A THEORETICAL ANALYSIS OF THE DECAY OF SECONDARY FLOW FOLLOWING A PIPE BEND

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Cornelius Chung-sheng Shih

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by

Cornelius Chung-sheng Shih

AN ABSTRACT

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ABSTRACT

This study was initiated with the suggestion that theoretical analysis of decaying secondary flow after the bend may contribute some information for the improvement of water distribution over the field by rotary irrigation sprinklers. The equations of motion and the continuity equation together with some assumptions and boundary conditions have been used to express the flow condition in the pipe, and were solved for the case of laminar flow.

The general solutions are expressed as asymptotic functions associated with the initial flow conditions at the entrance of the pipe.

Since the length of the transition segment is the main interest in this study, it was determined by applying the general solutions. The relationships between the length of the transition segment and the intensity of the initial flow or the roughness of the pipe were presented with some calculations.

In addition, the adaptation of the solutions for the condition of laminar flow to the condition of turbulent flow was attempted with a hope that the tendency in the relationships for turbulent flow can be assessed.

By using these relationships, the distances of transition segments were theoretically calculated on the basis of assumed sprinkler characteristics and flow conditions.

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REVIEW OF LITERATURE

Due to the great development of sprinkler irrigation in the past decade, the need for more uniform water distribution by this method has become more apparent. For the purpose of obtaining the necessary information for improving design, Bilanski and Kidder (1) have investigated various factors that affect the distribution of water from an intermediate pressure (30 - 60 psi) rotary irrigation sprinkler. Among the factors which affected the water distribution were the flow conditions preceding the nozzle. For example, when the distance between the nozzle and the main body of the sprinkler was varied by using extension tubes of different lengths, the longer extension tube resulted in an increase in the trajectory distance and lessened the amount of fall-out of water near the sprinkler. However, beyond a certain length, a further increase in the length of the extension tube did not further affect the trajectory distance or the amount of fall-out of water near the sprinkler. They also noted that the use of a short cylindrical tube in place of a sprinkler nozzle resulted in a more effective distribution of water, and that the most desirable distribution pattern was obtained when the tube length was two to four diameters (of the inside of the tube) as measured from the beginning of the bend in the sprinkler body to the discharge end. Their study indicated that the secondary motion caused by the bend of sprinkler is one of the major factors influencing the distribution of water. They suggested that further study of water flow through the bend of various shapes would be beneficial in predicting the characteristics of the bend and after-bend length necessary for optimum distribution of water.

As to the study of secondary flow in bends, most of the studies during the past three decades had their emphases on the problem of energy loss caused by bends, which is related to the effect of Reynolds number, relative radius, roughness of the pipe, deflection angle, and aspect ratio.

The theoretical explanation of the secondary flow in a horizontal curved pipe, or a bend, was first given by Thompson (2). He indicated that the centrifugal force on the fluid due to its curved trajectory, associated with the variations of pressure gradients over the cross-section of the pipe, made the occurrence of secondary flow possible.

Theoretical analyses were made by Dean (3) and Adler (4) for deriving a parameter to relate the resistance in a curved pipe to that in a similar straight pipe for small and large Reynolds numbers respectively. From their theoretical analyses associated with the experimental works by White (5), Taylor (6), and Keulegan and Beij (7), it was found that the critical Reynolds number for the transition from laminar to turbulent flow in bends is higher than for the straight pipe.

By approximate integration of the equations of motion, Dean (3) found that the theoretical expressions for velocity components of secondary flow across the circular cross-section of coiled pipe vary with the radius of curvature, Rc. His approximation gave a motion in qualitative agreement with that found experimentally by Eustice (8) and others.

The solutions of the equations of motion and continuity by Dean will be presented later for the application as boundary conditions in this analysis of the transition segment in a straight pipe after the bend.

In Dean's analysis, the fluid flow was assumed to be incompressible, laminar, viscous, and therefore, rotational. The secondary flow occurring in the coiled pipe was assumed to be fully developed and steady. He also introduced the assumption that the radius of the pipe is small in comparison to the radius of the bend, i.e. a/Rc is small, where a is the radius of the pipe.

As for the study of the transition segment in the pipe after a bend, Yarnell's (9) measurement indicated that lengths of from ten to twenty diameters are necessary for the spiral currents to decay for velocities increasing from five feet per second to twelve feet per second around a six inch ninety degree standard bend.

Anderson and Straub (10) concluded that the maximum transition length for a ninety degree miter bend was only ten diameters, while for a 180 degree reversed curve and for several special bends, a length of more than fifty diameters of straight pipe was required for the decay of spiral currents. In addition, they illustrated that the distance required to establish fully developed flow in the straight pipe depends on the flow pattern in the bend and on the configuration and roughness of the boundary. The end of the transition segment, they assumed to be that point where the pressure gradient downstream of the bend becomes constant and presumably the same as that of the normal flow in a straight pipe. However, theoretical analysis has not been made so far on this subject.

From the review of literature, it was felt that a complete analysis of the transition segment in the pipe after a bend, particularly the length of the segment, might help improve sprinkler design. Hence, the theoretical analysis was conducted primarily in this study. However, it is essential at this stage to explain the secondary flow at a bend and in a straight pipe after the bend.

When fluid flows through a horizontal pipe bend, there must be a pressure gradient across the pipe to balance the centrifugal force on the fluid due to its curved trajectory, the pressure being greatest at the outer side of the pipe and least at the inner side.

Near the wall all around the pipe the velocity is considerably reduced because of boundary resistance. Consequently, the pressure variation due to the centrifugal force is greater along the central plane between the inner and outer sides than the pressure difference near the upper or lower walls. Therefore, there is a pressure gradient along the wall from the upper or lower sides toward the inner side and along the wall from the outer side toward the upper or lower sides. These pressure gradients induce a transverse flow along the walls toward the inner side, then from the inner side along the central plane toward the outer wall.

The superposition of this transverse flow upon the primary longitudinal flow results in a diagonal flow along the walls toward the inner side and forms the so-called double spiral or longitudinal vortices. In the straight pipe following the bend, the secondary flow will gradually diminish in intensity along the pipe axis because of the disappearance of centrifugal force and the shearing stress at the wall associated with the secondary flow itself. The relationship between the length of the pipe and the intensity of the secondary flow might be expected to be asymptotic.

From previous experimental reports, it was confirmed that a significant intensity of the secondary flow at the pipe outlet affects the breakage of jet column of the water which in turn relates to the water distribution into the field.

INTRODUCTION

Presentation of the Problem

The remarkable increase in the use of irrigation sprinklers during the past ten years indicated their ever-increasing importance in opening new agricultural frontiers. Naturally, along with the development of this method of irrigation, there has been an urgent demand for technical and general information on sprinkler irrigation equipment.

Ideally, water should be uniformly distributed over the entire wetted area. However, as yet a sprinkler system and technique which will do this has not been developed.

Bilanski and Kidder investigated various factors affecting the distribution. Among those factors studied, it seemed that the distance from the bend in the body of the sprinkler to the nozzle, and the type of transition through this distance, greatly influences the distribution of water. This suggested that the intensity of secondary flow is important. As a suggestion for further study it was pointed out that a theoretical study of the decay of secondary flow after a bend might be valuable for the improvement of sprinkler design. Hence, the primary interest of this study was to determine analytically the length of a straight pipe after a bend required to reestablish normal flow.

Approach to the Problem

Since it was believed that the decay of secondary flow in the pipe mainly depends on the effect of shearing stress due to the viscosity of the fluid, equations of motion (Navier-Stokes) and continuity equation with proper boundary conditions were applied for solving the problem.

As a first approximation to the solution of the problem, the laminar case was solved. The main flow conditions at the initial section of the transition segment were assumed to be given by Dean's (3) analysis of flow in a bend. Other assumptions and approximations necessary to obtain the solution will be explained in detail in the development.

Since the solution was based on laminar flow, application for turbulent flow was attempted by replacing kinematic viscosity by a mean eddy viscosity in spite of the fact that the eddy viscosity would be variable over any section and along the transition segment. However, it was believed imperative that the probable tendency of the relationship for turbulent flow be assessed, since most of the practical flows in bends as well as in sprinklers may be supposed to be turbulent.

The results of the theoretical analysis were expressed as the relation between the length of transition segment and Reynold's number, radius of the pipe and curvature of the bend, and the friction factor in the pipe. The accuracy, although unchecked by experimental measurement, may be sufficient from

an engineering standpoint. Nevertheless, an experimental investigation both for laminar and turbulent flows would be worthwhile as a continuation of this study in order to confirm or modify this present analysis.

THE GENERAL SOLUTION OF THE EQUATIONS IN GOVERNING THE EFFECT OF SHEARING STRESS DUE TO THE FRICTION IN A STRAIGHT PIPE

I. Nomenclature

a Radius of the pipe, inch (in)

R_c Radius of curvature of the bend which may be connected to the upstream of the straight pipe, inch (in)

ρ Density of the fluid, slug per cubic feet ($lb\text{-}sec^2/ft^4$)

ν Kinematic viscosity of the fluid, square feet per second (ft^2/sec)

ε Eddy viscosity of the fluid for turbulent flow, square feet per second (ft^2/sec)

W_o Velocity component of the fluid along z axis at any point on the centerline of the pipe, feet per second (ft/sec), constant.

