z(C(t))$$

where $U(0) = 0$ and $B(t)$ and $C(t)$ are the integerals of $(\frac{b}{a})'$ and $(\frac{c}{a})'$, respectively. From here we get:

$$X_t = a(t)u(t) + Y\, a(t)B(t) + Z\, a(t)C(t),$$

therefore by Corollary (1.5.17) X_t is reciprocal. Imposing the boundary conditions we get that $a(t)$, $b(t)$ and $c(t)$ are satisfying the following relations:

$$a(0)B(0) = 1,\ a(T)C(T) = 1,\ C(0) = B(T) = 0.$$

In the next section we consider the Gaussian stationary reciprocal process and derive Jamison's result [5] by using the representation of the process.

1.6 Gaussian stationary reciprocal processes.

Let $X = \{X_t,\ t \in [0,T]\}$, $T > 0$ be a real continous stationary reciprocal Gaussian process. Here by stationarity on a bounded interval $[0,T]$ we mean that there is a stationary process on R such that on $[0,T]$ it coincides with X. We are assuming that $EX_t = 0$ and $EX_t^2 = 1$, for each t in $[0,T]$. By (1.5.17) we have the following representation for X_t:

$$X_t = Y_t + A(t)X_0 + B(t)X_T.$$

Let $r(t)$ be the covariance function of the process X, then we have

$$r(t) = EX_tX_0 = A(t) + B(t)EX_TX_0$$

$$= A(t) + B(t)r(T), \ t \in [0,T].$$

Now we consider the following cases:

(I) $A(t)X_0 + B(t)X_T = 0$, for all t in $[0,T]$, i.e. the process is independent of the two boundary random variables X_a, X_b. In this case $X(t) = Y(t)$ is a real Gaussian stationary Markov process, so its covariance functions is of the form:

$$r(t) = e^{-at}, \ t \text{ in } [0,T], \quad a > 0.$$

(II) $Y(t) = 0$, for all t in $[0,T]$, and X_0, X_T are independent. Let us first assume that $|r(t)| < 1$. We have:

$$1 = r(0) = EX_tX_t = A^2(t) + B^2(t),$$

therefore $A(t)$ and $B(t)$ are of the following forms:

$$A(t) = \cos(\varphi(t))$$

$$B(t) = \sin(\varphi(t))$$

for some real functions φ on $[0,T]$. On the other hand for t and $t + h$ in $[0,T]$ we have:

$$r(h) = EX_tX_{t+h} = E(A(t)X_0 + B(t)X_T)(A(t + h)X_0 + B(t + h)X_T)$$

$$= A(t)A(t + h) + B(t)B(t + h)$$

$$= \cos(\varphi(t)\cos(\varphi(t + h)) + \sin(\varphi(t))\sin(\varphi(t + h))$$

$$= \cos(\varphi(t + h) - \varphi(t)).$$

Therefore for each $s < t$ in $[0,T]$ we have:

$$r(t-s) = \cos(\varphi(t) - \varphi(s))$$

$$= \cos(\varphi(t))\cos(\varphi(s)) + \sin(\varphi(t)\sin(\varphi(s))$$

$$= \sum_{i=1}^{2} f_i(t)g_i(s)$$

where $f_1(t) = \cos(\varphi(t))$, $f_2(t) = \sin(\varphi(t))$ and $g_i(s) = f_i(s)$, $i = 1,2$. Here we show the following two facts about $\{f_1,f_2\}$ and $\{g_1,g_2\}$:

(i) g_1 and g_2 are linearly independent as elements of $L^2(o,c)$ for each c in $(0,T)$

(ii) $\det(f_i(t_j)) \neq 0$ $i,j = 1,2$ $t_1 < t_2$ in $(0,T)$.

To see (i), let $\alpha g_1(s) + \beta g_2(s) = 0$ for $s < c$, then by the continuity of the process we get that g_1 and g_2 are continuous, so by letting $s \to 0$ we get

$$\alpha \cos(\varphi(0)) + \beta \sin(\varphi(0)) = 0,$$

but $\cos(\varphi(0)) = 1$ and $\sin(\varphi(0) = 0$, therefore $\alpha = 0$ and $\beta \sin(\varphi(s) = 0$, for each $s < c$. But $\sin(\varphi(s)) \neq 0$ on $(0,T)$ (if $\sin(\varphi(s_0)) = 0$, then $\cos(\varphi(s_0)) = \pm 1$ and this implies that $|r(s_0)| = 1$), thus $\beta = 0$, this proves (i).

For (ii) we note that

$$\det(f_i(t_j)) = \begin{vmatrix} \cos(\varphi(t_1)) & \cos(\varphi(t_2)) \\ \sin(\varphi(t_1)) & \sin(\varphi(t_2)) \end{vmatrix}$$

$$= \sin(\varphi(t_2) - \varphi(t_1)).$$

Now if $\sin(\varphi(t_2) - \varphi(t_1) = 0$ then $\cos(\varphi(t_2) - \varphi(t_1)) = \pm 1$ which implies that $|r(t_2 - t_1)| = 1$, this proves (ii).
Therefore all the conditions of lemma [II. 1. [3]] are satisfied, so f_i's are the fundamental solution of a differential equation of order 2 and constant coefficient. Since f_i's are real trigonometic functions, the only possibility is that $f_1(t) = \cos(\alpha t)$. Hence in this case $r(t) = \cos(\varphi(t)) = \cos(\alpha t)$ for some $\alpha > 0$. The case $|r(t)| = 1$ will be discussed after the case (III).

