

FLEXURAL ELASTIC CHARACTERISTICS OF CROSS-SHAPED
STRUCTURAL JOINTS

by

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AN ABSTRACT

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A knowledge of the energy distribution characteristics of the joints is important for an analysis of indeterminate framed structures with deep-short members. This dissertation determines the characteristics of cross-shaped joints (internal joints of the framed structure) subjected to flexural action. The members of the frame are of rectangular cross section and the stress distribution is assumed to be plane.

The Airy's stress function ϕ inside the cross-shaped region is determined by solving the biharmonic differential equation by the numerical finite difference method. The stresses and the elastic energy per unit beam length are determined. The equivalent depth distribution is calculated, i.e., the depth distribution which when used in the evaluation of the energy by the conventional beam theory formulas will give the true elastic energy. The effects, of the fillets at the joint, of the dimensions of the cross shape, and of the variations in the Poisson's ratio, on the equivalent depth are studied.

The column portion of the cross shape is also analyzed with an assumed linear bending stress distribution and a uniform shear stress distribution at the beam to column junction. The analysis is made by taking the stress function in the form of a series. The comparison of equivalent depth curves, inside the column portion, calculated by the finite

difference method and by the series method shows a fair agreement as far as the shape of the equivalent depth curve is concerned. The series method is also used to investigate the effect of the different proportions of the cross shape on the equivalent depth inside the column.

It is concluded that the exact value of the equivalent depth depends upon the proportions of the cross shape, the type of the loading, and the radius of the fillet. For practical use, an approximate equivalent depth line is suggested, which can be used for any kind of loading and any proportions of the cross shape. In an example, worked out with the suggested approximation and the beam theory formulas, the total energy of the joint differs from the energy calculated by the finite difference method by less than 11%. This is a much smaller error than that which results from using either of the assumptions commonly made: that the equivalent depth at any section inside the joint is the same as the depth at the face of the joint or alternatively that the moment of inertia at any section inside the joint is infinity. Either of these assumptions leads to errors of about 100% in the total energy of the joint.

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To My
Late Father,
and
Younger Brother, Anu

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NOTATION

- A -- Cross-sectional area of the beam.
 A_0 -- Constant in Fourier series of shear stress τ_{xy} .
 A_m -- Constant in Fourier series of shear stress τ_{xy} .
 a_1, a_2, a_3 -- ζ values for corners of the cross shape.
 B_m -- Constant in Fourier series of normal stress σ_y .
 b -- Half width of the column.
 C_n -- Constants in the mapping series.
 $C_{1m}, C_{2m}, C_{3m}, C_{4m}$ -- Constants in stress function ϕ series.
 D -- Depth of the beam.
 d -- Half depth of the beam.
 d_e -- Equivalent depth of the beam.
 E -- Modulus of elasticity in tension or compression.
 e_x, e_y, e_z -- Unit extension (longitudinal strain) in x, y and z directions respectively.
 F -- Correction factor.
 F_y -- $Y/K_b h$
 $F(x)$ -- Function of x only, describing the distribution of σ_y on $y = b$.
 $f_m(y)$ -- Function of y only. Also function of integer m which varies from 1 to ∞
 $f_m''(y)$ -- Second derivative of $f_m(y)$.
 $f_m^{(4)}(y)$ -- Fourth derivative of $f_m(y)$.
 G -- Modulus of elasticity in shear.
 $G(x)$ -- Function of x only, describing the distribution of τ_{xy} on $y = b$.

- h -- Half height of the column of the cross shape.
- I -- Moment of inertia.
- K_1 -- Constant in the bending stress σ_y function.
- K_2 -- Constant in the shear stress τ_{xy} function.
- K_b -- $b = K_b h$
- K_d -- $d = K_d h$
- L -- Distance from the face of a column to the end of a beam of the cross shape.
- L_1 -- Distance from the face of a column to the point on the beam through which concentrated load is applied.
- L_2 -- Distance from the face of a column to the point of the beam through which applied loads are resisted.
- L_3 -- Portion of beam span greater than $1.333d$.
- M -- Bending moment.
- M_y -- Bending moment due to stress distribution σ_y .
- m -- Integer, varying from 1 to ∞ .
- N_x -- Normal force in X-direction.
- P_0 -- Constant in series of the stress function ϕ .
- R -- Ratio of the equivalent depth d_e to the depth $2d$ of the beam.
- r -- Radius of fillet.
- s -- Ungraded mesh size.
- U -- Elastic strain energy
- u -- Displacement in X-direction
- V -- Shear force.
- V_x -- Shear force in X-direction.
- v -- Displacement in Y-Direction.
- X, Y, Z -- Three dimensional axes.

- Z -- In Appendix A, it is complex variable plane.
- ΔS -- Small segment of the beam.
- dU -- Strain energy of the small volume element.
- $\sigma_1; \sigma_2$ -- Principal stresses.
- $\sigma_x, \sigma_y, \sigma_z$ -- Normal stress in x, y, and z directions, respectively.
- τ_{xy} -- Shear stress in Y-direction, acting on the plane perpendicular to X-axis. Similar meaning for τ_{yz}, τ_{zx} .
- $\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ -- Shearing strains.
- μ -- Poisson's Ratio.
- ∇^2 -- Laplace's operator = $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.
- ∇^4 -- $\frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$.
- θ -- $\xi = e^{i\theta}$
- $\theta_1; \theta_2$ -- Angle of relative rotation at the neutral axis.
- ϕ -- Airy's stress function.
- ϕ', ϕ'' -- First and second derivatives of ϕ , respectively.
- α_m -- $m\pi/h$.
- β -- An angle which approximated equivalent depth line makes with the horizontal line.
- ζ -- Complex number.
- $\pi\alpha_i$ -- Interior angles at vertices of the polygon.

Equations, tables, and figures in this dissertation are identified by the following notation: The character before full stop represents chapter or appendix number in which it appears. The number after full stop represents the sequence number of the equation, or table, or figure, of that particular chapter or appendix. As for example: Equation 2.3 is the third equation of Chapter II. Tables and figures of each chapter or appendix are located at the end of the chapter or appendix.

CHAPTER I

INTRODUCTION

Developments in concrete technology and connection methods in metal structures have created confidence among engineers in the validity of the assumption of rigidity of joints in frame structures. Hence, engineers design them in accordance with this assumption. Rigid frame structures are those which have beams and columns as the principal resisting members with the joints providing continuity. This dissertation describes the interaction between structural members rigidly connected at their joints. It is limited to frame structures in which the cross section of the members is rectangular and does not vary abruptly, except that there may be small fillets at the joints. The intersection angle is 90 degrees. This dissertation is further limited to frame structures in which the flexural deformation is the primary distortion. Shear and axial are secondary deformations. Procedures for the analysis of a frame structure which is of indeterminate nature have been known for many years. However, the flexural action of joints is not clearly understood even today. This dissertation presents a study of the flexural interaction at the joint of members rigidly connected. For this study, the structure has been assumed in state of plane stress.

For many years the bending deformation characteristics of beams has been understood. Theoretically the relative rotation of the end faces of the small segment ΔS of the structural member of uniform cross section subjected to pure bending moment M is equal to $M(\Delta S/EI)$. This relative rotation is often interpreted in terms of curvature, since M/EI is equal to curvature. The bending energy can be written as $(M)(\text{rotation})/2$. Hence the bending energy reduces to $M^2(\Delta S/2EI)$. In practice there will be few structures in which the beam is under a pure bending moment condition, i.e. in which the bending moment is constant along the span of the beam. Hence, shear energy will form part of the total energy. Also, axial energy will be a part of the total. Usually the shear energy and the axial energy will be insignificant compared with the bending energy. Hence, the last two factors are neglected for computing the total energy. So bending energy is usually taken as the total energy for practical purposes. Whatever method may be used to study the deformation characteristic of the structure necessary to analyze the indeterminate structure, the evaluation of the energy is a required step directly or indirectly.

For the evaluation of the total energy of members, the above discussed bending energy relationship has been used in the clear span zone by engineers. In the region of the joint two procedures are being used. The first

considers that the depth, i.e. the moment of inertia, at any point in the joint, to be used for evaluating total energy, is the same as at the face of the column. The second procedure uses the depth of the column as the effective depth from the face of the column through the joint, or in frames, since the column height is considerably larger than the depth of the beam, infinity is used as the value for the moment of inertia.

A review of engineering literature indicates that since 1900 many investigations have been conducted on rigid frames, joints, and knees. These investigations are focused on the validity of the assumptions of structural behavior as calculated by the elastic theory. The effect of various sizes and types of fillets on the distribution of the stresses in the joint and its effect on the other part of the frame have been studied. In concrete structures, factors such as the amount of reinforcement and its distribution have been studied. In steel structures, joint conditions and buckling properties have been taken into account. However, very little has been learned about the interaction of the column and the beam in the joint zone.

Inge Lyse and W. E. Black¹ have calculated the angular change of the knee faces from the observed principal

¹Inge Lyse and W. E. Black, "An Investigation of Steel Rigid Frames," Transactions of American Society of Civil Engineers, 107 (1942), pp. 143-144.

stress distribution. This angular change agreed closely with the one calculated from the information of the observed shear forces. They have also calculated the theoretical bending deformation on the assumption that the moment of inertia at any point in the knee zone is the same as that at the face. This calculated deformation is twice that of the one calculated from the observed principal stress distribution. These two authors have concluded that if, however, the effect of the shear is neglected through the frame (as would be done in the design) the large bending deformation assigned to the knee tends to offset the neglect of shear deformation in the frame as a whole.

This conclusion explains the reasons for the use of the first procedure. This matter of compensation is justified in frames in which the span length of the member is very large compared with the depth of the member, so that the effect of the shear is small. Does this compensation idea give the correct results for the frame analysis where the spans are very small? In other words is the contribution of the knee significant in such a case? Since it is known that the knee contributes very little to the bending deformation of the frame as a whole the application of the second procedure seems to be the natural choice. This will necessitate the need for taking into account the shear energy in the evaluation of total energy. However,

most engineers usually neglect shear energy thereby increasing the error in the total energy.

The second procedure was suggested by L. T. Evans.² In setting up the column coefficients for the slope-deflection he mentioned that if the beam is deep with respect to the height of the column, then it is evident that the column cannot bend in the knee zone. And this assumption is equivalent to assuming an infinite moment of inertia in the knee zone. Evans and others have prepared many tables and graphs charting different functions such as stiffness factors, carry-over factors and fixed-end moments for different loading conditions. Evans' assumption means that the moment of inertia increases abruptly at the face of the column.

Can there be any sudden change in the moment of inertia at the face of the knee? A similar question has been raised by Ralph E. Spaulding.³ Discussing the article, "An Analysis of Stepped-Column Mill Bents," by Daniel S. Ling, he pointed out that the effective moment of inertia does not change suddenly when the cross section changes

²L. T. Evans, "The Modified Slope Deflection Equations," Proceedings of American Concrete Institute, 28 (September 1931--April 1932), p. 118.

³Ralph E. Spaulding, Discussion on "An Analysis of Stepped-Column Mill Bents," by Daniel S. Ling, Transactions of American Society of Civil Engineers, 113 (1948), p. 1099.

suddenly. The effective cross section for M/EI analysis has been sketched by him. He did not furnish a computational analysis for this, but mentioned that the photo-elastic and the strain gauge analysis revealed that there are dead areas in the corner of the wider part.

Joseph A. Wise⁴ analyzed the inverted U-frame in which he assumed that the moment of inertia at any point in the knee is the moment of inertia at the face multiplied by the third power of the ratio of half the width of the column to the distance of the point under consideration from the center line of the column. He indicated the need for further investigation of frames with wide members.

This literature indicated that the followers of both procedures are in general agreement that the bending deformation of the knee is very small. Hence they neglect the knee or the joint entirely as far as the bending energy is concerned. The follower of the first procedure compensates for the shear energy of the frame while the follower of the second procedure will take the shear energy separately if it is of significant magnitude compared with the total. The questions remaining unanswered about the joint are: What is the shape of the moment of inertia curve within the joint? Will the neutral axis curvature within the joint

⁴Joseph A. Wise, "Corner Effects in Rigid Frames," Proceedings of American Concrete Institute, 35 (September 1938--June 1939), pp. 190-191.

zone represent the true effective moment of inertia to be used in the deformation analysis of the structure? If not, what is the true representation? What are the factors that affect the true effective moment of inertia? The answers to these questions are important when the spans are short. The aim of this dissertation is to bring forth answers by studying the interaction of the column to the beam subjected to pure bending moment conditions and variable bending moment conditions. The investigation made in this study is limited to cross-shaped joints although it is possible from the results to infer something about the behavior of "T" shaped and "L" shaped joints.

In Chapter II the principles involved in the analysis of indeterminate structures are discussed. It is concluded that the energy variation should be studied for analysis of indeterminate structures, which in turn reduces the problem to the evaluation of stresses. The problem has been specifically outlined for the evaluation of stresses by elasticity theory. Chapter III discusses the experimental method used to investigate the limits of the interaction zone of the cross-shaped joint. From photoelastic analysis it has been concluded that in the beam and in the column the limit extends a distance equal to half the column width and half the beam depth respectively from the faces of the joint. A numerical method for the evaluation of stresses

is discussed in Chapter IV. Using these stresses the internal energy has been calculated at various sections. The equivalent depth is defined as the one which if used in the evaluation of the energy by the conventional beam theory formulas would give the true elastic energy. The equivalent depth increases toward the center of the column up to a maximum of 1.40 times the beam depth for pure bending loading and up to 1.59 times the beam depth if shear forces and bending moment are transmitted across the joint. Chapter V discusses the Fourier series application to evaluate stresses and energy in the column zone. This method is used to compare the results with the numerical method and also to study the effect on the equivalent depth of changing the ratio of the beam depth to the column width. Chapter VI summarizes the results and conclusions of the study.

CHAPTER II

PRINCIPLES

For the analysis of indeterminate structures the only basic understanding required by the engineer is how to determine the deflection, linear or rotational, at any point. The determination of deflections generally reduces to the problem of computing the internal elastic energy of the structure. Because of the importance of the energy it will be appropriate to look into principles and the assumptions involved in the commonly used expressions for the energy. The bending energy of a small segment is $M^2 (\Delta S / 2EI)$ as mentioned in Chapter I. This expression of energy has been derived by using the following assumptions: (1) Hooke's Law is valid and the elastic limit is not exceeded during distortion. (2) Sections are plane after bending, i.e. the strain varies linearly across the cross section. The first assumption is valid since the most commonly used materials obey Hooke's Law and the analysis considered here applies to structures in which the elastic limit is not exceeded. The second one, although sufficiently accurate in most cases, is certainly not valid for sections within a joint.

Even in beams where the top and the bottom fibers deviate only a little from parallel, the distribution of the stresses as determined by the ordinary beam theory is not true as was shown by William R. Osgood.¹ He has derived the stress formulas for the beam with non-parallel surfaces by the application of the wedge theory. The use of the ordinary beam theory will not cause a large error in the stresses as long as the angle between the two non-parallel sides does not exceed 10 degrees. In the frame structure the beam from the face of the column inwards could be considered as a beam having a wedge angle of 180 degrees formed by the top and the bottom non-parallel surfaces. Therefore, the stress distribution determined by the ordinary beam theory will not be valid. Beginning at the face of the column the stresses will be distributed fan-like through the joint. This will result in the non-linearity of stresses in the joint zone.

Since the assumption of linearity is not satisfied, the ordinary formula for bending energy should not be used. This also indicates that the concept of geometric moment of inertia should not be used to evaluate the total energy. Also the conventional energy equation is often written in

¹William R. Osgood, "A Theory of Flexure for Beams with Non-Parallel Extreme Fibers," Transactions of American Society of Mechanical Engineers, 61 (1939), Journal of Applied Mechanics, pp. A-122--A-126.

terms of the neutral axis curvature. Many engineers evaluate the energy by using the neutral axis curvature and the external moment M irrespective of the stress distribution. The implication of this practice is illustrated by the assumed linear and non-linear stress distribution produced by the same moment M as shown in Figure 2.1. The angle of relative rotation at the neutral axis will not be equal in the two cases. Applying the conventional formulas the bending energy of the segment shown in Figure 2.1b will be evaluated as $M\theta_2$. The error involved in evaluating the energy by this approach is enormous. Hence, it is incorrect to think that the bending energy can always be computed in terms of neutral axis curvature.

In the beam of variable section, i.e. non-parallel surfaces, there will be shear stresses even in the case of pure bending moment, as shown by William R. Osgood.² Accordingly, there will be shear energy in the pure bending moment case, which should be taken into account to evaluate total energy. Its significance will not be considered at this stage. The 180 degrees wedge analogy discussed above suggests that there will be shear stresses in the column zone of the frame structure subjected to pure bending. Also in beams subjected to variable moment along the span

²Ibid., p. A-125.

there will be shear stresses. Due to non-linearity of bending stresses in the column zone the shear stresses will not be distributed in the parabolic shape even when the cross section is a rectangle. Hence, the shear energy if taken into account should not be evaluated by the conventional formula $1.2 V^2/2GA$.

If the stresses are not distributed as assumed in beam theory, then the total energy at any cross section can be evaluated by the integration of the strain energy density along the depth or by the summation of the strain energy of the small volume elements. The strain energy dU of the small volume element shown in Figure 2.2 can be written as follows:³

$$dU = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z + \tau_{xy} \gamma_{xy} + \tau_{yz} \gamma_{yz} + \tau_{zx} \gamma_{zx}) dx \cdot dy \cdot dz. \quad (2.1)$$

where $\sigma_x, \sigma_y, \sigma_z$ are the normal stresses, $\tau_{xy}, \tau_{yz}, \tau_{zx}$ are shear stresses, $\epsilon_x, \epsilon_y, \epsilon_z$ are unit extensions and

$\gamma_{xy}, \gamma_{yz}, \gamma_{zx}$ are shear strains. By using the stress-strain relationship the strain energy equation can be written in terms of stress as follows:

$$dU = \left[\frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\mu}{E}(\sigma_x \sigma_y + \sigma_y \sigma_z + \sigma_z \sigma_x) + \frac{1}{2G}(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) \right] dx \cdot dy \cdot dz. \quad (2.2)$$

³Chi-Teh Wang, Applied Elasticity (New York: McGraw-Hill Book Company, 1953), p. 37.

where E is Young's modulus, G is the shear modulus, and μ is Poisson's ratio.

When the width of the beam is small compared with the depth, the beam may be regarded as being an example of plane stress.⁴ In such a case the energy of the small segment per unit width will be reduced to

$$dU = \left[\frac{1}{2E} (\sigma_x^2 + \sigma_y^2 - 2\mu \sigma_x \sigma_y) + \frac{1}{2G} \tau_{xy}^2 \right] dx \cdot dy. \quad (2.3)$$

It is to be noted that by taking σ_x and τ_{xy} as zero and assuming the linear bending stress condition for σ_y , the conventional bending energy formula in terms of moment and the moment of inertia results from the plane stress formulation after integration over the depth. By the use of the strain energy formula of the plane stress condition the variation of the energy in the joint zone of the frame can be evaluated provided the actual stress condition can be determined at various points by some means. Information on the energy variation in the joint will be enough to determine the contribution of the various sections of the column to the deformation of the rest of the frame. Knowing the energy at any section in the joint zone and also the moment and the shear force, one can evaluate the equivalent moment of inertia or an equivalent depth. With this equivalent depth information the conventional formulas can be used to

⁴Ibid., p. 46.

evaluate the deflection, or the functions related to it, necessary for the analysis of an indeterminate structure. It is to be understood that the interpretation in terms of equivalent depth is a matter of convenience to explain the behavior in the conventional form.

The essential problem for evaluating the energy by the elasticity theory reduces to the determination of stresses in the structure. This can be done experimentally or theoretically. In either case it would be essential to investigate the boundary conditions which have to be imposed in order to produce the desired distortion in the structure. For the investigation of the interaction of columns with beams in the frame structures or specifically for the study of the energy variation in the internal joint subjected to flexural action, it will be enough to study the cross-shaped structure subjected to flexural action as shown in Figure 2.3. The σ_x stresses on ends are distributed so as to produce zero bending moment. The distribution of boundary stresses σ_x , σ_y , and τ_{xy} will be discussed in Chapter III. Due to the nature of the imposed boundary conditions as shown in Figure 2.3 the column part of the frame will act as part of the horizontal beam, i. e. there would not be any bending of the center line of the column itself. Referring to Figure 2.3, it is implied that there will not be any bending moment on horizontal cross sections of the column. The difference between the usual knee study which

is more a study of arch action and the study of the joint subjected to flexural action should be clearly noted. Even though the geometric shape of frames in the two studies appears to be the same the boundary conditions will be entirely different. The boundary forces shown in Figure 2.3 are the arbitrarily-imposed conditions used in this study.

In order to determine the state of the stress in an elastic body by the Theory of Elasticity it is necessary to solve the equations of equilibrium expressed in terms of stresses⁵ together with Beltrami-Michell compatibility equations⁶ subjected to proper boundary conditions. The alternative approach is to solve the equations of equilibrium expressed in terms of displacement⁷ subject to proper boundary conditions. Considering the case of plane stress and the plane strain condition in the X and Y directions and introducing the Airy's stress function such that the

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} \quad (2.4)$$

compatibility equations will be reduced to the biharmonic

⁵Wang, op. cit., p. 6.

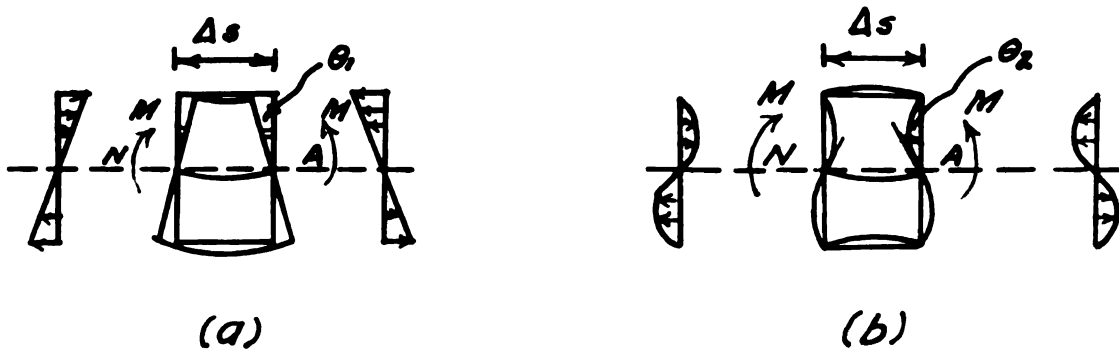
⁶Ibid., p. 33.

⁷Ibid., p. 34.

equation. This biharmonic equation is written as

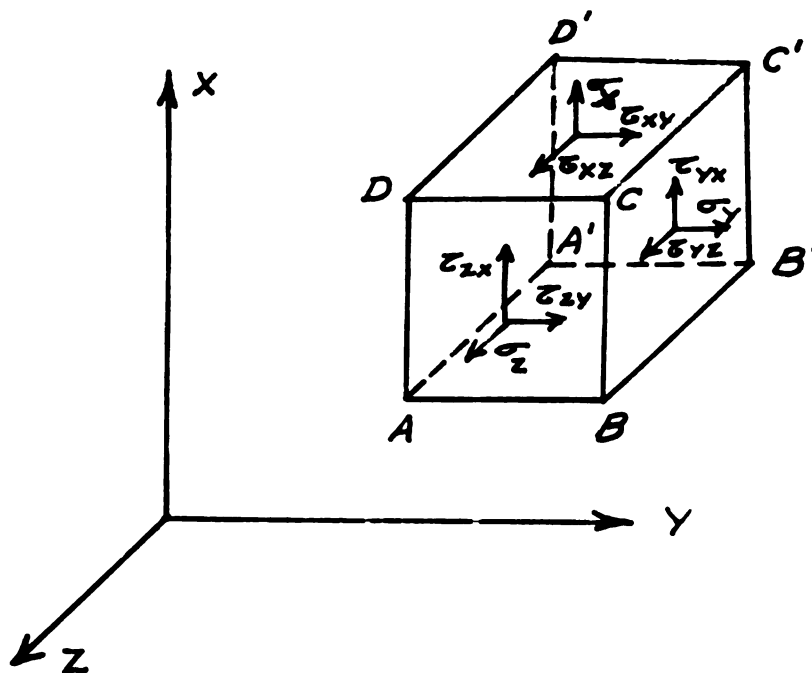
$$\nabla^4 \phi = 0 \quad (2.5)$$

where ∇^2 is Laplace's operator. The equations of equilibrium are satisfied identically. Hence, the problem of determining stresses in the elastic body reduces to finding the solution of the biharmonic equation satisfying the boundary conditions of the elastic body.



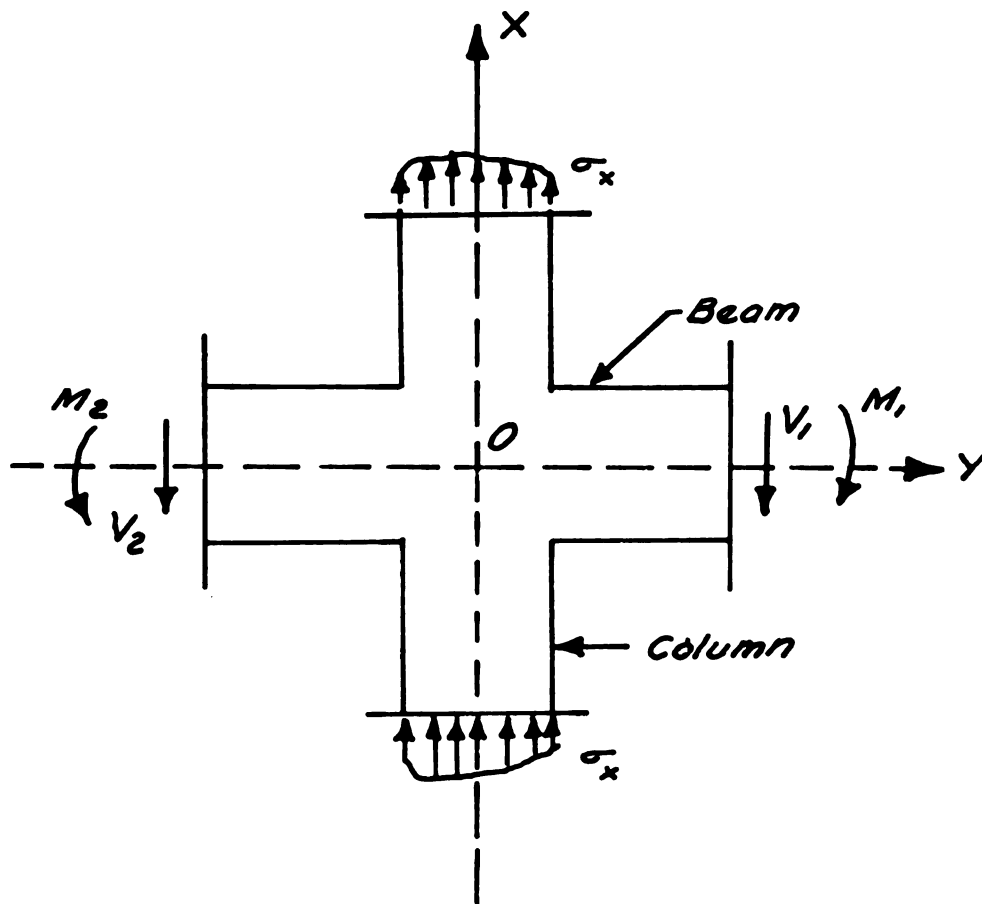
LINEAR AND NON-LINEAR STRESS DISTRIBUTION

FIGURE 2.1



STRESS BLOCK

FIGURE 2.2



CROSS-SHAPED JOINT SUBJECTED TO FLEXURAL ACTION

FIGURE 2.3

CHAPTER III

EXPERIMENTAL ANALYSIS

As discussed in Chapter II, the interaction of the column with the beam at an internal joint can be studied by analyzing the cross-shaped structure. This analysis requires basically the determination of stress distribution. This could be done experimentally by measuring strains or displacements, by determining stresses by optical methods such as photoelasticity, or theoretically by evaluating Airy's stress function ϕ or computing displacements u and v . In either case it is essential to investigate the limit of the interaction in the cross-shaped joint, i.e. the point beyond which the conventional (beam theory) mode of stress distribution is valid. Hence, the aim of this experiment is to investigate the limit. As the aim of the experiment is of qualitative nature, the photoelastic method has been selected for the purpose.

Photoelastic Method

The photoelastic method was developed by David Brewster in 1812. He discovered that an optically isotropic transparent solid becomes optically anisotropic upon forced deformation. It has been shown that in the plane stress condition the difference between the principal indices of

refraction is proportional to the difference between the principal stresses in the deformed material and that the optical axes of the deformed material coincide with the principal stress directions. To apply this principle, two optical systems are used in the photoelastic method. In one system a circular polariscope with monochromatic light is used to determine the lines of constant relative retardation, i.e. isochromatic lines, which are the lines of constant principal stress difference or lines of constant maximum shear stress in the photoelastic model. The second system consists of a cross polarizer and analyzer and uses white light which is suitable for the determination of isoclinic lines, i. e. the lines of constant inclination of principal stresses.

Equipment

The optical system known as a photoelastic polariscope consists of a polarizer, two quarter wave plates, model, analyzer, and the camera. It also includes an attachment for loading specimens. The arrangement of all parts has been standardized and presented in many books.¹ In this study a commercial polariscope, Chapman 5" Photoelastic Polariscope, as shown in Figure 3.1 is used. The loading

¹See for example, George Harmor Lee, An Introduction to Experimental Stress Analysis (New York: John Wiley and Sons, Inc., 1950), p. 163.

frame with the model mounted in it is shown in Figure 3.2. The loading frame was designed in such a way that the span of the beam of the cross-shaped frame could be varied. This feature was kept so as to investigate the effects of the span length.

Material of the Model

For determination of isochromatic fringes of the cross-shaped model, CR-39 was selected due to the following favorable properties:

1. It has polished surfaces.
2. It does not develop machining stresses even under relatively high cutting speeds. Also the machinability is fair.
3. Aging effect is small.
4. Elastic properties are good. Modulus of elasticity is 300,000 psi. Stress-strain curve is linear up to 3,000 psi. Ultimate strength is 6,000 psi.
5. Photoelastic constant is 84 psi. per fringe per inch of thickness. Hence for a $1/4$ " thick specimen the first order of the fringe will develop at a stress of 336 psi. and linearity will hold good up to a fringe order of 9.
6. The effect of temperature on physical and optical properties is very small.

Model Form

The model dimensions are shown in Figure 3.3. As shown in the figure, the horizontal part will act as the beam and the vertical portion will act as the column. The top and bottom parts of the column are unequal for equipment convenience. But as the lengths of the top column and the bottom column are large compared with the depth of the beam, the inequality will not affect the experiment. The small fillets of $r = 1/16$ " were provided to relax the stress concentration. Dimensions were selected in the model to make the maximum use of the space and the loading capacity of the Chapman Polariscopes. The loads are applied through points P, Figure 3.3, by means of a loading frame, and resisted at points Q. Figure 3.2 is a photograph of the set-up. It is to be noted that when the applied loads P are resisted only by loads Q there will not be any shear forces in the BC portion of the structure and the system is under pure bending moment in zone BC. When the shear forces are desired in the system, a pin is introduced in the hole made in the lower part of the column, which is rigidly connected to the loading frame. This will produce an axial force in the column and hence shear forces in the portion BC of the system.--The system in such a case will not be symmetrical about the horizontal axis due to unsymmetrical axial forces in the column.

Procedure

As discussed previously, for determination of isochromatic lines, a circular polariscope is used which contains a polarizer, analyzer, and two quarter wave (mica) plates. In the Chapman polariscope all parts are housed in the optical barrel. As illustrated in Figure 3.1, the optical barrel is arranged so that the loading frame with model will be between the mica plates. Spans L_1 and L_2 are 2.5" and 1", respectively. In the case of pure bending moment in portion BC, the load P is 41.25 lbs. A photograph taken of the isochromatic fringes with this loading condition is given in Figure 3.4. Span L_2 was varied to investigate the effect of it on the distribution of stresses. It was found that the use of a span L_2 greater than 1" did not change the mode of stress distribution from the one obtained in Figure 3.4. In the second case the shear forces were introduced in the portion BC of the system by introducing the pin in the column so that the applied loads P are resisted by supporting forces Q and an axial force through the pin. Spans L_1 and L_2 were kept the same and the load P was 55 lbs. The photograph of the isochromatic fringes is given in Figure 3.5.

Discussion and Conclusion

In Figure 3.4 for pure bending moment loading, the order for isochromatic fringes increases linearly from the

neutral axis at the beam section located $1/4''$ from the face of the column, i.e. at a section located away from the column face a distance equal to half the width of the column. This means, from the definition of isochromatic fringes, that $(\sigma_1 - \sigma_2)/2$ varies linearly. (σ_1 and σ_2 are the principal stresses.) At a beam section far away from the face of the column one cannot expect to find shear stresses or vertical normal stresses because this portion will behave like a beam of uniform section under pure bending moment. Since the isochromatic fringes at the section $1/4''$ from the face of the column appear identical to the ones far away from the face of the column, i.e. to the case of the uniform beam subjected to pure bending moment, it is to be concluded that there are no shear stresses or vertical normal stresses on the section $1/4''$ from the face of the column. Hence, σ_1 will be equal to σ_y and σ_2 will be equal to σ_x . Since the stress σ_x is zero, σ_y varies linearly, and the stress distribution on this section is of conventional form.

In the column zone at the horizontal sections $1/4''$ above or below the faces of the beam the order of isochromatic fringes is zero. This means that the section is stress free and that the flexural action of the beam is not carried into the column beyond a section located half the depth of the beam from the top or bottom of the beam.

From the linearity of the bending stresses beyond the section $1/4"$ from the face of the column it is deduced that when a shear force does exist there, then the shear stresses vary in a parabolic manner. Hence, the isochromatic fringes in this region should be the same as those of a beam similarly loaded but of constant cross section (not interrupted by an integrally-attached column). This is verified by comparing Figure 3.5 with results for the conventional case as given by Frocht.² The two agree outside the section $1/4"$ from the column face. In the lower column the order of the isochromatic fringes is constant except for some slight disturbance due to machining stresses. At the cross section $1/4"$ from the face of the joint it was found that in the pure bending moment case there were no bending stresses. It is to be expected there will not be any shear stresses at any cross section in the column beyond this line. It means that the horizontal section is a principal plane. Hence, σ_1 and σ_2 will be equal to σ_x and σ_y . σ_2 will be zero since σ_y is zero. Therefore, the same order of the fringe along the horizontal cross section, which means that $(\frac{\sigma_1 - \sigma_2}{2})$ is constant, implies that σ_x is distributed uniformly.

²Max Mark Frocht, Photoelasticity (New York: John Wiley and Sons, Inc., 1941), p. 148.

From the above experimental investigation and discussion it is concluded that the interaction of the column with the beam extends to the vertical cross section half the width of column from the face of the column, and the interaction of the beam with the column extends into the column to the horizontal cross section half the beam depth from the face of the beam. Hence, the conventional theories of stress distribution hold beyond this interaction zone.

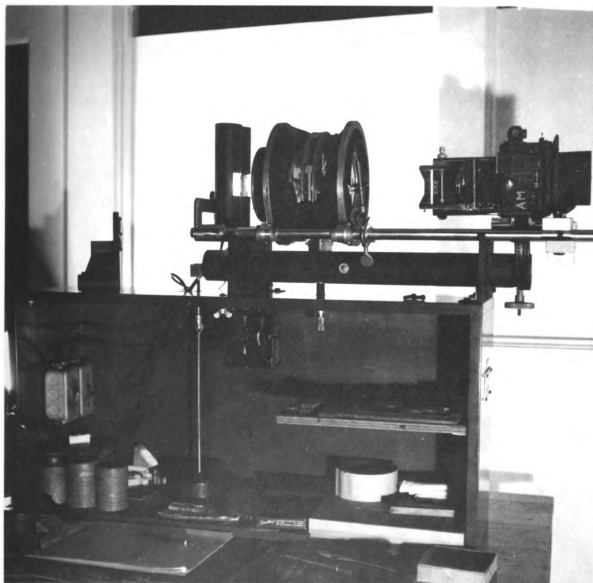


FIGURE 3.1 Polariscope set-up.

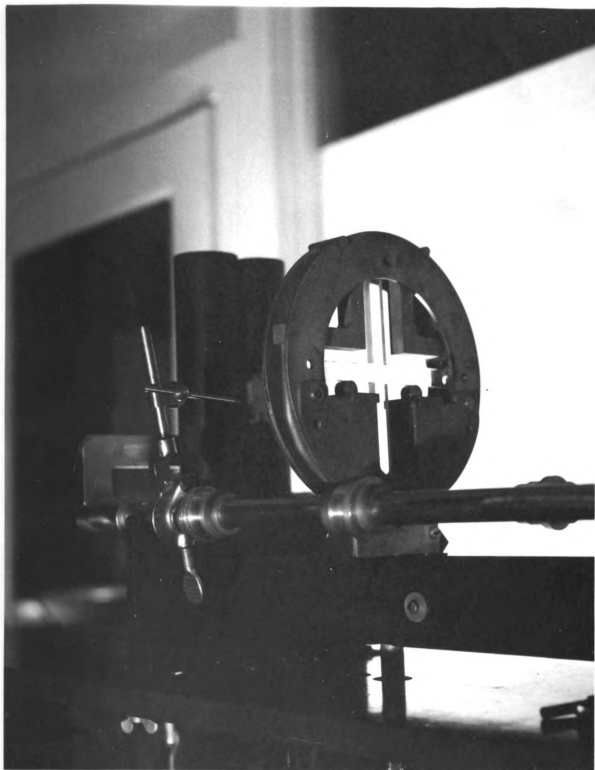
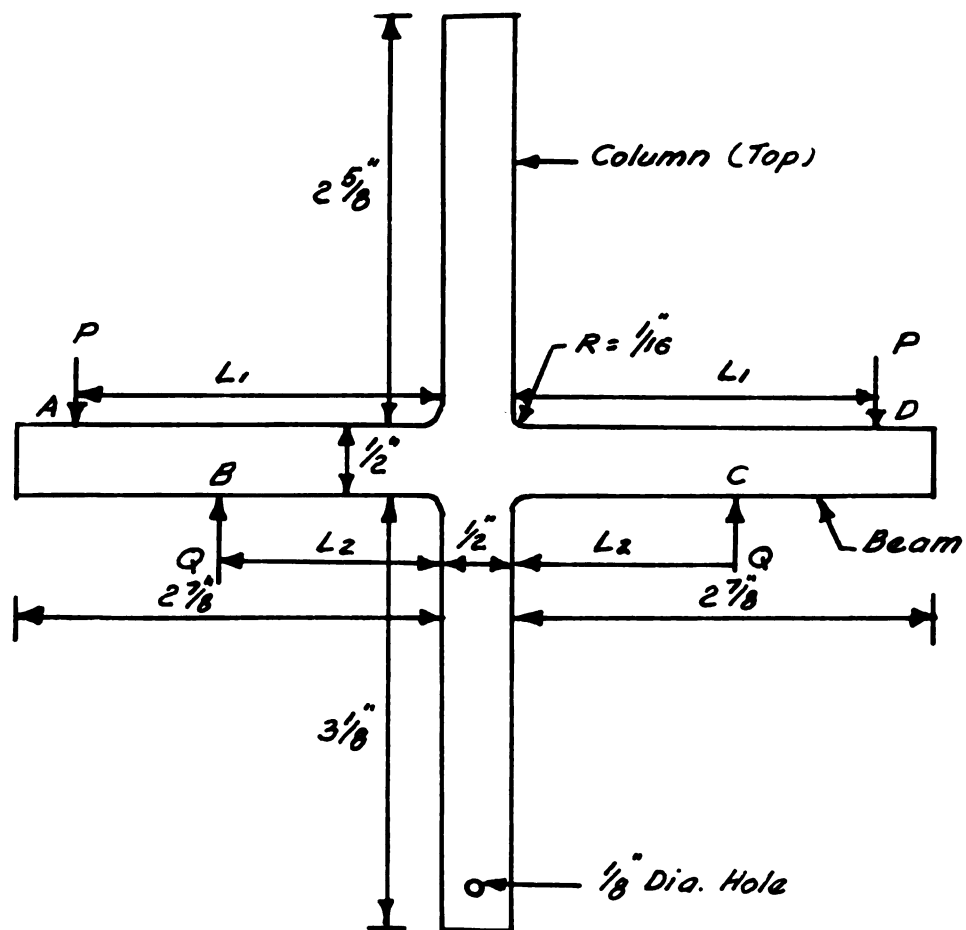


FIGURE 3.2 Model mounted in loading frame.