N Reynolds number, nondimensional

K Reciprocal of Reynolds number, nondimensional

r Radial coordinate in the cylindrical polar system, nondimensional

ϕ Angular coordinate in the cylindrical polar system, nondimensional

U Velocity component along r axis, nondimensional

V Velocity component perpendicular to U on r-ϕ plane, nondimensional

W Velocity component along z axis, nondimensional

P Fluid pressure, nondimensional

C_n Constants, n = 1, 2, 3, 4 . . . .

λ Eigen value, nondimensional

ξ Eigen value, nondimensional

G Negative constant for the expression in head loss due to friction

L Distance from the inlet of the pipe to a point where the velocity component, V, diminished to some small value, δ , nondimensional

δ Some small value of velocity component, V, in the process of decay, when z = L, nondimensional

The following diagram shows the schematic features of the flow in a straight pipe after a bend and the system of cylindrical polar coordinates.

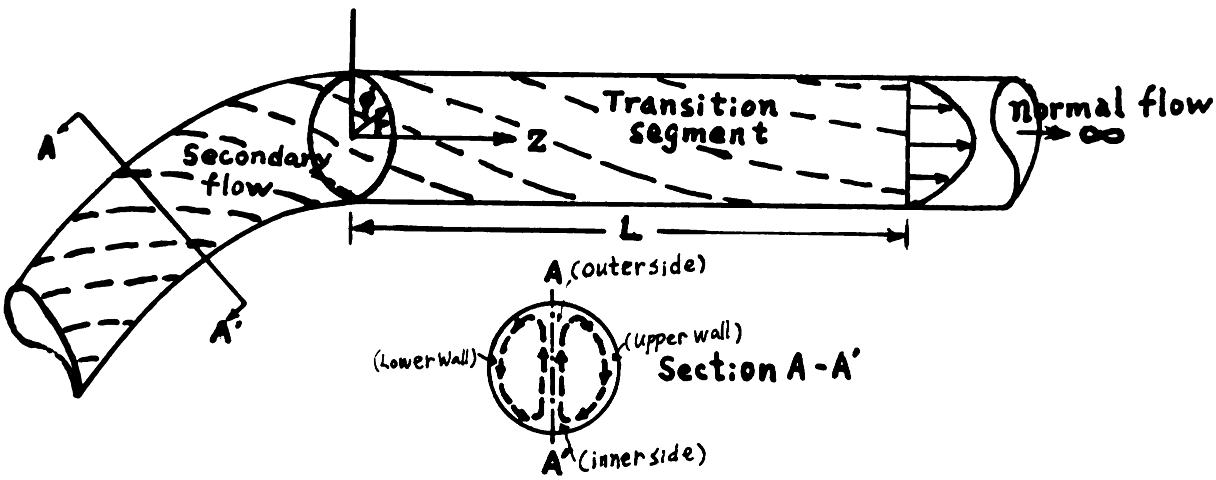

Figure 1

II. The Governing Equations in Cylindrical Polar Coordinates.

1. The Continuity Equation: if U', V', W', r', and z' are dimensional,

$$\frac{1}{r'}\frac{\partial}{\partial r'}(r'U') + \frac{1}{r'}\frac{\partial V'}{\partial \phi} + \frac{\partial W'}{\partial z'} = 0$$

Let $U(r, \phi, z) = \frac{U'}{W_0}$, $V(r, \phi, z) = \frac{V'}{W_0}$,

$W(r, \phi, z) = \frac{W'}{W_0}$, $P(r, \phi, z) = \frac{P'}{\rho W_0^2}$,

$r = \frac{r'}{a}$, $z = \frac{z'}{a}$,

then the non-dimensional continuity equation is

$$\frac{1}{r}\frac{\partial}{\partial r}(rU) + \frac{1}{r}\frac{\partial V}{\partial \phi} + \frac{\partial W}{\partial z} = 0 \quad \cdots\cdots (1)$$

2. The Equations of Motion (Navier-Stokes): When the flow is steady, i.e. $\frac{\partial U'}{\partial t} = \frac{\partial V'}{\partial t} = \frac{\partial W'}{\partial t} = 0$, and if P' is dimensional,

$$U'\frac{\partial U'}{\partial r'} + \frac{V'}{r'}\frac{\partial U'}{\partial \phi} + W'\frac{\partial U'}{\partial z'} - \frac{V'^2}{r'} = -\frac{1}{\rho}\frac{\partial P'}{\partial r'} + \nu\left(\nabla^2 U' - \frac{U'}{r'^2} - \frac{2}{r'^2}\frac{\partial V'}{\partial \phi}\right)$$

$$U'\frac{\partial V'}{\partial r'} + \frac{V'}{r'}\frac{\partial V'}{\partial \phi} + W'\frac{\partial V'}{\partial z'} + \frac{U'V'}{r'} = -\frac{1}{\rho r'}\frac{\partial P'}{\partial \phi} + \nu\left(\nabla^2 V' - \frac{V'}{r'^2} + \frac{2}{r'^2}\frac{\partial U'}{\partial \phi}\right)$$

$$U'\frac{\partial W'}{\partial r'} + \frac{V'}{r'}\frac{\partial W'}{\partial \phi} + W'\frac{\partial W'}{\partial z'} = -\frac{1}{\rho}\frac{\partial P'}{\partial z'} + \nu\left(\nabla^2 W'\right)$$

where $\nabla^2 = \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z'^2}$.

They are expressed nondimensionally in equations (2), (3) and (4).

$$U\frac{\partial U}{\partial r} + \frac{V}{r}\frac{\partial U}{\partial \phi} + W\frac{\partial U}{\partial z} - \frac{V^2}{r} = -\frac{\partial P}{\partial r} + \frac{\nu}{aW_0}\left(\nabla^2 U - \frac{U}{r^2} - \frac{2}{r^2}\frac{\partial V}{\partial \phi}\right) \cdots (2)$$

$$U\frac{\partial V}{\partial r} + \frac{V}{r}\frac{\partial V}{\partial \phi} + W\frac{\partial V}{\partial z} + \frac{UV}{r} = -\frac{1}{r}\frac{\partial P}{\partial \phi} + \frac{\nu}{aW_0}\left(\nabla^2 V - \frac{V}{r^2} + \frac{2}{r^2}\frac{\partial U}{\partial \phi}\right) \cdots (3)$$

$$U\frac{\partial W}{\partial r}+\frac{V}{r}\frac{\partial W}{\partial \phi}+W\frac{\partial W}{\partial z}=-\frac{\partial P}{\partial z}+\frac{\nu}{aW_0}(\nabla^2 W) \qquad \ldots (4)$$

where $\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\frac{\partial^2}{\partial z^2}$.

Let K = $\frac{\nu}{aW_0}=\frac{1}{N}$. Since N = $\frac{2aW_a}{\nu}$,

and average velocity, $W_a=\frac{W_0}{2}$

approximately in laminar flow, therefore N = $\frac{aW_0}{\nu}$.

III. Assumptions Underlying the Mathematical Analysis.

In order to solve the partial differential equations governing this problem, the following assumptions have been made. The validity and applicability of those assumptions in actual fluid flow problems, will be presented in the Discussion.

1. The fluid is viscous and incompressive, and the motion of the fluid is steady, laminar and rotational, i.e. kinematic viscosity, ν , appears in the equations presented above; density is constant: $\frac{\partial U}{\partial t}=\frac{\partial V}{\partial t}=\frac{\partial W}{\partial t}=0$ N does not exceed 2100.

2. It has been assumed that the number of independent variables of the velocity components can be reduced from three to two by fixing angular displacement, ϕ as a dummy variable or parameter, with a hope that the deviation of the solution from exact solution will be negligible.

Thus,
$$\begin{aligned}U &= U_1 \sin\phi + U_2\cos\phi\\ V &= V_1\cos\phi + V_2\sin\phi\\ W &= (1-r^2)+W_1\sin\phi+W_2\cos\phi\\ P &= Gz+P_1\sin\phi+P_2\cos\phi\end{aligned}$$

where U_1, U_2, V_1, V_2, W_1, W_2, P_1 and P_2 are functions of r and z, and G is a negative constant for the expression of head loss due to the friction in the pipe. In other words, the rate of head loss with respect to distance of the pipe, z, has been assumed to be constant if z is far away from the transition region.

3. Because the application of a perturbation method or an approximation method has been required for solving the differential equations, velocity components, U_1, U_2, V_1, V_2, W_1 and W_2 were assumed to be expanded in ascending power series of parameter, K, respectively. It has been noted that K is very much smaller than one but not less than zero ($0 < K << 1$).

