ON THE CHARACTERIZATION OF INERTIAL COEFFICIENT RINGS

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ABSTRACT

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By

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Let R be a commutative ring and A be an R-algebra which is finitely generated as an R-module and has Jacobson radical N. Ingraham defined R to be an _inertial coefficient ring_ if when A/N is a separable R-algebra there exists a separable R-subalgebra S of A such that $A = S + N$. (A,N) is called an _L.I. pair_ (lifting idempotent pair) if every idempotent in A/N is of the form $\bar{e}$ where e is an idempotent of A. Ingraham has conjectured that if for every finitely generated R-algebra A, (A,N) is an L.I. pair, then R is an inertial coefficient ring. The main result of Chapter II is that the converse of this conjecture is true: If A is a finitely generated algebra over an inertial coefficient ring R then (A,N) is an L.I. pair.

Let $X(R)$ denote the Pierce decomposition space of R and R_x denote the stalk of the sheaf over $X(R)$ at the point $x \in X(R)$. In Chapter III it is shown that R is an inertial coefficient ring if and only if R_x is an inertial coefficient ring for all $x \in X(R)$. This result is used to show several rings are inertial coefficient rings.

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CHAPTER I

PRELIMINARIES

Chapter I contains historical material and a description of the tools used in later chapters. We shall also fix the notation which will be used throughout.

§1. Separable Algebras and Inertial Coefficient Rings

All rings we shall consider contain an identity element; all subrings contain the identity of the overring; all homomorphisms preserve the identity. Throughout R denotes a commutative ring and A an R-algebra, where by an R-algebra A we mean a ring A along with a ring homomorphism θ of R into the center of A. This homomorphism induces a natural R-module structure on A by defining $r \cdot a = \theta(r) \cdot a$ for $r \in R$ and $a \in A$. If θ is a monomorphism, A is called a *faithful* R-algebra. An R-algebra is called *projective* or *finitely generated* if it is projective or finitely generated as an R-module. All R-algebras we shall consider are finitely generated. If A is a commutative ring, A is called a *commutative R-algebra*.

Example 1.1. Let $f(x)$ be a monic polynomial with coefficients in R. Then $A = \frac{R[x]}{\langle f(x) \rangle}$ is a finitely generated, faithful, free (as an R-module), commutative R-algebra. If the degree of $f(x)$ is n, then $\{1,x,\ldots,x^{n-1}\}$ is a free basis of A over R.

The notation rad B with be used to denote the Jacobson radical of a ring B. Throughout $N = \text{rad } A$. The following proposition relates the maximal ideals of R and the radical of A.

Proposition 1.2. [11, Lemma 1.1, p.78] Let A be a finitely generated R-algebra and let $\cap mA$ denote the intersection of the ideals mA as m runs over all maximal ideals of R.

(i) $\text{rad}(R)\cdot A \subseteq N$.

(ii) There exists a positive integer n such that $N^n \subseteq \cap mA$.

(iii) If A is projective, $\text{rad}(R)\cdot A = \cap mA$.

The following proposition due to Azumaya relates the radical of A to the radical of a subalgebra.

Proposition 1.3. [2, Corollary, p.126] Let A be a finitely generated algebra over R and B be an R-subalgebra of A. Then $N \cap B \subseteq \text{rad } B$.

Our interest is in the relationship between properties of R and the structure of finitely generated R-algebras. Our technique is to take data from A/N and "pull it back" to A. Our work here is concerned with pulling back two types of structure-idempotents and separability - and the relationship between them.

A finite dimensional algebra A over a field F is called a <u>separable</u> F-algebra if and only if A is isomorphic to a direct sum of full matrix rings over division rings having centers which are separable field extensions of F. In 1908 Wedderburn proved (in the case F has characteristic 0) that if A/N is separable over F then this separability can be "pulled back" to a subalgebra of A:

<u>Wedderburn Principal Theorem</u> [24]. If A is a finite dimensional F-algebra such that A/N is F-separable then there exists a separable F-subalgebra S of A such that $A = S + N$.

The Wedderburn Principal Theorem decomposes A into two parts: a separable algebra which is a direct sum of matrix algebras and the radical which is nilpotent.