Model thickness = $\frac{1}{4}$ "

MODEL FORM

FIGURE 3.3

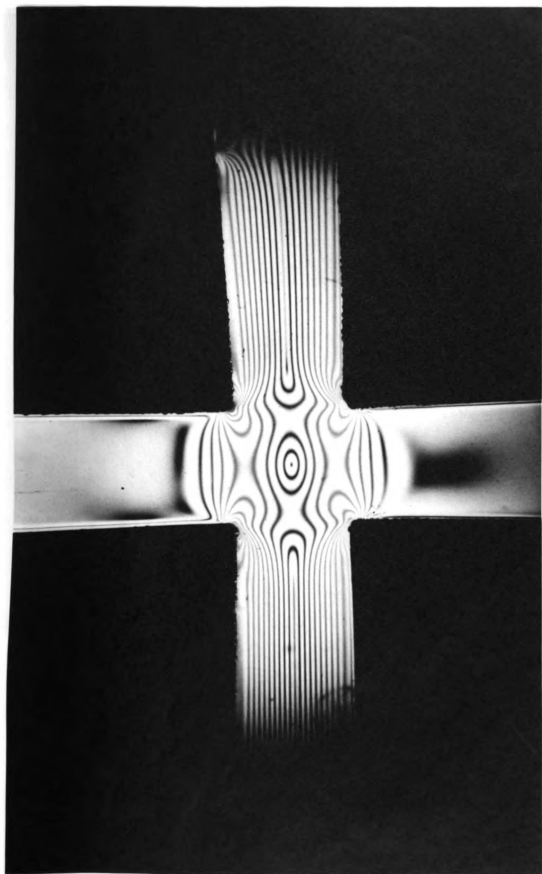


FIGURE 3.4 Isochromatic fringes of the cross shape subjected to pure bending moment.

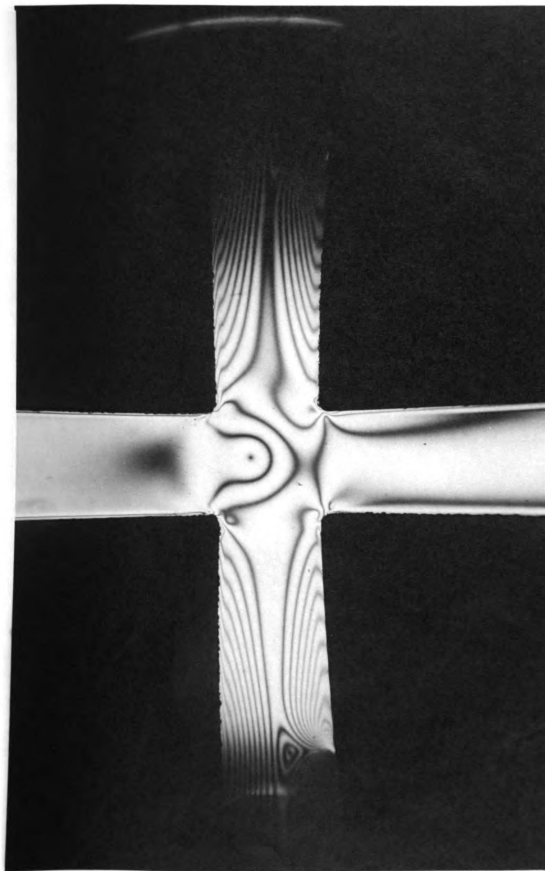


FIGURE 3.5 Isochromatic fringes of the cross shape subjected to variable bending moment (shear loading case).

CHAPTER IV

NUMERICAL METHOD

The general solution of the biharmonic differential equation, as discussed in Chapter II, satisfying the boundary conditions, is usually quite difficult. Some problems of practical interest can be solved by an inverse method, making some assumptions regarding the stress distribution. This will lead to an expression for Airy's stress function ϕ with some undetermined coefficients. Another method often used is to assume the stress function in terms of a series with undetermined coefficients. These undetermined coefficients are determined from the boundary conditions of the elastic body, which sometimes can be expanded in Fourier series.

In the absence of any notion of the stress function, the solution of the fundamental biharmonic boundary value problem can be made to depend upon a certain general representation of the biharmonic function by means of two analytic functions of a complex variable.¹ The original presentation of this method was made by E. Goursat in 1898.

¹I. S. Sokolnikoff, Mathematical Theory of Elasticity (New York: McGraw-Hill Book Co., Inc., 1956), p. 262.

While this method will present calculation difficulties for certain types of boundaries, general formulas have been written for the case when the boundary of the region is a circle.² It is possible to apply the formulas, derived for the case of a circular region, for any simply-connected region by introducing the mapping function which maps the particular region of complex Z -plane conformally onto the unit circle. An attempt to evaluate the mapping function for the cross shape is discussed in Appendix A. This method was abandoned because of convergence difficulties with the infinite series representation of the mapping function.

Because of the difficulty of the analytical solution for the elasticity problem, approximate numerical methods were used. The method of finite difference, which was first applied in elasticity problems by C. Rung in 1908, is a numerical method widely used in recent years. The wide use of this approach is due to the development of computers for the solution of simultaneous equations and the development of the relaxation method, which is used in the absence of computer facilities. In the method of finite differences one replaces the partial differential equation and the equation defining the boundary conditions by finite difference equations. Then the problem reduces to the solution of simultaneous linear algebraic equations.³

²Ibid., p. 145. ³Wang, op. cit., p. 106.

In elasticity problems the boundary conditions are expressed in terms of either the stresses or the displacements or a combination of both. For the plane stress condition the governing differential equation solved is

$\nabla^4 \phi = 0$, where ϕ is Airy's stress function. Hence, it will be necessary to express the boundary conditions in terms of Airy's stress function. This can be done by application of relations of Airy's stress function with the stresses,⁴ and the relations of the stresses to strains,⁵ and the strains to displacements.⁶ The cross-shaped region will be studied with two loading conditions. In part I of this chapter the boundary conditions are of such a nature that the entire structure is under a pure bending moment condition. In this part there is also given considerable detail on the numerical procedure. In part II, the boundary conditions are of such a nature that there will be external shear forces at any cross section of the beam. For each loading condition, the cross shape will be analyzed both with and without fillets at the re-entrant angles. The pure bending case will be analyzed for two different ratios of beam depth to column width. In part III, results are summarized and their interpretation is discussed.

⁴Ibid., p. 43.

⁵Ibid., p. 30.

⁶Ibid., p. 17.

I. PURE BENDING MOMENT CONDITION

The cross shape to be studied in this case is shown in Figure 4.1. The clear spans of the beam and the columns are taken equal to the width of the column inasmuch as it was concluded in Chapter III that the interaction range extends, at the most, approximately half the width of the column into the beam as well as into the column. Hence, on the boundary AB of the cross shape subjected to pure bending moment, the bending stress will vary linearly. Also the ratio of the beam depth to the column width will affect the interaction. For the present the ratio is taken as unity, and the effect of different ratios will be discussed later on. In many structures, specifically concrete structures, there will be construction fillets at the re-entrant angles formed by the junction of the beam and the column. These construction fillets will relieve the stress concentrations at the re-entrant angles, but will not have any radical effect upon the distribution of the stresses in the beam or the column far from the junction. The cross shape having the ratio of beam depth to column width as one and the ratio of the radius of the fillet to the half depth of the beam as $1/3$ has been analyzed at this point. The effect of the removal of the fillet on the interaction will be studied later on.

A. Cross Shape Having $L/b = 2.0$, $h/b = 3.0$, $d/b = 1.0$ and $r/d = 1/3$

Boundary conditions. Because of the loading conditions, the Airy's stress function ϕ in the cross shape is anti-symmetrical about the Y-axis and symmetrical about the X-axis. Therefore, it will only be necessary to study one-quarter of the entire region as shown in Figure 4.1b. The region has been divided as shown in the figure for setting up the finite difference equations. The boundary conditions can be summarized as follows:

On AB, $\sigma_y = K_1 x$; $\tau_{xy} = 0$.

On OA, $\sigma_y = 0$.

On OF, $\tau_{xy} = 0$.

On BCDEF the normal stress and the shear stresses are zero.

On such a traction-free boundary, the partial derivatives

$\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ of the stress function are constant. Since the stress function is in any case only determined up to an arbitrary additional linear function of x and y , it was possible without loss of generality to set the boundary conditions in such a way that $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$ on BCDEF.⁷ Then replacing the stresses by their expressions

⁷S. Timoshenko and J. N. Goodier, Theory of Elasticity (New York: McGraw-Hill Book Co., 1951), pp. 484-485.

in terms of Airy's stress function by Equation (2.4) and maintaining the continuity of all functions, such as Airy's function and its derivatives, on the boundary ABCDEF, the boundary condition can be reduced as follows:

On AB, $\phi = K_1 x^3/6 - K_1 x d^2/2$, Normal derivative is zero.

On BCDEF $\phi = -K_1 d^3/3$, Normal derivative is zero.

On OA, $\phi = 0$

On OF, Normal derivative is zero.

The implication of zero normal derivative in terms of finite difference equations can be explained by the following example:

$$\begin{aligned} \left(\frac{\partial \phi}{\partial Y} \right)_{6,3} &= 0 \\ \therefore (\phi_{6,4} - \phi_{6,2}) / 2 \Delta Y &= 0 \\ \therefore \phi_{6,4} &= \phi_{6,2} \end{aligned} \quad (4.1)$$

The subscript 6,3 used here identifies the node which is located at $X = 6$ and $Y = 3$ mesh units. The number before the comma represents the value of X and the number after the comma represents the value of Y . The same notation is used through the text.

From Figure 4.1b it can be seen that the ϕ value of point 4,4 can be evaluated by any of the normal derivative conditions at 3,4; 4,3 and n. In this solution it has been

810

12

1

22

27

3

10

2

5

10

1

2

6

1

8

2

1

1

evaluated by taking the average of the slope written in terms of forward difference and backward difference at point n and equating this average to zero, which is the normal derivative condition at n .

Biharmonic differential equation. The finite difference equation of the Biharmonic differential equation for a typical interior node 2,2 as shown in Figure 4.1b can be written as follows:⁸

$$\begin{aligned} (\nabla^4 \phi)_{2,2} = 0 \\ [20 \phi_{2,2} - 8 (\phi_{3,2} + \phi_{2,3} + \phi_{1,2} + \phi_{2,1}) \\ + 2 (\phi_{3,3} + \phi_{3,1} + \phi_{1,1} + \phi_{1,3}) \\ + (\phi_{4,2} + \phi_{2,4} + \phi_{0,2} + \phi_{2,0})] = 0 \quad (4.2) \end{aligned}$$

Similar equations can be set up for each interior node of the region shown in Figure 4.1b. At first these equations will have many unknowns of the nodes which are outside the region. By the application of the boundary conditions discussed above and the known functional values on the boundary all equations can be stated in terms of unknown functions at interior nodes only. Hence, there will be 37 equations with 37 unknowns. These equations have been solved by using the L2 program⁹ in the MISTIC digital

⁸Wang, op. cit., p. 111.

⁹Mistic Library, L2 Program (East Lansing: Computer Laboratory, Michigan State University, 1958).

computer at Michigan State University. The results of these equations are given in Table 4.1.

Bending stress and bending moment. From the ϕ values the bending stress σ_y at any point can be calculated by its relationship expressed in the finite difference form.¹⁰

$$\begin{aligned} \text{e.g. } (\sigma_y)_{2,1} &= \left(\frac{\partial^2 \phi}{\partial x^2} \right)_{2,1} \\ &= \frac{(\phi_{3,1} - 2\phi_{2,1} + \phi_{1,1})}{(\Delta x)^2} \end{aligned} \quad (4.3)$$

The results for the bending stress are recorded in Table 4.2. Its distribution on each section is given in graphical form in Figure 4.2. From the bending stress distribution the bending moment at any section can be calculated. Since the stress distribution is available in graphical form, the bending moment has been calculated by the graphical method. The X-axis of the stress distribution curves in Figure 4.2 has been divided into tenths of an inch. The bending moment at any section was calculated by the summation of products of the stress area of the one-tenth inch ordinate by the moment arm measured from the Y-axis to the middle of the stress segment. The results for the resisting

¹⁰Wang, op. cit., p. 110.

bending moments at various cross sections calculated by the described method are given in Table 4.3. Also the applied moments are given. The percentage errors compared with applied moment at various sections have been evaluated.

From Table 4.3 it appears that the errors at the face of the column and at the section where the fillet starts are the greatest. This indicates that there is considerable error in the stress distribution in the vicinity of the junction of the beam and the column. Due to the sudden change in the cross section at the junction the stress function will vary sharply. Hence, in order to determine the stress function more precisely it will be necessary to have a finer network for setting up the finite difference equations. It is to be expected that the stresses calculated will be considerably in error only in the vicinity of the fillet. Hence, the stress distribution obtained using a finer network throughout the region would not be much different from the one obtained by finer grading only in the vicinity of the fillet.

Graded net. The solution of simultaneous equations is done by MISTIC computer L2 program which is limited to a maximum of 39 equations. The finer grading around the fillet for the cross shape shown in Figure 4.1b would exceed the limit for MISTIC. Consequently it was necessary to make some modifications in the dimensions of the cross

shape shown in Figure 4.1. From Figure 4.2 it seems that at beam sections #6, #7, and #8 the σ_y stress distribution is very close to a straight line. Also in the column beyond the cross section parallel to Y-axis at 6 there are practically no stresses. This agrees with the photoelastic study in Chapter III. Hence, it will be appropriate to take the cross shape as shown in Figure 4.3 for the interaction study. The graded net around the fillet is shown in Figure 4.3b. While setting up the finite difference Equation like Equation 4.2 for points such as 3,23,¹¹ there will be many nodes like 34,2 which are not considered as unknowns. Hence, it will be necessary either to guess the value of such nodes or relate such nodes with neighboring unknown nodes. These means are possible only if some assumption is made. This difficulty can be avoided by replacing the operator ∇^4 by $(\nabla^2)(\nabla^2)$ and setting up the finite difference operators separately for each ∇^2 . The net size used for setting up the ∇^2 operators need not be the same for different nodes or even for the two successive operators at the same node. This procedure was

¹¹ Node 3,23 is identified as the one which is located at $X = 3$ units and $Y = 2.5$ units, i.e. located halfway between $Y = 2$ and $Y = 3$ units. The same notation applies when two numbers are before the comma. This notation is used throughout the text to identify the nodes which are between regular nodes.

first used by Allen and Dennis.¹² As an example the finite difference equation for point 23,23 shown in Figure 4.3b has been derived in Appendix B and is written as follows:

$$\begin{aligned}
 & 3^2 \phi_{23,23} - 2^2 \phi_{3,3} - 1^1 \phi_{3,2} - 1^0 \phi_{2,2} - 1^1 \phi_{2,3} \\
 & + 4 \phi_{34,3} + 4 \phi_{3,23} + 4 \phi_{23,3} + 4 \phi_{3,34} + \phi_{4,2} + \phi_{3,1} + \phi_{2,1} \\
 & + \phi_{1,2} + \phi_{1,3} + \phi_{2,4} = 0
 \end{aligned} \tag{4.4}$$

In a similar way the finite difference equation can be set up at each point where Equation 4.2 is not applicable. The boundary condition and its implication in terms of finite difference form is the same as that of the ungraded net except for point 34,34 which is evaluated in Appendix B. The finite difference equations for the graded region shown in Figure 4.3b were solved by the MISTIC computer. The values of the ϕ obtained are given in Table 4.4.

From the stress function values the stresses were calculated by the finite difference Equation 4.3 and from this information the bending moments at various cross sections were calculated by a graphical method. The results are given in Table 4.6.

¹²D. N. De. G. Allen and S. C. R. Dennis, "Graded Nets in Harmonic and Biharmonic Relaxation," Quart. Journal Mech. and Applied Maths., Vol. IV, Pt. 4 (1951), pp. 439-443.

Effect of graded net on bending moments. With the understanding that the distribution of the stresses in the region shown in Figure 4.3b will not be materially different from that of the region shown in Figure 4.1b the bending moment evaluations of Tables 4.3 and 4.6 can be compared to study the effect of the graded net. At beam sections #5, #6, and #7 the error has been increased at the most by 0.30%. At beam sections #3 and #4 the error is reduced by 5.49% and 10.20%, respectively. At beam section #2 the error is increased by 0.78%. The slight increase or decrease in the percentage error at sections #2, #5, #6, and #7 from Table 4.3 to Table 4.6 is not significant. The bending moments were calculated by a graphical method in which the stress values are read from the graph. Hence, any slight errors in the stresses when multiplied by the lever arm for evaluation of the moment may be responsible, but they are not significant. But from comparison of the results at sections #3 and #4 it can be concluded that the graded net improves the results considerably.

Higher order differences. Further improvement can be made in the finite difference approximation by taking a finer net. This will increase the number of simultaneous equations. An alternative approach in which higher order finite difference approximation formulas are used was

suggested by Fox.¹³ The finite difference equation, set up by considering the higher order finite difference approximation for the biharmonic equation could be used for evaluation of stress functions. This would involve considerable labor in setting up the equations. In general, by using the standard first order difference formulas, the accuracy in the stress function obtained is always better than in the derivatives of the function. Hence, even if the stress functions are obtained by standard first order formulas, the improvement in the results will be considerable with little extra labor if the derivatives are calculated by higher order differences and used for evaluation of stress.

For bending stress σ_y the finite difference formula using the higher order difference is as follows:¹⁴

$$\begin{aligned} \text{e.g. } (\sigma_y)_{2,1} &= \left(\frac{\partial^2 \phi}{\partial x^2} \right)_{2,1} \\ &= \frac{1}{(\Delta x)^2} \left[\nabla^2 \phi_{2,1} - \frac{1}{12} \nabla^4 \phi_{2,1} + \frac{1}{90} \nabla^6 \phi_{2,1} \dots \dots \right] \quad (4.5) \end{aligned}$$

The first term of the formula is the contribution of the standard first order difference application. The first

¹³L. Fox, "Some improvements in the use of relaxation methods for the solution of ordinary and partial differential Equations," Proceedings, Royal Society (London), A, Vol. 190 (1947), pp. 31-59.

¹⁴Wang, op. cit., p. 125.

factor has already been evaluated in Table 4.5. The second term is evaluated separately using the stress functions given in Table 4.4. Only the first two terms are taken in the formula (4.5) as the terms beyond this will be insignificant. The results of the bending stress evaluated by the formula (4.5) are given in Table 4.7. These results have been plotted in Figure 4.4. The bending moments have been calculated graphically using Figure 4.4. The comparison of the results is given in Table 4.8.

The effect of application of higher order difference in the stress distribution can be studied by looking into results of the percentage error in the calculated resisting moments given in Tables 4.6 and 4.8. There is practically no improvement in the result at beam section #3. Inside the column, i. e. at beam sections #0, #1, #2 the errors have been reduced almost to zero. The change in the errors at sections #4, #5, #6, and #7 is not important as far as the interaction of the column and beam is concerned. Looking into the values of the stresses in Table 4.5 and Table 4.7 it can be summarized that there is significant difference only at the node point of the maximum stress on a section. Inside the column zone there is quite a difference percentage-wise at the nodes close to the top boundary, but as far as the area of the stress diagram is concerned this will not be of any importance. Hence, in conclusion the only way to improve the error in the stresses near the

fillet is to have a very fine net around the fillet. This will increase considerably the number of equations. As the problem involves the joint region as a whole and the error in the stresses near the fillet will not greatly change the results, it will not be justifiable to spend more time for the solution of more simultaneous equations. In the subsequent calculations the values σ_y are calculated by higher order differences, while the normal stress σ_x and the shearing stress τ_{xy} are calculated by the standard first order difference.

Normal stress σ_x and shearing stress τ_{xy} . The finite difference formulas for σ_x and τ_{xy} stresses for the typical node 2,2 as shown in Figure 4.3b can be written as follows:^{15,16}

$$(\sigma_x)_{2,2} = \left(\frac{\partial^2 \phi}{\partial y^2} \right)_{2,2} = \frac{(\phi_{2,3} - 2\phi_{2,2} + \phi_{2,1})}{(\Delta y)^2}$$

$$(\tau_{xy})_{2,2} = - \left(\frac{\partial^2 \phi}{\partial x \cdot \partial y} \right)_{2,2} = - \frac{[(\phi_{3,3} - \phi_{3,1}) + (\phi_{1,1} - \phi_{1,3})]}{4 \cdot \Delta x \cdot \Delta y} \quad (4.6)$$

Using the Formula 4.6 the σ_x and τ_{xy} stresses are calculated with the help of the stress function ϕ given in Table 4.4. The results of these are given in Table 4.9.

¹⁵Ibid., p. 110.

¹⁶F. S. Shaw, An Introduction to Relaxation Methods (New York: Dover Publications, Inc., 1953), p. 36.

Energy. Knowing the stress distribution at various sections the energy can be evaluated by the formula discussed in Chapter II. The energy formula for a segment of unit length in the Y-direction is

$$U = \int \left[\frac{1}{2E} (\sigma_x^2 + \sigma_y^2 - 2\mu \sigma_x \sigma_y) + \frac{1}{2G} \tau_{xy}^2 \right] dx. \quad (4.7)$$

The integral along the entire depth in the X-direction can be evaluated by summation of the energy of small segments in the X-direction or by integration of the energy function in X-direction. As the stress information is available in numerical form it will be convenient to evaluate the energy by a graphical method. For this purpose σ_y^2 , σ_x^2 , τ_{xy}^2 were plotted, and the summation of the area between each curve and X-axis was carried out with a Planimeter. Figure 4.5 shows, for example, the curves of σ_y^2 at sections #0, #1, #2, #3 and the areas obtained. With the application of this area information to the Formula 4.7 the energy can be evaluated. The results of the computation are shown in Table 4.10. The value of Poisson's ratio μ is used as 0.30 throughout the text. The effect of μ on equivalent depth will be discussed later. Knowing the energy the equivalent depth can be calculated.

Equivalent depth. The equivalent depth at any section is defined as the one which, when used in the evaluation of the energy by the conventional beam theory formula will give

the actual energy. This can be illustrated in the following manner.

In the conventional energy formula, if the segment of the structure is under pure bending moment M , then the energy of the segment per unit length is given by $M^2/2EI$. The energy of the segment of rectangular cross section of unit length and of unit width and depth D will be $6M^2/ED^3$. According to the definition of equivalent depth, the following equation can be written for any section,

$$\frac{6M^2}{Ed_e^3} = \text{true energy at any section (Table 4.10),}$$

$$\text{e.g. for section \#3, } \frac{6M^2}{Ed_e^3} = \frac{0.403109}{2E} K_1^2 d^3 .$$

Note that d_e is the equivalent depth at the section while d is the half depth of the actual beam. M is the bending moment at the section. In this case $M = 0.66667 K_1 d^3$ and the equation yields

$$d_e^3 = \frac{5.3333d^3}{0.403109} .$$

Let R represent the ratio of equivalent depth to actual depth,

$$R = \frac{d_e}{2d} = \frac{\text{Equivalent depth}}{\text{Depth of beam}} .$$

$$\text{For section \#3, } R^3 = \frac{0.66667}{0.403109} = 1.6538 .$$

The results for R at various sections are given in Table 4.11. The graphical representation of the equivalent depth is given in Figure 4.6. The results indicate that there is a gradual increase in equivalent depth or moment of inertia and at the face the fillet is not fully effective as far as the energy evaluation by conventional beam theory is concerned. The equivalent depth, from Section #4 onwards, approaches 1.0 as expected.

B. Equivalent Depth Related to Ratio of d/b of Cross Shape

As mentioned earlier the ratio of the beam depth to the column width will affect the interaction of the column and the beam. In other words, the equivalent depth diagram will be affected by the ratio d/b . This can be viewed from two angles: (1) the effect of the ratio on the stress distribution curve at the face of the column, and (2) the effect of the ratio on the stress distribution inside the column even though the stress distribution on the face of the column is the same. The stress distribution on the face of the column will affect the equivalent depth at the face and also inside the column where the stress distribution is not that of the conventional theory even though the stress distribution on the face is linear. At this stage the primary interest will be in studying the effect of the ratio on the equivalent depth at the face of the column. How the stress distribution curve on the face of

the column will affect the equivalent depth inside the column will be discussed later. The stress distribution on the face depends upon the intensity of stress concentration near the corner. The effect of dimension changes on the stress concentration has been studied by several investigators and will be referred to below. The second effect will be studied in Chapter V under the reasonable assumption that for a given applied moment and beam depth the variations in the other dimensions which do not change the stress concentration factor also leave unchanged the stress distribution on the face of the column.

Stress concentration. In the neighborhood of the joint the stress concentration will be a function of the radius of the fillet, the clear height of the column, the clear span length of the beam, the depth of the beam, and the column width. According to the photoelasticity study in Chapter III, the stress distribution in the beam at a distance from the face of the column greater than half the width of the column coincides with the elementary beam theory distribution. This indicates that the same stress distribution in and around the joint would be obtained for any clear span length of the beam greater than half the column width. Hence, it is concluded that the stress concentration factor will be independent of the clear span length when the ratio of clear span length to half the

column width exceeds one, which covers all practical cases. It has been shown that with a given ratio of beam depth to column width, when the ratio of radius of fillet to the depth of the beam is greater than 0.10, i.e. $r/d > 0.20$, the stress concentration factor is independent of the ratio of clear height of the column to the radius of the fillet.¹⁷ This covers the case analyzed in the previous pages and also all practical cases. Hence, for practical purposes it can be concluded that the stress concentration factor is independent of the clear height of the column and the ratio of the height of the column to the radius of the fillet. So for a given r/d ratio the only parameter affecting the stress concentration factor is the ratio, d/b , of the depth of the beam to column width. The following paragraph will show that the stress concentration factor is independent of d/b when this ratio is less than about 0.30 for $r/d = 1/3$. In most cases of frame structures with short beams, this condition is not satisfied and it is necessary to study the effect of varying d/b .

The ratio d/b is equal to $d/h \cdot h/b$ where $2h$ is the total height of the column. In the range where the stress concentration factor is independent of d/h and h/b , it will

¹⁷S. Timoshenko, Strength of Materials, Part II (New York: D. Van Nostrand Co., Inc., 1956), p. 327.

be independent of d/b . It has been shown¹⁸ that for $d/h = 1/2$ and h/b smaller than $1/2$, the stress concentration is independent of h/b for values of r/d ranging from 0.30 to 2.0. The same investigation¹⁹ showed that the stress concentration factor is almost independent of d/h (within 6%) when this ratio is smaller than $3/5$ and h/b is smaller than $1/2$ with r/d varying from 0.3 to 2.0. Hence, for d/h equal to or smaller than $3/5$ and h/b equal to or smaller than $1/2$, the stress concentration factor is independent of d/b . Since in all cases of practical interest d/h is equal to or smaller than $3/5$, the stress concentration factor is independent of d/b when d/b is smaller than $3/10$.

For $r/d = 1/3$ and $d/h = 3/7$, the stress concentration factors for $d/b = 1.0, 1.5$ and 2.0 are 1.54, 1.50 and 1.47, respectively.²⁰ The equivalent depth analysis for the case of $d/b = 1.0$ is given in part IA. To show how the stress concentration factor affects the equivalent depth, another case is analyzed with $d/b = 1.5$.

¹⁸J. B. Hartman and M. M. Leven, "Factors of stress concentration for the bending case of fillets in flat bars and shafts with central enlarged section," Proceedings of the Society for Experimental Stress Analysis, IX, No. 1 (1951), p. 57.

¹⁹Ibid., p. 58.

²⁰R. E. Peterson, Stress Concentration Design Factors (New York: John Wiley and Sons, Inc., 1955), p. 71.

Cross shape having the ratio $L/b = 2.5$, $h/b = 3.5$, $d/b = 1.5$, and $r/d = 1/3$. The cross shape shown in Figure 4.7 was analyzed. The boundary conditions are the same as of the case shown in Figure 4.3. The same procedure was followed for the analysis. The resulting stress functions and stresses have been summarized in Tables 4.12 and 4.13. By the application of the numerical procedure explained before, the energy at various sections is calculated. Computations are given in Table 4.14. The equivalent depth information is in Table 4.15 and is presented graphically in Figure 4.8. Comparing the ratio at the face of the column of Table 4.15 (Section #2) to the one of Table 4.11 (Section #3), it is found that R has decreased from 1.1826 to 1.1388. It indicates that as d/b increases the R -value decreases. Interpretation of this fact will be discussed in Part III of this chapter.

Also a comparison of Figure 4.8 with Figure 4.6 indicates that the equivalent depth increases more slowly inside the column for the case of $d/b = 1.5$ than the case of $d/b = 1.0$. It means that the ratio of d/b affects the shape of the curve inside the column, i.e. even for the same d and the same mode of stress distribution on the face of the column, the alteration in the width of column will change the mode of distribution inside the column. This is the second effect as mentioned earlier in Part IB of this

chapter. The effect of various d/b ratios will be discussed in Chapter V.

C. Cross Shape Having $L/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$,
and $r/d = 0$

To study the effect of the removal of the fillet, an analysis was made of the cross shape as shown in Figure 4.3 with the fillet replaced by a corner. From the similarity of the boundary conditions it could be noted that the equations set up for the cross shape with fillet shown in Figure 4.3 can be used for the cross shape with a square corner. Due to elimination of the fillet, the ϕ values of the nodes $3^4, 3$; $3, 3$; and $3, 3^4$ are known now. Also the expression for $3^4, 3^4$ in terms of internal nodes will be changed as the boundary conditions which relate $3^4, 3^4$ are changed. Hence, after making these alterations the equations were solved. The results for the stress functions are given in Table 4.16. The resulting stresses, energy, and equivalent depth information are given in Tables 4.17, 4.18, and 4.19, respectively. The equivalent depth is presented graphically in Figure 4.9.

Calculation of the equivalent depth was carried out only as far as section #4 because Figure 4.6 indicates that, from section #4 on, the equivalent depth does not differ significantly from the actual depth. Comparison of results for equivalent depth from Table 4.19 with Table 4.11

indicates that the equivalent depth at the face has decreased. The reason is that in the case of a square corner the contact area at the face is smaller than the contact area in the case of fillet. Hence, for the same external moment at the section, the area between the graph of σ_y^2 and the X-axis will be greater, which will yield more energy and consequently less equivalent depth. The stress distribution of σ_y on the face of the column in the case with a fillet is flatter than in the case without a fillet, and this effect will be carried inside the column. The flatter the curve of the stress σ_y is, the less will be the energy. Hence, the energy due to σ_y at any section in the case of the fillet is less compared with the energy at the corresponding section in the case of the square corner. This indicates why the equivalent depth in the case of the fillet is greater for any section than it is at the same section in the square corner case.

D. Effect of Poisson's Ratio on Equivalent Depth

Poisson's ratio depends upon the kind of material. For steel it is between $1/2$ and $1/3$. For concrete it lies between 0.10 and 0.30. The most common value for concrete is between 0.15 to 0.20. As the elastic energy depends upon Poisson's ratio, another analysis is made with the value of Poisson's ratio of 0.15. The calculations are carried out with the information from Table 4.10. The

results are tabulated in Table 4.20. Comparison of equivalent depth of Table 4.20 with that of Table 4.11 indicates that the equivalent depth has increased inside the column by decreasing the value of Poisson's ratio. But this change is not significant compared with the value of the equivalent depth. Hence, for the pure bending case it is concluded that the equivalent depth is not sensitive to variations in Poisson's ratio.

II. VARIABLE BENDING MOMENT CONDITION (SHEAR-LOADING CASE)

A. Cross Shape Having $L/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$ and $r/d = 1/3$

The cross shape shown in Figure 4.3 is taken for this analysis. As it was concluded that the grading shown in Figure 4.3 gives good results, the same grading is used in this case. According to the conclusion regarding the extent of the interaction zone drawn from the photoelasticity study, the conventional beam theory stress distributions will act on the boundaries of the cross shape shown in Figure 4.3. The boundary forces are shown in Figure 4.10.

Boundary conditions. Because of the loading conditions, the Airy's stress function ϕ in the cross shape is anti-symmetrical about the Y-axis and symmetrical about the X-axis. Therefore, only one-quarter of the region

shown in Figure 4.10 has been analyzed. All grading and notations are as in Figure 4.3b. As all the boundary forces will be distributed in the conventional beam theory form, the boundary condition can be summarized as follows:

$$\text{On AB, } \tau_{xy} = -K_2 (d^2 - x^2); \quad \sigma_y = 0$$

On BCDE, Normal and shear stresses are zero.

$$\text{On EF, } \sigma_x = 2K_2 d^2/3; \quad \tau_{xy} = 0$$

$$\text{On OA, } \sigma_y = 0$$

$$\text{On OF, } \tau_{xy} = 0$$

where K_2 is a constant. These boundary conditions can be expressed in terms of Airy's stress function as follows:

$$\text{On AB, } \phi = 0; \quad \frac{\partial \phi}{\partial y} = K_2 (d^2 x - x^3/3).$$

$$\text{On BC, } \phi = 2/3(K_2 d^3 y) - 14/9 (K_2 d^4); \quad \frac{\partial \phi}{\partial x} = 0.$$

$$\text{On CD, } \phi = -2/9 (K_2 d^4) \sin(x, r) - 2/9 (K_2 d^4);$$

$$\frac{\partial \phi}{\partial x} = 0; \quad \frac{\partial \phi}{\partial y} = 2/3 (K_2 d^3).$$

$$\text{On DE, } \phi = -8/9 (K_2 d^4); \quad \frac{\partial \phi}{\partial y} = 2/3 (K_2 d^3).$$

$$\text{On EF, } \phi = 1/3 (K_2 d^2 y^2) - 11/9 (K_2 d^4); \quad \frac{\partial \phi}{\partial x} = 0.$$

where the angle (x, r) is the angle of inclination of the fillet radius drawn to the point in question measured from the X direction. All these boundary conditions are transformed into the finite difference form by the same method as in the case of pure bending moment with graded net.

Stress function ϕ and stresses. The finite difference equation of the biharmonic differential equation was set up for all interior nodes, and by use of the above boundary conditions the equations were transformed to contain only unknowns at interior nodes. These equations were solved by the L2 Program in the MISTIC computer. The results are given in Table 4.21. From the ϕ values the stresses σ_y , σ_x , τ_{xy} are calculated by using the finite difference forms. Results are recorded in Table 4.22. From the stress values, resisting moments and forces M_y , V_x , and N_x were calculated graphically at various sections; they are compared in Table 4.23 with the values of M_y , V_x , and N_x calculated by statics from the applied loads. The comparison shows that results are in fair agreement except for V_x at section #4. This discrepancy may be due to an incorrect value of stress τ_{xy} at 3,4, since the value given in Table 4.22 is inconsistent with the boundary conditions. But the error in τ_{xy} at node 3,4 and other evident errors in σ_y and τ_{xy} at 4,3 will not significantly affect the accuracy of the energy study inside the joint. The errors in maximum stress values at any section or errors in boundary stress values will not introduce significant errors into the energy calculation, because when the square of the stress is plotted as in Figure 4.5 the area change caused by the error in the stresses is very small compared with the total area between the curve and X-axis.

Energy and equivalent depth. The energy at various sections has been calculated from the results of the stress distribution by the formula and the method discussed before. The results are tabulated in Table 4.24. In the case of shear-loading also, the equivalent depth is defined as before, i.e. the one which if used in the conventional beam theory energy formula will give the true energy at the section. For a beam of uniform section subjected to variable moment the energy per unit length is written as,

$$M^2 / 2EI + 1.2 V^2 / 2GA.$$

where M and V are bending moment and shear force at the section, I and A are moment of inertia and area of the cross section, and G is the shear modulus. According to the definition of equivalent depth, for a rectangular cross section,

$6M^2 / Ed_e^3 + 1.2 V^2 / 2Gd_e = \text{True energy at the section,}$
e.g. for section 3,

$$6M^2 / Ed_e^3 + 1.2 V^2 / 2Gd_e = 5.809534 K_2^2 d^5 / 2E \quad \begin{matrix} \text{(from} \\ \text{Table} \\ \text{4.24)} \end{matrix}$$

From the known M and V at any section, the ratio of equivalent depth d_e to the depth $2d$ of the beam has been calculated and recorded in Table 4.25 and presented graphically in Figure 4.11. Comparison of Figure 4.11 with Figure 4.6 indicates that equivalent depth at the face is smaller in

the shear-loading case than in the pure bending moment case, but that the equivalent depth increases more rapidly with distance inside the column than it does in the pure bending case so that the maximum attained is greater in the shear-loading case by about 15%. Hence, it can be concluded that the shear energy due to shear force is dissipated more rapidly than the bending energy. As the bending energy in the shear-loading case depends upon the span length of the beam, the equivalent depth will be affected by it. This phase will be discussed in Part III and Chapter V.

B. Cross Shape Having $l/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$, and $r/d = 0$

The fillet was removed to study the effect of the fillet on the equivalent depth. The boundary conditions in this case are identical to those of Figure 4.10. Following the same approach, ϕ functions at all interior nodes have been evaluated and are given in Table 4.26. Stresses, energy, and equivalent depth are given in Tables 4.27, 4.28, and 4.29. The equivalent depth has been plotted in Figure 4.12. Comparison of results of Table 4.29 with 4.25 indicates that the equivalent depth at a section is smaller in case of a square corner than in the case with a fillet. This can be explained with the same reasoning as in the case of pure bending moment (Part I, C).

C. Effect of Poisson's Ratio on Equivalent Depth

It was concluded in the case of the pure bending moment that variations in Poisson's ratio do not affect greatly the equivalent depth. This was due to the fact that in that case elastic energy was principally contributed by bending stresses. In the case of shear-loading, the shear energy will be a significant part of the total energy. Hence, it was decided to analyze the case of shear-loading for equivalent depth with Poisson's ratio as 0.15. The calculations are carried out with the information of Table 4.24. The results of the equivalent depth are recorded in Table 4.30. Comparison of Table 4.30 with Table 4.25 indicates that the noticable differences are at beam sections #4, #3, #2, and #1. But these differences are not significant compared with the equivalent depth. The change in Poisson's ratio also affects, in approximately the same degree, the evaluation of the energy by the conventional beam formula. Hence, it is concluded that for all practical purposes the exact value of Poisson's ratio is unimportant for calculation of the equivalent depth in the case of shear-loading also.

III. INTERPRETATION OF RESULTS

A. Equivalent Depth Ratio R at the Face of the Column

Pure bending case. It was concluded in Part I--B that the stress concentration factor depends on d/b when d/b is

greater than 0.30 for ratio $r/d = 1/3$ and h/d greater than $5/3$. For investigation of the effect of different stress concentration factors on R (the ratio of equivalent depth to the depth of beam at the face of column) at the face, two cases of d/b were analyzed in Part I. The value of R at the face of the column with $r/d = 1/3$ and $d/b = 1.0$ was found to be 1.1826 (Table 4.11) compared with 1.1388 (Table 4.15) for d/b equal to 1.5. This decrease in R at the face can be attributed directly to the stress concentration factor which is decreased as mentioned earlier. Looking at the computation of energy in Tables 4.10 and 4.14 for section #3 and section #2 (vertical sections at the face) or #4 and #3, it can be concluded that the energy due to the bending stress σ_y is most of the total energy. So for analysis of the stress concentration effect, it can be considered that the equivalent depth ratio R is a function of the stress σ_y alone. With this in mind, the σ_y stresses are examined in both cases. The stress concentration factor at section #3 for $d/b = 1.5$ is 1.3804 (Table 4.13) compared with 1.4933 at section #4 for $d/b = 1.0$ (Table 4.7). The effect of stress concentration on the equivalent depth may be seen from the following consideration. The effect of different stress concentration factors on the distribution of the stress σ_y with the same resultant moment at the section is shown qualitatively in Figure 4.13. It can be shown that the area between the

σ_y^2 curve and the X-axis decreases as the stress concentration factor increases. Since the equivalent depth is a function primarily of the energy due to σ_y , the equivalent depth ratio R will increase as the stress concentration factor is increased. Hence, if d/b is increased, the ratio R will decrease.

Two different values of d/b have been considered here giving values of R equal to 1.1826 and 1.1388. It is recommended that for all values of d/b less than 2.0, and r/d equal to $1/3$, the value of R be taken as 1.16.

In the case of the cross-shaped joint without fillets, i.e., $r/d = 0$, theoretically there will be stresses of infinite magnitude at the re-entrant angle. This fact has not been observed in the calculated results because of the use of the finite difference method. But this will not make any significant error in the application of the results inasmuch as the practical limit for σ_y is determined by the strength of the structural material and this strength is far from infinite. For the case $r/d = 0$, no calculations have been made except for $d/b = 1$. It seems reasonable to assume that the dependence of R on d/b is no greater in the case of $r/d = 0$ than it was for $r/d = 1/3$, namely, a variation of about 4% for d/b less than 2.0.

Summarizing the results of R for the two cases, one with $r/d = 0$ and another with $r/d = 1/3$, it is concluded

that the value of R for any value of d/b smaller than 2.0 and r/d between 0 and $1/3$ will lie between about 1.10 and 1.16.

Variable bending moment case. In the case of shear-loading the ratio R at the face of the column for the joint with the fillet is 1.0689 (Table 4.25) compared with 1.1826 (Table 4.11) for the case of pure bending. As the span of the beam increases conditions at the joint approach the pure bending state as far as the total energy evaluation is concerned. That is, the shear energy due to shear force becomes less and less important compared with the bending energy. The same effect will be observed inside the joint, which will be studied in Chapter V. If one desires to obtain the ratio R at the face for different span lengths of the beam, i. e. L/d greater than 1.3333 with available calculation information it can be done by superimposing the results. In the cross shape of Figure 4.14, L/d is greater than 1.3333 and the height of the column is such that h/d is 2.3333. The stresses in the shaded area can be determined by analyzing the shaded cross shape. This can be done by superimposing the stress system due to Moment $M = VL_3$ (calculated with the use of pure bending moment stress information) on the stress system due to shear force on section AB. After superposition of the stresses, the energy can be evaluated and hence the ratio

R at the face. A similar approach could be followed for the joint without fillets and the ratio R which will vary from 0.9242 (Table 4.29) to 1.1010 (Table 4.19) for various span lengths can be determined.