Hence,

$$U_1 = KU_{11} + K^2U_{12} + K^3U_{13} + \cdots$$
$$U_2 = KU_{21} + K^2U_{22} + K^3U_{23} + \cdots$$
$$V_1 = KV_{11} + K^2V_{12} + K^3V_{13} + \cdots$$
$$V_2 = KV_{21} + K^2V_{22} + K^3V_{23} + \cdots$$
$$W_1 = KW_{11} + K^2W_{12} + K^3W_{13} + \cdots$$
$$W_2 = KW_{21} + K^2W_{22} + K^3W_{23} + \cdots$$

4. For application of secondary flow at the entrance of a straight pipe, Dean's analytical solutions for the secondary flow in the bend has been adopted as a part of the initial boundary condition when z is zero. According to Dean's notes, the assumptions for his solution have been listed as follows:

a. The ratio of radius of the curved pipe to radius of the curvature of the bend or coiled pipe is small, (about 1 to 5 percent).

b. U, V, and W (but not P, pressure) are independent of θ, which is another angular coordinate in a spherical orthogonal system.

c. The secondary flow is fully developed.

d. The flow conditions of the fluid are the same as the assumption 1.

IV. Boundary Conditions Around the Transition Segment in the Pipe.

(a) Since it is believed that the fully developed secondary flow is distorted at the entrance of the pipe, Dean's solutions with the product of ϕ functions and unknown functions of r have been applied for flow condition at the initial section of the pipe. The unknown functions of r; $F_1(r)$, $F_2(r)$, $F_3(r)$ and $F_4(r)$, can be determined by the measurement of velocity distribution across the initial section. Thus,

$$U(r,\phi,0) = \frac{a \sin\phi\,(1-r^2)^2(4-r^2)}{288\ K\ R_c} + F_1(r)\cos\phi$$

$$V(r,\phi,0) = \frac{a \cos\phi\,(1-r^2)(4-23r^2+7r^4)}{288\ K\ R_c} + F_2(r)\sin\phi$$

$$W(r,\phi,0) = (1-r^2)\left[1-\frac{3ra\sin\phi}{4R_c}+\frac{ra\sin\phi}{11520K^2R_c}(19-21r^2+9r^4-r^6)\right]+F_3(r)\cos\phi$$

$$P(r,\phi,0) = \frac{a\sin\phi\,(18r-12r^3+4r^5)}{24\ R_c} + F_4(r)\cos\phi$$

(b) It has been assumed that there is no flow at the boundary between fluid and inside surface of the pipe.

$$U(1,\phi,z) = 0,$$
$$V(1,\phi,z) = 0,$$
$$W(1,\phi,z) = 0,$$
$$P(1,\phi,z) \neq 0.$$

(c) When the secondary flow is diminished due to the effect of shearing stress by the friction at a large distance, L, along z axis, the boundary condition of fluid flow has been expressed as follows:

$$U(r,\phi,L) = 0,$$
$$V(r,\phi,L) = 0,$$
$$W(r,\phi,L) = 0,$$
$$P(r,\phi,L) = GL.$$

(d) After assumption 2 was made for providing the dummy variable, ϕ, the boundary conditions (a), (b) and (c) should be modified.

$$U_1(r,0) = \frac{a(1-r^2)^2(4-r^2)}{288\,K\,R_e},$$
$$U_2(r,0) = F_1(r),$$
$$V_1(r,0) = \frac{a(1-r^2)(4-23r^2+7r^4)}{288\,K\,R_e},$$
$$V_2(r,0) = F_2(r),$$
$$W_1(r,0) = (1-r^2)\left[-\frac{3ra}{4R_e} + \frac{ra}{11520\,K^2R_e}(19-21r^2+9r^4-r^6)\right],$$
$$W_2(r,0) = F_3(r),$$
$$P_1(r,0) = \frac{a(18r-12r^3+4r^5)}{24\,R_e},$$
$$P_2(r,0) = F_4(r).$$

(e)

$$U_1(1,z) = 0, \qquad U_2(1,z) = 0,$$

$$V_1(1,z) = 0, \qquad V_2(1,z) = 0,$$

$$W_1(1,z) = 0, \qquad W_2(1,z) = 0,$$

$$P_1(1,z) \neq 0, \qquad P_2(1,z) \neq 0. \qquad (\text{if } z < L)$$

(f)

$$U_1(r,L) = \delta \qquad U_2(r,L) = \delta$$

$$V_1(r,L) = \delta \qquad V_2(r,L) = \delta$$

$$W_1(r,L) = \delta \qquad W_2(r,L) = \delta$$

$$P_1(r,L) = \delta \qquad P_2(r,L) = \delta$$

and $\delta \longrightarrow 0$ or $L \longrightarrow \infty$.

V. Formulation of the Problem.

Substitution of assumption 2 into equations (1), (2), (3) and (4) has made it possible to eliminate the terms involving the derivative with respect to ϕ.

Thus, from continuity equation,

$$\sin\phi\left(\frac{U_1}{r}+\frac{\partial U_1}{\partial r}-\frac{V_1}{r}+\frac{\partial W_1}{\partial z}\right)+\cos\phi\left(\frac{U_2}{r}+\frac{\partial U_2}{\partial r}+\frac{V_2}{r}+\frac{\partial W_2}{\partial z}\right)=0 \quad .$$

Since it is obvious that functions of r and z are independent of ϕ, it is seen that:

$$\frac{U_1}{r}+\frac{\partial U_1}{\partial r}-\frac{V_1}{r}+\frac{\partial W_1}{\partial z}=0 \qquad \ldots\ldots\ldots\ldots(5)$$

$$\frac{U_2}{r}+\frac{\partial U_2}{\partial r}+\frac{V_2}{r}+\frac{\partial W_2}{\partial z}=0 \qquad \ldots\ldots\ldots\ldots(6)$$

From Navier-Stokes equations;

$$\sin^2\phi\left[U_1\frac{\partial U_1}{\partial r}-\frac{V_2U_2}{r}+W_1\frac{\partial U_1}{\partial z}-\frac{V_2^2}{r}\right]+(1-\sin^2\phi)\left[U_2\frac{\partial U_2}{\partial r}+\frac{V_1U_1}{r}+W_2\frac{\partial U_2}{\partial z}-\frac{V_1^2}{r}\right]$$
$$+\sin\phi\cos\phi\left[U_2\frac{\partial U_1}{\partial r}+U_1\frac{\partial U_2}{\partial r}+\frac{V_2U_1}{r}-\frac{V_1U_2}{r}+W_1\frac{\partial U_2}{\partial z}+W_2\frac{\partial U_1}{\partial z}-2\frac{V_1V_2}{r}\right]$$
$$+(1-r^2)\left[\sin\phi\frac{\partial U_1}{\partial z}+\cos\phi\frac{\partial U_2}{\partial z}\right]=$$

$$= -\left[\sin\phi\frac{\partial P_1}{\partial r} + \cos\phi\frac{\partial P_2}{\partial r}\right] + K\left[\sin\phi\left(\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\frac{\partial U_1}{\partial r} - \frac{2U_1}{r^2} + \frac{2V_1}{r^2} + \frac{\partial^2 U_1}{\partial z^2}\right)\right.$$
$$\left. + \cos\phi\left(\frac{\partial^2 U_2}{\partial r^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} - \frac{2U_2}{r^2} - \frac{2V_2}{r^2} + \frac{\partial^2 U_2}{\partial z^2}\right)\right].$$

Because of independency of r and z and ϕ, the following equations are obtained:

$$(1-r^2)\frac{\partial U_1}{\partial z} + \frac{\partial P_1}{\partial r} = K\left(\frac{\partial^2 U_1}{\partial r^2} + \frac{1}{r}\frac{\partial U_1}{\partial r} - \frac{2U_1}{r^2} + \frac{2V_1}{r^2} + \frac{\partial^2 U_1}{\partial z^2}\right) \qquad \ldots.(7)$$

$$(1-r^2)\frac{\partial U_2}{\partial z} + \frac{\partial P_2}{\partial r} - K\left(\frac{\partial^2 U_2}{\partial r^2} + \frac{1}{r}\frac{\partial U_2}{\partial r} - \frac{2U_2}{r^2} - \frac{2V_2}{r^2} + \frac{\partial^2 U_2}{\partial z^2}\right). \qquad \ldots.(8)$$

$$\cos^2\phi\left[U_2\frac{\partial V_1}{\partial r} + \frac{V_1 U_2}{r} + \frac{V_1 V_2}{r} + W_2\frac{\partial V_1}{\partial z}\right] + (1-\cos^2\phi)\left[U_1\frac{\partial V_2}{\partial r} - \frac{V_1 V_2}{r} + W_1\frac{\partial V_2}{\partial z} + \frac{U_1 V_2}{r}\right]$$
$$+ \cos\phi\sin\phi\left[U_1\frac{\partial V_1}{\partial r} + U_2\frac{\partial V_2}{\partial r} - \frac{V_1^2}{r} + \frac{V_2^2}{r} + W_1\frac{\partial V_1}{\partial z} + W_2\frac{\partial V_2}{\partial z} + \frac{U_1 V_1}{r} + \frac{U_2 V_2}{r}\right]$$
$$+ (1-r^2)\left[\cos\phi\frac{\partial V_1}{\partial z} + \sin\phi\frac{\partial V_2}{\partial z}\right]$$
$$= -\frac{1}{r}\left[\cos\phi P_1 - \sin\phi P_2\right] + K\left[\cos\phi\left(\frac{\partial^2 V_1}{\partial r^2} + \frac{1}{r}\frac{\partial V_1}{\partial r} - \frac{2V_1}{r^2} + \frac{\partial^2 V_1}{\partial z^2} + \frac{2U_1}{r^2}\right)\right.$$
$$\left. + \sin\phi\left(\frac{\partial^2 V_2}{\partial r^2} + \frac{1}{r}\frac{\partial V_2}{\partial r} - \frac{2V_2}{r^2} - \frac{2U_2}{r^2} + \frac{\partial^2 V_2}{\partial z^2}\right)\right].$$