(III) In this case all parts of the representation are present, and we are assuming that $r(T) = EX_0X_T = -1$. Since $A(t)X_0 + B(t)X_T - X_t$ is orthogonal to X_0 and X_T $(A(t)X_0 + B(t)X_T$ is the orthogonal projection of X_t on the space generated by $\{X_0, X_T\})$ we have:

$$E(A(t)X_0 + B(t)X_T - X_t)X_0 = 0$$

$$E(A(t)X_0 + B(t)X_T - X_t)X_T = 0$$

which gives us:

$$A(t) - B(t) = r(t)$$

$$-A(t) + B(t) = r(T-t)$$

Therefore by adding these two equation we get

$$r(t) + r(T-t) = 0.$$

One of the solution of this equation is

$$r(t) = 1 - a(t) \quad \text{with} \quad a = \frac{2}{T}.$$

Now we return to the case $|r(t_0)| = 1$, for some t_0 in $(0,T)$. In this case, as it is shown in [5] we have $r(t) = 1$ for all t in $(0,T)$ which is an special case of $e^{-\alpha t}$ with $\alpha = 0$.

CHAPTER 2

N-PLE MARKOV PROCESSES

AND

N-PLE RECIPROCAL PROCESSES

2.1. N-Ple Markov Processes

Let $\{X_t, t \in R\}$ be a real valued Gaussian process with mean zero and continous in quadratic mean, and having GFMP on the sets of the form $(-\infty,t)$, $t \in R$; i.e.

$$\sigma\{X_s: \ s > t\} \perp\!\!\!\perp \sigma\{X_s: \ s < t\} | \Gamma_t,$$

where Γ_t is the Germ Field and given by:

$$\Gamma_t = \bigcap_n \sigma\{X_s: \ |t-s| < \frac{1}{n}\}.$$

By [10] this property is equivialent to the following:

$$\sum_t^- \perp\!\!\!\perp \sum_t^+ \mid \Gamma_t$$

where

$$\sum_t^- = \bigcap_n \sigma\{X_s: \ s < t + \frac{1}{n}\}$$

$$\sum_t^+ = \bigcap_n \sigma\{X_s: \ s > t - \frac{1}{n}\}.$$

If the process is $N-1$ times continuously differentiable and the Germ Field Γ_t is generated by $X(t), X'(t),\ldots,X^{(N-1)}(t)$, the process has

N-ple Markov property in the sense of Doob [2]. Here it is understood that $X(t),\ldots,X^{(N-1)}(t)$ are linearly indpendent as elements of $L^2(\Omega,g_t P)$, where $\mathcal{g}_t = \bigcap_n \sigma\{X_s: \ |t-s| < \frac{1}{n}\}$. The following is a generalization of this notion.

2.1.1. Definition. A process $X = \{X_t,\ t \in T\}$ is called a Generalized N-ple Markov process with respect to the processes $\{Y_i(t),\ t \in T\}_{i=1,\ldots,N}$ if:

(i) for each t in T, $Y_1(t),\ldots,Y_N(t)$ are linearly independent as elements of $L^2(\Omega,g_t,P)$, where $\mathcal{g}_t = \bigcap_n \sigma\{X_s: \ |t-s| < \frac{1}{n}\}$,

(ii) $\sum_t^+ \amalg \sum_t^- | \Gamma_t$ where;

$$\textstyle\sum_t^+ = \bigcap_{\epsilon>0} \sigma\{X_u: \ u > t - \epsilon\}$$

$$\textstyle\sum_t^- = \bigcap_{\epsilon>0} \sigma\{X_u: \ u < t + \epsilon\}$$

$$\Gamma_t = \sigma\{Y_1(t),\ldots,Y_N(t)\},$$

and Γ_t is the Germ Field at t.

We have the following immediate result concerning the process $Z(t) = (Y_1(t),\ldots,Y_N(t))^*$ (* means the transopose of a matrix).

2.1.2. Theorem. If $\{X_t,\ t \in T\}$ is a generalized N-ple Markov process with respect to $\{Y_i(t)\}_{i=1,\ldots,N}$, then the process $Z(t) = (Y_1(t),\ldots,Y_N(t))^*$ is a Markov process.

Proof: By assumption we have

(2.1.3) $$\sigma\{X_u: \ u \geq s\} \amalg \sigma\{X_u: \ u \leq s\} | \sigma(Z(s)),$$

where $A \amalg B|G$ means that given G, A and B are conditionally independent. For each $\epsilon > 0$ we have:

$$\sigma\{Z_u: \ u \geq s + \epsilon\} \subset \sigma\{X_u: \ u \geq s\}$$

and

$$\sigma\{Z_u: \ u \leq s - \epsilon\} \subset \sigma\{X_u: \ u \leq s - \epsilon\},$$

therefore by (2.1.3) we have:

$$\sigma\{Z_u: \ u \geq s + \epsilon\} \amalg \sigma\{Z_u: \ u \leq s - \epsilon\}|\sigma\{Z(u)\}$$

so

$$\bigvee_{\epsilon>0} \sigma\{Z_u: \ u \geq s + \epsilon\} \amalg \bigvee_{\epsilon>0} \sigma\{Z_u: \ u \leq s - \epsilon\}|\sigma\{Z(u)\},$$

thus

$$\sigma\{Z_u: \ u > s\} \amalg \sigma\{Z_u: \ u < s\}|\sigma(Z(u)).$$

Finally by (1.1.1)(b) we get

$$\sigma\{Z_u: \ u \geq s\} \amalg \sigma\{Z_u: \ u \leq s\}|\sigma(Z(u)),$$

and this completes the proof.