In practice there will not be any cases of pure bending moment. Of course, the joint with a very long beam will approximate the pure bending moment case. But the determination of equivalent depth is important only for short spans inasmuch as the contribution of the joint to the total behavior of the beam becomes less important as spans increase. Hence, one will be interested in the equivalent depth information for the shear case. In this study actual values of R at the face of column for various values of L/d are not determined.

Also when d/b is greater than 1.0, the value of R will decrease due to reasons discussed for the pure bending moment case. In the case of pure bending moment this decrease was about 5%. With the thinking that the decrease in the shear case will not be too far from 5%, an analysis was not made in the shear case for $d/b = 1.5$. Summarizing the results of two cases of shear loading, one with $r/d = 0$ and another with $r/d = 1/3$, the value of R for $L/d = 1.3333$ and r/d between 0 and $1/3$ will lie between 0.9242 and 1.0689.

B. Contribution of Shear Stresses to the Total Energy

In most problems the shear energy is neglected. This cannot be done when the spans are small. From Table 4.24 it can be seen that at the face of the column the shear energy is even greater than the bending energy, and also inside the joint the contribution is significant. The same fact is observed in the results of Table 4.28. In short beams the assumption of infinite moment of inertia implies zero bending energy, which is not true as can be seen from the results. By taking infinite moment of inertia in the joint and also neglecting the shear energy two errors will be introduced, which do not compensate in any way. In the case of the joint subjected to pure bending moment there are shear stresses. The energy due to these shear stresses inside the joint is in some cases as high as 25% of the corresponding bending energy. In beam theory there is no way to find these shear stresses. In this case it was included because it was significant.

C. Stresses on the Face of the Column

The bending stress at the face of a column at a joint with square corners subjected to pure bending moment (beam section #3 in Table 4.17) is plotted in Figure 4.15. The stress concentration factor is obviously not correct, since it should be infinity. But in studying the general mode of stress distribution, it is to be noted that the stress

distribution is fairly linear in almost $3/4$ of the depth. The effect of such distribution on the energy can be observed with the help of Figure 4.13. Also the magnitude of shear energy on the face of the column is small compared with the total energy. Hence, it can be considered that as far as total energy evaluation is concerned, the actual stress distribution on the face of the column which includes

σ_y and τ_{xy} can be replaced by a linear distribution of σ_y attaining the maximum value FK_1d shown as dotted in Figure 4.15. The correction factor F is to be selected in such a way that energy due to the linear distribution with maximum value FK_1d on the face will be equal to true total energy. This suggests that for the study of energy variations inside the joint subjected to pure bending moment M , the stresses on the column face can be replaced by linear stress distribution of σ_y attaining maximum FK_1d . In Chapter V the stresses inside the column for such an assumed linear distribution on the face are calculated by series method. Comparison of results presented in that chapter shows that if the correction factor is selected to make the two equivalent depths agree at the face of the column, then the equivalent depth diagrams agree closely inside the column. The value of F in the case of a cross shape with square corner comes to 0.87. It is to be noted that the above modification of stress distribution is only for energy evaluations to calculate equivalent depth. When this

equivalent depth is used in the conventional beam theory formula, the moment used should be M and not FM . This is an arbitrary way to determine the mode of energy distribution inside the joint by the linear distribution of stress on the boundary.

The shear stress distribution on the face of the column also will be different from the conventional parabolic distribution. Even in the case of pure bending moment the shear stresses were found. Hence, in the case of the joint subjected to shear force, to find the distribution of the shear stress due to shear force V , it will be necessary to separate the effect of the corresponding moment from the total shear stress. Applying the principle of superposition, the separation can be done by using the results of the pure bending case. The shear stresses corresponding to the bending moment are evaluated at the face of the column. This shear stress is subtracted from the total shear stress obtained when the joint was subjected to external shear force. The remaining shear stress distribution is plotted in Figure 4.16. Again, as far as the magnitude is concerned, the net shear stress might not be correct at least in the stress concentration zone; the magnitude of the shear stress is certainly not correct at the re-entrant point in the case of a joint without fillets. But in general it appears that in almost $2/3$ of the beam depth the shear stress is approximately a uniform

distribution. This fact has been observed by Neuber.²¹ He analyzed a bar which has a deep notch subjected to shear force. On the section passing through the notch the shearing stress approaches uniformity except near the notch where there is stress concentration effect. Neglecting the effect of the area of the curve in the zone near the notch, with little error in the energy, the shearing stress distribution on the face of the column can be assumed as uniform. The uniform shear stress assumption will be considerably better than a parabolic distribution would be.

²¹N. Neuber, Theory of Notch Stresses (Ann Arbor, Michigan: J. W. Edwards, 1946), p. 44.

TABLE 4.1

ϕ VALUES FOR THE CROSS SHAPE HAVING $L/b = 2.0$,
 $h/b = 3.0$, $d/b = 1.0$, AND $r/d = 1/3$ SUBJECTED
 TO PURE BENDING MOMENT

Following values are to be multiplied by $-K_1 d^3/81$.

Node	ϕ
1,8	12.912304
1,7	12.812339
1,6	12.685156
1,5	12.432035
1,4	11.898325
1,3	11.032010
1,2	10.163045
1,1	9.556025
1,0	9.341692
2,8	22.870239
2,7	22.764690
2,6	22.646023
2,5	22.375402
2,4	21.680103
2,3	20.235799
2,2	18.761616
2,1	17.734676
2,0	17.373282
3,3	25.899469
3,2	24.518384
3,1	23.492986
3,0	23.125730
4,2	26.968565
4,1	26.509573
4,0	26.309274
5,2	27.364520
5,1	27.473239
5,0	27.478157
6,2	27.297142
6,1	27.543226
6,0	27.623471
7,2	27.154556
7,1	27.335881
7,0	27.404686
8,2	27.048032
8,1	27.116757
8,0	27.144883

TABLE 4.2

BENDING STRESS σ_y FOR THE CROSS SHAPE HAVING $d/b = 1.0$
AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT

Following values are to be multiplied by $K_1 d$.

Node	σ_y	Node	σ_y
1,8	0.32826	1,1	0.15304
2,8	0.64758	2,1	0.26893
3,8	0.91772	3,1	0.30463
1,7	0.31775	4,1	0.22810
2,7	0.63527	5,1	0.09930
3,7	0.94133	6,1	0.03081
1,6	0.30270	7,1	0.00131
2,6	0.62299	8,1	-0.01138
3,6	0.96755	9,1	-0.03732
1,5	0.27723	1,0	0.14557
2,5	0.58006	2,0	0.25274
3,5	1.02769	3,0	0.28543
1,4	0.23517	4,0	0.22385
2,4	0.49576	5,0	0.11373
3,4	0.96254	6,0	0.04045
1,3	0.20313	7,0	0.00456
2,3	0.39335	8,0	-0.01288
3,3	0.50701	9,0	-0.01609
1,2	0.17394		
2,2	0.31646		
3,2	0.36740		
4,2	0.22825		
5,2	0.05148		
6,2	0.00836		
7,2	-0.00406		
8,2	-0.00650		
9,2	-0.01067		

TABLE 4.3

COMPARISON OF RESISTING AND APPLIED MOMENTS M_y FOR THE
CROSS SHAPE HAVING $d/b = 1.0$ and $r/d = 1/3$
SUBJECTED TO PURE BENDING MOMENT

Moments are to be multiplied by $K_1 d^3$.

Beam-Section	Resisting M_y	Applied M_y	Percentage Error
#8*	0.6160	0.6667	- 7.60
#7	0.6340	0.6667	- 4.90
#6	0.6340	0.6667	- 4.90
#5	0.6520	0.6667	- 2.20
#4	0.5660	0.6667	-15.10
#3	0.5772	0.6667	-13.42
#2	0.6856	0.6667	+ 2.83
#1	0.7094	0.6667	+ 6.40
#0	0.6864	0.6667	+ 2.95

*Note: The beam section #8 identifies the cross section parallel to X-axis and located at $Y = 8$ mesh units, which means beam section #0 passes through the X-axis. The same notation is used throughout the text.

TABLE 4.4

ϕ VALUES FOR THE CROSS SHAPE WITH GRADED NET
 HAVING $L/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$,
 AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT

Following values are to be multiplied by $-K_1 d^3/81$.

Node	ϕ
1,6	12.890613
1,5	12.695906
1,4	12.279552
1,3	11.370442
1,2	10.339911
1,1	9.598510
1,0	9.335747
2,6	22.859918
2,5	22.677387
2,4	22.233970
2,3	20.752872
2,2	18.959068
2,1	17.692390
2,0	17.249545
23,45	25.751881
23,4	25.508220
23,34	24.952044
23,3	24.100923
23,23	23.103176
3,34	26.988954
3,3	26.259797
3,23	25.307320
3,2	24.460026
3,1	23.186521
3,0	22.743936
34,3	27.117069
34,23	26.410100
4,23	26.808571
4,2	26.466177
4,1	25.896843
4,0	25.666549
45,23	26.966356
5,2	27.003920
5,1	26.887627
5,0	26.821642
6,2	27.058373
6,1	27.069865
6,0	27.064627

TABLE 4.5

BENDING STRESS σ_y FOR THE CROSS SHAPE HAVING $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT (GRADED NET)

Following values are to be multiplied by $K_1 d$.

Node	σ_y
1,7	0.3333
2,7	0.6666
3,7	0.8888
1,6	0.3246
2,6	0.6477
3,6	0.9200
1,5	0.3016
2,5	0.6288
3,5	0.9606
1,4	0.2583
2,4	0.5765
23,4	0.7922
3,4	1.3260
1,3	0.2209
2,3	0.4306
23,3	0.5285
3,3	0.5829
34,3	0.4230
4,3	-0.0520*
1,2	0.1912
2,2	0.3465
3,2	0.3883
4,2	0.1631
5,2	0.0537
6,2	0.0125
7,2	-0.0130
1,1	0.1672
2,1	0.2889
3,1	0.3093
4,1	0.1911
5,1	0.0898
6,1	0.0280
7,1	-0.0155
1,0	0.1580
2,0	0.2688
3,0	0.2857
4,0	0.1963
5,0	0.1013
6,0	0.0337
7,0	-0.0144

*The value should be zero according to boundary conditions.

TABLE 4.6

COMPARISON OF RESISTING AND APPLIED MOMENTS M_y FOR A
CROSS SHAPE HAVING $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED
TO PURE BENDING MOMENT (GRADED NET)

Moments are to be multiplied by $K_1 d^3$.

Beam-Section	Resisting M_y	Applied M_y	Percentage Error
#7	0.6322	0.6667	- 5.17
#6	0.6332	0.6667	- 5.02
#5	0.6500	0.6667	- 2.50
#4	0.6340	0.6667	- 4.90
#3	0.7196	0.6667	+ 7.93
#2	0.6426	0.6667	- 3.61
#1	0.6810	0.6667	+ 2.14
#0	0.6496	0.6667	- 2.56

TABLE 4.7

BENDING STRESS σ_y FOR A CROSS SHAPE HAVING $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT
(HIGHER ORDER DIFFERENCE FORMULA, GRADED NET)

Following values are to be multiplied by $K_1 d$.

Node	σ_y	Node	σ_y	Node	σ_y
1,7	0.3333	1,3	0.2218	2,1	0.2973
2,7	0.6574	2,3	0.4389	3,1	0.3209
3,7	0.9259	3,3	0.5992	4,1	0.1897
1,6	0.3247	34,3	0.4513	5,1	0.0865
2,6	0.6519	1,2	0.1942	6,1	0.0265
3,6	0.9654	2,2	0.3559	7,1	-0.0228
1,5	0.2995	3,2	0.4105	1,0	0.1619
2,5	0.6284	4,2	0.1627	2,0	0.2766
3,5	1.0150	5,2	0.0489	3,0	0.2946
1,4	0.2533	6,2	0.0112	4,0	0.1968
2,4	0.5706	7,2	-0.0172	5,0	0.0990
3,4	1.4933	1,1	0.1710	6,0	0.0321
				7,0	-0.0224

TABLE 4.8

COMPARISON OF RESISTING AND APPLIED MOMENTS M_y FOR A
CROSS SHAPE HAVING $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED
TO PURE BENDING MOMENT (HIGHER ORDER DIFFERENCE
FORMULA, GRADED NET)

Moments are to be multiplied by $K_1 d^3$.

Beam-Section	Resisting M_y	Applied M_y	Percentage Error
#7	0.6392	0.6667	- 4.12
#6	0.6482	0.6667	- 2.77
#5	0.6498	0.6667	- 2.53
#4	0.6366	0.6667	- 4.51
#3	0.7184	0.6667	+ 7.75
#2	0.6682	0.6667	+ 0.23
#1	0.6702	0.6667	+ 0.52
#0	0.6698	0.6667	+ 0.46

TABLE 4.9

NORMAL STRESS σ_x AND SHEARING STRESS τ_{xy} FOR THE
CROSS SHAPE HAVING $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED
TO PURE BENDING MOMENT (GRADED NET)

Following values are to be multiplied by $K_1 d$.

Node	σ_x	Node	τ_{xy}
1,7	0.02431	0,6	0.01689
2,7	0.03113	1,6	0.00896
1,6	0.00948	2,6	-0.00845
2,6	0.00472	0,5	0.03395
1,5	0.02463	1,5	0.01738
2,5	0.02899	2,5	-0.01697
1,4	0.05474	0,4	0.07364
2,4	0.11530	1,4	0.05346
23,4	0.13980	2,4	-0.01625
3,4	0.00491*	23,4	-0.09877
1,3	0.01349	3,4	-0.27089*
2,3	0.03474	0,3	0.10776
23,3	0.06562	1,3	0.09097
3,3	0.09926	2,3	0.01667
34,3	0.19291	3,3	-0.09656
4,3	0.04245	34,3	-0.16558
5,3	-0.00087	4,3	-0.10888*
6,3	-0.01295	0,2	0.09844
1,2	-0.03212	1,2	0.08501
2,2	-0.05857	2,2	0.03615
3,2	-0.19998	3,2	-0.05437
4,2	0.00395	4,2	-0.08225
5,2	0.01336	5,2	-0.03258
6,2	0.00521	6,2	-0.00313
1,1	-0.05318	0,1	0.05579
2,1	-0.09154	1,1	0.04749
3,1	-0.09232	2,1	0.01978
4,1	-0.03747	3,1	-0.02527
5,1	-0.00559	4,1	-0.04261
6,1	0.00186	5,1	-0.02239
1,0	-0.05839	6,1	-0.00506
2,0	-0.09841		
3,0	-0.09835		
4,0	-0.05117		
5,0	-0.01466		
6,0	-0.00116		

*The value should be zero according to boundary conditions.

TABLE 4.10

COMPUTATION OF THE ENERGY FOR THE CROSS SHAPE HAVING
 $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO PURE BENDING
 MOMENT (POISSON'S RATIO = 0.30)

Following values are to be multiplied by $K_1^2 d^3 / 2E$.

Beam Section	$\int_{-h}^h \sigma_y^2 dx$ 1	$\int_{-h}^h \sigma_x^2 dx$ 2	$-2\mu \int_{-h}^h \sigma_x \sigma_y dx$ 3	$\frac{2(1+\mu)}{(\int_{-h}^h \tau_{xy}^2 dx)}$ 4	Energy = 1+2+3+4
#7	0.621860	0.001237	-0.012544	0.000000	0.610553
#6	0.595200	0.000149	-0.002432	0.000583	0.593500
#5	0.595200	0.001088	-0.011200	0.002108	0.587196
#4	0.653400	0.014507	-0.046208	0.034667	0.646365
#3	0.354140	0.016043	-0.032192	0.065118	0.403109
#2	0.237220	0.022237	-0.039296	0.042987	0.341740
#1	0.177066	0.014421	-0.029824	0.040546	0.261857
#0	0.160000	0.019029	-0.032256	0.000000	0.211285

TABLE 4.11

RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$ OF
 THE BEAM, FOR THE CROSS SHAPE HAVING $L/b = 1.333$,
 $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO
 PURE BENDING MOMENT (POISSON'S RATIO = 0.30)

Beam Section	Ratio R = $d_e / 2d$
#7	1.0298
#6	1.0395
#5	1.0432
#4	1.0104
#3	1.1826
#2	1.2495
#1	1.3654
#0	1.4667

TABLE 4.12

ϕ VALUES FOR THE CROSS SHAPE HAVING $L/b = 2.5$, $h/b = 3.5$, $d/b = 1.5$ AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT

Following values are to be multiplied by $-K_1 d^3/81$.

Node	ϕ
1,6	12.915553
1,5	12.816860
1,4	12.670708
1,3	12.323294
1,2	11.557676
1,1	10.827834
1,0	10.541960
12,23	17.257904
2,6	22.874733
2,5	22.775483
2,4	22.650430
2,3	22.278507
2,23	21.716453
2,2	20.983497
2,1	19.683635
2,0	19.182924
23,34	25.751823
23,3	25.534736
23,23	25.046649
23,2	24.282369
23,12	23.491438
3,23	26.957034
3,2	26.344213
3,12	25.627511
3,1	25.053340
3,0	24.561602
34,2	27.119253
34,12	26.667900
34,1	26.250702
34,01	25.983491
4,12	26.924957
4,1	26.735018
4,0	26.616489
45,12	27.003178
5,1	27.081935
5,0	27.111707
6,1	27.062630
6,0	27.092927

TABLE 4.13

STRESS VALUES FOR THE CROSS SHAPE HAVING $d/b = 1.5$
AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT

Following values are to be multiplied by $K_1 d$.

Node	σ_y	Node	σ_x	Node	τ_{xy}
1,6	0.32850	1,6	0.00158	0,6	0.01017
2,6	0.64833	2,6	-0.00289	1,6	0.00624
3,6	0.96151	1,5	0.00527	2,6	-0.00509
1,5	0.31742	2,5	0.00287	0,5	0.01360
2,5	0.64509	1,4	0.02236	1,5	0.00623
3,5	0.98906	2,4	0.02746	2,5	-0.00680
1,4	0.29670	1,3	0.04647	0,4	0.02742
2,4	0.62437	2,3	0.10256	1,4	0.01380
3,4	1.02340	23,3	0.12044	2,4	-0.01371
1,3	0.25851	3,3	0.01909**	0,3	0.06183
2,3	0.56908	1,2	-0.00397	1,3	0.04630
23,3	0.79598*	2,2	0.00054	2,3	-0.01270
3,3	1.38041	23,2	0.01184	23,3	-0.07834
1,2	0.23857	3,2	0.04617	3,3	-0.24794**
2,2	0.46367	34,2	0.12999	0,2	0.08308
23,2	0.54979*	4,2	0.05888	1,2	0.07208
3,2	0.58829	5,2	-0.01821	2,2	0.01253
34,2	0.42051	6,2	-0.01392	23,2	-0.00593
4,2	-0.05300**	1,1	-0.04933	3,2	-0.09600
1,1	0.22336	2,1	-0.08879	34,2	-0.14773
2,1	0.39949	3,1	-0.08880	4,2	-0.06780**
3,1	0.43344	4,1	-0.01627	0,1	0.05643
4,1	0.13548	5,1	0.00580	1,1	0.05001
5,1	0.03471	6,1	0.00359	2,1	0.02130
6,1	0.00338	1,0	-0.06353	3,1	-0.03936
7,1	-0.01704	2,0	-0.11127	4,1	-0.05262
1,0	0.21622	3,0	-0.10927	5,1	-0.01323
2,0	0.37451	4,0	-0.02634	6,1	-0.00310
3,0	0.38621	5,0	0.00662		
4,0	0.16664	6,0	0.00673		
5,0	0.05150				
6,0	0.00657				
7,0	-0.02546				

*Calculated by the application of standard difference formulas.

**The value should be zero according to boundary conditions.

TABLE 4.14

COMPUTATION OF THE ENERGY FOR THE CROSS SHAPE HAVING
 $d/b = 1.5$ AND $r/d = 1/3$ SUBJECTED TO PURE BENDING
 MOMENT (POISSON'S RATIO = 0.30)

Following values are to be multiplied by $K_1^2 d^3 / 2E$.

Beam Section	$\int_{-h}^h \sigma_y^2 \cdot dx$	$\int_{-h}^h \sigma_x^2 \cdot dx$	$-2M \int_{-h}^h \sigma_x \cdot \sigma_y \cdot dx$	$\frac{2(1+\mu)}{h} \int_{-h}^h \tau_{xy}^2 \cdot dx$	Energy = 1+2+3+4
	1	2	3	4	
#6	0.621866	0.000000	0.000000	0.000211	0.622077
#5	0.628266	0.000000	-0.001600	0.000316	0.626982
#4	0.612266	0.001003	-0.010560	0.001425	0.604134
#3	0.654934	0.011264	-0.041344	0.024688	0.649542
#2	0.417280	0.008086	-0.019328	0.045372	0.451409
#1	0.276054	0.012907	0.037200	0.016196	0.342336
#0	0.238934	0.019712	0.040832	0.000000	0.299478

TABLE 4.15

RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$ OF
 THE BEAM FOR THE CROSS SHAPE HAVING $L/b = 2.5$, $h/b =$
 3.5 , $d/b = 1.5$ and $r/d = 1/3$ SUBJECTED TO PURE BENDING
 MOMENT (POISSON'S RATIO = 0.30)

Beam Section	Ratio R = $d_e / 2d$
#6	1.0233
#5	1.0207
#4	1.0334
#3	1.0087
#2	1.1388
#1	1.2488
#0	1.3125

TABLE 4.16

ϕ VALUES FOR THE CROSS SHAPE HAVING $L/b = 1.333$,
 $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 0$ SUBJECTED TO
 PURE BENDING MOMENT

Following values are to be multiplied by $-K_1 d^3/81$.

Node	ϕ
1,6	12.894170
1,5	12.723336
1,4	12.390653
1,3	11.676058
1,2	10.709995
1,1	9.963409
1,0	9.691901
2,6	22.860477
2,5	22.698227
2,4	22.361995
2,3	21.334047
2,2	19.610856
2,1	18.304426
2,0	17.838595
23,45	25.741312
23,4	25.654421
23,34	25.459267
23,3	24.918531
3,23	26.168627
3,2	25.153184
3,1	23.807441
3,0	23.339476
34,23	26.976441
4,23	27.008374
4,2	26.701685
4,1	26.246298
4,0	26.044825
45,23	26.994946
5,2	27.077141
5,1	27.035479
5,0	26.995728
6,2	27.072446
6,1	27.107700
6,0	27.112256
23,23	23.986184

TABLE 4.17

STRESSES FOR THE CROSS SHAPE HAVING $d/b = 1.0$ AND
 $r/d = 0$ SUBJECTED TO PURE BENDING MOMENT

Following values are to be multiplied by $K_1 d$.

Node	σ_y	Node	σ_x	Node	τ_{xy}
1,4	0.26423	1,4	0.04243	0,4	0.05818
2,4	0.58307	2,4	0.07686	1,4	0.03789
23,4	0.86526*	23,4	0.04812	2,4	-0.02909
3,4	1.25120	1,3	0.02794	0,3	0.09337
1,3	0.22465	2,3	0.07725	1,3	0.07642
2,3	0.44634	23,3	0.17405	2,3	0.00461
23,3	0.66488	3,3	0.36950	23,3	-0.06096
3,3	1.02362	34,3	0.35809	3,3	0.24780
1,2	0.20345	4,3	-0.00744	0,2	0.09515
2,2	0.38163	45,3	0.00449	1,2	0.08416
3,2	0.47576	5,3	-0.01714	2,2	0.04111
4,2	0.11156	6,3	-0.01610	3,2	-0.06322
5,2	0.03779	1,2	-0.02439	4,2	-0.08967
6,2	0.00660	2,2	-0.04631	5,2	-0.02393
7,2	-0.02004	3,2	-0.05567	6,2	0.00098
1,1	0.18403	4,2	0.01745	0,1	0.05656
2,1	0.32449	5,2	0.01320	1,1	0.04923
3,1	0.35565	6,2	0.00413	2,1	0.02210
4,1	0.17884	1,1	-0.05279	3,1	-0.03098
5,1	0.07660	2,1	-0.09340	4,1	-0.04812
6,1	0.01869	3,1	-0.09753	5,1	-0.01935
7,1	-0.03125	4,1	-0.02821	6,1	-0.00226
1,0	0.17581	5,1	-0.00021		
2,0	0.30278	6,1	0.00341		
3,0	0.32164	1,0	-0.06033		
4,0	0.19382	2,0	-0.10352		
5,0	0.08980	3,0	-0.10399		
6,0	0.02401	4,0	-0.04477		
7,0	-0.03334	5,0	-0.00883		
		6,0	0.00051		

*Calculated by the application of standard difference formulas.

TABLE 4.18

COMPUTATION OF THE ENERGY FOR THE CROSS SHAPE
HAVING $d/b = 1.0$ AND $r/d = 0$ SUBJECTED TO PURE
BENDING MOMENT (POISSON'S RATIO = 0.30)

Following values are to be multiplied by $K_1 d^3/2E$.

Beam Section	$\int_{-h}^h \sigma_y^2 \cdot dx$	$\int_{-h}^h \sigma_x^2 \cdot dx$	$-2\mu \int_{-h}^h \sigma_x \cdot \sigma_y \cdot dx$	$\frac{2(1+\mu)}{\int_{-h}^h \tau_{xy}^2 \cdot dx}$	Energy = 1+2+3+4
	1	2	3	4	
#4	0.667733	0.005013	-0.028608	0.006739	0.650877
#3	0.419200	0.113707	-0.068352	0.034944	0.499499
#2	0.282453	0.004352	0.029028	0.043236	0.350959
#1	0.212907	0.015083	0.032960	0.013978	0.274928
#0	0.187733	0.019179	0.034240	0.000000	0.241152

TABLE 4.19

RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$
OF THE BEAM FOR THE CROSS SHAPE HAVING $L/b = 1.333$,
 $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 0$ SUBJECTED TO
PURE BENDING MOMENT (POISSON'S RATIO = 0.30)

Beam Section	Ratio R = $d_e/2d$
#4	1.0080
#3	1.1010
#2	1.2385
#1	1.3435
#0	1.4055

TABLE 4.20

RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$ OF THE BEAM, FOR THE CROSS SHAPE HAVING $L/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO PURE BENDING MOMENT (POISSON'S RATIO = 0.15)

Beam Section	Ratio $R = d_e/2d$
#7	1.0262
#6	1.0388
#5	1.0400
#4	0.9956
#3	1.1743
#2	1.2812
#1	1.4014
#0	1.5061

TABLE 4.21

ϕ VALUES FOR THE CROSS SHAPE HAVING $L/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO VARIABLE BENDING MOMENT

Following values are to be multiplied by $-K_2 d^4/27$.

Node	ϕ Value	Node	ϕ Value	Node	ϕ Value
1,6	2.836040	2,0	18.933481	34,23	25.380255
1,5	5.531027	23,45	14.351914	4,23	26.324080
1,4	7.841399	23,4	16.858783	4,2	27.862889
1,3	9.130839	23,34	18.800852	4,1	29.589936
1,2	9.744581	23,3	20.158745	4,0	30.095707
1,1	9.977754	23,23	21.105676	45,23	26.707999
1,0	10.033637	3,34	20.477396	5,2	28.831883
2,6	5.067647	3,3	22.350146	5,1	31.388433
2,5	10.015973	3,23	23.661692	5,0	32.180314
2,4	14.487830	3,2	24.543491	6,2	29.034005
2,3	17.055711	3,1	25.502780	6,1	31.954697
2,2	18.298496	3,0	25.770193	6,0	32.904771
2,1	18.803364	34,3	23.543537		

TABLE 4.22

STRESS VALUES FOR THE CROSS SHAPE HAVING $d/b = 1.0$
AND $r/d = 1/3$ SUBJECTED TO VARIABLE BENDING MOMENT

Following values are to be multiplied by $K_2 d^2$.

Node	σ_y	Node	σ_x	Node	τ_{xy}
1,6	0.19897	1,7	0.03525	0,7	-0.96296
2,6	0.43668	2,7	0.02898	1,7	-0.85185
3,6	0.62229	1,6	0.04702	2,7	-0.51852
1,5	0.33734	2,6	0.03964	0,6	-0.92184
2,5	0.83330	1,5	0.13720	1,6	-0.83466
3,5	1.40419	2,5	0.15882	2,6	-0.53908
1,4	0.35589	1,4	0.32229	0,5	-0.82973
2,4	1.00250	2,4	0.63464	1,5	-0.78418
23,4	1.63965*	23,4	0.75306	2,5	-0.58289
3,4	3.27706	3,4	0.69681**	0,4	-0.59997
1,3	0.39592	1,3	0.23423	1,4	-0.58664
2,3	0.88821	2,3	0.44170	2,4	-0.56253
23,3	1.21551*	23,3	0.54795	23,4	-0.64580
3,3	1.41426	3,3	0.74827	3,4	-0.65906**
34,3	0.98257*	34,3	1.23839	0,3	-0.32170
4,3	0.60617**	4,3	1.80245	1,3	-0.31755
1,2	0.39890	45,3	0.77867	2,3	-0.38444
2,2	0.78357	5,3	0.77875	3,3	-0.76580
3,2	1.00834	6,3	0.64400	34,3	-0.93857
4,2	0.81148	1,2	0.12686	4,3	-0.46592**
5,2	0.22637	2,2	0.24597	0,2	-0.14115
6,2	0.07244	3,2	0.41135	1,2	-0.10356
7,2	-0.03959	4,2	0.71195	2,2	-0.19214
1,1	0.38899	5,2	0.75844	3,2	-0.32019
2,1	0.72236	6,2	0.70444	4,2	-0.35298
3,1	0.89324	1,1	0.05910	5,2	-0.19706
4,1	0.78324	2,1	0.12492	6,2	-0.05096
5,1	0.40116	3,1	0.23062	0,1	-0.00931
6,1	0.16585	4,1	0.40709	1,1	-0.05291
7,1	0.00629	5,1	0.58822	2,1	-0.07814
1,0	0.38361	6,1	0.65687	3,1	-0.18607
2,0	0.70108	1,0	0.03725	4,1	-0.17681
3,0	0.85702	2,0	0.08674	5,1	-0.13649
4,0	0.76393	3,0	0.17827	6,1	-0.05430
5,0	0.44922	4,0	0.33718		
6,0	0.20164	5,0	0.52792		
7,0	0.00736	6,0	0.63338		

*Calculated by the application of standard difference formulas.

**The value should be zero according to boundary conditions.

TABLE 4.23

COMPARISON OF APPLIED AND RESISTING FORCES FOR THE
CROSS SHAPE HAVING $L/b = 1.333$, $h/b = 2.333$, $d/b =$
 1.0 AND $r/d = 1/3$ SUBJECTED TO VARIABLE BENDING
MOMENT

Moment M_y and shear V_x are to be multiplied by $K_2 d^4$ and $K_2 d^3$
respectively.

Type of Force	Beam Section	Applied	Resisting	Difference	Percentage Error Com- pared to the applied one
Moment M_y	#6	0.444444	0.41810	0.02634	5.92
	#5	0.88889	0.85108	0.03781	4.25
	#4	1.33333	1.30706	0.02627	1.97
	#3	1.77778	1.67638	0.10140	5.70
	#2	2.14815	2.14216	0.00599	0.28
	#1	2.37037	2.36270	0.00763	0.32
	#0	2.44444	2.41188	0.03256	1.33
Shear V_x	#7	1.33333	1.27147	0.06186	4.64
	#6	1.33333	1.25867	0.07466	5.60
	#5	1.33333	1.23307	0.10026	7.52
	#4	1.33333	1.16907	0.16426	12.32
	#3	1.33333	1.35253	0.01920	1.55
	#2	0.88889	0.85760	0.03129	3.52
	#1	0.44444	0.45653	0.01189	2.67

Normal force N_x is to be multiplied by $K_2 d^3$.

Column Section	Applied N_x	Resisting N_x	Difference	Percentage Error Compared to Applied N_x
#6*	1.33333	1.30987	0.02346	1.76
#5	1.33333	1.34400	0.01067	0.80
#4	1.33333	1.35253	0.01920	1.43
#3	1.33333	1.23733	0.09600	7.20
#2	1.13580	1.13067	0.00514	0.45
#1	0.64197	0.61867	0.02330	3.63

*The column section #6 identifies the cross section parallel
to Y-axis and located to at $X = 6$ mesh units, which means
that #0 passes through Y-axis.

TABLE 4.24

COMPUTATION OF THE ENERGY FOR THE CROSS SHAPE HAVING
 $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO VARIABLE BENDING
 MOMENT (POISSON'S RATIO = 0.30)

Following values are to be multiplied by $K_2^2 d^5 / 2E$.

Beam Section	$\int_{-h}^h \sigma_y^2 dx$	$\int_{-h}^h \sigma_x^2 dx$	$-2\mu \int_{-h}^h \sigma_x \sigma_y dx$	$\frac{2(1+\mu)}{(\int_{-h}^h \tau_{xy}^2 dx)}$	Energy = 1+2+3+4
	1	2	3	4	
#7	0.000000	0.000000	0.000000	2.496000	2.496000
#6	0.270933	0.005333	-0.014080	2.468266	2.729732
#5	1.134933	0.034133	-0.079360	2.268565	3.358271
#4	3.287958	0.531200	-0.684342	1.874773	5.009589
#3	1.958400	2.641106	-0.842240	2.052268	5.809534
#2	1.683120	1.395200	-0.587520	0.557400	3.048200
#1	1.529600	0.857600	-0.391680	0.177492	2.173012
#0	1.461334	0.716800	-0.345600	0.000000	1.832534

TABLE 4.25

RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$ OF
 THE BEAM, FOR THE CROSS SHAPE HAVING $L/b = 1.333$,
 $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO
 VARIABLE BENDING MOMENT (POISSON'S RATIO = 0.30)

Beam Section	Ratio R = $d_e/2d$
#7	0.9259
#6	0.9636
#5	1.0270
#4	0.9962
#3	1.0689
#2	1.4270
#1	1.6106
#0	1.6974

TABLE 4.26

ϕ VALUES FOR THE CROSS SHAPE HAVING $L/b = 1.333$,
 $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 0$ SUBJECTED TO
 VARIABLE BENDING MOMENT

Following values are to be multiplied by $-K_2 d^4/27$.

Node	ϕ
1,6	2.828842
1,5	5.546709
1,4	7.982591
1,3	9.741781
1,2	10.486299
1,1	10.692420
1,0	10.722905
2,6	5.052450
2,5	10.005810
2,4	14.651891
2,3	18.263531
2,2	19.611308
2,1	19.986017
2,0	20.051587
23,45	14.249657
23,4	17.048596
23,34	19.744479
23,3	21.783701
23,23	22.873326
3,23	25.345289
3,2	25.919802
3,1	26.645798
3,0	26.839549
34,23	26.392136
4,23	26.566125
4,2	28.155549
4,1	30.101899
4,0	30.668689
45,23	26.639651
5,2	28.876454
5,1	31.548696
5,0	32.389809
6,2	29.029805
6,1	31.980731
6,0	32.946813

TABLE 4.27

STRESSES FOR THE CROSS SHAPE HAVING $d/b = 1.0$ AND
 $r/d = 0$ SUBJECTED TO VARIABLE BENDING MOMENT

Following values are to be multiplied by $K_2 d^2$.

Node	σ_y	Node	σ_x	Node	τ_{xy}
1,4	0.41847	1,4	0.22556	0,4	-0.69918
2,4	1.06909	2,4	0.34481	1,4	-0.68814
23,4	1.96682	23,4	0.03435	2,4	-0.65041
3,4	2.63874	1,3	0.33822	23,4	-0.52667
1,3	0.39709	2,3	0.75462	3,4	-0.10961**
2,3	0.89600	23,3	1.26613	0,3	-0.41728
23,3	1.71273	3,3	2.20628	1,3	-0.41328
3,3	3.30270	34,3	2.04608	2,3	-0.45134
1,2	0.45115	4,3	1.15700	23,3	-0.65843
2,2	0.94437	45,3	0.96093	3,3	-1.06333
3,2	1.46353	5,3	0.74903	0,2	-0.15844
4,2	0.46021	6,3	0.64608	1,2	-0.14354
5,2	0.17355	1,2	0.17947	2,2	-0.14126
6,2	0.05712	2,2	0.32436	3,2	-0.36495
7,2	-0.03336	3,2	0.39793	4,2	-0.40866
1,1	0.47082	4,2	0.73640	5,2	-0.15657
2,1	0.89641	5,2	0.73440	6,2	-0.03752
3,1	1.11693	6,2	0.69296	0,1	-0.03943
4,1	0.66377	1,1	0.05854	1,1	-0.03669
5,1	0.32818	2,1	0.10305	2,1	-0.05693
6,1	0.13082	3,1	0.17741	3,1	-0.17274
7,1	-0.00789	4,1	0.45985	4,1	-0.21613
1,0	0.47162	5,1	0.60838	5,1	-0.11699
2,0	0.86714	6,1	0.66161	6,1	-0.04055
3,0	1.02152	1,0	0.02032		
4,0	0.70526	2,0	0.04371		
5,0	0.38016	3,0	0.12917		
6,0	0.16064	4,0	0.37786		
7,0	0.01062	5,0	0.56007		
		6,0	0.64405		

** The value should be zero according to boundary conditions.

TABLE 4.28

COMPUTATION OF THE ENERGY FOR THE CROSS SHAPE HAVING
 $d/b = 1.0$ AND $r/d = 0$ SUBJECTED TO VARIABLE BENDING
 MOMENT (POISSON'S RATIO = 0.30)

Following values are to be multiplied by $K_2^2 d^5 / 2E$.

