TIME DOMAIN ANALYSIS OF NETWORKS CONTAINING A UNIFORM TRANSMISSION LINE

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY
Harlow M. Judson

1964

THESIS

This is to certify that the

thesis entitled

TIME DOMAIN ANALYSIS OF
NETWORKS CONTAINING A UNIFORM
TRANSMISSION LINE

presented by

Harlow M. Judson

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in E.E.

Richard J. Reid
Major professor

Date April 15, 1964

O-169

ABSTRACT

TIME DOMAIN ANALYSIS OF NETWORKS CONTAINING A UNIFORM TRANSMISSION LINE

by Harlow M. Judson

In recent years, there has been considerable interest in the formulation of derivative-explicit equations as the mathematical model of lumped physical systems. This thesis deals with the formulation and solution of derivative-explicit equations for connected networks containing a uniform transmission line.

The incorporation of a transmission line, described by partial differential equations, into a set of ordinary differential equations associated with the remainder of the network is accomplished by utilization of the Laplace transform in a manner not dependent on the linearity of other components in the network. The basic steps in the preliminary analysis are based on the concepts of oriented linear graphs.

A sufficient condition for the applicability of the analysis is that the transmission line operate in the conventional two-port manner, that is, in accordance with the simultaneous equations

$$\frac{\partial v}{\partial x} = - Ri - L \frac{\partial i}{\partial t}$$

$$\frac{\partial i}{\partial x} = - Gv - C \frac{\partial v}{\partial t}$$

and that all R, C, and L elements be positive and finite.

The solution of the derivative-explicit equations is shown to be sufficient for the solution of the network.

The complete general formulation is carried out for linear time invariant networks containing a distortionless transmission line, and several special cases are considered.

The derivative-explicit equations are in the form of series of terms representing the multiple reflections and time delays associated with the transmission line. It is found that the solutions must be obtained in a step-by-step process due to the time delayed terms.

The general formulation, several special cases, and the general solution process are illustrated by example.

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A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Electrical Engineering

1964

ACKNOWLEDGEMENT

The author wishes to express his indebtedness to committee chairman, Dr. R. J. Reid for many valuable suggestions and patient encouragement; to Drs. R. C. Dubes, L. J. Giacoletto, and H. E. Koenig for their many helpful conversations; and to department chairman, Dr. L. W. Von Tersch for his unfailing support while this thesis was being written.

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INTRODUCTION

The time-domain analysis of linear, lumped parameter, electrical networks has been the subject of many authors (1,2,3) and the transient analysis of transmission lines has also been extensively treated (4,5,6). Transmission lines are normally classified as distributed parameter components which can be analyzed from the viewpoint of either field theory or circuit theory.

Under very few restrictions, derivative-explicit equations can be derived for RLC networks such that solution of the equations is sufficient to complete the solution of the networks (1,3). A network solution implies that the current and voltage have been obtained for all components of the network.

The two primary questions considered in this thesis are:

(1) Under what conditions can a set of derivative-explicit equations be obtained for RLC networks which contain a uniform transmission line?

(2) Can the linear transmission line be handled in a manner which is independent of the linearity of the rest of the network?

In Chapter I the restrictions on the network are stated and the oriented linear graph representation of the transmission line is introduced. A set of modified derivative-explicit equations is then derived and a discussion on the reduction of the modified equations into the normal derivative-explicit form is given.

In Chapter II the four cases which can arise with respect to placement of the transmission line graph elements in the tree are considered. A transmission line problem can be associated with each of the four cases and the Laplace transform is used to obtain general solutions for the four problems. If the transmission line is distortionless, general time domain solutions can be obtained in terms of assumed arbitrary driving functions. If the transmission line is not distortionless, the time domain solutions cannot be obtained for general driving functions except in terms of rather complex integrals. The time domain solutions obtained for distortionless lines are in the form of series of time-delayed terms corresponding to the multiple reflections which occur at the transmission line terminals.

In Chapter III, the series solutions of Chapter II are combined with the modified derivative-explicit equations of Chapter I to produce derivative-explicit equations in normal form. Each resulting equation is in the form of a series of terms. Each term contains a delayed unit step as a factor so that the terms become sequentially non-zero. As a consequence solutions must be obtained in a step-by-step manner since in general the equations change whenever a term of the series becomes non-zero. Recursion formulas are given for the calculation of coefficients appearing in the final equations.

In Chapter IV, the general formulation and solution processes are illustrated by example. Several special cases are also illustrated while the general solution is being obtained.

In Chapter V, the major results of the thesis are reviewed and some topics warranting further investigation are mentioned.

I. PRELIMINARY ANALYSIS

Given a connected network composed of a uniform transmission line of arbitrary length and a finite set of R,L,C, e(t) and h(t) elements such that:

(1) The network contains no circuit of e(t) elements.

(2) The network contains no cut-set of h(t) elements.

(3) The transmission line operates in accordance with the transmission line Eqs. 1.0 and 1.1.

$$\frac{\partial v}{\partial x} = - Ri - L \frac{\partial i}{\partial t} \qquad 1.0$$

$$\frac{\partial i}{\partial x} = - Gv - C \frac{\partial v}{\partial t} \qquad 1.1$$

(4) Certain matrix inverses exist; this will always be the case for networks with positive, finite, R,L, and C elements.

Then the following results will be demonstrated.

(1) The transmission line, operationally described by simultaneous partial differential equations, will be incorporated into a system of ordinary differential equations in a manner not dependent on the linearity of the remainder of the network.

(2) If the transmission line is distortionless a system of derivative-explicit equations will be derived. The solution of these

equations is shown to be sufficient to complete the solution of the network.

The general problem to be considered is illustrated by Fig. 1.0 where N is the entire network except for the transmission line. Note that N may be disconnected without violating the condition that the entire network be connected.

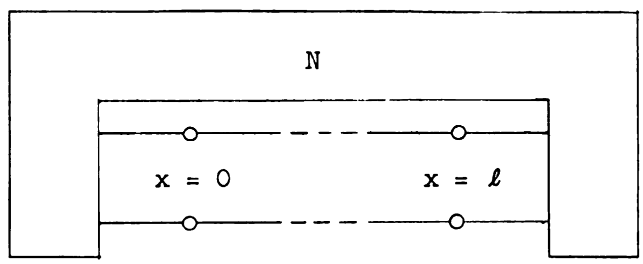

Figure 1.0 A general network containing a transmission line.

One of the possible oriented linear graph representations of a four-terminal component (7) is illustrated in Fig. 1.1. For the special case in which the four-terminal component is a transmission line operating in accordance with Eqs. 1.0 and 1.1, the element in Fig. 1.1 labeled v_3, i_3 may be omitted because i_3 is identically equal to zero. Figure 1.2 illustrates the oriented linear graph representation used throughout this thesis for the transmission line. The graph given in Fig. 1.2 will be called the two-port representation.

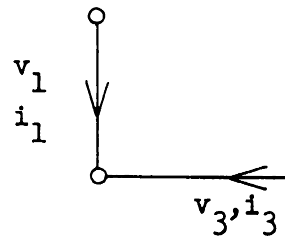

Figure 1.1 Linear graph representation of a four terminal component

Figure 1.2 Two-port graph representation of a transmission line

Two important points can be illustrated by considering the network of Fig. 1.3 and the corresponding linear graph of Fig. 1.4.

(1) The network currents and voltages obtained by using the two-port representation are not always valid.

(2) Use of the two-port representation, even when it is not valid, does give rise to a problem which can be analyzed.

Examination of Fig. 1.4 leads to the immediate conclusion that no current can exist in any of the elements since each element is a cut-set. This result illustrates point (2). Point (1) follows from the fact that a current would actually be present in the voltage driver due to the capacitance between the transmission line conductors.

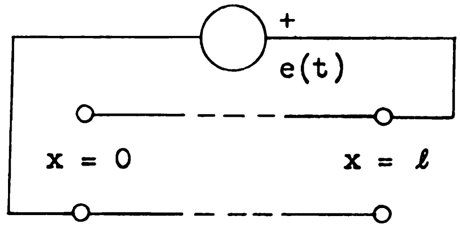

Figure 1.3 An example network for which the two-port transmission line representation is not valid

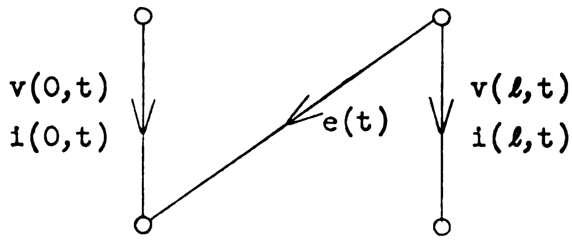

Figure 1.4 Graph of the example network using the two-port transmission line representation

The following definition and theorem specify one class of networks N for which the two-port transmission line representation is always valid.

<u>Definition 1.0</u>: A graph is <u>disconnected</u> if a proper subgraph and its complement have no vertex in common.

<u>Theorem 1.0</u>: If the network N of Fig. 1.0 is disconnected then the two-port transmission line representation is valid.

<u>Proof</u>: If an oriented linear graph is drawn for the entire network and the three element representation of Fig. 1.1 is used for the transmission line, then the element v_3, i_3 is a cut-set;

hence $i_3 \equiv 0$, which completes the proof.

The following definition and theorem are given as a possible aid to recognition of disconnected networks:

Definition 1.1: A graph is separable if a proper subgraph and its complement have exactly one vertex in common.

Theorem 1.1: The graph G of a network N is disconnected if and only if for some proper subgraph G_1 of G, the vertex matrix[1] of G can be written in the diagonal form[2];

$$A_a = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix} \qquad 1.2$$

where the columns of A_a associated with A_{11} correspond to the elements of G_1 and the columns of A_{22} correspond to elements in the complement of G_1.

Proof: Follows directly from definition 1.2.

The general analysis in the following paragraphs is carried out

---

1 See definition A(1).

2 Both the f-circuit and f-cut-set matrices of a graph G can be written in the diagonal form of Eq. 1.2 if G is either separable or disconnected.

for a general connected network N. The results are valid for networks in which the two-port representation of the transmission line is valid.

The graph elements associated with specific measurements at a transmission line port will be called T elements in this thesis. A subscript $0(\ell)$ will be appended to the T designation, when appropriate, to indicate a particular port.

For a given network of the class under consideration, an appropriate linear graph representation is obtained and a tree is selected by the following rules[1] used in the order given:

(1) All e(t) elements are placed in the tree.

(2) All possible C elements are placed in the tree.

(3) The tree is completed with R and/or T elements, if possible.

(4) If necessary, the tree is completed with L elements.

Several general statements can be made concerning these rules, with proof of validity readily available (2,3).

(1) If the graph has no circuit (cut-set) of voltage (current) drivers, it is always possible to place all voltage (current) drivers in a tree (cotree).

(2) Any C element not in the tree is in a circuit of e(t) and C elements.

---

[1] These rules are very similar to the rules used in RLC graph analysis (3).

(3) Any L element not in the cotree is in a cut-set of h(t) and L elements.

In the course of selecting a tree by the given rules, four possible assignments may be made with the T elements. These assignments will be called the four cases:

Case (1): $T_\ell(T_o)$ is a branch (chord).

Case (2): Both T elements are chords.

Case (3): T_o (T_ℓ) is a branch (chord).

Case (4): Both T elements are branches.

The notational convention for the equations in the following analysis is specified by definitions 1.2 and 1.3.

Definition 1.2: The branch (chord) L, G, and C element matrices of a graph G are the diagonal element value matrices L_1, G_1, C_1 (L_2, G_2, C_2) where the subscript 1 (2) indicates the elements to be branches (chords) of a tree (cotree).

Definition 1.3: The branch (chord) C, R, T, and L current and voltage matrices of a graph G are the column voltage and current matrices i_{C_1}, i_{R_1}, i_{T_1}, i_{C_1} (i_{C_2}, i_{R_2}, i_{T_2}, i_{L_2}) and v_{C_1}, v_{R_1}, v_{T_1}; v_{L_1} (v_{C_2}, v_{R_2}, v_{T_2}, v_{L_2}) where the subscript 1 (2) indicates the currents and voltages to be associated with the branches (chords) of a tree (cotree).

Three basic sets of equations associated with the graph of an electrical network are the f-cut-set, f-circuit, and element equations. The general forms of these equations for the class of networks under consideration are, respectively:

$$\left[\begin{array}{ccccc|ccccc} U & & \bigcirc & & & S_{11} & S_{12} & S_{13} & S_{14} & S_{15} \\ & U & & & & S_{21} & S_{22} & S_{23} & S_{24} & S_{25} \\ & & U & & & 0 & S_{32} & S_{33} & S_{34} & S_{35} \\ & \bigcirc & & U & & 0 & S_{42} & S_{43} & S_{44} & S_{45} \\ & & & & U & 0 & 0 & 0 & S_{54} & S_{55} \end{array}\right] \left[\begin{array}{c} i_e \\ i_{C_1} \\ i_{R_1} \\ i_{T_1} \\ i_{L_1} \\ \hline i_{C_2} \\ i_{R_2} \\ i_{T_2} \\ i_{L_2} \\ h(t) \end{array}\right] = 0 \qquad 1.3$$

$$\left[\begin{array}{ccccc|ccccc} B_{11} & B_{12} & 0 & 0 & 0 & U & & & \bigcirc & \\ B_{21} & B_{22} & B_{23} & B_{24} & 0 & & U & & & \\ B_{31} & B_{32} & B_{33} & B_{34} & 0 & & & U & & \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} & & \bigcirc & & U & \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} & & & & & U \end{array}\right] \left[\begin{array}{c} e(t) \\ v_{C_1} \\ v_{R_1} \\ v_{T_1} \\ v_{L_1} \\ \hline v_{C_2} \\ v_{R_2} \\ v_{T_2} \\ v_{L_2} \\ v_h \end{array}\right] = 0 \qquad 1.4$$

$$\begin{bmatrix} i_{C_1} \\ i_{R_1} \\ v_{L_1} \\ i_{C_2} \\ i_{R_2} \\ v_{L_2} \end{bmatrix} = \begin{bmatrix} C_1 & & & & & \\ & G_1 & & & \bigcirc & \\ & & L_1 & & & \\ & & & C_2 & & \\ & \bigcirc & & & G_2 & \\ & & & & & L_2 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} v_{C_1} \\ v_{R_1} \\ \frac{d}{dt} i_{L_1} \\ \frac{d}{dt} v_{C_2} \\ v_{R_2} \\ \frac{d}{dt} i_{L_2} \end{bmatrix} \qquad 1.5$$

Equations 1.3 and 1.4 are often written in more concise form as

$$[U \mid S_c] \begin{bmatrix} i_1 \\ \hline i_2 \end{bmatrix} = 0 \qquad 1.6$$

$$[B_T \mid U] \begin{bmatrix} v_1 \\ \hline v_2 \end{bmatrix} = 0 \qquad 1.7$$

When the columns of the f-cut-set and f-circuit equations are in the same order, the following relationship holds (Th. A(1)):

$$S_C = -B_T' \qquad 1.8$$

where the prime superscript indicates the transpose of the matrix. Applied to the submatrices of S_C and B_T, Eq. 1.8 implies:

$$S_{ij} = -B_{ji}' \qquad 1.9$$

The zero submatrices of $S_C(B_T)$ in Eq. 1.3 (1.4) are a direct consequence of statements (2) and (3) following the tree selection rules, and Eq. 1.9.

The general form of the derivative-explicit equations obtained for RLC graphs (1,3) can be written as

$$\frac{d}{dt}\begin{bmatrix} v_{C_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} A_o \end{bmatrix}\begin{bmatrix} v_{C_1} \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} A_1 \end{bmatrix}\begin{bmatrix} e(t) \\ h(t) \end{bmatrix} + \begin{bmatrix} A_2 \end{bmatrix}\begin{bmatrix} \frac{d}{dt}\, e(t) \\ \frac{d}{dt}\, h(t) \end{bmatrix} \qquad 1.10$$

where the coefficient matrices, $[A_i]$, are specific algebraic combinations of submatrices from the f-cut-set, f-circuit, and element equations. It is also shown that the solution of Eq. 1.10 when appropriately substituted into the three basic sets of equations is sufficient[1] to complete the solution of the network.

One of the major goals of this thesis is the derivation of equations similar in form and properties to Eq. 1.10 for RLC networks which contain a uniform transmission line of arbitrary length. The equations obtained in the latter case are comewhat more complex than Eq. 1.10 due to the reflections and time delays associated with the transmission line.

The variables in the left hand column matrix of Eq. 1.5 can be incorporated into a subset of the matrix Eqs. 1.3 and 1.4, as indicated in Eq. 1.11.