This simple fact leads us to a Goursat type ([8], p. 74) representation of Generalized N-ple Markov processes.

2.1.4. Theorem: Let $\{X_t, t \in T\}$ be a Gaussian Generalized N-ple Markov process with respect to the Gaussian processes $\{Y_i(t), t \in T\}_{i=1,\ldots,N}$. If the covariance matrix $\Gamma(t,s) = E(Z(t)Z^*(s))$ of $Z(t) = (Y_1(t),\ldots,Y_N(t))^*$ is nonsingular, then:

(2.1.5) $$X_t = \sum_{i=1}^{N} \psi_i(t)u_i(t)$$

where $\psi_i(t)$, $i = 1,\ldots,N$, are N real functions and $\underline{U}(t) = (U_1(t),\ldots,U_N(t))$ is an N-variate martingale [ ].

Proof. From (2.1.2), $Z(t)$ is an N-variate Gaussian Markov process. Therefore by (3.1 [7]) it has the following representation:

$$Z(t) = \Phi(t)\underline{U}(t)$$

where $\Phi(t)$ is an $N\times N$ non-singular matrix and $\underline{U}(t)$ is an N-variate martingale. On the other hand by the Markov property of $\{X_t\}$ we have:

$$\begin{aligned} X_t &= E(X_t | X_u: \; u \leq t) \\ &= E(X_t | Z(t)) \\ &= A(t)Z(t) \end{aligned}$$

where $A(t)$ is a IXN matrix, so we have:

$$\begin{aligned} X_t &= A(t)\Phi(t)\underline{U}(t) \\ &= \psi(t)\underline{U}(t) = \sum_{i=1}^{N} \psi_i(t)u_i(t) \end{aligned}$$

where $\psi(t) = (\psi_1(t),\ldots,\psi_N(t)) = A(t)\Phi(t)$, and $\underline{U}(t) = (U_1(t),\ldots,U_N(t))^*$.

This completes the proof.

2.2. N-Ple Reciprocal processes.

In this section we are giving a definition for an N-ple reciprocal process and study its relation with the Generalized N-ple Markov process. Here again all the processes are Gaussian with mean zero and continous in quadratic mean on some probability space (Ω,F,P). We are also assuming that all the σ-fields involved contain all sets of measure zero. In the following $\mathcal{g}_t$ is the same as in (2.1.1).

2.2.1. Definition. A process $X = \{X_t, t \in T\}$ is called an N-ple reciprocal process with respect to the processes $\{Y_i(t)\}$, $i = 1,\ldots,N$, if:

(i) for each t, $Y_1(t),\ldots,Y_N(t)$ are linearly independent in $L^2(\Omega,\mathcal{g}_t,P)$,

(ii) $\Sigma^+_{u,v} \perp\!\!\!\perp \Sigma^-_{u,v} | \Gamma_{u,v}$ for all $u \leq v$, where:

$$\Sigma^+_{u,v} = \bigcap_{\epsilon>0} \sigma\{X_t: \ t \in (u - \epsilon, v + \epsilon)\}$$

$$\Sigma^-_{u,v} = \bigcap_{\epsilon>0} \sigma\{X_t: \ t \notin (u + \epsilon, v - \epsilon)\}$$

$$\Gamma_{u,v} = \sigma\{Y_1(u),\ldots,Y_N(u); Y_1(v),\ldots,Y_N(v)\}.$$

Parallel to the Generalized N-ple Markov processes, Theorem (2.1.2), we have the following. Its proof is essentially the same as the one in (2.1.2):

2.2.2 Theorem: If $\{X_t, t \in T\}$ is an N-ple reciprocal process with respect to $\{Y_i(t)\}_{i=1,\dots,N}$, then $Z(t) = (Y_1(t),\dots,Y_N(t))^*$ is a reciprocal process.

Proof: By the assumption we have

$$\sigma\{X_t:\ t \in [u,v]\} \amalg \sigma\{X_t:\ t \notin [u,v]\} | \sigma\{Z(u),Z(v)\}$$

Therefore for each $\epsilon > 0$ we have:

$$\sigma\{Z_t:\ t \in (u+\epsilon, v-\epsilon)\} \amalg \sigma\{Z_t:\ t \notin [U-\epsilon, v+\epsilon]\} | \sigma\{Z(u),Z(v)\}$$

so $\bigvee_{\epsilon>0} \sigma\{Z_t:\ t \in (u+\epsilon, v-\epsilon)\} \amalg \bigvee_{\epsilon>0} \sigma\{Z_t:\ t \notin [u-\epsilon, v+\epsilon]\} | \sigma\{Z(u),Z(v)\}$,

thus $\sigma\{Z_t:\ t \in (u,v)\} \amalg \sigma\{Z_t:\ t \notin (u,v)\} | \sigma\{Z(u),Z(v)\}$,

and this completes the proof.

In the following we give a representation for the N-ple reciprocal processes. For this we need a similar result to [ROY & NGOC] for the N-ple reciprocal processes. We will use the following notations:

$$F^+_{u,v} = \sigma\{X_t:\ t \notin (u,v)\},\ F^-_u = \sigma\{X_t:\ t \le u\}$$

$$F_\infty = \bigcap_{u<v} F^+_{u,v},\ F_{-\infty} = \bigcap_u \sigma\{X_t;\ t \le u\},\ F_{+\infty} = \bigcap_u \sigma\{X_t:\ t \ge u\},$$

as well as the notations in definition (2.1.1) and (2.2.1).