Azumaya (1951) obtained a generalization of Wedderburn's Theorem for a class of rings called Hensel local rings (which will be defined in the following section).

<u>Azumaya's Theorem</u> [2]. If A is a finitely generated algebra over a Hensel local ring R with maximal ideal m and if A/N is separable over R/m, then there exists an R-subalgebra S of A such that $A = S + N$ and S/mS is separable over R/m.

Auslander and Goldman (1960) [1] generalized the notion of a separable algebra to algebras over an arbitrary commutative ring R. For finitely generated R-algebras their definition is equivalent to the following:

Definition. A finitely generated R-algebra A is a separable R-algebra if A/mA is a separable R/m-algebra for every maximal ideal m of R.

Example 1.4. Let $A = R_{n\times n}$ be the ring of $n\times n$ matrices over a commutative ring R. $A/mA \cong (R/m)_{n\times n}$, and thus A is separable.

Ingraham [11] has considered rings all of whose finitely generated algebras satisfy the analogue of the Wedderburn Principal Theorem under Auslander and Goldman's generalized notion of separability:

Definition. A ring R is called an inertial coefficient ring if for every finitely generated R-algebra A such that A/N is a separable R-algebra, there exists a separable R-subalgebra S of A such that $A = S + N$. S is called an inertial subalgebra of A.

Known inertial coefficient rings include the following: a field, a Hensel local ring, a von Neumann regular ring [4, Theorem 1, p.370] (R is von Neumann regular if and only if for every $r \in R$ there exists an $s \in R$ such that $r^2s = r$), and a Noetherian Hilbert ring [13, Corollary 2, p.553] (R is a

Hilbert ring if and only if every prime ideal is the intersection of the maximal ideals containing it). In Chapter III we shall give some new examples of inertial coefficient rings. The following theorem of Ingraham motivated the work in Chapter II.

Theorem 1.5. [13, Theorem 2, p.554] If for any finitely generated, commutative R-algebra A each idempotent in A/N is the image of an idempotent in A, then if B is a finitely generated, commutative R-algebra such that $B/\text{rad } B$ is R-separable then B contains an R-inertial subalgebra.

We shall next describe and relate two numerical tools, $\text{rank}_R(A)$ and $\mu_m(A)$, which will be used in Chapter II. For any prime ideal p of R, let R_p denote the localization of R at p. R_p is a local ring (a commutative ring with a unique maximal ideal). Let M be a finitely generated, projective R-module. Since projective modules over a local ring are free, the R_p-module $M \otimes_R R_p \cong M_p$ is free of finite rank $\text{rank}_p(M)$. If there is a fixed integer n such that $\text{rank}_p(M) = n$ for all prime ideals p of R, we say the rank of M is defined and equals n and we write $\text{rank}_R(M) = n$. For reasons which will become clear in section 3, in much of our work we can assume the ring R has no idempotents but 0 and 1. (Such a ring is called a connected ring). If R is a connected ring and M is a finitely generated, projective R-module, then $\text{rank}_R(M)$ is defined [6, Theorem 4.12, p.32].

Let A be a finitely generated, faithful, commutative R-algebra and let m be a maximal ideal of R. Then m has only finitely many maximal ideals M of A such that $M \cap R = m$, so we can define $\mu_m(A) = \sum_M [A/M:R/m]$, where $[A/M:R/m]$ is the dimension of the field extension A/M over the field R/m. $\mu_m(A)$ is easily evaluated for algebras A of the type in Example 1.1:

<u>Lemma 1.6</u>. Let $A = \frac{R[x]}{\langle f(x)\rangle}$ where $f(x)$ is a monic polynomial contained in $R[x]$. Let $\overline{f(x)}$ denote the polynomial in $R/m[x]$ obtained by reducing the coefficients of $f(x)$ modulo m. Suppose $\overline{f(x)} = \prod_{i=1}^{n} (\overline{p_i(x)})^{e_i}$ is the factorization of $\overline{f(x)}$ into positive integer powers of monic polynomials $\overline{p_i(x)}$ which are irreducible in $R/m[x]$. Then $\mu_m(A) = \sum_{i=1}^{n}$ degree $(\overline{p_i(x)})$.