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[1] The number of Eqs. necessary and sufficient for the solution of the RLC network is equal to the rank of $[A_o]$. Full rank does not result when capacitors (inductors) in series (parallel) are in the tree (cotree), nor under a variety of other conditions.

$$\begin{bmatrix} U & 0 & 0 & S_{21} & S_{22} & 0 \\ 0 & U & 0 & 0 & S_{32} & 0 \\ 0 & 0 & B_{45} & 0 & 0 & U \\ 0 & 0 & 0 & 0 & U & 0 \end{bmatrix} \begin{bmatrix} i_{C_1} \\ i_{R_1} \\ v_{L_1} \\ i_{C_2} \\ i_{R_2} \\ v_{L_2} \end{bmatrix} = - \begin{bmatrix} S_{23} & S_{24} & S_{25} & 0 & 0 & 0 & 0 & 0 \\ S_{33} & S_{34} & S_{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{41} & B_{42} & B_{43} & B_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -U \end{bmatrix} \begin{bmatrix} i_{T_2} \\ i_{L_2} \\ h(t) \\ e(t) \\ v_{C_1} \\ v_{R_1} \\ v_{T_1} \\ i_{R_2} \end{bmatrix} \qquad 1.11$$

The variables of the right hand column matrix of Eq. 1.5 can similarly be expressed as

$$\begin{bmatrix} \frac{d}{dt} v_{C_1} \\ v_{R_1} \\ \frac{d}{dt} i_{L_1} \\ \frac{d}{dt} v_{C_2} \\ v_{R_2} \\ \frac{d}{dt} i_{L_2} \end{bmatrix} = \begin{bmatrix} U & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & U & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -S_{54} & -S_{55} & 0 & 0 & 0 & 0 \\ -B_{12} & 0 & 0 & 0 & -B_{11} & 0 & 0 & 0 \\ 0 & -B_{23} & 0 & 0 & 0 & -B_{21} & -B_{22} & -B_{24} \\ 0 & 0 & U & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{d}{dt} v_{C_1} \\ v_{R_1} \\ \frac{d}{dt} i_{L_2} \\ \frac{d}{dt} h(t) \\ \frac{d}{dt} e(t) \\ e(t) \\ v_{C_1} \\ v_{T_1} \end{bmatrix} \qquad 1.12$$

Substitution of Eqs. 1.11 and 1.12 into Eq. 1.5 yields, after multiplication of coefficient matrices and some application of Eq. 1.9:

$$-\begin{bmatrix} S_{23} & S_{24} & S_{25} & 0 & 0 & 0 & 0 & 0 \\ S_{33} & S_{34} & S_{35} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{41} & B_{42} & B_{43} & B_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -U \end{bmatrix} \begin{bmatrix} i_{T_2} \\ i_{L_2} \\ h(t) \\ e(t) \\ v_{C_1} \\ v_{R_1} \\ v_{T_1} \\ i_{R_2} \end{bmatrix} = \begin{bmatrix} C_1+S_{21}C_2S'_{21} & 0 \\ 0 & 0 \\ 0 & L_2+B_{45}L_1B'_{45} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{d}{dt}v_{C_1} \\ \frac{d}{dt}i_{L_2} \end{bmatrix} +$$

$$\begin{bmatrix} S_{22}G_2S'_{32} & 0 & S_{21}C_2S'_{11} & S_{22}G_2S'_{12} & S_{22}G_2S'_{22} & S_{22}G_2S'_{42} \\ G_1+S_{32}G_2S'_{32} & 0 & 0 & S_{32}G_2S'_{12} & S_{32}G_2S'_{22} & S_{32}G_2S'_{42} \\ 0 & B_{45}L_1B'_{55} & 0 & 0 & 0 & 0 \\ -G_2B_{23} & 0 & 0 & -G_2B_{21} & -G_2B_{22} & -G_2B_{24} \end{bmatrix} \begin{bmatrix} v_{R_1} \\ \frac{d}{dt}h(t) \\ \frac{d}{dt}e(t) \\ e(t) \\ v_{C_1} \\ v_{T_1} \end{bmatrix} \quad 1.13$$

The matrix equation represented by the second row of submatrices of Eq. 1.13 is

$$-S_{33}i_{T_2}-S_{34}i_{L_2}-S_{35}h(t) = [G_1+S_{32}G_2S'_{32}]\ v_{R_1}+S_{32}G_2S'_{12}e(t)+S_{32}G_2S'_{22}v_{C_1}+ S_{32}G_2S'_{42}v_{T_1} \quad 1.14$$

When the matrix $[G_1+S_{32}G_2S'_{32}]$ is non-singular[1] Eq. 1.14 can be solved for v_{R_1} as

---

[1] When all network resistors are positive and finite, the matrix is positive definite (Def. A(2)) and hence non-singular.

$$v_{R_1} = - A^{-1} \Big(S_{33} i_{T_2} + S_{34} i_{L_2} + S_{35} h(t) + S_{32} G_2 S'_{12} e(t) + S_{32} G_2 S'_{22} v_{C_1} + S_{32} G_2 S'_{42} v_{T_1} \Big) \qquad 1.15$$

where

$$A = [G_1 + S_{32} G_2 S'_{32}] \qquad 1.16$$

and the -1 superscript indicates the matrix inverse.

The matrix equation obtained from the fourth row of submatrices of Eq. 1.13 yields an expression for i_{R_2}. Substitution of Eq. 1.15 into this expression gives

$$i_{R_2} = G_2 B_{23} A^{-1} \Big(S_{33} i_{T_2} + S_{34} i_{L_2} + S_{35} h(t) + [S_{32} G_2 S'_{12} + G_2 S'_{12}]\ e(t) + [S_{32} G_2 S'_{22} + G_2 S'_{22}]\ v_{C_1} + [S_{32} G_2 S'_{42} + G_2 S'_{42}]\ v_{T_1} \Big) \qquad 1.17$$

Substitution of Eq. 1.15 into the matrix equations obtained by expanding the first and third rows of Eq. 1.13 yields a set of modified[1] derivative-explicit equations which can be written as

---

[1] The word modified is used because of the presence of the unspecified quantities v_{T_1} and i_{T_2} in the Eqs.

$$\frac{d}{dt}\begin{bmatrix} v_{C_1} \\ i_{L_2} \end{bmatrix} = -B^{-1}\begin{bmatrix} S_{21}C_2S'_{11} & 0 \\ 0 & B_{45}L_1B_{55} \end{bmatrix}\begin{bmatrix} \frac{d}{dt}\,e(t) \\ \frac{d}{dt}\,h(t) \end{bmatrix} +$$

$$\begin{bmatrix} S_{22}G_2S'_{32}A^{-1}S_{32}G_2S'_{22}-S_{22}G_2S'_{22} & S_{22}G_2S'_{32}A^{-1}S_{34}-S_{24} \\ S'_{24}-S'_{34}A^{-1}S_{32}G_2S'_{22} & -B_{43}A^{-1}B'_{43} \end{bmatrix}\begin{bmatrix} v_{C_1} \\ i_{L_2} \end{bmatrix} +$$

$$\begin{bmatrix} S_{22}G_2S'_{32}A^{-1}S_{32}G_2S'_{12}-S_{22}G_2S'_{12} & S_{22}G_2S'_{32}A^{-1}S_{35}-S_{25} \\ S'_{14}-S'_{34}A^{-1}S_{32}G_2S'_{12} & -B_{43}A^{-1}B'_{53} \end{bmatrix}\begin{bmatrix} e(t) \\ h(t) \end{bmatrix} +$$

$$\begin{bmatrix} S_{22}G_2S'_{32}A^{-1}S_{32}G_2S'_{42}-S_{22}G_2S'_{42} & S_{22}G_2S'_{32}A^{-1}S_{33}-S_{23} \\ S'_{44}-S'_{34}A^{-1}S_{32}G_2S'_{42} & -B_{43}A^{-1}B'_{33} \end{bmatrix}\begin{bmatrix} v_{T_1} \\ i_{T_2} \end{bmatrix} \qquad 1.18$$

where the matrix B, which is positive definite for positive, finite, L and C elements, is given by

$$B = \begin{bmatrix} C_1+S_{21}C_2S'_{21} & 0 \\ 0 & L_2+B_{45}L_1B'_{45} \end{bmatrix} \qquad 1.19$$

Writing Eq. 1.18 with notation similar to that used in Eq. 1.10 yields

$$\frac{d}{dt}\begin{bmatrix} v_{C_1} \\ i_{L_2} \end{bmatrix} = \begin{bmatrix} B_o \end{bmatrix}\begin{bmatrix} v_{C_1} \\ i_{L_2} \end{bmatrix} + \begin{bmatrix} B_1 \end{bmatrix}\begin{bmatrix} e(t) \\ h(t) \end{bmatrix} + \begin{bmatrix} B_2 \end{bmatrix}\frac{d}{dt}\begin{bmatrix} e(t) \\ h(t) \end{bmatrix} + \begin{bmatrix} B_3 \end{bmatrix}\begin{bmatrix} v_{T_1} \\ i_{T_2} \end{bmatrix} \qquad 1.20$$

It can be readily concluded by examination of Eqs. 1.10 and 1.20

that if expressions can be found for v_{T_1} and i_{T_2} such that

$$v_{T_1} = f_1(v_{C_1}, i_{L_2}, e(t), h(t)) \qquad 1.21$$

$$i_{T_2} = f_2(v_{C_1}, i_{L_2}, e(t), h(t)) \qquad 1.22$$

where the functions f_1 and f_2 are linear combinations of the variables in the arguments; then substitution of these expressions into Eq. 1.20 will yield an equation which is in the form of Eq. 1.10.[1]

The formal procedure for evaluating v_{T_1} and i_{T_2} in terms of v_{C_1}, i_{L_2}, e(t), and h(t) will be:

(1) Obtain expressions for i_{T_1} and v_{T_2} from Eqs. 1.3 and 1.4, respectively.

(2) Eliminate resistor currents and voltages from the expressions obtained in the preceding step by use of Eqs. 1.15 and 1.17.

(3) Use the resulting expressions as drivers at the appropriate transmission line ports.

(4) Solve the resulting transmission line problem for v_{T_1} and i_{T_2}.

(5) Eliminate undesired quantities by substitution.

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1 Eqs. which have the form of Eq. 1.10 are said to be in normal form.

Three questions require consideration:

(1) Under what conditions can steps 1 through 5 be taken?

(2) If the five steps can be taken and the solution of the resulting equations can be obtained, can the solution of the network be completed?

(3) Since the proposed analysis involves obtaining rather general solutions for the transmission line terminal quantities, can these solutions be utilized in an alternative formulation procedure to reduce the number of equations which must be solved?

Question (1) will be considered in Chapters II and III.

Question (2) is readily answered. The quantities v_{C_1} and i_{L_2} are obtained in the solution of the equations. Then i_{C_1} and v_{L_2} are obtained from Eq. 1.5. Since v_{T_1} and i_{T_2} are assumed to be expressed in terms of v_{C_1}, i_{L_2}, e(t), and h(t), these quantities are known, and the resistor currents and voltages can be obtained from Eqs. 1.15, 1.17, and 1.5. Since v_{C_2} (i_{L_1}) can be obtained from the first (last) row of Eq. 1.4 (1.3), i_{C_2} (v_{L_1}) can be found from Eq. 1.5. Finally, i_e and v_h can be obtained from Eqs. 1.3 and 1.4 since all chord currents and branch voltages have been evaluated.

Question (3) is somewhat more subtle and may require some additional explanation. Suppose that a C(L) element in the tree (cotree) is in parallel (series) with a T element. Under these conditions, does step (4) in the proposed procedure effectively eliminate the equation associated with this C(L) element? Consider the formulation which would result if the T element rather than the C(L) element were placed in the tree (cotree). This would imply that the f-circuit (f-cut-set) containing C(L) would consist of T_1 and C (T_2 and L), and that C(L) would be a chord (branch). Under these conditions, the formulation step associated with Eq. 1.12 would involve derivatives of v_{T_1} and i_{T_2}, in additional to the quantities which appear in the right hand column matrix of Eq. 1.12. This implies that the resulting equation which is equivalent to Eq. 1.18 would contain derivatives of v_{T_1}, and i_{T_2}. It is concluded that the differential equation for C(L) has been replaced by a differential equation for v_{T_1} (i_{T_2}); hence the number of equations has not changed.

It should be noted that the actual number of equations which must be solved simultaneously can be less than the number specified by Eq. 1.20. In general, both the interconnection pattern and the component values can effect the number of equations requiring simultaneous solution.

An example network illustrating the reduction of the number of simultaneous equations is considered in Chapter IV. The example also illustrates the important point that the equations in the reduced set are generally more complex than the original derivative-explicit equations. Therefore, reducing the number of equations may not appreciably simplify the network solution.

II. THE TRANSMISSION LINE ANALYSIS

2.0 Introduction

In this chapter, general frequency domain solutions are obtained for transmission line problems representing the four cases which arose in Chapter I, and general time domain solutions are obtained for distortionless lines. In section 2.1, the problems are restricted to uniform transmission lines with zero initial conditions, and the general frequency domain solutions of the transmission line equations are obtained. In section 2.2, the frequency domain solutions are obtained for the four cases. These solutions are in the form of series of terms representing the multiple reflections at the transmission line terminals. In section 2.3, the two general terms of the series solutions are discussed, and the inverse transform of the general terms is given. In section 2.4, the general time domain solutions are given, for each of the four cases, for distortionless transmission lines.

2.1 Transformation and solution of the transmission line equations.

The voltage and current distributions on the uniform transmission line are assumed to correspond to equations 1.0 and 1.1, and the initial distributions are

$$v(x,0) = 0, \quad 0 < x < \ell \qquad 2.1.0$$

$$i(x,0) = 0, \quad 0 < x < \ell \qquad 2.1.1$$

Taking the Laplace transform of 1.0 and 1.1, subject to equations 2.1.0 and 2.1.1, gives

$$\frac{d\,V(x,s)}{dx} = -\,(R + sL)\,I(x,s) \qquad 2.1.2$$

$$\frac{d\,I(x,s)}{dx} = -\,(G + sC)\,V(x,s) \qquad 2.1.3$$

Taking the derivative of Eq. 2.1.2 (2.1.3) with respect to x and substituting Eq. 2.1.3 (2.1.2) into the resulting expression gives

$$\frac{d^2\,V(x,s)}{dx^2} = (R + sL)\,(G + sC)\,V(x,s) \qquad 2.1.4$$

$$\frac{d^2\,I(x,s)}{dx^2} = (R + sL)\,(G + sC)\,I(x,s) \qquad 2.1.5$$

The general solutions of Eqs. 2.1.4 and 2.1.5, respectively, can be written as:

$$V(x,s) = Ae^{\gamma x} + Be^{-\gamma x} \qquad 2.1.6$$

$$I(x,s) = De^{\gamma x} + Fe^{-\gamma x} \qquad 2.1.7$$

where A, B, D, and F may be functions of s, and

$$\gamma = (R + sL)^{1/2}\,(G + sC)^{1/2} \qquad 2.1.8$$

Substitution of Eqs. 2.1.6 and 2.1.7 into Eq. 2.1.2 yields

$$A = -Z_o D \qquad 2.1.9$$

$$B = Z_o F \qquad 2.1.10$$

where

$$Z_o = \left[\frac{R + sL}{G + sC}\right]^{1/2} \qquad 2.1.11$$

Thus Eq. 2.1.7 may be written as

$$I(x,s) = \frac{1}{Z_o}\ [\ -Ae^{\gamma x} + Be^{-\gamma x}] \qquad 2.1.12$$

2.2 Frequency domain solutions for the four cases.

In general, A and B of Eqs. 2.1.6 and 2.1.12 can be evaluated if two independent terminal quantities are specified. In the four problems considered in this section, the specified terminal quantities are v_{T_2} and i_{T_1}, as indicated in Chapter I.

In the following equations, the notation used is defined by Eq. 2.2.0.

$$F_i(s) = \mathcal{L}(f_i(t)) \qquad 2.2.0$$

Case (1) will be considered in some detail, while only the results are given for the remaining cases. Figure 2.2.0 illustrates the general transmission line problem of case (1), in which T_ℓ is a branch and T_o is a chord.

From Eqs. 2.1.6, 2.1.12, and 2.2.0, we have

$$E(s) = V(0,s) = A + B \qquad 2.2.1$$

$$H(s) = I(\ell,s) = \frac{1}{Z_o}\ [\ -Ae^{\gamma \ell} + Be^{-\gamma \ell}] \qquad 2.2.2$$

This last pair of equations can be solved for A and B, giving

$$A = \frac{E(s)\, e^{-2\gamma} \; -Z_o \; H(s)\, e^{-\gamma \ell}}{1 + e^{-2\gamma \ell}} \qquad 2.2.3$$

$$B = \frac{E(s) + Z_o \; H(s)\, e^{-\gamma \ell}}{1 + e^{-2\gamma \ell}} \qquad 2.2.4$$

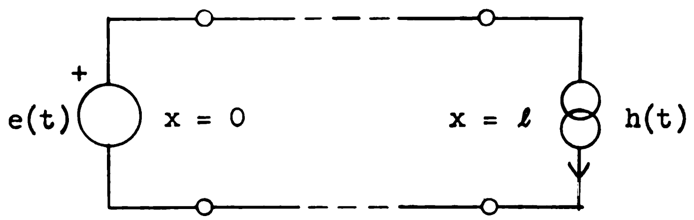

Figure 2.2.0 General representation of case (1).

Using Eqs. 2.2.3 and 2.2.4, the unspecified pair of terminal quantities can be expressed as

$$V(\ell,s) = [2E(s)\, e^{-\gamma \ell} - Z_o \; H(s)\,(1 - e^{-2\gamma \ell})] \; \frac{1}{1 + e^{-2\gamma \ell}} \qquad 2.2.5$$

$$I(0,s) = [\frac{E(s)}{Z_o}\,(1 - e^{-2\gamma \ell}) + 2\; H(s)\, e^{-\gamma \ell}] \; \frac{1}{1 + e^{-2\gamma \ell}} \qquad 2.2.6$$

If $|e^{-2\gamma \ell}| < 1$, the fractional factor can be expanded in an infinite series as

$$\frac{1}{1 + e^{-2\gamma \ell}} = 1 - e^{-2\gamma \ell} + e^{-4\gamma \ell} - e^{-6\gamma \ell} + \dots \qquad 2.2.7$$

Since $Re(\gamma) > 0$, the series expansion is valid unless $\ell = 0$. For $\ell = 0$, Eqs. 2.2.5 and 2.2.6 become, respectively:

$$V(\ell,s) = E(s) \qquad 2.2.8$$

$$I(0,s) = H(s) \qquad 2.2.9$$

This last pair of equations are merely expressions of Kirchhoff's circuit and vertex equations.

Substitution of Eq. 2.2.7 into Eqs. 2.2.5 and 2.2.6 gives, after some manipulation to collect terms,

$$V(\ell,s) = 2E(s) \, [\, e^{-\gamma\ell} - e^{-3\gamma\ell} + e^{-5\gamma\ell} \; \ldots \,] - Z_o \, H(s) + 2Z_o \, H(s) \, [e^{-2\gamma\ell} - e^{-4\gamma\ell} + e^{-6\gamma\ell} \ldots] \qquad 2.2.10$$

$$I(0,s) = \frac{E(s)}{Z_o} - \frac{2E(s)}{Z_o} [e^{-2\gamma\ell} - e^{-4\gamma\ell} + e^{-6\gamma\ell} \ldots] + 2H(s) \, [e^{-\gamma\ell} - e^{-3\gamma\ell} + e^{-5\gamma\ell} \ldots] \qquad 2.2.11$$

Equations 2.2.10 and 2.2.11 are the general s-domain solutions for v_{T_1} and i_{T_2} of case (1). The inverse transform will not be considered until comparable results have been given for the remaining cases.

