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A METHOD OF GENERATING INTEGRAL REPRESENTATIONS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

Walter William Turner

1963

THESIS

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A METHOD OF GENERATING INTEGRAL REPRESENTATIONS

By

Walter William Turner

AN ABSTRACT OF A THESIS

Submitted to the School of Graduate Studies of Michigan
State University of Agriculture and Applied Science
in fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1963

Approved

ABSTRACT

A METHOD OF GENERATING INTEGRAL REPRESENTATIONS

by Walter William Turner

Imposing the condition that the Schrödinger equation $\nabla^2 U + \phi U = 0$ be simultaneously separable in at least two coordinate systems sharing a coordinate, one obtains functional equations whose solution completely determines ϕ. The special functions obtained by the separated ordinary operators can be related through integral relations by using a well-known integral theorem. With this theorem one can predict the value of the integral involving special functions, and in this way some new integral representations are discovered, which contain as special cases some of the existing integral representations. Thus, a unified theory of these integral representations is obtained.

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ACKNOWLEDGMENTS

The author wishes to express his most sincere thanks to Dr. Alfred Leitner whose deep interest, and devoted supervision have made this investigation possible. He also wishes to thank Dr. Charles P. Wells for his interest in this thesis, and also acknowledge the more indirect influence of the ideas of Dr. Josef Meixner of Aachen, Germany.

The writer is deeply indebted to the National Science Foundation for the financial support that has made this investigation possible.

DEDICATION

To Patricia, Constance, Anne,

Roberta, Nancy, and Rose.

TABLE OF CONTENTS

INTRODUCTION

This represents an attempt to initiate a unifying concept for generating many of the existing integral representations, as well as generating some new integral representations for special functions of Mathematical Physics. Three main ideas are involved; (i) simultaneous separability of linear partial differential equations, (ii) solutions of the Schrödinger equation,

$$(1)\ \nabla^2 U + \phi\,(u_1, u_2, u_3)\, U = 0$$

and (iii) a theorem concerning a definite integral representing the solution of a linear differential equation of two variables (Meixner-7).

In order that (1) be separable $\phi(u_1, u_2, u_3)$ has a definite form as exhibited in Table 1, depending on the coordinate system under consideration. Equation (1) can then be solved by the method of separation of variables in various coordinate systems for various forms of the function ϕ.

It is well known that the scalar Helmholtz equation obtained by letting $\phi = k^2$ is separable in all eleven orthogonal coordinate systems involving ellipsoidal surfaces and their degeneracies, which are listed in Table 1. The case $\phi = k^2$ is discussed quite thoroughly in the literature and the use of the integral theorem is summarized by Meixner (Meixner-7).

In this thesis we ask what is the most general form of ϕ in order that (1) be separable in two coordinate systems, and this is called simultaneous separability. This yields functional equations for each pair considered and these equations determine the form of ϕ.

We will impose the restriction that the pair of coordinate systems in which (1) is to be simultaneously separable share a coordinate. In that case equation (1) can be reduced in each of the two coordinate systems to a partial differential equation involving only two variables, by separating out the common variable.

The form of the integral theorem we wish to apply in this thesis requires that the above restriction be imposed (see page 27).

When each of the reduced partial differential equations is solved by the separation of variables we are led to two special functions. The product of either pair will serve as the kernel of an integral. By using the theorem and integrating over suitable paths, integrals relating the special functions in the other pair are obtained. In such a way it is possible to obtain integral representations.

In 1958 A. Leitner and J. Meixner investigated cylindrical, spherical, and prolate spheroidal coordinates and obtained new integral representations (Leitner-4). We will investigate other pairs of systems of coordinates to obtain new integral representations, which appear in the thesis as (42) page 33, (49) page 43 and (50) page 46. Special cases of these integral representations are also derived and appear as (43) page 34 and (51) page 47. We believe (42), (49), and (50) are new integral representations.

II. SIMULTANEOUS SEPARABILITY

The original idea behind this thesis was to investigate all possible pairs of coordinates in Table 1 which share a common variable but we soon realized this was too ambitious a project to undertake, and furthermore such a project would be of little practical advantage. We could not obtain solutions for a few of the functional equations encountered. Moreover consideration of all possible pairs of coordinates led to special functions that were not of the hypergeometric class, such as Mathieu functions, Lame' polynomials, Spheroidal wave functions and other functions whose theory is complex. We, therefore, limited our investigations to those pairs of coordinates which lead to classes of most interest, namely the hypergeometric class of special functions, and their confluences.

The new integral representations we found contained as special cases many of the existing integral representations and in this way they are an attempt to initiate a unifying concept into the broad area of integral relations.

Our first concern is that of simultaneous separability so we begin by considering pairs of coordinates in Table 1 which share a common variable. The systems all having z in common are rectangular, circular cylindrical, parabolic cylindrical, and elliptic cylindrical. Systems having ϕ as a common coordinate are circular cylindrical, spherical, paraboloidal, prolate spheroidal, and oblate spheroidal. Only two systems have the variable r in common, namely, spherical and

TABLE 1

	u_1	u_2	u_3	$\emptyset \ni \nabla^2 U + \emptyset U = 0$ is separable
1. Rectangular	x	y	z	$A_1(x) + A_2(y) + A_3(z)$
2. Circular Cylindrical	$\rho\cos\phi$	$\rho\sin\phi$	z	$B_1(\rho) + \frac{B_2(\phi)}{\rho^2} + B_3(z)$
3. Spherical	$r\sin\theta\cos\phi$	$r\sin\theta\sin\phi$	$r\cos\theta$	$C_1(r) + \frac{C_2(\theta)}{r^2} + \frac{C_3(\phi)}{r^2\sin^2\theta}$
4. Parabolic Cylindrical	$\xi\eta$	$\frac{1}{2}(\xi^2-\eta^2)$	z	$\frac{D_1(\xi)+D_2(\eta)}{\xi^2+\eta^2} + D_3(z)$
5. Sphero-conal	$\frac{r}{\beta}\sqrt{(\beta^2+\zeta^2)(\beta^2-\eta^2)}$	$\frac{r\zeta\eta}{\alpha\beta}$	$\frac{r}{\alpha}\sqrt{(\alpha^2-\zeta^2)(\alpha^2-\eta^2)}$	$E_1(r) + \frac{E_2(\eta)+E_3(\zeta)}{r^2(\eta^2+\zeta^2)}$
6. Paraboloidal	$\xi\eta\cos\phi$	$\xi\eta\sin\phi$	$\frac{1}{2}(\xi^2-\eta^2)$	$\frac{F_1(\xi)+F_2(\eta)}{\xi^2+\eta^2} + \frac{F_3(\phi)}{\xi^2\eta^2}$
7. Elliptic Cylindrical	$c\xi\eta$	$c\sqrt{(\xi^2-1)(1-\eta^2)}$	z	$\frac{G_1(\xi)+G_2(\eta)}{c^2(\xi^2-\eta^2)} + G_3(z)$
8. Prolate Spheroidal	$a\sqrt{(\xi^2-1)(1-\eta^2)}\cos\phi$	$a\sqrt{(\xi^2-1)(1-\eta^2)}\sin\phi$	$a\xi\eta$	$\frac{H_1(\xi)+H_2(\eta)}{a^2(\xi^2-\eta^2)} + \frac{H_3(\phi)}{a^2(\xi^2-1)(1-\eta^2)}$
9. Oblate Spheroidal	$a\sqrt{(\xi^2+1)(1-\eta^2)}\cos\phi$	$a\sqrt{(\xi^2+1)(1-\eta^2)}\sin\phi$	$a\xi\eta$	$\frac{I_1(\xi)+I_2(\eta)}{a^2(\xi^2+\eta^2)} + \frac{I_3(\phi)}{a^2(\xi^2+1)(1-\eta^2)}$
10. Ellipsoidal	$\sqrt{\frac{(\xi+\alpha)(\eta+\alpha)(\zeta+\alpha)}{(\beta-\alpha)(\gamma-\alpha)}}$	$\sqrt{\frac{(\xi+\beta)(\eta+\beta)(\zeta+\beta)}{(\alpha-\beta)(\gamma-\beta)}}$	$\sqrt{\frac{(\xi+\gamma)(\eta+\gamma)(\zeta+\gamma)}{(\alpha-\gamma)(\beta-\gamma)}}$	
11. Parabolic Ellipsoidal	$\frac{\xi+\eta+\zeta-a-b}{2}$	$\sqrt{\frac{(a-\xi)(a-\eta)(a-\zeta)}{b-a}}$	$\sqrt{\frac{(b-\xi)(b-\eta)(b-\zeta)}{a-b}}$	

sphero-conal. These can be considered pairwise in thirteen ways, but as we previously noted, not all of these combinations can be separated simultaneously.

We will consider, as an example, the simultaneous separability of rectangular and cylindrical coordinates in detail and merely list the results of other pairs. Referring to Table 1 we see the forms of ϕ for rectangular and cylindrical coordinates are respectively:

$$A_1(x) + A_2(y) + A_3(z)$$

$$\text{and } B_1(\rho) + \frac{B_2(\phi)}{\rho^2} + B_3(z)$$

Since z is the common variable we have $A_3(z) = B_3(z)$, but otherwise arbitrary, and we must solve the functional equation

$$(2) \quad B_1(\rho) + \frac{B_2(\phi)}{\rho^2} = A_1(x) + A_2(y)$$

We proceed by differentiating (2) with respect to x obtaining

$$\frac{dA_1}{dx} = \frac{dB_1}{d\rho}\frac{\partial\rho}{\partial x} + \frac{d}{d\rho}\left(\frac{1}{\rho^2}\right)\frac{\partial\rho}{\partial x} B_2 + \frac{1}{\rho^2}\frac{dB_2}{d\phi}\frac{\partial\phi}{\partial x}$$

Now since $\rho^2 = x^2 + y^2$ and $\phi = \tan^{-1}\frac{y}{x}$ it is obvious that

$$\frac{\partial\rho}{\partial x} = \frac{x}{\rho}, \quad \frac{\partial\rho}{\partial y} = \frac{y}{\rho}, \quad \frac{\partial\phi}{\partial x} = \frac{-y}{\rho^2} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = \frac{x}{\rho^2}$$

We now use these relations to obtain

$$\frac{dA_1}{dx} = \frac{x}{\rho}\frac{dB_1}{d\rho} + \frac{-2x}{\rho^4} B_2 - \frac{y}{\rho^4}\frac{dB_2}{d\phi}$$

Now we take the derivative with respect to y to obtain

$$0 = x\frac{d}{d\rho}\left(\frac{1}{\rho}\right)\frac{\partial\rho}{\partial y}\;\frac{dB_1}{d\rho} + \frac{x}{\rho}\;\frac{d^2B_1}{d\rho^2}\;\frac{\partial\rho}{\partial y} - 2x\frac{d}{d\rho}\left(\frac{1}{\rho^4}\right)\frac{\partial\rho}{\partial y}B_2 - \frac{2x}{\rho^4}\frac{dB_2}{d\phi}\frac{\partial\phi}{\partial y}$$

$$-\frac{1}{\rho^4}\;\frac{dB_2}{d\phi} - y\;\frac{d}{d\rho}\left(\frac{1}{\rho^4}\right)\;\frac{\partial\rho}{\partial y}\frac{dB_2}{d\phi} - \frac{y}{\rho^4}\;\frac{d^2B_2}{d\phi^2}\;\frac{\partial\phi}{\partial y}$$

$$0 = \frac{-xy}{\rho^3}\;\frac{dB_1}{d\rho} + \frac{xy}{\rho^2}\;\frac{d^2B_1}{d\rho^2} + \frac{8xy}{\rho^6}B_2 - \frac{2x^2}{\rho^6}\;\frac{dB_2}{d\phi} - \frac{1}{\rho^4}\frac{dB_2}{d\phi} + \frac{4y^2}{\rho^6}\frac{dB_2}{d\phi} - \frac{xy}{\rho^6}\frac{d^2B_2}{d\phi^2}$$

$$0 = \frac{xy}{\rho^2}\left[\frac{d^2B_1}{d\rho^2} - \frac{1}{\rho}\;\frac{dB_1}{d\rho}\right] + \frac{xy}{\rho^6}\left[8\,B_2 - \frac{d^2B_2}{d\phi^2}\right] + \frac{dB_2}{d\phi}\left[\frac{3}{\rho^4} - \frac{6\rho^2\cos^2\phi}{\rho^6}\right]$$

Now we separate the variables to obtain

$$(3)\quad \rho^4\;\frac{d^2B_1}{d\rho^2} - \rho^3\;\frac{dB_1}{d\rho} = \frac{d^2B_2}{d\phi^2} + \frac{6\cos^2\phi - 3}{\cos\phi\,\sin\phi}\;\frac{dB_2}{d\phi} - 8\,B_2$$

Now (3) yields two ordinary differential equations which are

$$\frac{d^2B_1}{d\rho^2} - \frac{1}{\rho}\;\frac{dB_1}{d\rho} = \frac{A}{\rho^4}$$

$$\frac{d^2B_2}{d\phi^2} + \frac{6\cos^2\phi - 3}{\cos\phi\,\sin\phi}\;\frac{dB_2}{d\phi} - 8\,B_2 = A$$

The solutions to these equations are:

$$B_1(\rho) = -a^2\rho^2 + k^2 + \frac{A}{8}\;\frac{1}{\rho^2}$$

and $B_2(\phi) = \dfrac{-A}{8} + \dfrac{\frac{1}{4} - 4\sigma^2}{\cos^2\phi} + \dfrac{\frac{1}{4} - 4\tau^2}{\sin^2\phi}$

This determines the particular form that (2) must assume in order that the Schrödinger equation be separable in both rectangular and cylindrical coordinates simultaneously.

$$S(1,2)\quad \begin{cases} B_1(\rho) + \dfrac{B_2(\phi)}{\rho^2} = k^2 - a^2\rho^2 + \dfrac{\frac{1}{4} - 4\sigma^2}{\cos^2\phi} + \dfrac{\frac{1}{4} - 4\tau^2}{\sin^2\phi} \\[2ex] A_1(x) + A_2(y) = k^2 - a^2(x^2 + y^2) + \dfrac{\frac{1}{4} - 4\sigma^2}{x^2} + \dfrac{\frac{1}{4} - 4\tau^2}{y^2} \end{cases}$$

The problem of solving the functional equation in other pairs of coordinate systems is similar to the one just illustrated and the detailed calculations will not be given. The results will be found in Table 2.

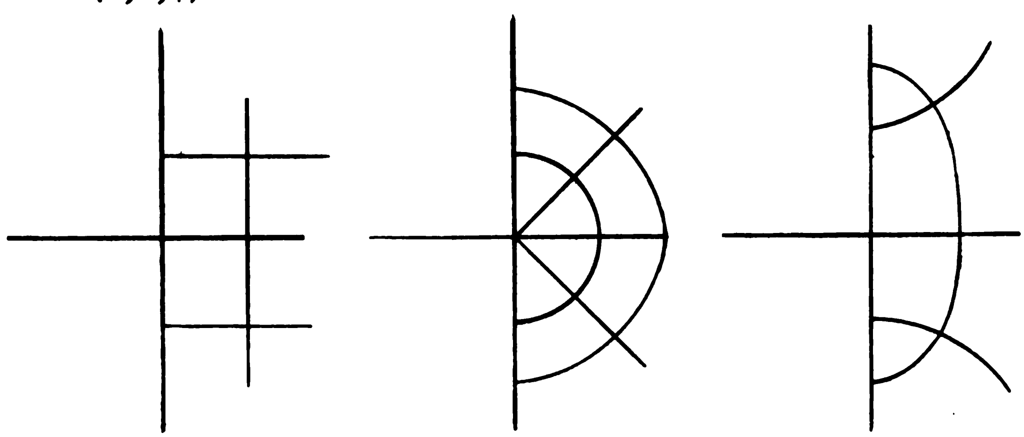
To denote various coordinate systems we shall use the numbers as they appear in the first column of Table 1. Whenever the same greek letters are used for two distinct pairs of coordinates, for reasons of tradition, we shall subscript the variables according to the numbering in Table 1. For example (ξ_7, η_7, z) are the elliptic cylindrical coordinates.