<u>Proof</u>: Any maximal ideal M of A such that $M \cap R = m$ contains the ideal mA. Since $\frac{A}{mA} \cong \frac{R/m[x]}{\langle\overline{f(x)}\rangle}$, then by the Chinese Remainder Theorem $\frac{A}{mA} \cong \oplus \sum_{i=1}^{n} \frac{R/m[x]}{\langle\overline{p_i(x)}^{e_i}\rangle}$. Each

$B_i = \frac{R/m[x]}{\langle\overline{p_i(x)}^{e_i}\rangle}$ is a local ring with unique maximal ideal

$M_i' = \frac{\langle\overline{p_i(x)}\rangle}{\langle\overline{p_i(x)}^{e_i}\rangle}$. Thus the maximal ideals M_i/mA of A/mA are

of the form $\frac{M_i}{mA} = B_1 \oplus \cdots \oplus B_{i-1} \oplus M_i' \oplus B_{i+1} \oplus \cdots \oplus B_n$,

$A/M_i \cong \frac{A/mA}{M_i/mA} \cong \frac{R/m[x]}{\langle\overline{p_i(x)}\rangle}$, and $[A/M_i:R/m] =$ degree $(\overline{p_i(x)})$.

Ingraham and W.C. Brown computed $\mu_m(S)$ for inertial subalgebras of certain algebras.

Proposition 1.7. [5, see proof of Lemma 1, p.11] Let A be a finitely generated, projective, faithful, commutative R-algebra. If A has an inertial subalgebra S, then for each maximal ideal m of R,

$$\mu_m(A) = \mu_m(S) = \mathrm{rank}_m[S].$$

The following properties will be used in Chapters II and III and are listed here for reference:

(I) Let $\mathfrak{A}$ be an ideal of R. An $R/\mathfrak{A}$-algebra A has an R-algebra structure induced by the homomorphism of R onto $R/\mathfrak{A}$. A is a separable R-algebra if and only if it is a separable $R/\mathfrak{A}$-algebra [6, p.45].

(II) If S is a separable R-algebra and I is a two-sided ideal of S, then S/I is a separable R-algebra [6, Proposition 1.11, p.46].

(III) If S_1 and S_2 are separable R-algebras then $S_1 \oplus S_2$ is a separable R-algebra [6, Proposition 1.13, p.47].

(IV) Projective Lifting Property. If S is a separable R-algebra, then every S-module which is R-projective is S-projective [6, Proposition 2.3, p.48].

(V) Inertial subalgebras of finitely generated algebras are finitely generated [21, Theorem 5, p.5].

(VI) Inertial subalgebras of finitely generated, commutative, projective R-algebras are projective [11, Proposition 2.8, p.80].

(VII) Let R be an inertial coefficient ring. If R' is a finitely generated, commutative, R-algebra, where $R'/\mathrm{rad}\, R'$ is a separable R-algebra, then R' is an inertial coefficient ring [11, see proof of Proposition 3.3, p.85].

(VIII) If R is an inertial coefficient ring and $\mathfrak{A}$ is an ideal of R, then $R/\mathfrak{A}$ is an inertial coefficient ring [11, Corollary 3.4, p.86].

(IX) If $\mathrm{rank}_R(M_1)$ and $\mathrm{rank}_R(M_2)$ are defined for finitely generated, projective R-modules M_1 and M_2, then $\mathrm{rank}_R(M_1 \oplus M_2)$ is defined and equals $\mathrm{rank}_R(M_1) + \mathrm{rank}_R(M_2)$.

(X) If A is a commutative, finitely generated, projective R-algebra and M is a finitely generated, projective A-module such that $\mathrm{rank}_A(M)$ and $\mathrm{rank}_R(A)$ are defined, then $\mathrm{rank}_R(M)$ is defined and equals $\mathrm{rank}_R(A) \cdot \mathrm{rank}_A(M)$ [6, Exercise 2, p.35].