Case (2): neither T element is in the tree. Figure 2.2.1 illustrates the general problem and the placement of the drivers. The solutions obtained for the unspecified terminal quantities are given by

Eqs. 2.1.12 and 2.1.13.

$$I(0,s) = \frac{E_1(s)}{Z_o} + \frac{2E_1(s)}{Z_o} [e^{-2\gamma \ell} + e^{-4\gamma \ell} + e^{-6\gamma \ell} + \ldots] -$$

$$\frac{2E_2(s)}{Z_o} [e^{-\gamma \ell} + e^{-3\gamma \ell} + e^{-5\gamma \ell} + \ldots] \qquad 2.1.12$$

$$- I(\ell,s) = \frac{E_2(s)}{Z_o} + \frac{2E_2(s)}{Z_o} [e^{-2\gamma \ell} + e^{-4\gamma \ell} + e^{-6\gamma \ell} + \ldots] -$$

$$\frac{2E_1(s)}{Z_o} [e^{-\gamma \ell} + e^{-3\gamma \ell} + e^{-5\gamma \ell} + \ldots] \qquad 2.1.13$$

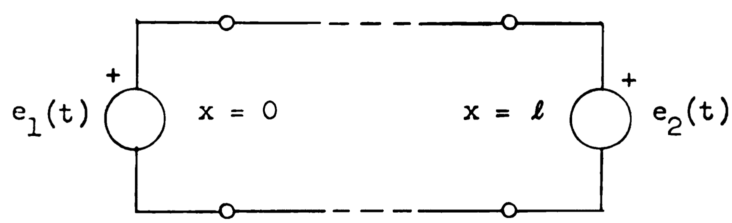

Figure 2.2.1 General representation of case (2).

Case (3): T_o is a branch and T_ℓ is a chord. Figure 2.2.2 illustrates the general problem and the placement of the drivers. The solutions obtained for the unsepcified terminal quantities are given by Eqs. 2.1.14 and 2.1.15.

$$V(0,s) = 2E(s) [e^{-\gamma \ell} - e^{-3\gamma \ell} + e^{-5\gamma \ell} \ldots] + Z_o H(s) -$$

$$2Z_o H(s) [e^{-2\gamma \ell} - e^{-4\gamma \ell} + e^{-6\gamma \ell} \ldots] \qquad 2.1.14$$

$$-I(\ell,s) = \frac{E(s)}{Z_o} - \frac{2E(s)}{Z_o}\,[e^{-2\gamma\ell} - e^{-4\gamma\ell} + e^{-6\gamma\ell} \ldots] -$$

$$2\,H(s)\,[e^{-\gamma\ell} - e^{-3\gamma\ell} + e^{-5\gamma\ell} \ldots] \qquad 2.1.15$$

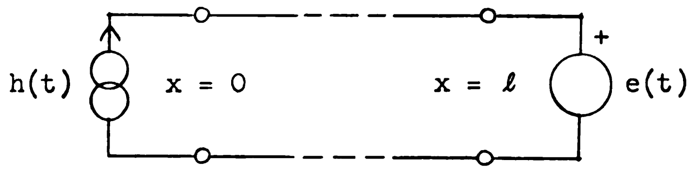

Figure 2.2.2 General representation of case (3).

Case (4): both T elements are in the tree. Figure 2.2.3 illustrates the general problem and the placement of the drivers. The solutions obtained for the unspecified terminal quantities are given by Eqs. 2.1.16 and 2.1.17.

$$V(0,s) = Z_o\,H_1(s) + 2Z_o\,H_1(s)\;[e^{-2\gamma\ell} + e^{-4\gamma\ell} + e^{-6\gamma\ell} + \ldots] -$$

$$2Z_o\,H_2(s)\,[e^{-\gamma\ell} + e^{-3\gamma\ell} + e^{-5\gamma\ell} + \ldots] \qquad 2.1.16$$

$$V(\ell,s) = 2Z_o\,H_1(s)\,[e^{-\gamma\ell} + e^{-3\gamma\ell} + e^{-5\gamma\ell} + \ldots] - Z_o\,H_2(s)$$

$$2Z_o\,H_2(s)\,[e^{-2\gamma\ell} + e^{-4\gamma\ell} + e^{-6\gamma\ell} + \ldots] \qquad 2.1.17$$

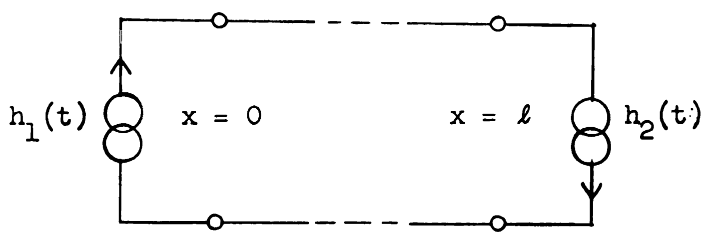

Figure 2.2.3 General representation of case (4).

Equations 2.1.10 through 2.1.17 are the general s-domain solutions for v_{T_1} and i_{T_2} in the four cases. In obtaining these solutions by means of the Laplace transform, the problem has not been limited to linear networks, although the Laplace transform is a linear operator. The only condition imposed on the network N by this formulation is that the linear combinations of element voltages and currents corresponding to v_{T_2} and i_{T_1} be Laplace transformable (8). It can be concluded that this analysis of the transmission line is valid, in most cases, for nonlinear electrical networks.

2.3 The two general terms and their inverse transform.

Equations 2.1.8 and 2.1.11 can be rewritten as

$$\gamma = \frac{1}{v_d}\sqrt{\left(s + \frac{R}{L}\right)\left(s + \frac{G}{C}\right)} \qquad 2.3.0$$

$$Z_o = R_o\sqrt{\frac{s + \frac{R}{L}}{s + \frac{G}{C}}} \qquad 2.3.1$$

where R_o and v_d are given by Eqs. 2.3.2 and 2.3.3.

$$R_o = \sqrt{\frac{L}{C}} \qquad 2.3.2$$

$$v_d = \frac{1}{\sqrt{LC}} \qquad 2.3.3$$

The quantities R_o and v_d correspond to the characteristic resistance and propagation velocity, respectively, on distortionless transmission lines.

By using the notation of Eqs. 2.3.0 and 2.3.1, it is readily apparent that only the following two distinct forms occur in the series solutions of section 2.2:

Form (1) $\quad K_n\, F(s)\, e^{-A_n\sqrt{(s+\rho)(s+\sigma)}}$

Form (2) $\quad K_n'\, F(s)\sqrt{\frac{s+\rho}{s+\sigma}}\; e^{-A_n\sqrt{(s+\rho)(s+\sigma)}}$

where:

(1) $F(s)$ is the Laplace transform of a specified function.

(2) K_n and K_n' are constants which in general depend on n.

(3) $A_n = \frac{n\ell}{v_d}$

(4) $\rho(\sigma) = \frac{R}{L}\,(\frac{G}{C})$ if R_o is a factor of K_n'.

(5) $\rho(\sigma) = \frac{G}{C}\,(\frac{R}{L})$ if $\frac{1}{R_o}$ is a factor of K_n'.

The inverse Laplace transforms of the two general forms can be calculated by use of the convolution theorem and transform tables (8,9) as:

$$\mathcal{L}^{-1}\left\{K_nF(s)\, e^{-A_n\sqrt{(s+\rho)(s+\sigma)}}\right\} = \Bigg\{K_n\int_{A_n}^{t} e^{-\frac{A_n}{2}(\rho+\sigma)}\,\delta_{A_n} f(t-t_1)\, dt_1 \,+$$

$$K_n\int_{A_n}^{t} \tfrac{1}{2}(\rho-\sigma)\, e^{-\frac{1}{2}(\rho+\sigma)t_1}\left[\frac{t_1}{(t_1^2-A_n^2)^{\frac{1}{2}}}\; I_1\!\left(\tfrac{1}{2}(\rho-\sigma)(t_1^2-A_n^2)^{\frac{1}{2}}\right)\right.$$

$$\left. +\; I_o\left(\tfrac{1}{2}(\rho-\sigma)(t_1^2-A_n^2)^{\frac{1}{2}}\right)\right] f(t-t_1)\;\, dt_1\Bigg\}\, U(t-A_n) \qquad 2.3.4$$

$$\mathcal{L}^{-1}\left\{K_n' F(s)\left(\frac{s+\rho}{s+\sigma}\right)^{\frac{1}{2}} e^{-A_n\sqrt{(s+\rho)(s+\sigma)}}\right\} = \left[K_n' \int_{A_n}^{t} e^{-\frac{A_n}{2}(\rho+\sigma)} \delta_{A_n} f(t-t_1)\, dt_1 + K_n' \int_{A_n}^{t} A_n(\rho-\sigma)\, e^{-\frac{(\rho+\sigma)}{2} t_1} I_1\left(\frac{1}{2}(\rho-\sigma)(t_1^2-A_n^2)^{\frac{1}{2}}\right) f(t-t_1)\, dt_1\right] U(t-A_n) \quad 2.3.5$$

where I_i is the modified Bessel function of the first kind of order i, $U(t-A_n)$ is the unit step at $t=A_n$, and δ_{A_n} is the unit impulse at $t_1=A_n$. The integrals involving the Bessel functions cannot be evaluated in general form due to the arbitrary driving function in the integrands.

The integrals involving the impulse function can be readily evaluated for a general driving function f(t) as

$$U(t-A_n)\, K_n \int_{A_n}^{t} e^{-\frac{A_n}{2}(\rho+\sigma)} \delta_{A_n} f(t-t_1)\, dt_1 = K_n\, e^{-\frac{A_n}{2}(\rho+\sigma)} f(t-A_n)\, U(t-A_n) \quad 2.3.6$$

$$U(t-A_n)\, K_n' \int_{A_n}^{t} e^{-\frac{A_n}{2}(\rho+\sigma)} \delta_{A_n} f(t-t_1)\, dt_1 = K_n'\, e^{-\frac{A_n}{2}(\rho+\sigma)} f(t-A_n)\, U(t-A_n) \quad 2.3.7$$

For distortionless lines, $\rho=\sigma$. Under this condition, the integrands of all terms in Eqs. 2.3.4 and 2.3.5 which involve Bessel functions are identically zero, and Eqs. 2.3.6 and 2.3.7 give the complete inverse transforms for forms (1) and (2), respectively.

The voltage distribution which results from a square voltage pulse of finite amplitude and arbitrary duration has been evaluated for a general uniform line of infinite length ((8), p. 294)[1]. Any finite waveform could be constructed as a succession of square pulses of infinitesimal duration, and a general solution for such a waveform could be formulated in terms of integrals. When integral solutions are obtained for the transmission line terminal quantities, step 5 of the formal elimination procedure cannot be performed. Therefore, only distortionless lines will be considered in the remainder of this thesis.

2.4 General time domain solutions for distortionless lines:

The general time domain solutions for the four cases are obtained by using Eqs. 2.3.6 and 2.3.7 on the series of section 2.2. The notation used in these solutions is defined by the following equations:

$$f_{i_n} = f_i(t - n\tau), \text{ where } n \geq 0 \qquad 2.4.0$$

$$\epsilon_n = e^{-\frac{n\tau}{2}(\rho+\sigma)}, \text{ where } n \geq 0 \qquad 2.4.1$$

$$\tau = \frac{\ell}{v_d} \qquad 2.4.2$$

Note that τ is the time period required for a signal to propagate over the transmission line of length ℓ, and that $\epsilon_n = 1$ for the special case of lossless lines. The general solutions are:

[1] The formulation used in obtaining the series solutions of section 2.2 allows results obtained for an infinite line to be applied to the terms of the series, with the distance of propagation being $n\ell$.

Case (1)

$$v(\ell,t) = - R_o\, h_o\, \epsilon_o\, U_o + 2R_o\, [h_2\, \epsilon_2\, U_2 - h_4\, \epsilon_4\, U_4 \;\ldots] +$$

$$2\, [e_1\, \epsilon_1\, U_1 - e_3\, \epsilon_3\, U_3 + e_5\, \epsilon_5\, U_5 \ldots] \qquad 2.4.3$$

$$i(0,t) = \frac{e_o\, \epsilon_o\, U_o}{R_o} - \frac{2}{R_o}\, [e_2\, \epsilon_2\, U_2 - e_4\, \epsilon_4\, U_4 \;\ldots] +$$

$$2\, [h_1\, \epsilon_1\, U_1 - h_3\, \epsilon_3\, U_3 + h_5\, \epsilon_5\, U_5 \ldots] \qquad 2.4.4$$

Case (2)

$$i(0,t) = \frac{e_{1_o}\, U_o\, \epsilon_o}{R_o} + \frac{2}{R_o}\, [e_{1_2}\, \epsilon_2\, U_2 + e_{1_4}\, \epsilon_4\, U_4 + \ldots] -$$

$$\frac{2}{R_o}\, [\, e_{2_1}\, \epsilon_1\, U_1 + e_{2_3}\, \epsilon_3\, U_3 + \ldots] \qquad 2.4.5$$

$$i(\ell,t) = \frac{2}{R_o}\, [e_{1_1}\, \epsilon_1\, U_1 + e_{1_3}\, \epsilon_3\, U_3 + \ldots] - \frac{e_{2_o}\, \epsilon_o\, U_o}{R_o} -$$

$$\frac{2}{R_o}\, [e_{2_2}\, \epsilon_2\, U_2 + e_{2_4}\, \epsilon_4\, U_4 + \ldots] \qquad 2.4.6$$

Case (3)

$$v(0,t) = R_o\, h_o\, \epsilon_o\, U_o - 2R_o\, [h_2\, \epsilon_2\, U_2 - h_4\, \epsilon_4\, U_4 \;\ldots] +$$

$$2\, [e_1\, \epsilon_1\, U_1 - e_3\, \epsilon_3\, U_3 \;\ldots] \qquad 2.4.7$$

$$i(\ell,t) = - \frac{e_o\, \epsilon_o\, U_o}{R_o} + \frac{2}{R_o}\, [e_2\, \epsilon_2\, U_2 - e_4\, \epsilon_4\, U_4 \;\ldots]$$

$$+ 2\, [h_1\, \epsilon_1\, U_1 - h_3\, \epsilon_3\, U_3 \;\ldots] \qquad 2.4.8$$

Case (4)

$$v(0,t) = R_o (h_{1_o} \epsilon_o U_o) + 2R_o [h_{1_2} \epsilon_2 U_2 + h_{1_4} \epsilon_4 U_4 + \ldots]$$

$$-2R_o [h_{2_1} \epsilon_1 U_1 + h_{2_3} \epsilon_3 U_3 + \ldots] \qquad 2.4.9$$

$$v(\ell,t) = 2R_o [h_{1_1} \epsilon_1 U_1 + h_{1_3} \epsilon_3 U_3 + \ldots] - R_o (h_{2_o} U_o \epsilon_o)$$

$$- 2R_o [h_{2_2} \epsilon_2 U_2 + h_{2_4} \epsilon_4 U_4 + \ldots] \qquad 2.4.10$$

Equations 2.4.3 through 2.4.10 could obviously be written more concisely by using summation notation, but the manipulation of these equations, which is the subject of Chapter III, is more readily followed when the expanded forms are used.

III. THE DERIVATIVE-EXPLICIT EQUATIONS

3.0 Introduction

In the first two chapters, modified derivative-explicit equations have been derived for networks containing a uniform transmission line, and general t-domain solutions have been obtained for transmission line problems associated with the four cases which arise with respect to placement of the T elements in the tree. The t-domain solutions are restricted to distortionless transmission line problems.

In section 3.1, the formal procedure for removing the explicit presence of transmission line terminal quantities from the modified derivative-explicit equations is carried out in detail for case (1). In section 3.2, derivative-explicit equations are given for the remaining three cases. In section 3.3, the application of the analysis to several special cases is considered.

3.1 The derivative-explicit equations for case (1).

As indicated in Chapter I, the basic procedure for removing v_{T_1} and i_{T_2} from Eq. 1.18 requires expressions for v_{T_1} and i_{T_2} in terms of v_{C_1}, i_{L_2}, e(t), and h(t).

Step (1) of the formal procedure is accomplished with the writing of Eqs. 3.1.0 and 3.1.1.

$$-i_{T_1} = S_{42}\, i_{R_2} + S_{43}\, i_{T_2} + S_{44}\, i_{L_2} + S_{45}\, h(t) \qquad 3.1.0$$

$$-v_{T_2} = B_{31}\, e(t) + B_{32}\, v_{C_1} + B_{33}\, v_{R_1} + B_{34}\, v_{T_1} \qquad 3.1.1$$

The explicit dependence of these two equations on the R elements of the graph can be eliminated by using Eqs. 1.15 and 1.17. Substitution and rearrangement yields

$$-i_{T_1} = [S_{42}G_2B_{23}A^{-1}S_{33}+S_{43}]\; i_{T_2} + [S_{42}G_2B_{23}A^{-1}S_{34}+S_{44}]\; i_{L_2} + [S_{42}G_2B_{23}A^{-1}S_{35}+S_{45}]\; h(t) + S_{42}[G_2B_{23}A^{-1}S_{32}G_2S'_{12}+G_2S'_{12}]\; e(t) + S_{42}[G_2B_{23}A^{-1}S_{32}G_2S'_{22}+G_2S'_{22}]\; v_{C_1} + S_{42}[G_2B_{23}A^{-1}S_{32}G_2S'_{42}+G_2S'_{42}]\; v_{T_1} \qquad 3.1.2$$

$$-v_{T_2} = [B_{31}+S'_{33}A^{-1}S_{32}G_2S'_{12}]\; e(t) + [B_{32}+S'_{33}A^{-1}S_{32}G_2S'_{22}]\; v_{C_1} + [B_{34}+S'_{33}A^{-1}S_{32}G_2S'_{42}]\; v_{T_1} + S'_{33}A^{-1}S_{33}i_{T_2}+S'_{33}A^{-1}S_{34}i_{L_2}+S'_{33}A^{-1}S_{55}h(t) \qquad 3.1.3$$

In Eq. 3.1.3 S'_{33} has been substituted for $-B_{33}$. The linear combinations of currents and voltages specified by Eqs. 3.1.2 and 3.1.3 are generalized forms of the driving functions used in Chapter II. The substitution of Eqs. 3.1.2 and 3.1.3 into the t-domain series solutions of section 2.4 completes the first four steps of the procedure given in Chapter I. The only remaining step to be taken is the solution of the equations resulting from this last substitution for v_{T_1} and i_{T_2} in terms of v_{C_1}, i_{L_2}, e(t), and h(t).