2.2.3 Lemma. Let $\{X_t, t \in R\}$ be a Gaussian N-ple reciprocal process with respect to the process $\{Y_i(t), i = 1,\dots,N\}$, if either $F_{-\infty}(Y)$ or $F_{+\infty}(Y)$ is trivial, then $\{X_t, t \in R\}$ is an N-ple Markov process with respect to $\{Y_i(t), i = 1,\dots,N\}$.

Proof. Let $Z(t) = (Y_1(t),\dots,Y_N(t))^*$, then by (2.2.2) $\{Z(t), t \in R\}$ is a reciprocal process with trivial tail, therefore from ROYER & NGOC [12] we get:

$$\lim_{n\to\infty} \sigma\{Z_t\} \vee \sigma\{Z_n, Z_{n+1},\dots\} = \sigma\{Z_t\} \tag{2.2.4}$$

Therefore for f bounded and measurable function with respect to $\sigma\{X_t:\ t \in (u,v)\}$ and an integer n with $u < v < n$ we have:

$$\begin{aligned} E(f|\textstyle\sum^-_{u,n}(X)) &= E(f|Z_u, Z_n) \qquad \text{(by reciprocal property)} \\ &= E(f|\sigma\{Z_u\} \vee \sigma\{Z_n, Z_{n+1},\dots\}). \end{aligned} \tag{2.2.5}$$

The last equality is because of the reciprocal property and the fact that $\sigma\{Z_n, Z_{n+1},\dots\} \subset \sigma\{X_t:\ t \geq u\}$ as we showed in the proof of the Theorems (2.1.2) and (2.2.2).

Therefore by Martingale theorem, (2.2.4) and (2.2.5) we get

$$\lim_{n\to\infty} E(f|\textstyle\sum^-_{u,n}(X)) = E(f|\sigma(Z_u)),$$

taking conditional expection with respect to $F^-_u(X)$ and use dominated convergence theorem for conditional expection we get

$$E(f|\textstyle\sum^-_u) = E(f|Z_u).$$

This completes the proof.

For the case of $T = [a,b]$ the proof being similar is omitted.

Now we are in a position to give a representation for the N-ple reciprocal processes.

2.2.6. Theorem: Let $X = \{X_t, t \in T\}$ be a Gaussian N-ple reciprocal process, then it has the following representation:

$$X_t = U_t + V_t$$

where U_t is at most an N-ple Markov process and independent of V_t. $F_\infty^U = \{\emptyset,\Omega\}$ and V_t is measurable with respect to F_∞^X.

Proof:

$$\text{Let } U_t = X_t - P_{H_\infty}^{X_t}$$

where $H_\infty = \bigcap_{u>0} \overline{sp}\{X_t: |t| > u\}$. It is clear that U_t is orthogonal to $V_t = P_{H_\infty}^{X_t}$. We note that

$$\bigcap_u \overline{sp}\{U_t: |t| > u\} = \bigcap_u \overline{sp}\{X_t: |t| > u\} \ominus \bigcap_u \overline{sp}\{P_{H_\infty}^{X_t}: |t| > u\}$$

$$= H_\infty \ominus H_\infty = \{0\}$$

This implies that $F_\infty^U = \bigcap_u \sigma\{U_t: |t| > u\} = \{\emptyset,\Omega\}$. Now we show that U_t is at most an N-ple reciprocal process. Let $u < t < v$, and $H(X)_{u,v} = \overline{sp}\{X_s: s \notin (u,v)\}$:

$$E(U_t|V_s: s \notin (u,v)) = P_{H_{u,v}(U)}^{U_t} = P_{H_{u,v}(X) \ominus H_\infty}^{U_t} = P_{H_{u,v}(X)}^{U_t}$$

$$= P_{H_{u,v}(X)}^{X_t - V_t} = P_{H_{u,v}(X)}^{X_t} - P_{H_\infty}^{X_t},$$

by reciprocal property of X_t we get:

$$E(U_t|U_s: s \notin (u,v)) = P_{\overline{sp}\{\underline{Y}_{(u)},\underline{Y}_{(v)}\}}^{X_t} - P_{H_\infty}^{X_t}.$$

where $\underline{Y}(u) = \{Y_1(u),\dots,Y_N(u)\}$ is the process that X_t is N-ple reciprocal with respect to that. So we have:

$$E(U_t | U_s: \; s \notin (u,v)) = A\,\underline{Y}(u) + B\,\underline{Y}(v) - P_{H_\infty}^{X_t}.$$

Also we have

$$P_{H_\infty}^{X_t} = P_{H^{(X)}_{u,v}} P_{H_\infty}^{X_t} = P_H \, P_{HX_{(u,v)}}^{X_t} = P_{H_\infty}^{\underline{Y}(u)A + \underline{Y}(v)B}$$

therefore we get

$$E(U_t | U_s: \; s \notin (u,v)) = A\,\underline{Y}(u) + B\,\underline{Y}(v) - P_{H_\infty}^{A\,\underline{Y}(u) + B\,\underline{Y}(v)}$$

$$= A(\underline{Y}(u) - P_{H_\infty}^{\underline{Y}(u)}) + B(\underline{Y}(v) - P_{H_\infty}^{\underline{Y}(v)}).$$

This equation shows that U_t is at most N-ple reciprocal with respect to the process $\{Y_i(u) - P_{H_\infty}^{Y_i(u)}\}_{i=1,\dots N}$.