§2. Hensel Rings and the Idempotent Lifting Property

Let $f(x)$ be a polynomial in $R[x]$ and $\overline{f(x)}$ be the polynomial in $R/\mathfrak{A}[x]$, where $\mathfrak{A}$ is an ideal of R, obtained by reducing the coefficients of $f(x)$ modulo $\mathfrak{A}$. Suppose $\overline{f(x)} = g_0(x)\cdot h_0(x)$ in $R/\mathfrak{A}[x]$. A problem historically of interest to algebraists is the existence of polynomials $g(x), h(x) \in R[x]$ such that $f(x) = g(x)\cdot h(x)$ and $\overline{g(x)} = g_0(x)$, $\overline{h(x)} = h_0(x)$. In 1902 Hensel proved a lemma [10] stating that if R is the p-adic numbers and if $\mathfrak{A}$ is its unique maximal ideal then certain factorizations, $\overline{f(x)} = g_0(x)\cdot h_0(x)$, can be "lifted" to corresponding factorizations of $f(x)$ in $R[x]$

In 1951 Azumaya [2] called a local ring satisfying Hensel's lemma a <u>Hensel ring</u> - i.e. a local ring R with maximal ideal m is a Hensel ring if for every monic polynomial $f(x) \in R[x]$ such that $\overline{f(x)} = g_0(x)h_0(x)$ where $g_0(x), h_0(x)$ are monic, relatively prime polynomials in $R/m[x]$, there exist monic polynomials $g(x), h(x)$ in $R[x]$ such that $f(x) = g(x)\cdot h(x)$, $\overline{g(x)} = g_0(x)$, and $\overline{h(x)} = h_0(x)$. Azumaya considered algebras over Hensel local rings and showed that the Hensel property of the ring was entirely reflected in the algebras over the ring; that is a local ring is Hensel if and only if idempotents can be lifted from A/I to A in all finitely generated R-algebras A, for any two-sided ideal I of A. This allows one to lift families of pairwise orthogonal idempotents and matrix ring decompositions. Azumaya used these results to determine the structure of finitely generated algebras over a Hensel local ring.

In 1963 Lafon [15] extended the definition of a Hensel ring to nonlocal rings. Greco (1968) [7,8,9] used Lafon's definition to generalize certain results Azumaya had obtained for Hensel local rings.

<u>Definition</u>. A ring R is a <u>Hensel ring</u> if for all monic polynomials $f(x) \in R[x]$ and every decomposition $\bar{f} = g_0 \cdot h_0$ in $\frac{R}{\text{rad } R}[x]$ with g_0 and h_0 monic and $\langle g_0 \rangle + \langle h_0 \rangle = \frac{R}{\text{rad } R}[x]$, there exists a pair of monic polynomials $g, h \in R[x]$ such that $f = g \cdot h$ and $\bar{g} = g_0$, $\bar{h} = h_0$. (It can be proved that g and h are unique [15, Proposition 1, p.80]).

Any ring R having $\text{rad}(R) = 0$ is trivially a Hensel ring. We now give another example.

<u>Example 1.8</u>. A commutative ring which is Hausdorff and complete with respect to a linear topology in which $\text{rad } R$ is a closed ideal with every element topologically nilpotent is a Hensel ring [3, Theorem 1, p.215-6]; e.g. rings with nilpotent radical, the p-adic numbers, and the formal power series rings over a Noetherian ring.

<u>Definition</u>. Let I be a two-sided ideal of a ring A. We call (A,I) and <u>L.I. pair</u> (lifting idempotent pair) if every idempotent of A/I is of the form $\bar{e}$ where e is an idempotent of A. We say A has the <u>idempotent lifting property</u> if (A,N)

is an L.I. pair. (In a commutative ring A, if an idempotent in A/N can be lifted to A there is exactly one lift in A [7, Lemma 1.2, p.46].)

Example 1.9. Jacobson proved that if N is a nil ideal of A, A has the idempotent lifting property [14, Proposition 4, p.54].

Lemma 1.10. Let $\mathfrak{A}$ be a two-sided ideal of A, $\mathfrak{A} \subseteq N$.

(i) If $(A,\mathfrak{A})$ and $(A/\mathfrak{A},N/\mathfrak{A})$ are L.I. pairs, then A has the idempotent lifting property.