Case (1): T_ℓ is a branch and T_o is a chord. Hence

$$i_{T_1}(v_{T_1}) = -i(\ell,t)\ (v(\ell,t)) \qquad 3.1.4$$

$$v_{T_2}(i_{T_2}) = v(0,t)\ (i(0,t)) \qquad 3.1.5$$

To simplify the writing of the rather considerable number of lengthy equations which appear in the following pages, several notational simplifications are defined:

$$v(\ell,t)\ (i(\ell,t)) = v_\ell(t)\ (i_\ell(t)) \qquad 3.1.6$$

$$v(0,t)\ (i(0,t)) = v_o(t)\ (i_o(t)) \qquad 3.1.7$$

$$C_o = -\left[S_{42}G_2B_{23}A^{-1}S_{33}+S_{43}\right] \qquad 3.1.8$$

$$C_1 = -S_{42}\left[G_2B_{23}A^{-1}S_{32}G_2S_{42}'+G_2S_{42}'\right] \qquad 3.1.9$$

$$N(t) = i_{T_1} - C_o i_{T_2} - C_1 v_{T_1} \qquad 3.1.10$$

$$D_o = -S_{33}'A^{-1}S_{33} \qquad 3.1.11$$

$$D_1 = -\left[B_{34}+S_{33}'A^{-1}S_{32}G_2S_{42}'\right] \qquad 3.1.12$$

$$M(t) = v_{T_2} - D_o i_{T_2} - D_1 v_{T_1} \qquad 3.1.13$$

It should be observed that $N(t)$ and $M(t)$ involve only linear combinations of the quantities i_{L_2}, v_{C_1}, $e(t)$, and $h(t)$. Substitution of

$v_{T_2}(-i_{T_1})$ for e(t) (h(t)) in Eqs. 2.4.3 and 2.4.4 yields

$$v_{\ell_o} = \frac{R_o \epsilon_o U_o}{1-C_1 R_o} [C_o i_{o_o} + N_o] +$$

$$\frac{2}{1-C_1 R_o} [(D_o i_{o_1} + D_1 v_{\ell_1} + M_1) \quad \epsilon_1 U_1 - (D_o i_{o_3} + D_1 v_{\ell_3} + M_3) \quad \epsilon_3 U_3 + \ldots] +$$

$$\frac{2R_o}{1-C_1 R_o} [(C_o i_{o_2} + C_1 v_{\ell_2} + N_2) \quad \epsilon_2 U_2 - (C_o i_{o_4} + C_1 v_{\ell_4} + N_4) \quad \epsilon_4 U_4 + \ldots] \qquad 3.1.14$$

$$i_{o_o} = \frac{\epsilon_o U_o}{R_o - D_o} [D_1 v_{\ell_o} + M_o] - \frac{2}{R_o - D_o} [(D_o i_{o_2} + D_1 v_{\ell_2} + M_2) \epsilon_2 U_2 - (D_o i_{o_4} + D_1 v_{\ell_4} + M_4) \epsilon_4 U_4 + \ldots] -$$

$$\frac{2R_o}{R_o - D_o} [(C_o i_{o_1} + C_1 v_{\ell_1} + N_1) \epsilon_1 U_1 - (C_o i_{o_3} + C_1 v_{\ell_3} + N_3) \epsilon_3 U_3 + \ldots] \qquad 3.1.15$$

where the notation of section 2.4 has been used for time and exponential functions.

Substitution of Eq. 3.1.15 into Eq. 3.1.14 yields, after some algebraic manipulation, the following expression for v_{ℓ_o}:

$$v_{\ell_o} = \left[\frac{R_o (R_o - D_o)}{D} N_o + \frac{R_o C_o}{D} M_o \right] \epsilon_o U_o +$$

$$\frac{2(R_o - D_o)}{D} [(D_o i_{o_1} + D_1 v_{\ell_1} + M_1) \epsilon_1 U_1 - (D_o i_{o_3} + D_1 v_{\ell_3} + M_3) \epsilon_3 U_3 + \ldots] -$$

$$\frac{2R_o (R_o - D_o)}{D} [(C_o i_{o_2} + C_1 v_{\ell_2} + N_2) \epsilon_2 U_2 - (C_o i_{o_4} + C_1 v_{\ell_4} + N_4) \epsilon_4 U_4 + \ldots] -$$

$$\frac{2R_o C_o}{D} [(D_o i_{o_2} + D_1 v_{\ell_2} + M_2) \epsilon_2 U_2 - (D_o i_{o_4} + D_1 v_{\ell_4} + M_4) \epsilon_4 U_4 + \ldots] -$$

$$\frac{2R_o^2 C_o}{D} [(C_o i_{o_1} + C_1 v_{\ell_1} + N_1) \epsilon_1 U_1 - (C_o i_{o_3} + C_1 v_{\ell_3} + N_3) \epsilon_3 U_3 + \ldots] \qquad 3.1.16$$

where

$$D = (1-C_1R_o)(R_o-D_o) - R_oC_oD_1 \qquad 3.1.17$$

Substitution of Eq. 3.1.16 into Eq. 3.1.15 gives the following expression for i_{o_o}:

$$i_{o_o} = \left[\frac{1-C_1R_o}{D} N_o + \frac{D_1R_o}{D} N_o\right] \epsilon_o U_o -$$

$$\frac{2(1-C_1R_o)}{D} [(D_o i_{o_2} + D_1 v_{\ell_2} + M_2)\, \epsilon_2 U_2 - (D_o i_{o_4} + D_1 v_{\ell_4} + M_4)\, \epsilon_4 U_4 + \ldots] -$$

$$\frac{2R_oD_1}{D} [(C_o i_{o_2} + C_1 v_{\ell_2} + N_2)\, \epsilon_2 U_2 - (C_o i_{o_4} + C_1 v_{\ell_4} + N_4)\, \epsilon_4 U_4 + \ldots] -$$

$$\frac{2R_o(1-C_1R_o)}{D} [(C_o i_{o_1} + C_1 v_{\ell_1} + N_1)\, \epsilon_1 U_1 - (C_o i_{o_3} + C_1 v_{\ell_3} + N_3)\, \epsilon_3 U_3 + \ldots] +$$

$$\frac{2D_1}{D} [(D_o i_{o_1} + D_1 v_{\ell_1} + M_1)\, \epsilon_1 U_1 - (D_o i_{o_3} + D_1 v_{\ell_3} + M_3)\, \epsilon_3 U_3 + \ldots] \qquad 3.1.18$$

Equations 3.1.17 and 3.1.18 are valid for all values of t, and can be generalized by replacing t by $t-n\tau$. The resulting expressions can be written as

$$v_{\ell_n} = \left[\frac{R_o(R_o-D_o)}{D} N_n + \frac{R_oC_o}{D} M_n\right] \epsilon_o U_n +$$

$$\frac{2(R_o-D_o)}{D}[(D^*_{n+1}+M_{n+1})\ \epsilon_1 U_{n+1} - (D^*_{n+3}+M_{n+3})\ \epsilon_3 U_{n+3} + \ldots] -$$

$$\frac{2R_oC_o}{D}[(D^*_{n+2}+M_{n+2})\ \epsilon_2 U_{n+2} - (D^*_{n+4}+M_{n+4})\ \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2R_o(R_o-D_o)}{D}[(C^*_{n+2}+N_{n+2})\ \epsilon_2 U_{n+2} - (C^*_{n+4}+N_{n+4})\ \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2R_o^2C_o}{D}[(C^*_{n+1}+N_{n+1})\ \epsilon_1 U_{n+1} - (C^*_{n+3}+N_{n+3})\ \epsilon_3 U_{n+3} + \ldots] \qquad 3.1.19$$

$$i_{o_n} = \left[\frac{1-C_1R_o}{D} M_n + \frac{D_1R_o}{D} N_n\right] \epsilon_o U_n +$$

$$\frac{2D_1}{D}[(D^*_{n+1}+M_{n+1})\ \epsilon_1 U_{n+1} - (D^*_{n+3}+M_{n+3})\ \epsilon_3 U_{n+3} + \ldots] -$$

$$\frac{2(1-C_1R_o)}{D}[(D^*_{n+2}+M_{n+2})\ \epsilon_2 U_{n+2} - (D^*_{n+4}+M_{n+4})\ \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2R_oD_1}{D}[(C^*_{n+2}+N_{n+2})\ \epsilon_2 U_{n+2} - (C^*_{n+4}+N_{n+4})\ \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{-2R_o(1-C_1R_o)}{D}[(C^*_{n+1}+N_{n+1})\ \epsilon_1 U_{n+1} - (C^*_{n+3}+N_{n+3})\ e_3 U_{n+3} + \ldots] \qquad 3.1.20$$

where

$$D^*_n = D_o i_{o_n} + D_1 v_{\ell_n} \qquad 3.1.21$$

$$C^*_n = C_o i_{o_n} + C_1 v_{\ell_n} \qquad 3.1.22$$

By using Eqs. 3.1.19 and 3.1.20 in Eqs. 3.1.21 and 3.1.22, D_n^* and C_n^* are evaluated as

$$D_n^* = \left[\frac{J}{D} M_n + \frac{D_1 R_o^2}{D}\right] N_n \quad \epsilon_o U_n +$$

$$\frac{2D_1R_o}{D} [(D_{n+1}^*+M_{n+1}) \epsilon_1 U_{n+1} - (D_{n+3}^*+M_{n+3}) \epsilon_3 U_{n+3} + \ldots] -$$

$$\frac{2D_1R_o^2}{D} [(C_{n+2}^*+N_{n+2}) \epsilon_1 U_{n+2} - (C_{n+4}^*+N_{n+4}) \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2J}{D} [(D_{n+2}^*+M_{n+2}) \epsilon_2 U_{n+2} - (D_{n+4}^*+M_{n+4}) \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2R_oJ}{D} [(C_{n+1}^*+N_{n+1}) \epsilon_1 U_{n+1} - (C_{n+3}^*+N_{n+3}) \epsilon_3 U_{n+3} + \ldots] \qquad 3.1.23$$

$$C_n^* = \left[\frac{C_o}{D} M_n + \frac{R_oK}{D}\right] \epsilon_o U_n +$$

$$\frac{2K}{D} [(D_{n+1}^*+M_{n+1}) \epsilon_1 U_{n+1} - (D_{n+3}^*+M_{n+3}) \epsilon_3 U_{n+3} + \ldots] -$$

$$\frac{2R_oK}{D} [(C_{n+2}^*+N_{n+2}) \epsilon_2 U_{n+2} - (C_{n+4}^*+N_{n+4}) \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2C_o}{D} [(D_{n+2}^*+M_{n+2}) \epsilon_2 U_{n+2} - (D_{n+4}^*+M_{n+4}) \epsilon_4 U_{n+4} + \ldots] -$$

$$\frac{2R_oC_o}{D} [(C_{n+1}^*+N_{n+1}) \epsilon_1 U_{n+1} - (C_{n+3}^*+N_{n+3}) \epsilon_3 U_{n+3} + \ldots] \qquad 3.1.24$$

where

$$J = D_o(1-C_1R_o) + D_1C_oR_o \qquad 3.1.25$$

$$K = C_1(R_o-D_o) + C_oD_1 \qquad 3.1.26$$

The remainder of the procedure is a repeated substitution of Eqs. 3.1.23 and 3.1.24 into Eqs. 3.1.16 and 3.1.18 to find expressions for $v(\ell,t)$ and $i(0,t)$ in terms of M and N only. The calcuations for evaluating $v(\ell,t)$ will be given in some detail.

The initial step is the expansion of C_1^* and D_1^* in Eq. 3.1.16, which gives

$$
\begin{aligned}
v_{\ell_o} = {} & \left[\frac{R_o(R_o-D_o)}{D} N_o + \frac{R_oC_o}{D} M_o \right] U_o - \frac{2R_o(R_o-D_o)}{D} [\bar{C}_2 - \bar{C}_4 \ldots] - \\
& \frac{2R_oC_o}{D} [\bar{D}_2 - \bar{D}_4 \ldots] + \frac{2(R_o-D_o)}{D} \Bigg[\Bigg[\frac{J+D}{D} M_1 + \frac{D_1R_o^2}{D} N_1 + \\
& \frac{2D_1R_o}{D} \Big[[\bar{D}_2/\epsilon_1 - \bar{D}_4/\epsilon_1 \ldots] - R_o [\bar{C}_3/\epsilon_1 - \bar{C}_5/\epsilon_1 \ldots] \Big] - \\
& \frac{2J}{D} \Big[[\bar{D}_3/\epsilon_1 - \bar{D}_5/\epsilon_1 \ldots] + R_o [\bar{C}_2/\epsilon_1 - \bar{C}_4/\epsilon_1 \ldots] \Big] \Bigg] \epsilon_1U_1 \\
& - \bar{D}_3 + \bar{D}_5 - \bar{D}_7 \ldots \Bigg] \quad - \frac{2R_o^2C_o}{D} \Bigg[\Bigg[\frac{R_oK}{D} N_1 + \frac{C_o}{D} M_1 + N_1 + \\
& \frac{2K}{D} \Big[[\bar{D}_2/\epsilon_1 - \bar{D}_4/\epsilon_1 \ldots] - R_o [\bar{C}_3/\epsilon_1 - C_5/\epsilon_1 \ldots] \Big] - \\
& \frac{2C_o}{D} \Big[[\bar{D}_3/\epsilon_1 - \bar{D}_5/\epsilon_1 \ldots] + R_o [\bar{C}_2/\epsilon_1 - \bar{C}_4/\epsilon_1 \ldots] \Big] \Bigg] \epsilon_1U_1 \\
& - \bar{C}_3 + \bar{C}_5 - \bar{C}_7 \ldots \Bigg]
\end{aligned}
\tag{3.1.27}
$$

where

$$\bar{C}_n = (C_n^* + N_n)\,\epsilon_nU_n \tag{3.1.28}$$

$$\bar{D}_n = (D_n^* + M_n)\,\epsilon_nU_n \tag{3.1.29}$$

Equation 3.1.27 is rewritten by collecting terms, expanding the sums J + D and $R_oK + D$, and making use of the following two identities:

$$U_n \, U_{n+p} = U_{n+p}, \; p \geq 0 \qquad 3.1.30$$

$$\epsilon_i \epsilon_j = e_{i+j} \qquad 3.1.31$$

The resulting expression can be written as

$$v_{\ell_o} = \left[\frac{R_o(R_o-D_o)}{D} N_o + \frac{R_oC_o}{D} M_o\right]U_o + \left[\frac{2(R_o-D_o)}{D}\left[\frac{R_o(1-C_1R_o)}{D} M_1 + \frac{D_1R_o^2}{D} N_1\right] - \frac{2R_o^2C_o}{D}\left[\frac{R_o-D_o}{D} N_1 + \frac{C_o}{D} M_1\right]\right]\epsilon_1 U_1 +$$
$$\left[\frac{2(R_o-D_o)}{D} \quad \frac{2D_1R_o}{D} - \frac{2R_oC_o}{D} - \frac{2R_o^2C_o}{D} \quad \frac{2K}{D}\right]\left[[\bar{D}_2-\bar{D}_4 \; \ldots] - R_o \, [\bar{C}_3-\bar{C}_5 \; \ldots]\right] -$$
$$\left[\frac{2(R_o-D_o)}{D} \quad \frac{2J}{D} + \frac{2(R_o-D_o)}{D} - \frac{2R_o^2C_o}{D} \quad \frac{2C_o}{D}\right]\left[[\bar{D}_3-\bar{D}_5 \; \ldots] + R_o \, [\bar{C}_2-\bar{C}_4 \; \ldots]\right] \qquad 3.1.32$$

It is readily apparent that v_{ℓ_o} in Eq. 3.1.32 is dependent on only M and N for $t < 2\tau$, whereas in Eq. 3.1.16 this limited dependence was true only for $t < \tau$. In general it is found that the expansion of the lowest order C_n^* and D_n^* will add one time interval τ to the period over which v_{ℓ_o} is dependent only on M and N. Repeated substitution of Eqs. 3.1.23 and 3.1.24 to evaluate coefficients is a laborious process, and therefore a recursion formula for the calculation of coefficients would be most helpful.

To aid in the development of recursion formulas, the following notation is adopted:

$$S_1 = \frac{2(R_o - D_o)}{D} \qquad 3.1.33$$

$$T_1 = \frac{2R_oC_o}{D} \qquad 3.1.34$$

$$S_2 = S_1 \frac{2D_1R_o}{D} - T_1(1 + \frac{2KR_o}{D}) \qquad 3.1.35$$

$$T_2 = S_1 \; (1 + \frac{2J}{D}) - T_1 \frac{2R_oC_o}{D} \qquad 3.1.36$$

$$P_n = \frac{R_o(1-C_1R_o)}{D} \; M_n + \frac{D_1R_o^2}{D} \; N_n \qquad 3.1.37$$

$$Q_n = \frac{R_o(R_o-D_o)}{D} N_n + \frac{R_oC_o}{D} \; M_n \qquad 3.1.38$$

Expansion of C_2^* and D_2^* in Eq. 3.1.30 leads to the following expression for v_{ℓ_o}:

$$v_{\ell_o} = Q_oU_o + [S_1P_1 - T_1Q_1] \; \epsilon_1U_1 + [S_2P_2 - T_2Q_2] \; \epsilon_2U_2 +$$
$$S_3 \left[[\bar{D}_3 - \bar{D}_5 \ldots] - R_o \; [\bar{C}_4 - \bar{C}_6 \ldots]\right] \; -$$
$$T_3 \left[[\bar{D}_4 - \bar{D}_6 \ldots] + R_o \; [\bar{C}_3 - \bar{C}_5 \ldots]\right] \qquad 3.1.39$$

where

$$S_3 = S_2 \frac{2D_1R_o}{D} - T_2 \; (1 + \frac{2KR_o}{D}) \qquad 3.1.40$$

$$T_3 = S_2 \left(1 + \frac{2J}{D}\right) - T_2 \frac{2R_oC_o}{D} \qquad 3.1.41$$

Since S_3 and T_3 are related to S_2 and T_2 in exactly the same way that S_2 and T_2 are related to S_1 and T_1, while Eq. 3.1.39 is related to Eq. 3.1.32 in exactly the same way that Eq. 3.1.32 is related to Eq. 3.1.16, it is possible to write the following general expression for v_{ℓ_o}:

$$\begin{aligned} v_{\ell_o} = v(\ell,t) = v_{T_1} = Q_oU_o + & [S_1P_1 - T_1Q_1] \ \epsilon_1U_1 + \\ & [S_2P_2 - T_2Q_2] \ \epsilon_2U_2 + \\ & \vdots \\ & [S_nP_n - T_nQ_n] \ \epsilon_nU_n + \\ & \vdots \end{aligned} \qquad 3.1.42$$

where, for $n \geq 1$,

$$S_{n+1} = S_n \frac{2D_1R_o}{D} - T_n\left(1 + \frac{2KR_o}{D}\right) \qquad 3.1.43$$

$$T_{n+1} = S_n \left(1 + \frac{2J}{D}\right) - T_n \frac{2R_oC_o}{D} \qquad 3.1.44$$

Before the general derivative-explicit equations for case (1) can be written, $i(0,t)$ must be expressed in terms of M and N only. The procedure followed is the same as was used for $v(\ell,t)$, and the final result can be written as

$$i_{o_o} = i(0,t) = i_{T_2} = \frac{P_o}{R_o} U_o + [X_1P_1 - V_1Q_1] \epsilon_1U_1 + [X_2P_2 - V_2Q_2] \epsilon_2U_2 + \cdots + [X_nP_n - V_nQ_n] \epsilon_nU_n + \cdots \quad 3.1.45$$

where

$$X_1 = \frac{2D_1}{D} \quad 3.1.46$$

$$V_1 = \frac{2(1-C_1R_o)}{D} \quad 3.1.47$$

$$X_{n+1} = X_n \frac{2D_1R_o}{D} - V_n \left(1 + \frac{2KR_o}{D}\right) \quad n \geq 1 \quad 3.1.48$$

$$V_{n+1} = X_n \left(1 + \frac{2J}{D}\right) - V_n \frac{2R_oC_o}{D} \quad n \geq 1 \quad 3.1.49$$

Substitution of Eqs. 3.1.42 and 3.1.45 into Eq. 1.18 completes the derivation of derivative-explicit equations for case (1). Since the numerous notational changes made in this section tend to obscure the form of the results, no direct substitution will be made at this time. However, it may be instructive to consider some general observations which can be made about the derivative-explicit equations for case (1)[1]:

---

[1] These observations also hold for the Eqs. obtained for the remaining cases.