The solution to the problem of simultaneous separability of any given pair of coordinate systems is denoted $S(i,j)$, where $i, j = 1, 2, \ldots 10$. So since rectangular coordinates in Table 1 correspond to the number 1 and cylindrical to the number 2, we denote the solution of the problem of simultaneous separability in rectangular and cylindrical coordinates by $S(1,2)$. Likewise $S(2,6)$ refers to the solution of the problem of simultaneous separability of cylindrical and paraboloidal coordinates and so forth.

Furthermore we found that the same function of ϕ arose as the solution to the problem of simultaneous separability for pairs of coordinates more than once. In fact we found that S(1,2), S(1,7) and S(2,7) have the same solution, as do S(2,3), S(2,8) and S(3,8). We denoted these triplets by S(1,2,7) and S(2,3,8) respectively in Table 2.

Table 2 has been so arranged as to call attention to the fact that ϕ has the same form when the table is read horizontally. This symmetry differs from the one discussed above, since the previous symmetry deals with the same function of space whereas this new symmetry involved equality of form and the variables are different functions of space. This equality of form can easily be explained by geometrical considerations.

Consider (2,3,8): when the azimuth ϕ is held fixed in each of these systems, we obtain three two dimensional coordinate graphs which are illustrated in Figure 1. We obtain the same graphs when we hold z fixed in (1,2,7).



(a) (b) (c)

Figure 1. Similarities between (2,3,8) and (1,2,7)

Azimuth ϕ held constant in (2,3,8) and z held constant in (1,2,7)

(a) Picture of lines of constant ρ,z (or x and y)

(b) Picture of lines of constant θ, r (or ϕ, ρ).

(c) Picture of lines of constant ξ_8, η_8 (or ξ_7, η_7)

TABLE 2

$\mathcal{S}(1,2,7)$

$$A_1(x) + A_2(y) = k^2 - a^2(x^2+y^2) + \frac{1/4 - 4\sigma^2}{x^2} + \frac{1/4 - 4\tau^2}{y^2}$$

$$B_1(\rho) + \frac{B_2(\phi)}{\rho^2} = k^2 - a^2\rho^2 + \frac{1/4 - 4\sigma^2}{\rho^2\cos^2\phi} + \frac{1/4 - 4\tau^2}{\rho^2\sin^2\phi}$$

$$\frac{G_1(\xi) + G_2(\eta)}{\xi^2+\eta^2} = k^2 - a^2c^2(\xi^2+\eta^2-1) + \frac{1/4 - 4\sigma^2}{c^2\xi^2\eta^2} - \frac{1/4 - 4\tau^2}{c^2(\xi^2-1)(1-\eta^2)}$$

$\mathcal{S}(2,3,8)$

$$B_1(\rho) + \frac{B_2(\phi)}{\rho^2} = k^2 + b(\rho^2+z^2) + \frac{c}{z^2} + \frac{d}{\rho^2}$$

$$C_1(r) + \frac{C_2(\theta)}{r^2} = k^2 + br^2 + \frac{c}{r^2\cos^2\theta} + \frac{d}{r^2\sin^2\theta}$$

$$\frac{H_1(\xi) + H_2(\eta)}{\xi^2+\eta^2} = k^2 + ba^2(\xi^2+\eta^2-1) + \frac{c}{a^2\xi^2\eta^2} - \frac{d}{a^2(\xi^2-1)(1-\eta^2)}$$

$\mathcal{S}(1,4)$

$$A_1(x) + A_2(y) = k^2 + 2dy + \frac{b}{x^2} - 4\tau^2y^2 - \tau^2x^2$$

$$\frac{D_1(\xi) + D_2(\eta)}{\xi^2+\eta^2} = k^2 + d(\xi^2-\eta^2) + \frac{b}{\xi^2\eta^2} - \tau^2(\xi^2-\eta^2)^2 - \tau^2\xi^2\eta^2$$

$\mathcal{S}(2,6)$

$$B_1(\rho) + B_3(z) = k^2 + 2dz + \frac{b}{\rho^2} - 4\tau^2z^2 - \tau^2\rho^2$$

$$\frac{F_1(\xi) + F_2(\eta)}{\xi^2+\eta^2} = k^2 + d(\xi^2-\eta^2) + \frac{b}{\xi^2\eta^2} - \tau^2(\xi^2-\eta^2)^2 - \tau^2\xi^2\eta^2$$

$\mathcal{S}(2,4)$

$$B_1(\rho) + \frac{B_2(\phi)}{\rho^2} = k^2 + \frac{a}{\rho} + \frac{b\sin\phi + d}{\rho^2\cos^2\phi}$$

$$\frac{D_1(\xi) + D_2(\eta)}{\xi^2+\eta^2} = k^2 + \frac{2a}{\xi^2+\eta^2} + \frac{b}{\xi^2\eta^2}\left(\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right) + \frac{d}{\xi^2\eta^2}$$

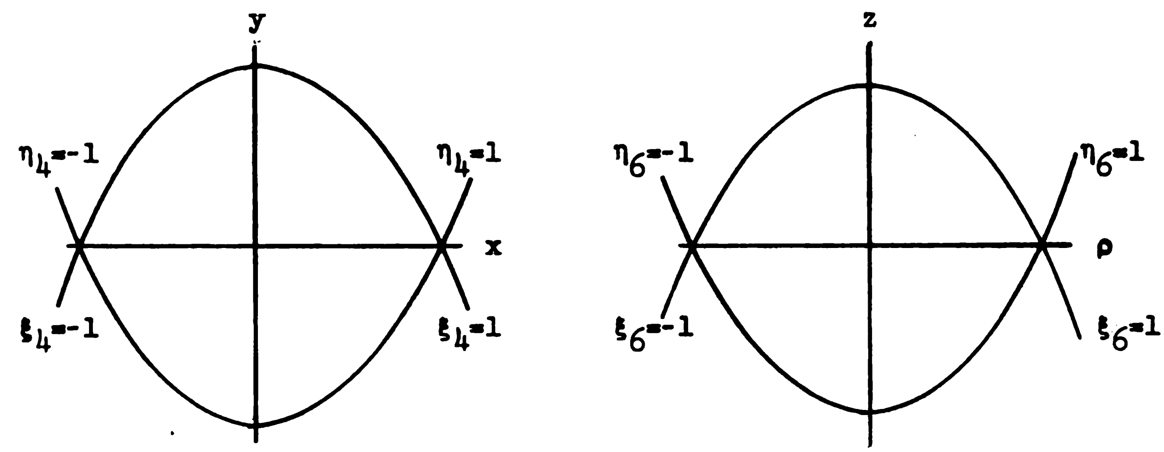
$\mathcal{S}(3,6)$

$$C_1(r) + \frac{C_2(\theta)}{r^2} = k^2 + \frac{a}{r} + \frac{b\cos\theta + d}{r^2\sin^2\theta}$$

$$\frac{F_1(\xi) + F_2(\eta)}{\xi^2+\eta^2} = k^2 + \frac{2a}{\xi^2+\eta^2} + \frac{b}{\xi^2\eta^2}\left(\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right) + \frac{d}{\xi^2\eta^2}$$

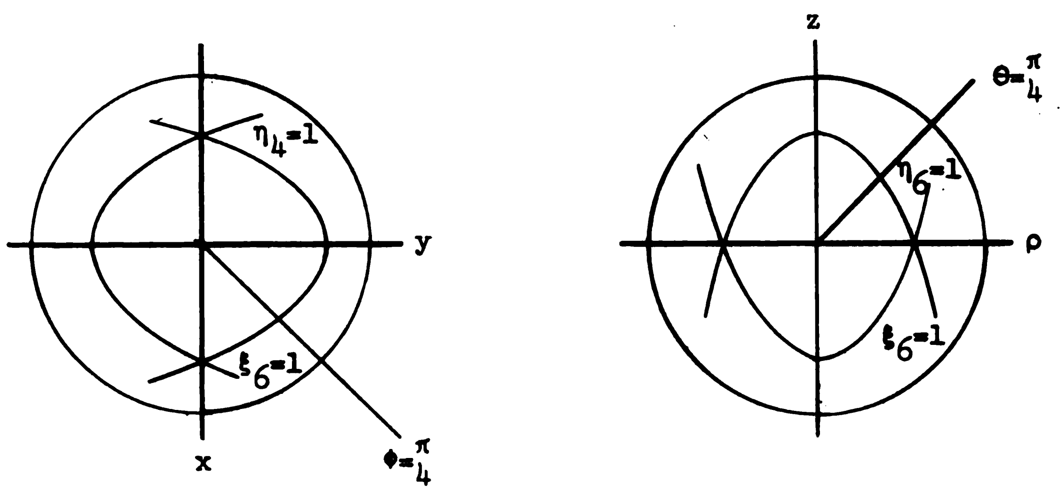
Figure 2 indicates why S(1,4) and S(2,6) likewise yield the same functional form for ϕ even though the variables are not the same functions of space. Figure 3 indicates why S(2,4) and S(3,6) also yield the same functional form for ϕ. Also notice that in Figure 3 θ corresponds to $90^\circ - \Phi$.

We have not been able to solve the functional equations for the coordinate pairs (6,8) and (4,7) but they too have geometrical similarities.



Plane of constant z　　　　Plane of constant Φ

Figure 2. Similarities Between (1,4) and (2,6)



Plane of constant z　　　　Plane of constant Φ

Figure 3. Similarities Between (2,4) and (3,6)

III. SOLUTIONS OF THE SCHRÖDINGER EQUATION

In this section we will solve the various forms of the Schrödinger equation, which are obtained when we use the results of the various simultaneous separabilities. Usually we will indicate two or more solutions of the ordinary differential equations that occur after we assume a separated solution of the partial differential equation. Finally we will summarize the results of this section in Tables 3 through 5.

A. S(1,2)

The simultaneous separability form of this equation is:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \left[k^2 - a^2(x^2+y^2) + \frac{\frac{1}{4} - 4\sigma^2}{x^2} + \frac{\frac{1}{4} - 4\tau^2}{y^2} + A_3(z)\right] U = 0$$

We assume a solution of the form $U = k_1(x)k_2(y)\, e^{i\mu z}$ to obtain

$$\frac{d^2 k_1}{dx^2} + \left[k^2 - \mu^2 - c_o - a^2x^2 + \frac{\frac{1}{4} - 4\sigma^2}{x^2}\right] k_1 = 0 \quad ,$$

and $$\frac{d^2 k_2}{dy^2} + \left[c_o - a^2y^2 + \frac{\frac{1}{4} - 4\tau^2}{y^2}\right] k_2 = 0 \quad .$$

These two equations can be reduced to the form of Whittaker's differential equation. Thus we can find $k_1(x)$ and $k_2(y)$ to be:

(4) $$k_1(x) = \frac{1}{\sqrt{x}} W_{\nu,\sigma}(ax^2) \text{ or } \frac{1}{\sqrt{x}} M_{\nu,\sigma}(ax^2), \quad \nu = \frac{k^2-\mu^2-c_o}{4a} \quad ;$$

(5) $$k_2(y) = \frac{1}{\sqrt{y}} W_{\gamma,\tau}(ay^2) \text{ or } \frac{1}{\sqrt{y}} M_{\gamma,\tau}(ay^2), \quad \gamma = \frac{c_o}{4a} \quad .$$

In cylindrical coordinates the same equation is:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2}+\frac{\partial^2 U}{\partial z^2}+\left[k^2-a^2\rho^2+\frac{\frac{1}{4}-4\sigma^2}{\rho^2\cos^2\phi}+\frac{\frac{1}{4}-4\tau^2}{\rho^2\sin^2\phi}+A_3(z)\right]U=0$$

We assume a solution of the form $U = g_1(\rho)g_2(\phi)e^{i\mu z}$ to obtain

$$\frac{d^2g_1}{dp^2}+\frac{1}{\rho}\frac{dg_1}{d\rho}+\left[k^2-\mu^2-a^2\rho^2-\frac{c_1}{\rho^2}\right]g_1=0 \quad ,$$

and $$\frac{d^2g_2}{d\phi^2}+\left[c_1+\frac{\frac{1}{4}-4\sigma^2}{\cos^2\phi}+\frac{\frac{1}{4}-4\tau^2}{\sin^2\phi}\right]g_2=0 \quad .$$

The first of these equations can be reduced to the form of Whittaker's differential equation, and has as a solution

$$(6)\quad g_1(\rho)=\frac{1}{\rho}\,W_{\nu+\gamma,\epsilon}\,(a\rho^2)\ \text{or}\ \frac{1}{\rho}M_{\nu+\gamma,\epsilon}\,(a\rho^2),\ \epsilon=\frac{c_1}{2} \quad .$$

The second equation can be transformed to a generalized hypergeometric function (Leitner and Meixner - 5). The solution is

$$(7)\ g_2\,(\phi)=(1-\cos^2\phi)^{\frac{1}{4}}\ \tilde{\not{g}}\,{}^{-2\tau,\ 2\sigma-\frac{1}{2}}_{2\epsilon-\frac{1}{2}}\ (\cos\phi) \quad .$$

B. S(2,4)

In parabolic cylinder coordinates the equation is:

$$\frac{1}{\xi^2+\eta^2}\left[\frac{\partial^2 U}{\partial\xi^2}+\frac{\partial^2 U}{\partial\eta^2}\right]+\frac{\partial^2 U}{\partial z^2}+\left[k^2+\frac{2a}{\xi^2+\eta^2}+\frac{b}{\xi^2\eta^2}\,\frac{\xi^2-\eta^2}{\xi^2+\eta^2}+\frac{d}{\xi^2\eta^2}+B_3(z)\right]U=0$$

We assume a solution of the form $U = k_1(\xi)\ k_2\ (\eta)\ e^{i\mu z}$ to obtain

$$\frac{d^2k_1}{d\xi^2}+\left[k^2+2a-c_o-\mu^2\xi^2+\frac{d-b}{\xi^2}\right]k_1=0 \quad ,$$

$$\text{and } \frac{d^2k_2}{d\eta^2} + \left[c_o - \mu^2\eta^2 + \frac{d+b}{\eta^2}\right]k_2 = 0 \quad .$$

Both of these differential equations can be reduced to the form of Whittaker's differential equations,

$$(8)\quad k_1(\xi) = \frac{1}{\sqrt{\xi}} W_{\alpha,\tau}(\mu\xi^2) \text{ or } \frac{1}{\sqrt{\xi}} M_{\alpha,\tau}\,(\mu\xi^2)\ ;\ \alpha = \frac{k^2+2a-c_o}{4\mu},\ \tau = \frac{\sqrt{1+4b-4d}}{4},$$

$$(9)\quad k_2(\eta) = \frac{1}{\sqrt{\eta}} W_{\nu,\sigma}(\mu\eta^2) \text{ or } \frac{1}{\sqrt{\eta}} M_{\nu,\sigma}\,(\mu\eta^2);\ \nu = \frac{c_o}{4\mu},\ \sigma = \frac{\sqrt{1-4b-4d}}{4} \quad .$$

In cylindrical coordinates the Schrödinger equation is:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2} + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + \frac{a}{\rho} + \frac{b\sin\phi}{\rho^2\cos^2\phi} + \frac{d}{\rho^2\cos^2\phi} + B_3(z)\right]U = 0$$

We assume a solution of the form $U = g_1(\rho)g_2(\phi)e^{i\mu z}$ to obtain

$$\frac{d^2g_1}{d\rho^2} + \frac{1}{\rho}\frac{dg_1}{d\rho} + \left[k^2 - \mu^2 + \frac{a}{\rho} - \frac{\epsilon^2}{\rho^2}\right]g_1 = 0\ ,$$

$$\text{and } \frac{d^2g_2}{d\phi^2} + \left[\epsilon^2 + \frac{b\sin\phi + d}{\cos^2\phi}\right]g_2 = 0 \quad .$$

The first of these can be reduced to the form of Whittaker's differential equations:

$$(10)\quad g_1(\rho) = \frac{1}{\sqrt{\rho}}\, W_{\lambda,\epsilon}(2\sqrt{\mu^2-k^2}\,\rho) \text{ or } \frac{1}{\sqrt{\rho}}\, M_{\lambda,\epsilon}\,(2\sqrt{\mu^2-k^2}\,\rho),\ \lambda = \frac{4}{2\sqrt{\mu^2-k^2}}$$