Beam Section	$\int_{-h}^h \sigma_y^2 dx$	$\int_{-h}^h \sigma_x^2 dx$	$-2M \int_{-h}^h \sigma_x \sigma_y dx$	$\frac{2(1+\mu)}{\int_{-h}^h \tau_{xy}^2 dx}$	Energy = 1+2+3+4
	1	2	3	4	
#4	3.029333	0.099200	-0.168320	2.043946	5.004149
#3	2.922667	5.162667	-1.248000	1.442132	8.279466
#2	2.396667	1.448533	-0.584960	0.701653	3.961893
#1	1.911447	0.864000	-0.368000	0.166400	2.573847
#0	1.802667	0.742400	-0.309760	0.000000	2.235307

TABLE 4.29

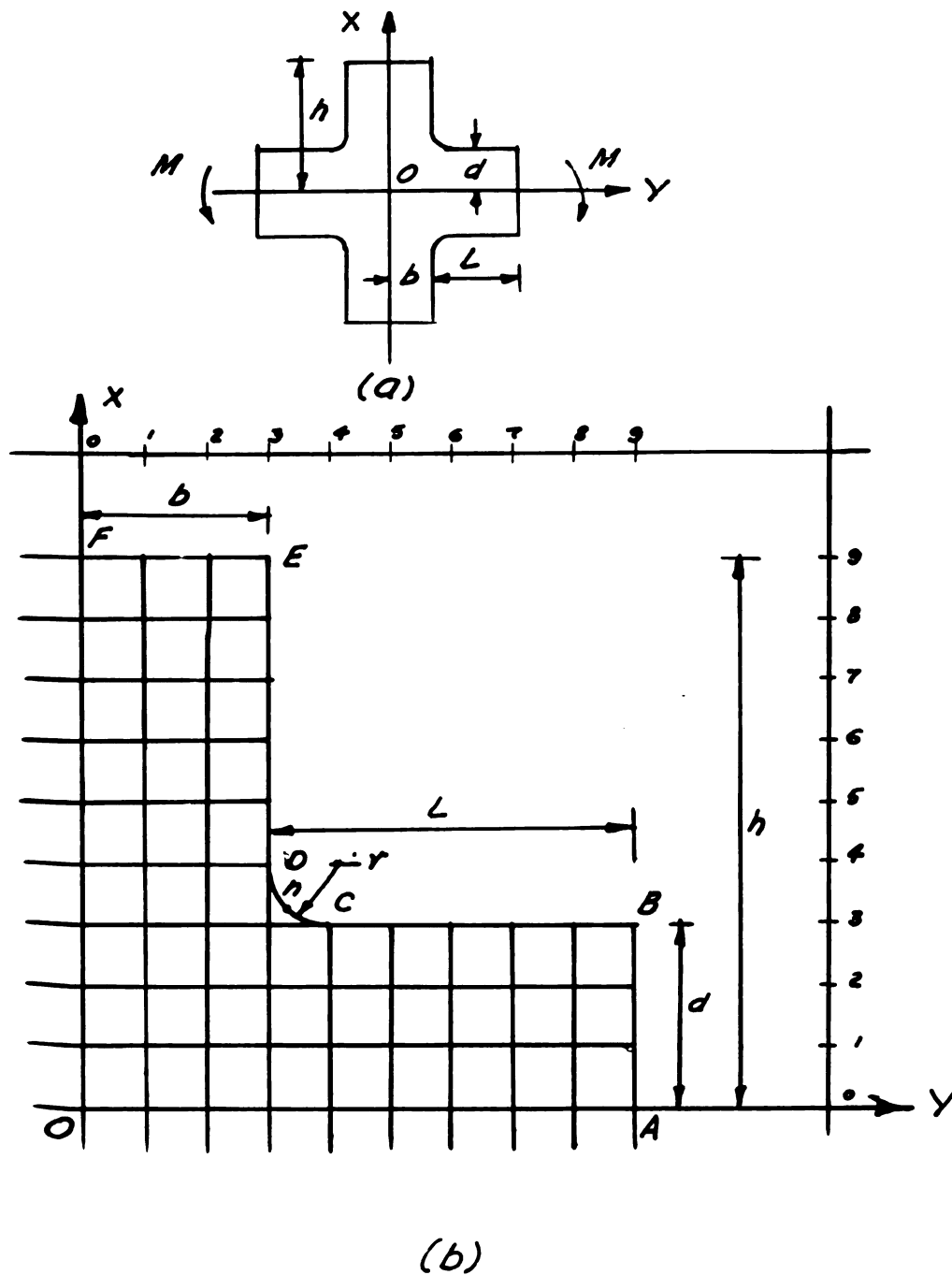
RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$ OF
 THE BEAM, FOR THE CROSS SHAPE HAVING $L/b = 1.333$,
 $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 0$ SUBJECTED TO
 VARIABLE BENDING MOMENT (POISSON'S RATIO = 0.30)

Beam Section	Ratio R = $d_e / 2d$
#4	0.9684
#3	0.9242
#2	1.2911
#1	1.5184
#0	1.5877

TABLE 4.30

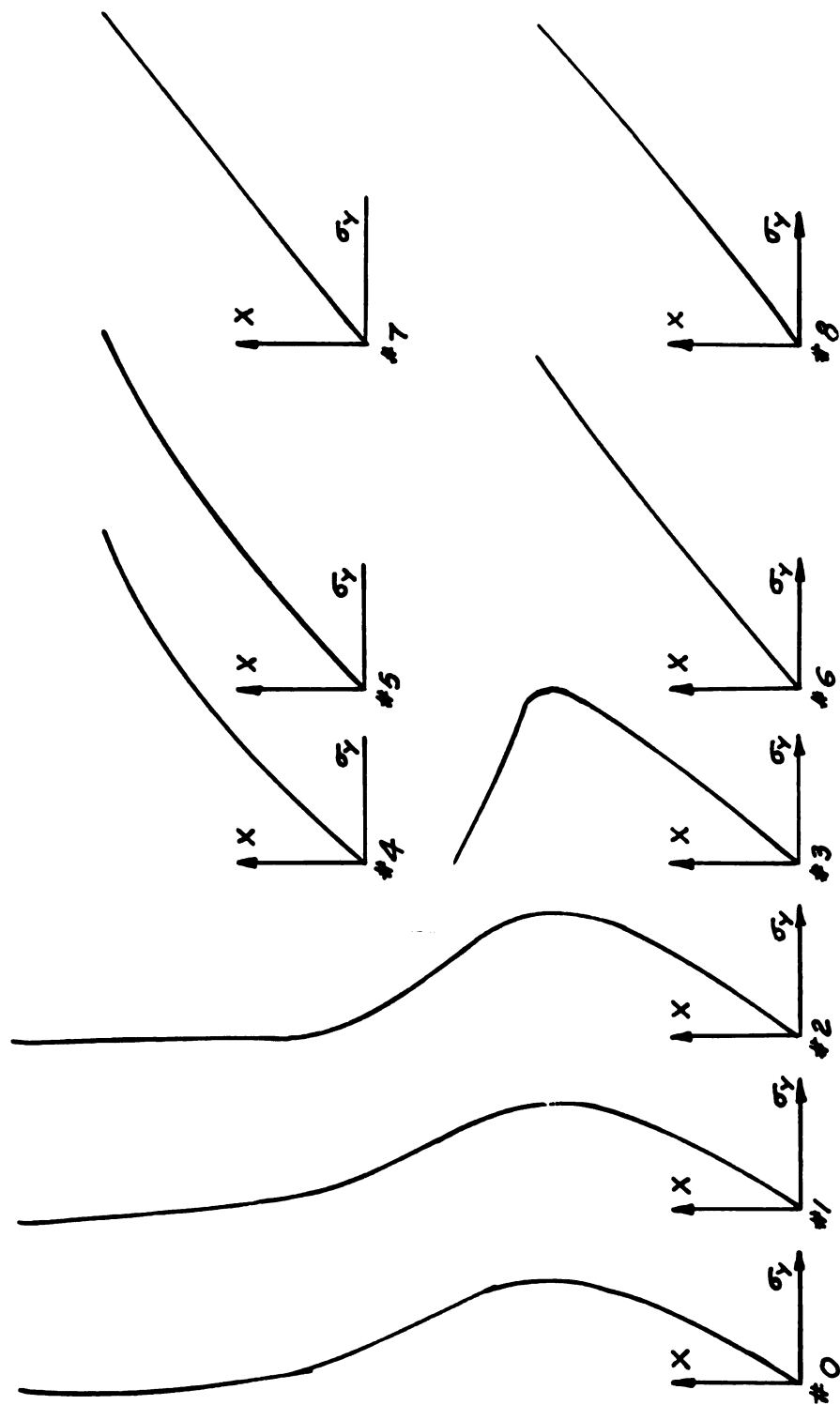
RATIO R, THE EQUIVALENT DEPTH d_e TO THE DEPTH $2d$ OF THE BEAM, FOR THE CROSS SHAPE HAVING $L/b = 1.333$, $h/b = 2.333$, $d/b = 1.0$ AND $r/d = 1/3$ SUBJECTED TO VARIABLE BENDING MOMENT (POISSON'S RATIO = 0.15)

Beam Section	Ratio R = $d_e/2d$
#6	0.9637
#5	1.0265
#4	0.9389
#3	1.0400
#2	1.3756
#1	1.5632
#0	1.7093



Cross Shape having $\frac{L}{b} = 2.0$, $\frac{h}{d} = 3.0$, $\frac{d}{b} = 1.0$ and $\frac{r}{d} = \frac{1}{3}$

FIGURE 4.1

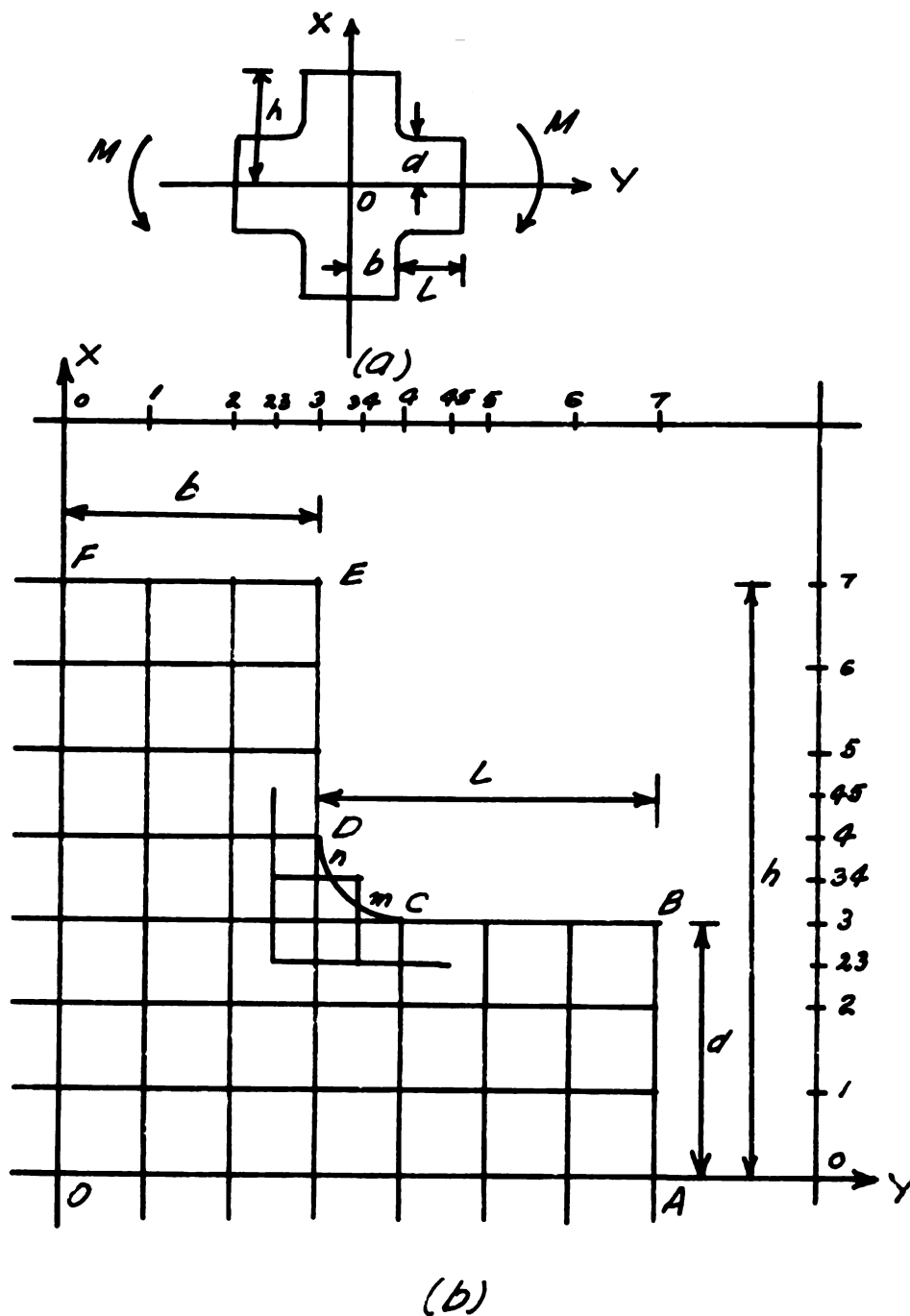


Scale: X axis $1'' = \frac{2}{3} \cdot d$ σ_y axis $1'' = 0.50 K/d$

BENDING STRESS σ_y CURVES FOR VARIOUS BEAM SECTIONS OF THE CROSS SHAPE HAVING $b=d=0.3333h$

FIGURE 4.2

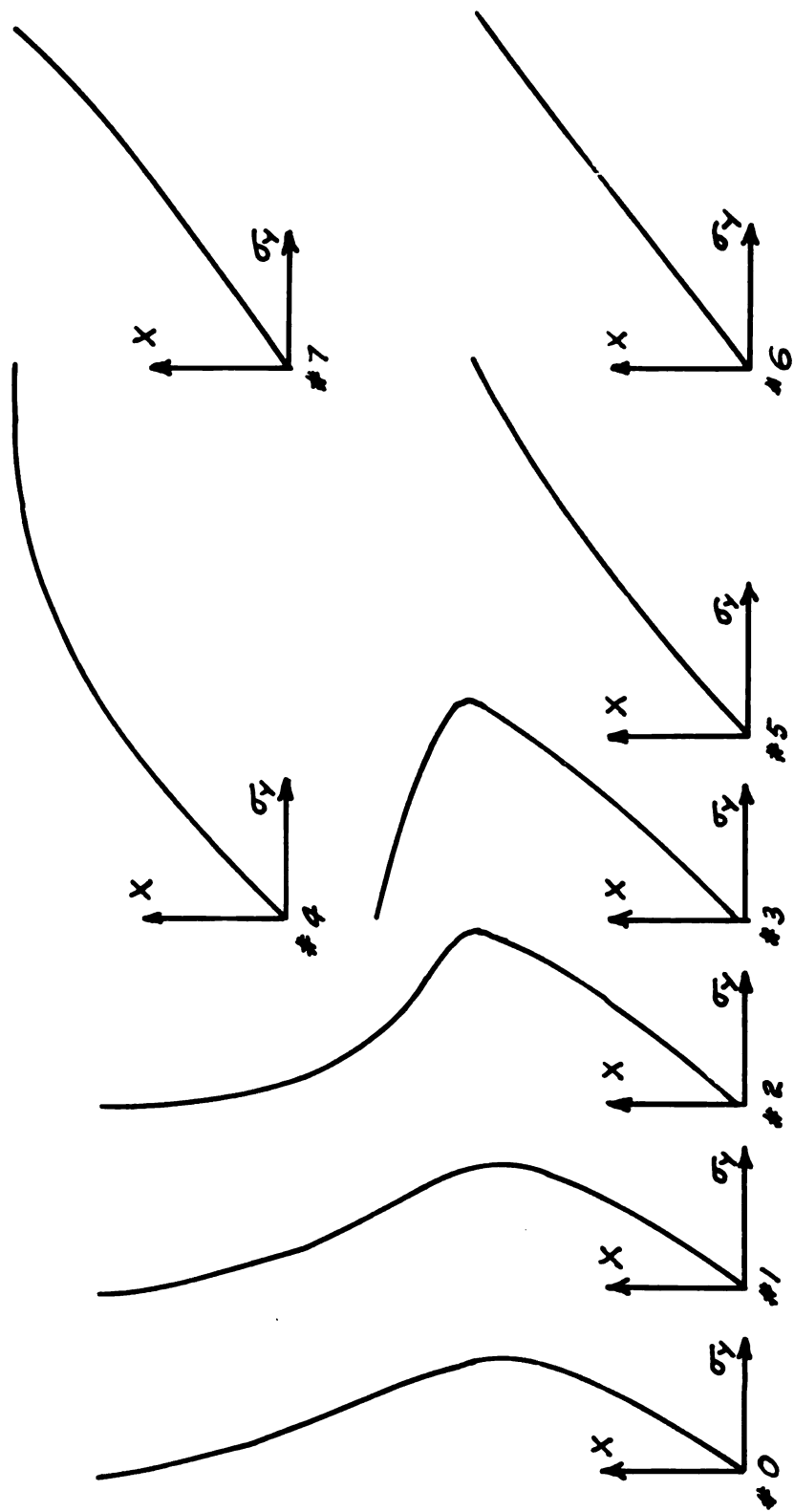
(PURE BENDING MOMENT)



CROSS SHAPE WITH GRADED NET HAVING

$$\frac{L}{b} = 1.333, \frac{h}{b} = 2.333, \frac{d}{b} = 1.0 \text{ AND } \frac{r}{d} = \frac{1}{3}$$

FIGURE 4.3



BENDING STRESS σ_y CURVES FOR VARIOUS BEAM SECTIONS OF A CROSS SHAPE HAVING

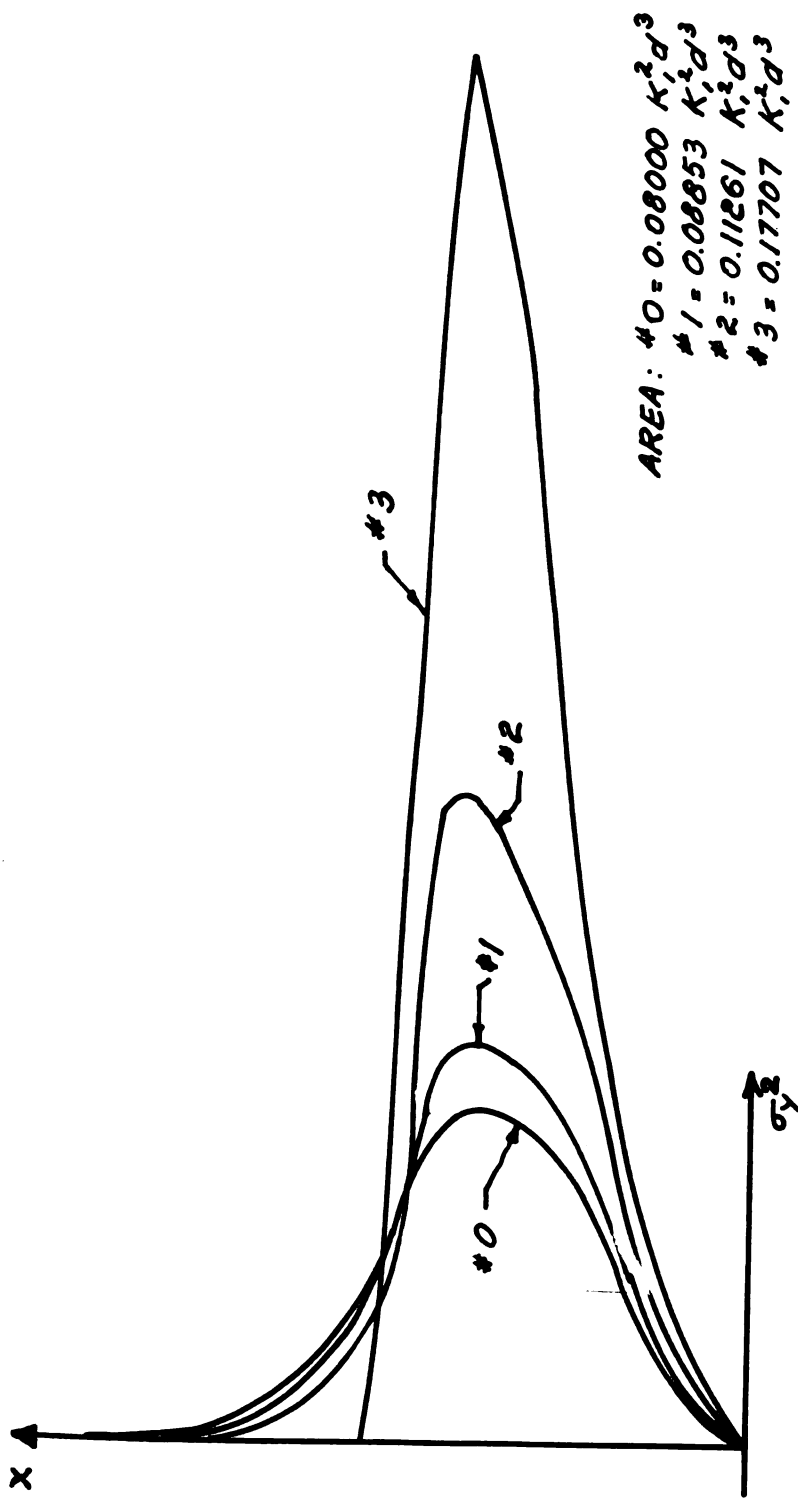
$b \cdot d = 0.42856h$.

(PURE BENDING MOMENT. GRADED NET)

Scale: x axis $1'' = \frac{2}{3}d$.

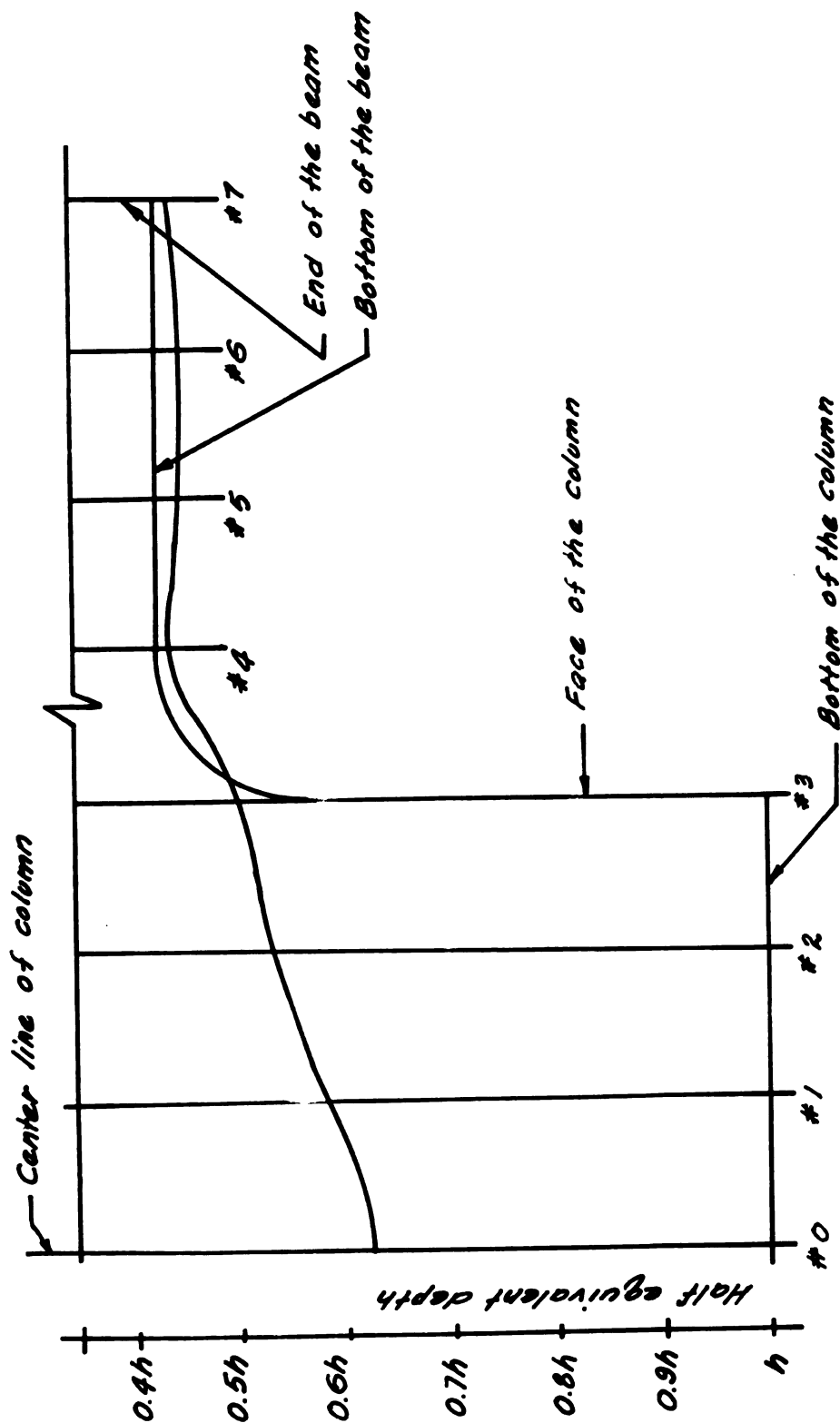
σ_y axis $1'' = 0.50K/d$

FIGURE 4.4



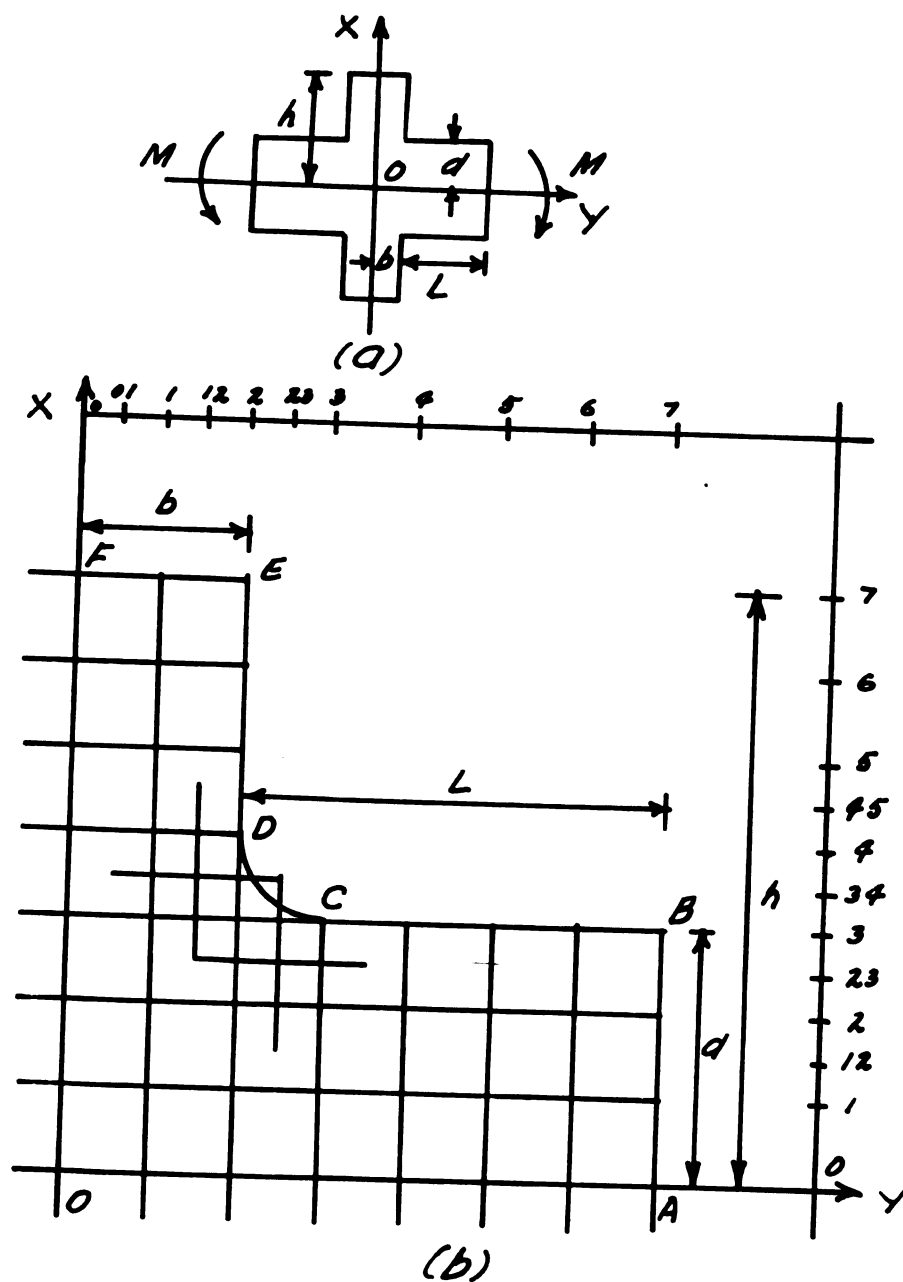
σ_y^2 CURVES FOR BEAM SECTIONS FOR A CROSS SHAPE HAVING $b=d=0.42856h$.
 (PURE BENDING MOMENT)

FIGURE 4.5 Scale: x axis $1'' = 2 \frac{d}{3}$
 σ_y^2 axis $1'' = 0.05 K_1^2 d^2$



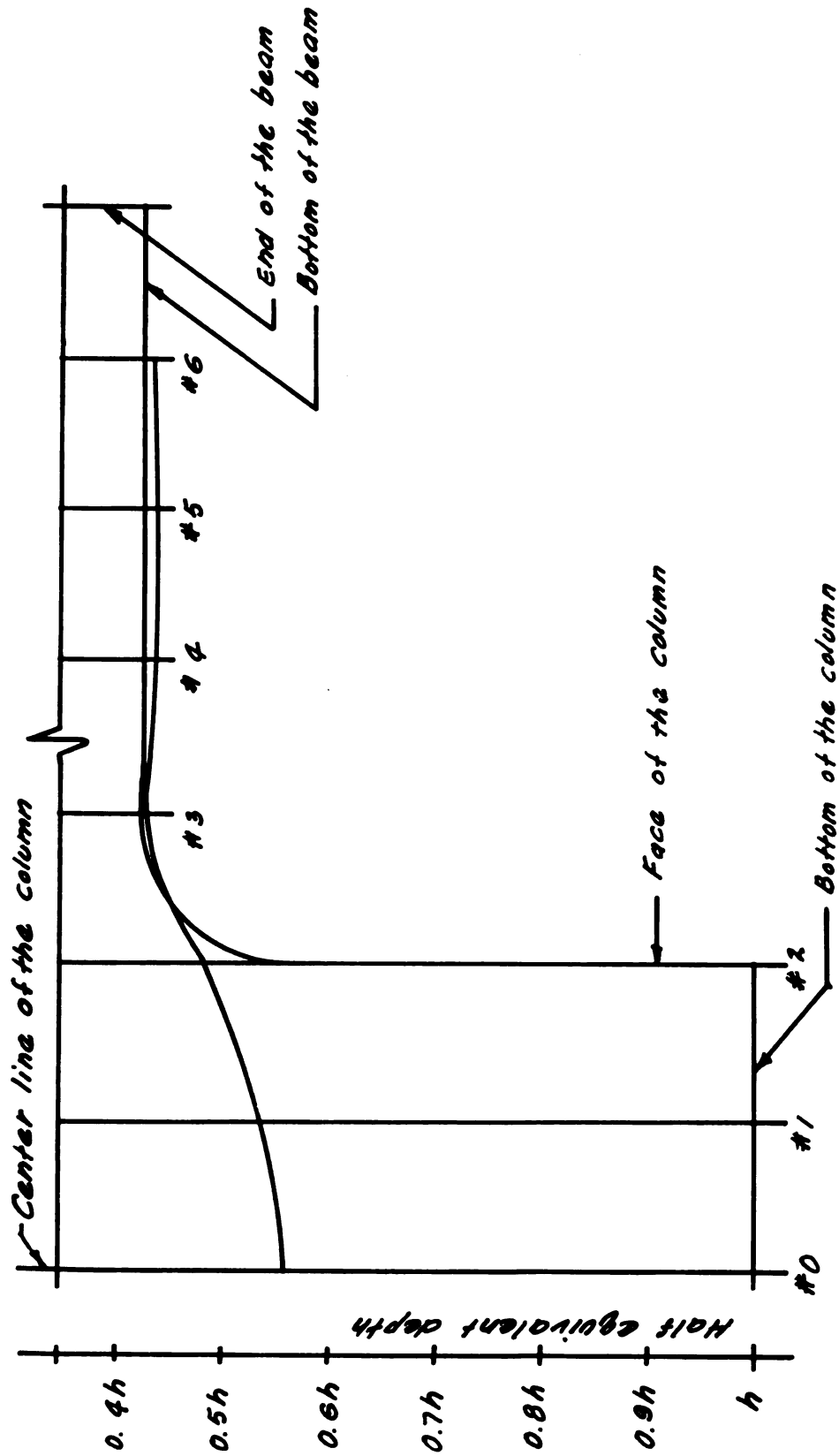
EQUIVALENT DEPTH DIAGRAM FOR A CROSS SHAPE HAVING $\frac{d}{b} = 1.0$ AND $\frac{V}{Q} = \frac{1}{3}$ SUBJECTED TO PURE BENDING MOMENT

FIGURE 4.6



CROSS SHAPE HAVING $\frac{L}{b} = 2.5$, $\frac{h}{b} = 3.5$, $\frac{d}{b} = 1.5$ AND $\frac{r}{a} = \frac{1}{3}$

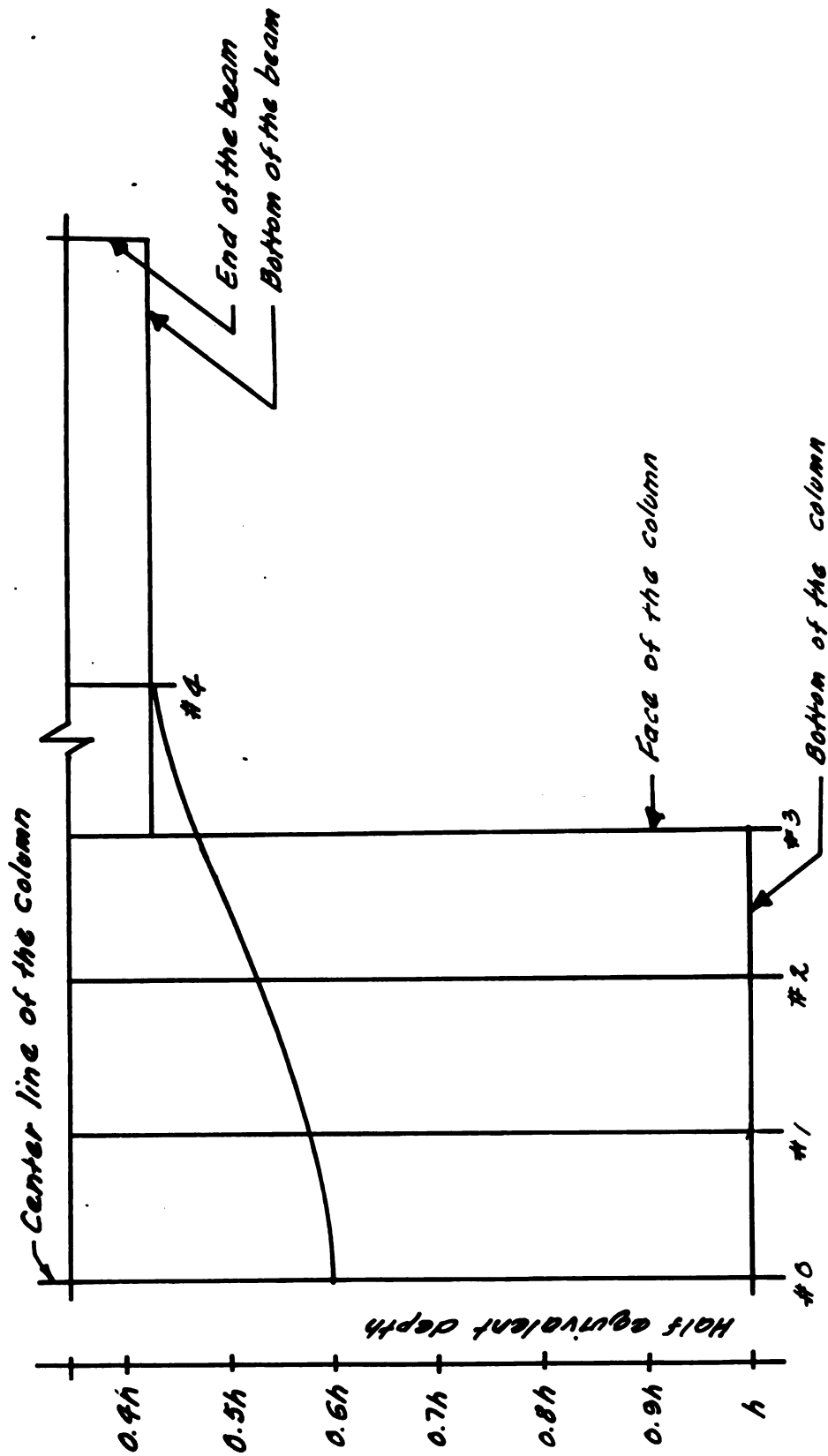
FIGURE 4.7



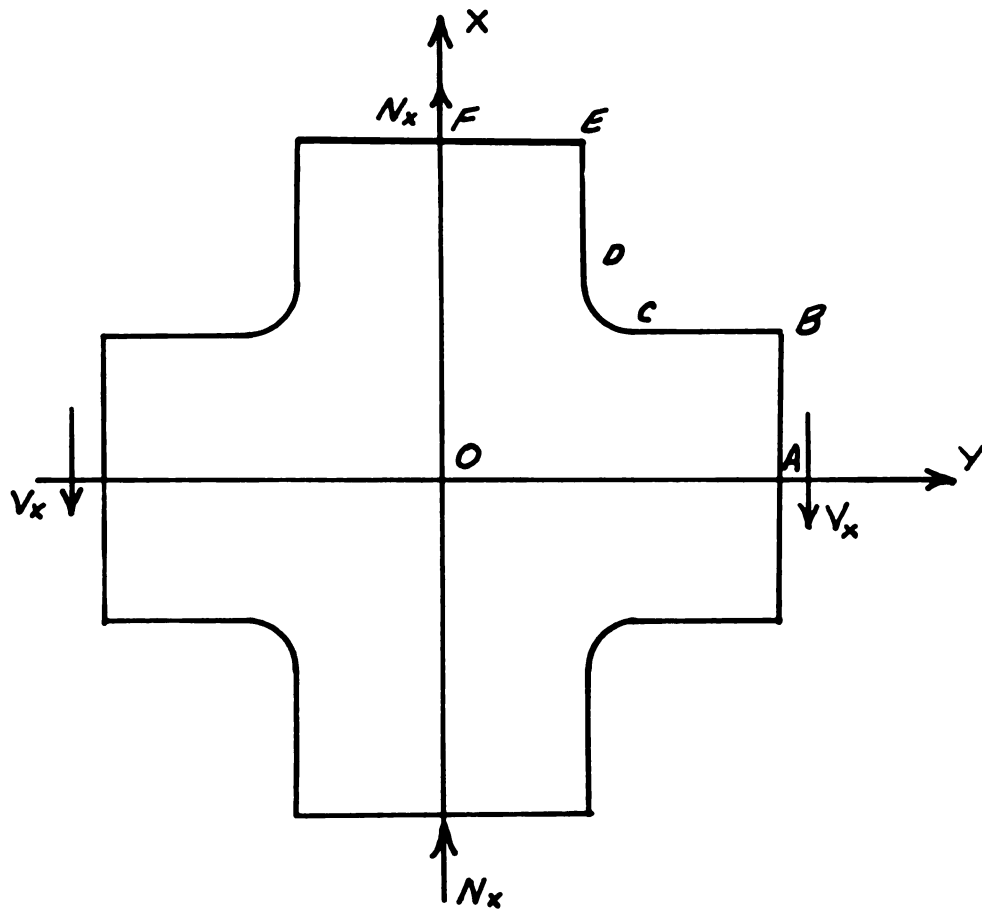
EQUIVALENT DEPTH DIAGRAM FOR A CROSS SHAPE HAVING $\frac{d}{b} = 1.5$ AND $\frac{r}{d} = \frac{1}{3}$

SUBJECTED TO PURE BENDING MOMENT

FIGURE 4B

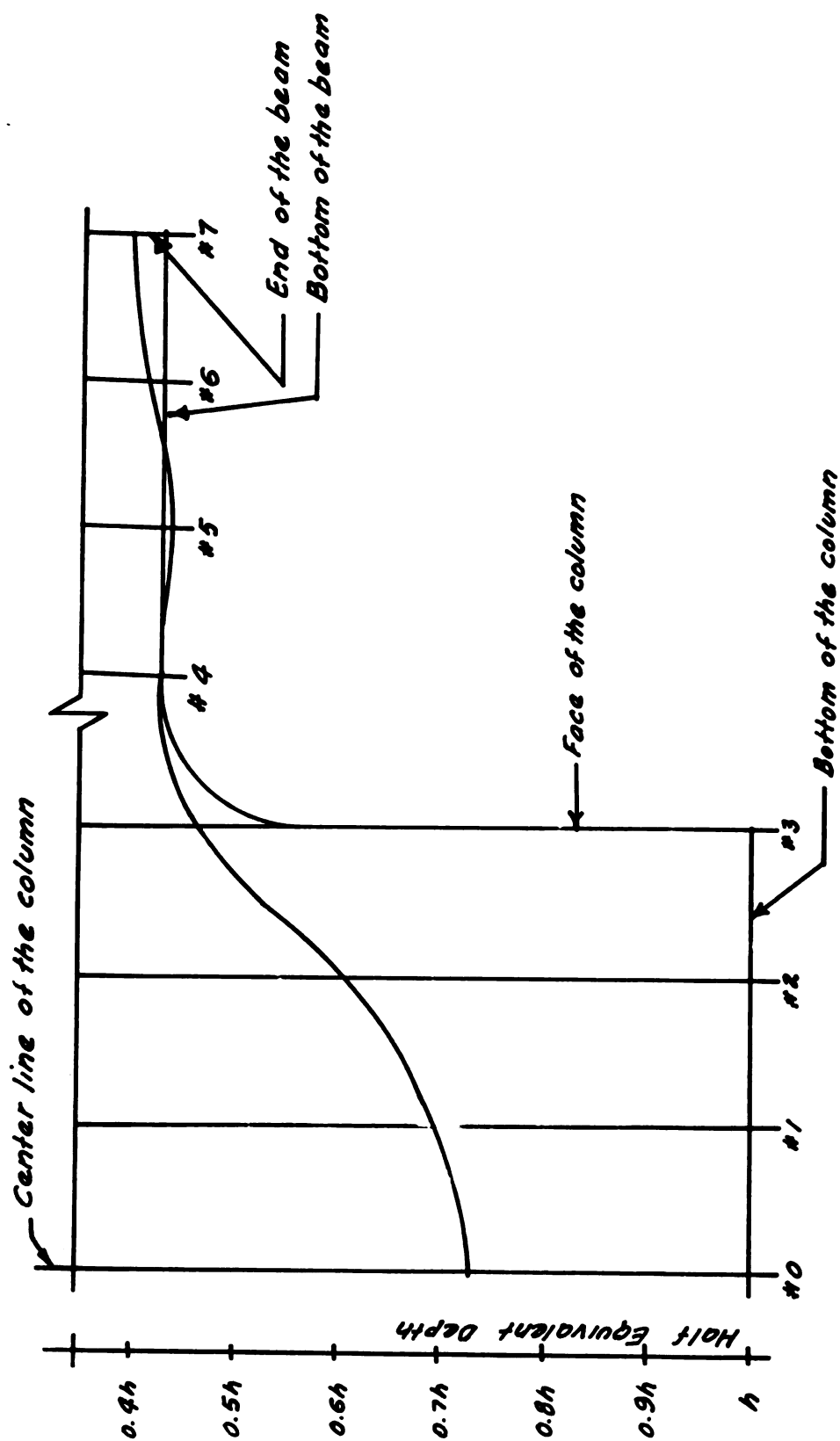


EQUIVALENT DEPTH DIAGRAM FOR A CROSS SHAPE HAVING $\frac{d}{b} = 1.0$ AND $\frac{r}{d} = 0$
 SUBJECTED TO PURE BENDING MOMENT
 FIGURE 4.9.