(1) For $t < \tau$, the equations are in normal form.

(2) For $\tau \leq t < 2\tau$, a term is added to the equations which resulted for $t < \tau$. This term involves a linear combination of the functions $v_{C_1}(t-\tau)$, $i_{L_2}(t-\tau)$, $e(t-\tau)$, and $h(t-\tau)$. If the equations for $t < \tau$ have been solved the linear combination is a specified function and can be grouped with the term of specified driving functions. Thus the equation for the time interval $\tau < t \leq 2\tau$ has the same mathematical form as Eq. 1.10.

(3) For each sucessive interval of duration τ a similar reasoning can be applied to show that the equations for each interval will have the normal form if the equations for all preceding intervals have been solved[1].

(4) Every coefficient for each and every interval is a specified algebraic combination of R_o and submatrices of Eqs. 1.3, 1.4, 1.5.

In conclusion, the solution of the derivative-explicit equations for RLC networks containing a uniform distortionless transmission line must be carried out in a step-by-step manner due to the reflection associated with the transmission line terminals. In each step of the solution process the equations are in normal form.

---

[1] Reference (1) contains an excellent discussion of the properties of these Eqs., including theorems on the existence of solutions.

3.2 The general expressions for v_{T_1} and i_{T_2} in cases (2), (3) and (4).

The three remaining cases are handled in a manner similar to that used for case (1). The algebraic details are omitted and only the results and notational convention are given.

Case (2): neither T element is in the tree, hence the T_1 matrices are null. The general expression for i_{T_2} can be written as

$$i_{T_2} = \begin{bmatrix} i(0,t) \\ -i(\ell,t) \end{bmatrix} = \begin{bmatrix} \frac{P_o}{R_o} & S_1P_1+T_1Q_1 & \cdots & S_nP_n+T_nQ_n & \cdots \\ \frac{-Q_o}{R_o} & -(X_1P_1+V_1Q_1) & \cdots & -(X_nP_n+V_nQ_n) & \cdots \end{bmatrix} \begin{bmatrix} U_o \\ \epsilon_1 U_1 \\ \vdots \\ \epsilon_n U_n \\ \vdots \end{bmatrix} \qquad 3.2.0$$

where

$$P_n = \frac{C_1R_o}{D} M_n - \frac{R_o(R_o+D_1)}{D} N_n \qquad 3.2.1$$

$$Q_n = \frac{R_o(R_o+C_o)}{D} M_n - \frac{R_oC_o}{D} N_n \qquad 3.2.2$$

$$D = (R_o+C_o)\,(R_o+D_1) - C_1D_o \qquad 3.2.3$$

$$S_1 = \frac{2C_1}{D} \qquad 3.2.4$$

$$T_1 = \frac{2(R_o+D_1)}{D} \qquad 3.2.5$$

$$X_1 = \frac{2(R_o+C_o)}{D} \qquad 3.2.6$$

$$V_1 = \frac{2D_o}{D} \qquad 3.2.7$$

$$S_{n+1}(X_{n+1}) = S_n(X_n)\,\frac{2C_1R_o}{D} + T_n(V_n)\,[1+\frac{2K}{D}] \qquad 3.2.8$$

$$T_{n+1}(V_{n+1}) = S_n(X_n)\,[1+\frac{2J}{D}] + T_n(V_n)\,\frac{2D_oR_o}{D} \qquad 3.2.9$$

$$J = -C_o\,(R_o+D_1) + C_1D_o \qquad 3.2.10$$

$$K = -D_1\,(R_o+C_o) + C_1D_o \qquad 3.2.11$$

$$\begin{bmatrix} C_o & C_1 \\ D_o & D_1 \end{bmatrix} = S'_{33}\,A^{-1}\,S_{33} \qquad 3.2.12$$

$$\begin{bmatrix} N(t) \\ M(t) \end{bmatrix} = S'_{33}A^{-1}\,[S_{34}i_{L_2}+S_{35}h(t)] + [S'_{33}A^{-1}S_{32}G_2S'_{12}-S'_{13}]\,e(t) + [\,S'_{33}A^{-1}S_{32}G_2S'_{22}-S'_{23}]\,v_{C_1} \qquad 3.2.13$$

Case (3): T_o is a branch and T_ℓ is a chord. The general expressions for v_{T_1} and i_{T_2} can be written as

$$v_{T_1} = v(0,t) = Q_oU_o + \sum_{i=1}^{\infty} \left(S_iP_i + T_iQ_i\right)\,\epsilon_iU_i \qquad 3.2.14$$

$$i_{T_2} = -i(\ell,t) = \frac{P_o}{R_o} U_o - \sum_{i=1}^{\infty} [X_i P_i + V_i Q_i] \epsilon_i U_i \qquad 3.2.15$$

where

$$P_n = \frac{R_o(1-C_1R_o)}{D} M_n + \frac{D_1R_o^2}{D} N_n \qquad 3.2.16$$

$$Q_n = - \frac{R_oC_o}{D} M_n + \frac{R_o(R_o+D_o)}{D} N_n \qquad 3.2.17$$

$$D = (1-R_oC_1)(R_o+D_o) + R_oC_oD_1 \qquad 3.2.18$$

$$S_1 = \frac{2(R_o+D_o)}{D} \qquad 3.2.19$$

$$T_1 = \frac{2R_oC_o}{D} \qquad 3.2.20$$

$$X_1 = - \frac{2D_1}{D} \qquad 3.2.21$$

$$V_1 = \frac{2(1-C_1R_o)}{D} \qquad 3.2.22$$

$$S_{n+1} (X_{n+1}) = S_n (X_n) \frac{2D_1R_o}{D} + T_n (V_n) [1 + \frac{2KR_o}{D}] \qquad 3.2.23$$

$$T_{n+1} (V_{n+1}) = S_n (X_n) [\frac{2J}{D} - 1] + T_n (V_n) \frac{2R_oC_o}{D} \qquad 3.2.24$$

$$J = D_o (1-C_1R_o) + D_1C_oR_o \qquad 3.2.25$$

$$K = C_1 (R_o+D_o) - C_oD_1 \qquad 3.2.26$$

$$D_o = S'_{33}A^{-1}S_{33} \qquad 3.2.27$$

$$C_o = S_{43}-S_{42}G_2S'_{32}A^{-1}S_{33} \qquad 3.2.28$$

$$D_1 = S'_{43}-S'_{33}A^{-1}S_{32}G_2S'_{42} \qquad 3.2.29$$

$$C_1 = S_{42}[G_2S'_{32}A^{-1}S_{32}G_2S'_{42}- G_2S'_{42}] \qquad 3.2.30$$

$$M(t) = v_{T_2} + D_o i_{T_2} - D_1 v_{T_1} \qquad 3.2.31$$

$$N(t) = i_{T_1} + C_o i_{T_2} - C_1 v_{T_1} \qquad 3.2.32$$

Case (4): both T_o and T_ℓ are in the tree, hence the T_2 matrices are null. The general expression for v_{T_1} can be written as

$$v_{T_1} = \begin{bmatrix} v(0,t) \\ v(\ell,t) \end{bmatrix} = \begin{bmatrix} -R_oP_o & S_1P_1-T_1Q_1 & \cdots & S_nP_n-T_nQ_n & \cdots \\ -R_oQ_o & X_1P_1-V_1Q_1 & \cdots & X_nP_n-V_nQ_n & \cdots \end{bmatrix} \begin{bmatrix} U_o \\ U_1\epsilon_1 \\ \vdots \\ U_n\epsilon_n \\ \vdots \end{bmatrix} \qquad 3.2.33$$

$$P_n = -\frac{C_1R_o}{D}M_n + \frac{1+R_oD_1}{D}N_n \qquad 3.2.34$$

$$Q_n = \frac{1+R_oC_o}{D}M_n - \frac{R_oD_o}{D}N_n \qquad 3.2.35$$

$$D = (1+R_oD_1)(1+R_oC_o) - R_o^2\, C_1D_o \qquad 3.2.36$$

$$S_1 = \frac{2R_o^2 C_1}{D} \qquad 3.2.37$$

$$T_1 = \frac{2R_o(1+D_1R_o)}{D} \qquad 3.2.38$$

$$X_1 = -\frac{2R_o(1+R_oC_o)}{D} \qquad 3.2.39$$

$$V_1 = \frac{2R_o^2 D_o}{D} \qquad 3.2.40$$

$$S_{n+1}(X_{n+1}) = -S_n(X_n)\frac{2C_1R_o}{D} - T_n(V_n')[1 + \frac{2K}{D}] \qquad 3.2.41$$

$$T_{n+1}(V_{n+1}) = S_n(X_n)[1 + \frac{2J}{D}] + T_n(V_n)\frac{2R_oD_o}{D} \qquad 3.2.42$$

$$J = R_o^2 D_o C_1 - C_o R_o (1 + R_o D_1) \qquad 3.2.43$$

$$K = R_o^2 D_o C_1 - D_1 R_o (1 + R_o C_1) \qquad 3.2.44$$

$$\begin{bmatrix} C_o & C_1 \\ D_o & D_1 \end{bmatrix} = S_{42}[-G_2S_{32}'A^{-1}S_{32}G_2S_{42}' + G_2S_{42}'] \qquad 3.2.45$$

$$\begin{bmatrix} N(t) \\ M(t) \end{bmatrix} = -i_{T_1} + \begin{bmatrix} C_o & C_1 \\ D_o & D_1 \end{bmatrix} . v_{T_1} \qquad 3.2.46$$

3.3 Some special cases.

General derivative-explicit equations for RLC networks which contain a uniform distortionless transmission line of arbitrary length have been derived in the preceding sections. The equations are valid if the transmission line operation is described by Eqs. 1.0 and 1.1 and if certain matrices are non-singular.

In general the derivative-explicit equations must be solved in a step-by-step fashion, with new equations, new conditions, and new solutions for each time interval τ. The general solution process is lengthy for even simple networks, but there are a number of special cases in which the solution is relatively simple. These special cases are:

(1) The network N is disconnected:

(a) For all disconnected networks N, one pair of the C_i and D_i coefficients is zero in each of the four cases. In case (3), C_1 and D_o are zero, while in cases (1), (2), and (4), C_o and D_1 are zero. The null coefficients represent a coupling between T_o and T_ℓ in the graph. When N is disconnected, the network graph is disconnected and exactly one of the T elements is in each of the parts of the graph. It can be readily concluded that there is no coupling between T_o and T_ℓ when N is disconnected[1].

(b) If N is disconnected and one part of N is resistive, then the terminal quantity (v_{T_1} or i_{T_2}) of Eq. 1.18 which is in the resistive part of N will not appear in the derivative-explicit equations.

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[1] The mathematical basis for this conclusion is contained in footnote 2 on page 8.

The truth of this statement will be demonstrated for case (1).

Suppose the resistive part of N includes T_ℓ in the network graph and the formulation falls into case (1). T_ℓ is a chord, and Eq. 3.1.3 becomes

$$-v_{T_2} = S'_{33}A^{-1}S_{33}i_{T_2} \qquad 3.30$$

since none of the other terms in Eq. 3.1.3 can exist. Also, B_{31}, B_{32}, and B_{34} have only zero entries. By comparing the vanishing coefficients of Eq. 3.1.3 with the coefficients of i_{T_2} in Eq. 1.18, it is found that i_{T_2} will not appear in any of the differential equations. A similar argument can be used if T_ℓ is a branch, or if any of the other three cases are investigated.

For all disconnected networks N with one resistive part, it is found that either M(t) or N(t) is zero, and either P_n or Q_n is zero. For these networks, the resistive part appears in the derivative-explicit equations only through a reflection coefficient. In the special case where the resistive part of the network N is equivalent to R_o, this part of the network does not enter the differential equations at all, and the differential-equations take a form which describes an RLC network in which one of the resistors has the value R_o. Thus the step-by-step solution process is not necessary for this class of networks.

(2) The line length ℓ is zero or infinity.

(a) $\ell = \infty$; consider the derivative-explicit equations which result for $\ell = \infty$. For finite propagation velocities, we have

$$\lim_{\ell \to \infty} \tau = \infty \qquad 3.3.1$$

Therefore,

$$U_i \equiv 0, \ i \geq 1 \qquad 3.3.2$$

and the equations of sections 3.1 and 3.2 which specify i_{T_2} and v_{T_1}, in each of the cases, contain a single term. In each of the four cases, the single term equation implies that the elements T_o and T_ℓ are equivalent to resistors of resistance R_o. The validity of this last statement will be demonstrated for case (1). Eq. 3.1.45 becomes, for $\tau = \infty$,

$$i_{T_2} = \frac{P_o}{R_o} \qquad 3.3.3$$

Substituting the general forms for P_o, M_o, and N_o, Eq. 3.3.3 becomes

$$(R_o \ i(0,t) - v(0,t)) \ (1-C_1 R_o) = D_1 \ (R_o \ i(\ell,t) + v(\ell,t)) \qquad 3.3.4$$

For any given network C_1 and D_1 are specified numbers. Since one of the variables on each side of Eq. 3.3.4 can be independently varied, it is concluded that each side must be independently constant. If the independent pair of variables is set equal to zero then the value of the constant is seen to be zero. Since this result is independent of the values of C_1 and D_1, the terms involving the terminal quantities must be zero. Hence

$$R_o = \frac{v(0,t)}{i(0,t)} = \frac{v_{T_o}}{i_{T_o}} \qquad 3.3.5$$

$$R_o = \frac{v(\ell,t)}{-i(\ell,t)} = \frac{v_{T_\ell}}{i_{T_\ell}} \qquad 3.3.6$$

Equations 3.3.5 and 3.3.6 are simply defining equations for resistors of value R_o.

(b) $\ell = 0$; if derivative-explicit equations have been obtained for a network which contains a uniform transmission line and the equations are then examined under the condition of varying line length, the case $\ell = 0$ must be handled with caution. In general networks, reducing the line length to zero may give rise to circuits of voltage drivers, cut-sets of current drivers, or possibly may result in a topology such that the tree used as the basis for formulating the equations may no longer conform to the rules specified in Chapter I. If a general network is to be analyzed for the case $\ell = 0$ the general procedure should be as follows:

(1) Redraw the linear graph for the network with the transmission line removed.

(2) Check the resulting graph for conformity with the postulated network restrictions.

(3) Formulate and solve the derivative-explicit equations for the graph.

While in general the case $\ell = 0$ may lead to difficulty, there is at least one class of networks in which $\ell = 0$ may be considered without reformulation. When N (of Fig. 1.0) is disconnected and one part of N is entirely composed of positive finite resistors, then the case $\ell = 0$ may be handled by direct substitution of $\ell = 0$ into the solutions for v_{T_1} and i_{T_2} prior to subsituting these solutions into the equations.

The validity of this last statement will be shown for case (1), and an example is given in Chapter IV.

Assume that the resistive part of N is connected to the $x = \ell$ port of the transmission line. Since the resistive part of N is simply a two-terminal resistive network, this part can be represented in the graph by a single equivalent resistor R^*. Note that R^* is in the cotree for case (1). If the derivative-explicit equations are formulated for the network with special notation used to identify equations relevant to R^*, then the three basic sets of equations can be written in the following form:

$$\left[\begin{array}{ccccc|cccccc} U & 0 & 0 & 0 & 0 & S_{11} & S_{12_1} & 0 & S_{13} & S_{14} & S_{15} \\ 0 & U & 0 & 0 & 0 & S_{21} & S_{22_1} & 0 & S_{23} & S_{24} & S_{25} \\ 0 & 0 & U & 0 & 0 & 0 & S_{32_1} & 0 & S_{33} & S_{34} & S_{35} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & U & 0 & 0 & 0 & 0 & S_{54} & S_{55} \end{array}\right] \left[\begin{array}{c} i_e \\ i_{C_1} \\ i_{R_1} \\ i_{T_1} \\ i_{L_1} \\ \hline i_{C_2} \\ i_{R_2} \\ i_{R*} \\ i_{T_2} \\ i_{L_2} \\ h(t) \end{array}\right] = 0 \qquad 3.3.7$$

Equation 3.3.7 can be written as

$$[U \mid S_C] \left[\begin{array}{c} i_1 \\ \hline i_2 \end{array}\right] = 0 \qquad 3.3.8$$

The circuit equations can be written as

$$[-S_C' \;\vdots\; U] \begin{bmatrix} v_1 \\ \hdashline v_2 \end{bmatrix} = 0 \qquad 3.3.9$$

where S_C is given by Eqs. 3.3.7 and 3.3.8.

The element equations are the same as in equation 1.5 except for the cotree resistors. The representation used is

$$\begin{bmatrix} i_{R_2} \\ i_{R^*} \end{bmatrix} = \begin{bmatrix} G_2 & 0 \\ 0 & \frac{1}{R^*} \end{bmatrix} \begin{bmatrix} v_{R_2} \\ v_{R^*} \end{bmatrix} \qquad 3.3.10$$

In the matrix equations, G_2, v_{R_2}, and i_{R_2} refer to all cotree resistors except R^*, and the matrix S_{12} of Chapter I becomes $[S_{12_1} \;\vdots\; S_{12_{R^*}}]$ in the equations in this section.