The second equation can be transformed to a generalized hypergeometric function(**Leitner-4**):

$$(11)\quad g_2(\phi) = (1-\sin^2\phi)^{\frac{1}{4}}\ \tilde{\phi}\begin{matrix} 2\sigma,\ 2\tau-\frac{1}{2} \\ 2\epsilon - \frac{1}{2}\end{matrix}\ (\sin\phi)$$

This function can also be reduced to the Gauss hypergeometric equation,

$$(12)\quad g_2(\phi) = \frac{{}_2F_1\left[\frac{1}{2} - \epsilon - \sigma - \tau, \frac{1}{2} + \epsilon - \sigma - \tau;\ 1-2\tau;\ \cos^2(\frac{\pi}{4} - \frac{\phi}{2})\right]}{\cos^{-\frac{1}{2}}\phi\,(1+\sin\phi)^{\tau}\,(1-\sin\phi)^{\sigma}},$$

$$\text{or}\quad \frac{{}_2F_1\left[\frac{1}{2} - \epsilon + \tau - \sigma;\ \frac{1}{2} + \epsilon + \tau - \sigma;\ 1 + 2\tau;\ \cos^2(\frac{\pi}{4} - \frac{\phi}{2})\right]}{\cos^{-\frac{1}{2}}\phi\,(1+\sin\phi)^{-\tau}\,(1-\sin\phi)^{\sigma}}$$

C. S(3,6)

In paraboloidal coordinates the equation is:

$$\frac{1}{\xi+\eta^2}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}(\xi\frac{\partial U}{\partial\xi}) + \frac{1}{\eta}\frac{\partial}{\partial\eta}(\eta\frac{\partial U}{\partial\eta})\right] + \frac{1}{\xi^2\eta^2}\frac{\partial^2 U}{\partial\phi^2} + F(\xi,\eta,\phi)\,U = 0$$

where $F(\xi,\eta,\phi) = k^2 + \dfrac{2a}{\xi^2+\eta^2} + \dfrac{b}{\xi^2\eta^2}\left(\dfrac{\xi^2\eta^2}{\xi^2+\eta^2}\right) + \dfrac{d + c_3(\phi)}{\xi^2\eta^2}$

We assume a solution of the form $U = k_1(\xi)k_2(\eta)\,e^{i\mu\phi}$ to obtain

$$\frac{d^2k_1}{d\xi^2} + \frac{1}{\xi}\frac{dk_1}{d\xi} + \left[2a - c_o + k^2\xi^2 + \frac{d - b - \mu^2}{\xi^2}\right]k_1 = 0\ ,$$

$$\text{and}\quad \frac{d^2k_2}{d\eta^2} + \frac{1}{\eta}\frac{dk_2}{d\eta} + \left[c_o + k^2\eta^2 + \frac{d + b - \mu^2}{\eta^2}\right]k_2 = 0\ .$$

Both of these equations can be reduced to the form of Whittaker's differential equation,

$$(13)\quad k_1(\xi) = \frac{1}{\xi}W_{\nu,\sigma}(ki\xi^2) \text{ or } \frac{1}{\xi}M_{\nu,\sigma}(ki\xi^2);\quad \nu = \frac{(c_2-2a)i}{4k},\quad \sigma = \frac{\sqrt{\mu^2+b-d}}{2},$$

(14) $k_2(\eta) - \frac{1}{\eta} W_{\gamma,\tau}\ (ki\eta^2)$ or $\frac{1}{\eta} M_{\gamma,\tau}\ (ki\eta^2)$; $\tau = \frac{\sqrt{\mu^2-b-d}}{2}$, $\gamma = \frac{-c_2 i}{4k}$.

For spherical coordinates the Schrödinger equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r^2 \frac{\partial U}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial U}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 U}{\partial\phi^2} + C(r,\theta,\phi)\,U = 0,$$

where $C(r,\theta,\phi) = \frac{a}{r} + k^2 + \frac{b\cos\theta + d + C_3(\phi)}{r^2\sin^2\theta}$

We assume a solution of the form $U = g_1(r) g_2(\theta) e^{i\mu\phi}$ to obtain

$$r^2 \frac{d^2 g_1}{dr^2} + 2r\frac{dg_1}{dr} + \left[ar + k^2 r^2 - c_1\right] g_1 = 0,$$

and $$\frac{d^2 g_2}{d\theta^2} + \frac{\cos\theta}{\sin\theta}\frac{dg_2}{d\theta} + \left[c_1 + \frac{d-\mu^2 + b\cos\theta}{\sin^2\theta}\right] g_2 = 0 .$$

The first of these equations can be reduced to the form of Whittaker's differential equation:

(15) $g_1(r) = \frac{1}{r} W_{\nu+\gamma,\epsilon}\ (2kir)$ or $\frac{1}{r} M_{\nu+\gamma,\epsilon}\ (2kir)$, $\epsilon = \frac{\sqrt{4c_1+1}}{2}$

The second equation can be reduced to a generalized hypergeometric equation (Leitner-5):

(16) $g_2(\theta) = \tilde{\phi}^{\,2\tau,2\sigma-\frac{1}{2}}_{\,2\epsilon-\frac{1}{2}}\ (\cos\frac{\theta}{2})$

D. S(1,4)

The form that ϕ assumes in the case S(1,4) is quite complicated, because the separated ordinary equations cannot be reduced to recognized special functions. Certain special cases can be readily solved and so this section is subdivided into three special cases.

$$S(1,4)\quad \phi = \begin{cases} A_1(x) + A_2(y) = k^2 + 2dy + \dfrac{b}{x^2} - 4\tau^2y^2 - \tau^2x^2 \\[2ex] D_1(\xi) + D_2(\eta) = k^2(\xi^2+\eta^2)+d(\xi^4-\eta^2)+\dfrac{b}{\xi^2}+\dfrac{b}{\eta^2} - \tau^2(\xi^6+\eta^6) \end{cases}$$

Case (1) $d = 0 \quad \tau = 0 \quad b = \frac{1}{4} - \sigma^2$

In this case for rectangular coordinates the Schrödinger equation becomes:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + \frac{\frac{1}{4} - \sigma^2}{x^2} + A_3(z) \right] U = 0$$

We assume a solution of the form $U = k_1(x)k_2(y)e^{i\mu z}$ to obtain

$$\frac{d^2k_1}{dx^2} + \left[k^2 - \mu^2 - \gamma^2 + \frac{\frac{1}{4} - \sigma^2}{x^2} \right] k_1 = 0 \,,$$

and $\dfrac{d^2k_2}{dy^2} + \gamma^2 k_2 = 0$.

The first equation can be reduced to the form of Bessel's differential equation,

(17) $k_1(x) = x^{\frac{1}{2}} Z_\sigma (i \sqrt{a^2+\gamma^2}\, x)$, $a = \sqrt{\mu^2-k^2}$,

where Z_σ is any cylinder function.

The second equation is easily solved.

(18) $k_2(y) = \sin\gamma y$ or $\cos\gamma y$

For parabolic cylinder coordinates we have:

$$\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2}\right) + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + \frac{\frac{1}{4}-\sigma^2}{\xi^2+\eta^2}\left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right) + A_3(z) \right] U = 0$$

We assume a solution of the form $U = g_1(\xi)g_2(\eta)e^{i\mu z}$ to obtain

$$\frac{d^2 g_1}{d\xi^2} + \left[c_1 + (k^2-\mu^2)\ \xi^2 + \frac{\frac{1}{4}-\sigma^2}{\xi^2} \right] g_1 = 0 \quad ,$$

and $$\frac{d^2 g_2}{d\eta^2} + \left[-c_1 + (k^2-\mu^2)\ \eta^2 + \frac{\frac{1}{4}-\sigma^2}{\eta^2} \right] g_2 = 0 \ .$$

Both of these equations can be reduced to Whittaker's differential equation,

(19) $g_1(\xi) = \frac{1}{\sqrt{\xi}}\, W_{\epsilon,\frac{\sigma}{2}}\ (a\xi^2)$ or $\frac{1}{\sqrt{\xi}}\, M_{\epsilon,\frac{\sigma}{2}}\ (a\xi^2)$; $a = \sqrt{\mu^2-k^2}$, $\epsilon = \frac{c_1}{4a}$,

(20) $g_2(\eta) = \frac{1}{\sqrt{\eta}}\, W_{-\epsilon,\frac{\sigma}{2}}\ (a\eta^2)$ or $\frac{1}{\sqrt{\eta}}\, M_{-\epsilon,\frac{\sigma}{2}}\ (a\eta^2)$.

Case (2) $d = 0$ $\quad c_1 = 0$ $\quad b = 1 - 16\sigma^2$

In rectangular coordinates the Schrödinger equation becomes:

$$\frac{\partial^2 U}{dx^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + \frac{1-16\sigma^2}{x^2} - 4\tau^2 y^2 - \tau^2 x^2 + A_3(z) \right] U = 0$$

We assume a solution of the form $U = k_1(x)k_2(y)e^{i\mu z}$ to obtain

$$\frac{d^2 k_1}{dx^2} + \left[k^2-\mu^2-\gamma^2-\tau^2 x^2 + \frac{1-16\sigma^2}{x^2} \right] k_1 = 0 \ ,$$

and $$\frac{d^2 k_2}{dy^2} + \left[\gamma^2 - 4\tau^2 y^2 \right] k_2 = 0 \quad .$$

The first equation can be reduced to Whittaker's differential equation

(21) $k_1(x) = \frac{1}{\sqrt{x}} W_{-\nu-i\epsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau x^2)$ or $\frac{1}{\sqrt{x}} M_{-\nu-i\epsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau x^2)$,

$\epsilon = \frac{\mu^2 k_2}{\tau}$, $\nu = \frac{\gamma^2}{4\tau}$. The second equation can be reduced to the parabolic cylinder equation

(22) $k_2(y) = D_{\nu-\frac{1}{2}}(-2\sqrt{i\tau}\, y)$ or $D_{\nu-\frac{1}{2}}(2\sqrt{i\tau}\, y)$.

The differential equation for $k_2(y)$ can also be reduced to the confluent hypergeometric equation of Kummer, and another choice for $k_2(y)$ would be

(23) $k_2(y) = e^{-i\tau y^2}\, {}_1F_1(\frac{-\nu}{2}+\frac{1}{4};\frac{1}{2};2i\tau y^2)$ or $e^{-i\tau y^2} y\, {}_1F_1(\frac{-\nu}{2}+\frac{3}{4};\frac{3}{2};2i\tau y^2)$.

For parabolic cylinder coordinates we have:

$$\frac{1}{\xi^2+\eta^2}\left(\frac{\partial^2 U}{\partial \xi^2}+\frac{\partial^2 U}{\partial \eta^2}\right)+\frac{\partial^2 U}{\partial z^2}+\left[k^2+\frac{1-16\sigma^2}{\xi^2+\eta^2}\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)-\frac{\tau^2(\xi^6+\eta^6)}{\xi^2+\eta^2}+A_3(z)\right]U=0$$

We assume a solution $U = g_1(\xi)g_2(\eta)e^{i\mu z}$ to obtain

$$\frac{d^2 g_1}{d\xi^2}+\left[(k^2-\mu^2)\,\xi^2-\tau^2\xi^6+\frac{1-16\sigma^2}{\xi^2}\right]g_1 = 0\ ,$$

and $$\frac{d^2 g_2}{d\eta^2}+\left[(k^2-\mu^2)\,\eta^2-\tau^2\eta^6+\frac{1-16\sigma^2}{\eta^2}\right]g_2 = 0\ .$$

Both of these equations can be reduced to Whittaker's differential equation,

(24) $g_1(\xi) = \xi^{-\frac{3}{2}} W_{\epsilon,\sigma}(\frac{-\tau}{2}\xi^4)$ or $\xi^{-\frac{3}{2}} M_{\epsilon,\sigma}(\frac{-\tau}{2}\xi^4)$;

(25) $g_2(\eta) = \eta^{-\frac{3}{2}} W_{\epsilon,\sigma}(\frac{-\tau}{2}\eta^4)$ or $\eta^{-\frac{3}{2}} M_{\epsilon,\sigma}(\frac{-\tau}{2}\eta^4)$.

Case (3) $c_1 = 0 \quad b = \frac{-3}{4}$

In rectangular coordinates the Schrödinger equation is:

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + 2dy - \frac{3}{4x^2} - 4\tau^2 y^2 - \tau^2 x^2 + A_3(z) \right] U = 0$$

We assume a solution of the form $U = k_1(x) k_2(y) e^{i\mu z}$ to obtain

$$\frac{d^2 k_1}{dx^2} + \left[k^2 - \mu^2 - \gamma^2 - \tau^2 x^2 - \frac{3}{4x^2} \right] k_1 = 0 ,$$

and $$\frac{d^2 k_2}{dy^2} + \left[\gamma^2 + 2dy - 4\tau^2 y^2 \right] k_2 = 0 .$$

The first equation can be reduced to Whittaker's differential equation

(26) $k_1(x) = \frac{1}{\sqrt{x}} W_{\sigma-\nu, \frac{i\sqrt{5}}{4}} (\tau x^2)$ or $\frac{1}{\sqrt{x}} M_{\sigma-\nu, \frac{i\sqrt{5}}{4}} (\tau x^2)$, $\sigma = \frac{k^2 - \mu^2}{4\tau}$,

$\nu = \frac{\gamma^2}{4\tau}$, $d^2 = 16\tau^3 \epsilon^2$. The second equation can be reduced to the parabolic cylinder equation

(27) $k_2(y) = D_{+\nu-\epsilon^2-\frac{1}{2}} (2\sqrt{\tau} y - 2\epsilon)$ or $D_{\nu-\epsilon^2-\frac{1}{2}} (-2\sqrt{\tau} y + 2\epsilon)$.

In parabolic cylinder coordinates the Schrödinger equation is:

$$\frac{1}{\xi^2+\eta^2} \left(\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} \right) + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + d(\xi^2 - \eta^2) - \frac{3}{4\xi^2 \eta^2} - \frac{\tau^2(\xi^6 + \eta^6)}{\xi^2 + \eta^2} + A_3(z) \right] U = 0$$

We assume a solution of the form $U = g_1(\xi)\, g_2(\eta)\, e^{i\mu z}$ to obtain

$$\frac{d^2 g_1}{d\xi^2} + \left[(k^2 - \mu^2)\xi^2 + d\xi^4 - \tau^2 \xi^6 - \frac{3}{4\xi^2} \right] g_1 = 0 ,$$

$$\text{and } \frac{d^2 g_2}{d\eta^2} + \left[(k^2-\mu^2)\eta^2 - d\eta^4 - \tau^2\eta^6 - \frac{3}{4\eta^2} \right] g_2 = 0 \ .$$

Both of these equations can be reduced to parabolic cylinder equations,

$$(28) \quad g_1(\xi) = \frac{1}{\sqrt{\xi}} D_{\sigma+\epsilon^2} \left(\sqrt{\tau} \ \xi + \epsilon \right);$$

$$(29) \quad g_2(\eta) = \frac{1}{\sqrt{\eta}} D_{\sigma-\epsilon^2} \left(\sqrt{\tau} \ \eta + \epsilon \right) .$$

E. S(2,6)

This case is somewhat similar to the S(1,4) case so we will consider two cases, which we will now indicate.

$$S(2,6) \quad \phi = \begin{cases} B_1(\rho) + B_3(z) = k^2 + 2dz + \dfrac{b}{\rho^2} - 4\tau^2 z^2 - \tau^2\rho^2 \\ \\ \dfrac{F_1(\xi)+F_2(\eta)}{\xi^2+\eta^2} = k^2 + d(\xi^2-\eta^2) + \dfrac{b}{\xi^2\eta^2} - \tau^2(\xi^2-\eta^2)^2 - \tau^2\xi^2\eta^2 \end{cases}$$

Case (1) $d = 0 \quad \tau = 0$

For cylindrical coordinates the Schrödinger equation is:

$$\frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial U}{\partial \rho}\right) + \frac{1}{\rho^2} \frac{\partial^2 U}{\partial \phi^2} + \frac{\partial^2 U}{\partial z^2} + \left[k^2 + \frac{b}{\rho^2} + \frac{B_2(\phi)}{\rho^2} \right] U = 0$$

We assume a solution of the form $U = k_1(\rho)k_2(z)e^{i\mu\phi}$ to obtain

$$\frac{d^2 k_1}{d\rho^2} + \frac{1}{\rho}\frac{dk_1}{d\rho} + \left[k^2 - \gamma^2 + \frac{b - \mu^2}{\rho^2} \right] k_1 = 0 \ ,$$

$$\text{and } \frac{d^2 k_2}{dz^2} + \gamma^2 k_2 = 0 \ .$$

The first of these can be reduced to the form of Bessel's differential equation,

(30) $k_1(\rho) = Z_\sigma(\nu\rho)$, $\sigma = \sqrt{\mu^2 - b}$, $\nu = \sqrt{k^2 - \gamma^2}$.