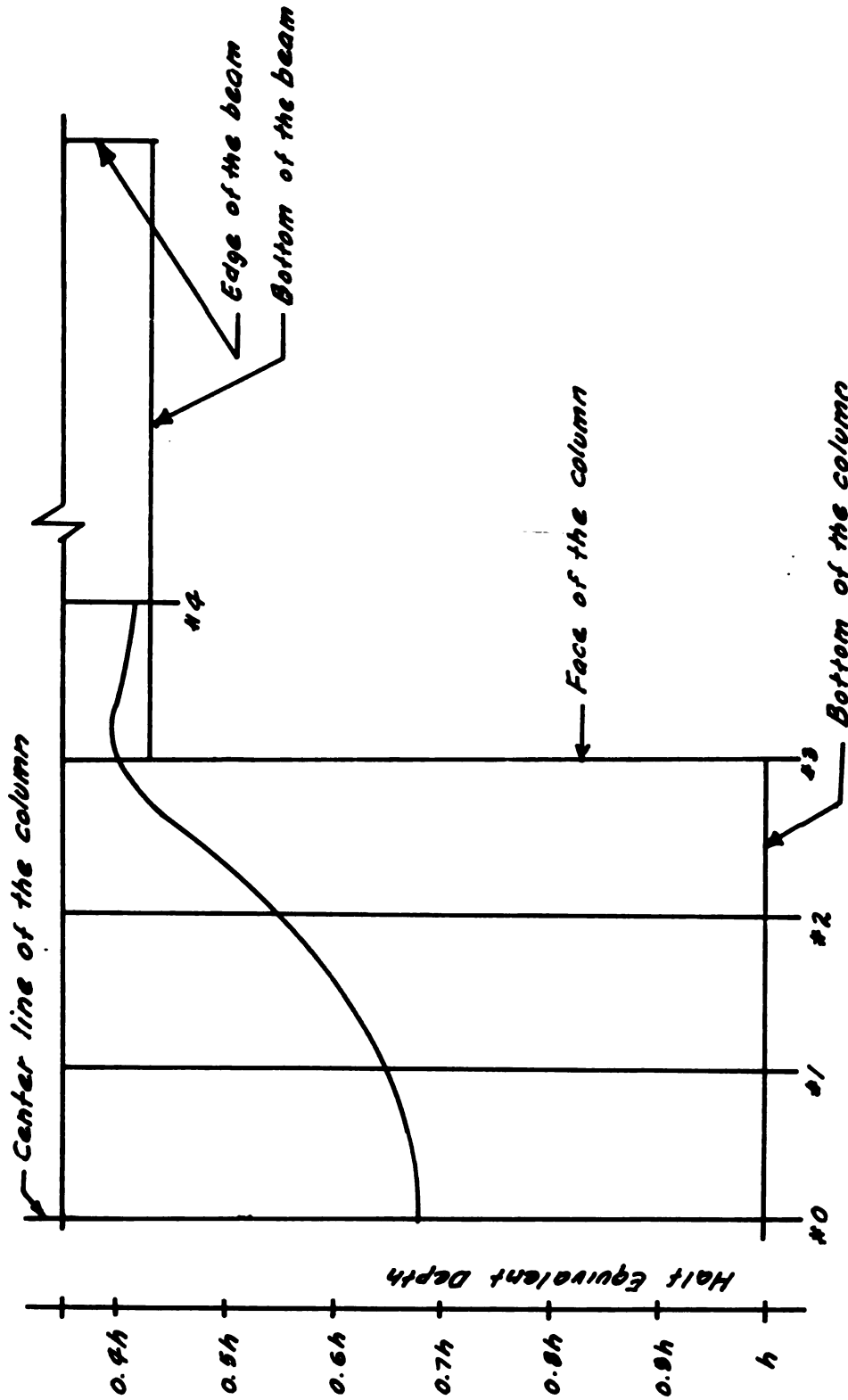
CROSS SHAPE SUBJECTED TO VARIABLE BENDING MOMENT

FIGURE 4.10



EQUIVALENT DEPTH DIAGRAM FOR A CROSS SHAPE HAVING $\frac{d}{b}=1.0$ AND $\frac{r}{a}=\frac{1}{3}$ SUBJECTED TO VARIABLE BENDING MOMENT

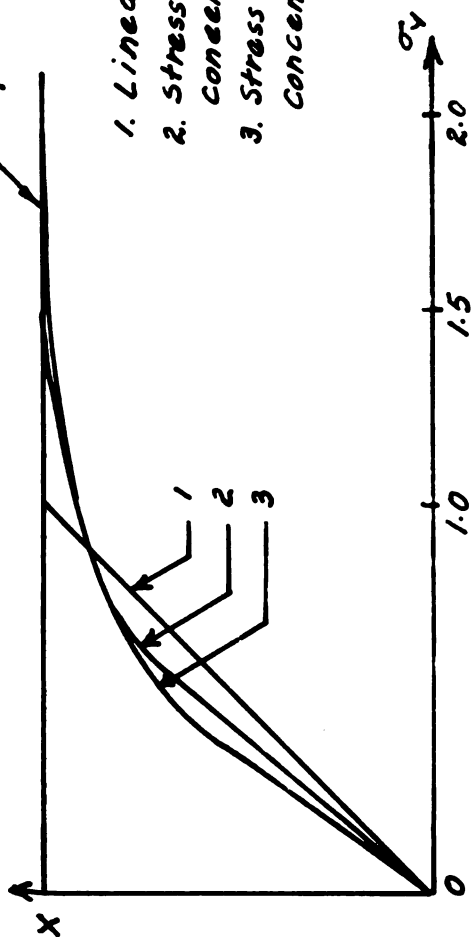
FIGURE 4.11



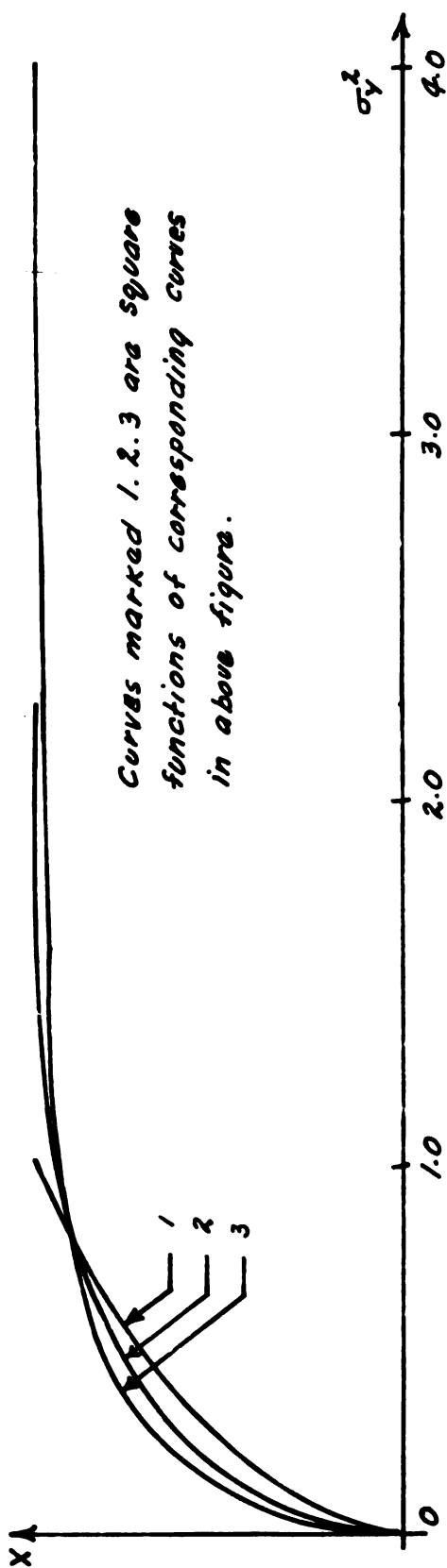
EQUIVALENT DEPTH DIAGRAM FOR A CROSS SHAPE HAVING $\frac{d}{b}=1.0$ AND $\frac{V}{d}=0$ SUBJECTED TO VARIABLE BENDING MOMENT

FIGURE 4.12

Top of beam



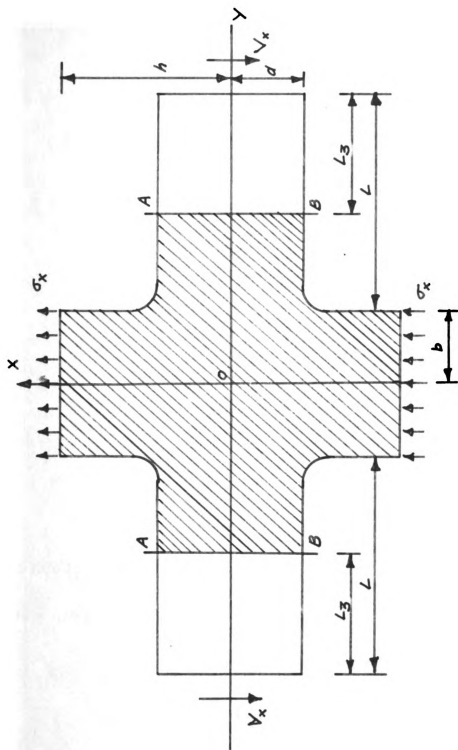
1. Linear Stress distribution.
2. Stress distribution having stress concentration factor 1.5
3. Stress distribution having stress concentration factor 2.0



Curves marked 1.2.3 are square functions of corresponding curves in above figure.

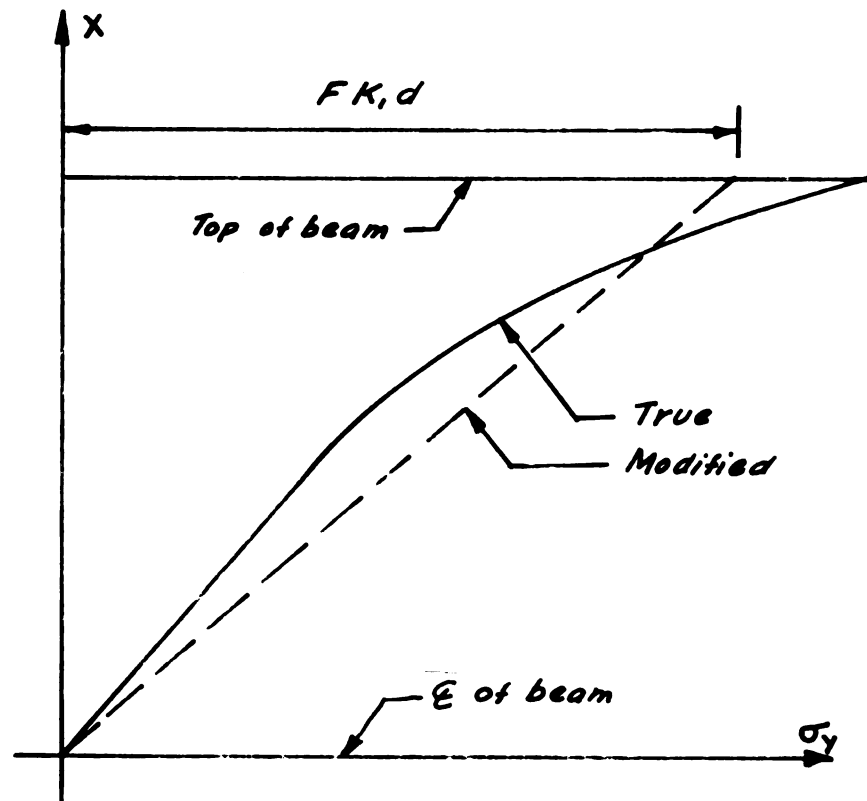
QUALITATIVE DISTRIBUTION OF σ_y AND σ_y^2 ON BEAM SECTION AT FACE OF COLUMN

FIGURE 4.13



CROSS SHAPE HAVING $\frac{L}{b} = 1.0$, $\frac{L}{d} = 2.333$, $\frac{L}{a} = \frac{1}{3}$ AND $\frac{L}{d}$ GREATER THAN 1.333 SUBJECTED TO VARIABLE BENDING MOMENT

FIGURE 4.14



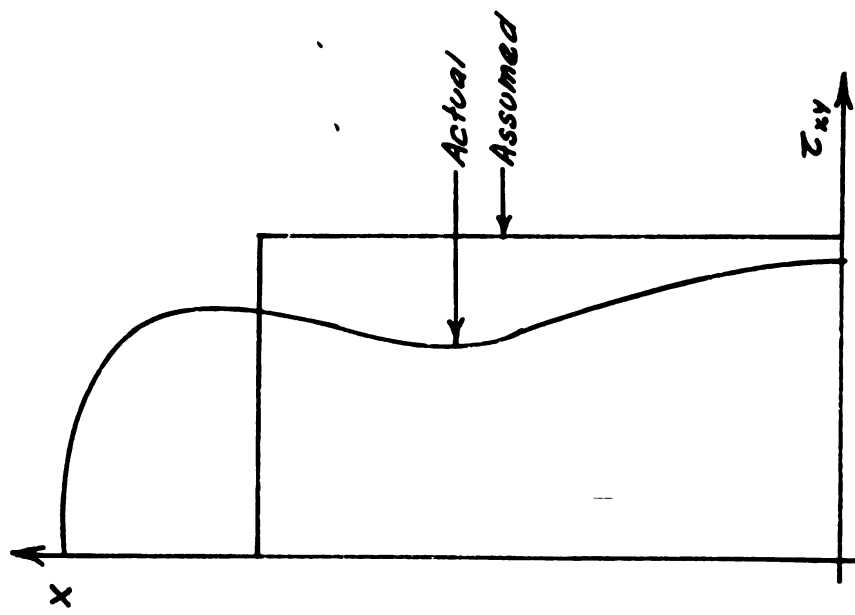
Scale: x axis, $1'' = d/3$; σ_y axis, $1'' = 0.25K, d$.

σ_y STRESS DISTRIBUTION FOR BEAM SECTION #3 FOR A

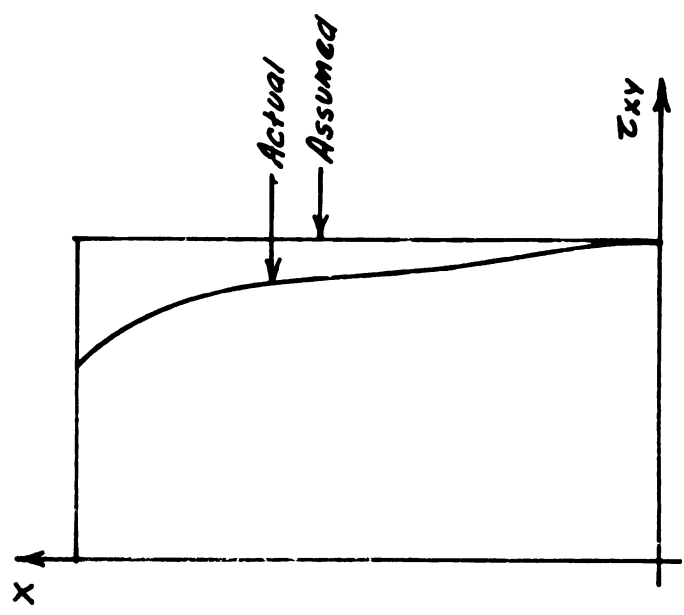
CROSS SHAPE HAVING $b = d = 0.42856 h$ AND $\frac{V}{d} = 0$

(PURE BENDING MOMENT)

FIGURE 4.15



Cross shape with $\frac{d}{b} = 1.0$ AND $\frac{v}{d} = \frac{1}{3}$



Cross shape with $\frac{d}{b} = 1.0$ AND $\frac{v}{d} = 0$

ACTUAL AND ASSUMED SHEAR STRESS DISTRIBUTION ON THE FACE OF A COLUMN
DUE TO SHEAR FORCE V

Scale: X axis: $1'' = \frac{d}{3}$
 z_{xy} axis: $1'' = 0.40 K_x d^2$

FIGURE 4.16

CHAPTER V

SERIES SOLUTION

An analytical solution of the cross shape as a whole is cumbersome, but as the principal interest is to determine the stress distribution within the column, it will be appropriate to look for some analytical solution which can furnish the stress information in the column zone. It is possible to analyze the column zone separately if the stresses on the column face can be known by other means. In part I of this chapter, a general series solution is discussed for the stress function ϕ in a column subjected to moment and shear forces due to a beam connection. The formulas for energy, displacement, slope, and curvature are derived. In part II, the problem is solved for the pure bending moment case. In part III, the problem is solved for the variable bending moment case. In parts II and III the effect of different proportions of the cross shape on the equivalent depth is discussed. In part IV, the method to be used for combination of numerical method and series method results is discussed. In part V, the approximate equivalent depth curve, which is to be used for practical purposes, is discussed.

I. GENERAL SOLUTION

Boundary Conditions

The loading conditions for the column are illustrated in Figure 5.1. It is assumed that the stress function ϕ is symmetrical about the X-axis and anti-symmetrical about the Y-axis. The moment M is imposed on the boundary by some mode of stress distribution of σ_y on the area $x = 0$ to $x = \pm d$. Also shearing stress τ_{xy} are distributed in some form with the resultant shear force V . The mode of distribution of σ_y and τ_{xy} is not specified at this stage except for the assumed symmetry conditions. There will be some σ_x stresses on the top and the bottom edges of the column to resist the downward forces V . If $h \geq 2d$, as is usually the case in frame structures, it will be reasonable to assume that σ_x will be distributed uniformly on the top and the bottom. This was also justified by the photoelasticity study. Also, in order to get a symmetrical problem it is assumed that the magnitudes of σ_x on $x = \pm h$ are the same. It would be easy to superpose a uniform state of uniaxial σ_x stress on the results obtained, to get case where the two values are not equal.

Evaluation of Stress Function ϕ

To solve the elasticity problem for the determination of stresses in the structure, assuming a state of plane

stress to exist, one has to determine the stress function ϕ which satisfies the biharmonic differential equation,

$$\nabla^4 \phi = 0 \quad (5.1)$$

and the boundary conditions. As the stress is discontinuous on the boundaries $y = \pm b$, the stress function is assumed in the form of a trigonometric series in x^1 . The stress function

$$\phi = \frac{P_0 x y^2}{2} + \left[\sum_{m=1}^{\infty} f_m(y) \cdot \sin \frac{m\pi x}{h} \right] \quad (5.2)$$

will satisfy the Equation 5.1. Here m is an integer and each $f_m(y)$ is a function of y only. Substitution of Equation 5.2 into Equation 5.1, using the notation $m\pi/h = \alpha_m$, reduces Equation 5.1 to

$$\alpha_m^4 f_m(y) - 2\alpha_m^2 f_m''(y) + f_m^{IV}(y) = 0, \quad (5.3)$$

which is to be solved for $f_m(y)$. The general solution of this differential equation is

$$f_m(y) = C_{1m} \cosh \alpha_m y + C_{2m} \sinh \alpha_m y \\ + C_{3m} y \cosh \alpha_m y + C_{4m} y \sinh \alpha_m y. \quad (5.4)$$

¹The series methods used in this chapter follow the procedures given in S. Timoshenko and J. N. Goodier, Theory of Elasticity (New York: McGraw-Hill Book Company, Inc., 1951), pp. 46-50.

Substituting Equation 5.4 into Equation 5.2,

$$\phi = \frac{P_0 xy^2}{2} + \sum_{m=1}^{\infty} [C_{1m} \cosh \alpha_m y + C_{2m} \sinh \alpha_m y + C_{3m} y \cosh \alpha_m y + C_{4m} y \sinh \alpha_m y] \sin \alpha_m x. \quad (5.5)$$

As the stress function is symmetrical about X-axis and anti-symmetrical about Y-axis, C_{2m} and C_{3m} should be zero. Hence, the stress function will be as follows:

$$\phi = \frac{P_0 xy^2}{2} + \sum_{m=1}^{\infty} [C_{1m} \cosh \alpha_m y + C_{4m} y \sinh \alpha_m y] \sin \alpha_m x. \quad (5.6)$$

Evaluation of Constants

In Equation 5.6, the constants P_0 , C_{1m} and C_{4m} are to be determined from boundary conditions of the structure. As the stress distribution on the boundaries $y = \pm b$ is discontinuous, it will be convenient to express the given stress distribution in terms of Fourier series. The moments on faces $y = \pm b$ are produced by some form of σ_y distribution which is anti-symmetrical about the Y-axis, and the shearing stress τ_{xy} are distributed symmetrically about the Y-axis in some form on the boundaries $y = \pm b$ with the resultant shear force V . Therefore, σ_y and τ_{xy} , on the boundaries can be expressed in terms of Fourier series as follows:

$$\sigma_y = \sum_{m=1}^{\infty} B_m \cdot \sin \alpha_m x \quad (5.7)$$

$$\tau_{xy} = \pm \frac{A_0}{2} \pm \sum_{m=1}^{\infty} A_m \cos \alpha_m x \quad (5.8)$$

where

$$B_m = \frac{1}{h} \int_{-h}^h F(x) \sin \alpha_m x \cdot dx \quad (5.9)$$

$$A_m = \frac{1}{h} \int_{-h}^h G(x) \cos \alpha_m x \cdot dx \quad (5.10)$$

$$A_0 = \frac{1}{h} \int_{-h}^h G(x) \cdot dx \quad (5.11)$$

$F(x)$ and $G(x)$ are given functions prescribing the distribution of σ_y and τ_{xy} , respectively, on boundaries $y = b$ along $x = -h$ to h . In Equation 5.8, the plus signs apply to $y = +b$, and minus signs to $y = -b$. At this stage it will be considered that B_m , A_m and A_0 are evaluated from the known stress distribution on the boundary. The stress components corresponding to the stress function, Equation 5.6, are as follows:

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} = P_0 x + \sum_{m=1}^{\infty} [C_{1m} \alpha_m^2 \cosh \alpha_m y + 2C_{4m} \alpha_m \cosh \alpha_m y + C_{4m} \alpha_m^2 y \sinh \alpha_m y] \sin \alpha_m x \quad (5.12)$$

$$\sigma_y = \frac{\partial^2 \phi}{\partial x^2} = - \sum_{m=1}^{\infty} [C_{1m} \alpha_m^2 \cosh \alpha_m y + C_{4m} \alpha_m^2 y \sinh \alpha_m y] \cdot [\sin \alpha_m x] \quad (5.13)$$

$$\tau_{xy} = - \frac{\partial^2 \phi}{\partial x \cdot \partial y} = - \sum_{m=1}^{\infty} [C_{1m} \alpha_m^2 \sinh \alpha_m y + C_{4m} \alpha_m \sinh \alpha_m y + C_{4m} \alpha_m^2 y \cosh \alpha_m y] \cos \alpha_m x - P_0 y. \quad (5.14)$$

By substituting $y = \pm b$ in Equations 5.13 and 5.14 and comparing with Equations 5.7 and 5.8, C_{1m} and C_{4m} can be evaluated as follows:

$$C_{1m} = \frac{2}{\alpha_m^2} \left[\frac{A_m b \alpha_m \sinh \alpha_m b - B_m (\sinh \alpha_m b + b \alpha_m \cosh \alpha_m b)}{(2b \alpha_m + \sinh 2 \alpha_m b)} \right],$$

$$C_{4m} = \frac{2}{\alpha_m} \left[\frac{B_m \sinh \alpha_m b - A_m \cosh \alpha_m b}{(2b \alpha_m + \sinh 2 \alpha_m b)} \right],$$

$$P_0 = -\frac{A_0}{2b}. \quad (5.15)$$

When C_{1m} , C_{4m} and P_0 are known the stress function ϕ is completely known. From ϕ the stresses, energy, displacements, slope, and curvature can be evaluated.

Evaluation of Energy, Displacement, Slope and Curvature

Formulas

The energy of a small segment of Figure 5.1 per unit length in the Y direction and unit length in Z direction can be written according to Equation 2.3 as

$$\text{Energy} = \frac{1}{2E} \int_{-h}^h [\sigma_x^2 + \sigma_y^2 - 2\nu \sigma_x \sigma_y + 2(1+\nu) \tau_{xy}^2] dx. \quad (5.16)$$

Substituting Equations 5.12, 5.13, and 5.14 into Equation 5.16 and simplifying after integration, will reduce Equation 5.16 to

$$\begin{aligned}
\text{Energy} = \frac{1}{2E} \left[\left\{ \frac{2P_0^2 h^3}{3} + h \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \cosh \alpha_m \gamma + 2C_{4m} \alpha_m \cosh \alpha_m \gamma \right. \right. \\
\left. \left. C_{4m} \alpha_m^2 \gamma \sinh \alpha_m \gamma)^2 \right. \right. \\
\left. - 4P_0 h^2 \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \cosh \alpha_m \gamma + 2C_{4m} \alpha_m \cosh \alpha_m \gamma \right. \\
\left. + C_{4m} \alpha_m^2 \gamma \sinh \alpha_m \gamma) \frac{(-1)^m}{m\pi} \right\} \\
+ \left\{ h \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \cosh \alpha_m \gamma + C_{4m} \alpha_m^2 \gamma \sinh \alpha_m \gamma)^2 \right\} \\
- 4M \left\{ (C_{1m} \alpha_m^2 \cosh \alpha_m \gamma + C_{4m} \alpha_m^2 \gamma \sinh \alpha_m \gamma) \frac{P_0 (-1)^m h^2}{m\pi} \right. \\
\left. - \frac{h}{2} \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \cosh \alpha_m \gamma + 2C_{4m} \alpha_m \cosh \alpha_m \gamma \right. \\
\left. + C_{4m} \alpha_m^2 \gamma \sinh \alpha_m \gamma) (C_{1m} \alpha_m^2 \cosh \alpha_m \gamma + C_{4m} \alpha_m^2 \gamma \sinh \alpha_m \gamma) \right\} \\
+ 2(1+M) \left\{ h \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \sinh \alpha_m \gamma + C_{4m} \gamma \alpha_m^2 \cosh \alpha_m \gamma \right. \\
\left. C_{4m} \alpha_m \sinh \alpha_m \gamma)^2 + 2P_0^2 \gamma^2 h \right\} \Bigg] . \quad (5.17)
\end{aligned}$$

From the energy formula the equivalent depth can be evaluated. The displacement, slope, and curvature can be evaluated by expressing the displacement derivatives in terms of stresses as follows:

$$e_x = \frac{\partial u}{\partial x}, \quad e_y = \frac{\partial v}{\partial y}, \quad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad (5.18)$$

$$e_x = \frac{1}{E}(\sigma_x - \mu \sigma_y), \quad e_y = \frac{1}{E}(\sigma_y - \mu \sigma_x), \quad \gamma_{xy} = \frac{2(1+\mu)}{E} \tau_{xy}. \quad (5.19)$$

After substitution of Equations 5.12, 5.13, and 5.14 into Equation 5.19, proper integrations are carried out to evaluate u and v . The results are written as,

$$u = \frac{1}{E} \left[\frac{P_0 x^2}{2} - \sum_{m=1}^{\infty} (C_{1m} \alpha_m \cosh \alpha_m y + 2 C_{4m} \cosh \alpha_m y + C_{4m} \alpha_m y \sinh \alpha_m y) \cos \alpha_m x \right. \\ \left. - \mu \sum_{m=1}^{\infty} (C_{1m} \alpha_m \cosh \alpha_m y + C_{4m} \alpha_m y \sinh \alpha_m y) \cos \alpha_m x - \frac{P_0 y^2}{2} (2 + \mu) + \alpha_2 \right],$$

$$v = \frac{1}{E} \left[- \sum_{m=1}^{\infty} (C_{1m} \alpha_m \sinh \alpha_m y + C_{4m} \alpha_m y \cosh \alpha_m y - C_{4m} \sinh \alpha_m y) \sin \alpha_m x \right. \\ \left. - \mu P_0 x y - \mu \sum_{m=1}^{\infty} (C_{1m} \alpha_m \sinh \alpha_m y + 2 C_{4m} \sinh \alpha_m y + C_{4m} \alpha_m y \cosh \alpha_m y - C_{4m} \sinh \alpha_m y) \sin \alpha_m x + \alpha_1 \right],$$

$$\frac{\partial u}{\partial y} = \frac{1}{E} \left[- \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \sinh \alpha_m y + 2 C_{4m} \alpha_m \sinh \alpha_m y + C_{4m} \alpha_m \sinh \alpha_m y + C_{4m} \alpha_m^2 y \cosh \alpha_m y) \cos \alpha_m x \right. \\ \left. - \mu \sum_{m=1}^{\infty} (C_{1m} \alpha_m^2 \sinh \alpha_m y + C_{4m} \alpha_m \sinh \alpha_m y + C_{4m} \alpha_m^2 y \cosh \alpha_m y) \cos \alpha_m x - P_0 (2 + \mu) y \right],$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{E} \left[- \sum_{m=1}^{\infty} (C_{1m} \alpha_m^3 \cosh \alpha_m y + 4 C_{4m} \alpha_m^2 \cosh \alpha_m y + C_{4m} \alpha_m^3 y \sinh \alpha_m y) \cos \alpha_m x \right. \\ \left. - \mu \sum_{m=1}^{\infty} (C_{1m} \alpha_m^3 \cosh \alpha_m y + 2 C_{4m} \alpha_m^2 \cosh \alpha_m y + C_{4m} \alpha_m^3 y \sinh \alpha_m y) \cos \alpha_m x - P_0 (2 + \mu) \right], \quad (5.20)$$

where a_1 and a_2 are constants. The center line deflection u , its slope $\frac{\partial u}{\partial y}$ and curvature $\frac{\partial^2 u}{\partial y^2}$ can be evaluated from Equation 5.20 by substituting $x = 0$.

All results discussed above are in terms of C_{1m} and C_{4m} and P_0 , which in turn depend on A_0 , A_m , B_m , i.e. on the boundary conditions of the structure. In the following pages the problem will be discussed with specific boundary conditions. As in the previous chapter, the analysis of the column zone will be made with two sets of boundary conditions. One is the case of pure bending moment, and the second one will be of such a nature that there will be external shear forces.

II. PURE BENDING MOMENT CONDITION

Boundary Conditions

In view of the discussion, in Chapter IV, of the true bending stress distribution and the effect of modifying it to the linear stress distribution, the bending stress σ_y on the face of the column is assumed as linear function in the present case, i. e. on $y = \pm b$ from $x = 0$ to $x = \pm d$

$\sigma_y = K_1 x$. The boundary conditions can be summarized as follows:

On $y = \pm b$, $\sigma_y = K_1 x$ from $x = 0$ to $x = \pm d$,
 $\sigma_y = 0$ from $x = \pm d$ to $x = \pm h$, and
 $\tau_{xy} = 0$ from $x = 0$ to $x = \pm h$.

On $x = \pm h$, $\sigma_x = \tau_{xy} = 0$ from $y = 0$ to $y = \pm b$.

Substitution of the boundary conditions of $y = \pm b$ into Equations 5.9, 5.10, and 5.11 for the Fourier coefficients yields the results that constants A_0 and A_m are equal to zero, and

$$\begin{aligned} B_m &= \frac{1}{h} \int_{-d}^d K_1 x \cdot \sin \alpha_m x \cdot dx \\ &= \frac{2K_1}{h\alpha_m^2} (\sin \alpha_m d - d\alpha_m \cos \alpha_m d) \end{aligned} \quad (5.21)$$

Now that A_0 , A_m , and B_m are known the constants of Equation 5.15 can be evaluated. P_0 will be zero in this case.

Equation 5.12 yields σ_x equal to zero on $x = \pm h$. There are shearing stresses τ_{xy} on $x = \pm h$, which are numerically insignificant.

Evaluated Results

As the stress function has been assumed in the form of an infinite series, all functions related to it will be infinite series. In practice one takes enough terms of the series to give reasonable results. With discontinuous values of σ_y and τ_{xy} assumed, the coefficients A_m and B_m will be of the order $1/\alpha_m$. The coefficients C_{1m} and C_{4m} will be of the order $(1/\alpha_m^2)(e^{-\alpha_m b})$. The coefficients of the trigonometric functions in the series for the stresses will be of order $e^{-(b-|y|)\alpha_m}$. Hence, convergence is assured for all interior points (where $|y|$ is less than b).

The convergence on the face $y = b$ is known for the series for σ_y and τ_{xy} because these series reduce to the boundary expansions, and the 50 term partial sum represents quite accurately the given boundary stress distribution, as shown by Figure 5.2. The series for the displacement components and the series for the energy are more rapidly convergent and converge also on $y = b$, while the series for the slope $\frac{\partial u}{\partial y}$ is of the same order as that for the stresses. The coefficients in the series for the curvature $\frac{\partial^2 u}{\partial y^2}$ are of the order $\alpha_m \cdot e^{-(b-|y|)\alpha_m}$, giving slower convergence in the interior and not converging at all on the face. In view of the above discussion, all calculations in this chapter have been carried out with at most $m = 50$.

Due to repetition of the arithmetic process for all functions, all computations in this chapter were carried out with the MISTIC computer. Various programs were prepared in the floating decimal form. The programs were designed in such a way that the dimensions of the cross shape are variable parameters. The program for the evaluation of the energy is given in Appendix C as an example.

In the case of the pure bending moment loading, according to the conclusions in Chapter II, the stresses on the face of the column will be independent of the beam span L_2 (Figure 3.3), and the height of the column for L/b and h/d greater than 1.0 and 2.0, respectively. Hence, for the analysis of the mode of the stress distribution in

the column portion of the joint the only variables are b , d , and h . Two independent dimensionless parameters can be formed from these variables, say b/h and d/b . As an example, for $b = d = 0.42856h$ all quantities discussed above are evaluated. Stresses are recorded in Table 5.1 and bending stress σ_y is plotted in Figure 5.2. The energy and the equivalent depth results are given in Table 5.2. The displacement u , slope $\frac{\partial u}{\partial y}$, and curvature $\frac{\partial^2 u}{\partial y^2}$ for the beam center line (Y-axis) are recorded in Table 5.3 and presented graphically in Figure 5.3. Equivalent depth based on center line curvature is calculated by $M = EI \frac{\partial^2 u}{\partial y^2}$. This equivalent depth is recorded in the last column of Table 5.3. The two equivalent depth curves calculated on the energy and center line curvature concepts are plotted in Figure 5.4. Figure 5.4 indicates that the equivalent depth based on the curvature is nowhere near that based on the elastic energy. Hence, an energy calculation derived from the equivalent depth based on center line curvature would be greatly in error. To study the effect of the various ratios of the b and d dimensions of the cross shape on the equivalent depth inside the column, equivalent depths are evaluated for different proportions of b and d and are presented in Table 5.4 and in Figures 5.5 and 5.6

Discussion of Results

The deflection u at the face of the column can be made zero by selecting the constant a_2 in the deflection

Equation 5.20 as 0.203703. Comparison of the deflection and the slope diagrams indicates that the deflection is fairly constant in the central half of the column. Hence, the slope in that region is almost zero compared with the slope in the vicinity of the face of the column.

The equivalent depth diagrams inside the column calculated by the numerical method of Chapter IV and one calculated by the series method of the chapter (Table 5.2) are plotted in Figure 5.7 for comparison. From a study of Figure 5.7a it is concluded that the shape of the equivalent depth diagram for the case with the fillet agrees closely with the one calculated by series application in the middle $2/3$ width of the column. But the value of the equivalent depth does not agree. These two curves can be made to coincide (within 7%) by adding a constant amount to all of the ordinates of the series method curve, raising or lowering it enough to make the two curves agree at one point. This suggests that the shape of the stress distribution in that region at various sections is not altered by replacing the true stress distribution on the face of the column of the cross shape by a linear stress distribution of σ_y on the face of the column. The equivalent depth diagram in the outer $1/3$ portion of the column does not agree because in the case of the cross shape with the fillet the stress is distributed over more than the depth of the beam. Hence, the contact area for transferring the

stresses from the beam to the column is not the same in two cases. Also in Figure 5.7b there is agreement in the shapes of the curves in the middle half of the column. Comparison of the results of Figures 5.7a and 5.7b indicates that the portion of the column in which there is disagreement is the same in both cases. Hence, it is concluded that even though the R value (the ratio of the equivalent depth d_e to the actual depth $2d$ of the beam), is different at the face of the column in the two cases of Figures 5.7a and 5.7b due to difference in stress concentration, the stress concentration factor does not disturb the mode of stress distribution inside the column away from the face of the column. In other words, the local stress concentration effect does not extend to the entire width of the column.

The suggested identity of stress distributions, based on the agreement in the shape of the equivalent depth diagrams in the middle $2/3$ portion of the column can be checked by comparing the stress distributions in that region calculated by the numerical method and by the series method. As the σ_y stress contributes most of the energy, only σ_y stresses are compared. Comparison of Figure 4.4 with Figure 5.2 shows the agreement of the mode of stress distribution in the middle $2/3$ of the column.

The shape of the equivalent depth diagram calculated by the numerical method for a cross shape without fillets

agrees closely with that calculated by the series method throughout the column width as can be seen from Figure 5.7. This again supports the argument presented in Chapter IV that the true stress distribution existing at the face of the column of the cross shape subjected to pure bending moment loading condition can be replaced by a linear stress distribution without altering the mode of stress distribution throughout the width of the column. The application of series results in combination with the numerical method will be discussed in Part IV.

Comparison of results in Table 5.4 for $b = d = 0.42856h$ with those for $b = d = 0.2h$ indicates that when the ratio d/b remains constant, the height of the column does not have any effect upon R . Indirectly this supports the conclusion of interaction limit in the column given in Chapter III. Of course, this lack of dependence on the height of the column has some limitation. This can be seen by taking $d = h$. The ratio R at all sections inside the column will be 1.0 as is shown by the conventional beam theory and not the one calculated in Table 5.4. With the help of the conclusion in Chapter III, about the interaction limit in the column it can be concluded that when d is greater than $0.5h$, then the equivalent depth information in Table 5.4 cannot be used. The lack of dependence of R on the height of the column can be observed by comparing

the results of $b = 0.42856h$, $d = 0.3h$ ($d/b = 0.70$) with the results for $b = 0.3h$, $d = 0.2h$ ($d/b = 0.667$). Of course, the results for the ratio R in the two cases do not agree closely because d/b ratios are a little different in the two cases, but the trend is apparent. Here also there will be a limit to the lack of dependence on the height. In general for any d/b ratio if d exceeds $0.5h$, then the ratio R will no longer be independent of the height. As d varies from $0.5h$ to h the ratio R will approach the value 1.0.

In Figure 5.5 the depth d of the beams is kept constant and equal to $0.2h$ and the column width is varied. As the width of the column is increased the equivalent depth curve becomes steeper attaining different maximum equivalent depths. This is due to the fact that as the column becomes wider there is more transition zone available and consequently the stress distribution curve at any particular beam section becomes flatter. It is to be noted that all the curves coincide near the face of the column. They spread out as they approach the center line of the column. In Figure 5.6 the width of the column is kept constant and the depth of the beam is decreased. The same characteristics are observed as in Figure 5.5.

III. VARIABLE BENDING MOMENT CONDITION (SHEAR-LOADING CASE)

A cross shape subjected to shear loading by a shear force V having a parabolic distribution on the ends of the beam and a uniform normal stress distribution on the top and the bottom ends of the column as shown in Figure 4.14 will be replaced by a column acted upon by the shear force V on the face of the column and the bending moment $M = VL$ as shown in Figure 5.1, where L is the span length of the beam. Since it was concluded in Chapter IV that on the face of the column the assumption of uniform shear stress distribution due to shear force V will be better than the assumption of a parabolic distribution, a uniform stress distribution is assumed in the following analysis. The bending stress distribution is assumed to be linear. This assumption will not alter significantly the mode of stress distribution inside the column. The analysis of this case can be done by the application of results derived in Part I. For this the constants A_0 , A_m , and B_m are to be evaluated by using the boundary conditions.

Boundary Conditions

If the shear stress τ_{xy} is given by $-K_2(d^2 - x^2)$ on the end of the beam, then the moment M acting on the face of the column is equal to $(4/3) K_2 d^3 L$. The boundary conditions are summarized as follows:

On $y = \pm b$, $\sigma_y = 2K_2 Lx$ from $x = 0$ to $x = \pm d$ and $\sigma_y = 0$
from $x = \pm d$ to $x = \pm h$.

$\tau_{xy} = (2/3) K_2 d^2$ from $x = 0$ to $x = \pm d$ and

$\tau_{xy} = 0$ from $x = \pm d$ to $x = \pm h$.

On $x = +h$, $\tau_{xy} = 0$ and $\sigma_x = (2/3) K_2 d^3/b$ from $y = 0$ to
 $y = \pm b$.

On $x = -h$, $\tau_{xy} = 0$ and $\sigma_x = -(2/3) K_2 d^3/b$ from $y = 0$ to
 $y = \pm b$.

By using these boundary conditions in the Formulas 5.9, 5.10 and 5.11 for Fourier coefficients, A_0 , A_m , and B_m are evaluated. Hence, the constants C_{1m} and C_{4m} are known, and the stress function is known.