If the various coefficients of Chapter III are evaluated for the case under consideration, the results are

$$D_1 = C_o = 0 \qquad 3.3.11$$

$$C_1 = -\frac{1}{R^*} \qquad 3.3.12$$

$$D_o = -S_{33}' A^{-1} S_{33} \qquad 3.3.13$$

$$D = (1 + \frac{R_o}{R^*}) (R_o - D_o) \qquad 3.3.14$$

$$N_n = 0 \qquad 3.3.15$$

$$Q_n = 0 \tag{3.3.16}$$

$$1 + \frac{2KR_o}{D} = \frac{R^* - R_o}{R^* + R_o} \tag{3.3.17}$$

$$1 + \frac{2J}{D} = \frac{R_o + D_o}{R_o - D_o} \tag{3.3.18}$$

It can be shown that D_o is a negative definite quadratic **form,** and therefore

$$\left| \frac{R_o + D_o}{R_o - D_o} \right| \leq 1 \tag{3.3.19}$$

Substitution of $\ell = 0$ into the equations for $v(\ell,t)$ and $i(0,t)$ yields:

$$v(\ell,t) \Big|_{\ell = 0} = P_o \; T\left(\sum_{i=o}^{\infty} (K_o K_1)^i \right) \tag{3.3.20}$$

$$i(0,t) \Big|_{\ell = 0} = P_o \quad \frac{1}{R_o} + \frac{2K_o}{R_o - D_o} \left(\sum_{i=0}^{\infty} (K_o K_1)^i \right) \tag{3.3.21}$$

where

$$T = \frac{2R^*}{R_o + R^*} \tag{3.3.22}$$

$$K_o = \frac{R_o - R^*}{R_o + R^*} \tag{3.3.23}$$

$$K_1 = \frac{R_o + D_o}{R_o - D_o} \tag{3.3.24}$$

$$P_o = \frac{R_o}{R_o - D_o} M_o \qquad 3.3.25$$

Since $|K_o K_1| < 1$, the infinite series converge. Using the result that

$$\sum_{i=0}^{\infty} (K_o K_1)^i = \frac{1}{1-K_o K_1} \qquad 3.3.26$$

the expressions for the terminal quantities can be written as

$$v(\ell,t) \Big|_{\ell = 0} = \frac{R^*}{R^* - D_o} M_o \qquad 3.2.27$$

$$i(0,t) \Big|_{\ell = 0} = \frac{M_o}{R^* - D_o} \qquad 3.3.28$$

From Eq. 3.1.13,

$$M_o = v(0,t) - D_o\, i(0,t) \qquad 3.3.29$$

Solving Eq. 3.3.29 for v(0,t) and evaluating for $\ell = 0$ gives

$$v(0,t) \Big|_{\ell = 0} = M_o + D_o\, i(0,t) \Big|_{\ell = 0} \qquad 3.3.30$$

From Eqs. 3.3.27 and 3.3.28,

$$v(0,t) \Big|_{\ell = 0} = v(\ell,t) \Big|_{\ell = 0} \qquad 3.3.31$$

From Eq. 3.1.10, since $N_o = 0$,

$$i_{T_1} = - \frac{v_{T_1}}{R^*} \qquad 3.3.32$$

Since $i_{T_1} = - i(\ell,t)$, Eq. 3.3.32 yields

$$i(\ell,t) \Big|_{\ell = 0} = i(0,t) \Big|_{\ell = 0} \qquad 3.3.33$$

It can be concluded that substitution of $\ell = 0$ into the expressions for $v(\ell,t)$ and $i(0,t)$ has had the effect of replacing the element T_0 by the resistor R^*, and has not violated Kirchhoff's laws.

IV. EXAMPLES OF THE ANALYSIS

4.0 Introduction

In the preceding chapters, derivative-explicit equations have been derived for RLC networks which contain a uniform distortionless transmission line. Several special cases in which the solution of the equations is relatively simple have been mentioned in Chapter III. In section 4.1, an example problem which illustrates the general formulation and several special cases is considered. In section 4.2, the formulation of a necessary and sufficient set of equations is discussed. In section 4.3, the alternative formulation discussed in Chapter I is illustrated.

4.1 An example of the analytical procedure.

The network of Figure 4.1.0 will be analyzed by using the derivative-explicit formulation presented in the preceding chapters. The choice of the network used in this example was based on the ease with which special cases can be illustrated as the general analysis is performed.

The following features of the network should be noted:

(1) N is disconnected,

(2) One part of N is resistive.

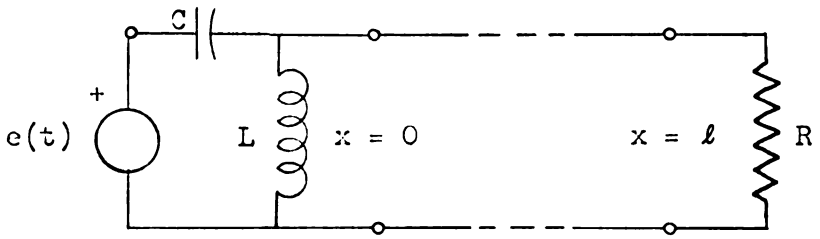

Figure 4.1.0 The example network.

Since N is disconnected, the two-port graphical representation of the transmission line is known to be valid. Figure 4.1.1 gives the oriented linear graph which is used as the basis for the formulation.

The voltage driver and initial conditions are specified by the following equations:

$$e(t) = 1, \quad t > 0 \qquad 4.1.0$$

$$e(t) = 0, \quad t < 0$$

$$v_C(0) = 0 \qquad 4.1.1$$

$$i_L(0) = 0 \qquad 4.1.2$$

The initial step in the analysis is the selection of a tree by application of the rules in section 1.1. The tree must include e(t), C, and either T_ℓ or R. In this example T_ℓ is selected as a branch to make

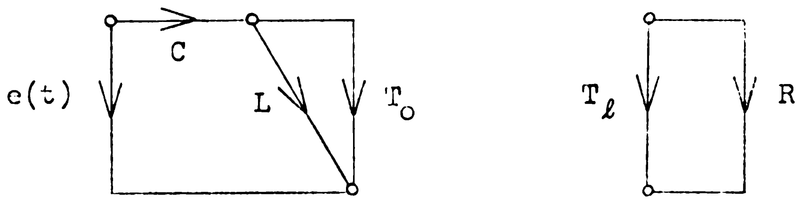

Figure 4.1.1 Oriented linear graph for the example network.

the formulation fit into case (1) since this case has been more thoroughly detailed than the other cases in preceding sections.

The f-cut-set, f-circuit, and element equations are, respectively:

$$\begin{array}{c} \\ 1 \\ 2 \\ 4 \\ \\ \end{array}
\begin{array}{cccccc}
e(t) & C & T_\ell & R & T_o & L \\
\end{array}$$

$$\begin{array}{c} 1 \\ 2 \\ 4 \end{array}
\left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & -1 & -1 \\
0 & 0 & 1 & 1 & 0 & 0
\end{array}\right]
\left[\begin{array}{c} i_e \\ i_C \\ i_{T_\ell} \\ \hline i_R \\ i_{T_o} \\ i_L \end{array}\right] = 0 \qquad 4.1.3$$

$$\begin{array}{ccc} & & \end{array}\qquad \begin{array}{ccc} 2 & 3 & 4 \end{array}$$

$$\begin{array}{c} 2 \\ 3 \\ 4 \end{array}
\left[\begin{array}{ccc|ccc}
0 & 0 & -1 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & 0 & 1
\end{array}\right]
\left[\begin{array}{c} e(t) \\ v_C \\ v_{T_\ell} \\ \hline v_R \\ v_{T_o} \\ v_L \end{array}\right] = 0 \qquad 4.1.4$$

$$\begin{array}{ccc} 1 & 2 & 4 \end{array}$$

$$\begin{bmatrix} i_C \\ i_R \\ v_L \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & G & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} \dot{v}_C \\ v_R \\ \dot{i}_L \end{bmatrix} \qquad 4.1.5$$

where the integers below and to the left of Eqs. 4.1.3 and 4.1.4 indicate the j and i of S_{ij} and B_{ij} as these submatrices appear in the preceding chapters. The dot over the entries in the right hand column matrix of Eq. 4.1.5 denotes the time derivative.

Substitution of the appropriate submatrices into Eq. 1.18 yields the modified derivative explicit equations as

$$\begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & L \end{bmatrix}^{-1} \left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} e(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_{T_\ell} \\ i_{T_o} \end{bmatrix} \right] \qquad 4.1.6$$

Performing the indicated matrix operations in Eq. 4.1.6 gives

$$\dot{v}_C = \frac{1}{C} [i_L + i_{T_o}] \qquad 4.1.7$$

$$\dot{i}_L = \frac{1}{L} [e(t) - v_C] \qquad 4.1.8$$

Note that v_{T_ℓ} does not appear in the modified derivative-explicit equations. This result is in accord with the discussion in section 3.3.

The next step in the analysis is the calculation of i_{T_o} from Eq. 3.1.45. The various coefficients are evaluated from their respective defining equations by substitution from Eqs. 4.1.3, 4.1.4 and 4.1.5. The results are

$$C_o = D_o = D_1 = 0 \qquad 4.1.9$$

$$C_1 = -\frac{1}{R} \qquad 4.1.10$$

$$N_n = 0 \qquad 4.1.11$$

$$M_n = (e(t-n\tau) - v_C(t-n\tau))\, U(t-n\tau) \qquad 4.1.12$$

$$D = R_o \left(1 + \frac{R_o}{R}\right) \qquad 4.1.13$$

$$P_n = M_n \qquad 4.1.14$$

$$Q_n = 0 \qquad 4.1.15$$

$$J = 0 \qquad 4.1.16$$

$$K = -\frac{R_o}{R} \qquad 4.1.17$$

$$X_1 = 0 \qquad 4.1.18$$

$$V_1 = \frac{2}{R_o} \qquad 4.1.19$$

$$X_{n+1} = V_n \left(\frac{R_o - R}{R_o + R}\right) \qquad 4.1.20$$

$$V_{n+1} = X_n \qquad 4.1.21$$

In accordance with the conclusions reached in section 3.3, note that $C_o = D_1 = N_n = Q_n = 0$.

Since the coefficient of V_n in Eq. 4.1.20 has the general form of a current reflection coefficient, the following notation will be used in the remainder of this example:

$$K_o = \frac{R_o - R}{R_o + R} \qquad 4.1.22$$

Based on the value of the coefficients calculated for this example, i_{T_o} can be written as

$$i_{T_o} = \frac{1}{R_o} [e(t)-v_C] + \frac{2}{R_o} \sum_{i=1}^{\infty} K_o^i [e(t-2i\ \tau) - v_C\ (t-2i\ \tau)]\ U(t-2i\ \tau) \qquad 4.1.23$$

The derivative-explicit equations for the example network can now be written in expanded form as:

$$\dot{v}_C = \frac{1}{C} [i_L + \frac{1}{R_o} (e(t)-v_C(t)) + \frac{2K_o}{R_o} (e(t-2\tau) - v_C(t-2\tau))\ \ U(t-2\tau)$$

$$+ \frac{2K_o^2}{R_o}\ (e(t-4\tau) - v_C(t-4\tau))\ \ U(t-4\tau) + \ldots] \qquad 4.1.24$$

$$\dot{i}_L = \frac{1}{L} (e(t) - v_C(t)) \qquad 4.1.8$$

Before considering the general solution of Eqs. 4.1.24 and 4.1.8 these equations will be examined under some of the special circumstances discussed in section 3.3.

(1) If $\ell = \infty$, the derivative-explicit equations become

$$\dot{v}_C = \frac{1}{C} [i_L + \frac{1}{R_o} (e(t) - v_C(t))] \qquad 4.1.25$$

$$\dot{i}_L = \frac{1}{L} (e(t) - v_C(t)) \qquad 4.1.26$$

Equations 4.1.25 and 4.1.26 describe the network of Fig. 4.1.2. In essence, T_o has been replaced by a resistor R_o.

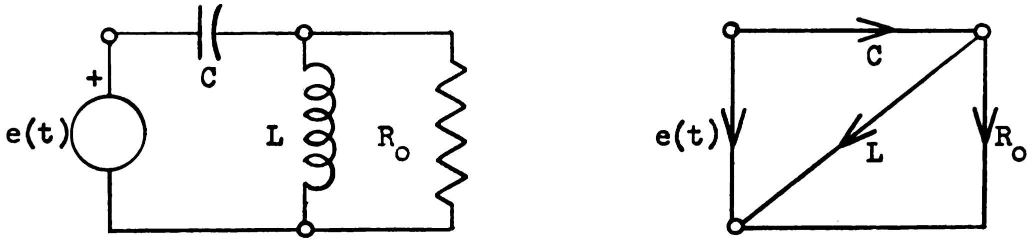

Figure 4.1.2 Reduced example network and graph for special cases.

(2) If $R = R_o$ Eq. 4.1.22 indicates $K_o = 0$. Thus the derivative-explicit equations are Eqs. 4.1.25 and 4.1.26 for this case.

(3) If $\ell = 0$, then $\tau = 0$, and Eqn. 4.1.24 can be written as

$$\dot{v}_C = \frac{1}{C} [i_L - \frac{1}{R_o} (e(t)-v_C(t)) + \frac{2}{R_o} (e(t)-v_C(t)) \sum_{i=0}^{\infty} K_o^i] \qquad 4.1.27$$

where $\frac{1}{R_o}$ $(e(t) - v_C(t))$ has been added and subtracted inside the bracket. When $|K_o| < 1$, the sum in Eqn. 4.1.27 converges:

$$\sum_{i=0}^{\infty} K_o^i = \frac{1}{1-K_o} \qquad 4.1.28$$

Thus, when $0 < R < \infty$, the derivative-explicit equations for $\ell = 0$ can be written as

$$\dot{v}_C = \frac{1}{C} (i_L + \frac{1}{R} (e(t) - v_C(t))) \qquad 4.1.29$$

$$\dot{i}_L = \frac{1}{L} (e(t) - v_C(t)) \qquad 4.1.30$$

By comparing Eq. 4.1.29 with Eq. 4.1.25, the network represented by Eqs. 4.1.29 and 4.1.30 is seen to be the network of Fig. 4.1.2 when R_o is replaced by R.

These results conform to the discussion in section 3.3, with the special case $D_o = 0$ appearing in this example.

It is instructive to note the results when R is allowed to approach zero and infinity in Eq. 4.1.29. For $R = \infty$, the differential equations become

$$\dot{v}_C = \frac{1}{C} \; i_L \qquad 4.1.31$$

$$\dot{i}_L = \frac{1}{L} \left(e(t) - v_C(t)\right) \qquad 4.1.32$$

Equations 4.1.31 and 4.1.32 describe the network of Fig. 4.1.2 with R_o removed.

When $R = 0$, Eq. 4.1.29 becomes undefined. Note that with R in parallel with L, the condition $v_L = v_R$ implies $\dot{i}_L = 0$ when $R = 0$. Thus Eq. 4.1.30 gives

$$v_C(t) = e(t) \qquad 4.1.33$$

This last equation is merely an expression of Kirchhoff's circuit equation, and is obviously valid.

Part of the difficulty associated with the values $R = 0$ and $R = \infty$ is that the element $R = 0$ (∞) should be treated as a voltage (current) driver with $e(t)$ ($h(t)$) = 0 (0) in the formulation.

The solution of Eqs. 4.1.24 and 4.1.8 will now be considered. The basic point illustrated by this example is the step-by-step solution process which is necessary due to the form of the derivative-explicit equations. Numerical values for the elements of the network will be assigned on the basis of simplicity rather than reality.

For the time interval $0 \leq t < 2\tau$ the differential equations are

$$\dot{v}_C = \frac{1}{C}\left[i_L + \frac{1}{R_o}\ (e(t) - v_C)\right] \qquad 4.1.34$$

$$\dot{i}_L = \frac{1}{L}\left[e(t) - v_C\right] \qquad 4.1.35$$

If Eq. 4.1.35 is substituted into the derivative of Eq. 4.1.34 the resulting expression is

$$\ddot{v}_C = \frac{1}{LC}\ (e(t) - v_C) - \frac{1}{R_oC}\ \dot{v}_C \qquad 4.1.36$$

Since $e(t) = 1$ for $t > 0$, Eq. 4.1.36 can be written in terms of the differential operator D as

$$(D^2 + \frac{D}{R_oC} + \frac{1}{LC})\ v_C = \frac{1}{LC} \qquad 4.1.37$$

The roots of the characteristic equation are

$$D = \frac{-\frac{1}{R_oC} \pm \sqrt{(\frac{1}{R_oC})^2 - \frac{4}{LC}}}{2} \qquad 4.1.38$$

Three cases can occur with respect to the roots. The particular case which does arise is determined by the value of $(\frac{1}{R_oC})^2 - \frac{4}{LC}$. The cases are:

(1) Roots real and distinct for $(\frac{1}{R_oC})^2 > \frac{4}{LC}$.

(2) Roots are complex conjugates for $(\frac{1}{R_oC})^2 < \frac{4}{LC}$.

(3) Roots are real and repeated for $(\frac{1}{R_o C})^2 = \frac{4}{LC}$.

The values assigned below to the network elements for this example give rise to real repeated roots.