For the second equation we have

(31) $k_2(z) = \sin \gamma z$ or $\cos \gamma z$.

For paraboloidal coordinates the Schrödinger equation is:

$$\frac{1}{\xi^2+\eta^2}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial U}{\partial\xi}\right)+\frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial U}{\partial\eta}\right)\right]+\frac{\partial^2 U}{\partial\phi^2}+\left[k^2+\frac{b+B_2(\phi)}{\xi^2\eta^2}\right]U = 0$$

We assume a solution of the form $U = g_1(\xi)g_2(\eta)e^{i\mu\phi}$ to obtain

$$\frac{d^2 g_1}{d\xi^2}+\frac{1}{\xi}\frac{dg_1}{d\xi}+\left[c_1+k^2\xi^2+\frac{b-\mu^2}{\xi^2}\right]g_1 = 0 ,$$

and $$\frac{d^2 g_2}{d\eta^2}+\frac{1}{\eta}\frac{dg_2}{d\eta}+\left[-c_1+k^2\eta^2+\frac{b-\mu^2}{\eta^2}\right]g_2 = 0 .$$

Both of these equations can be reduced to the form of Whittaker's differential equation,

(32) $g_1(\xi) = \frac{1}{\xi}W_{\epsilon,\frac{\sigma}{2}}(ik\xi^2)$ or $\frac{1}{\xi}M_{\epsilon,\frac{\sigma}{2}}(ik\xi^2)$, $\epsilon = \frac{-ic_1}{4k}$;

(33) $g_2(\eta) = \frac{1}{\eta}W_{-\epsilon,\frac{\sigma}{2}}(ik\eta^2)$ or $\frac{1}{\eta}M_{-\epsilon,\frac{\sigma}{2}}(ik\eta^2)$.

Case (2) $d = 0$ $c_1 = 0$

In cylindrical coordinates the Schrödinger equation is:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial U}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 U}{\partial\phi^2}+\frac{\partial^2 U}{\partial z^2}+\left[k^2+\frac{b}{\rho^2}-\tau^2\rho^2-4\tau^2 z^2+\frac{B_2(\phi)}{\rho^2}\right]U = 0$$

We assume a solution of the form $U = k_1(\rho)\, k_2(z)\, e^{i\mu\phi}$ to obtain

$$\frac{d^2k_1}{d\rho^2} + \frac{1}{\rho}\,\frac{dk_1}{d\rho} + \left[k^2 - \gamma^2 - \tau^2\rho^2 + \frac{b-\mu^2}{\rho^2} \right] k_1 = 0 \ ,$$

and $$\frac{d^2k_2}{dz^2} + \left[\gamma^2 - 4\tau^2 z^2 \right] k_2 = 0 \ .$$

The first of these equations can be reduced to the form of Whittaker's differential equation,

$$(34)\quad k_1(\rho) = \frac{1}{\rho} W_{2\epsilon-\nu,\sigma}\,(\tau\rho^2) \ \text{or}\ \frac{1}{\rho} M_{2\epsilon-\nu,\sigma}\,(\tau\rho^2),\ \epsilon = \frac{k^2}{8\tau},\ \nu = \frac{\gamma^2}{4\tau},$$

$\sigma = \sqrt{b-\mu^2}$. The second equation can be reduced to the form of the parabolic cylinder equation,

$$(35)\quad k_2(z) = D_{\nu-\frac{1}{2}}\,(2\sqrt{\tau}\,z) \ \text{or}\ D_{\nu-\frac{1}{2}}\,(-2\sqrt{\tau}\,z) \ .$$

In paraboloidal coordinates the Schrödinger equation is:

$$\frac{1}{\xi^2+\eta^2}\left[\frac{1}{\xi}\,\frac{\partial}{\partial\xi}\left(\xi\,\frac{\partial U}{\partial\xi}\right) + \frac{1}{\eta}\,\frac{\partial}{\partial\eta}\left(\eta\frac{\partial U}{\partial\eta}\right)\right] + \frac{1}{\xi^2\eta^2}\,\frac{\partial^2 U}{\partial\phi^2} + F(\xi,\eta,\phi)\, U = 0$$

where $$F(\xi,\eta,\phi) = k^2 + \frac{b}{\xi^2\eta^2} - \tau^2(\xi^2-\eta^2)^2 - \tau^2\xi^2\eta^2 + \frac{B_2(\phi)}{\xi^2\eta^2} \ .$$

We assume a solution of the form $U = g_1(\xi) g_2(\phi) e^{i\mu\phi}$ to obtain

$$\frac{d^2g_1}{d\xi^2} + \frac{1}{\xi}\,\frac{dg_1}{d\xi} + \left[\frac{b-\mu^2}{\xi^2} + k^2\xi^2 - \tau^2\xi^6 \right] g_1 = 0 \ ,$$

and $$\frac{d^2g_2}{d\eta^2} + \frac{1}{\eta}\,\frac{dg_1}{d\eta} + \left[\frac{b-\mu^2}{\eta^2} + k^2\eta^2 - \tau^2\eta^6 \right] g_2 = 0 \ .$$

Both of these equations can be reduced to the form of Whittaker's differential equation,

(36) $$g_1(\xi) = \frac{1}{\xi^2} W_{\epsilon,\frac{\sigma}{4}} \left(\frac{\tau}{2}\xi^4\right) \quad \text{or} \quad \frac{1}{\xi^2} M_{\epsilon,\frac{\sigma}{4}} \left(\frac{\tau}{2}\xi^4\right) ;$$

(37) $$g_2(\eta) = \frac{1}{\eta^2} W_{\epsilon,\frac{\sigma}{4}} \left(\frac{\tau}{2}\eta^4\right) \quad \text{or} \quad \frac{1}{\eta^2} M_{\epsilon,\frac{\sigma}{4}} \left(\frac{\tau}{2}\eta^4\right) .$$

TABLE 3

$$
S(1,2)\left\{
\begin{array}{lll}
(4) & k_1(x)=\dfrac{1}{\sqrt{x}}W_{\nu,\sigma}(ax^2) & \text{or } \dfrac{1}{\sqrt{x}}M_{\nu,\sigma}(ax^2) \\
(5) & k_2(y)=\dfrac{1}{\sqrt{y}}W_{\gamma,\tau}(ay^2) & \text{or } \dfrac{1}{\sqrt{y}}M_{\gamma,\tau}(ay^2) \\
(6) & g_1(\rho)=\dfrac{1}{\rho}W_{\nu+\gamma,\varepsilon}(a\rho^2) & \text{or } \dfrac{1}{\rho}M_{\nu+\gamma,\varepsilon}(a\rho^2) \\
(7) & g_2(\phi)=(1-\cos^2\phi)^{1/4}\,\tilde{\emptyset}^{2\tau,2\sigma-1/2}_{2\varepsilon-1/2}(\cos\phi) &
\end{array}
\right.
$$

$$
S(2,4)\left\{
\begin{array}{lll}
(8) & k_1(\xi)=\dfrac{1}{\sqrt{\xi}}W_{\alpha,\tau}(\mu\xi^2) & \text{or } \dfrac{1}{\sqrt{\xi}}M_{\alpha,\tau}(\mu\xi^2) \\
(9) & k_2(\eta)=\dfrac{1}{\sqrt{\eta}}W_{\nu,\sigma}(\mu\eta^2) & \text{or } \dfrac{1}{\sqrt{\eta}}M_{\nu,\sigma}(\mu\eta^2) \\
(10) & g_1(\rho)=\dfrac{1}{\sqrt{\rho}}W_{\lambda,\varepsilon}(2\sqrt{\mu^2-k^2}\,\rho) & \text{or } \dfrac{1}{\sqrt{\rho}}M_{\lambda,\varepsilon}(2\sqrt{\mu^2-k^2}\,\rho) \\
(11) & g_2(\phi)=(1-\sin^2\phi)^{1/4}\,\tilde{\emptyset}^{2\sigma,2\tau-1/2}_{2\varepsilon-1/2}(\sin\phi) & \\
(12) & g_2(\phi)=\cos^{1/2}\phi\,(1+\sin\phi)^{-\tau}(1-\sin\phi)^{-\sigma}\,{}_2F_1\left(\tfrac12-\varepsilon-\sigma-\tau,\tfrac12+\varepsilon-\sigma-\tau;1-2\tau;\cos^2(\tfrac{\pi}{4}-\tfrac{\phi}{2})\right) &
\end{array}
\right.
$$

$$
S(3,6)\left\{
\begin{array}{lll}
(13) & k_1(\xi)=\dfrac{1}{\xi}W_{\nu,\sigma}(ki\xi^2) & \text{or } \dfrac{1}{\xi}M_{\nu,\sigma}(ki\xi^2) \\
(14) & k_2(\eta)=\dfrac{1}{\eta}W_{\gamma,\tau}(ki\eta^2) & \dfrac{1}{\eta}M_{\gamma,\tau}(ki\eta^2) \\
(15) & g_1(r)=\dfrac{1}{r}W_{\gamma+\nu,\varepsilon}(2kir) & \dfrac{1}{r}M_{\gamma+\nu,\varepsilon}(2kir) \\
(16) & g_2(\theta)=\tilde{\emptyset}^{2\tau,2\sigma-1/2}_{2\varepsilon-1/2}(\cos\tfrac{\theta}{2}) &
\end{array}
\right.
$$

TABLE 4

$$S(1,4)$$

case (1): $d=\tau=0$, $b=\frac{1}{4}-\sigma^2$

(17) $k_1(x) = x^{1/2} Z_\sigma\left(i\sqrt{a^2+\gamma^2}\, x\right)$

(18) $k_2(y) = \sin\gamma y$ or $\cos\gamma y$

(19) $g_1(\xi) = \frac{1}{\sqrt{\xi}} W_{\varepsilon,\frac{\sigma}{2}}(a\xi^2)$ or $\frac{1}{\sqrt{\xi}} M_{\varepsilon,\frac{\sigma}{2}}(a\xi^2)$

(20) $g_2(\eta) = \frac{1}{\sqrt{\eta}} W_{-\varepsilon,\frac{\sigma}{2}}(a\eta^2)$ or $\frac{1}{\sqrt{\eta}} M_{-\varepsilon,\frac{\sigma}{2}}(a\eta^2)$

case (2): $d=c_1=0$, $b=1-16\sigma^2$

(21) $k_1(x) = \frac{1}{\sqrt{x}} W_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau x^2)$ or $\frac{1}{\sqrt{x}} M_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau x^2)$

(22) $k_2(y) = D_{\nu-1/2}(-2\sqrt{i\tau}\, y)$ or $D_{\nu-1/2}(2\sqrt{i\tau}\, y)$

(23) $k_2(y) = e^{-i\tau y^2} {}_1F_1\left(-\frac{\nu}{2}+\frac{1}{2};\frac{1}{2};2i\tau y^2\right)$ or $e^{-i\tau y^2} y\, {}_1F_1\left(-\frac{\nu}{2}+\frac{3}{4};\frac{3}{2};2i\tau y^2\right)$

(24) $g_1(\xi) = \xi^{-3/2} W_{\varepsilon,\sigma}\left(-\frac{\tau}{2}\xi^4\right)$ or $\xi^{-3/2} M_{\varepsilon,\sigma}\left(-\frac{\tau}{2}\xi^4\right)$

(25) $g_2(\eta) = \eta^{-3/2} W_{\varepsilon,\sigma}\left(-\frac{\tau}{2}\eta^4\right)$ or $\eta^{-3/2} M_{\varepsilon,\sigma}\left(-\frac{\tau}{2}\eta^4\right)$

case (3): $c_1=0$, $b=-\frac{3}{4}$

(26) $k_1(x) = \frac{1}{\sqrt{x}} W_{\sigma-\nu,\frac{i\sqrt{5}}{4}}(\tau x^2)$ or $\frac{1}{\sqrt{x}} M_{\sigma-\nu,\frac{i\sqrt{5}}{4}}(\tau x^2)$

(27) $k_2(y) = D_{\nu-\varepsilon^2-1/2}(2\sqrt{\tau}\, y-2\varepsilon)$ or $D_{\nu-\varepsilon^2-1/2}(-2\sqrt{\tau}\, y+2\varepsilon)$

(28) $g_1(\xi) = \frac{1}{\sqrt{\xi}} D_{\sigma+\varepsilon^2}(\sqrt{\tau}\,\xi+\varepsilon)$ or $\frac{1}{\sqrt{\xi}} D_{\sigma+\varepsilon^2}(-\sqrt{\tau}\,\xi-\varepsilon)$

(29) $g_2(\eta) = \frac{1}{\sqrt{\eta}} D_{\sigma-\varepsilon^2}(\sqrt{\tau}\,\eta+\varepsilon)$ $\frac{1}{\sqrt{\eta}} D_{\sigma-\varepsilon^2}(-\sqrt{\tau}\,\eta-\varepsilon)$

TABLE 5

$\mathcal{S}(2,6)$

Case	No.	Solution	or
case(1) $\tau = d = 0$	(30)	$k_1(\rho) = Z_\sigma(\nu\rho)$	
	(31)	$k_2(z) = \sin\gamma z$	or $\cos\gamma z$
	(32)	$g_1(\xi) = \frac{1}{\xi} W_{\varepsilon,\frac{\sigma}{2}}(ik\xi^2)$	or $\frac{1}{\xi} M_{\varepsilon,\frac{\sigma}{2}}(ik\xi^2)$
	(33)	$g_2(\eta) = \frac{1}{\eta} W_{-\varepsilon,\frac{\sigma}{2}}(ik\eta^2)$	or $\frac{1}{\eta} M_{-\varepsilon,\frac{\sigma}{2}}(ik\eta^2)$
case(2) $d = c_1 = 0$	(34)	$k_1(\rho) = \frac{1}{\rho} W_{2\varepsilon-\nu,\sigma}(\tau\rho^2)$	or $\frac{1}{\rho} M_{2\varepsilon-\nu,\sigma}(\tau\rho^2)$
	(35)	$k_2(z) = D_{\nu-\frac{1}{2}}(2\sqrt{\tau}\,z)$	or $D_{\nu-\frac{1}{2}}(2\sqrt{\tau}\,z)$
	(36)	$g_1(\xi) = \frac{1}{\xi^2} W_{\varepsilon,\frac{\sigma}{4}}(\frac{\tau}{2}\xi^4)$	or $\frac{1}{\xi^2} M_{\varepsilon,\frac{\sigma}{4}}(\frac{\tau}{2}\xi^4)$
	(37)	$g_2(\eta) = \frac{1}{\eta^2} W_{\varepsilon,\frac{\sigma}{4}}(\frac{\tau}{2}\eta^4)$	or $\frac{1}{\eta^2} M_{\varepsilon,\frac{\sigma}{4}}(\frac{\tau}{2}\eta^4)$

IV. A METHOD OF GENERATING INTEGRAL REPRESENTATIONS

In this section we shall introduce the theorem which enables us to generate integral representations between the various special functions, appearing in Tables 3 and 4. First we will introduce appropriate notation. Let x_1, x_2, x_3 represent mutually orthogonal coordinates (any system appearing in Table 1). The Schrödinger equation, $\nabla^2 U + \phi U = 0$, for the various forms of ϕ in Table 2, can be solved by the method of separation of variables. Thus $U(x_1, x_2, x_3) = k_1(x_1)\, k_2(x_2)\, k_3(x_3)$. The associated ordinary operators will all be selfadjoint, e.g. of the form

$$L_i = \frac{\partial}{\partial x_i} p_i(x_i) \frac{\partial}{\partial x_i} + q_i(x_i) \qquad i = 1, 2, 3.$$

When we write $U(x_1, x_2, x_3) = K(x_1, x_2)\, k_3(x_3)$ we obtain a partial differential equation of two variables of the form

$$(38) \quad F_1(x_1)\, L_1\, K(x_1, x_2) = F_2(x_2)\, L_2\, K(x_1, x_2)$$

The theorem is:

Let

(a) C be a path in the x_2 plane with endpoints a, b.