(ii) If A is a commutative ring having the idempotent lifting property, then $(A,\mathfrak{A})$ and $(A/\mathfrak{A},N/\mathfrak{A})$ are L.I. pairs.

Proof: (i) is an immediate consequence of the definition of L.I. pairs. See [7, Corollary 1.3, p.46] for a proof of (ii).

The following theorem relates Hensel rings and L.I. pairs.

Theorem 1.11. [7, Theorem 4.1, p.55 and 8, Theorem 2.2, p.51] The following are equivalent properties of a commutative ring R:

(i) R is a Hensel ring.

(ii) For every finitely generated R-algebra A, $(A,\mathrm{rad}(R)\cdot A)$ is an L.I. pair.

(iii) For every finitely generated, commutative, free R-algebra A, $(A,\mathrm{rad}(R)\cdot A)$ is an L.I. pair.

Corollary 1.12. R is a Hensel ring if and only if every finitely generated, projective R-algebra A has the idempotent lifting property.

Proof: If R is a Hensel ring and A is a finitely generated, projective R-algebra then $(A, \mathrm{rad}(R)\cdot A)$ is an L.I. pair by Theorem 1.11 (ii). By Proposition 1.2 (ii) and (iii) $\frac{N}{\mathrm{rad}(R)\cdot A}$ is a nilpotent ideal of $\frac{A}{\mathrm{rad}(R)\cdot A}$ and by Example 1.9, $(\frac{A}{\mathrm{rad}(R)\cdot A}, \frac{N}{\mathrm{rad}(R)\cdot A})$ is an L.I. pair. Thus by Lemma 1.10 (i), (A,N) is an L.I. pair. Conversely, we shall show R is Hensel by proving that (iii) of Theorem 1.11 holds. Let A be a finitely generated, commutative, free R-algebra. By the hypothesis (A,N) is an L.I. pair. Since A is commutative, $(A, \mathrm{rad}(R)\cdot A)$ is an L.I. pair by Lemma 1.10 (ii).

All finitely generated algebras over a Hensel ring need not have the idempotent lifting property, e.g. let $R = Z_{(p)}[x]$, where $Z_{(p)}$ is the subring of the rational numbers having denominators relatively prime to a fixed prime integer p. R has $\mathrm{rad}(R) = 0$ and therefore is a Hensel ring. Let $A = \frac{Z_{(p)}[x]}{\langle x^2-x+p\rangle}$. Then $\bar{x}$ is an idempotent in A/N, but A has no idempotents other than 0 and 1 [18, Theorem 43.14, p.184].

If A is a finitely generated algebra over a Hensel local ring, a von Neumann regular ring, or a Noetherian Hilbert ring, then idempotents can be lifted from A/N to A [13, see proof of Corollary 2, p.553]. It is of interest to find necessary and sufficient conditions on R such that every finitely generated R-algebra has the idempotent lifting property. We have seen [Corollary 1.12] that such a ring must necessarily be a Hensel ring, but that R being a Hensel ring is not sufficient to guarantee that every finitely generated algebra has the idempotent lifting property; in Chapter II we shall see that R being an inertial coefficient ring is a sufficient condition.

We conclude this section with a new example of a ring all of whose finitely generated algebras have the idempotent lifting property.

Example 1.13. Let U be the ring of formal series in one indeterminate over the rational numbers and S be the subring of formal power series over the integers. Since U and S are Noetherian rings, their radical topologies satisfy the topological criteria of Example 1.8, and thus U and S are Hensel rings. Let T and R be the subrings of U and S, respectively, consisting of power series which are convergent at zero ($g(x) \in U$ is called convergent at zero if there exists an open interval N of the real line such that $0 \in N$ and $g(r)$ is an absolutely convergent series for every $r \in N$ [25, p.142]). It can be shown that the radicals of U, S, T, and R are each generated by x, and it follows that the canonical homomorphisms $U \to U/\text{rad } U$, $S \to S/\text{rad } S$, $T \to T/\text{rad } T$, and $R \to R/\text{rad } R$ all can be described as evaluation at $x = 0$. T is a Hensel ring [18, Theorem 45.5, p.193].