Then separately,

$$(1-r^2)\frac{\partial V_1}{\partial z} + \frac{P_1}{r} = K\left(\frac{\partial^2 V_1}{\partial r^2} + \frac{1}{r}\frac{\partial V_1}{\partial r} - \frac{2V_1}{r^2} + \frac{2U_1}{r^2} + \frac{\partial^2 V_1}{\partial z^2}\right) \qquad \ldots.(9)$$

$$(1-r^2)\frac{\partial V_2}{\partial z} - \frac{P_2}{r} = K\left(\frac{\partial^2 V_2}{\partial r^2} + \frac{1}{r}\frac{\partial V_2}{\partial r} - \frac{2V_2}{r^2} - \frac{2U_2}{r^2} + \frac{\partial^2 V_2}{\partial z^2}\right). \qquad \ldots.(10)$$

Therefore,

$$P_1 = Kr\left(\frac{\partial^2 V_1}{\partial r^2} + \frac{1}{r}\frac{\partial V_1}{\partial r} - \frac{2V_1}{r^2} + \frac{2U_1}{r^2} + \frac{\partial^2 V_1}{\partial z^2}\right) - r(1-r^2)\frac{\partial V_1}{\partial z} \qquad \ldots.(11)$$

$$P_2 = -Kr\left(\frac{\partial^2 V_2}{\partial r^2} + \frac{1}{r}\frac{\partial V_2}{\partial r} - \frac{2V_2}{r^2} - \frac{2U_2}{r^2} + \frac{\partial^2 V_2}{\partial z^2}\right) + r(1-r^2)\frac{\partial V_2}{\partial z}. \qquad \ldots.(12)$$

Also,

$$\sin^2\phi\left[U_1\frac{\partial W_1}{\partial r} - \frac{V_2 W_2}{r} + W_1\frac{\partial W_1}{\partial z}\right] + \cos^2\phi\left[U_2\frac{\partial W_2}{\partial r} + \frac{V_1 W_1}{r} + W_2\frac{\partial W_1}{\partial z}\right]$$
$$+ \sin\phi\cos\phi\left[U_2\frac{\partial W_1}{\partial r} + U_1\frac{\partial W_2}{\partial r} - \frac{V_2 W_1}{r} - \frac{V_1 W_2}{r} + W_2\frac{\partial W_1}{\partial z} + W_1\frac{\partial W_2}{\partial z}\right]$$
$$+ \sin\phi\left[(1-r^2)\frac{\partial W_1}{\partial z} - 2U_1 r\right] + \cos\phi\left[(1-r^2)\frac{\partial W_2}{\partial z} - 2U_2 r\right] =$$

$$= -\left[G + \sin\phi\frac{\partial P_1}{\partial z} + \cos\phi\frac{\partial P_2}{\partial z}\right] + K\left[\sin\phi\left(\frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r}\frac{\partial W_1}{\partial r} - \frac{W_1}{r^2} + \frac{\partial^2 W_1}{\partial z^2}\right)\right.$$

$$\left. + \cos\phi\left(\frac{\partial^2 W_2}{\partial r^2} + \frac{1}{r}\frac{\partial W_2}{\partial r} - \frac{W_2}{r^2} + \frac{\partial^2 W_2}{\partial z^2}\right)\right] .$$

Then separately,

$$(1-r^2)\frac{\partial W_1}{\partial z} - 2rU_1 + \frac{\partial P_1}{\partial z} = K\left(\frac{\partial^2 W_1}{\partial r^2} + \frac{1}{r}\frac{\partial W_1}{\partial r} - \frac{W_1}{r^2} + \frac{\partial^2 W_1}{\partial z^2}\right) \qquad \ldots.(13)$$

$$(1-r^2)\frac{\partial W_2}{\partial z} - 2rU_2 + \frac{\partial P_2}{\partial z} = K\left(\frac{\partial^2 W_2}{\partial r^2} + \frac{1}{r}\frac{\partial W_2}{\partial r} - \frac{W_2}{r^2} + \frac{\partial^2 W_2}{\partial z^2}\right) . \qquad \ldots.(14)$$

Substitution of equation (11) and (12) into equation (7), (8), (13) and (14) has made it possible to eliminate the pressure terms. By application of the perturbation method as given by assumption 3, to equation (5), (6), (7), (8), (13) and (14) the coefficients of the first power of K give, respectively,

$$\frac{U_{11}}{r} + \frac{\partial U_{11}}{\partial r} - \frac{V_{11}}{r} + \frac{\partial W_{11}}{\partial z} = 0 \qquad \ldots.(15)$$

$$\frac{U_{21}}{r} + \frac{\partial U_{21}}{\partial r} - \frac{V_{21}}{r} + \frac{\partial W_{21}}{\partial z} = 0 \qquad \ldots.(16)$$

$$(1-r^2)\frac{\partial U_{11}}{\partial z} + (3r^2-1)\frac{\partial V_{11}}{\partial z} + r(r^2-1)\frac{\partial^2 V_{11}}{\partial r\partial z} = 0 \qquad \ldots.(17)$$

$$(1-r^2)\frac{\partial U_{21}}{\partial z} + (1-3r^2)\frac{\partial V_{21}}{\partial z} + r(1-r^2)\frac{\partial^2 V_{21}}{\partial r\partial z} = 0 \qquad \ldots.(18)$$

$$(1-r^2)\frac{\partial W_{11}}{\partial z} - 2U_{11}r + r(r^2-1)\frac{\partial^2 V_{11}}{\partial z^2} = 0 \qquad \ldots.(19)$$

$$(1-r^2)\frac{\partial W_{21}}{\partial z} - 2U_{21}r + r(1-r^2)\frac{\partial^2 V_{21}}{\partial z^2} = 0 \qquad \ldots.(20)$$

Let equations (17) and (18) be integrated with respect to z, thus,

$$U_{11} = r\frac{\partial V_{11}}{\partial r} + \left(\frac{1-3r^2}{1-r^2}\right)V_{11} + f_1(r) \qquad \ldots.(21)$$

$$U_{21} = -r\frac{\partial V_{21}}{\partial r} - \left(\frac{1-3r^2}{1-r^2}\right)V_{21} + f_2(r) \qquad \ldots.(22)$$

Since $U_{11}(r,z) = U_{21}(r,z) = V_{11}(r,z) = V_{21}(r,z) = \frac{\partial V_{11}}{\partial r} = \frac{\partial V_{21}}{\partial r} = 0$ as z is far away from transition segment, then $f_1(r)$ and $f_2(r)$ are shown to be equal to zero.

Then let equations (15), (16), (21) and (22) be substituted into equations (19) and (20) respectively; the results will be presented as follows:

$$\frac{\partial^2 V_{11}}{\partial z^2} + \frac{\partial^2 V_{11}}{\partial r^2} + \frac{3}{r}\frac{\partial V_{11}}{\partial r} - \frac{4(1+r^2)}{(1-r^2)^2}V_{11} = 0 \qquad \ldots.(23)$$

$$\frac{\partial^2 V_{21}}{\partial z^2} + \frac{\partial^2 V_{21}}{\partial r^2} + \frac{3}{r}\frac{\partial V_{21}}{\partial r} - \frac{4(1+r^2)}{(1-r^2)^2}V_{21} = 0 \qquad \ldots.(24)$$

VI. Solutions of the Partial Differential Equations

If it has been assumed that the function of r and z can be separated, so that V_{11} and V_{21} in equations (23) and (24) can be expressed as

$$V_{11} = R_{11}Z_{11}, \qquad V_{21} = R_{21}Z_{21},$$

where,

$$R_{11} = R_{11}(r), \qquad Z_{11} = Z_{11}(z),$$

$$R_{21} = R_{21}(r), \qquad Z_{21} = Z_{21}(z),$$

then the equations (23) and (24) can be transformed into the following ordinary differential equations:

$$\frac{1}{Z_{11}}\frac{d^2 Z_{11}}{dz^2} = -\frac{1}{R_{11}}\frac{d^2 R_{11}}{dr^2} - \frac{3}{rR_{11}}\frac{dR_{11}}{dr} + \frac{4(r^2+1)}{(1-r^2)^2} = \lambda^2 \qquad \ldots.(25)$$

$$\frac{1}{Z_{21}}\frac{d^2 Z_{21}}{dz^2} = -\frac{1}{R_{21}}\frac{d^2 R_{21}}{dr^2} - \frac{3}{rR_{21}}\frac{dR_{21}}{dr} + \frac{4(r^2+1)}{(1-r^2)^2} = \zeta^2 \qquad \ldots.(26)$$