Evaluated Results

It is to be noted that the parameters in the stress function ϕ will be b , d , L , and h . All these parameters will affect the energy. The equivalent depth as defined in Part III of Chapter IV is calculated with various ratios of b , d , and L to h . In Table 5.5 the results are recorded for the case in which L is kept constant at the value 0.57143h, and b and d are variables. In Table 5.6 the results are recorded for the case in which b and d are kept constant at 0.42856h and L is a variable. In Figure 5.8 the equivalent depth calculated for the case of $b = d = 0.42856h$ and $L = 0.57143h$ is plotted. Also in this figure

are plotted the results of equivalent depth calculated by the numerical method for shear-loaded joints both with and without fillets. In Figures 5.9 and 5.10 part of the results from Table 5.5 are plotted. Figure 5.11 represents the results of Table 5.6.

Discussion of Results

Figure 5.8 indicates that for joints with fillets the shape of the equivalent depth curve as calculated by the series method agrees closely in the middle $2/3$ of the column with that determined by the numerical method. Hence, the true mode of stress distribution at various sections inside the column is not altered by replacing the true stress distribution at the face of the column by the linear distribution of σ_y and uniform distribution of τ_{xy} . The difference due to contact area on the face of the column brings out the disagreement near the face.

In the case of the joint without fillets the shape of the equivalent-depth curve as calculated by the numerical method agrees only in the middle half of the column width with that calculated by the series method. The agreement in the shear loading case is not as good as in the case of pure bending. This means that in the case without fillets the shear stress distribution due to shear force V is different from the assumed uniform one.

Figure 5.9 shows the curves of equivalent depth in which d and L of the cross shape are kept the same and b is varied. The behavior is the same as in the case of the pure bending moment. The reasoning given for this behavior in the case of pure bending moment applies here too.

Figure 5.10 shows the curves in which b and L are kept constant and d is varied. The effect is the same as in Figure 5.9. Comparison of Figures 5.5 and 5.9 indicates that for $b = d = 0.2h$ the curves for the pure bending moment case and the shear loading case are almost the same, and the variation with b/d is similar in the two cases. Although the agreement is not so close for larger values of b/d as it is for $b/d = 1.0$. Figures 5.6 and 5.10 indicate that for $b = d = 0.42856h$, the curves are not the same for the two cases of loading. The curve in the shear-loading case is steeper, and it is expected that the disagreement would even be greater for larger values of b/d . Concluding, as d increases with respect to h , the pure bending moment and shear-loading case curves spread apart. This conclusion can also be drawn from Tables 5.4 and 5.5. The agreement of the equivalent depth curves in the case of pure bending moment and the case of shear loading for $d = 0.2h$ can be explained as follows: The relatively high loss of energy in the case of shear loading is offset by the energy due to uniform normal stress σ_x existing in the column. For long columns (small d/h) this energy in the column due to

σ_x will be a significant part of the total whereas it will not be so significant for short columns.

The equivalent depth d_e is found by solving the equation $M^2/2EI + 1.2 V^2/2GA = \text{Elastic energy}$, in which I and A are expressed in terms of d_e . In the case of shear loading the bending moment on a section is a linear function of the beam span length. As the beam span length becomes larger, the effect of shear becomes smaller and the equivalent depth as calculated in the shear case will approach the one for the case of pure bending moment. This fact is observed by noticing the trend of the equivalent-depth curve as L increases, Figure 5.11. Comparing the results of the equivalent depth for the case of $b = d = 0.42856h$ and $L = 9.14288h$ from Table 5.6 with the one of the pure bending moment case with $b = d = 0.42856h$ shows that the equivalent depth values for these two cases differ by less than 2.0%. Similar behavior is to be expected also for other ratios of b/h and d/h , so that the results given in Table 5.5 approach those in Table 5.4 as L increases. It was shown previously that for $d = 0.2h$ the equivalent depth curves in pure bending and shear loading case are almost the same. Hence, for that particular d/h ratio, L is not important, but as d/h increases, L will be significant as far as agreement between pure bending and shear loading is concerned.

IV. INTERPRETATION OF RESULTS OF SERIES METHOD AND NUMERICAL METHOD

A. Pure Bending Moment

It was noted in Part III of Chapter IV that the true stress distribution on the face of the column can be replaced by a linear stress distribution having a maximum stress FK_1d , where F is a correction factor. It was also noted that the series-method solution with assumed linear distribution having maximum σ_y stress equal to K_1d results in a shape of the equivalent depth diagram similar to that obtained from the numerical solution in most of the column. Hence, the value of the correction factor F is to be selected in such a way that equivalent depths at the column face as calculated by the numerical method and the series method are the same. Concluding, as far as the shape of the equivalent depth curve is concerned the equivalent depth as calculated by the series method can be used for all purposes. Hence, the equivalent depth curve inside the column can be plotted by selecting the proper curve of the series method and locating this curve by raising or lowering it enough to have at the face of the column the proper value of the ratio R as calculated by the numerical method (1.16 in case of $r/d = 1/3$ and 1.10 in case of $r/d = 0$). These values of the ratio R of equivalent depth to the actual depth are discussed in Part III of Chapter IV.

B. Variable Bending Moment (Shear-Loading Case)

As in the case of pure bending moment, the series method results in the shear case also can be used in combination with the results of the numerical method. In the shear-loading case the R value at the face of the column will depend upon the proportional contribution of the shear energy due to shear force to the total energy. According to the type of the loading, the R value will depend upon the beam span length as discussed in Chapter IV. By selecting the proper R value at the face of the column and combining it with the proper curve of the series method, depending upon d , b , and L values, the equivalent depth diagram can be constructed.

V. APPROXIMATION OF RESULTS IN FORM CONVENIENT FOR USE IN DESIGN

The correct equivalent depth diagram inside the column can be constructed for any d , b , and L proportions of the joint by the approach discussed in the preceding pages. The results indicate that the equivalent depth diagram depends upon the loading condition and proportions of the joint. Hence, for precise calculations an extensive set of curves must be prepared which can be used by the designer. This can be done by constructing the various sets of curves for different proportions of the joint and

different types of loading, by the series method, and combining these results properly with the values of R at the face of the column evaluated by the numerical method. For practical purposes, however, an approximate equivalent depth diagram has been constructed as explained in the following paragraph which can be used for all proportions and any kind of loading without introducing any serious error.

First the selection of the R value at the face of the column will be considered. For pure bending-moment loading it was concluded in Chapter IV that for practical purposes the R value at the face of the column could be taken as 1.16 for d/b less than 2.0 and $r/d = 1/3$. In the case of the shear loading for $r/d = 1/3$ and $d/b = 1.0$, the R value was found to be 1.07. As the span becomes larger, the R value at the face of the column approaches the value for the case of the pure bending moment. For $r/d = 0$, R at the face is 1.10 in the case of the pure bending moment and 0.9242 in the case of shear loading. In all these cases the ratio of the span length L to the beam depth $2d$ was 0.667. In all practical problems this ratio is greater than 2.0. Hence, for values of $L/2d$ greater than 2 the R value at the face of the column in the case of shear loading will be between 1.07 and 1.16 for $r/d = 1/3$ and between 0.9242 and 1.10 for $r/d = 0$. In view of the previous discussion it is suggested that if the

R value at the face of the column is taken as 1.0 for $r/d = 0$ and as 1.10 for $r/d = 1/3$, for any kind of loading and any joint proportions, it will not introduce any significant error, keeping in mind the many uncertainties in design practice.

The equivalent-depth curves calculated by the series method are approximated by dashed straight lines in Figures 5.5, 5.6, 5.9, and 5.10. The angles which these lines make with the horizontal line are noted. In the case of pure bending for $d = 0.2h$, as the column width b increases from $0.2h$ to $0.42856h$ the angle ϕ which the approximating line makes with the horizontal line increases from 20 degrees to 31 degrees. In the case of the shear loading for $d = 0.2h$ it varies from 22 degrees to 35 degrees. In Figures 5.5 and 5.9 the ratio of the beam span length L to the beam depth $2d$ is 1.428. Almost all practical problems will have this ratio greater than 2.0. As the ratio $L/2d$ increases the approximate equivalent depth lines in the case of the shear loading will approach to the pure bending moment case. This behavior has been discussed before. In view of the results of Figures 5.5 and 5.9 it can be generalized that for any kind of loading and beam span length the approximate equivalent depth line will be at an angle ϕ varying from 20 to 35 degrees for the ratio of $d/b = 1.0$ to $d/b = 0.4667$.

For $b = d = 0.42856h$, which might be the most extreme case, angle β varies from 20 degrees for the pure bending moment case to 30 degrees for the shear-loading case. In reality b will not be greater than $0.42856h$. It can be inferred from Figures 5.5 and 5.9 that for b smaller than $0.42856h$ and $d = 0.42856h$, β will be smaller than 20 degrees for the pure bending case and smaller than 30 degrees for the shear loading case. Concluding, it is suggested that the angle β be taken as 30 degrees for any kind of loading and practical proportions of the cross-shaped joint of the frame structure.

The error involved in the suggested approximate equivalent depth line is examined by comparing the total energy inside the joint, which is evaluated by the true and approximated equivalent depth curves for the cross shape having $b = d = 0.42856h$. (The true energy is evaluated from the results of Chapter IV.) For the pure bending moment case these errors are 10.8% and 5.6% for the case of $r/d = 1/3$ and $r/d = 0$, respectively. For the shear-loading case these are 8.75% and 5.32%, respectively. It is of interest to contrast this small error introduced by the approximation with much larger error introduced by the two assumptions in common practice. As was pointed out in Chapter I, the two assumptions that have been made are either (1) that the effective depth in the column is equal to the depth at the face of the joint or (2) that

the moment of inertia is infinite in the column zone. For $b = d = 0.42856h$ and $r/d = 0$, the true total energy inside the joint is $0.333 K_1^2 d^4/2E$ or $3.75 K_2^2 d^6/2E$ for pure bending moment or shear-loading cases, respectively. The total energy, calculated by assuming that the equivalent depth at any section inside the joint is the same as that at the face, is $0.6667 K_1^2 d^4/2E$ or $8.12 K_2^2 d^6/2E$, for pure bending moment or shear-loading cases, respectively. The total energy calculated by assuming infinite moment of inertia inside the joint is zero in both cases. The error involved in either of the present practices is about 100%. It is to be noted again that this error will be significant for frame structures only when the spans of the beams are small.

TABLE 5.1

STRESSES FOR A COLUMN HAVING $b = d = 0.42856h$
SUBJECTED TO PURE BENDING MOMENT

Stress values are to be multiplied by $K_1 h$.

$\frac{y}{b}$	$\frac{x}{d}$	σ_y	σ_x	τ_{xy}
0	0	0.00000	0.00000	0.00000
	2/8	0.08626	-0.02962	0.00000
	4/8	0.14975	-0.05052	0.00000
	6/8	0.17138	-0.05392	0.00000
	7/8	0.16442	-0.04789	0.00000
	1	0.14770	-0.03784	0.00000
	10/8	0.09925	-0.01372	0.00000
	12/8	0.05374	0.00372	0.00000
	14/8	0.02408	0.00984	0.00000
	16/8	0.00879	0.00791	0.00000
	7/3	0.00000	0.00000	0.00000
2/8	0	0.00000	0.00000	0.02981
	2/8	0.09034	-0.02640	0.02609
	4/8	0.15862	-0.04646	0.01357
	6/8	0.18203	-0.05060	0.00711
	7/8	0.17283	-0.04427	-0.01708
	1	0.15188	-0.03329	-0.02362
	10/8	0.09501	-0.00816	-0.02235
	12/8	0.04770	0.00689	-0.01057
	14/8	0.02021	0.01026	-0.00064
	16/8	0.00708	0.00733	0.00450
	7/3	0.00000	0.00000	0.00642
4/8	0	0.00000	0.00000	0.05081
	2/8	0.09987	-0.01263	0.04629
	4/8	0.18272	-0.02713	0.02793
	6/8	0.21677	-0.03577	-0.01252
	7/8	0.20135	-0.03041	-0.03618
	1	0.16434	-0.01710	-0.05147
	10/8	0.07786	0.00847	-0.04134
	12/8	0.02995	0.01216	-0.01411
	14/8	0.01068	0.00791	0.00201
	16/8	0.00329	0.00390	0.00877
	7/3	0.00000	0.00000	0.01100

TABLE 5.1 (Continued)

$\frac{x}{b}$	$\frac{x}{d}$	σ_y	σ_x	τ_{xy}
6/8	0	0.00000	0.00000	0.04739
	2/8	0.10694	0.02131	0.04621
	4/8	0.20860	0.03291	0.03754
	6/8	0.28096	0.01963	-0.00310
	7/8	0.26874	0.00958	-0.04955
	1	0.18499	0.02094	-0.08767
	10/8	0.03471	0.02438	-0.03852
	12/8	0.00761	0.00055	-0.00414
	14/8	0.00205	-0.00687	0.00636
	16/8	0.00045	-0.00580	0.00950
	7/3	0.00000	0.00000	0.01029
7/8	0	0.00000	0.00000	0.03077
	2/8	0.10748	0.04669	0.03046
	4/8	0.21401	0.08807	0.02949
	6/8	0.31301	0.10026	0.01235
	7/8	0.33381	0.07197	-0.03123
	1	0.19847	0.05768	-0.11015
	10/8	0.00822	0.00352	-0.01587
	12/8	0.00131	-0.02189	0.00187
	14/8	0.00030	-0.02113	0.00560
	16/8	0.00002	-0.01325	0.00645
	7/3	0.00000	0.00000	0.00654
1	0	0.00000	0.00000	0.00000
	2/8	0.10326	0.07025	0.00000
	4/8	0.21787	0.15851	0.00000
	6/8	0.32533	0.25084	0.00000
	7/8	0.38284	0.30553	0.00000
	1	0.21141	0.13424	0.00000
	10/8	-0.00356	-0.07304	0.00000
	12/8	-0.00280	-0.05739	0.00000
	14/8	0.00237	-0.03629	0.00000
	16/8	-0.00058	-0.02219	0.00000
	7/3	0.00000	0.00000	0.00000

TABLE 5.2
 ENERGY AND RATIO R FOR A COLUMN HAVING
 $b = d = 0.42856h$ SUBJECTED TO PURE
 BENDING MOMENT

$\frac{y}{b}$	Energy	Ratio $R = d_e/2d$
0.0	0.025299	1.2756
0.1	0.025689	1.2687
0.2	0.026852	1.2513
0.3333	0.029535	1.2117
0.4	0.031403	1.1867
0.5	0.034345	1.1517
0.6	0.037616	1.1172
0.6667	0.039762	1.0969
0.8	0.043730	1.0627
0.9	0.047361	1.0347
1.0	0.061145	0.9503

TABLE 5.3

CENTER LINE DEFLECTION u , SLOPE $\frac{\partial u}{\partial y}$ CURVATURE $\frac{\partial^2 u}{\partial y^2}$; AND RATIO R BASED ON CURVATURE*

$\frac{y}{b}$	Deflection u	Slope $\frac{\partial u}{\partial y}$	Curvature $\frac{\partial^2 u}{\partial y^2}$	Ratio R Based on Curvature
0.0	0.029031	0.000000	-0.175401	1.7855
0.1	0.028868	-0.007766	-0.193022	1.7303
0.2	0.028342	-0.017082	-0.247946	1.5917
0.3333	0.026898	-0.034842	-0.387464	1.3716
0.4	0.025731	-0.047328	-0.490688	1.2679
0.5	0.023195	-0.072503	-0.694452	1.1292
0.6	0.019373	-0.107267	-0.963117	1.0127
0.6667	0.015872	-0.138319	-1.179628	0.9464
0.8	0.005776	-0.219948	-1.693661	0.8389
0.9	-0.005336	-0.301710	-2.123542	0.7780
1.0	-0.020373	-0.410173		

*Deflection, slope, and curvature are to be multiplied by $K_1 h^2/E$, $K_1 h/E$, and K_1/E , respectively.

TABLE 5.4

VALUES OF RATIO R FOR VARIOUS b, d, AND h. PURE BENDING MOMENT CONDITION

$\frac{y}{b}$	b=0.42856h d=0.42856h	b=0.42856h d=0.3h	b=0.42856h d=0.2h	b=0.28556h d=0.42856h	b=0.3h d=0.2h	b=0.2h d=0.2h	b=0.1h d=0.2h
0.0	1.2756	1.5452	2.0736	1.1272	1.5957	1.2789	1.0786
0.1	1.2687	1.5296	2.0418	1.1252	1.5784	1.2721	1.0777
0.2	1.2513	1.4855	1.9525	1.1241	1.5300	1.2529	1.0752
0.3333	1.2117	1.3959	1.7702	1.1068	1.4291	1.2127	1.0695
0.4	1.1867	1.3438	1.6679	1.0091	1.3735	1.1890	1.0659
0.5	1.1517	1.2640	1.5081	1.0865	1.2858	1.1523	1.0598
0.6	1.1172	1.1894	1.3559	1.0736	1.2038	1.1176	1.0532
0.6667	1.0969	1.1458	1.2645	1.0654	1.1558	1.0970	1.0488
0.8	1.0627	1.0786	1.1205	1.0493	1.0817	1.0621	1.0399
0.9	1.0347	1.0374	1.0506	1.0333	1.0380	1.0337	1.0325
1.0	0.9503	0.8735	0.9121	0.9809	0.9266	0.9547	0.9738

TABLE 5.5

VALUES OF RATIO R FOR VARIOUS b, d, and h KEEPING L CONSTANT. SHEAR LOADING CONDITION

L = 0.57143h for all following cases

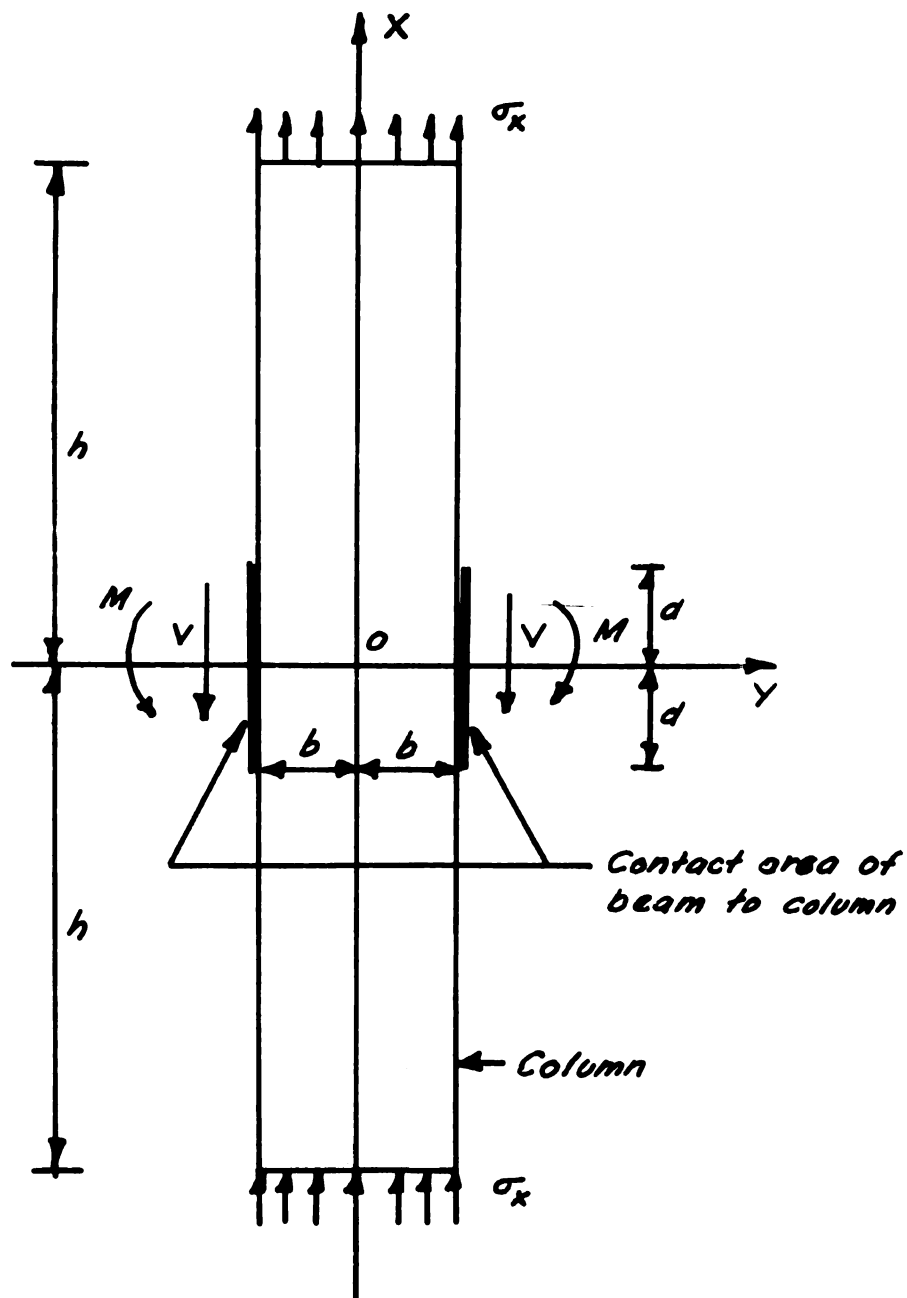
$\frac{y}{b}$	b=0.42856h d=0.42856h	b=0.42856h d=0.3h	b=0.42856h d=0.2h	b=0.2h d=0.2h	b=0.1h d=0.2h
0.0	1.3878	1.6852	2.2620	1.2805	0.9454
0.1	1.3807	1.6702	2.2313	1.2749	0.9446
0.2	1.3600	1.6271	2.1438	1.2587	0.9421
0.3333	1.3143	1.5356	1.9597	1.2236	0.9364
0.4	1.2852	1.4799	1.8492	1.2017	0.9325
0.5	1.2358	1.3907	1.6735	1.1660	0.9254
0.6	1.1808	1.3004	1.4986	1.1287	0.9167
0.6667	1.1411	1.2420	1.3889	1.1034	0.9101
0.8	1.0496	1.1269	1.1991	1.0496	0.8943
0.9	0.9576	1.0213	1.0731	0.9949	0.8787
1.0	0.8048	0.8266	0.8437	0.8831	0.8464

TABLE 5.6

VALUES OF RATIO R FOR DIFFERENT SPAN LENGTHS L, KEEPING b, d, and h CONSTANT, SHEAR
LOADING CONDITION

b = d = 0.42856h for all following cases

$\frac{y}{b}$	L=0.57143h	L=1.14286h	L=2.28572h	L=4.57144h	L=9.14288h
0.0	1.3878	1.3742	1.3376	1.3097	1.2933
0.1	1.3807	1.3670	1.3305	1.3029	1.2867
0.2	1.3600	1.3461	1.3103	1.2833	1.2675
0.3333	1.3143	1.3009	1.2671	1.2419	1.2272
0.4	1.2852	1.2731	1.2411	1.2171	1.2031
0.5	1.2358	1.2281	1.1997	1.1781	1.1655
0.6	1.1808	1.1814	1.1586	1.1402	1.1293
0.6667	1.1411	1.1502	1.1326	1.1170	1.1074
0.8	1.0496	1.0852	1.0832	1.0753	1.0696
0.9	0.9576	1.0212	1.0375	1.0388	1.0374
1.0	0.8048	0.8919	0.9287	0.9419	0.9467



BOUNDARY FORCES ON THE COLUMN

FIGURE 5.1

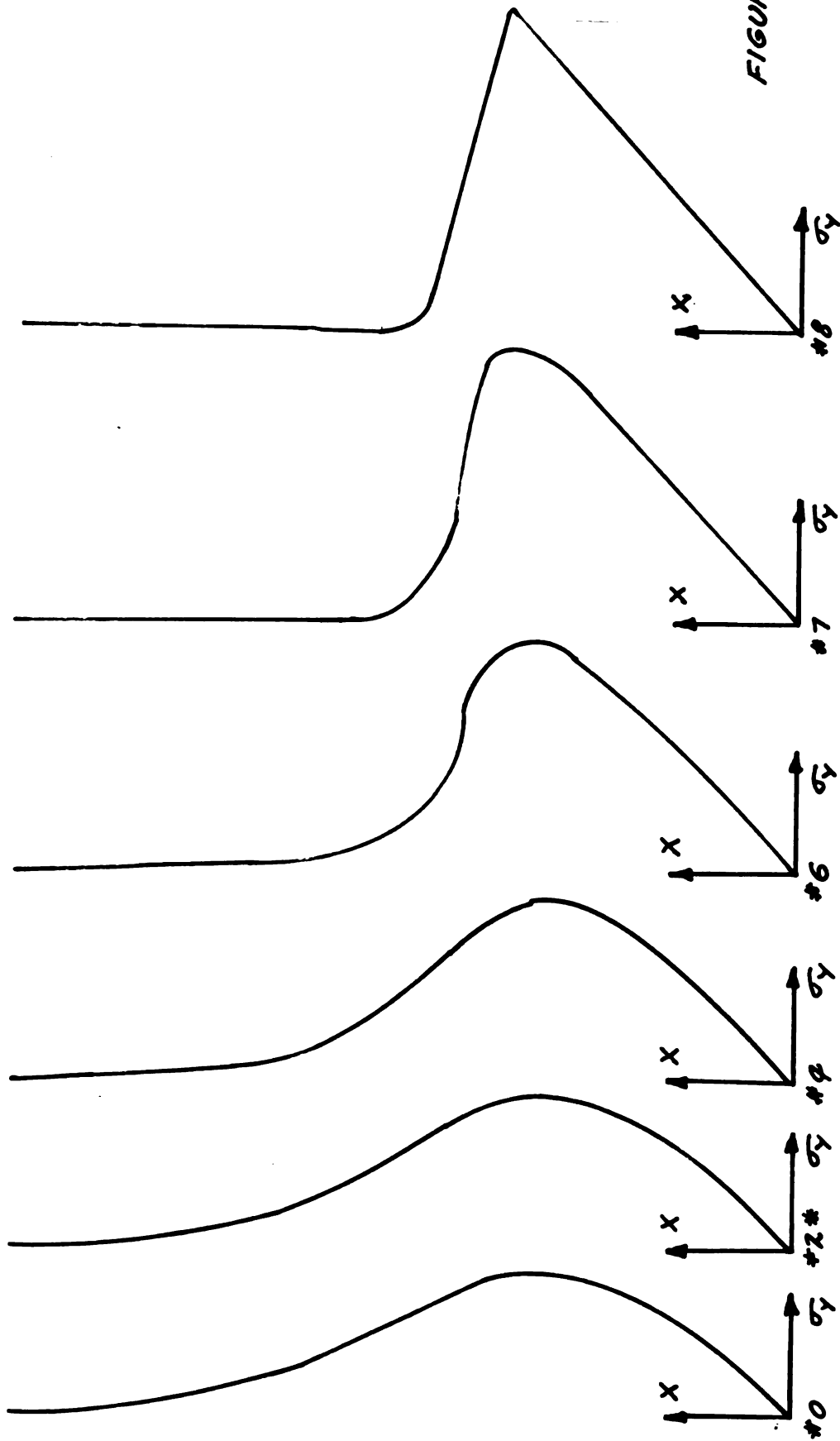


FIGURE 5.2

BENDING STRESS σ_y CURVES FOR VARIOUS BEAM SECTIONS OF A COLUMN HAVING

$b = d = 0.92856 h$, SUBJECTED TO PURE BENDING MOMENT. Scale: X axis, $l'' = 0.5d$

Beam Section #2 is located at $\frac{y}{b} = \frac{2}{8}$ Similar meaning is for all sections σ_y axis, $l'' = 0.20 \frac{K_2 h}{K_1 h}$

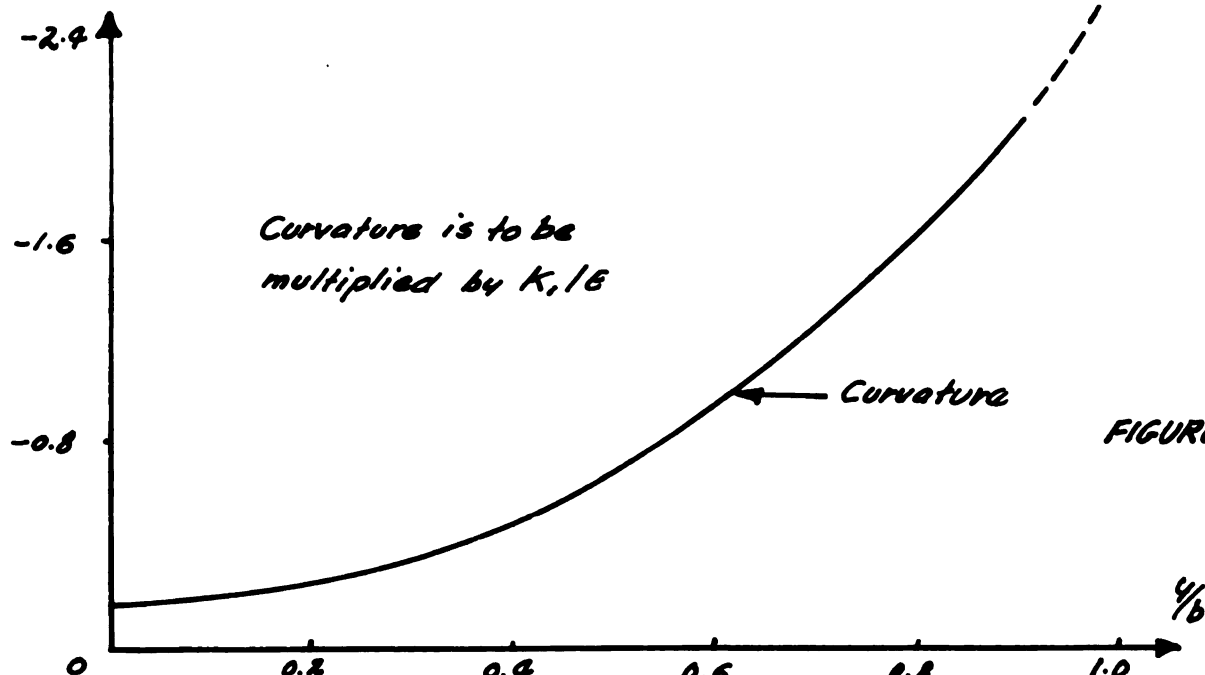
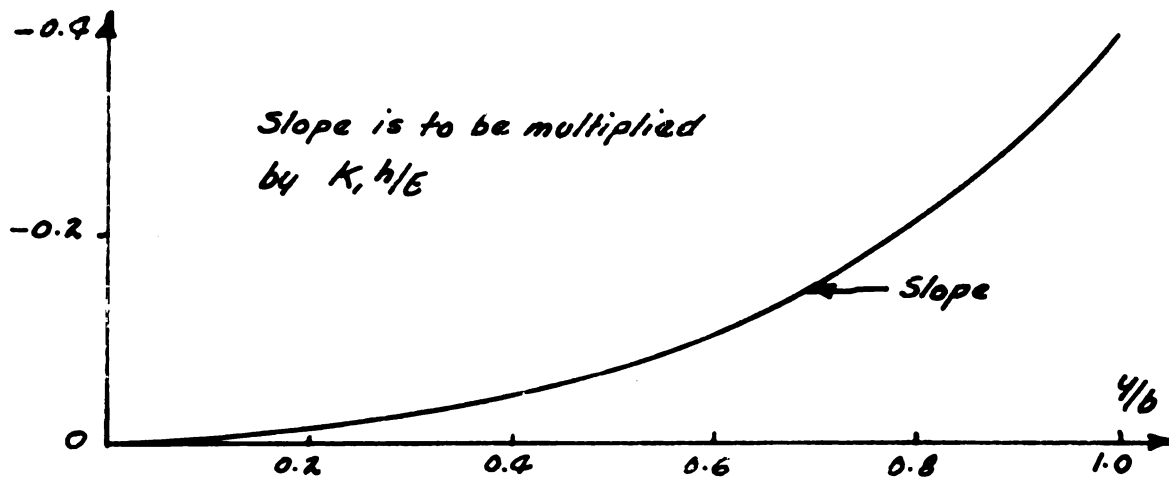
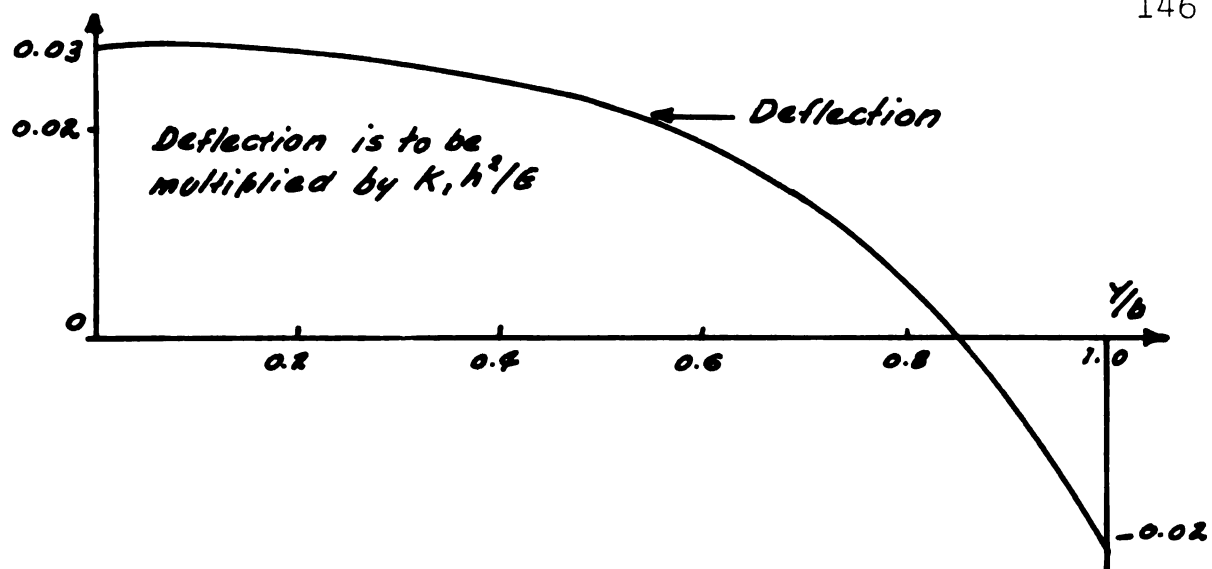
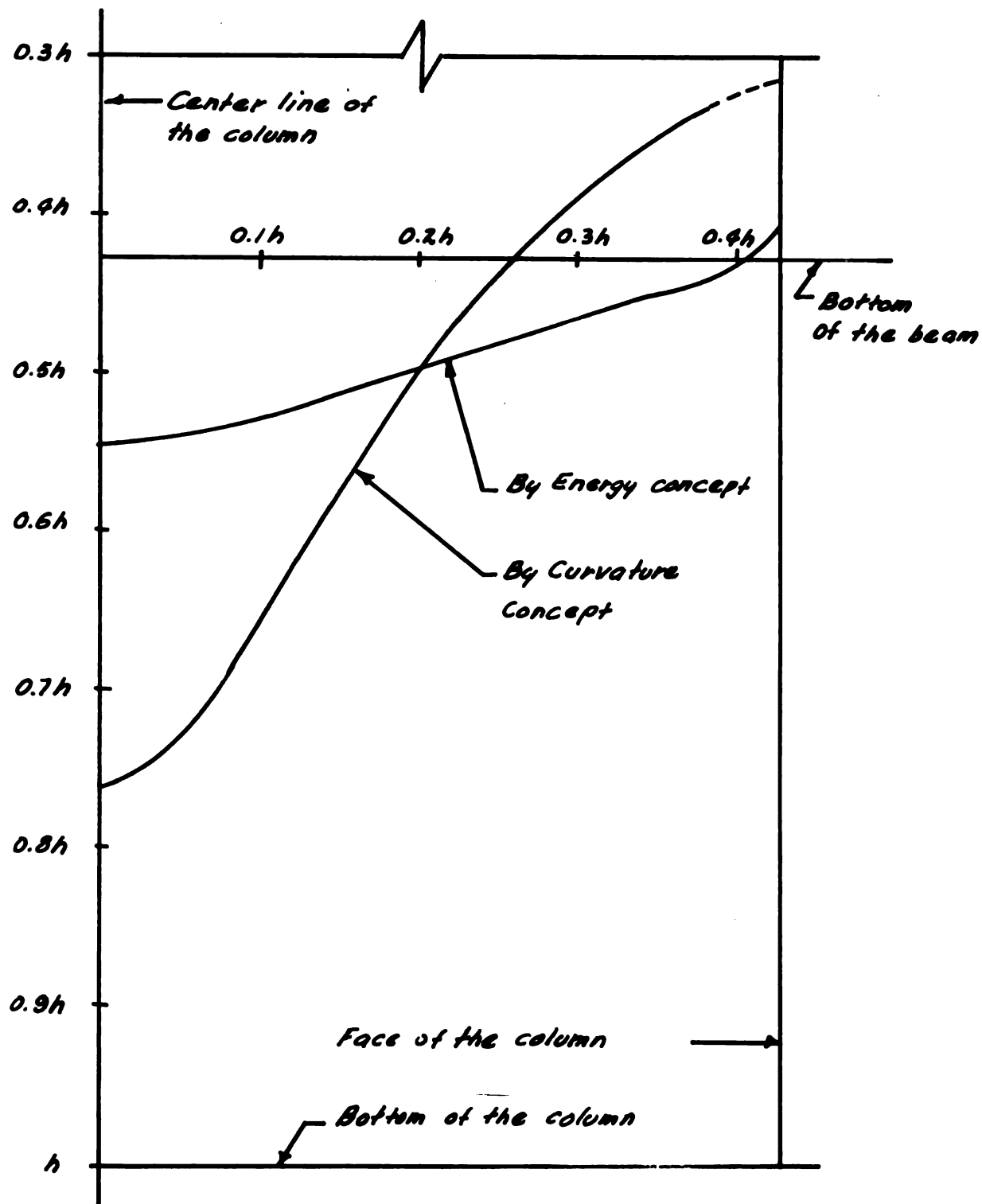
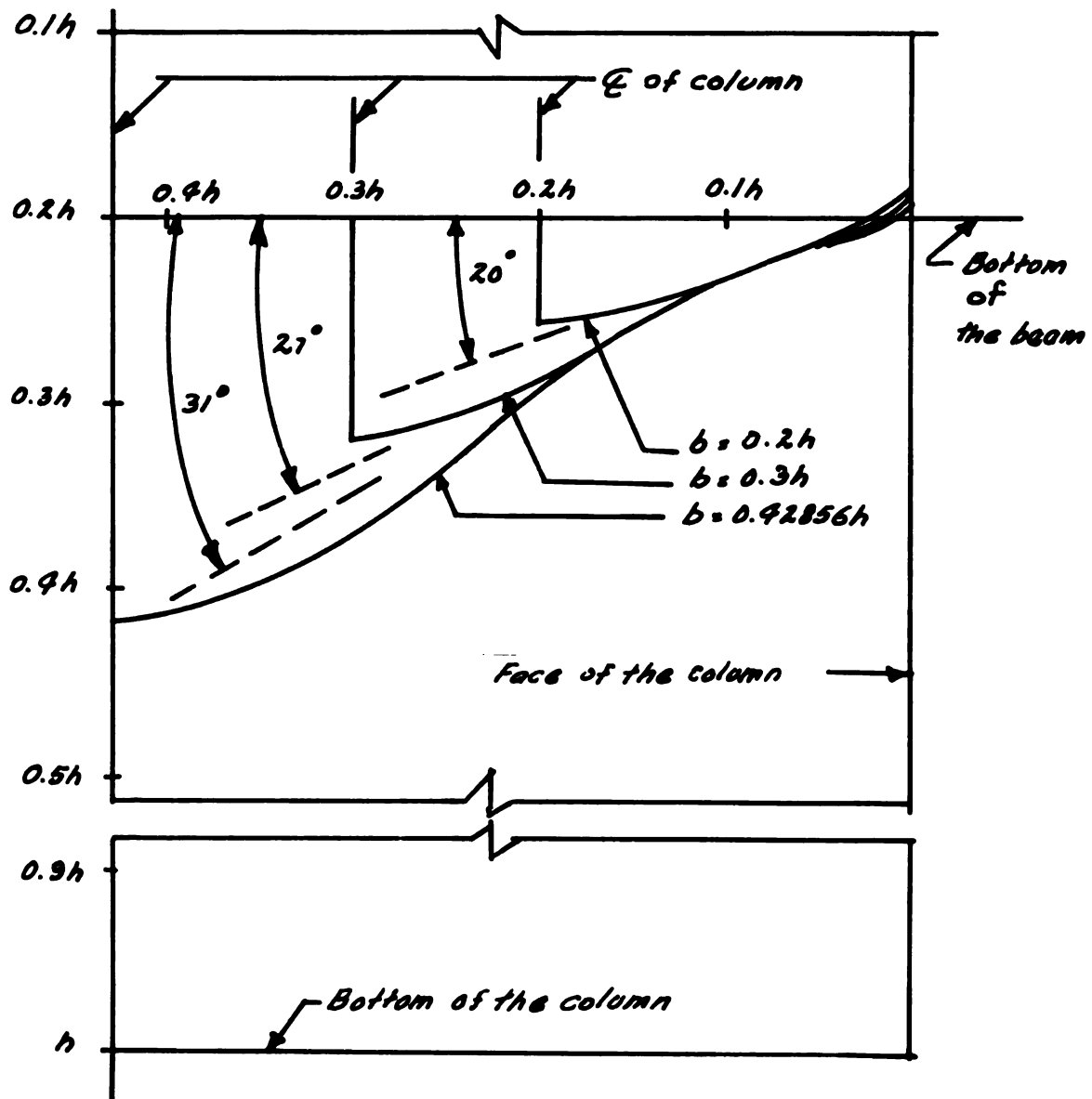