We shall show that R is a Hensel ring. Let $f(y) \in R[y]$ be a monic polynomial such that $\overline{f(y)} = \overline{g_0(y)}\ \overline{h_0(y)}$ where $g_0(y), h_0(y) \in R[y]$ are such that $\overline{g_0(y)}$, $\overline{h_0(y)}$ are monic polynomials in $(R/\operatorname{rad} R)[y]$ with $\langle\overline{g_0(y)}\rangle + \langle\overline{h_0(y)}\rangle = (R/\operatorname{rad} R)[y]$.

$$\begin{tikzcd}[column sep=small] & U[y] \arrow[dl, dash] \arrow[dr, dash] & \\ S[y] & & T[y] \\ & R[y] \arrow[ul, dash] \arrow[ur, dash] & \end{tikzcd}$$

Since $R[y] \subseteq S[y]$ and $T[y]$, and since S and T are both Hensel rings, it is easily seen that the factorization of $\overline{f(y)}$ in $(R/\operatorname{rad} R)[y]$ can be lifted to a corresponding factorization of $f(y)$ in both $S[y]$ and $T[y]$. Since $S[y]$ and $T[y]$ are both contained in $U[y]$ and since U is a Hensel ring and thus the lift of a factorization is unique, we therefore have a factorization $f(y) = g(y)\cdot h(y)$ in $S[y] \cap T[y] = R[y]$ where $\overline{g(y)} = \overline{g_0(y)}$ and $\overline{h(y)} = \overline{h_0(y)}$. Thus R is a Hensel ring and by Theorem 1.11 (ii) for any finitely generated R-algebra A, $(A, \operatorname{rad}(R)\cdot A)$ is an L.I. pair.

Since $R/\operatorname{rad} R$ is isomorphic to the integers and the integers form a Noetherian Hilbert ring, $(A/\operatorname{rad}(R)\cdot A, N/\operatorname{rad}(R)\cdot A)$ is an L.I. pair. Hence by Lemma 1.10 (i) for any finitely generated R-algebra A, (A,N) is an L.I. pair.

§3. The Decomposition Space

The Pierce decomposition space $X(R)$ of a commutative ring R was first described by Pierce [20] using sheaf-theoretic methods. The description of $X(R)$ given here avoids the use of sheaf-theoretic language and is due to Villamayer and Zelinsky [23] (they call $X(R)$ the Boolean spectrum of R). Magid's book [17] to which we will frequently refer is an easily accessible, complete source on this approach to $X(R)$.

The decomposition space $X(R)$ is a tool useful in proving that certain results which are known to hold for connected, commutative rings also hold for arbitrary commutative rings. The technique is to study certain connected homomorphic images of R and "patch together" these results to obtain the result for R.

Let $\mathrm{Spec}(R)$ denote the set of prime ideals of R endowed with the Zariski topology (a basis of closed sets of $\mathrm{Spec}(R)$ is $\{V(I)\}$ where I ranges over all ideals of R and where $V(I) = \{p \in \mathrm{Spec}(R): p \supseteq I\}$). We define $X(R)$ to be the quotient space of $\mathrm{Spec}(R)$ obtained by identifying connected components of $\mathrm{Spec}(R)$. It can be shown that $X(R)$ is a totally disconnected, compact, Hausdorff space or equivalently a <u>profinite space</u> (an inverse limit of finite discrete spaces) [17, Corollary II.4, p.26]. The space $X(R)$ has the following useful topological property.

<u>Proposition 1.14</u>. (The Partition Property) [17, Lemma I.7, p.3] Any open cover of $X(R)$ has a refinement which is a partition (a finite family of disjoint open subsets of $X(R)$ which covers $X(R)$).

It can be shown that two prime ideals of R belong to the same connected component of $\mathrm{Spec}(R)$ if and only if they contain the same idempotents [17, Proposition II.3, p.26]. Thus if R has no idempotents but 0 or 1, $\mathrm{Spec}(R)$ is connected (justifying our calling such a ring a connected ring).