From observation of boundary conditions (d) and (f), the signs of eigen values are so chosen that the conditions are satisfied. Hence, from equation (25)

$$Z_{11} = C_1 e^{-\lambda z} + C_2 e^{\lambda z}$$

When $z \longrightarrow \infty$, $Z_{11} \neq \infty$, so that $C_2 = 0$

therefore $$Z_{11} = C_1 e^{-\lambda z} \qquad \text{.................. (27)}$$

It is assumed that function of R_{11} can be expressed as:

$$R_{11} = (1 - r^2)\, y(r) \qquad \text{.................. (28)}$$

Substituting equation (28) into the right hand side of the equation (25), thus,

$$\frac{d^2 y}{dr^2} + \frac{1}{r}\left(\frac{3 - 7r^2}{1 - r^2}\right)\frac{dy}{dr} + \left[\lambda - \frac{(10 - 2r^2)}{(1 - r^2)^2}\right] y = 0 \qquad \text{........ (29)}$$

The use of series solution for equation (29), derives the solution for y. As a by-product, the eigen value, λ^2, is found to be 2.

$$\lambda^2 = 2 , \qquad \lambda = \pm\sqrt{2} .$$

Then, $$y(r) = C_3\left(\frac{1}{r^2} + \frac{5}{4}r^2 + \frac{19}{12}r^4 + \frac{193}{96}r^6 + \frac{93}{30}r^8 + \frac{16453}{5760}r^{10} + ----\right)$$
$$+ C_4\left(1 + r^2 + \frac{17}{12}r^4 + \frac{43}{24}r^6 + \frac{173}{80}r^8 + \frac{1217}{480}r^{10} + ----\right).$$

Since, $y(r) \neq \infty$ when $r = 0$, therefore $C_3 = 0$

and $$R_{11} = (1 - r^2)\, C_4\left(1 + r^2 + \frac{17}{12}r^4 + \frac{43}{24}r^6 + \frac{173}{80}r^8 + \frac{1217}{480}r^{10} + \cdots\right) \qquad \text{... (30)}$$

Then, if $C_4 \cdot C_1 = C_5$,

$$V_{11} = C_5(1 - r^2)\left(1 + r^2 + \frac{17}{12}r^4 + \frac{43}{24}r^6 + \frac{173}{80}r^8 + \frac{1217}{480}r^{10} + \cdots\right) e^{-\sqrt{2} z} \qquad \text{....... (31)}$$

Substituting equation (31) into (21), thus,

$$U_{11} = C_5(1 - r^2)\left(1 - r^2 - \frac{4}{3}r^4 - \frac{37}{34}r^6 - \frac{429}{240}r^8 - \frac{865}{480}r^{10} + \cdots\right) e^{-\sqrt{2} z} \qquad \text{........ (32)}$$

By substitution of equation (31) and (32) into (15), W_{11} is obtained after integration with respect to z,

$$W_{11} = -\frac{C_5}{\sqrt{2}}\left(6r + \frac{25}{12}r^3 + \frac{29}{24}r^5 + \frac{31}{72}r^7 + \frac{113}{140}r^9 + \frac{23123}{240}r^{11} + \cdots\right)e^{-\sqrt{2}z} \quad \cdots\cdots\cdots (33)$$

From equation (26), the left hand side can be presented as follows:

$$\frac{d^2 Z_{21}}{dz^2} - \xi^2 Z_{21} = 0 \quad \cdots\cdots\cdots (34)$$

The solution of equation (34) is:

$$Z_{21} = C_6 e^{-\xi z} + C_7 e^{\xi z}$$

When $z \longrightarrow \infty$, $Z_{21} \neq \infty$ so that $C_7 = 0$,

therefore $Z_{21} = C_6 e^{-\xi z}$. $\quad \cdots\cdots\cdots\cdots\cdots\cdots (35)$

The method of series solution has been used to obtain the solution of R_{21} in equation (26). As the by-product, the eigen value, ξ, is found to be equal to $\pm\sqrt{2}$: $\xi = \pm\sqrt{2}$; positive ξ is chosen. It is noted that the solution of R_{21} is identical with that of R_{11} except the change of arbitrary constant;

$$R_{21} = (1 - r^2)C_8\left(1 + r^2 + \frac{17}{12}r^4 + \frac{43}{24}r^6 + \frac{173}{80}r^8 + \frac{1217}{480}r^{10} + \cdots\right) \quad \cdots\cdots (36)$$

Then, let $C_8 \cdot C_6 = C_{10}$,

$$V_{21} = C_{10}(1 - r^2)\left(1 + r^2 + \frac{17}{12}r^4 + \frac{43}{24}r^6 + \frac{173}{80}r^8 + \frac{1217}{480}r^{10} + \cdots\right)e^{-\sqrt{2}z} \quad \cdots\cdots (37)$$

Substituting equation (37) into (22),

$$U_{21} = -C_{10}(1 - r^2)\left(1 - r^2 - \frac{4}{3}r^4 - \frac{37}{34}r^6 - \frac{429}{240}r^8 - \frac{965}{480}r^{10} - \cdots\right)e^{-\sqrt{2}z} \quad \cdots\cdots (38)$$

By substitution of equation (37) and (38) into (16), W_{21} is obtained after integration with respect to z,

$$W_{21} = \frac{C_{10}}{\sqrt{2}}\left(6r + \frac{25}{12}r^3 + \frac{29}{24}r^5 + \frac{31}{72}r^7 + \frac{113}{40}r^9 + \frac{23123}{240}r^{11} + \cdots\right)e^{-\sqrt{2}z} \quad \cdots\cdots (39)$$

The second equations which are found from coefficient of second power of parameter K for U_{11}, V_{11}, W_{11}, W_{21}, V_{21}, U_{21}, U_{12}, V_{12}, W_{12}, U_{22}, V_{22}, and W_{22} are obtained from equations (5), (6), (7), (8), (13), and (14) by application of assumption 3, in a similar manner as for the equations from first power of K.

$$\frac{U_{12}}{r}+\frac{\partial U_{12}}{\partial r}-\frac{V_{12}}{r}+\frac{\partial W_{12}}{\partial z}=0 \qquad \ldots\ldots\ldots(40)$$

$$\frac{U_{22}}{r}+\frac{\partial U_{22}}{\partial r}+\frac{V_{22}}{r}+\frac{\partial W_{22}}{\partial z}=0 \qquad \ldots\ldots\ldots(41)$$

$$(1-r^2)\frac{\partial U_{12}}{\partial z}+(3r^2-1)\frac{\partial V_{12}}{\partial z}+r(r^2-1)\frac{\partial^2 V_{12}}{\partial r\partial z}=f_3(r)e^{-\sqrt{2}z} \qquad \ldots.(42)$$

$$(1-r^2)\frac{\partial U_{22}}{\partial z}+(1-3r^2)\frac{\partial V_{22}}{\partial z}+r(1-r^2)\frac{\partial^2 V_{22}}{\partial r\partial z}=f_4(r)e^{-\sqrt{2}z} \qquad \ldots.(43)$$

$$(1-r^2)\frac{\partial W_{12}}{\partial z}-2U_{12}r+r(r^2-1)\frac{\partial^2 V_{12}}{\partial z^2}=f_5(r)e^{-\sqrt{2}z} \qquad \ldots.(44)$$

$$(1-r^2)\frac{\partial W_{22}}{\partial z}-2U_{22}r+r(r^2-1)\frac{\partial^2 V_{22}}{\partial z^2}=f_6(r)e^{-\sqrt{2}z} \qquad \ldots.(45)$$

If equations (42) and (43) are integrated with respect to z, then,

$$U_{12}=r\frac{\partial V_{12}}{\partial r}+\left(\frac{1-3r^2}{1-r^2}\right)V_{12}+f_7(r)e^{-\sqrt{2}z}+f_8(r) \qquad \ldots.(46)$$

$$U_{22}=-r\frac{\partial V_{22}}{\partial r}+\left(\frac{1-3r^2}{1-r^2}\right)V_{22}+f_9(r)e^{-\sqrt{2}z}+f_{10}(r) \qquad \ldots.(47)$$

Since $f_8(r)$ and $f_{10}(r)$ are independent of z in these equations, therefore $f_8(r)$ and $f_{10}(r)$ should be equal to zero, ($f_8(r)=f_{10}(r)=0$). Then the substitution of equations (40), (41), (46) and (47) into equations (44) and (45) will result as follows:

$$\frac{\partial^2 V_{12}}{\partial z^2}+\frac{\partial^2 V_{12}}{\partial r^2}+\frac{3}{r}\frac{\partial V_{12}}{\partial r}-\frac{4(1+r^2)}{(1-r^2)^2}V_{12}=f_{11}(r)e^{-\sqrt{2}z} \qquad \ldots.(48)$$

$$\frac{\partial^2 V_{22}}{\partial z^2}+\frac{\partial^2 V_{22}}{\partial r^2}+\frac{3}{r}\frac{\partial V_{22}}{\partial r}-\frac{4(1+r^2)}{(1-r^2)^2}V_{22} = f_{12}(r)\,e^{-\sqrt{2}z} . \qquad \ldots.(49)$$