Since we are assuming that the involved processes are Gaussian and for the Gaussian processes the conditional expectations are orthogonal projections on some sub-Hilbert spaces we could write the definitions (2.1.1) and (2.2.1) in terms of projections instead of conditional expectations as follows:

$$P_{H_t^+} P_{H_t^-} = P_{H_t^-} P_{H_t^+} = P_t \qquad \text{(Markov property)}$$

where

$$H_t^+ = \bigcap_{\epsilon>0} \overline{sp}\{X_u: \; u \geq t - \epsilon\}$$

$$H_t^- = \bigcap_{\epsilon>0} \overline{sp}\{X_u: \; u \leq t + \epsilon\}$$

and

$$\Gamma_t = \overline{sp}\{Y_1(t),\ldots,Y_N(t)\}$$

where $\overline{sp}\{\ldots\}$ is the linear span closure of $\{\ldots\}$ under the norm of $L^2(\Omega,F,P)$. For the reciprocal property we have

$$P_{H^+_{(u,v)}} P_{H^-_{(u,v)}} = P_{H^-_{(u,v)}} P_{H^+_{(u,v)}} = P_{\Gamma(u,v)} .$$

with

$$H^+_{(u,v)} = \bigcap_{\epsilon>0} \overline{sp}\{X_t: \ t \in (u-\epsilon,\ v+\epsilon)\}$$

$$H^-_{(u,v)} = \bigcap_{\epsilon>0} \overline{sp}\{X_t: \ t \notin (u-\epsilon,\ v+\epsilon)\}$$

$$\Gamma_{(u,v)} = \overline{sp}\{Y_1(u),\ldots,Y_N(u);\ Y_1(v),\ldots,Y_N(v)\}.$$

This is the motivation for giving the definition of N-ple Markov and N-ple reciprocal properties in the case of HSO-valued processes in the next section.

2.3. HSO-valued N-ple Markov and N-ple Reciprocal Processes.

Let H and K be two separable Hilbert spaces and $X = \{X_t,\ t \in T\}$ be an HSO-valued process on H into K as introduced in section (1.4). Also we assume (1.4.13).

2.3.1. Definition: Let $Y = \{Y_i(t),\ t \in T\}$ $i=1,\ldots,N$, be N linearly independent HSO-valued processes in $\bigcap_n G\{X_s: \ |t-s| < \frac{1}{n}\}$. We say with respect to Y, X is an:

(i) N-ple Markov process if

$$P_{G_t^+} P_{G_t^-} = P_{G_t^-} P_{G_t^+} = P_{\Gamma_t} \quad , \quad t \in T$$

(ii) N-ple reciprocal process if

$$P_{G^+_{(u,v)}} P_{G^-_{(u,v)}} = P_{G^-_{(u,v)}} P_{G^+_{(u,v)}} = P_{\Gamma_{(u,v)}}$$

where

$$G_t^+ = \bigcap_{\epsilon>0} G\{X_u:\ u \geq t - \epsilon\},\ G_t^- = \bigcap_{\epsilon>0} G\{X_u:\ u \leq t + \epsilon\},$$

$$G^+_{(u,v)} = \bigcap_{\epsilon>0} G\{X_t:\ t \notin (u - \epsilon, v + \epsilon)\},\ G^-_{(u,v)} = \bigcap_{\epsilon>0} G\{X_t:\ t \in (u + \epsilon, v - \epsilon)\},$$

$$\Gamma_t = G\{Y_1(t),\dots,Y_N(t)\},\ \Gamma_{(u,v)} = G\{Y_1(u),\dots,Y_N(u);\ Y_1(v),\dots,Y_N(v)\}.$$

Now we are going to establish results similar to (2.1.2), (2.1.4), (2.2.2), (2.2.3) and finally (2.2.6) for HSO-valued processes.

2.3.2. Theorem. Let $X = \{X_t:\ t \in T\}$ be an N-ple HSO-valued Markov process with respect to $Y = \{Y_1(t),\dots,Y_N(t)\}$, then the process $(Y_1(t),\dots,Y_N(t))$ is a Markov process.

Proof: Let $t > s$ be two points in T, since $Y_1(u) \in G_u^-$ and G_u^+, $i=1,\dots,N$, $u \in T$, we have:

$$(Y_i(t)|\overset{-}{\underset{s}{M}}(Y)) = ((Y_i(t)|G_s^-)|\overset{+}{\underset{s}{M}}(Y)),$$

but by Markov property of X we get:

$$(Y_i(t)|G_s^-) = (Y_i(t)|\Gamma_s),$$

therefore:

$$\begin{aligned}(Y_i(t)|M_s^-(Y)) &= ((Y_i(t)|\Gamma_s)|M_s^-(Y)) \\ &= (Y_i(t)|\Gamma_s) \\ &= (Y_i(t)|G(Y_1(s),\dots,Y_N(s)),\end{aligned}$$

this completes the proof.

Next is a representation for N-ple HSO-valued Markov processes. We recall that we are making the assumption (1.4.13).