Let e be an idempotent of R and let $N(e) = \{x \in X(R) : x \subseteq V(R(1-e))\}$. The sets $N(e)$ have the following useful properties:

<u>Proposition 1.15</u>. [17, Proposition II.12, p.30]

(i) $N(0) = \emptyset$ and $N(1) = X(R)$.

(ii) $N(e) \cap N(f) = N(ef)$.

(iii) $N(e) = N(f)$ if and only if $e = f$.

(iv) The sets $\{N(e)\}$ form a basis of open, closed sets for the topology on $X(R)$. Furthermore, any open, closed subset of $X(R)$ is of the form $N(e)$ for some idempotent e of R.

Let $I(x)$ be the ideal of R generated by the set of idempotents in any prime ideal contained in a point $x \in X(R)$ and define $R_x = R/I(x)$. R_x is a connected ring [17, Corollary II.21, p.34].

When computing $X(R)$ and R_x for a particular ring R it is usually easier to view $X(R)$ in an equivalent formulation as the collection of maximal Boolean ideals of R [17, pp.27-28]. A set of idempotents x of R is called a

<u>maximal Boolean ideal</u> if

(i) For every idempotent e of R either $e \in x$ or $1-e \in x$, but not both; and,

(ii) If e and f are idempotents of R then $ef \in x$ if and only if $e \in x$ or $f \in x$.

It can be shown that $R_x = R/I(x)$ where $I(x)$ is the ideal of R generated by the elements of x [17, Proposition II.9, p.28].

Let M be an R-module and let $m \in M$. Throughout let $m_x = m + I(x)M$ denote the image of m in $M_x = M \otimes_R R_x \cong M/I(x)M$.

<u>Proposition 1.16</u>. [17, Proposition II.16, p.32] Let a and b belong to the R-module M, let $x \in X(R)$, and suppose $a_x = b_x$. Then there exists a neighborhood $N(e)$ of x in $X(R)$ for some idempotent $e \in R$ such that $a_y = b_y$ in M_y for all $y \in N(e)$, $e_x = 1_x$, and $ae = be$.

<u>Proposition 1.17</u>. [17, Proposition II.17, p.32] Let a and b be elements of the R-module M such that $a_x = b_x$ for all $x \in X(R)$. Then $a = b$.

Propositions 1.16 and 1.17 along with the Partition Property will be the principal tools used to "patch together" results from the connected rings R_x to obtain results for R.

CHAPTER II

INERTIAL COEFFICIENT RINGS AND THE IDEMPOTENT LIFTING PROPERTY

E.C. Ingraham has conjectured that a ring R is an inertial coefficient ring if idempotents can be lifted from A/N to A in all finitely generated R-algebras A. Both he and Azumaya have used the technique of lifting idempotents to produce inertial subalgebras. The main result of this chapter is that idempotents can be lifted from A/N to A in all finitely generated algebras A over an inertial coefficient ring, and thus the converse of Ingraham's conjecture is true.

The result is proved in three steps. First it is shown that if R is a connected inertial coefficient ring, idempotents can be lifted from $R/\text{rad } R$ to R. Next the decomposition space is used to show idempotents can be lifted from $R/\text{rad } R$ to R in any inertial coefficient ring R. Finally we show that idempotents can be lifted from A/N to A in any finitely generated R-algebra A.

The first step will be proved by contradiction; we will assume R has a nonliftable idempotent and produce a finitely generated R-algebra A such that $\frac{A}{N}$ is R-separable but A contains no inertial subalgebra.

Lemma 2.1. Let $f(x)$ be a monic polynomial in $R[x]$ such that $\overline{f(x)} = x^r(x-1)^s$ for r,s positive integers, where $\overline{f(x)} \in (R/\mathrm{rad}\,R)[x]$. Suppose there do not exist monic polynomials $g_0(x), h_0(x)$ in $R[x]$ such that $f(x) = g_0(x)\cdot h_0(x)$ with $\overline{g_0(x)} = x^r$ and $\overline{h_0(x)} = (x-1)^s$. Then $\overline{xf(x)} = x^{r+1}(x-1)^s$, and there do not exist monic polynomials $g(x), h(x)$ in $R[x]$ with $xf(x) = g(x)h(x)$, $\overline{g(x)} = x^{r+1}$, and $\overline{h(x)} = (x-1)^s$.