It is found actually that the equations (48) and (49) are respectively the nonhomogeneous case of equation (23) and (24). If separation technique is applied for the second equations, similarly as for the first equations, i.e.,

$$V_{12} = R_{12}\, Z_{12} \quad \text{and} \quad V_{22} = R_{22}\, Z_{22}$$

where $R_{12} = R_{12}(r)$, $Z_{12} = Z_{12}(z)$, $R_{22} = R_{22}(r)$

and $Z_{22} = Z_{22}(z)$

Therefore,

$$Z_{12}\left\{-\frac{4(1+r^2)}{(1-r^2)^2}R_{12}+\frac{3}{r}\frac{\partial R_{12}}{\partial r}+\frac{\partial^2 R_{12}}{\partial r^2}\right\}+R_{12}\frac{\partial^2 Z_{12}}{\partial z^2}=f_{11}(r)\,e^{-\sqrt{2}z} \quad \ldots.(50)$$

$$Z_{22}\left\{-\frac{4(1+r^2)}{(1-r^2)^2}R_{22}+\frac{3}{r}\frac{\partial R_{22}}{\partial r}+\frac{\partial^2 R_{22}}{\partial r^2}\right\}+R_{22}\frac{\partial^2 Z_{22}}{\partial z^2}=f_{12}(r)\,e^{-\sqrt{2}z} \quad \ldots.(51)$$

where

$$\frac{\partial^2 R_{12}}{\partial r^2}+\frac{3}{r}\frac{\partial R_{12}}{\partial r}-\frac{4(1+r^2)}{(1-r^2)^2}R_{12} = -\lambda_1^2 R_{12} \qquad \ldots\ldots\ldots(52)$$

$$\frac{\partial^2 R_{22}}{\partial r^2}+\frac{3}{r}\frac{\partial R_{22}}{\partial r}-\frac{4(1+r^2)}{(1-r^2)^2}R_{22} = -\lambda_2^2 R_{22} . \qquad \ldots\ldots\ldots(53)$$

Rearranging equations (50) and (51):

$$\frac{\partial^2 Z_{12}}{\partial z^2}-\lambda_1^2 Z_{12} = f_{13}(r)\,e^{-\sqrt{2}z} \qquad \ldots\ldots\ldots(54)$$

$$\frac{\partial^2 Z_{22}}{\partial z^2}-\lambda_2^2 Z_{22} = f_{14}(r)\,e^{-\sqrt{2}z} \qquad \ldots\ldots\ldots(55)$$

where $f_{13}(r) = \dfrac{f_{11}(r)}{R_{12}}$, $f_{14}(r) = \dfrac{f_{12}(r)}{R_{22}}$.

The general solutions of the equation (54) and (55) are,

$$Z_{12} = C_{11}e^{\lambda_1 z}+C_{12}e^{-\lambda_1 z}+\frac{f_{13}(r)}{2-\lambda_1^2}e^{-\sqrt{2}z} \qquad \ldots\ldots\ldots(56)$$

$$Z_{22} = C_{13}e^{\lambda_2 z}+C_{14}e^{-\lambda_2 z}+\frac{f_{14}(r)}{2-\lambda_2^2}e^{-\sqrt{2}z} . \qquad \ldots\ldots\ldots(57)$$

Since Z_{12} and Z_{22} are not to become infinite when z approaches infinity, then, $C_{11} = C_{13} = 0$.

It is found that the only solution for equation (52) or (53) would be obtained if the eigen value λ_1^2, or λ_2^2 is 2.

Thus, $\lambda_1 = \lambda_2 = \pm\sqrt{2}$.

If the positive values of λ_1 and λ_2 are chosen, equations (56) and (57) can be expressed as follows:

$$Z_{12} = C_{15}\, e^{-\sqrt{2}\, z} \qquad \text{........(58)}$$

$$Z_{22} = C_{16}\, e^{-\sqrt{2}\, z} \quad . \qquad \text{........(59)}$$

Because the only eigen value of λ_1^2 or λ_2^2 is 2, $f_{13}(r)$ and $f_{14}(r)$ must be equal to zero in order to satisfy the condition that value of Z does not increase to infinity. L'hospital's rule is applied for the proof. From equations (52) and (53), solutions for R_{12} and R_{22} are obtained, and it is found that they are proportional with some constant. This relationship is also true for the infinite number of R functions.

Hence, $\frac{R_{11}}{C_4} = \frac{R_{21}}{C_8} = \frac{R_{12}}{C_{17}} = \frac{R_{22}}{C_{18}} = \cdots\cdots$.

Also, the same statement can be applied to the solutions of Z functions, i.e.,

$$e^{-\sqrt{2}\, z} = \frac{Z_{11}}{C_1} = \frac{Z_{21}}{C_7} = \frac{Z_{12}}{C_{15}} = \frac{Z_{22}}{C_{16}} = \cdots\cdots$$

Therefore,

$$V_1 = K V_{11} + K^2 V_{12} + K^3 V_{13} + \cdots = Z_{11} R_{11} \left[K + K^2 \frac{C_{17} C_{15}}{C_1 C_4} + \cdots \right]$$

$$V_2 = Z_{21} R_{21} \left[K + K^2 \frac{C_{16} C_{18}}{C_7 C_8} + \cdots \right] \; .$$

From boundary condition (d), when $z = 0$,

$$V_1(r, 0) = \frac{a(1 - r^2)(4 - 23r^2 + 7r^4)}{288\, K R_c}$$

$$V_2(r, 0) = F_2(r)$$

$$V_1(r, z) = \frac{a(1 - r^2)(4 - 23r^2 + 7r^4)}{288\, K R_c}\, e^{-\sqrt{2}\, z} \qquad \text{....(60)}$$

and if, $C_{19} = C_7 \left[K + K^2 \frac{C_{16} C_{18}}{C_7 C_8} + \cdots \right]$ and $C_{19} R_{21} = F_2(r)$, then, $$V_2(r,z) = F_2(r)\, e^{-\sqrt{2}\,z} \qquad \text{........(61)}$$

Then, $$V = V_1 \cos\phi + V_2 \sin\phi$$

$$= \left[\frac{a(1-r^2)(4-23r^2+7r^4)}{288\,K R_c}\cos\phi + F_2(r)\sin\phi\right] e^{-\sqrt{2}\,z} \qquad \text{....(62)}$$

As a suggestion for the experimental analysis, precise measurements of velocity distribution for each component at the initial section of the pipe would be sufficient for finding the function of r, $F_2(r)$.

As a particular case, the solution of V is available from equation (62) if ϕ is zero,

$$V(r,0,z) = \frac{a(1-r^2)(4-23r^2+7r^4)}{288\,K R_c}\, e^{-\sqrt{2}\,z} \qquad \text{........(63)}$$

Since the length of transition segment can be determined only by using the solution of V, the solutions of U and W were neglected.

DETERMINATION OF LENGTH OF TRANSITION SEGMENT

The solution for the decay of the secondary flow as indicated in equation (63) is an asymptotic function, so that the length of transition segment, L, is an infinity for a complete decay of V to zero.

A more realistic appraisal is to assume that velocity component, V, approaches some small value, δ, when the flow travels to a distance, L, along z axis. The dimensionless value of δ can be made practically equal to zero from the engineering standpoint.

If $\quad V(r,0,L) = \delta \quad$ and $\quad \lim_{z\to\infty} V(r,0,z) = 0$.

From equation (62):

$$\delta = \frac{a(1-r^2)(4-23r^2+7r^4)}{288\, R_c} N e^{-\sqrt{2}L},$$

then,

$$e^{\sqrt{2}L} = \frac{a(1-r^2)(4-23r^2+7r^4)}{288\, R_c \delta} N \qquad \text{........(64)}$$

The relation between L and the various factors affecting secondary flow was shown in the above equation.

In order to appreciate the actual flow case as much as possible, the effect due to radius factor is eliminated from the expression for the relationship. Thus, only V at center axis of the pipe is applied, as an about average value of V, i.e. to set r = 0.

Then,

$$e^{\sqrt{2}L} = \frac{a}{72 R_c \delta} N \qquad \text{........(65)}$$

Equation (65) is noted to be only for the case of laminar flow, i.e., N is less than 2100.

Since most flow cases in practice are so turbulent that the relationship expressed in equation (65) may seldom find application in engineering problems, an attempt to assess the tendency in the relationship for turbulent flow has been made.