2.3.3. Theorem: Let $X = \{X_t, t \in T\}$ be an N-ple HSO-valued Markov process with respect to $Y = \{Y_1(t),\dots,Y_N(t), t \in T\}$ and the covariance functions $\Gamma_i(t,s)$ of Y_1, $i = 1,\dots N$, have the property that $\Gamma_i^-(s,s)\Gamma_i(t,s)$ is one-to-one on $\overline{R}\{\Gamma_i(t,t)\}$ onto $\overline{R}\{\Gamma_i(s,s)\}, i = 1,\dots N$, and for all $s \le t$, then X has the following representation:

$$X_t = \sum_{i=1}^{N} u_i(t)\psi_i(t),$$

where $\{u_i(t), t \in T\}$, $i = 1,\dots N$ are HSO-valued Martingales, and $\psi_i(t)$, $i = 1,\dots,N$, are in $B(H,H)$.

Proof: As a result of Theorem (2.3.2) we get that each $\{Y_i(t), t \in T\}$ is a Markov process, therefore by the assumption on the covariance of $\{Y_i(t)\}$ and (Theorem 2.11 [11]) we have

$$(2.3.4) \qquad Y_i(t) = u_i(t)\Phi_i(t),$$

where $u_i(t)$'s are HSO-valued Martingales and $\Phi_i(t)$'s are in $B(H,H)$.

On the other hand by N-ple Markov property of X we have:

$$X_t = (X_t | M_t^-) = (X_t | G\{Y_1(t), \ldots, Y_N(t)\}),$$

thus by assumption (1.4.13) we have

$$X_t = \sum Y_i(t) A_i(t)$$

for some $A_i(t)$ in $B(H,H)$. Now we substitute for $Y_i(s)$ from (2.3.4) we get

$$X_t = \sum_{i=1}^{N} u_i(t) \Phi_i(t) A_i(t)$$

$$= \sum_{i=1}^{N} u_i(t) \psi_i(t)$$

where $\psi_i(t) = \Phi_i(t) A_i(t)$ is in $B(H,H)$, $i = 1, \ldots, N$.

Now we study HSO-valued reciprocal processes and give a representation for them.

(2.3.5). Theorem. Let $X = \{X_t: \ t \in T\}$ be an N-ple HSO-valued reciprocal process with respect to $\{Y_i(t), \ t \in T\}$, $i = 1, \ldots, N$, then $(Y_1(t), \ldots, Y_N(t))$ is a reciprocal process.

Proof: Let $u < v$ and $t \in (u,v)$, then for each $i = 1, \ldots, N$;

$$(Y_i(t) | M_{u,v}^+(Y)) = ((Y_i(t) | G_{u,v}^+) | M_{u,v}^+(Y)),$$

therefore by reciprocal property of X we get

$$(Y_i(t) | M_{u,v}^+(Y)) = ((Y_i(t) | \Gamma_{(u,v)}) | M_{u,v}^+(Y))$$

$$= (Y_i(t) | \Gamma_{u,v})$$

$$= (Y_i(t)|G\{Y_1(u),\ldots,Y_N(u);\ Y_1(v),\ldots,Y_N(v)\})$$

and this completes the proof.

For the next theorem we need the following lemma which states under some conditions we get the Markov property of a reciprocal process.

2.3.6. Lemma. Let $X = \{X_t:\ t \in T\}$ be Gaussian Hilbert-space valued reciprocal process with respect to $\{Y_1(t),\ldots,Y_N(t)\}$ and $M_\infty^Y = \bigcap_{u<v} M_{u,v}^Y = \{0\}$, then $\{X_t:\ t \in T\}$ is an N-ple Markov process with respect to $\{Y_1(t),\ldots,Y_N(t)\}$.

Proof: First we show the following:

$$G\{Y_1(u),\ldots,Y_N(u)\} \vee G\{Y_1(n),\ldots,Y_N(n);\ Y_1(n+1),\ldots,Y_N(n+1),\ldots\}$$

converges to $G\{Y_1(u),\ldots,Y_N(u)\}$ as $n \to \infty$.

Let

$$M = G\{Y_1(u),\ldots,Y_N(u)\} \text{ and}$$

$$M_n = G\{Y_1(n),\ldots,Y_N(n);\ Y_1(n+1),\ldots,Y_N(n+1),\ldots\} \vee M.$$

we have:

$$M_n = M \oplus (M_n \ominus M).$$

we note that $M_n \ominus M \subset G\{Y_1(n),\ldots,Y_N(n);\ Y_1(n+1),\ldots,Y_N(n+1),\ldots\}$ which converges to $\{0\}$ as $n \to 0$. So we get:

$$\bigcap M_n = M \oplus \bigcap_r (M_n \ominus M) = M.$$

Now let $u < t < n$, then by reciprocal property of X we have:

$$(X_t|M^+_{u,n}) = (X_t|G\{Y_1(u),\ldots,Y_N(u),Y_1(n),\ldots,Y_N(n)\})$$

$$= (X_t|G\{Y_1(u),\ldots,Y_N(u);\ Y_1(n),\ldots,Y_N(n);\ Y_1(n+1),\ldots,Y_N(n+1),\ldots\})$$

$$= (X_t|M_n)$$

Now let $n \to \infty$ we get:

$$\lim_{n\to\infty} (X_t|M^+_{u,n}) = (X_t|G\{Y_1(u),\ldots,Y_N(u)\}),$$

by projecting this equation on M^-_u we get:

$$(X_t|M^X_u) = (X_t|G\{Y_1(u),\ldots,Y_N(u)\}),$$

and this completes the proof.

Finally we have the following representation theorem.