(b) B be a domain in the x_1 plane.

(c) $f_2(x_2)$ be a separated solution of $L_2 f_2(x_2) = 0$.

(d) $K(x_1, x_2)$ be a regular analytic solution of (38) for $x_1 \in B$ and x_2 on C.

(39) Then $t(x_1) = \int_C K(x_1, x_2)\, F_2^{-1}(x_2)\, f_2(x_2)\, dx_2$

is uniformly convergent for $x_1 \in B$ and <u>$t(x_1)$ will be a solution of the differential equation $L_1 f_1(x_1) = 0$</u> provided the bilinear concomitant vanishes, that is to say provided

$$p_2(x_2) \left(\frac{\partial K(x_1, x_2)}{\partial x_2} f_2(x_2) - K(x_1, x_2) \frac{\partial f_2(x_2)}{\partial x_2} \right) \Bigg]_a^b = 0 .$$

This theorem is the very essence of the idea of generating many integral representations. We begin with a linear second order partial differential equation of three variables which can be solved by the method of separation of variables. Furthermore each of the separated ordinary equations are of the self adjoint form, $L_i f_i (x_i) = 0$ for $i = 1, 2, 3$. Then we eliminate the variable x_3 by assuming a solution of the form $u = f_1(x_1) f_2(x_2) f_3(x_3)$ so that the remaining equation has the form of (38).

Next we introduce a second coordinate system sharing one coordinate variable, say x_3, with the first. Let the other two variables be denoted x_1' x_2'. Suppose that the Schrödinger equation is simultaneously separable in both systems. Then there is another partial differential equation of a form equivalent to (38) but involving x_1' and x_2', e.g.

(40) $F_1'(x_1') L_1' K'(x_1', x_2') = F_2' (x_2') L_2' K'(x_1', x_2')$

where $L_i' = (\frac{\partial}{\partial x_1'}) p_i'(x_1') (\frac{\partial}{\partial x_1'}) + q_i'(x_i')$, $i = 1, 2$.

Now $K'(x_1' x_2') f_3(x_3')$ is also a solution of the original Schrödinger equation.

We can now choose for our kernel $K(x_1, x_2)$ in (39) a solution of (40). Then if we integrate the product of the kernel and $f_2(x_2)/F_2(x_2)$ along a path for which the bilinear concomitant vanishes we are assured that this integral will satisfy $L_1 f_1(x_1) = 0$. Of course the beauty of the method now becomes clear, for we know $f_1(x_1)$ and we need only determine what linear combination of solutions of $L_1 f_1(x_1) = 0$ will actually represent the integral.

The power of this method lies in the fact that we can choose various kernels subject to paths for which the bilinear concomitant vanishes. Furthermore, we can interchange roles of $f_1(x_1)$, $f_2(x_2)$ and even choose our kernel from the first coordinate system. This then should exhibit the power of the method and subsequent sections will contain a few integral representations worked in detail.

V. INTEGRAL REPRESENTATION USING S(1,2)

We are now ready to consider the results of previous section for the case S(1,2) and generate a new integral representation. By examining Section III we have:

$$F_1(x)L_1 = \frac{d^2}{dx^2} + k^2-\mu^2-a^2x^2 + \frac{\frac{1}{4} - 4\sigma^2}{x^2} \qquad F_1(x) = 1 \quad p_1(x) = 1$$

$$F_2(y)L_2 = \frac{-d^2}{dy^2} + a^2y^2 - \frac{\frac{1}{4} - 4\tau^2}{y^2} \qquad F_2(y) = 1 \quad p_2(y) = 1$$

$$F_1'(\rho)L_1' = \frac{1}{\rho}\frac{d}{d\rho}(\rho\frac{d}{d\rho}) + k^2-\mu^2+a^2\rho^2 \qquad F_1'(\rho) = \frac{1}{\rho} \quad p_1'(\rho) = \rho$$

$$F_2'(\phi)L_2' = \frac{-d^2}{d\phi^2} - \frac{\frac{1}{4} - 4\sigma^2}{\cos^2\phi} - \frac{\frac{1}{4} - 4\tau^2}{\sin^2\phi} \qquad F_2'(\phi) = 1 \quad p_2'(\phi) = 1$$

The separated ordinary differential equations corresponding to these operators are also solved in section III. We assumed that $U = k_1(x)k_2(y)e^{i\mu z}$ or $U = g_1(\rho)g_2(\phi)e^{i\mu z}$ so we simply list the solutions we will use in this section, which are determined by our choice of path and the vanishing of the bilinear concomitant. Our kernel will be the product of (4) and (5)

$$K(x,y) = \frac{1}{\sqrt{x}} W_{\nu,\sigma}(ax^2) \frac{1}{\sqrt{y}} W_{\gamma,\tau}(ay^2) .$$

Also we choose (6) for

$$f_2(x_2) = \frac{1}{\rho} W_{\nu+\gamma,\epsilon}(a\rho^2) .$$

Now by the theorem in section IV and especially equation (39) we have the integral

$$t(\phi) = \int_C \frac{W_{\nu,\sigma}(ax^2)}{\sqrt{x}} \frac{W_{\gamma,\tau}(ay^2)}{\sqrt{y}} \frac{W_{\nu+\gamma,\epsilon}(a\rho^2)}{\rho} \rho \, d\rho$$

and we realize this integral represents a solution of $F_2'(\phi)L_2'g_2(\phi) = 0$. That is to say it is one or possibly a linear combination of two of the various functions denoted by $(1-\cos^2\phi)^{1/4}\ \tilde{\emptyset}^{-2\tau,\,2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)$.

We must still choose a path C. When we take it along the real axis from 0 to ∞ the bilinear concomitant vanishes, e.g.

$$\rho\left\{\frac{\partial}{\partial\rho}\left(\frac{\partial_{\nu,\sigma}(a\rho^2\cos^2\phi)}{\rho\sin^{\frac{1}{2}}\phi\ \cos^{\frac{1}{2}}\phi}\ W_{\gamma,\tau}(a\rho^2\sin^2\phi)\right)\frac{W_{\gamma+\nu,\epsilon}(a\rho^2)}{\rho}\ -\right.$$

$$\left.\left.\frac{W_{\nu,\sigma}(a\rho^2\cos^2\phi)\ W_{\gamma,\tau}(a\rho^2\sin^2\phi)}{\rho\sin^{\frac{1}{2}}\phi\ \cos^{\frac{1}{2}}\phi}\ \frac{\partial}{\partial\rho}\left(\frac{W_{\gamma+\nu,\epsilon}(a\rho^2)}{\rho}\right)\right\}\right|_0^\infty = 0$$

subject to the condition $|\mathrm{Re}\ \sigma| + |\mathrm{Re}\ \tau| + |\mathrm{Re}\ \epsilon| < \frac{1}{2}$ (Whittaker-8).

Our integral takes the form

$$t(\phi) = \int_0^\infty \frac{W_{\nu,\sigma}(ax^2)}{\sqrt{x}}\ \frac{W_{\gamma,\tau}(ay^2)}{\sqrt{y}}\ W_{\nu+\gamma,\epsilon}(a\rho^2)\ d\rho$$

$$t(\phi) = \int_0^\infty \frac{W_{\nu,\sigma}(a\rho^2\cos^2\phi)\ \ W_{\gamma,\tau}(a\rho^2\sin^2\phi)}{\rho\ \cos^{\frac{1}{2}}\phi\ \sin^{\frac{1}{2}}\phi}\ W_{\nu+\gamma,\epsilon}\ (a\rho^2)\ d\rho$$

This integral converges provided $|\mathrm{Re}\ \sigma| + |\mathrm{Re}\ \tau| + |\mathrm{Re}\ \epsilon| < \frac{3}{2}$

To determine which functions $\tilde{\emptyset}$ we actually have express $W_{\nu,\sigma}(a\rho^2\cos^2\phi)$ in terms of confluent hypergeometric functions (Magnus-6) and choose $a = 1$.

$$t(\phi) = \frac{\cos^{2\sigma+\frac{1}{2}}\phi}{\sin^{\frac{1}{2}}\phi}\,\frac{\Gamma(-2\sigma)}{\Gamma(\frac{1}{2}-\sigma-\nu)}\int_0^\infty \frac{{}_1F_1\left[\sigma-\nu+\frac{1}{2};\,1+2\sigma;\,\rho^2\cos^2\phi\right] W_{\gamma,\tau}(\rho^2\sin^2\phi)\, W_{\nu+\gamma,\varepsilon}(\rho^2)}{e^{\frac{\rho^2\cos^2\phi}{2}}\,\rho^{-2\sigma}}\,d\rho$$

$$+\ \frac{\cos^{\frac{1}{2}-2\sigma}\phi}{\sin^{\frac{1}{2}}\phi}\,\frac{\Gamma(2\sigma)}{\Gamma(\frac{1}{2}+\sigma-\nu)}\int_0^\infty \frac{{}_1F_1\left[-\sigma-\nu+\frac{1}{2};\,1-2\sigma;\,\rho^2\cos^2\phi\right] W_{\gamma,\tau}(\rho^2\sin^2\phi)\, W_{\nu+\gamma,\varepsilon}(\rho^2)}{e^{\frac{\rho^2\cos^2\phi}{2}}\,\rho^{2\sigma}}\,d\rho$$

Let $z = \rho^2$ and examine this expression for small $\cos\phi$.

$$t(\phi) \approx \frac{\cos^{2\sigma+\frac{1}{2}}\phi}{\sin^{\frac{1}{2}}\phi}\,\frac{\Gamma(-2\sigma)}{\Gamma(\frac{1}{2}-\sigma-\nu)}\,\frac{1}{2}\int_0^\infty z^{\sigma-\frac{1}{2}}\, W_{\gamma,\tau}(z)\, W_{\nu+\gamma,\varepsilon}(z)\,dz\ +$$

$$\frac{\cos^{\frac{1}{2}-2\sigma}\phi}{\sin^{\frac{1}{2}}\phi}\,\frac{\Gamma(2\sigma)}{\Gamma(\frac{1}{2}+\sigma-\nu)}\,\frac{1}{2}\int_0^\infty z^{-\sigma-\frac{1}{2}}\, W_{\gamma,\tau}(z)\, W_{\nu+\gamma,\varepsilon}(z)\,dz$$

These integrals can be evaluated (Bateman-1, Vol. 2, Page 410 (42)).

$$(41)\quad t(\phi) \approx \frac{\cos^{2\sigma+\frac{1}{2}}\phi}{2\sin^{\frac{1}{2}}\phi}\,\frac{\Gamma(-2\sigma)}{\Gamma(\frac{1}{2}-\sigma-\nu)}\left(\frac{\Gamma(\frac{3}{2}+\varepsilon+\tau+\sigma)\,\Gamma(\frac{3}{2}-\varepsilon+\tau+\sigma)\,\Gamma(-2\tau)}{\Gamma(\frac{1}{2}-\gamma-\tau)\,\Gamma(2-\nu-\gamma+\tau+\sigma)}\ {}_3F_2\left[\begin{matrix}\frac{3}{2}+\varepsilon+\tau+\sigma,\ \frac{3}{2}-\varepsilon+\tau+\sigma,\ \frac{1}{2}-\gamma+\tau\\ 1+2\tau,\ 2-\nu-\gamma+\tau+\sigma\end{matrix};1\right]\right.$$

$$\left.+\ \frac{\Gamma(\frac{3}{2}+\varepsilon-\tau+\sigma)\,\Gamma(\frac{3}{2}-\varepsilon-\tau+\sigma)\,\Gamma(2\tau)}{\Gamma(\frac{1}{2}-\gamma-\tau)\,\Gamma(2-\nu-\gamma-\tau+\sigma)}\ {}_3F_2\left[\begin{matrix}\frac{3}{2}+\varepsilon-\tau+\sigma,\ \frac{3}{2}-\varepsilon-\tau+\sigma,\ \frac{1}{2}-\gamma-\tau\\ 1-2\tau,\ 2-\nu-\gamma-\tau+\sigma\end{matrix};1\right]\right)$$

$$+\ \frac{\cos^{\frac{1}{2}-2\sigma}\phi}{2\sin^{\frac{1}{2}}\phi}\,\frac{\Gamma(2\sigma)}{\Gamma(\frac{1}{2}+\sigma-\nu)}\left(\frac{\Gamma(\frac{3}{2}+\varepsilon+\tau-\sigma)\,\Gamma(\frac{3}{2}-\varepsilon+\tau-\sigma)\,\Gamma(-2\tau)}{\Gamma(\frac{1}{2}-\gamma-\tau)\,\Gamma(2-\nu-\gamma+\tau-\sigma)}\ {}_3F_2\left[\begin{matrix}\frac{3}{2}+\varepsilon+\tau-\sigma,\ \frac{3}{2}-\varepsilon+\tau-\sigma,\ \frac{1}{2}-\gamma+\tau\\ 1+2\tau,\ 2-\nu-\gamma+\tau-\sigma\end{matrix};1\right]\right.$$

$$\left.+\ \frac{\Gamma(\frac{3}{2}+\varepsilon-\tau-\sigma)\,\Gamma(\frac{3}{2}-\varepsilon-\tau-\sigma)\,\Gamma(2\tau)}{\Gamma(\frac{1}{2}-\gamma+\tau)\,\Gamma(2-\nu-\gamma-\tau-\sigma)}\ {}_3F_2\left[\begin{matrix}\frac{3}{2}+\varepsilon-\tau-\sigma,\ \frac{3}{2}-\varepsilon-\tau-\sigma,\ \frac{1}{2}-\gamma-\tau\\ 1-2\tau,\ 2-\nu-\gamma-\tau-\sigma\end{matrix};1\right]\right)$$

When we examine the various forms of $(1-\cos^2\phi)^{1/4}\,\tilde{\phi}^{-2\tau,\,2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)$

for small cos Φ and compare them with the previous expression for $t(\Phi)$ we find that we have a linear combination of the functions

$(1-\cos^2\phi)^{1/4}\,\tilde{r}^{-2\tau,\,2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)$ and $(1-\cos^2\phi)^{1/4}\,\tilde{r}^{-2\tau,\,-2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)$ where

$$\tilde{r}^{-2\tau,\,2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)=\frac{\pi\, e^{i\pi(\epsilon+\tau-\sigma-1/2)}\sin^{2\tau}\phi\,\cos^{2\sigma+1/2}\phi}{2^{2\epsilon+1/2}\Gamma(1+2\sigma)\Gamma(\epsilon-\tau-\sigma+\frac{1}{2})\Gamma(\epsilon+\tau-\sigma+\frac{1}{2})}\;{}_2F_1\left[\begin{matrix}\tau+\sigma+\epsilon+\frac{1}{2},\ \tau+\sigma-\epsilon+\frac{1}{2}\\ 1+2\sigma\end{matrix};\cos^2\phi\right]$$

So if we define

$$T(\nu,\gamma,\sigma,\tau,\epsilon)=\frac{2^{2\epsilon+1/2}\Gamma(-2\sigma)\Gamma(1+2\sigma)\Gamma(\epsilon+\tau-\sigma+\frac{1}{2})\Gamma(\epsilon-\tau+\frac{1}{2}-\sigma)}{2\pi\; e^{i\pi(\epsilon-\sigma+\tau-1/2)}\;\Gamma(\frac{1}{2}-\sigma-\nu)}\cdot$$

$$\left(\frac{\Gamma(-2\tau)\Gamma(\frac{3}{2}+\epsilon+\tau+\sigma)\Gamma(\frac{3}{2}-\epsilon+\tau+\sigma)}{\Gamma(\frac{1}{2}-\gamma-\tau)\Gamma(2-\nu-\gamma+\tau+\sigma)}\;{}_3F_2\left[\begin{matrix}\sigma+\tau+\epsilon+\frac{3}{2},\ \sigma+\tau-\epsilon+\frac{3}{2},\ \frac{1}{2}-\gamma+\tau\\ 1+2\tau,\ 2-\nu-\gamma+\tau+\sigma\end{matrix};1\right]\right.$$

$$\left.+\;\frac{\Gamma(2\tau)\Gamma(\frac{3}{2}+\epsilon-\tau+\sigma)\Gamma(\frac{3}{2}-\epsilon-\tau+\sigma)\,e^{-2\pi i\tau}}{\Gamma(\frac{1}{2}-\gamma+\tau)\Gamma(2-\nu-\gamma-\tau+\sigma)}\;{}_3F_2\left[\begin{matrix}\sigma-\tau+\epsilon+\frac{3}{2},\ \sigma-\tau-\epsilon+\frac{3}{2},\ \frac{1}{2}-\gamma-\tau\\ 1-2\tau,\ 2-\nu-\gamma-\tau+\sigma\end{matrix};1\right]\right)$$

we can write

$$(42)\qquad \int_0^\infty \frac{W_{\nu,\sigma}(x^2)\,W_{\gamma,\tau}(y^2)}{\sqrt{xy}}\,W_{\nu+\gamma,\epsilon}(\rho^2)\,d\rho\;=$$

$$T(\nu,\gamma,\sigma,\tau,\epsilon)\,(1-\cos^2\phi)^{1/4}\,\tilde{r}^{-2\tau,\,2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)\;+\;T(\nu,\gamma,-\sigma,\tau,\epsilon)\,(1-\cos^2\phi)^{1/4}\,\tilde{r}^{-2\tau,\,-2\sigma-1/2}_{2\epsilon-1/2}(\cos\phi)$$

provided $|\mathrm{Re}\,\sigma| + |\mathrm{Re}\,\tau| + |\mathrm{Re}\,\epsilon| < \frac{1}{2}$.