FIGURE 5.3

CENTER LINE (Y-AXIS) DEFLECTION, SLOPE AND CURVATURE FOR A COLUMN HAVING $b = d = 0.42856h$ SUBJECTED TO PURE BENDING MOMENT



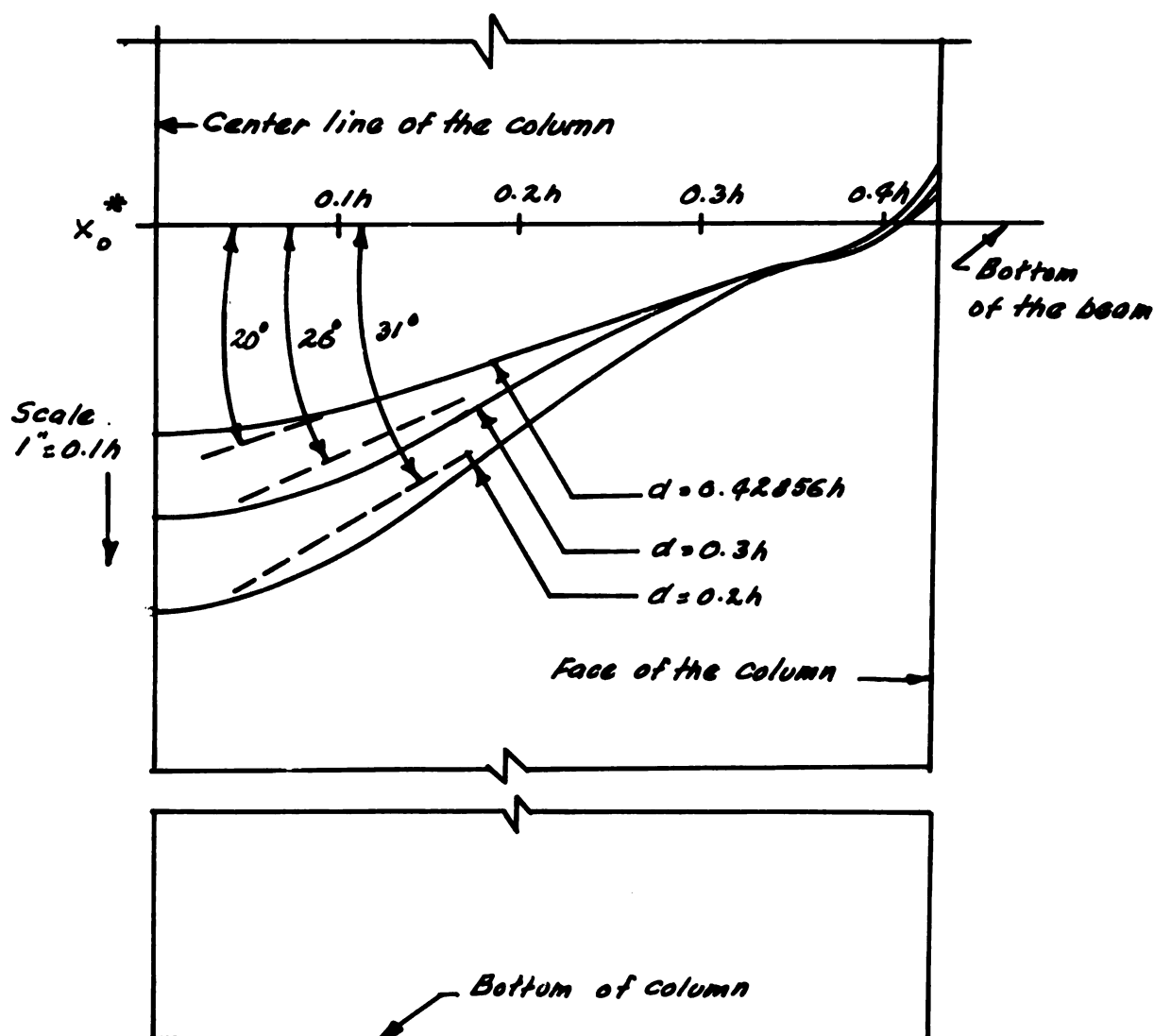
EQUIVALENT DEPTH AS CALCULATED BY ENERGY AND CURVATURE CONCEPTS FOR $b=d=0.42856h$. (PURE BENDING MOMENT)
FIGURE 5.4



NOTATION: — TRUE EQUIVALENT DEPTH CURVE
 ---- APPROXIMATED EQUIVALENT DEPTH LINE

EQUIVALENT DEPTH CURVES FOR DIFFERENT COLUMN WIDTHS WITH $d = 0.2h$ (PURE BENDING MOMENT)

FIGURE 5.5



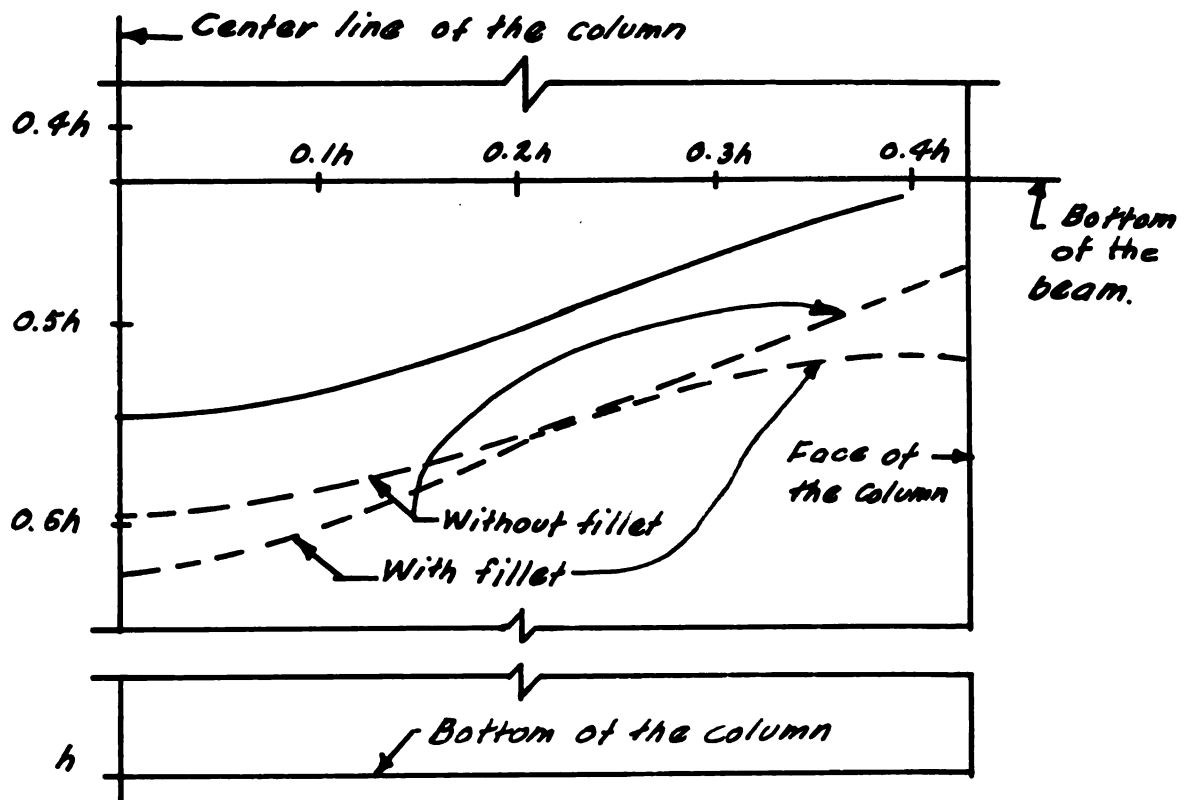
* $x_o = 0.42856h$, $0.3h$ AND $0.2h$ FOR EQUIVALENT DEPTH CURVES HAVING $d = 0.42856h$, $0.3h$ AND $0.2h$ RESPECTIVELY.

NOTATION. — TRUE EQUIVALENT DEPTH CURVE.

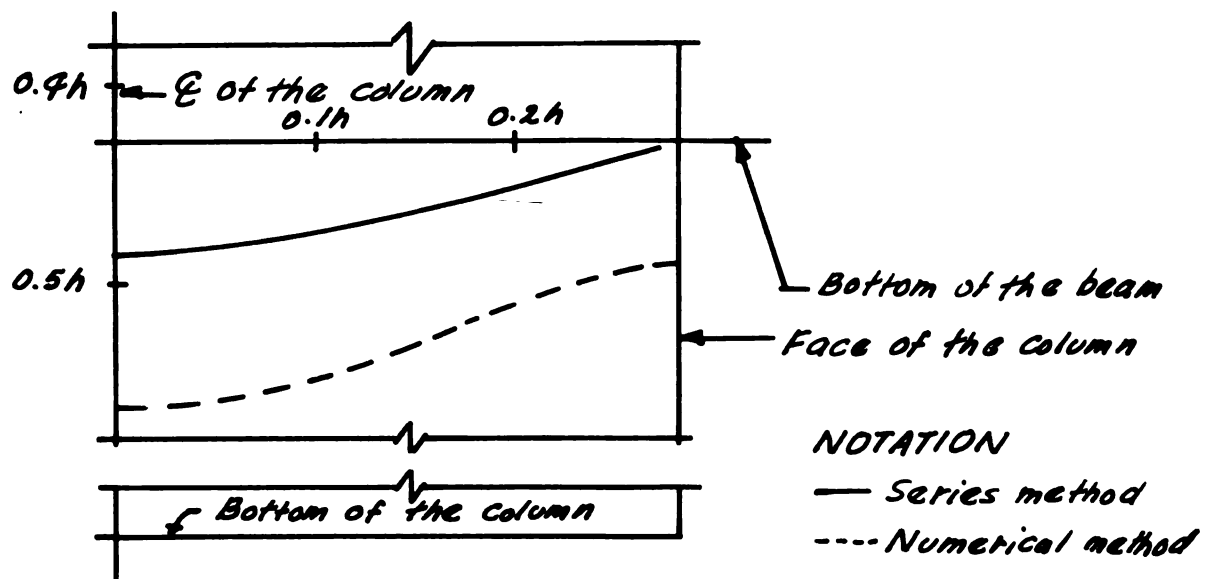
---- APPROXIMATED EQUIVALENT DEPTH LINE

EQUIVALENT DEPTH CURVES FOR DIFFERENT BEAM DEPTHS
WITH $b = 0.42856h$ (PURE BENDING MOMENT)

FIGURE 5.6



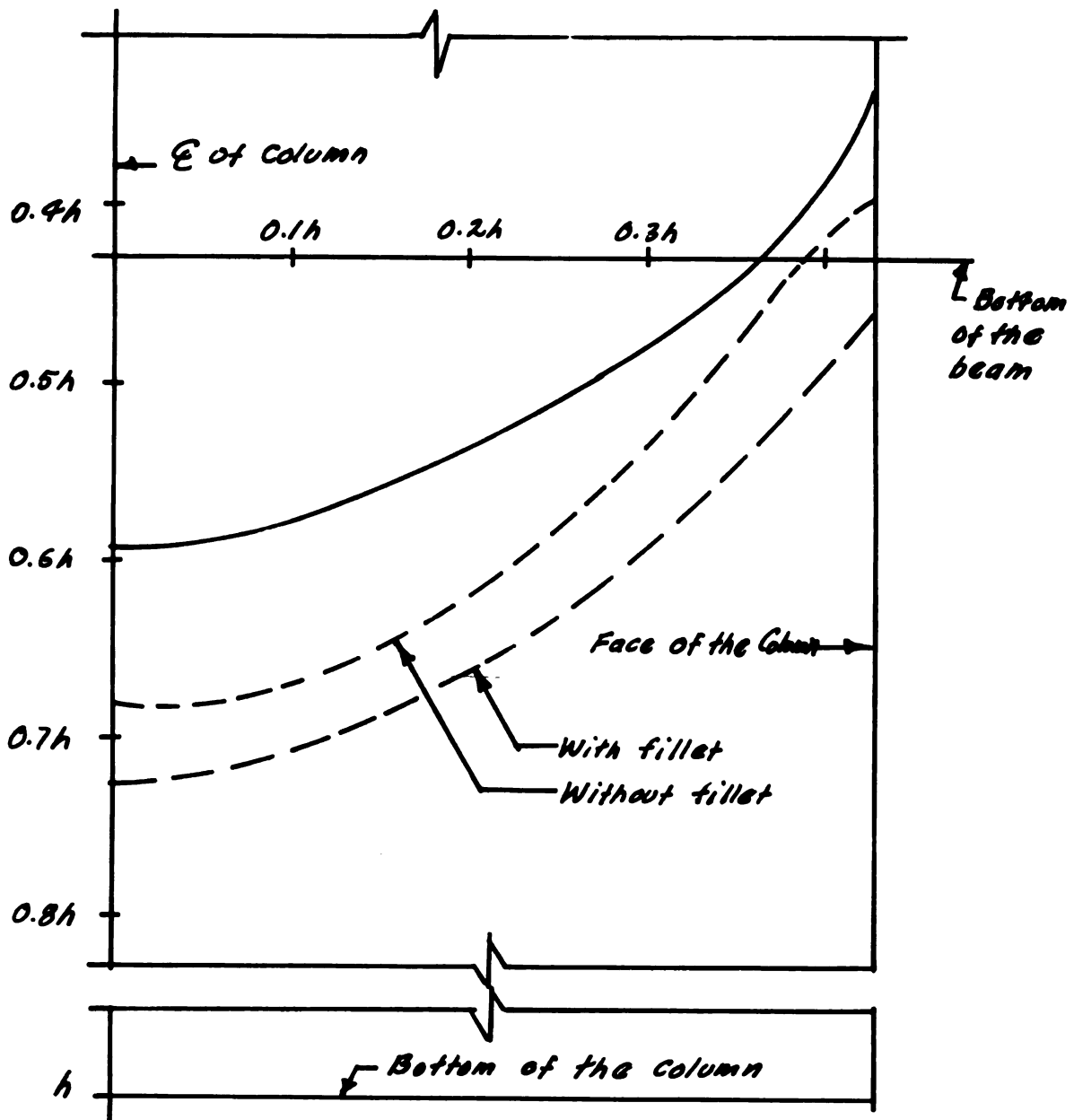
(a) $b = d = 0.42856h$



(b) $b = 0.28556h$ $d = 0.42856h$

EQUIVALENT DEPTH CURVES AS CALCULATED BY SERIES
METHOD AND NUMERICAL METHOD (PURE BENDING MOMENT)

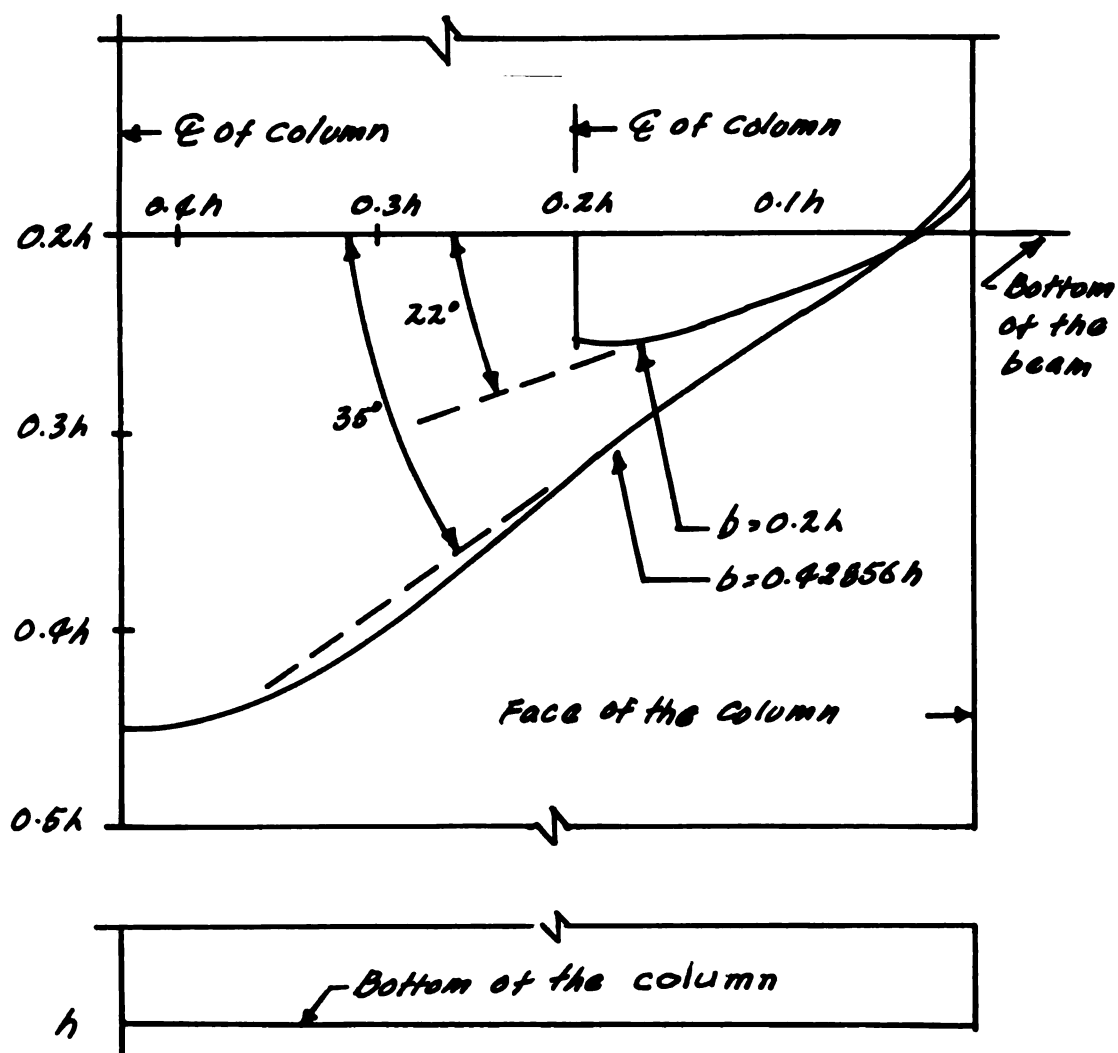
FIGURE 5.7.



NOTATION. — Series method.
 ---- Numerical method.

EQUIVALENT DEPTH CURVES AS CALCULATED BY SERIES METHOD AND NUMERICAL METHOD (SHEAR LOADING.)

FIGURE 5.8

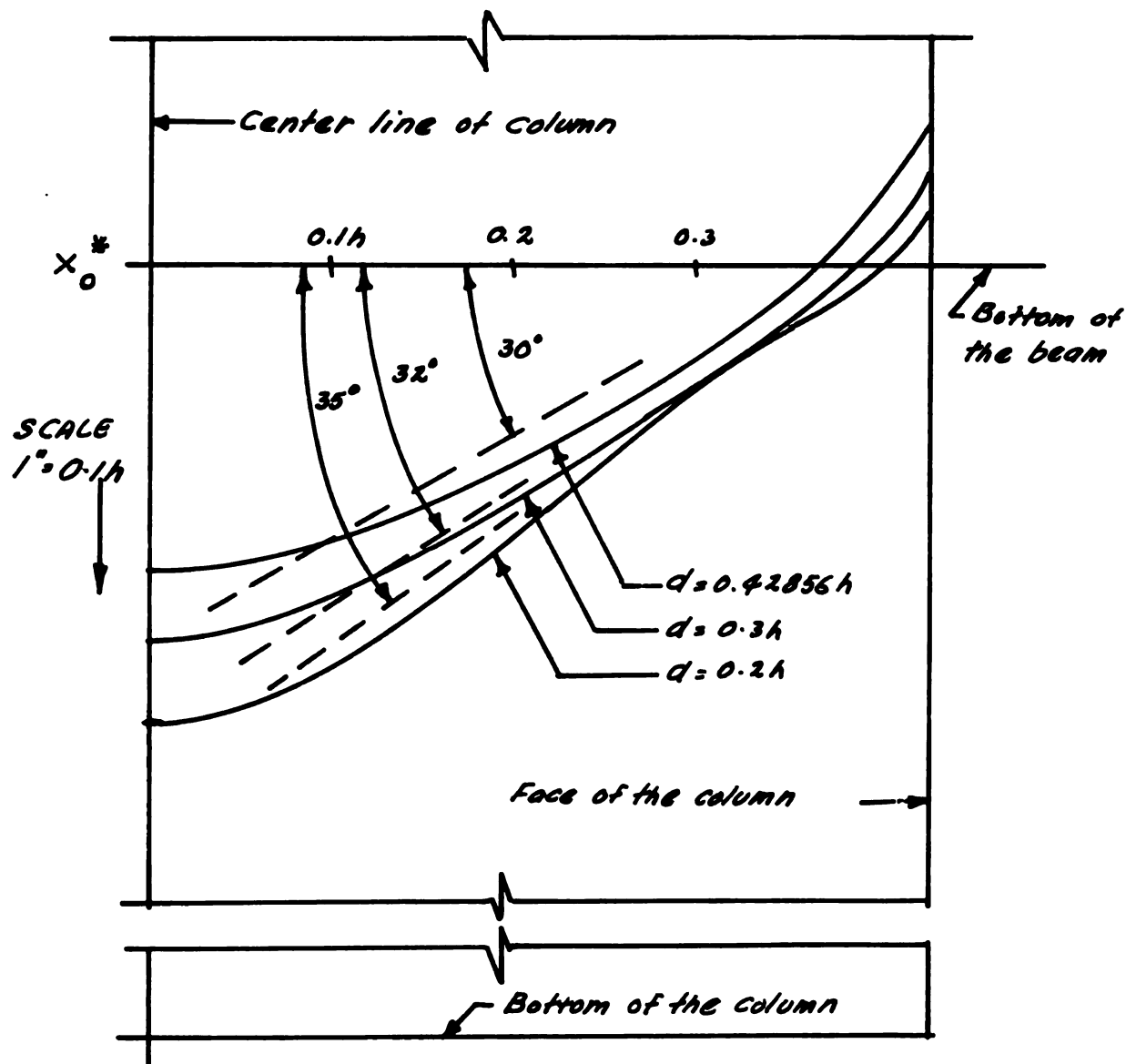


NOTATION — True equivalent depth curve.
 ---- Approximated equivalent depth line.

EQUIVALENT DEPTH CURVES FOR DIFFERENT COLUMN

WIDTHS WITH $d = 0.2h$ AND $l = 0.57143h$. (SHEAR LOADING)

FIGURE 5.9.

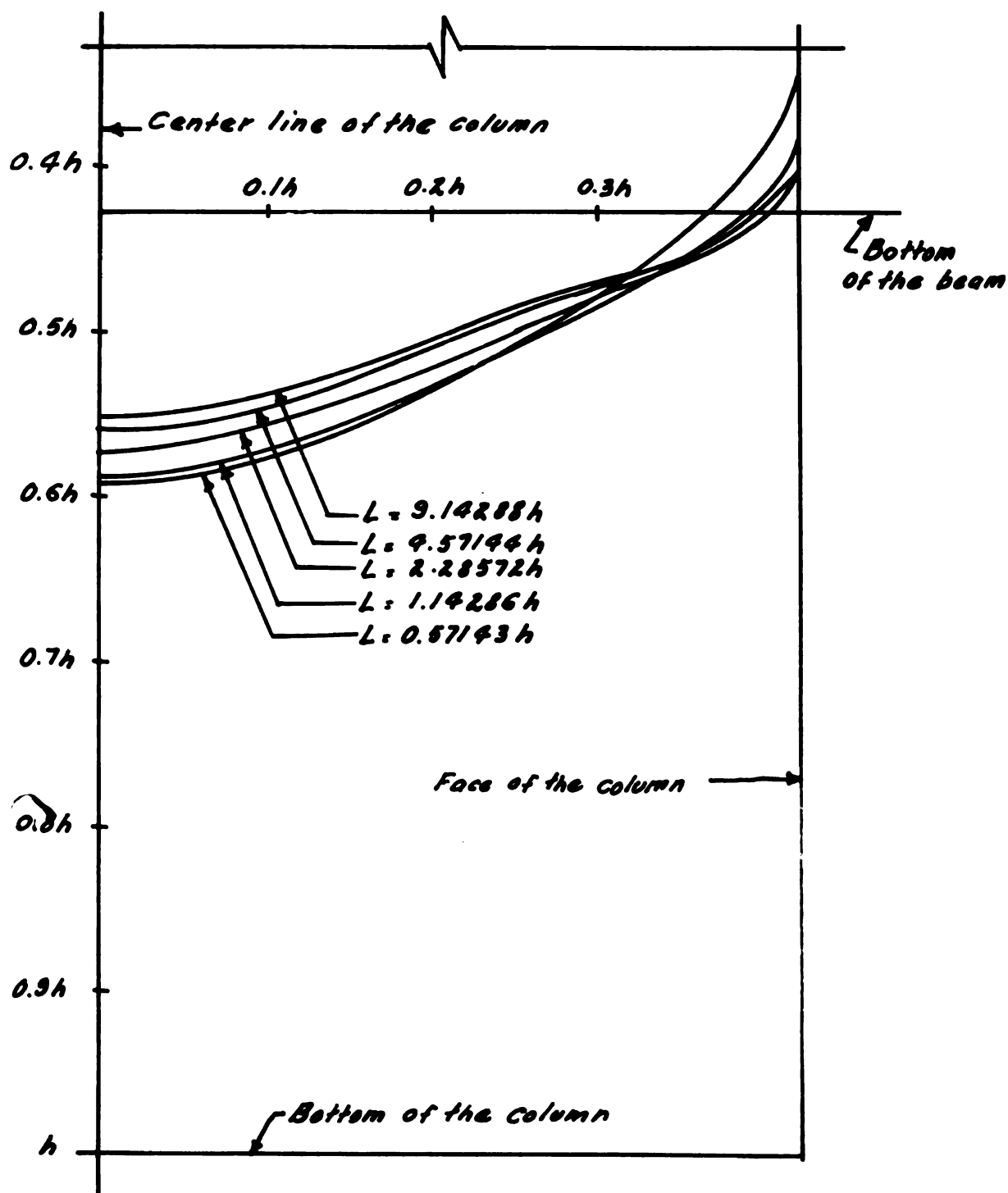


* $X_0 = 0.42856h, 0.3h$ AND $0.2h$ FOR EQUIVALENT DEPTH CURVES HAVING $d = 0.42856h, 0.3h, 0.2h$ RESPECTIVELY

NOTATION. — True equivalent depth curve
---- Approximated equivalent depth line

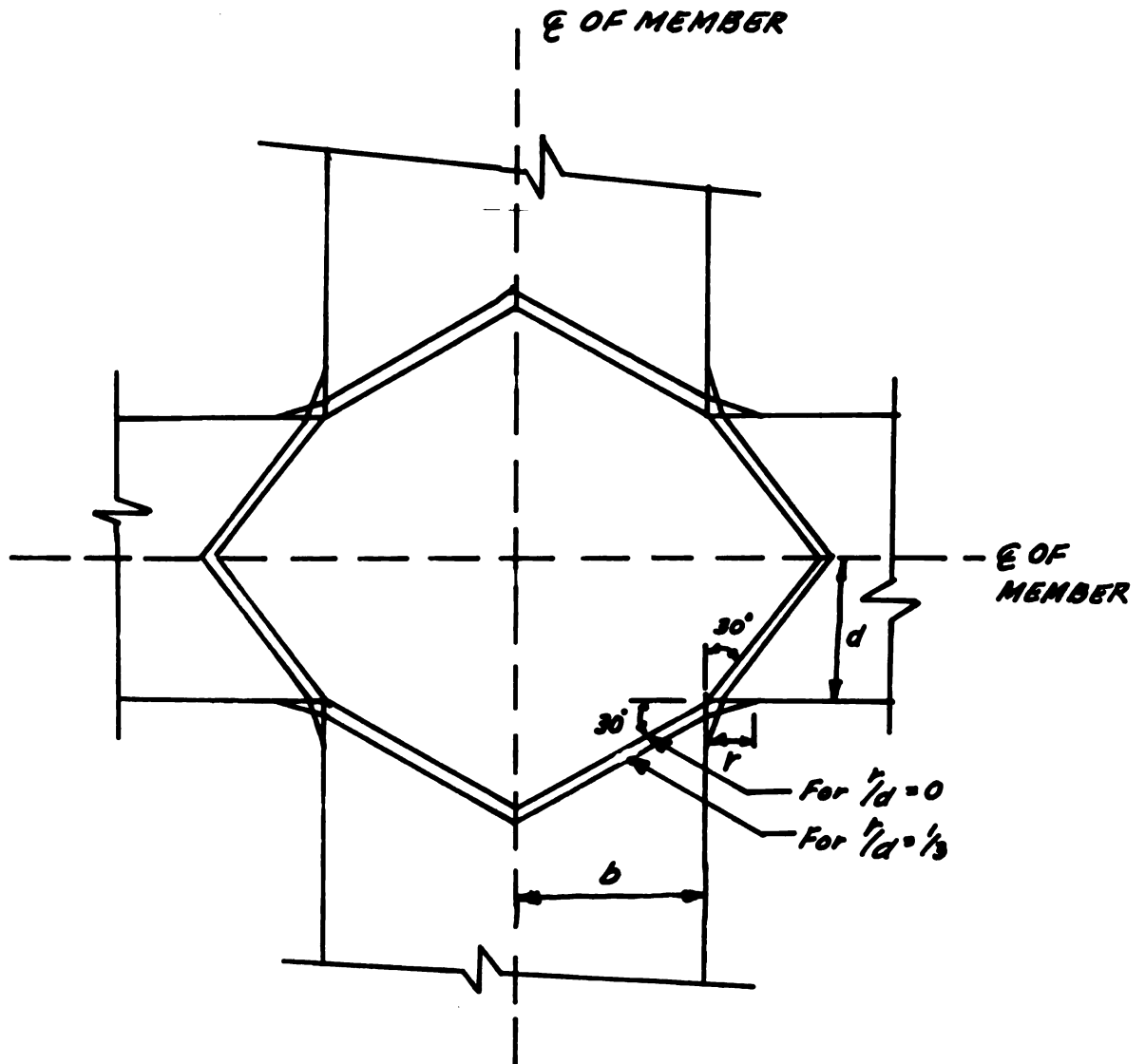
EQUIVALENT DEPTH CURVES FOR DIFFERENT BEAM DEPTHS WITH $b = 0.42856h$ (SHEAR LOADING).

FIGURE 5.10



EQUIVALENT DEPTH CURVES FOR DIFFERENT BEAM SPAN LENGTHS WITH $b = d = 0.42856h$. (SHEAR LOADING)

FIGURE 5.11



***APPROXIMATED EQUIVALENT DEPTH DIAGRAM FOR RECTANGULAR
FLEXURAL MEMBERS INTERSECTING AT A RIGID JOINT***

FIGURE 5.12

CHAPTER VI

SUMMARY AND CONCLUSION

For the analysis of indeterminate frame structures in which the flexural deformation is the primary one, the energy variation in the beam including the joint subjected to flexural action is the essential information required to be known. Basically this reduces the problem to the determination of the stress distribution. Except in and around the joint the stress distribution is that of the conventional beam theory. Inside the joint the stress distribution can be determined by analytical or experimental methods. Photoelasticity investigations of the cross shape subjected to flexural action were carried out to determine the limits beyond which the stress distribution is that of conventional beam theory. It was concluded that the limit extends into the beam and the column portion of the cross shape a distance equal to half the column width and half the beam depth, respectively, from the face of the joint. Using this information the stress distribution in and around the joint subjected to flexural action is determined by the numerical finite difference method. From the stress distribution information the elastic energy per unit beam span length was calculated at the various beam sections.

From this was calculated an equivalent depth, which if used in the conventional beam theory energy formula would give the true elastic energy at the section. The cross shape was analyzed with two loading conditions. In one case the joint was subjected to pure bending moment. In the second case the joint was subjected to variable bending moment (shear loading). The effect of small fillets was also studied. A series method was used to determine the effect of different proportions of the cross shape on the shape of the equivalent depth curve inside the joint. An approximate equivalent depth line was suggested, which is to be used for practical purposes.

The following conclusions can be summarized from the analysis:

1. The knowledge of the energy variations in and around the joint of the frame structure subjected to flexural action is required for the analysis of the frame structures in which the flexural deformation is the primary one.
2. Under pure bending moment condition, the equivalent depth at the face of the column or the joint depends upon the fillet radius, beam depth, column width, and column height. Under variable bending moment condition (shear loading), the equivalent depth at the face of the column depends upon these things and in addition on beam span length.

3. The shape of the curve inside the joint depends upon the loading conditions and the proportions of the cross shape.
4. For exact analysis a series of equivalent depth curves must be prepared with different loading conditions and different proportions of the cross shape.
5. For practical purposes, the equivalent depth is not sensitive to variations in Poisson's ratio.
6. For practical purposes, an approximate equivalent depth line, as suggested in Part V of Chapter V, can be used for any loading condition and any proportions of the cross shape.

The analysis was limited to the cross shape with symmetry about the beam and column center lines. The tee shaped joint in which the symmetry is about one axis and the knee joint in which there is no symmetry can be analyzed for the equivalent depth information by following the procedure of this analysis. It is to be expected that in the tee shape and the knee shape joints subjected to flexural action the equivalent depth will be smaller than the corresponding beam section in the cross shape.

APPENDIX A

CONFORMAL MAPPING OF A CROSS-SHAPED POLYGON

For the region bounded by rectilinear polygon of n sides, such as the cross shape, the mapping function that maps the interior of the polygon onto the unit circle has the form*

$$z = \omega(\xi) = P \int_0^\xi [(\xi - \xi_1)^{(\alpha_1-1)} (\xi - \xi_2)^{(\alpha_2-1)} \dots (\xi - \xi_n)^{(\alpha_n-1)}] d\xi + Q$$

where the ξ_i are the points on the boundary of the unit circle in the ξ -plane, corresponding to the vertices of the polygon in the z -plane, and the numbers $\pi\alpha_i$ are the interior angles at the vertices of the polygon. P and Q are constants. This formula is known as the Schwarz-Christoffel transformation. Applying the formula to the symmetrical cross shape as shown in Figure A.1 the mapping function reduces to

$$z = P \int_0^\xi [(\xi - a_1)^{-1/2} (\xi - a_2)^{1/2} (\xi - a_3)^{-1/2} (\xi + \bar{a}_1)^{-1/2} (\xi + \bar{a}_2)^{1/2} (\xi + \bar{a}_3)^{-1/2} \\ (\xi + a_1)^{-1/2} (\xi + a_2)^{1/2} (\xi + a_3)^{-1/2} (\xi - \bar{a}_1)^{-1/2} (\xi - \bar{a}_2)^{1/2} (\xi - \bar{a}_3)^{-1/2}] d\xi + Q$$

The cross shape region is symmetrical about X and Y axis. Due to symmetry on the unit circle $\bar{a}_1 = 1/a_1, \bar{a}_2 = 1/a_2,$

* Sokolnikoff, op. cit., p. 57.

$\bar{a}_3 = 1/a_3$. Hence, the mapping function reduces to

$$z = P \int_0^{\xi} \left(1 - \frac{\xi^2}{a_1^2}\right)^{-1/2} \left(1 - \frac{\xi^2}{a_2^2}\right)^{1/2} \left(1 - \frac{\xi^2}{a_3^2}\right)^{-1/2} \left(1 - \xi^2 a_1^2\right)^{-1/2} \left(1 - \xi^2 a_2^2\right)^{1/2} \left(1 - \xi^2 a_3^2\right)^{-1/2} d\xi + Q$$

The mapping function as shown is an elliptical integral. This makes it difficult to use in the stress formulas of the Muskhelishvili method.* Each factor of the integrand can be expanded by an infinite series. These series are multiplied together and the result integrated term by term to obtain the mapping function in the form

$$z = \sum_{n=1}^{\infty} C_n \xi^n \quad \text{where the } C_n \text{ are the constants de-}$$

pending upon a_1, a_2, a_3 . To evaluate the series one must take a finite number of the terms in the series. This certain number has to be decided in such a way that the mapping function obtained by this limitation should map the region of the circle onto the desired shape with reasonable accuracy.

Computer Program

The multiplication of six binomial series is very cumbersome by long hand because of the large number of terms required. A program was therefore prepared so that the tedious computations could be done by the MISTIC computer. The program for the mapping function was prepared in the floating decimal form for use with the MISTIC digital

* Ibid., pp. 262-280.

computer located at Michigan State University. The program was prepared in such a way that the odd coefficients for the mapping function were calculated up to the 49th power for any given set values of a_1, a_2, a_3 . This allows one to evaluate the different mapping functions for different dimensions of the cross shape with the same program. A second program was made, which calculated the value of Z corresponding to any point on the unit circle for a particular cross shape. The data necessary for the second program are the 24 values of $\Theta (\zeta = e^{i\theta})$ and the coefficients evaluated by the first program for the series of the mapping function for the particular value of a_1, a_2, a_3 . The program was made in such a way that the Z value will be calculated for the same point for powers of n up to $n = 21$, $n = 33$, or $n = 49$. Comparison of the three results gives some idea of the convergence of the series.

Results

The values for the mapping function for the three different proportions are given in Table A.1. Since $a_2 = e^{0.251\pi}$, the cross shape is symmetrical about the 45 degree line of the quadrant. Hence, the Z values are evaluated for various values of θ from 0 to 45 degrees ($\zeta = e^{i\theta}$), which are given in Tables A.2, A.3, and A.4. The transformed shapes to 1/8th of the circle to corresponding part of the cross shapes are shown in Figure A.2, A.3, and A.4.

Discussion of Results

From the mapped figures it appears that the corners are too far off from the actual conditions. Also the rate of the improvement in the mapped shape by taking more terms of the series is very slow. By taking more terms in the series one could hope to get a sufficiently accurate explicit solution of the mapping function in the polynomial form. But it is to be noted that in the method* for the solution of the biharmonic equation with the help of conformal mapping one has to solve a number of simultaneous equations of the same magnitude as the degree of the polynomial of the mapping function. The capability for the solution of the simultaneous equations in a practicable time will restrict the usefulness of the method even though theoretically there is a solution by this method. The MISTIC computer is limited to 39 equations at the present. Since it appeared that more than 39 equations would be needed, this approach was abandoned. It is remarked also that there is a possibility that the mapping series will not converge at all at some boundary points, although the calculations seem to show a slow convergence.

*Ibid., pp. 278-279.

TABLE A.1
COEFFICIENTS OF THE SERIES OF THE MAPPING FUNCTION

Power n	C_n = Coefficients of ξ^n		
	Case 1	Case 2	Case 3
1	1.0000 00	1.0000 00	1.0000 00
5	0.2618 03	0.2902 11	0.2996 05
9	0.0845 86	0.1341 44	0.1520 11
13	0.0240 38	0.0843 09	0.1094 60
17	-0.0116 96	0.0492 78	0.0807 94
21	-0.0227 44	0.0294 33	0.0660 83
25	-0.0232 51	0.0135 60	0.0540 57
29	-0.0148 76	0.0035 80	0.0465 22
33	-0.0052 21	-0.0041 57	0.0398 50
37	+0.0038 01	-0.0085 30	0.0352 01
41	+0.0085 91	-0.0113 46	0.0309 07
45	+0.0095 84	-0.0120 25	0.0277 08
49	+0.0067 78	-0.0117 15	0.0246 79

Coefficients C_n in cases 1, 2, and 3 are for cross shapes defined by the following positions of a_1 , a_2 , a_3 .

Case 1. $a_1 = e^{0.1i\pi/2}$, $a_2 = e^{0.5i\pi/2}$, $a_3 = e^{0.90i\pi/2}$;
Case 2. $a_1 = e^{0.05i\pi/2}$, $a_2 = e^{0.5i\pi/2}$, $a_3 = e^{0.95i\pi/2}$;
Case 3. $a_1 = e^{0.01i\pi/2}$, $a_2 = e^{0.5i\pi/2}$, $a_3 = e^{0.99i\pi/2}$.