2.3.7. (Representation Theorem). Let $\{X_t,\ t \in T\}$ be an N-ple Gaussian Hilber-spece valued reciprocal process with respect to $\{Y_1(u),\ldots,Y_n(u)\}$, then X_t has the following representation:

$$X_t = U_t + V_t$$

where U_t is an at most N-ple Markov process and orthogonal to the process V_t which lies in M^X_∞.

Proof: Let $V_t = (X_t|M^X_\infty)$ and $U_t = X_t - V_t$. The only thing that we have to show is that U_t is at most an N-ple reciprocal process and then in view of Lemma (2.3.6) it suffices to show that $M^U_\infty = \{0\}$.

Let $u < v$, then

$$G^-_{(u,v)}(U) = \bigcap_{\epsilon>0} G\{U_t:\ t \notin (u+\epsilon,\ v-\epsilon)\}$$

$$= \bigcap_{\epsilon>0} G\{X_t - V_t: \ t \notin (U + \epsilon, v - \epsilon)\}$$

$$= \bigcap_{\epsilon>0} G\{X_t: \ t \notin (U + \epsilon, v - \epsilon)\} \ominus \bigcap_{\epsilon>0} G\{V_t: \ t \notin (U + \epsilon, v - \epsilon)\}$$

$$= G^-(X)_{(u,v)} \ominus G^-(V)_{(u,v)} ,$$

therefore for $u < t < v$, we have:

$$(U_t|G^-(U)_{(u,v)}) = (U_t|G^-(X)_{(u,v)}) - (U_t|G^-(V)_{(u,v)}$$

$$= (U_t|G^-(X)_{(u,v)}$$

$$= (X_t|G^-(X)_{(u,v)}) - V_t$$

therefore by reciprocal property of X we get:

$$(U_t|G^-(U)_{(u,v)}) = (X_t|G\{Y_i(u), Y_i(v), i = 1,\dots,N\}) - V_t$$

$$= \sum_{i=1}^{N} Y_i(u)A_i + \sum_{i=1}^{N} Y_i(v)B_i - V_t,$$

for some A_i, B_i in $B(H,H)$ (these are functions of u,v,t). On the other hand

$$V_t = (X_t|M_\infty^X) = ((X_t|G^+(X)_{(u,v)})|M_\infty^X)$$

$$= ((X_t|G\{Y_i(u), Y_i(v), i = 1,\dots,N\})|M_\infty^X)$$

$$= (\sum_{i=1}^{N} Y_i(u)A_i + \sum_{i=1}^{N} Y_i(v)B_i|M_\infty^X)$$

$$= \sum_{i=1}^{N} (Y_i(u)|M_\infty^X)A_i + \sum_{i=1}^{N} (Y_i(v)|M_\infty^X)B_i .$$

Hence:

$$(U_t | G^+_{(u,v)}(u)) = \sum_{i=1}^{N} [Y_i(u) - (Y_i(u)|M_\infty^X)]A_i + \sum_{i=1}^{N} [Y_i(v) - (Y_i(v)|M_\infty^X]B_i.$$

This relation shows that $\{U_t,\ t \in T\}$ has reciprocal property with respect to the process $\{Y_i(t) - (Y_i(t)|M_\infty^X)\}_{i=1,\ldots,N}$. Since these N processes may not be linearly independent we will not get exactly N-ple reciprocal process. Now we show that $M_\infty^U = \{0\}$. Indeed:

$$M_\infty^U = \bigcap_{u<v} G\{U_t:\ t \in (u,v)\} =$$

$$\bigcap_{u<v} G\{X_t:\ t \notin (u,v)\} \ominus \bigcap_{u<v} G\{V_t:\ t \notin (u,v)\}$$

$$= M_\infty^X \ominus M_\infty^X = \{0\},$$

and this completes the proof.

CHAPTER 3

3.1. INFINITE ORDER MARKOV PROCESSES

Let $X = \{X_t, t \in T\}$ be a Gaussian processes with mean 0 and continous in quadratic mean. For each $u \in R$, let $\{Y_u(t), t \in T\}$ be another Gaussian process with mean 0 and jointly continous in (u,t). Saying the family $\{Y_i^{(t)},\ldots,Y_n^{(t)}\}$ of n stochastic processes is linearly independent is equivalent to the fact that if for each t

$$\int_R Y_u(t)\, G\,(du) = 0,$$

then $G \equiv 0$, where G is a finite Borel measure with supp $G \subset \{1,2,\ldots,n\}$. Motivated from this we have the following definition.

3.1.1. Definition. We say that the process $\{Y_u(t), t \in T\}$, $u \in R$ is "Free" if for each finite Borel measure G

$$\int Y_u(t)\, G\,(du) = 0$$

implies that $G \equiv 0$. [We note that G might not be a positive measure].

Now we have the following definition of infinite order Markov processes.

3.1.2. Definition. Let $X = \{X_t, t \in T\}$ be a Gaussian process of mean 0 and continous in the mean. We say that X has infinite order Markov property with respect to the process $\{Y_u(t), u \in R, t \in T\}$, if,

(i) for each $t \in T$, $\{Y_u(t), u \in R\}$ is a free mean zero Gaussian process in $L^2(\Omega, \mathcal{G}_t, p)$, where $\mathcal{G}_t = \bigcap_n \sigma\{X_s: |t-s| < \frac{1}{n}\}$. Also $\{Y_u(t), u \in R, t \in T\}$ is jointly continous in (u,t).