We shall now check this result against known cases by assigning special values to the parameters. Let $\nu = \sigma+\frac{1}{2}$, and $\gamma = \tau+\frac{1}{2}$. The integral of (42) becomes

$$\cos^{2\sigma+\frac{1}{2}}\phi \ \sin^{2\tau+\frac{1}{2}}\phi \ \frac{1}{2}\int_0^\infty z^{\sigma+\tau} e^{\frac{-z}{2}} W_{\sigma+\tau+1,\varepsilon}(z)\,dz \quad \text{where } \rho^2 = z.$$

This integral can be evaluated (Bateman-1, Vol. 1, Page 337, (8)), and yields

$$(43) \quad \frac{1}{2}\cos^{2\sigma+\frac{1}{2}}\phi \ \sin^{2\tau+\frac{1}{2}}\phi \ \Gamma(\sigma+\tau+\varepsilon+\tfrac{3}{2})\,\Gamma(\sigma+\tau-\varepsilon+\tfrac{3}{2})$$

We now must show that the right hand side of (42) reduces to (43). First we note that $T(\sigma+\frac{1}{2},\tau+\frac{1}{2},-\sigma,\tau,\varepsilon)\to 0$ and also the last half of $T(\sigma+\frac{1}{2},\tau+\frac{1}{2},\sigma,\tau,\varepsilon)$ vanishes due to the factor $\Gamma(\frac{1}{2}-\gamma+\tau)$.

We have

$$T(\sigma+\tfrac{1}{2},\tau+\tfrac{1}{2},\sigma,\tau,\varepsilon) = \frac{2^{2\varepsilon+\frac{1}{2}}\,\Gamma(1+\sigma)\,\Gamma(\tau-\sigma+\varepsilon+\frac{1}{2})\,\Gamma(-\tau-\sigma+\varepsilon+\frac{1}{2})\,\Gamma(\frac{3}{2}+\varepsilon+\tau+\sigma)\,\Gamma(\frac{3}{2}-\varepsilon+\tau+\sigma)}{2\pi\, e^{i\pi(\varepsilon-\sigma+\tau-\frac{1}{2})}},$$

hence $$(1-\cos^2\phi)^{\frac{1}{4}}\,\tilde{\gamma}^{-2\tau,2\sigma-\frac{1}{2}}_{2\varepsilon-\frac{1}{2}}(\cos\phi)\,T(\sigma+\tfrac{1}{2},\tau+\tfrac{1}{2},\sigma,\tau,\varepsilon) =$$

$$\cos^{2\sigma+\frac{1}{2}}\phi \ \sin^{2\tau+\frac{1}{2}}\phi \cdot \frac{1}{2}\,\Gamma(\sigma+\tau+\varepsilon+\tfrac{3}{2})\,\Gamma(\sigma+\tau-\varepsilon+\tfrac{3}{2}).$$

VI. INTEGRAL REPRESENTATIONS USING S(1,4)

Referring to Section III we recall the case S(1,4) was subdivided into three special cases. In this section we will obtain integral representations for cases (1) and (2).

A. For case (1) we have:

$$F_1(\xi)L_1 = \frac{d^2}{d\xi^2} + c_1 + (k^2-\mu^2)\xi^2 + \frac{\frac{1}{4}-\sigma^2}{\xi^2} \qquad F_1(\xi) = 1 \quad p_1(\xi) = 1$$

$$F_1(\eta)L_2 = \frac{-d^2}{d\eta^2} + c_1 - (k^2-\mu^2)\eta^2 - \frac{\frac{1}{4}-\sigma^2}{\eta^2} \qquad F_2(\eta) = 1 \quad p_2(\eta) = 1$$

$$F_1'(x)L_1' = \frac{d^2}{dx^2} + k^2-\mu^2-\gamma^2 + \frac{\frac{1}{4}-\sigma^2}{x^2} \qquad F_1'(x) = 1 \quad p_1'(x) = 1$$

$$F_2'(y)L_2' = \frac{-d^2}{dy^2} - \gamma^2 \qquad F_2'(y) = 1 \quad p_2'(y) = 1$$

The separated ordinary differential equations corresponding to these operators are solved in Section III. We assumed that $U = k_1(x)k_2(y)e^{i\mu z}$ or $U = g_1(\xi)g_2(\eta)e^{i\mu z}$, so we simply list the solutions we will use in this section, which are determined by our choice of path and the vanishing of the bilinear concomitant. For our kernel we choose the product of (19) and (20)

$$K(\xi,\eta) = \frac{1}{\sqrt{\xi}} W_{\epsilon,\frac{\sigma}{2}}(a\xi^2) \frac{1}{\sqrt{\eta}} W_{-\epsilon,\frac{\sigma}{2}}(a\eta^2) \ , \ a = \sqrt{\mu^2-k^2} \ ,$$

and formula (18) for $f_2(x_2) = \sin\gamma y$.

If we choose a suitable path and use $K(\xi,\eta)\sin\gamma y$ as our integrand, the integral will then represent Bessel functions of order $\pm\sigma$ and argument $\sqrt{k^2-\mu^2-\gamma^2}\ x$, except for a factor $x^{\frac{1}{2}}$. It is convenient to use the notation $a^2 = \mu^2-k^2$ introduced earlier.

The integral $$t(x) = \int_C \frac{1}{\sqrt{\xi}} W_{\epsilon,\frac{\sigma}{2}}(a\xi^2) \frac{1}{\sqrt{\eta}} W_{-\epsilon,\frac{\sigma}{2}}(a\eta^2) \sin\gamma y \, dy$$

will be some linear combination of $x^{1/2} I_{\pm\sigma}(\sqrt{a^2+\gamma^2}\, x)$, where I is a modified Bessel function. Our choice of path C is the real axis from $-\infty$ to $+\infty$, since this causes the bilinear concomitant to vanish, e.g.

$$\sin\gamma y \frac{\partial}{\partial y}\left[\frac{W_{\epsilon,\frac{\sigma}{2}}(a\xi^2) W_{-\epsilon,\frac{\sigma}{2}}(a\eta^2)}{\sqrt{x}}\right] - \frac{W_{\epsilon,\frac{\sigma}{2}}(a\xi^2) W_{-\epsilon,\frac{\sigma}{2}}(a\eta^2)}{\sqrt{x}} \frac{\partial}{\partial y}\left[\sin\gamma y\right] \Bigg]_{-\infty}^{+\infty} = 0.$$

provided $|\mathrm{Re}\, a| > 0$.

The integral can now be written

$$(44) \quad t(x) = \int_0^{\infty} \frac{1}{\sqrt{x}} W_{\epsilon,\frac{\sigma}{2}}(ay + a\sqrt{x^2+y^2}) \, W_{-\epsilon,\frac{\sigma}{2}}(-ay + a\sqrt{x^2+y^2}) \sin\gamma y \, dy$$
$$+ \int_{-\infty}^{0} \frac{1}{\sqrt{x}} W_{\epsilon,\frac{\sigma}{2}}(ay - a\sqrt{x^2+y^2}) \, W_{-\epsilon,\frac{\sigma}{2}}(-ay - a\sqrt{x^2+y^2}) \sin\gamma y \, dy.$$

To determine which solution of the modified Bessel equation t(x) actually is we consider only the first part, namely

$$\mathcal{A}(x) = \int_0^{\infty} \frac{1}{\sqrt{x}} W_{\epsilon,\frac{\sigma}{2}}(ay + a\sqrt{x^2+y^2}) \, W_{-\epsilon,\frac{\sigma}{2}}(-ay + a\sqrt{x^2+y^2}) \sin\gamma y \, dy .$$

Examine $\mathcal{A}(x)$ for small x. We obtain

$$\mathcal{A}(x) \approx \frac{\Gamma(-\sigma)\, x^{\sigma+\frac{1}{2}}}{\Gamma(\frac{1-\sigma}{2}+\epsilon)} \left(\frac{a}{2}\right)^{\frac{\sigma+1}{2}} \int_0^{\infty} W_{\epsilon,\frac{\sigma}{2}}(2ay) \frac{\sin\gamma y}{y^{\frac{\sigma+1}{2}}} \, dy$$
$$+ \frac{\Gamma(\sigma)\, x^{\frac{1}{2}-\sigma}}{\Gamma(\frac{1+\sigma}{2}+\epsilon)} \left(\frac{a}{2}\right)^{\frac{1-\sigma}{2}} \int_0^{\infty} W_{\epsilon,\frac{\sigma}{2}}(2ay) \frac{\sin\gamma y}{y^{\frac{1-\sigma}{2}}} \, dy ,$$

provided $|\mathrm{Re}\ \sigma| < 2$, which insures convergence of the integrals.

These integrals are known (Bateman-1, Vol. I, Page 337 (8)). Using the Euler identity for $\sin\gamma y$, and with small x we have,

$$d(x) \approx \frac{\Gamma(-\sigma)\Gamma(1-\sigma)\, a^{1+\sigma}}{\Gamma\left(\frac{1-\sigma}{2}+\varepsilon\right)\Gamma\left(\frac{3-\sigma}{2}-\varepsilon\right)} \frac{x^{\sigma+\frac{1}{2}}}{2i} \left[\frac{{}_2F_1\left[\begin{matrix} 1, \frac{\sigma+1}{2}-\varepsilon \\ \frac{3-\sigma}{2}-\varepsilon \end{matrix}; \frac{-(a+i\gamma)^2}{a^2+\gamma^2}\right]}{a-i\gamma} - \frac{{}_2F_1\left[\begin{matrix} 1, \frac{\sigma+1}{2}-\varepsilon \\ \frac{3-\sigma}{2}-\varepsilon \end{matrix}; \frac{-(a-i\gamma)^2}{a^2+\gamma^2}\right]}{a+i\gamma} \right]$$

$$+ \frac{\Gamma(\sigma)\Gamma(1+\sigma)\, a\, 2^{\sigma}}{\Gamma\left(\frac{1+\sigma}{2}+\varepsilon\right)\Gamma\left(\frac{3+\sigma}{2}-\varepsilon\right)} \frac{x^{\frac{1}{2}-\sigma}}{2i} \left[\frac{{}_2F_1\left[\begin{matrix} \sigma+1, \frac{1+\sigma}{2}-\varepsilon \\ \frac{3+\sigma}{2}-\varepsilon \end{matrix}; \frac{-(a+i\gamma)^2}{a^2+\gamma^2}\right]}{(a-i\gamma)^{\sigma+1}} - \frac{{}_2F_1\left[\begin{matrix} \sigma+1, \frac{1+\sigma}{2}-\varepsilon \\ \frac{3+\sigma}{2}-\varepsilon \end{matrix}; \frac{-(a-i\gamma)^2}{a^2+\gamma^2}\right]}{(a+i\gamma)^{\sigma+1}} \right].$$

Using the definition of the hypergeometric functions as Gauss series, we have

$$d(x) \approx \frac{\Gamma(1-\sigma)\Gamma(-\sigma)\, a^{1+\sigma} x^{\sigma+\frac{1}{2}}}{\Gamma\left(\frac{1-\sigma}{2}+\varepsilon\right)\Gamma\left(\frac{3-\sigma}{2}-\varepsilon\right)(a^2+\gamma^2)^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\left(\frac{\sigma+1}{2}-\varepsilon\right)_n (-1)^n}{\left(\frac{3-\sigma}{2}-\varepsilon\right)_n} \sin(2n+1)\nu$$

$$+ \frac{\Gamma(1+\sigma)\Gamma(\sigma)\, a\, 2^{\sigma} x^{\frac{1}{2}-\sigma}}{\Gamma\left(\frac{1+\sigma}{2}+\varepsilon\right)\Gamma\left(\frac{3+\sigma}{2}-\varepsilon\right)(a^2+\gamma^2)^{\frac{1+\sigma}{2}}} \sum_{n=0}^{\infty} \frac{(1+\sigma)_n}{n!} \frac{\left(\frac{1+\sigma}{2}-\varepsilon\right)_n (-1)^n}{\left(\frac{3+\sigma}{2}-\varepsilon\right)_n} \sin(2n+1+\sigma)\nu,$$

$$\operatorname{Re}\sigma < 1, \quad e^{i\nu} = \frac{a+i\gamma}{\sqrt{a^2+\gamma^2}} = \cos\nu + i\sin\nu.$$

The new form of the restriction on σ is necessary to insure convergence of the Fourier series.

The analysis for the second integral appearing in (44) is similar, and leads to the same restriction on σ.