R_o = 1 ohm

C = 1 farad

L = 4 henrys

R = $\frac{1}{3}$ ohm

τ = 1 second

From Eq. 4.1.22 and the specified values of R and R_o, K_o is evaluated as

$$K_o = \frac{1}{2} \qquad 4.1.39$$

The general solution of Eq. 4.1.36 can be written as

$$v_C = K_1 e^{-\frac{t}{2}} + K_2 t e^{-\frac{t}{2}} + 1 \qquad 4.1.40$$

where K_1 and K_2 are constants which can be evaluated by using the initial conditions given in Eqs. 4.1.1 and 4.1.2. The solution obtained for v_C can be written as

$$v_C = e^{-\frac{t}{2}} (\frac{t}{2} - 1) + 1 \quad 0 \leq t < 2 \qquad 4.1.41$$

Substituting Eq. 4.1.41 into Eq. 4.1.34 gives

$$i_L = \frac{t}{4} e^{-\frac{t}{2}} \quad 0 \leq t < 2 \qquad 4.1.42$$

At t = 2 another term in Eq. 4.1.24 becomes non-zero. The differential equations for $2 \leq t < 4$ are

$$\dot{v}_C = i_L + 1 - v_C + 1 - v_C(t-2) \qquad 4.1.43$$

$$\dot{i}_L = \frac{1}{4}[1 - v_C] \qquad 2 \leq t < 4 \qquad 4.1.44$$

In the time interval for which Eq. 4.1.43 is valid, the time delayed function is obtained from Eq. 4.1.41 as

$$v_C(t-2) = e^{-\frac{(t-2)}{2}}\left(\frac{t-2}{2} - 1\right) + 1 \qquad 2 \leq t < 4 \qquad 4.1.45$$

The conditions imposed to evaluate constants of integration which arise in the solution of Eqs. 4.1.43 and 4.1.44 are that v_C and i_L must be continuous at t = 2. The complete solutions for Eqs. 4.1.43 and 4.1.44 are

$$v_C = e^{-\frac{t}{2}}\left(\frac{t}{2} - 1\right) + 1 + e^{-\frac{1}{2}(t-2)}\left(\frac{t^3}{24} - \frac{3t^2}{4} + \frac{7}{2}t - \frac{13}{3}\right) \qquad 2 \leq t < 4 \qquad 4.1.46$$

$$i_L = \frac{t}{4}\, e^{-\frac{t}{2}} + e^{-\frac{1}{2}(t-2)}\left(\frac{t^3}{48} - \frac{t^2}{4} + \frac{3t}{4} - \frac{2}{3}\right) \qquad 2 \leq t < 4 \qquad 4.1.47$$

Another term enters Eq. 4.1.24 at t = 4. The set of differential equations for the time interval $4 \leq t < 6$ is

$$\dot{v}_C = i_L + 1 - v_C(t) + 1 - v_C(t-2) + \frac{1}{2}\,(1 - v_C(t-4)) \qquad 4.1.48$$

$$\dot{i}_L = \frac{1}{4}[1 - v_C(t)] \qquad 4.1.49$$

The solution of Eqs. 4.1.48 and 4.1.49 lead to the following expressions which are valid for $0 \leq t < 6$

$$v_C = v_{C_1} + v_{C_2}\, U(t-2) + v_{C_3}\, U(t-4) \qquad 4.1.50$$

$$i_L = i_{L_1} + i_{L_2}\, U(t-2) + i_{L_3}\, U(t-4) \qquad 4.1.51$$

where v_{C_1} (i_{L_1}) is given by Eq. 4.1.41 (4.1.42), $v_{C_1} + v_{C_2}$ ($i_{L_1} + i_{L_2}$) is given by Eq. 4.1.46 (4.1.47) and where

$$v_{C_3} = e^{-\frac{1}{2}(t-4)} \left[\frac{t^5}{960} - \frac{5t^4}{96} + \frac{15t^3}{16} - \frac{23}{3}t^2 + \frac{173}{6}t - \frac{202}{5}\right] \qquad 4.1.52$$

$$i_{L_3} = e^{-\frac{1}{2}(t-4)} \left[\frac{t^5}{1920} - \frac{t^4}{48} + \frac{29}{96}t^3 - \frac{97}{48}t^2 + \frac{19}{3}t - \frac{113}{15}\right] \qquad 4.1.53$$

The general solution of the differential equations will not be carried out for any additional time intervals. To complete the network solution for the time interval $0 \leq t < 6$, the remaining unspecified network voltages and currents must be calculated. From the basic network equations (4.1.3, 4.1.4, 4.1.5) we have

$$v_L = e(t) - v_C(t) = 4\,\frac{di_L}{dt} \qquad 4.1.54$$

$$i_C = -\,i_e = \frac{dv_C}{dt} \qquad 4.1.55$$

$$v_R = Ri_R = v(\ell,t) \qquad 4.1.56$$

Obviously v_L, i_C, and i_e can be obtained from the solutions to the derivative-explicit equations. To obtain v_R and i_R, $v(\ell,t)$ must be calculated from Eq. 3.1.42.

$$v(\ell,t) = Q_o + \sum_{i=1}^{\infty} (S_iP_i - T_iQ_i)\, \epsilon_i U_i \qquad 3.1.42$$

For this example network, we have

$$v(\ell,t) = \sum_{i=1}^{\infty} S_iP_iU_i \qquad 4.1.57$$

$$S_1 = \frac{2R}{R_o+R} \qquad 4.1.58$$

$$T_1 = 0 \qquad 4.1.59$$

$$S_{n+1} = T_nK_o \qquad 4.1.60$$

$$T_{n+1} = S_n \qquad 4.1.61$$

The solution for $v_R(t)$ can be written as follows, for $0 \leq t < 7$:

$$v_R(t) = \frac{2R}{R_o+R}\left[(1-v_C(t-1))\,U(t-1) + \frac{1}{2}\,(1-v_C(t-3))\,U(t-3) + \frac{1}{4}\,(1-v_C(t-5))\,U(t-5)\right] \qquad 4.1.62$$

The numerical evaluation of $v_C(t)$, $i_L(t)$, $i_C(t)$, $v_L(t)$, and $i_R(t)$ are given for $0 \leq t < 6$ in Figs. 4.1.3 through 4.1.7 respectively.

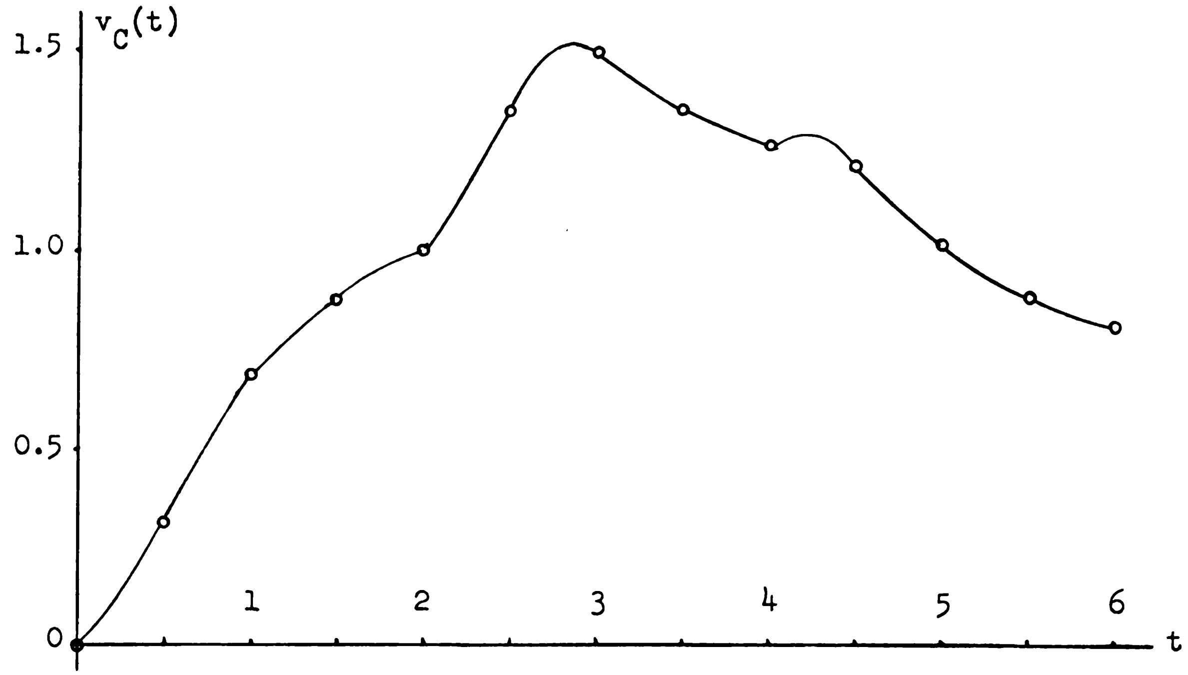

Figure 4.1.3 Capacitor voltage for t < 6.

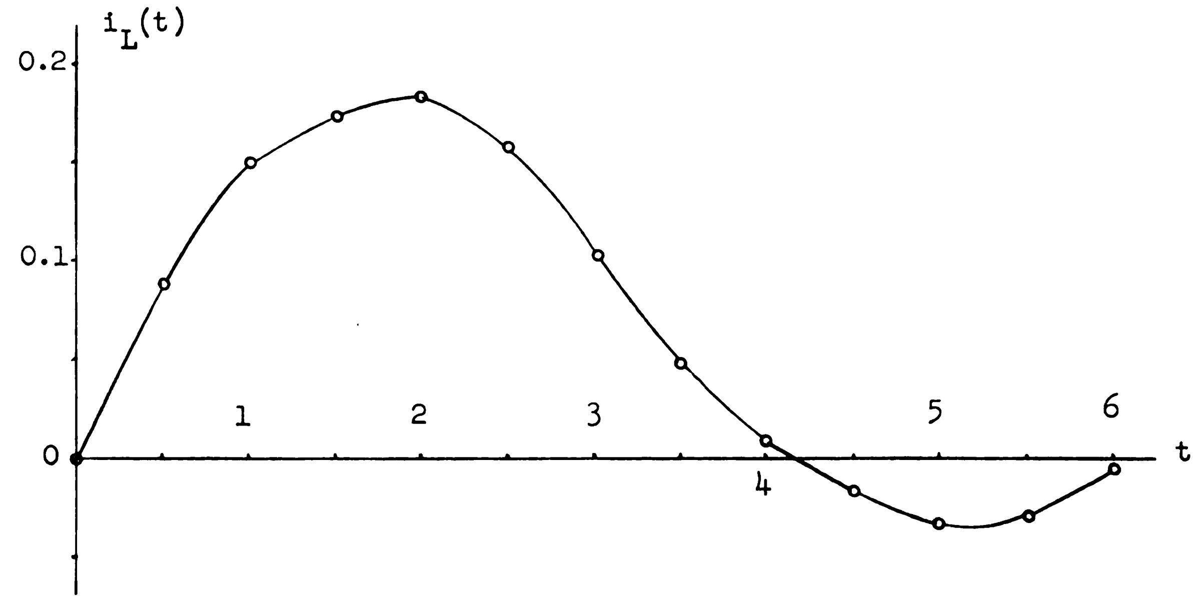

Figure 4.1.4 Inductor current for t < 6.

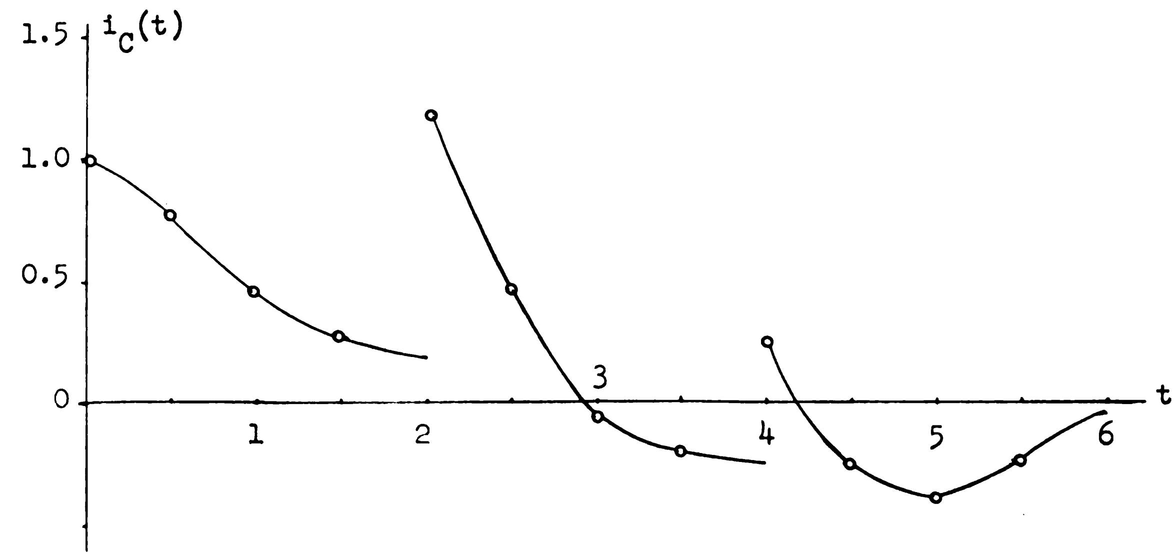

Figure 4.1.5 Capacitor current for $t < 6$.

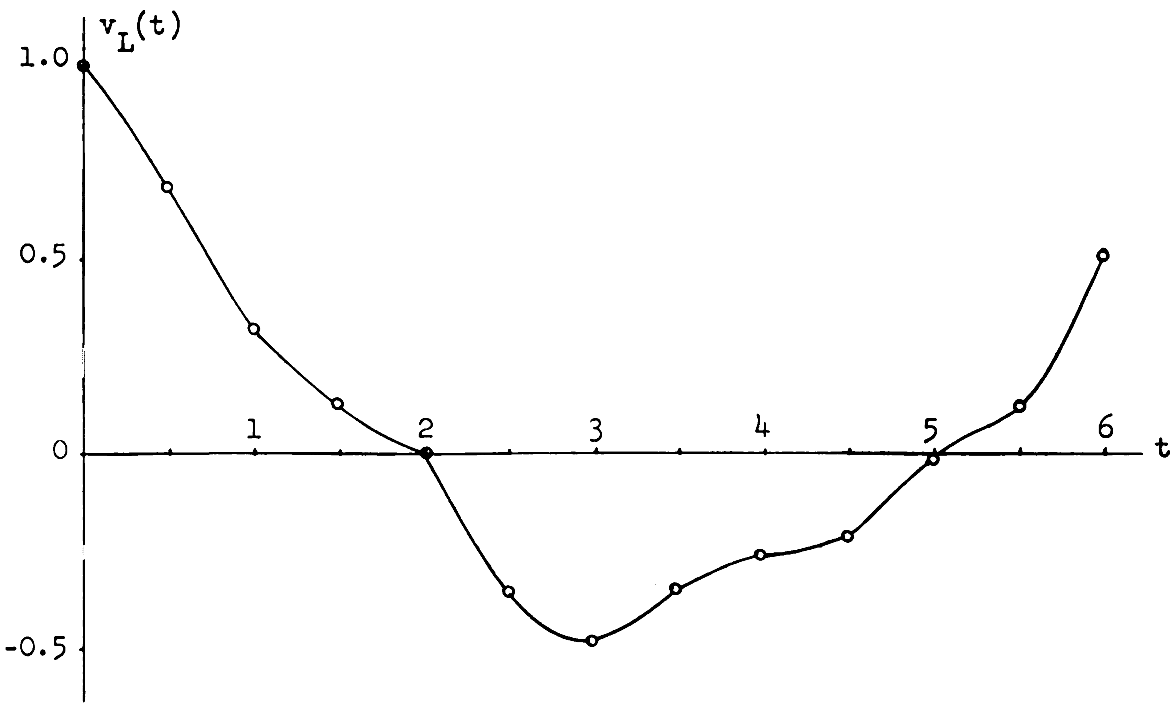

Figure 4.1.6 Inductor voltage for $t < 6$.

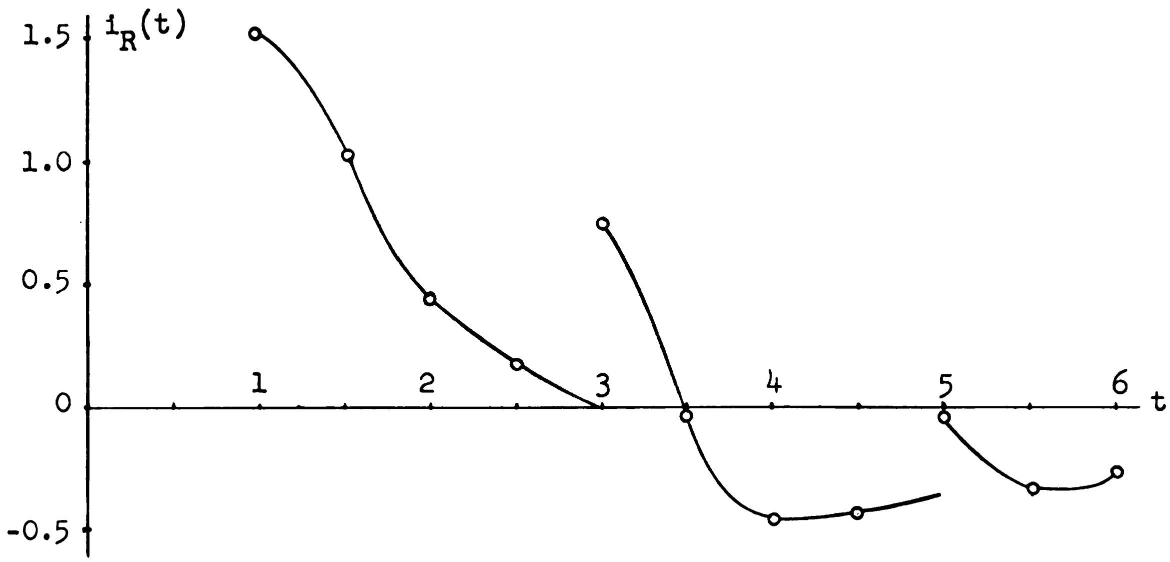

Figure 4.1.7 Resistor current for $t < 6$.

4.2 The formulation of a reduced set of equations.

Prior to considering a specific example network in which the normal form derivative-explicit equations are such that det $[A_o] = 0$, some general equations will be presented.

The reduction of Eq. 1.20 to normal form implies that i_{T_2} and v_{T_1} from Chapter III have been substituted into Eq. 1.20 and coefficients have been collected. The result can be written as

$$\frac{d}{dt}[X] = [B_o^*]\,[X] + [B_1^*]\,f(t) \qquad 4.2.0$$

where f(t) includes time delayed terms and derivatives of driving functions as specified functions of time, and X is a column matrix of the dependent variables.

If det $[B_o^*] = 0$, assume that the rows and columns of the matrices have been manipulated such that Eqs. 4.2.0 can be written as

$$\frac{d}{dt} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} f(t) \qquad 4.2.1$$

where B_{11} is non singular and has the rank of B_o^*. Using the dot convention to denote time derivatives, the equations obtained from the first and second rows of Eq. 4.2.1 are

$$\dot{X}_1 = B_{11} X_1 + B_{12} X_2 + B_1 \ f(t) \qquad 4.2.2$$

$$\dot{X}_2 = B_{21} X_1 + B_{22} X_2 + B_2 \ f(t) \qquad 4.2.3$$

Eq. 4.2.2 can be solved for X_1 since B_{11}^{-1} exists. Substitution of the solution for X_1 into Eq. 4.2.3 yields

$$\dot{X}_2 - B_{21}B_{11}^{-1}\dot{X}_1 = [B_{22}-B_{21}B_{11}^{-1}B_{12}] \ X_2 + [B_2-B_{21}B_{11}^{-1}B_1] \ f(t) \qquad 4.2.4$$

The coefficient of X_2 in Eq. 4.2.4 is zero (Th. A(2)), and therefore, integration of this equation yields the variables X_2 in terms of X_1 as

$$X_2 = B_{21}B_{11}^{-1} [X_1 - X_1(0)] + X_2(0) + [B_2-B_{21} \ B_{11}^{-1}B_1] \int_0^t f(t)\,dt \quad 4.2.5$$

where $X_1(0)$ is a column matrix of the initial conditions on the variables X_1.