(ii) $\Sigma_t^- \perp\!\!\!\perp \Sigma_t^+ | \Gamma_t$

where,

$$\Sigma_t^- = \bigcap_{\epsilon>0} \sigma\{X_s: s < t + \epsilon\}$$

$$\Sigma_t^+ = \bigcap_{\epsilon>0} \sigma\{X_s: s > t - \epsilon\}$$

$$\Gamma_t = \sigma\{Y_u(t), u \in R\}.$$

If the process is infinitely many defferentiable and all derivatives form a splilling field, then $\{X_t^{(n)}, n \in N\}$ can serve as an example for $\{Y_u(t), u \in R\}$.

Since in the case of Gaussian processes the conditional expectations are orthogonal projections, we note that for each $t > s$, we have:

$$E(X_t|\Sigma_s^-) = P_{H_s^-}^{X_t}$$

where $H_s^- = \bigcap_{\epsilon>0} \overline{sp}\{X_u: u < s + \epsilon\}$. Having infinite Markov property gives that

$$E(X_t|\Sigma_s^-) = P_{H(s)}^{X_t}$$

where

$$H(s) = \overline{sp}\{Y_u(s), u \in R\}.$$

Now we observe that:

(3.1.3) $\overline{sp}\{Y_u(s),\ u \in R\} = \{\int Y_u(s)\ G\ (du):\ G$ a finite Borel measure$\}$.

It is obvious that the right hand side is a subset of left hand side, to see the other way around, we note that:

$$Y_u(s) = \int Y_v(s)\ G_u\ (dv)$$

where

$$G_u(dv) = \begin{cases} 0 & \text{if} \quad v \neq u, \\ 1 & \text{if} \quad v = u. \end{cases}$$

So we have (3.1.3).

Using (3.1.3) the definition 3.1.2 (ii) can be written in the form:

(3.1.4) $$E(X_t|\Sigma_s^-) = \int Y_u(s)\ G\ (t,s,du),$$

and also,

(3.1.5) $$E(Y_u(t)|\Gamma_s) = \int Y_v(s)\ g_u\ (t,s,dv),$$

for some finite Borel measures G and g. By putting $t = s$ in (3.1.4) and (3.1.5) and using the assumption on $\{Y_u(t),\ u \in R\}$, then for all $t \in T$ we get:

(3.1.6) $$X_t = \int Y_u(t)\ G\ (t,t,du),$$

and

$$g_u(t,t,dv) = \begin{cases} 1 & \text{if} \quad u = v \\ 0 & \text{if} \quad v \neq u. \end{cases}$$

Next we prove that $\{Y_u(t), t \in T, u \in R\}$ has Markov property. More explicitly:

3.1.7. Theorem. Let $X = \{X_t, t \in T\}$ be an infinite order Markov process with respect to $\{Y_u(t), u \in R, t \in T\}$, then

$$\sigma\{Y_u(s): u \in R, s \leq t\} \amalg \sigma\{Y_u(s): u \in R, s \geq t\} | \Gamma_t.$$

Proof. For each $\epsilon > 0$ we have:

$$\sigma\{Y_u(s): u \in R, s < t - \epsilon\} \subset \sigma\{X(s): s < t\}$$

3.1.8

$$\sigma\{Y_u(s): u \in R, s > t + \epsilon\} \subset \sigma\{X(s): s > t\}.$$

By assumption we have:

$$\sigma\{X(s): s < t\} \amalg \sigma\{X(s): s > t\} | \Gamma_t.$$

Therefore by (3.1.8) for each $\epsilon > 0$ we have:

$$\sigma\{Y_u(s): u \in R, s < t - \epsilon\} \amalg \sigma\{Y_u(s): u \in R, s > t + \epsilon\} | \Gamma_t,$$

or

$$\bigvee_{\epsilon>0} \sigma\{Y_u(s): u \in R, s < t - \epsilon\} \amalg \bigvee_{\epsilon>0} \sigma\{Y_u(s): u \in R, s > t + \epsilon\} | \Gamma_t,$$

and finally:

$$\sigma\{Y_u(s): u \in R, s < t\} \amalg \sigma\{Y_u(s): u \in R, s > t\} | \Gamma_t.$$

This completes the proof.

By this Theorem we have:

$$E(Y_u(t)|Y_v(\tau),\ v \in R,\ \tau < s) = E(Y_u(t)|Y_v(s),\ v \in R)$$

$$= \int Y_v(s)\ g_u\ (t,s,dv),$$

for some finite Borel measure g.

Remark (1). For the Markov processes and N-ple Markov processes a representation is given, [7], [8] and Theorem (2.1.4). Here a representation of infinite order Markov processes is under consideration.

Remark (2). A generalization to the simple stationary Markov processes is the notion of T-positivity [4]. By definition a process $X = \{X_t,\ t \in R\}$ is called T-positive if for the times reflection operator T on $\overline{sp}\{X_t,\ t \in R\}$ given by

$$T1=1 \quad \text{and} \quad TX(t) = X(-t),\ t \in R$$

we have the following T-positivity property:

$$(*) \quad P_+TP_+ \geq 0.$$

where P_+ is the projection onto $\overline{sp}\{X_s:\ s \geq 0\}$. In the stationary Gaussian case (*) is equivialent to:

$$\sum_{v,u\in I} a_v\bar{a}_u\ r(t_u+t_v) \geq 0,$$

where I is any finite index set and $r(\cdot)$ is the covariance function of the processes. For the infinite order stationary Markov process $X = \{X_t,\ t \in R\}$ under certain conditions on $\{Y_u(t),\ u \in R,\ t \in R\}$ we have the T-positivity of X.

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