By interchanging kinematic viscosity, ν , in equation (65) with average eddy viscosity, ε , for turbulent flow, thus,

$$e^{\sqrt{2}L} = \frac{a}{72 R_c \delta} N \left(\frac{\nu}{\varepsilon}\right) \qquad \text{........(66)}$$

As for the expression for ε in the pipe flow:

$$\varepsilon = \frac{k d \sqrt{g S d}}{9.2} q \qquad \text{........(67)}$$

where $S = \frac{h_L}{L} = \frac{f}{d} \cdot \frac{W_a^2}{2g}$: slope of hydraulic gradient

$q = \frac{y}{a}\left(1 - \frac{y}{a}\right)$:

k : Von Karman's universal constant (usually $= 0.4$)

d : diameter of the pipe, feet

g : gravitational acceleration (ft/sec^2)

y : the distance measured from pipe boundary toward the center along the radius, feet

Since, $\sqrt{gSd} = W_a\sqrt{\frac{f}{2}}$ where f is the friction factor in Darcy's equation $\left(h_L = f \frac{L}{d} \frac{W_a^2}{2g}\right)$

Therefore equation (67) becomes

$$\varepsilon = \frac{k\, d\, W_a \sqrt{f}}{9.2\sqrt{2}} q \qquad \text{........(68)}$$

For engineering purposes, q may be replaced by its average value, q_a : considering $0 \leq \frac{y}{a} \leq 1$ and $\frac{y}{a} = x$, hence, $q_a = \int_0^1 x(1-x)\,dx = \frac{1}{6}$

Then $\qquad \varepsilon = 0.005125\sqrt{f}\,d\,W_a$ . ...(69)

Substituting equation (69) into (66)

$$e^{\sqrt{2}L} = \frac{a\nu N}{(72R_e\delta)(0.005125\sqrt{f}\,d\,W_a)} - 2.71\,\frac{a}{R_e\delta\sqrt{f}} \qquad ....(70)$$

The above equation may be applied for turbulent flow in wholly rough pipes, since pipe friction in wholly rough pipes at high Reynolds numbers will be governed primarily by the size and pattern of the roughness, since the disruption of the laminar film will render viscous action negligible.

For turbulent flow in smooth pipes, Vennard (11) has suggested an approximate relation between f and N that can be substituted into equation (70):

$$f = 0.0032 + \frac{0.221}{N^{0.237}}$$

Then

$$e^{\sqrt{2}L} = 2.71\,\frac{a}{R_e\delta\sqrt{0.0032 + \frac{0.221}{N^{0.237}}}} \qquad ....(71)$$

This equation should be applicable for turbulent flow in smooth pipes.

CALCULATION

Using equations (65), (70) and (71), calculations are made for various values of the parameters, δ and a/R_c:

When $\delta = 0.01$, $a/R_c = 0.25$, and by equation (65),

$$e^{\sqrt{2}L} = 0.347\,N \qquad \text{........(72)}$$

When $\delta = 0.01$, $a/R_c = 0.01$, and by equation (65),

$$e^{\sqrt{2}L} = 0.0139\,N. \qquad \text{........(73)}$$

Equations (72) and (73) are provided for laminar flow, and are plotted as curves A_1 and A_2 in Figure 2.

When $\delta = 0.01$, $a/R_c = 0.25$, and by equation (70),

$$e^{\sqrt{2}L} = 67.8\,\frac{1}{\sqrt{f}}. \qquad \text{........(74)}$$

When $\delta = 0.01$, $a/R_c = 0.01$ and by equation (70),

$$e^{\sqrt{2}L} = 2.7\,\frac{1}{\sqrt{f}}. \qquad \text{........(75)}$$

Equations (74) and (75) are provided for turbulent flow in wholly rough pipes, and are plotted as curves B_1 and B_2 in Figure 2.

Since the relationships in equation (74) and &5) are independent of Reynolds number, the limit in Reynolds number for application of each value of the friction factor, f, was selected from Figure 84, p.191, "Elementary Fluid Mechanics" (11).

In Figure 2, only three values of f were selected with its limitations in Reynolds numbers:

For $f = 0.06$; $N > 20000$.

For $f = 0.035$; $N > 100000$.

For $f = 0.02$; $N > 500000$.

When $\delta = 0.01$, $a/R_c = 0.25$, and by equation (71),

$$e^{\sqrt{2}L} = \frac{67.75}{\sqrt{0.0032 + \frac{0.221}{N^{0.237}}}} \quad \text{.........(76)}$$

When $= 0.01$, $a/R_c = 0.01$, and by equation (71)

$$e^{\sqrt{2}L} = \frac{2.71}{\sqrt{0.0032 + \frac{0.221}{N^{0.237}}}} \quad \text{.........(77)}$$

Equations (76) and (77) are provided for turbulent flow in a smooth pipe, and are plotted as curves D_1 and D_2 in Figure 2. The dotted lines in Figure 2 between B curves or A curves and D curves were drawn in an attempt to premise the relationship in the transition region of those curves.

From equation (65) the relationship between the distance of transition segment and residue, δ, is derived, while the other parameters are fixed as constants.

If the product, $\frac{aN}{72R_c}$, is set as a constant, then

$$e^{\sqrt{2}L} = V(0,0)\frac{1}{\delta}$$

where $V(0,0) = \frac{aN}{72R_c}$ is the velocity component V at the center of inlet section of the pipe.

Let $\frac{V(0,0)}{\delta} = H$, then

$$e^{\sqrt{2}L} = H \quad \text{.........(78)}$$

where H is the ratio of $V(0,0)$ to velocity component V at a distance L from inlet section on the center line of the pipe.

Equation (78) is plotted in Figure 3.

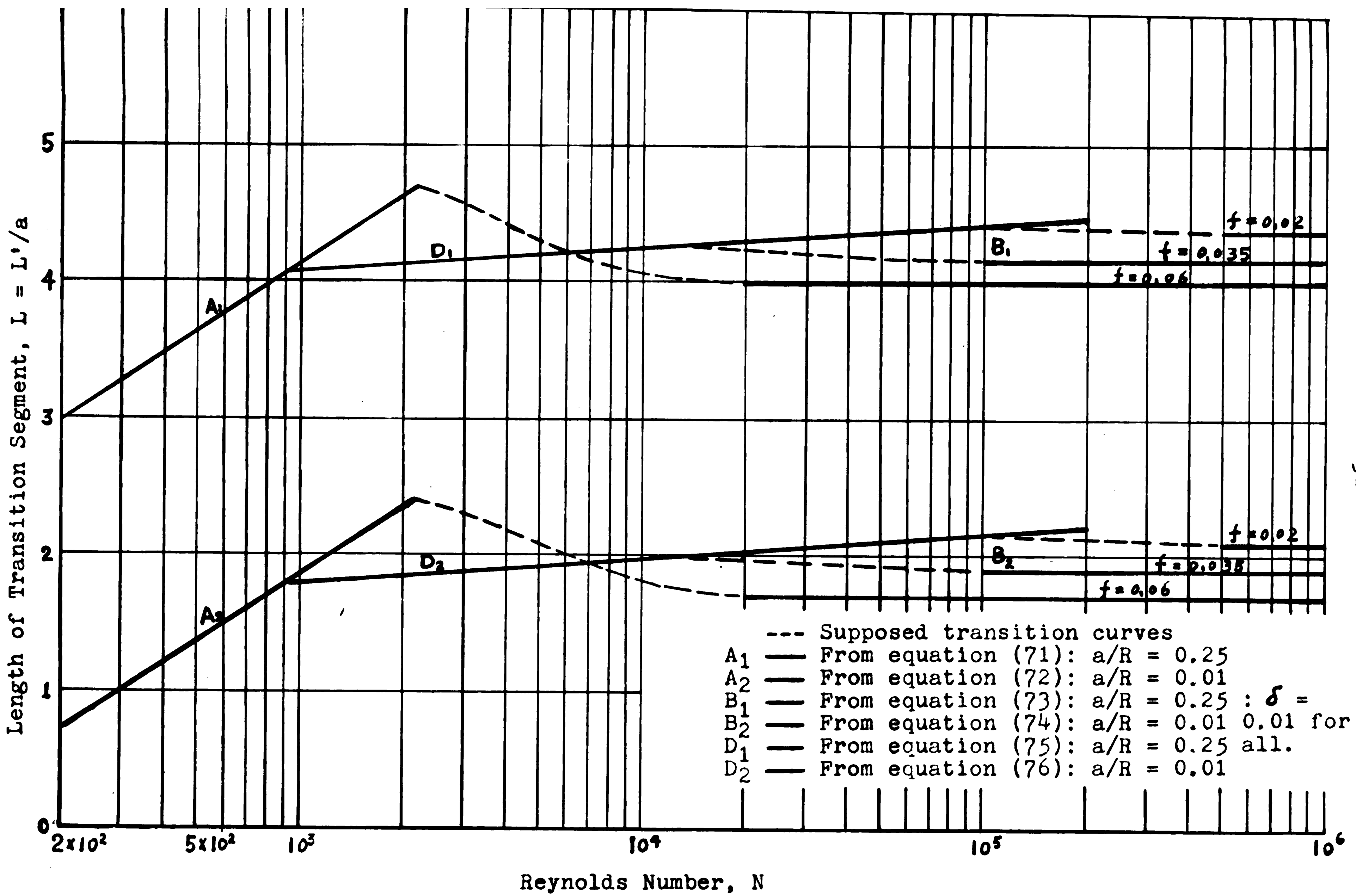

Figure 2. Relationship between the length of transition segment and the Reynolds number or the friction factor.