For small x

$$(45).\ t(x) \approx \frac{\Gamma(1-\sigma)\Gamma(\sigma)\ a^{1+\sigma}\ x^{\sigma+\frac{1}{2}}}{\Gamma(\frac{1-\sigma}{2}-\epsilon)\,\Gamma(\frac{1-\sigma}{2}+\epsilon)\,(a^2+\gamma^2)^{\frac{1}{2}}} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\left(\frac{\sigma+1}{2}+\epsilon\right)_n}{\left(\frac{3-\sigma}{2}-\epsilon\right)_n \left(\frac{1-\sigma}{2}-\epsilon\right)} - \frac{\left(\frac{\sigma+1}{2}+\epsilon\right)_n}{\left(\frac{3-\sigma}{2}+\epsilon\right)_n \left(\frac{1-\sigma}{2}+\epsilon\right)} \right] \sin(2n+1)\nu$$

$$+ \frac{\Gamma(1+\sigma)\Gamma(\sigma)\ a 2^{\sigma}\ x^{\frac{1}{2}-\sigma}}{\Gamma(\frac{1+\sigma}{2}+\epsilon)\,\Gamma(\frac{1+\sigma}{2}-\epsilon)\,(a^2+\gamma^2)^{\frac{\sigma+1}{2}}} \sum_{n=0}^{\infty} (-1)^n \left[\frac{\left(\frac{1+\sigma}{2}-\epsilon\right)_n}{\left(\frac{1+\sigma}{2}-\epsilon\right)\left(\frac{3+\sigma}{2}-\epsilon\right)_n} - \frac{\left(\frac{1+\sigma}{2}+\epsilon\right)_n}{\left(\frac{3+\sigma}{2}+\epsilon\right)_n \left(\frac{1+\sigma}{2}+\epsilon\right)} \right] \frac{(1+\sigma)_n}{n!} \sin(2n+1+\sigma)\nu$$

Consider each series separately. First we have

$$s_1(\nu) = \sum_{n=0}^{\infty} \left[\frac{\left(\frac{\sigma+1}{2}-\epsilon\right)_n}{\left(\frac{1-\sigma}{2}-\epsilon\right)\left(\frac{3-\sigma}{2}-\epsilon\right)_n} - \frac{\left(\frac{\sigma+1}{2}+\epsilon\right)_n}{\left(\frac{3-\sigma}{2}+\epsilon\right)_n \left(\frac{1-\sigma}{2}+\epsilon\right)} \right] (-1)^n \sin(2n+1)\nu$$

This series can be summed as a Fourier sine series and the functional dependence of s_1 upon ν can be determined explicitly as follows. Modify $s_1(\nu)$ by using the definition of the Barnes symbol.

$$s_1(\nu) = \sum_{n=0}^{\infty} \left[\frac{\Gamma(\frac{1-\sigma}{2}-\epsilon)\Gamma(\frac{1-\sigma}{2}+\epsilon)}{\Gamma(\frac{2-\sigma}{2}+\epsilon-n-\frac{1}{2})\Gamma(\frac{2-\sigma}{2}-\epsilon+n+\frac{1}{2})} - \frac{\Gamma(\frac{1-\sigma}{2}-\epsilon)\,\Gamma(\frac{1-\sigma}{2}+\epsilon)}{\Gamma(\frac{2-\sigma}{2}+\epsilon+n+\frac{1}{2})\,\Gamma(\frac{2-\sigma}{2}-\epsilon-n-\frac{1}{2})} \right] \sin(2n+1)\nu$$

The bracket can be represented as an integral (Gröbner-2, Page 108,9(c)).

$$s_1(\nu) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1-\sigma}{2}-\epsilon)\Gamma(\frac{1-\sigma}{2}+\epsilon)2^{2-\sigma}}{\pi\ \Gamma(1-\sigma)} \left[\int_0^{\pi/2} \cos^{-\sigma}x\ \sin\epsilon x\ \sin(2n+1)x\ dx \right] \sin(2n+1)\nu$$

$$S_1(\nu) = \sum_{n=0}^{\infty} \frac{\Gamma(\frac{1-\sigma}{2}-\varepsilon)\Gamma(\frac{1-\sigma}{2}+\varepsilon)2^{2-\sigma}}{\pi\Gamma(1-\sigma)}\left[\int_0^{\pi}|\cos x|^{-\sigma}\sin\varepsilon x\,\sin(2n+1)x\,dx\right]\sin(2n+1)\nu$$

By the Fourier theorem we obtain

$$(46)\qquad S_1(\nu) = \frac{\Gamma(\frac{1-\sigma}{2}-\varepsilon)\Gamma(\frac{1-\sigma}{2}+\varepsilon)\,2^{2-\sigma}}{\pi\Gamma(1-\sigma)}|\cos\nu|^{-\sigma}\sin\varepsilon\nu$$

The second series in (45)

$$(47)\qquad S_2(\nu) = \sum_{n=0}^{\infty}\left[\frac{\left(\frac{1+\sigma}{2}-\varepsilon\right)_n}{\left(\frac{1+\sigma}{2}-\varepsilon\right)\left(\frac{3+\sigma}{2}-\varepsilon\right)_n} - \frac{\left(\frac{1+\sigma}{2}+\varepsilon\right)_n}{\left(\frac{3+\sigma}{2}+\varepsilon\right)_n\left(\frac{1+\sigma}{2}+\varepsilon\right)}\right]\frac{(1+\sigma)_n}{n!}(-1)^n\sin(2n+1+\sigma)\nu$$

can be shown to have the value

$$(48)\qquad S_2(\nu) = \frac{\Gamma(\frac{\sigma+1}{2}-\varepsilon)\Gamma(\frac{\sigma+1}{2}+\varepsilon)}{\Gamma(1+\sigma)}\sin 2\varepsilon\nu\,.$$

First of all we note that (47) reduces to

$$S_2(\nu) = \sum_{n=0}^{\infty}\left[\frac{2}{2n+1+\sigma-2\varepsilon} - \frac{2}{2n+1+\sigma+2\varepsilon}\right]\frac{(-1)^n(1+\sigma)_n}{n!}\sin(2n+1+\sigma)\nu\,.$$

This series may now be summed by the following steps. Making use of the identity

$$\sin(2n+1+\sigma-2\varepsilon+2\varepsilon)\nu = \sin(2n+1+\sigma-2\varepsilon)\nu \cos 2\varepsilon\nu + \cos(2n+1+\sigma-2\varepsilon)\nu \sin 2\varepsilon\nu$$

we have

$$S_2(\nu) = 2\cos 2\varepsilon\nu\left[\sum_{n=0}^{\infty}\frac{(-1)^n(1+\sigma)_n}{n!}\frac{\sin(2n+1+\sigma-2\varepsilon)\nu}{2n+1+\sigma-2\varepsilon} - \sum_{n=0}^{\infty}\frac{(-1)^n(1+\sigma)_n}{n!}\frac{\sin(2n+1+\sigma+2\varepsilon)\nu}{2n+1+\sigma+2\varepsilon}\right]$$

$$+ 2\sin 2\varepsilon\nu\left[\sum_{n=0}^{\infty}\frac{(-1)^n(1+\sigma)_n}{n!}\frac{\cos(2n+1+\sigma-2\varepsilon)\nu}{2n+1+\sigma-2\varepsilon} - \sum_{n=0}^{\infty}\frac{(-1)^n(1+\sigma)_n}{n!}\frac{\cos(2n+1+\sigma+2\varepsilon)\nu}{2n+1+\sigma+2\varepsilon}\right].$$

For the four series in the brackets we introduce the notation

$$S_2(\nu) = 2\cos 2\varepsilon\nu\left[A(\nu)-C(\nu)\right] + 2\sin 2\varepsilon\nu\left[B(\nu)+D(\nu)\right].$$

We will now prove that the first bracket vanishes and that the second bracket has a value independent of ν. To show this, differentiate each series with respect to ν.

We have

$$A'(\nu) = \sum_{n=0}^{\infty}\frac{(-1)^n(1+\sigma)_n}{n!}\;\frac{e^{i(2n+1+\sigma-2\varepsilon)\nu}+e^{-i(2n+1+\sigma-2\varepsilon)\nu}}{2}$$

$$= \frac{1}{2}e^{i(1+\sigma-2\varepsilon)\nu}\,{}_2F_1(1+\sigma,1;1;-e^{2i\nu}) + \frac{1}{2}e^{-i(1+\sigma-2\varepsilon)\nu}\,{}_2F_1(1+\sigma,1;1;-e^{-2i\nu})$$

$$= \frac{\cos 2\nu\varepsilon}{2^{1+\sigma}\cos^{1+\sigma}\nu}$$

Similarly

$$B'(\nu) = \frac{\sin 2\epsilon\nu}{2^{1+\sigma}\cos^{1+\sigma}\nu}\ , \quad C'(\nu) = \frac{\cos 2\epsilon\nu}{2^{1+\sigma}\cos^{1+\sigma}\nu}\ , \text{ and } D'(\nu) = \frac{-\sin 2\epsilon\nu}{2^{1+\sigma}\cos^{1+\sigma}\nu}\ .$$

Also $A(0) = C(0) = 0$. It follows that $A(\nu) = C(\nu)$.

Furthermore $D'(\nu) = -B'(\nu)$.

$$D(\nu) - D(0) = B(0) - B(\nu) = -\int_0^\nu \frac{\sin 2\epsilon\tau}{2^{1+\sigma}\cos^{1+\sigma}\tau}\, d\tau$$

$$S_2(\nu) = 2\cos 2\epsilon\nu\left[A(\nu) - C(\nu)\right] + 2\sin 2\epsilon\nu\left[B(0) + D(0)\right]$$

$$S_2(\nu) = 2\sin 2\epsilon\nu \sum_{n=0}^{\infty} \frac{(-1)^n (1+\sigma)_n}{n!}\left[\frac{1}{2n+1+\sigma-2\epsilon} - \frac{1}{2n+1+\sigma+2\epsilon}\right]$$

$$S_2(\nu) = \sin 2\epsilon\nu \sum_{n=0}^{\infty}\left[\frac{1}{n+1+\frac{\sigma-1}{2}-\epsilon} - \frac{1}{n+1+\frac{\sigma-1}{2}+\epsilon}\right]\frac{\Gamma(-\sigma)}{\Gamma(1+n)\,\Gamma(-\sigma-n)}$$

$$S_2(\nu) = \sin 2\epsilon\nu \sum_{n=0}^{\infty}\left[\frac{1}{n+1+\frac{\sigma-1}{2}-\epsilon} - \frac{1}{n+1+\frac{\sigma-1}{2}+\epsilon}\right]\binom{-\sigma-1}{n}$$

Note that this form for $S_2(\nu)$ is far simpler than form (47). We now show how the infinite series immediately above can be summed as Gamma functions involving σ and ϵ. First we use the identity $\int_0^1 \sum_{n=0}^{\infty}\binom{\alpha}{n} x^{n+\beta}\, dx = \sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{1}{n+1+\beta} = \int_0^1 (1+x)^\alpha x^\beta\, dx$.

$$S_2(\nu) = \sin 2\epsilon\nu \int_0^1 \left[(1+x)^{-\sigma-1} x^{\frac{\sigma-1}{2}-\epsilon} + (1+x)^{-\sigma-1} x^{\frac{\sigma-1}{2}+\epsilon}\right] dx$$

$$S_2(\nu) = \sin 2\varepsilon\nu \int_0^1 \left[\frac{x^{\frac{\sigma+1}{2}-\varepsilon-1} + x^{\frac{\sigma+1}{2}+\varepsilon-1}}{(1+x)^{\sigma+1}} \right] dx$$

This integral is known (Gröbner-2, Page 171,9(d)).

$$S_2(\nu) = \sin 2\varepsilon\nu\, \beta\left(\tfrac{\sigma+1}{2}-\varepsilon, \tfrac{\sigma+1}{2}+\varepsilon\right) = \sin 2\varepsilon\nu \frac{\Gamma\left(\frac{\sigma+1}{2}-\varepsilon\right)\Gamma\left(\frac{\sigma+1}{2}+\varepsilon\right)}{\Gamma(\sigma+1)}$$

The proof of (48) is now complete. Now that the series $S_1(\nu)$ and $S_2(\nu)$ have been summed, and are represented by (46) and (48) respectively, these functions continue the series for $S_1(\nu)$ and $S_2(\nu)$ beyond Re $\sigma < 1$, and this restriction upon σ may be removed.

Returning to equation (45) and using (46) and (48) we obtain

$$t(x) \approx \frac{\Gamma(\sigma)\, a^{1+\sigma} x^{\sigma+\frac{1}{2}}\, 2^{2-\sigma}}{\pi\, (a^2+\gamma^2)^{\frac{1}{2}}} |\cos\nu|^{-\sigma} \sin\varepsilon\nu$$

$$+ \frac{\Gamma(\sigma)\, a\, 2^{\sigma}\, x^{\frac{1}{2}-\sigma}}{(a^2+\gamma^2)^{\frac{\sigma+1}{2}}} \sin 2\varepsilon\nu$$

Remember that this is an approximate form for $t(x)$ when x is small, and that, for any x, $t(x)$ must be a linear combination of modified Bessel functions. Now

$$x^{\sigma} \approx \frac{I_{\sigma}(\sqrt{a^2+\gamma^2}\, x)\, 2^{\sigma}\, \Gamma(1+\sigma)}{(a^2+\gamma^2)^{\sigma/2}}$$

when x is small.

We conclude

$$t(x) = \frac{\Gamma(1+\sigma)\Gamma(\sigma)\, 4\, a^{1+\sigma}}{\pi\, (a^2+\gamma^2)^{\frac{\sigma+1}{2}}} |\cos\nu|^{-\sigma} \sin\varepsilon\nu\; x^{\frac{1}{2}} I_{\sigma}(\sqrt{a^2+\gamma^2}\, x)$$

$$+ \frac{\Gamma(1-\sigma)\Gamma(\sigma)\, a}{(a^2+\gamma^2)^{\frac{1}{2}}} \sin 2\varepsilon\nu\; x^{\frac{1}{2}} I_{-\sigma}(\sqrt{a^2+\gamma^2}\, x)$$

Using the Weierstrass identity $\Gamma(-\sigma)\Gamma(1+\sigma) = \dfrac{-\pi}{\sin\pi\sigma}$ we obtain

$$t(x) = \frac{-4x^{\frac{1}{2}}}{\sin\pi\sigma}(\cos\nu)^{1+\sigma}|\cos\nu|^{-\sigma}\sin\epsilon\nu\, I_{\sigma}(\sqrt{a^2+\gamma^2}\,x) + \frac{\pi x^{\frac{1}{2}}}{\sin\pi\sigma}(\cos\nu)\sin 2\epsilon\nu\, I_{-\sigma}(\sqrt{a^2+\gamma^2}\,x)$$

Our final integral representation in this case is

$$(49)\quad \int_{-\infty}^{\infty} W_{\epsilon,\frac{\sigma}{2}}(a\xi^2)\, W_{-\epsilon,\frac{\sigma}{2}}(a\eta^2)\sin\gamma y\, d\gamma =$$

$$\frac{-4x}{\sin\pi\sigma}(\cos)^{1+\sigma}|\cos\nu|^{-\sigma}\sin\epsilon\nu\, I_{\sigma}(\sqrt{a^2+\gamma^2}\,x) + \frac{\pi x}{\sin\pi\sigma}\cos\nu\sin 2\epsilon\nu\, I_{-\sigma}(\sqrt{a^2+\gamma^2}\,x),$$

provided $|\mathrm{Re}\,a| > 0$, $|\mathrm{Re}\,\sigma| < 2$.

B. For case (2) we have

$$F_1(\xi)L_1 = \frac{d^2}{d\xi^2} + (k^2-\mu^2)\xi^2 - \tau^2\xi^6 + \frac{1-16\sigma^2}{\xi^2}, \qquad F_1(\xi) = 1 \quad p_1(\xi) = 1$$

$$F_2(\eta)L_2 = \frac{-d^2}{d\eta^2} + (\mu^2-k^2)\eta^2 + \tau^2\eta^6 - \frac{1-16\sigma^2}{\eta^2} \qquad F_2(\eta) = 1 \quad p_2(\eta) = 1$$

$$F_1'(x)L_1' = \frac{d^2}{dx^2} + k^2-\mu^2-\gamma^2-\tau^2x^2 + \frac{1-16\sigma^2}{x^2} \qquad F_1'(x) = 1 \quad p_1'(x) = 1$$

$$F_2'(y)L_2' = \frac{-d^2}{dy^2} - \gamma^2 + 4\tau^2y^2 \qquad F_2'(y) = 1 \quad p_2'(y) = 1$$

The separated ordinary differential equations corresponding to these operators are solved in Section III. We assumed that $U = g_1(\xi)g_2(\eta)e^{i\mu z}$ or $U = k_1(x)k_2(y)e^{i\mu z}$ so we simply list the solutions we will use, which are determined by our choice of path and the vanishing of the bilinear concomitant. For our kernel we choose the product of (24) and (25)

$$K(\xi,\eta) = \xi^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\xi^4\right) \eta^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\eta^4\right) \quad , \quad \varepsilon = \frac{\mu^2 - k^2}{8\tau} ,$$

and (21) for

$$f_2(x_2) = \frac{1}{\sqrt{x}} \; W_{-\nu-2\varepsilon, \frac{\sqrt{16\sigma^2 - 3}}{2}}\left(\tau x^2\right) \quad , \quad \nu = \frac{\gamma^2}{4\tau} .$$

If we choose a suitable path and use $K(\xi,\eta)\, f_2(x_2)$ as our integrand, the integral will then represent parabolic cylinder functions of order $\nu - \frac{1}{2}$. The integral

$$t(y) = \int_C \xi^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\xi^4\right) \eta^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\eta^4\right) \frac{1}{\sqrt{x}} \; W_{-\nu-2\varepsilon, \frac{\sqrt{16\sigma^2-3}}{2}}\left(\tau x^2\right) dx$$

will be a linear combination of $e^{-i\tau y^2}\ {}_1F_1\left(\frac{-\nu}{2}+\frac{1}{4};\frac{1}{2};2i\tau y^2\right)$ and $e^{-i\tau y^2}\ y\ {}_1F_1\left(\frac{-\nu}{2}+\frac{3}{4};\frac{3}{2};2i\tau y^2\right)$. (See (23) Section III).