Substitution of Eq. 4.2.5 into Eq. 4.2.2 yields the following necessary set of normal form equations:

$$\dot{X}_1 = [B_{11} + B_{12} B_{21} B_{11}^{-1}] X_1 - B_{12} B_{21} B_{11}^{-1} X_1(0) + B_{12} X_2(0)$$

$$+ B_{12} [B_2 - B_{21} B_{11}^{-1} B_1] \int_0^t f(t)\, dt + B_1 f(t) \qquad 4.2.6$$

Note that this reduced set of equations has terms involving initial conditions and integrals which did not appear in the original full set of equations. The practical advantage of using the reduced set of equations must be decided on the basis of the actual equations for any given network.

The network of Fig. 4.2.0 will be used to illustrate the formulation of a necessary and sufficient set of derivative-explicit equations. In this example network the series connected capacitors C_A and C_B and the parallel inductors L_A and L_B are the elements which give rise to the "excess" equations.

Using the linear graph representation of Fig. 4.2.1 as the basis for formulation, the f-cut-set, f-circuit, and element equations are respectively:

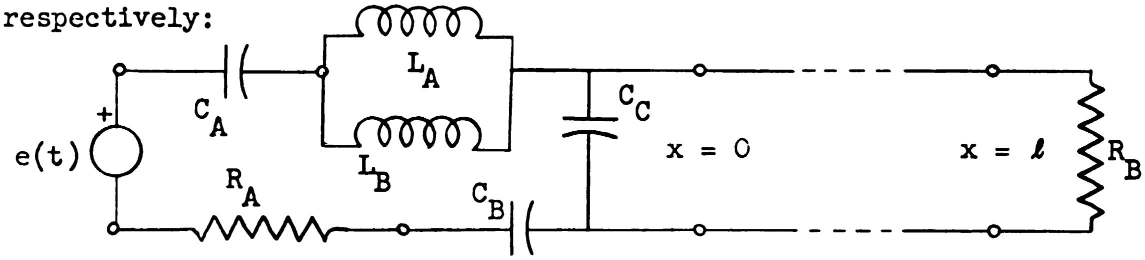

Figure 4.2.0 Example network for illustration of redundant equations.

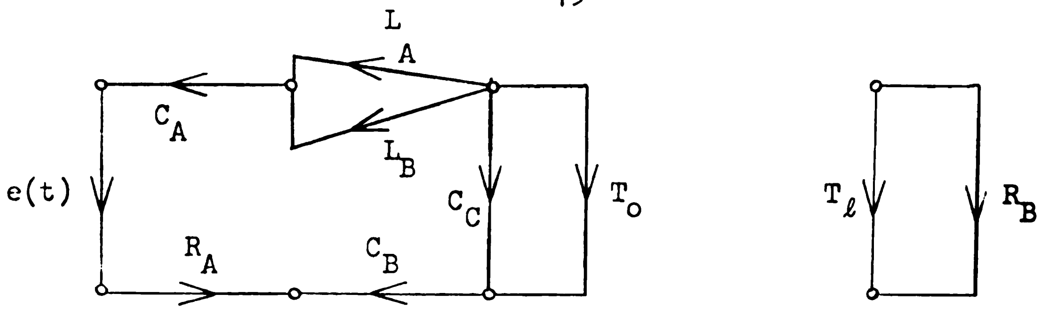

Figure 4.2.1 Oriented linear graph for the example network.

$$
\begin{array}{c}
\\ 1 \\ \\ 2 \\ \\ \\ 3
\end{array}
\begin{array}{c}
\begin{matrix} e(t) & C_A & C_B & C_C & R_A & R_B & T_o & T_\ell & L_A & L_B \end{matrix} \\
\left[\begin{array}{cccccc:cccc}
1 & & & & & & 0 & 0 & -1 & -1 \\
 & 1 & & & \bigcirc & & 0 & 0 & -1 & -1 \\
 & & 1 & & & & 0 & 0 & 1 & 1 \\
 & & & 1 & & & 1 & 0 & 1 & 1 \\
 & \bigcirc & & & 1 & & 0 & 0 & -1 & -1 \\
 & & & & & 1 & 0 & 1 & 0 & 0
\end{array}\right]
\end{array}
\left[\begin{array}{c} i_1 \\ \hdashline i_2 \end{array}\right] = 0 \qquad 4.2.7
$$

3 4

$$
[-S_C' \;\vdots\; U] \left|\begin{array}{c} v_1 \\ \hdashline v_2 \end{array}\right| = 0 \qquad 4.2.8
$$

$$
\begin{bmatrix} i_{C_A} \\ i_{C_B} \\ i_{C_C} \\ i_{R_A} \\ i_{R_B} \\ v_{L_A} \\ v_{L_B} \end{bmatrix}
=
\begin{bmatrix}
C_A & & & & & & \\
 & C_B & & & \bigcirc & & \\
 & & C_C & & & & \\
 & \bigcirc & & \frac{1}{R_A} & & & \\
 & & & & \frac{1}{R_B} & & \\
 & & & & & L_A & \\
 & & & & & & L_B
\end{bmatrix}
\begin{bmatrix} \dot{v}_{C_A} \\ \dot{v}_{C_B} \\ \dot{v}_{C_C} \\ v_{R_A} \\ v_{R_B} \\ \dot{i}_{L_A} \\ \dot{i}_{L_B} \end{bmatrix} \qquad 4.2.9
$$

where $i_1(v_1)$ and $i_2(v_2)$ correspond to tree and cotree currents (voltages) respectively. The integers below and to the left of the matrix in Eq. 4.2.7 specify the j and i respectively of S_{ij} as given in preceding chapters.

The modified derivative-explicit equations can be readily obtained as

$$\begin{bmatrix} C_A & & & & \\ & C_B & & \bigcirc & \\ & & C_C & & \\ & \bigcirc & & L_A & \\ & & & & L_B \end{bmatrix} \begin{bmatrix} \dot{v}_{C_A} \\ \dot{v}_{C_B} \\ \dot{v}_{C_C} \\ \dot{i}_{L_A} \\ \dot{i}_{L_B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 & -1 \\ -1 & 1 & 1 & -R_A & -R_A \\ -1 & 1 & 1 & -R_A & -R_A \end{bmatrix} \begin{bmatrix} v_{C_A} \\ v_{C_B} \\ v_{C_C} \\ i_{L_A} \\ i_{L_B} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ -1 \end{bmatrix} e(t)$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} i_{T_o} \\ i_{T_\ell} \end{bmatrix} \qquad 4.2.10$$

After obtaining and substituting the solution for i_{T_o} in case (2) from Chapter III, manipulation of Eq. 4.2.10 yields

$$\begin{bmatrix} \dot{v}_{C_B} \\ \dot{v}_{C_C} \\ \dot{i}_{L_A} \\ \dot{v}_{C_A} \\ \dot{i}_{L_B} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{C_B} & 0 & -\frac{1}{C_B} \\ 0 & -\frac{1}{R_o C_C} & -\frac{1}{C_C} & 0 & -\frac{1}{C_C} \\ \frac{1}{L_A} & \frac{1}{L_A} & -\frac{R_A}{L_A} & -\frac{1}{L_A} & -\frac{R_A}{L_A} \\ 0 & 0 & \frac{1}{C_A} & 0 & +\frac{1}{C_A} \\ \frac{1}{L_B} & \frac{1}{L_B} & -\frac{R_A}{L_B} & -\frac{1}{L_B} & -\frac{R_A}{L_B} \end{bmatrix} \begin{bmatrix} v_{C_B} \\ v_{C_C} \\ i_{L_A} \\ v_{C_A} \\ i_{L_B} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{L_A} \\ 0 \\ -\frac{1}{L_B} \end{bmatrix} e(t) + \begin{bmatrix} 0 \\ f(v_{C_C}(t-n\tau)) \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad 4.2.11$$

where $f(v_{C_C}(t-n\tau))$ represents the time delayed terms in i_{T_o}.

Since Eq. 4.2.11 is in the form of Eq. 4.2.1, Eqs. 4.2.4 and 4.2.6 can be used to obtain

$$\begin{bmatrix} v_{C_A} \\ i_{L_B} \end{bmatrix} = \begin{bmatrix} -\frac{C_B}{C_A} & 0 & 0 \\ 0 & 0 & \frac{L_A}{L_B} \end{bmatrix} \begin{bmatrix} v_{C_B} \\ v_{C_C} \\ i_{L_A} \end{bmatrix} + \begin{bmatrix} 1 & +\frac{C_B}{C_A} & 0 & 0 \\ 0 & 0 & -\frac{L_A}{L_B} & 1 \end{bmatrix} \begin{bmatrix} v_{C_A}(0) \\ v_{C_B}(0) \\ i_{L_A}(0) \\ i_{L_B}(0) \end{bmatrix} \quad 4.2.12$$

$$\begin{bmatrix} \dot{v}_{C_B} \\ \dot{v}_{C_C} \\ \dot{i}_{L_A} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{C_B}(1+\frac{L_A}{L_B}) \\ 0 & -\frac{1}{R_o C_C} & -\frac{1}{C_C}(1+\frac{L_A}{L_B}) \\ \frac{1}{L_A}(1+\frac{C_B}{C_A}) & \frac{1}{L_A} & -R_A(\frac{1}{L_A}+\frac{1}{L_B}) \end{bmatrix} \begin{bmatrix} v_{C_B} \\ v_{C_C} \\ i_{L_A} \end{bmatrix} +$$

$$\begin{bmatrix} 0 & +\frac{L_A}{C_B L_B} & 0 & -\frac{1}{C_B} & 0 & 0 \\ 0 & +\frac{L_A}{C_C L_B} & 0 & -\frac{1}{C_C} & 0 & 1 \\ -\frac{C_B}{L_A C_A} & +\frac{R_A}{L_B} & -\frac{1}{L_A} & -\frac{R_A}{C_A} & -\frac{1}{L_A} & 0 \end{bmatrix} \begin{bmatrix} v_{C_B}(0) \\ i_{L_A}(0) \\ v_{C_A}(0) \\ i_{L_B}(0) \\ e(t) \\ f(v_{C_C}(t-n\tau)) \end{bmatrix} \qquad 4.2.13$$

Equation 4.1.13 specifies a necessary and sufficient set of derivative-explicit equations for the network of Fig. 4.2.0.

4.3 An example of the alternate formulation.

In Chapter I the possibility of selecting a tree by rules other than those postulated was mentioned. The conclusion reached was that while a somewhat different system of derivative-explicit equations could be obtained, there was no possibility of reducing the number of equations in the system by the alternate formulation. The second possible tree is based on the following rules:

(1) All e(t) elements are in the tree.

(2) All possible C elements are placed in the tree, unless T_o and/or T_ℓ is in parallel with any of these capacitors. In this case, either terminal elements or the parallel capacitors may be put into the tree.

(3) All possible L elements are placed in the cotree, unless T_o and/or T_ℓ is in series with any of these inductors. In this case, either the terminal elements or the series inductors may be placed in the cotree.

(4) All h(t) elements are in the cotree.

By considering the network of Fig. 4.3.1, this alternative forumlation is briefly illustrated. The oriented linear graph used in this example is the graph of Fig. 4.1.1 with the C and L elements interchanged. Since C is now in parallel with T_o, a tree composed of e(t), R, and T_o will be used in the formulation.

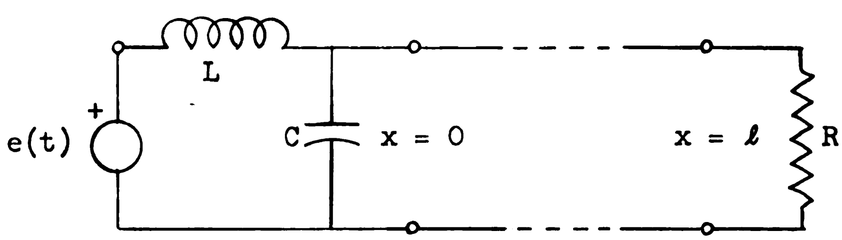

Figure 4.3.0 Example network for illustration of alternative formulation.

The corresponding f-cut-set, f-circuit, and element equations are, respectively:

$$\begin{array}{cccccc} e(t) & R & T_o & C & T_\ell & L \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 & 0 & 1 \\ 0 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_{e(t)} \\ i_R \\ i_{T_o} \\ i_C \\ i_{T_\ell} \\ i_L \end{bmatrix} = 0 \qquad 4.3.0$$

$$\begin{bmatrix} 0 & 0 & -1 & | & 1 & 0 & 0 \\ 0 & -1 & 0 & | & 0 & 1 & 0 \\ -1 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} e(t) \\ v_R \\ v_{T_o} \\ v_C \\ v_{T_\ell} \\ v_L \end{bmatrix} = 0 \qquad 4.3.1$$

$$\begin{bmatrix} i_R \\ i_C \\ v_L \end{bmatrix} = \begin{bmatrix} G & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & L \end{bmatrix} \begin{bmatrix} v_R \\ \frac{d}{dt} v_C \\ \frac{d}{dt} i_L \end{bmatrix} = 0 \qquad 4.3.2$$

If each of the vectors of Eq. 4.3.2 is expressed in terms of a different set of variables as was done in Chapter I, the equation analogous to Eq. 1.12 is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_{T_\ell} \\ i_{T_o} \\ v_{T_o} \\ e(t) \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ C & 0 \\ 0 & L \end{bmatrix} \begin{bmatrix} \frac{d}{dt} v_{T_o} \\ \frac{d}{dt} i_L \end{bmatrix} + \begin{bmatrix} G \\ 0 \\ 0 \end{bmatrix} v_{R_1} \qquad 4.3.3$$

The modified derivative-explicit equations obtained from Eq. 4.3.3 are

$$\frac{d}{dt}\begin{bmatrix} v_{T_o} \\ i_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_{T_o} \\ i_L \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} [e(t)] + \begin{bmatrix} -\frac{1}{C} \\ 0 \end{bmatrix} [i_{T_o}] \qquad 4.3.4$$

The results obtained in section 4.2 imply that the rank of the coefficient matrix B_o^* determines the number of necessary and sufficient simultaneous equations for a given network.

The normal form for Eq. 4.3.4 can be expressed as

$$\frac{d}{dt}\begin{bmatrix} v_{T_o} \\ i_L \end{bmatrix} = \begin{bmatrix} -\frac{1}{R_o} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_{T_o} \\ i_L \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} e(t) \\ f(v_{T_o}(t-n\tau)) \end{bmatrix} \qquad 4.3.5$$

and since B_o^* has full rank, the necessary number of equations is two. Therefore, no reduction in the number of equations results from the alternate formulation.

V. CONCLUSIONS

5.0 Conclusion

In the first three chapters of this thesis, systems of derivative-explicit equations have been derived for RLC networks containing a uniform, distortionless, transmission line. The solution of the equations has been shown to be sufficient to complete the solution of the network.

The general form of the derivative-explicit equations is a series of time-delayed terms which become sequentially non-zero. The solution of the equations must be obtained by a step-by-step process since the equations for various time intervals involve the solutions for previous time intervals.

The formulation of a minimum order set of equations from the general sufficient set could conceivably result in a significant saving of labor. Since networks do not ordinarily contain series capacitors nor parallel inductors, the real significance of the reduced set formulation would be where component values are responsible for the "excess" equations.

5.1 Additional problems.

At least two topics worthy of further investigation have arisen in this thesis.

(1) In Chapter I, the two-port representation was only shown to

be valid when N is disconnected. It would be very helpful if all networks in which the two-port representation is valid could be characterized. One possible starting point for such an investigation could be the consistency of results obtained when the transmission line is replaced by a generalized equivalent circuit. The equivalent circuit itself would have to be consistent with Eqs. 1.0 and 1.1.

(2) The appearance of time-delayed terms in the derivative-explicit equations has a rather general effect on the solutions of the equations. Examination of Eq. 4.2.0 leads to the conclusion that the characteristic equations will be the same for every time interval. In every time interval after the first, the characteristic roots will appear in the term f(t) since f(t) contains time-delayed terms which correspond to solutions for previous intervals. Calculation of the particular integrals will therefore result in polynomials which increase in order as the solution is carried out for additional time intervals. In general, the rate of increase of the polynomial order is related to the multiplicity of the characteristic roots. The solutions obtained for v_C and i_L in section 4.1 illustrate these points. A further investigation into the polynomials might lead to relationships which would allow computer calculation of the polynomials. This result would yield an appreciable reduction in labor if solutions were to be obtained for a large number of time intervals.

APPENDIX A

SOME THEOREMS AND DEFINITIONS

Theorem A(1): (2, p. 98)[1] If the columns of the f-circuit and f-cut-set matrices of an oriented graph G are arranged in the same order of branches and chords for a defining tree so that

$$B_f = [B_T \mid U] \quad \text{and} \quad S_f = [U \mid S_C]$$

we have the relation

$$B_T = - S_C'$$

Theorem A(2): If the square matrix A of order n has rank $r \leq n$, and if the matrix A is written as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

where A_{11} is of order r and non-singular, then $A_{22} - A_{21} A_{11}^{-1} A_{12} = 0$.

[1] Parenthesis give reference and page number where equivalent, or more general, theorems or defs. are stated.

Proof: The matrix A can be factored as indicated in the identity A.1.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \equiv \begin{bmatrix} U & 0 \\ A_{21}A_{11}^{-1} & U \end{bmatrix} \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22}-A_{21}A_{11}^{-1}A_{12} \end{bmatrix} \begin{bmatrix} U & A_{11}^{-1}A_{12} \\ 0 & U \end{bmatrix} \quad \text{A.1}$$

The rank of the matrix product must be r. Since the first and last matrices in the product are non-singular, the rank of the matrix in the middle of the product must be r. This conclusion is based on the fact that when an arbitrary matrix B is pre-(or post-) multiplied by a conformable non-singular matrix, the product has the rank of B (11, p. 109). By hypothesis, A_{11} has rank r, and hence the first r rows of the middle matrix are linearly independent. Since 0 submatrices appear above and to the left of the submatrix $A_{22} - A_{21} A_{11}^{-1} A_{12}$, any non-zero entry in $A_{22} - A_{21} A_{11}^{-1} A_{12}$ will result in a row which is linearly independent of the first r rows of the matrix. This violates the hypothesis, and therefore $A_{22} - A_{21} A_{11}^{-1} A_{12} = 0$.

Definition A(1): (2, p. 89) The vertex matrix A_a of an oriented graph is defined by

$A_a = [a_{ij}]$ is of order v x e for a graph with v vertices and e elements, where

$a_{ij} = 1$ if element j is incident at vertex i and is oriented away from vertex i

$a_{ij} = -1$ if element j is incident at vertex i and is oriented toward vertex i, and

$a_{ij} = 0$ if element j is not incident at vertex i.

Definition A(2): (10, p. 256) A real symmetric matrix A is called a positive definite matrix if and only if the corresponding quadratic form X'AX is positive definite.

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