TABLE A.2

Z-VALUES OF THE TRANSFORMED SHAPE FOR A CROSS SHAPE HAVING
 $a_1 = e^{0.1i\pi/2}$, $a_2 = e^{0.5i\pi/2}$, $a_3 = e^{0.9i\pi/2}$

θ in Degrees	n = 21		n = 33		n = 49	
	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z
0	1.3359 88	0.0000 00	1.2926 39	0.0000 00	1.3213 94	0.0000 00
2	1.3326 70	0.0952 78	1.3077 18	0.0600 82	1.3090 18	0.0885 39
4	1.3200 51	0.1944 96	1.3341 03	0.1543 47	1.3065 22	0.1568 15
6	1.2917 67	0.2975 34	1.3316 63	0.2859 67	1.3281 00	0.2597 99
8	1.2418 04	0.3983 74	1.2733 57	0.4232 41	1.2976 42	0.4188 64
10	1.1688 55	0.4865 42	1.1671 98	0.5249 80	1.1721 07	0.5469 98
12	1.0786 63	0.5512 51	1.0482 63	0.5714 11	1.0287 92	0.5765 39
14	0.9830 43	0.5862 25	0.9508 97	0.5744 55	0.9458 69	0.5577 00
16	0.8958 82	0.5928 36	0.8867 77	0.5623 71	0.9007 58	0.5577 45
18	0.8279 76	0.5799 45	0.8451 93	0.5563 22	0.8491 52	0.5675 80
20	0.7831 08	0.5604 76	0.8097 05	0.5596 15	0.8010 23	0.5627 01
22	0.7572 02	0.5462 99	0.7728 36	0.5645 22	0.7707 59	0.5581 94
24	0.7408 71	0.5438 22	0.7372 93	0.5650 43	0.7415 53	0.5640 30
26	0.7240 71	0.5522 83	0.7076 66	0.5623 19	0.7076 41	0.5648 29
28	0.7005 82	0.5653 76	0.6840 22	0.5605 85	0.6829 30	0.5596 26
30	0.6702 44	0.5752 31	0.6629 92	0.5613 89	0.6647 13	0.5613 89
32	0.6380 63	0.5766 63	0.6423 91	0.5628 72	0.6415 93	0.5651 31
34	0.6108 73	0.5696 06	0.6229 76	0.5631 41	0.6204 36	0.5618 03
36	0.5934 31	0.5586 84	0.6062 60	0.5624 63	0.6079 34	0.5599 13
38	0.5858 79	0.5503 49	0.5923 24	0.5620 42	0.5946 24	0.5639 02
40	0.5844 39	0.5492 89	0.5804 29	0.5620 57	0.5784 83	0.5638 81
42	0.5829 54	0.5560 49	0.5708 05	0.5619 44	0.5696 35	0.5599 68
44	0.5772 83	0.5671 14	0.5646 27	0.5621 77	0.5666 10	0.5617 74
45	0.5726 09	0.5726 09	0.5629 95	0.5629 95	0.5643 98	0.5643 98

TABLE A.3

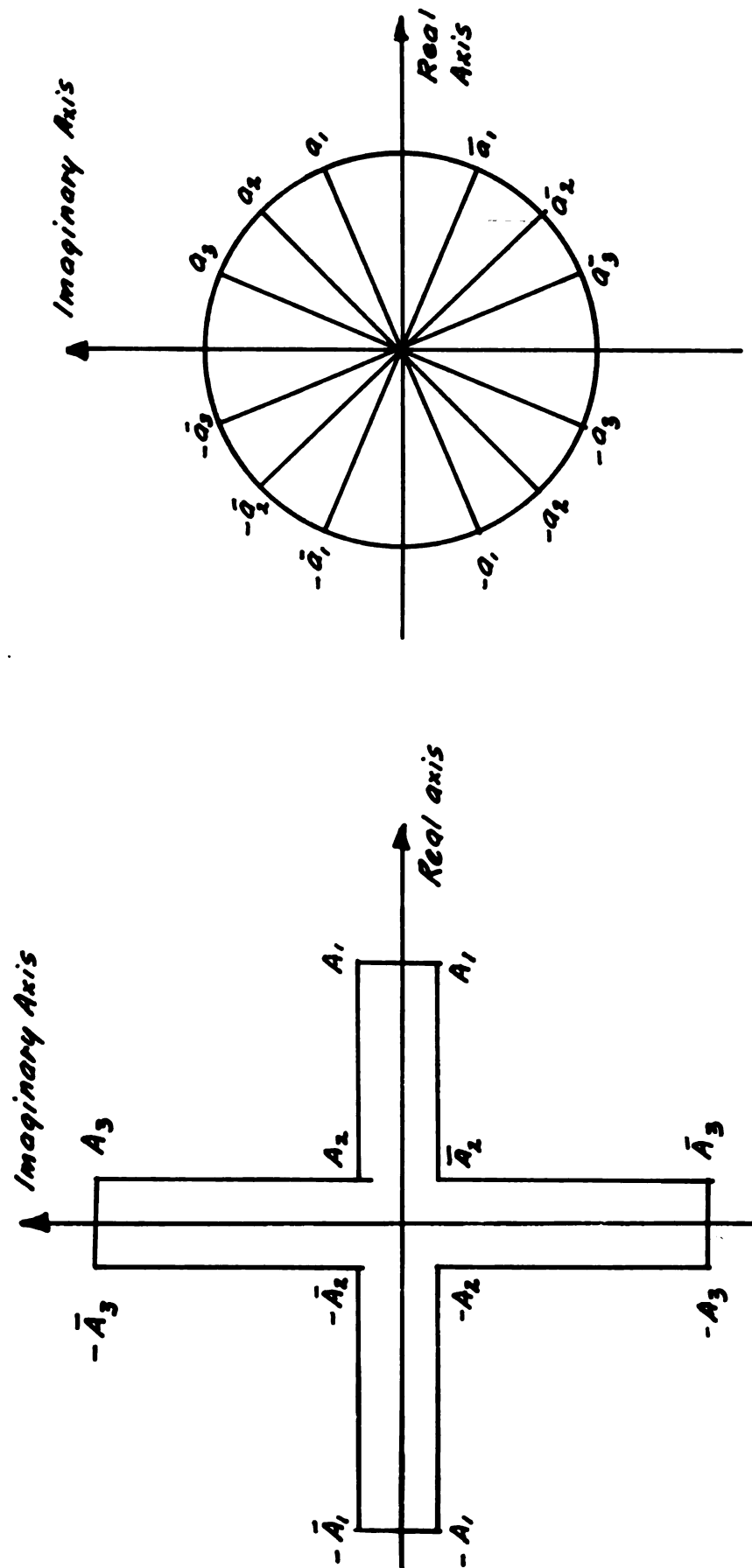
Z-VALUES OF THE TRANSFORMED SHAPE FOR A CROSS SHAPE HAVING
 $a_1 = e^{0.05i\pi/2}$, $a_2 = e^{0.5i\pi/2}$, $a_3 = e^{0.95i\pi/2}$

θ in Degrees	n = 21				n = 33				n = 49			
	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z
0	1.5873	76	0.0000	00	1.6003	59	0.0000	00	1.5567	42	0.0000	00
2	1.5512	75	0.2109	56	1.5601	97	0.2205	82	1.5578	97	0.1775	20
4	1.4522	41	0.3892	61	1.4510	98	0.4027	42	1.4925	26	0.3983	24
6	1.3146	84	0.5126	39	1.3033	33	0.5210	77	1.3095	23	0.5598	78
8	1.1694	02	0.5754	52	1.1548	91	0.5721	27	1.1195	69	0.5795	99
10	1.0431	40	0.5885	32	1.0361	27	0.5745	05	1.0279	63	0.5433	32
12	0.9506	68	0.5730	77	0.9575	86	0.5581	46	0.9841	51	0.5499	36
14	0.8923	07	0.5515	35	0.9090	11	0.5476	90	0.9166	15	0.5694	16
16	0.8573	00	0.5394	88	0.8708	86	0.5508	13	0.8540	03	0.5572	04
18	0.8308	80	0.5417	02	0.8299	18	0.5597	31	0.8252	54	0.5474	80
20	0.8015	37	0.5534	30	0.7863	29	0.5634	45	0.7943	43	0.5608	90
22	0.7652	81	0.5655	32	0.7482	92	0.5591	97	0.7485	19	0.5635	18
24	0.7254	91	0.5704	87	0.7206	97	0.5533	35	0.7194	26	0.5511	95
26	0.6891	58	0.5663	01	0.6998	53	0.5527	80	0.7042	15	0.5538	65
28	0.6619	87	0.5567	37	0.6784	01	0.5573	68	0.6756	52	0.5636	35
30	0.6450	32	0.5483	23	0.6536	75	0.5610	49	0.6459	68	0.5572	79
32	0.6344	72	0.5461	84	0.6298	68	0.5595	48	0.6341	04	0.5509	80
34	0.6242	56	0.5511	36	0.6122	01	0.5551	71	0.6209	82	0.5594	36
36	0.6098	61	0.5595	54	0.6004	82	0.5534	96	0.5964	87	0.5618	26
38	0.5909	23	0.5658	81	0.5896	60	0.5564	00	0.5824	15	0.5528	35
40	0.5712	87	0.5661	79	0.5763	42	0.5599	51	0.5794	52	0.5543	42
42	0.5565	61	0.5606	09	0.5632	01	0.5595	88	0.5667	47	0.5623	26
44	0.5506	73	0.5534	48	0.5559	08	0.5562	35	0.5533	76	0.5574	48
45	0.5511	68	0.5511	68	0.5552	85	0.5552	85	0.5535	13	0.5535	13

TABLE A.4

Z-VALUES OF THE TRANSFORMED SHAPE FOR A CROSS SHAPE HAVING
 $a_1 = e^{0.01i\pi/2}$, $a_2 = e^{0.5i\pi/2}$, $a_3 = e^{0.99i\pi/2}$

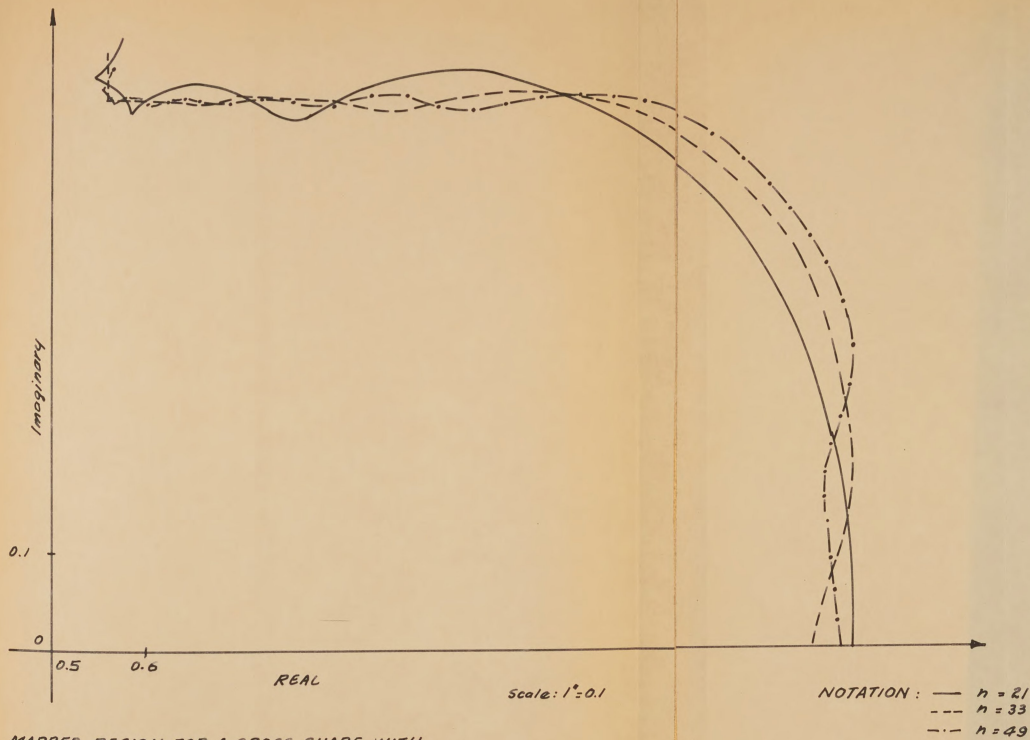
θ in Degrees	n = 21				n = 33				n = 49			
	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z	Real Part of Z	Imaginary Part of Z
0	1.7079	0.0000	1.8483	0.0000	1.8483	0.0000	1.9668	0.0000	1.9668	0.0000	1.9668	0.0000
2	1.6534	0.2712	1.7290	0.2712	1.7290	0.3885	1.7396	0.3885	1.7396	0.5051	1.7396	0.5051
4	1.5066	0.4884	1.4501	0.4884	1.4501	0.6131	1.3392	0.6131	1.3392	0.6335	1.3392	0.6335
6	1.3104	0.6168	1.1794	0.6168	1.1794	0.6364	1.1507	0.6364	1.1507	0.5344	1.1507	0.5344
8	1.1175	0.6523	1.0339	0.6523	1.0339	0.5576	1.1241	0.5576	1.1241	0.5226	1.1241	0.5226
10	0.9702	0.6200	1.0021	0.6200	1.0021	0.5055	1.0408	0.5055	1.0408	0.5819	1.0408	0.5819
12	0.8867	0.5607	0.9915	0.5607	0.9915	0.5276	0.9302	0.5276	0.9302	0.5673	0.9302	0.5673
14	0.8580	0.5137	0.9353	0.5137	0.9353	0.5767	0.8975	0.5767	0.8975	0.5311	0.8975	0.5311
16	0.8568	0.5010	0.8480	0.5010	0.8480	0.5892	0.8784	0.5892	0.8784	0.5558	0.8784	0.5558
18	0.8524	0.5220	0.7848	0.5220	0.7848	0.5582	0.8114	0.5582	0.8114	0.5747	0.8114	0.5747
20	0.8256	0.5588	0.7684	0.5588	0.7684	0.5292	0.7642	0.5292	0.7642	0.5473	0.7642	0.5473
22	0.7749	0.5881	0.7678	0.5881	0.7678	0.5368	0.7591	0.5368	0.7591	0.5423	0.7591	0.5423
24	0.7146	0.5943	0.7423	0.5943	0.7423	0.5665	0.7298	0.5665	0.7298	0.5676	0.7298	0.5676
26	0.6638	0.5769	0.6913	0.5769	0.6913	0.5788	0.6808	0.5788	0.6808	0.5620	0.6808	0.5620
28	0.6360	0.5486	0.6487	0.5486	0.6487	0.5604	0.6672	0.5604	0.6672	0.5417	0.6672	0.5417
30	0.6313	0.5268	0.6379	0.5268	0.6379	0.5372	0.6630	0.5372	0.6630	0.5549	0.6630	0.5549
32	0.6383	0.5238	0.6430	0.5238	0.6430	0.5390	0.6280	0.5390	0.6280	0.5682	0.6280	0.5682
34	0.6406	0.5399	0.6322	0.5399	0.6322	0.5616	0.6013	0.5616	0.6013	0.5506	0.6013	0.5506
36	0.6270	0.5645	0.5983	0.5645	0.5983	0.5750	0.6047	0.5750	0.6047	0.5452	0.6047	0.5452
38	0.5975	0.5823	0.5662	0.5823	0.5662	0.5625	0.5925	0.5625	0.5925	0.5642	0.5925	0.5642
40	0.5625	0.5830	0.5600	0.5830	0.5600	0.5411	0.5624	0.5411	0.5624	0.5616	0.5624	0.5616
42	0.5368	0.5668	0.5725	0.5668	0.5725	0.5386	0.5595	0.5386	0.5595	0.5441	0.5595	0.5441
44	0.5309	0.5444	0.5760	0.5444	0.5760	0.5584	0.5688	0.5584	0.5688	0.5539	0.5688	0.5539
45	0.5357	0.5357	0.5692	0.5357	0.5692	0.5692	0.5640	0.5692	0.5640	0.5640	0.5640	0.5640



Complex S-Plane

Cross-shaped region on complex Z and S planes

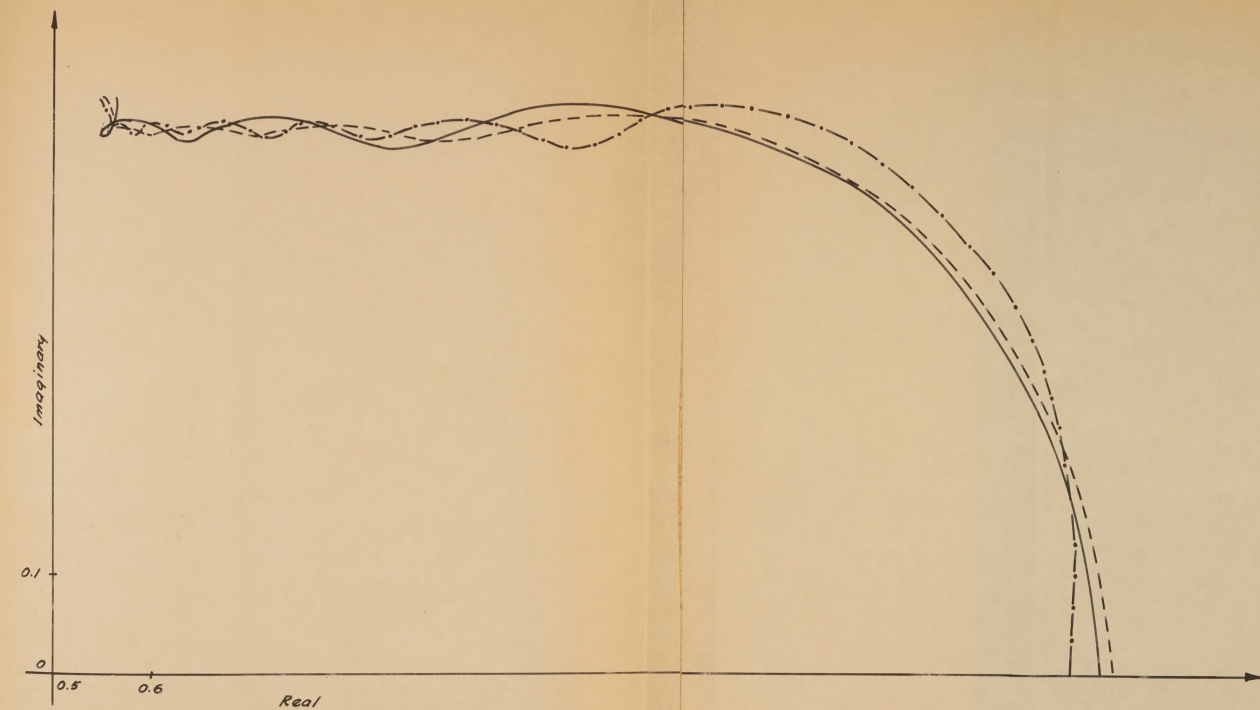
Figure. A-1



MAPPED REGION FOR A CROSS SHAPE WITH

$$Q_1 = e^{i \cdot 10^\circ \pi/2}, Q_2 = e^{i \cdot 5^\circ \pi/2}, Q_3 = e^{i \cdot 90^\circ \pi/2} \quad (\frac{1}{8} \text{th region shown in figure.})$$

FIGURE A-2.

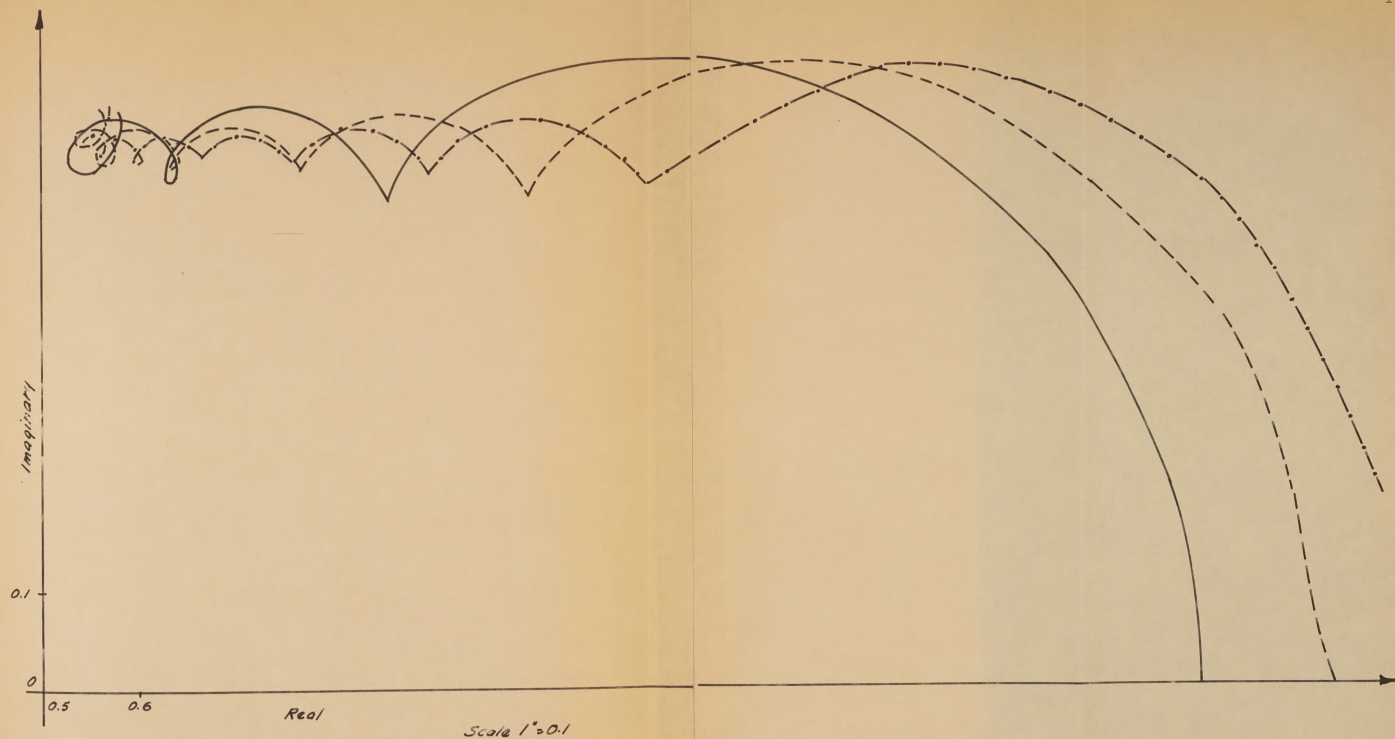


Scale $l'' = 0.1$

MAPPED REGION FOR A CROSS SHAPE WITH $\alpha_1 = e^{i0.5; \pi/2}$, $\alpha_2 = e^{i5; \pi/2}$
 $\alpha_3 = e^{i95; \pi/2}$ ($1/8$ th region shown in figure)

FIGURE A.3

NOTATION: — $h = 21$
 --- $h = 33$
 - · - $h = 49$



MAPPED REGION FOR A CROSS SHAPE WITH

$$a_1 = e^{.01; \pi/2}, a_2 = e^{.5; \pi/2}, a_3 = e^{.99; \pi/2} \text{ (1/8th region shown in figure)}$$

FIGURE A.4

NOTATION: — $h = 21$

--- $h = 33$

-.- $h = 45$

APPENDIX B

FINITE DIFFERENCE EQUATIONS

I. Finite Difference Equation for the Biharmonic Differential Equation for an Irregular Node

In finer grading of a region in which the biharmonic differential equation is to be solved by the finite difference method, a problem arises in applying the standard finite difference formula, Equation 4.2, for a node in the intermediate region such as 23,23 in Figure 4.3b. This difficulty is resolved if the procedure which was discussed and referred to in Chapter IV is followed as suggested by Allen and Dennis. This procedure is illustrated below in setting up the finite difference equation for the biharmonic differential equation for the node 23,23 of Figure 4.3b.

The biharmonic differential equation for the node 23,23 is as follows:

$$(\nabla^4 \phi)_{23,23} = 0.$$

The finite difference equation can be derived as follows:

$$\begin{aligned} & (\nabla^2) (\nabla^2 \phi)_{23,23} = 0 \\ & (\nabla^2) (\phi_{3,3} + \phi_{3,2} + \phi_{2,2} + \phi_{2,3} - 4\phi_{23,23}) / (s/\sqrt{2})^2 = 0 \\ & (\phi_{34,3} + \phi_{3,23} + \phi_{23,3} + \phi_{3,34} - 4\phi_{3,3}) / (s/2)^2 \\ & + (\phi_{4,2} + \phi_{3,1} + \phi_{2,2} + \phi_{3,3} - 4\phi_{3,2}) / (s)^2 \\ & + (\phi_{3,2} + \phi_{2,1} + \phi_{1,2} + \phi_{2,3} - 4\phi_{2,2}) / (s)^2 \end{aligned}$$

$$\begin{aligned}
& + (\phi_{3,3} + \phi_{2,2} + \phi_{1,3} + \phi_{2,4} - 4\phi_{2,3})/(s)^2 \\
& -4 (\phi_{3,3} + \phi_{3,2} + \phi_{2,2} + \phi_{2,3} - 4\phi_{23,23})/(s\sqrt{2})^2 \\
& = 0
\end{aligned}$$

where s = ungraded mesh size.

This will reduce to

$$\begin{aligned}
& 32\phi_{23,23} - 22\phi_{3,3} - 11\phi_{3,2} - 10\phi_{2,2} - 11\phi_{2,3} + 4\phi_{34,3} \\
& + 4\phi_{3,23} + 4\phi_{23,3} + 4\phi_{3,34} + \phi_{4,2} + \phi_{3,1} + \phi_{2,1} + \phi_{1,2} + \phi_{2,4} \\
& + \phi_{1,3} = 0
\end{aligned}$$

It is to be noticed that the mesh size is not the same throughout the derivation. The mesh size taken at any stage of derivation is one suitable for setting up the finite difference equation for the Laplace operator at the node without introducing new intermediate points. Also net directions are chosen as suitable for the node.

II. Approach for Evaluation of $\phi_{34,34}$ (Refer Figure 4.3)

The ϕ value for the node $34,34$ can be expressed in terms of $\phi_{3,34}$ or in terms of $\phi_{34,3}$ by evaluating the finite difference equations for the boundary conditions at m and n , respectively. In reality the singularity of $\phi_{34,34}$ is to be observed. In the present analysis $\phi_{34,34}$ has been taken as the average of the two values of $\phi_{34,34}$ evaluated by satisfying the boundary conditions at m and n ,

respectively. For that, first ϕ is expanded in terms of Taylor series in the X or Y direction, respectively, at points m or n. In the Taylor series, the boundary conditions are substituted. Then the value of $\phi_{34,34}$ can be evaluated. The discussed procedure is illustrated as follows:

Applying the Taylor series formula,

$$\begin{aligned} \phi_{34,34} = & \phi_m + (X_{34,34} - X_m)(\phi'_m) + (1/2)(X_{34,34} - X_m)^2(\phi''_m) \\ & + \dots \end{aligned}$$

Also,

$$\begin{aligned} \phi_{3,34} = & \phi_m + (X_{3,34} - X_m)(\phi'_m) + (1/2)(X_{3,34} - X_m)^2(\phi''_m) \\ & + \dots \end{aligned}$$

where ϕ'_m and ϕ''_m are first and second partial derivatives of ϕ with respect to x, evaluated at m. By substituting the values of ϕ'_m , ϕ''_m and the difference of X coordinates, the required value of $\phi_{34,34}$ can be evaluated in terms of $\phi_{3,34}$ from the above relations. In a similar manner by expanding ϕ in a series in the Y direction $\phi_{34,34}$ can be evaluated in terms of $\phi_{34,3}$. Then the average is taken of the two $\phi_{34,34}$ values.

APPENDIX C

MISTIC COMPUTER PROGRAM FOR EVALUATION OF TOTAL ELASTIC ENERGY ALONG THE BEAM SECTION OF THE COLUMN SUBJECTED TO PURE BENDING MOMENT ACCORDING TO THE SERIES SOLUTION

Problem Outline

For a column subjected to pure bending moment condition, the total energy across the beam section per unit length in the Y direction is given by the Equation 5.17 in which the constants P_0 , C_{1m} and C_{4m} are evaluated according to boundary conditions mentioned in Part II of Chapter V. In the formula for such a case the variable parameters are dimensions d and b; and the Poisson's ratio μ , m the number of terms for a series and the value of y. The variables d, b, and y are expressed as follows:

$$d = K_d h,$$

$$b = K_b h,$$

$$y = F_y K_b h,$$

where F_y will vary from 0 to 1.

Data and Answer Form

In the following program the data to be supplied is to be in the sequence of π , K_d , K_b , μ and eleven values of F_y . The answers will be printed out in the sequence of

$$\int_{-h}^h \sigma_y^2 \cdot dx, \quad \int_{-h}^h \sigma_x^2 \cdot dx, \quad \int_{-h}^h \sigma_x \cdot \sigma_y \cdot dx, \quad \int_{-h}^h \tau_{xy}^2 \cdot dx,$$

and total energy (in terms of $2E$ where E is modulus of elasticity) for each value of F_y . Eleven sets of such values will be printed out in the order in which F_y values are fed in the machine. This program manipulates the numbers in the floating decimal form,* i.e. the numbers represented in the form of $A(10)^p$, where $1 \geq A \geq 1/10$ and $64 > p \geq -64$. Hence, the numbers to be fed in the data should be in the floating decimal form. The answers will be printed out in the floating decimal form.

Master Program

The master program, i.e. complete program, is composed partly of various subroutines. The order pairs of these subroutines are not written in the following program, as all these subroutines are available in the MISTIC library.** Hence, only the designation and the title as used in the library is given wherever these subroutines are used. The program is prepared for eleven values of F_y and the maximum value of m as 50. But a little modification in the following program will allow one to change the number of values of F_y and the maximum value of m . The value of m as 50 is used for the reasons set forth in the discussion of the convergence of series in Part II of Chapter V.

*"Illiac Programming," Digital Computer Laboratory (Urbana, Illinois: University of Illinois, 1955), pp. 4-9.

**MISTIC Library is located at Computer Laboratory, Michigan State University, East Lansing, Michigan.

Orders or Subroutine Designation	Notes
Subroutine XI 218	Decimal order Input (25)
00 3K	Directive
00F 009F	Locations 9 and 10 are floating Accumulators
00F 0011F	
00F 00180F	
00F 00210F	
00F 00240F	
00F 00260F	
00 11K	Directive
Subroutine A1 63	Floating decimal arithmetic routine (168)
00 180K	Directive
Subroutine A3 125	Convert a number from floating decimal representation to <u>normal</u> machine form (27)
00 210K	Directive
Subroutine SA2 127	Exponential Auxiliary for floating decimal (16)
00 240K	Directive
Subroutine SA2-M	Hyperbolic sine and cosine floating point Auxiliary (18)
00 260K	Directive
Subroutine TA1 126	Sine Auxiliary for floating decimal (26)

00 305K Directive

Location	Orders	Notes
0	22L 50L	Transfer control to R.H. of L. Standard subroutine entry.
1	26 S4 88F	Bring the number from the data tape into Acc.
2	8S 290F OK 2F	Store contents of Acc. into 290F. Set the loop for Box 0 by setting $g_0 = 0$ and $c_0 = -2$.
3	88 F 0S 291F	Bring the number from the data tape into Acc. Store the contents of Acc. into designated location.
4	03 3L OK 2F	If $(c_0 + 1) \leq 0$, transfer control to L.H. of 3L. Set the loop for Box 0 by setting $g_0 = 0$ and $c_0 = -2$.
5	05 291F 87 290F	Bring contents of designated location into Acc. Multiply contents of Acc. by π .
6	0S 291F 03 5L	Store contents of Acc. at designated location. If $(c_0 + 1) \leq 0$, transfer control to L.H. of 5L.
7	8K 2F 8S 293F	Put 2 into Acc. Store number 2 at 293F.
8	8K F 8S 294F	Put number 0 into Acc. Store number 0 at 294F.
9	8S 295F 8S 296F	Store number 0 at 295F. Store number 0 at 296F.
10	OK 50F 85 294F	Set the loop for Box 0 by setting $g_0 = 0$ and $c_0 = -50$. Bring contents of 294F into Acc.
11	84 291F 8S 294F	Add contents of 291F to contents of Acc. Store contents of Acc. at 294F.
12	8J S8 8S 297F	Enter Sine subroutine. Store contents of Acc (Sine value) at 297F.

Location	Orders	Notes
13	85 290F	Bring π into Acc.
	86 293F	$\pi/2$ into Acc.
14	84 294F	Add contents of 294F to the Acc.
	8J S8	Enter Sine subroutine.
15	87 294F	Multiply contents of 294F by that of Acc.
	8S 298F	Store contents of Acc. at 298F.
16	85 297F	Bring contents of 297F into Acc.
	80 298F	Subtract contents of 298F from that of Acc.
17	8S 297F	Store contents of Acc. at 297F.
	85 296F	Bring contents of 296F into Acc.
18	84 290F	Add contents of 290F to the Acc.
	8S 296F	Store contents of Acc. at 296F.
19	85 297F	Bring contents of 297F into Acc.
	87 293F	Multiply contents of 293F by that of Acc.
20	86 296F	Divide contents of Acc. by that of 296F.
	86 296F	Divide contents of Acc. by that of 296F.
21	8S 297F	Store evaluated value of B_m at 297F.
	85 295F	Bring contents of 295F into Acc.
22	84 292F	Add contents of 292F to the Acc.
	8S 295F	Store contents of Acc. at 295F.
23	8S 295F	Store order. Waste order.
	8J 24L	Transfer control to L.H. of 24L.
24	22 24L	Transfer control to R.H. of 24L.
	50 24L	Standard subroutine entry.
25	26 S7	Enter Hyperbolic Sine and Cosine subroutine.
	50 25L	Standard subroutine entry.
26	26 S4	Re-enter A1 subroutine.
	85 14S7	Bring Hyperbolic sine function into Acc.

Location	Orders	Notes
27	8S 298F 85 15S7	Store contents of Acc. at 298F. Bring Hyperbolic cosine function into Acc.
28	87 295F 84 298F	Multiply contents of Acc. by that of 295F. Add contents of 298F to the Acc.
29	8S 299F 85 295F	Store contents of Acc. at 299F. Bring contents of 295F into Acc.
30	87 293F 8S 300F	Multiply contents of Acc. by that of 293F. Store contents of Acc. at 300F.
31	8S 300F 8J 32L	Store order. Waste order. Transfer control to L.H. of 32L.
32	22 32L 50 32L	Transfer control to R.H. of 32L. Standard subroutine entry.
33	26 S7 50 33L	Enter Hyperbolic sine and cosine subroutine. Standard subroutine entry.
34	26 S4 85 14S7	Reenter A1 routine Bring hyperbolic sine function into Acc.
35	84 300F 8S 300F	Add contents of 300F to the Acc. Store contents of Acc. at 300F.
36	81 299F 87 297F	Bring negative contents of 299F into Acc. Multiply contents of 297F by that of Acc. --
37	86 300F 87 293F	Divide contents of Acc. by that of 300F. Multiply contents of Acc. by that of 293F.
38	0S 350F 85 298F	Store C_{1m} starting from 350F. Bring contents of 298F into Acc.
39	87 297F 87 293F	Multiply contents of 297F by that of Acc. Multiply contents of 293F by that of Acc.

Location	Orders	Notes
40	86 300F 0S 450F	Divide contents of Acc. by that of 300F. Store C_{2m} starting from 450F.
41	02 10L 8J 42L	If $(c_0 + 1) \leq 0$, transfer control to R.H. of 10L. Transfer control to L.H. of 42L.
42	26 600F 0F F	Transfer control to L.H. of 600F. Stop order.
	00 600K	Directive
0	22 L 50 L	Transfer control to R.H. of L. Standard subroutine entry.
1	26 S4 OK 12F	Reenter A1 subroutine. Set the loop for Box 0 by setting $g_0 = 0$ and $c_0 = -12$.
2	88 F 0S 699F	Bring the number from the tape. Store contents of Acc. at designated location.
3	03 2L OK 11F	If $(c_0 + 1) \leq 0$, transfer control to L. H. of 2L. Set the loop for Box 0 by setting $g_0 = 0$ and $c_0 = -11$.
4	1K 50F 2K 5F	Set the loop for Box 1 by setting $g_1 = 0$ and $c_1 = -50$. Set the loop for Box 2 by setting $g_2 = 0$ and $c_2 = -5$.
5	8K F 2S 715F	Put 0 into Acc. Store number 0 from 715F onwards.
6	22 5L 05 700F	If $(c_2 + 1) \leq 0$, transfer control to R.H. of 5L. Bring F_y into Acc.
7	87 292F 8S 720F	Multiply contents of 292F by that of Acc. Store contents of Acc. at 720F.
8	85 715F 84 720F	Bring contents of 715F into Acc. Add contents of 720F to that of Acc.

Location	Orders	Notes
9	8S 715F 8J 10L	Store contents of Acc. at 715F. Transfer control to L.H. of 10L.
10	22 10L 50 10L	Transfer control to R.H. of 10L. Standard subroutine entry.
11	26 S7 50 11L	Enter hyperbolic sine and cosine sub- routine Standard subroutine entry.
12	26 S4 85 15S7	Reenter subroutine A1. Bring hyperbolic function into Acc.
13	17 350F 8S 721F	Multiply contents of designated location by that of Acc. Store contents of Acc. at 721F.
14	85 14S7 17 450F	Bring hyperbolic sine function into Acc. Multiply contents of designated location by that of Acc.
15	8S 722F 87 715F	Store contents of Acc. at 722F. Multiply contents of Acc. by that of 715F.
16	8S 723F 85 15S7	Store contents of Acc. at 723F. Bring hyperbolic function into Acc.
17	17 450F 8S 724F	Multiply contents of Acc. by that of designated location. Store content of Acc. at 724F.
18	87 715F 8S 725F	Multiply contents of 715F by that of Acc. Store contents of Acc. at 725F.
19	85 14S7 17 350F	Bring hyperbolic sine function into Acc. Multiply contents of Acc. by that of designated location.
20	8S 726F 85 721F	Store contents of Acc. at 726F. Bring contents of 721F into Acc.
21	84 723F 8S 727F	Add contents of 723F to that of Acc. Store contents of Acc. at 727F.
22	87 727F 8S 728F	Multiply contents of 727F by that of Acc. Store contents of Acc. at 728F.

Location	Orders	Notes
23	85 727F 84 724F	Bring contents of 727F into Acc. Add contents of 724F to that of Acc.
24	84 724F 8S 729F	Add contents of 724F to that of Acc. Store contents of Acc. at 729F.
25	87 729F 8S 730F	Multiply contents of 729F by that of Acc. Store contents of Acc. at 730F.
26	85 726F 84 725F	Bring contents of 726F into Acc. Add contents of 725F to that of Acc.
27	84 722F 8S 731F	Add contents of 722F to that of Acc. Store contents of Acc. at 731F.
28	87 731F 8S 732F	Multiply contents of 731F by that of Acc. Store contents of Acc. at 732F.
29	81 727F 87 729F	Bring negative contents of 727F into Acc. Multiply contents of 729F by that of Acc.
30	8S 733F 85 716F	Store contents of Acc. at 733F. Bring contents of 716F into Acc.
31	84 728F 8S 716F	Add contents of 728F to that of Acc. Store contents of Acc. at 716F. $\sum f(\sigma_y^2) \cdot dx$.
32	85 717F 84 730F	Bring contents of 717 F into Acc. Add contents of 730F to that of Acc.
33	8S 717F 85 718F	Store contents of Acc. at 717F. $\sum f(\sigma_x^2) \cdot dx$. Bring contents of 718F into Acc.
34	84 733F 8S 718F	Add contents of 733F to that of Acc. Store contents of Acc. at 718F. $\sum f(\sigma_x)(\sigma_y) \cdot dx$
35	85 719F 84 732F	Bring contents of 719F into Acc. Add contents of 732F to that of Acc.
36	8S 719F 13 8L	Store contents of Acc. at 719F. $\sum f(\tau_{xy}^2) \cdot dx$. If $(c_1 + 1) \leq 0$, transfer control to L. H. of 8L.

Location	Orders	Notes
37	8F 1F 3K 4F	Give a carriage return and line feed. Set the loop for Box 3 by setting $g_3 = 0$ and $c_3 = -4$.
38	35 716F 89 9F	Bring contents of designated location into Acc. Print contents of Acc.
39	33 38L 81 718F	If $(c_3 + 1) \leq 0$, transfer control to L.H. of 38L. Bring contents of 718F into Acc.
40	87 293F 87 699F	Multiply contents of Acc. by 2. Multiply contents of Acc. by that of 699F.
41	8S 718F 8K 1F	Store contents of Acc. at 718F. Put number 1 into Acc.
42	84 699F 87 293F	Add contents of 699F to Acc. Multiply contents of Acc. by 2.
43	87 719F 84 718F	Multiply contents of 719F by that of Acc. Add contents of 718F to Acc.
44	84 717F 84 716F	Add contents of 717F to Acc. Add contents of 716F to Acc.
45	89 9F 8F 1F	Print out the number from Acc. (Energy) • (2E). Give a carriage return and line feed.
46	03 4L 8J 47L	If $(c_3 + 1) \leq 0$, transfer control to L.H. of 4L. Transfer control to L.H. of 47L.
47	0F F 0F F	Stop order. Stop order.
	24 305N	When the program is read in the machine and it comes to symbol N, the machine will be ready to execute the order 24 305F. Hence, after starting with START switch control will be transferred to L.H. of 305F.

Operation of Program

The program is read into the machine. When the program is read in, the machine will stop at 24 305F. Then the program tape is removed and the data tape is put into the reader. Then START switch is put on. It will take the data in and will do the calculations. Machine will go on until eleven sets of answers are printed out.

Calculation Time

It takes about 6 minutes for calculation of energy and its components for eleven sets of F_y values. This time includes the time for reading in the program and data tape and printing out answers.

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