Our choice of path C is the real axis from $-\infty$ to $+\infty$, since this causes the bilinear concomitant to vanish, e.g.

$$\left. \frac{\partial}{\partial x}\left[\xi^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\xi^4\right) \eta^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\eta^4\right)\right] \frac{1}{\sqrt{x}} W_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}\left(\tau x^2\right) - \xi^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\xi^4\right) \eta^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\eta^4\right) \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x}} W_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}\left(\tau x^2\right)\right] \right|_{-\infty}^{+\infty} = 0 ,$$

provided $\mathrm{Re}\,\nu > -\frac{1}{2}$ (Magnus-6).

The integral can be written

$$t(x) = \int_{-\infty}^{+\infty} \xi^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\xi^4\right) \eta^{-\frac{3}{2}} W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}\eta^4\right) \frac{1}{\sqrt{X}}\; W_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau X^2)\, dX .$$

This converges absolutely for $\mathrm{Re}\,\nu > -\frac{1}{2}$ and when $x = 0$ this is valid for all real values of y provided $|\mathrm{Re}\; 4\sigma| + |\mathrm{Re}\sqrt{16\sigma^2-3}| < 2$. By the theorem of Section IV we may assume $t(x)$ has the following form:

$$t(x) = A(\varepsilon,\sigma,\nu,\tau)\, e^{-i\tau y^2}\, {}_1F_1\left(\tfrac{1-\nu}{2}; \tfrac{1}{2}; 2i\tau y^2\right) + B(\varepsilon,\sigma,\nu,\tau)\, e^{-i\tau y^2}\, y\, {}_1F_1\left(\tfrac{3}{4}-\tfrac{\nu}{2}; \tfrac{3}{2}; 2i\tau y^2\right)$$

It remains only to determine $A(\epsilon,\sigma,\nu,\tau)$ and $B(\epsilon,\sigma,\nu,\tau)$. To determine $B(\epsilon,\sigma,\nu,\tau)$ we consider the $\lim\limits_{y\to 0}\left(\frac{\partial t(x)}{\partial y}\right)$.

$$B(\varepsilon,\sigma,\nu,\tau) = \lim_{y\to 0}\left[\int_{-\infty}^{+\infty} -\frac{\tau}{2}\left(\frac{\partial \xi^4}{\partial y} + \frac{\partial \eta^4}{\partial y}\right) W'_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}X^2\right) W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}X^2\right) W_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau X^2)\, \frac{dX}{X^2}\right.$$

Since $\dfrac{\partial \xi^4}{\partial y} + \dfrac{\partial \eta^4}{\partial y} = 8y$, the integrand tends to zero hence

$B(\varepsilon,\sigma,\nu,\tau) \equiv 0$.

To determine $A(\epsilon,\sigma,\nu,\tau)$ we let y approach zero.

$$A(\varepsilon,\sigma,\nu,\tau) = 2\int_0^{\infty} \left[W_{\varepsilon,\sigma}\left(-\tfrac{\tau}{2}X^2\right)\right]^2 W_{-\nu-2\varepsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau X^2)\, \frac{dX}{X^2}$$

We have not been able to evaluate this integral so our answer takes the final form,

$$(50)\quad \int_{-\infty}^{+\infty} \xi^{-3/2} W_{\epsilon,\sigma}\left(\tfrac{-\tau}{2}\xi^4\right) \eta^{-3/2} W_{\epsilon,\sigma}\left(\tfrac{-\tau}{2}\eta^4\right) W_{-\nu-2\epsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau x^2)\frac{dx}{\sqrt{x}} =$$

$$A(\epsilon,\sigma,\nu,\tau)\, e^{-i\tau y^2} {}_1F_1\left(\tfrac{1-\nu}{2};\tfrac{1}{2};2i\tau y^2\right) \quad \text{where } A(\epsilon,\sigma,\nu,\tau) = 2\int_0^{\infty}\left[W_{\epsilon,\sigma}\left(\tfrac{-\tau}{2}x^4\right)\right]^2 W_{-\nu-2\epsilon,\frac{\sqrt{16\sigma^2-3}}{2}}(\tau x^2)\frac{dx}{x^2}$$

We notice that Kummer's function does not depend on σ, and ϵ so we consider the special case where $\sigma = \epsilon - \frac{1}{2}$. Our integral for $A(\epsilon,\epsilon-\frac{1}{2},\nu,\tau)$ becomes

$$A(\epsilon,\epsilon-\tfrac{1}{2},\nu,\tau) = \sqrt{2\tau}\int_0^{\infty}(-\varsigma^2)^{2\epsilon} e^{\varsigma^2}\, W_{-\nu-2\epsilon,\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}}}(2\varsigma^2)\,\frac{d\varsigma}{\varsigma^2}$$

where $\varsigma = \sqrt{\frac{\tau}{2}}\, x$. Let $2\varsigma^2 = u$ to obtain

$$A(\epsilon,\epsilon-\tfrac{1}{2},\nu,\tau) = \frac{\sqrt{\tau}}{2^{2\epsilon}}\int_0^{\infty} u^{2\epsilon-\frac{3}{2}}\, e^{\frac{u}{2}}\, W_{-\nu-2\epsilon,\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}}}(u)\, du$$

This integral is known (Bateman-1, Vol. 2, Page 406, (25)).

$$A(\epsilon,\epsilon-\tfrac{1}{2},\nu,\tau) = \lim_{\lambda\to 0}\frac{-\lambda a^{\lambda}\sqrt{\tau}\; G_{34}^{33}\left(a\,\middle|\,{\frac{1}{2},1,\frac{1}{2}-\nu \atop 2\epsilon+\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}},\,2\epsilon-\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}},-\lambda,\lambda}\right)}{\pi^{\frac{1}{2}}\, 2^{2\epsilon}\,\Gamma\left(\frac{1}{2}+\nu+2\epsilon+\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}}\right)\Gamma\left(\frac{1}{2}+\nu+2\epsilon-\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}}\right)}$$

Let $R = \sqrt{4\epsilon^2 - 4\epsilon + \frac{1}{4}}$ and use the definition of the Meier G-Function (Bateman-1) to obtain

$$A(\epsilon,\epsilon-\tfrac{1}{2},\nu,\tau) = \frac{\sqrt{\tau}\; 2^{-2\epsilon}\,\pi^{-\frac{1}{2}}}{\Gamma(\frac{1}{2}+\nu+2\epsilon+R)\,\Gamma(\frac{1}{2}+\nu+2\epsilon-R)}\,.$$

$$\lim_{\lambda\to 0}\left(\frac{\Gamma(-2R)\Gamma(-\lambda-2\epsilon-R)\Gamma(\frac{1}{2}+2\epsilon+R)\Gamma(2\epsilon+R)\,\Gamma(\frac{1}{2}+\nu+2\epsilon+R)}{a^{2\epsilon-R}\,\Gamma(1+2\epsilon-\lambda+R)}\;{}_3F_3\left[\begin{matrix}\frac{1}{2}+2\epsilon+R,\frac{1}{2}+\nu+2\epsilon+R,2\epsilon+R\\ 1+2R,1+2\epsilon+\lambda+R,1+2\epsilon-\lambda+R\end{matrix};-a\right]\right.$$

$$\frac{\Gamma(2R)\,\Gamma(-\lambda-2\epsilon+R)\Gamma(\frac{1}{2}+2\epsilon-R)\Gamma(2\epsilon-R)\,\Gamma(\frac{1}{2}+\nu+2\epsilon-R)}{a^{-2\epsilon+R}\;\Gamma(1+2\epsilon-R-\lambda)}\;{}_3F_3\left[\begin{matrix}\frac{1}{2}+2\epsilon-R,2\epsilon-R,\frac{1}{2}+\nu+2\epsilon-R\\ 1-2R,1+2\epsilon+\lambda-R,1+2\epsilon-\lambda-R\end{matrix};-a\right]$$

$$\left.\frac{\Gamma(-\lambda)\,\Gamma(\frac{1}{2}-\lambda)\;\Gamma(\frac{1}{2}+\nu-\lambda)\Gamma(\lambda+2\epsilon-R)\Gamma(\lambda+2\epsilon+R)}{a^{\lambda}\;\Gamma(1-2\lambda)}\;{}_3F_3\left[\begin{matrix}\frac{1}{2}-\lambda,-\lambda,\frac{1}{2}-\lambda+\nu\\ 1-2\lambda,1-\lambda-2\epsilon+R,1-\lambda-2\epsilon-R\end{matrix};-a\right]\right)$$

Since $\Gamma(-\lambda) = \frac{-1}{\lambda}$ in the neighborhood of $\lambda = 0$, we obtain

$$A(\epsilon,\epsilon-\tfrac{1}{2},\nu,\tau) = \frac{\sqrt{\tau}}{2^{2\epsilon}}\;\frac{\Gamma(2\epsilon+R)\Gamma(2\epsilon-R)\,\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}+\nu+2\epsilon+R)\;\Gamma(\frac{1}{2}+\nu+2\epsilon-R)}\,.$$

For the special case $\sigma = \epsilon - \frac{1}{2}$ our integral becomes

$$(51)\qquad \int_{-\infty}^{+\infty} W_{\epsilon,\epsilon-\frac{1}{2}}(-\tfrac{\tau}{2}\xi^4)\,W_{\epsilon,\epsilon-\frac{1}{2}}(-\tfrac{\tau}{2}\eta^4)\;W_{-\nu-2\epsilon,\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}}}(\tau x^2)\;\frac{dx}{x^2} =$$

$$\frac{\sqrt{\tau}}{2^{2\epsilon}}\;\frac{\Gamma(2\epsilon+\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}})\Gamma(2\epsilon-\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}})\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2}+\nu+2\epsilon+\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}})\Gamma(\frac{1}{2}+\nu+2\epsilon-\sqrt{4\epsilon^2-4\epsilon+\frac{1}{4}})}\;e^{-i\tau y^2}\;{}_1F_1(\tfrac{1-\nu}{2};\tfrac{1}{2};2i\tau y^2).$$

VII. EQUIVALENCE OF FORM FOR THE INTEGRAL REPRESENTATIONS USING S(3,6) AND S(1,2)

To show the equivalence of form we will begin by considering S(1,3). Referring to Section III we have

$$F_1(\xi)L_1 = \frac{1}{\xi}\frac{d}{d\xi}\left(\xi\frac{d}{d\xi}\right) + \quad 2a - c_o + k^2\xi^2 + \frac{d-b-\mu^2}{\xi^2} \qquad F_1(\xi)=1 \quad p_1(\xi)=\xi$$

$$F_2(\eta)L_2 = -\frac{1}{\eta}\frac{d}{d\eta}\left(\eta\frac{d}{d\eta}\right) + \quad -c_o - k^2\eta^2 - \frac{d+b-\mu^2}{\eta^2} \qquad F_2(\eta)=1 \quad p_2(\eta)=\eta$$

$$F_1'(r)L_1' = \frac{d}{dr}\left(r^2\frac{d}{dr}\right) + ar + k^2r^2 - c_1 \qquad F_1'(r)=1 \quad p_1'(r)=r^2$$

$$F_2'(\theta)L_2' = \frac{-d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\right) - c_1\sin\theta - \frac{d-\mu^2+b\cos\theta}{\sin\theta} \qquad F_2'(\theta)=1 \quad p_2'(\theta)=\sin\theta$$

The separated ordinary differential equation corresponding to these operators are solved in Section III. We assumed that $U = k_1(\xi)k_2(\eta)e^{i\mu\phi}$ or $U = g_1(r)g_2(\theta)e^{i\mu\phi}$ so we simply list the solutions we will use in this section, which are determined by our choice of path and the vanishing of the bilinear concomitant. For our kernel we choose the product of (13) and (14),

$$K(\xi,\eta) = \frac{1}{\xi}W_{\nu,\sigma}(ki\xi^2)\frac{1}{\eta}W_{\gamma,\tau}(ki\eta^2)$$

and (15) for $F_2(x_2) = \frac{1}{r}W_{\nu+\gamma,\epsilon}(2kir)$

By the integral theorem of Section IV we have

$$t(\theta) = \int_C \frac{1}{\xi} W_{\nu,\sigma}(ki\xi^2) \, \frac{1}{\eta} W_{\gamma,\tau}(ki\eta^2) \, \frac{1}{r} W_{\nu+\gamma,\varepsilon}(2kir) \, dr$$

where $t(\theta)$ represents one or possibly a linear combination of the various functions denoted by

$$\tilde{\emptyset}^{-2\tau,-\frac{1}{2}\pm 2\sigma}_{-\frac{1}{2}\pm 2\varepsilon} \left(\cos^2 \frac{\theta}{2}\right).$$

If we choose a suitable path and use $K(\xi,\eta)\, f_2(x_2)$ as our integrand, the integral will represent a generalized hypergeometric equation (Leitner-4). For our path we choose the real axis from 0 to ∞ since this causes the bilinear concomitant to vanish, e.g.

$$r^2 \left(\frac{\partial}{\partial r} \left[\frac{1}{\xi\eta} W_{\nu,\sigma}(ki\xi^2) W_{\gamma,\tau}(ki\eta^2) \right] \frac{1}{r} W_{\nu+\gamma,\varepsilon}(2kir) - \frac{1}{\xi\eta} W_{\nu,\sigma}(ki\xi^2) W_{\gamma,\tau}(ki\eta^2) \, \frac{\partial}{\partial r} \left[\frac{1}{r} W_{\nu+\gamma,\varepsilon}(2kir) \right] \right) \Bigg|_0^{\infty} = 0$$

provided $|\operatorname{Re}\sigma| + |\operatorname{Re}\tau| + |\operatorname{Re}\varepsilon| < \frac{1}{2}$.

Our integral takes the form

$$t(\theta) = \int_0^{\infty} \frac{1}{\xi\eta r} W_{\nu,\sigma}(ki\xi^2) W_{\gamma,\tau}(ki\eta^2) W_{\nu+\gamma,\varepsilon}(2kir) \, dr$$

Using $r \sin\theta = \xi\eta$ and $r\cos\theta = \frac{1}{2}(\xi^2 - \eta^2)$ we have $\eta^2 = r(1 - \cos\theta)$, $\xi^2 = r(1 + \cos\theta)$, so our integral becomes

$$t(\theta) = \int_0^{\infty} \frac{1}{r^2 \sin\theta} W_{\nu,\sigma}(kir[1+\cos\theta]) W_{\gamma,\tau}(kir[1-\cos\theta]) W_{\nu+\gamma,\varepsilon}(2kir) \, dr,$$

and this integral converges when $|\operatorname{Re}\sigma| + |\operatorname{Re}\tau| + |\operatorname{Re}\varepsilon| < \frac{3}{2}$.

Let $\cos\frac{\theta}{2} = w$ and $r = \frac{\rho^2}{2ki}$, we have

$$(52)\quad t(\theta) = \frac{1}{2kiw\sqrt{1-w^2}} \int_0^\infty \frac{1}{\rho^3} W_{\nu,\sigma}(\rho^2 w^2)\, W_{\gamma,\tau}(\rho^2[1-w^2])\, W_{\gamma+\tau,\varepsilon}(\rho^2)\, d\rho .$$

This integral is equivalent to (42) where θ corresponds to $90^\circ - \Phi$. The integral (42) was obtained by considering S(1,2) and we realize that (52) is equivalent to (42) and is obtained from S(3,6).

VIII. CONCLUSION

The objective of this thesis was to present a unified method of generating integral representations in special function theory, using the idea of simultaneous separability of

$$\nabla^2 U + \phi U = 0$$

in orthogonal curvilinear coordinate systems.

Not all possible cases were pursued. For those in which the functional equations for ϕ could be solved and in which the special functions were of the hypergeometric class, several integrals were obtained.

Many more integral representations could be generated using S(2,4), S(2,6) and case (3) of S(1,4), which we did not find time to investigate. Furthermore, it is conceivable that other simultaneous separabilities exist which we did not find. We did not investigate all possible special cases of our main results (42), (49) and (50).

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