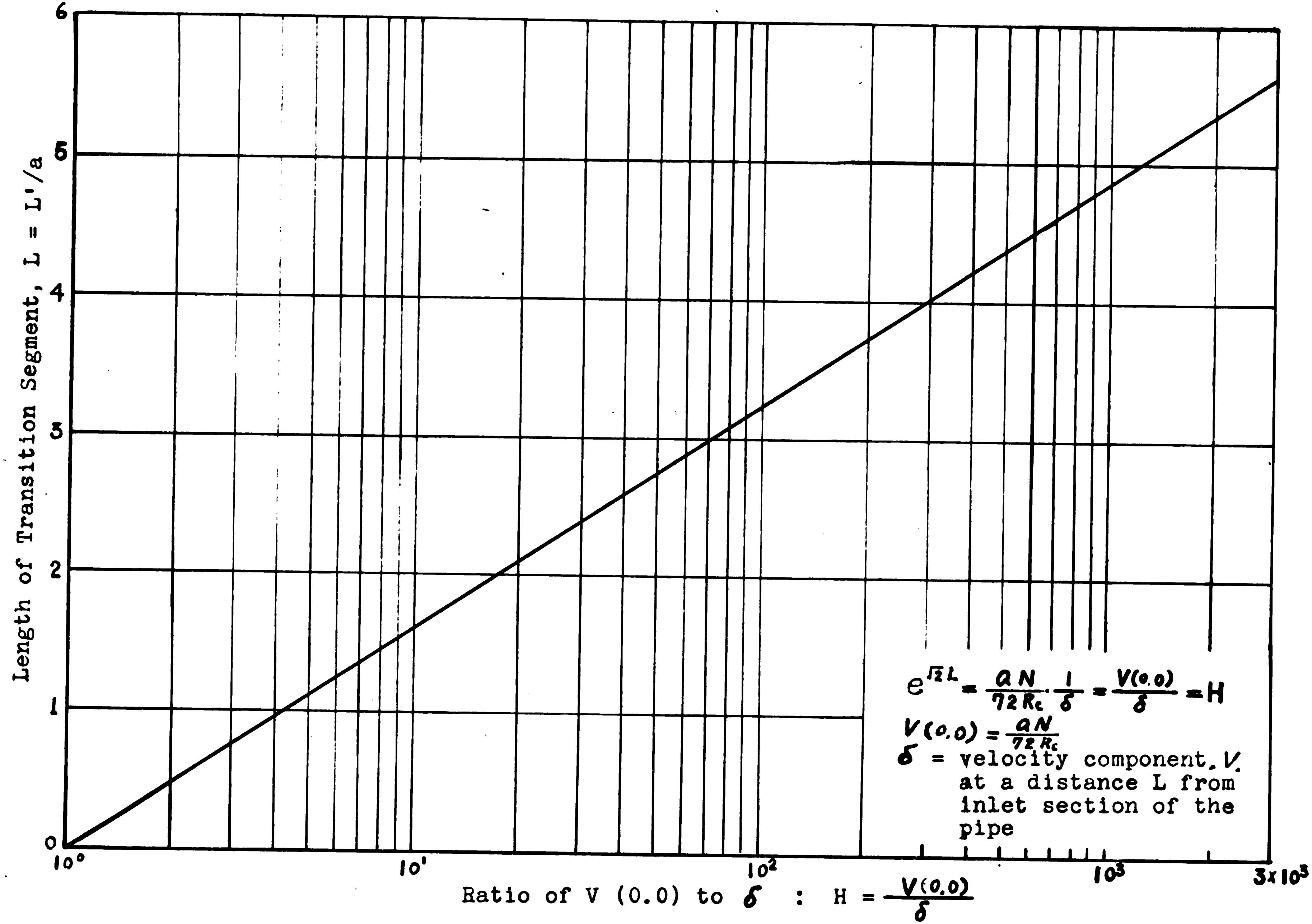

Figure 3. Relationship between the length of transition segment and the ratio of V(0.0) to δ.

DISCUSSION

In the above treatment, several assumptions have been made to simplify the equations of motion describing the flow in the pipe, in order that they might be solved. Most of the assumptions are good approximations when applied to a limited range of Reynolds numbers and the ratios of radius of the pipe to radius of the curvature of the bend which is connected upstream of the straight pipe. In other words, if a limited range of secondary flow intensity is applied on the initial section of the pipe, the approximation is reasonable. Outside these limitations, the equations may give correctly the general nature of the tendency but close numerical agreement should not be expected.

In assumption 2, ϕ was so fixed as a dummy variable, that the number of independent variables was reduced from three to two, and thereby the equations were solved.

For the function of ϕ in the solution was assumed to be the product of trigonometric function, the change of the magnitude of ϕ function would be small along the change of z downstream in comparison with those of the other variables, r and z.

However, it was noticed that the error due to the assumption 2 caused the failure in satisfying the continuity equation. This failure should not be vital to the solution since

since the main interest of this study is concerned with the decaying distance along z axis.

The setup of the boundary conditions was generally considered to be reasonable and proper with an exception of boundary conditions (a) and (d). From a physical consideration of the flow phenomenon at the connection between the bend and the pipe, it is clear that the secondary flow from the bend should be distorted before getting to the connecting section. Moreover, the secondary flow may not be fully developed in a bend of 90 degrees or less. Therefore, adopting the fully developed secondary flow by Dean, associated with an unknown function of r for subordinate flow, F(r), is a necessity, but by no means a completely satisfying boundary condition.

In the process of the solution, it was shown that infinite terms of Z functions or R functions of the velocity component, V, differ only by constants. Therefore, the V function could be expressed with unique function of Z and R by summation of an infinite number of constants which include parameters, K, and arbitrary constants. Consequently, the exact solution of V was derived from approximate solutions obtained by perturbation method.

As mentioned in the boundary condition (a), the function of r, $F_2(r)$, is unknown and can be determined by the experiment to measure the velocity profile of the initial section of the pipe.

For $F_2(r)$ is undetermined, a complete solution has been obtained, only when ϕ is equal to zero (i.e., a direction perpendicular to the plane on which the bend and the pipe are laid).

In order to determine the length of transition segment, based on the solution obtained in the above analysis, the central axis of the pipe (i.e., $r = 0$) is chosen for convenience. Since the secondary flow intensity at the center line or its neighborhood will be distorted less than near the boundary, at the connecting section of the bend and the pipe, one may hope that the error due to the application of improper boundary condition should be minimized. In the determination of transition length, the relationship between the distance, L, required to decay the secondary flow componemt, V, to a residue, δ, and the intensity of secondary flow at the initial section of the pipe, is presented for laminar flow with a limitation that the ratio, a/R_c, is small (about 1 to 5 percent).

Outside these limitations, some attempts were made to indicate the general nature of the tendency in the relationship without expectation of numerical agreement. The relationship with Reynolds number for turbulent flow in a smooth pipe and the relationship with friction factor, f, in wholly rough pipes for turbulent flow were derived in equation (71) and (70) respectively. With a purpose of detecting the tendency of the relationship in actual sprinkler bends, higher ratio (a/R_c=0.25) is applied into the above three equations (65), (70), and (71), and those relationships were presented in the calculation, and illustrated in Figure 2. Some experimental analysis may prove the validity of the attempts.

CONCLUSIONS

A representative relationship between the length of the transition segment, L, and the intensity of secondary flow superimposed was obtained by deriving the function of V at r = 0, d = 0, and z = L, thus,

$$V(0,0,z) = \frac{a}{72 R_c} N e^{-\sqrt{2} z}.$$

If V becomes some small value, δ , when z is L, then

$$\delta = \frac{a}{72 R_c} N e^{-\sqrt{2} L}$$

or $$e^{\sqrt{2} L} = \frac{a N}{72 R_c \delta}$$, for laminar flow.

The tendencies of the relationship for turbulent flow were shown in equations (70) and (71) respectively for wholly rough pipe and smooth pipe.

$$e^{\sqrt{2} L} = \frac{2.71 a}{R_c \delta \sqrt{f}} \qquad \text{........(70)}$$

$$e^{\sqrt{2} L} = \frac{2.71 a}{R_c \delta \sqrt{0.0032 + \frac{0.221}{N^{0.237}}}}. \qquad \text{........(71)}$$

Based on the above equations, the relationships between the length of the transition segment and the Reynolds number, N, the friction factor, f, and residue, δ , were shown in Figure 2 and 3 where equations (72), (73), (74), (75), (76), (77), and (78) are plotted.

In order to meet some practical application for irrigation sprinkler, equations (72), (74) and (76) were provided with $a/R_c = 0.25$ as a characteristics of sprinkler bend, although the ratio might be outside the limitations of analysis.

It is difficult, at this stage of study, to make any evaluation of the result of this theoretical analysis, since there is no proper and reliable experiment which can be used for comparison.

RECOMMENDATIONS FOR FURTHER STUDY

1. Make measurement of velocity distribution at the initial section of the pipe for various intensities of flow. The measurement in the laminar flow should be intensive and precise so that the comparison between the assumed boundary condition (a) and the experimental result will be possible, and F(r) can be determined. Three components of the velocity should be measured individually, if possible.
2. Experimental analysis of the length of the transition segment in the pipe for various flow conditions, roughness of the pipe, and the ratio of a and R_c, is recommended for the evaluation and adjustment of this theoretical analysis.
3. The possibility of applying this analysis for the improvement of sprinkler design should be further investigated.

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