Proof: Let $f(x) = x^{r+s} + \sum_{i=0}^{r+s-1} a_i x^i$ with $a_i \in R$. Then $xf(x) = x^{r+s+1} + \sum_{i=0}^{r+s-1} a_i x^{i+1}$. Suppose $xf(x) = g(x)h(x)$ with $\overline{g(x)} = x^{r+1}$ and $\overline{h(x)} = (x-1)^s$. Then $g(x) = x^{r+1} + \sum_{k=0}^{r} n_k x^k$ with $n_k \in \mathrm{rad}\,R$ and $h(x) = x^s + \sum_{k=0}^{s-1} [\binom{s}{k}(-1)^{s-k} + n_k']x^k$ with $n_k' \in \mathrm{rad}\,R$. Equating constant terms of $xf(x) = g(x)h(x)$ gives $n_0((-1)^s + n_0') = 0$. $(-1)^s + n_0'$ is a unit of R, and thus $n_0 = 0$. Therefore $xf(x) = x(x^r + \sum_{k=1}^{r} n_k x^{k-1})\cdot h(x)$. Since x is not a zero divisor in $R[x]$ we have $f(x) = (x^r + \sum_{k=1}^{r} n_k x^{k-1})\cdot h(x)$. Thus

$$g_0(x) = x^r + \sum_{k=1}^{r} n_k x^{k-1}$$

and $h_0(x) = h(x)$ contradict the hypotheses.

Theorem 2.2. Let R be a connected ring. If $R/\mathrm{rad}\,R$ has an idempotent not equal to $\bar{0}$ or $\bar{1}$ then R is not an inertial coefficient ring.

Proof: By [14, Proposition 4, p.54] there exists a $p \in \text{rad } R$ such that $x^2 - x + p \in R[x]$ has no root in rad R. Then $x^2 - x + p \equiv x(x-1)$ modulo rad R. If there exist monic polynomials $g(x), h(x)$ contained in $R[x]$ such that $x^2 - x + p = g(x)\cdot h(x)$ with $\overline{g(x)} = x$ and $\overline{h(x)} = x - 1$ then $g(x) = x + n$ for some $n \in \text{rad } R$ and $-n$ is a root of $x^2 - x + p$ in Rad R.

Let $f(x) = x^3 - x^2 + px$. Then $\overline{f(x)} = x^2(x-1)$, but by Lemma 2.1 there do not exist monic polynomials $g(x)$ and $h(x)$ contained in $R[x]$ such that $f(x) = g(x)\cdot h(x)$ and $\overline{g(x)} = x^2$, $\overline{h(x)} = x - 1$.

Let A be the finitely generated, faithful, free, commutative R-algebra $A = \dfrac{R[x]}{\langle x^3-x^2+px\rangle}$. Then $A/(\text{rad}(R)\cdot A) = \dfrac{(R/\text{rad}R)[x]}{\langle x^2(x-1)\rangle}$; since $\langle x\rangle$ and $\langle x-1\rangle$ are comaximal ideals, the Chinese Remainder Theorem gives $A/(\text{rad}(R)\cdot A) \cong \dfrac{(R/\text{rad}R)[x]}{\langle x^2\rangle} \oplus \dfrac{(R/\text{rad}R)[x]}{\langle x-1\rangle}$. Thus $\dfrac{A}{N} \cong \dfrac{A/(\text{rad}(R)\cdot A)}{N/(\text{rad}(R)\cdot A)} \cong R/\text{rad } R \oplus R/\text{rad } R$. Therefore by Chapter I, properties II and III, p.7, A/N is R-separable. Furthermore, for any maximal ideal m of R, $\mu_m(A) = 2$ [Lemma 1.6].

We will show that the assumption that A has an inertial subalgebra S leads to a contradiction. By Chapter I, property VI, p.8, if such an S exists, it must be a projective R-module.

Case 1. Assume S is connected. Since R is connected, $\operatorname{rank}_R(S)$ is defined and by Proposition 1.7 $\mu_m(A) = \operatorname{rank}_R(S)$. By the "projective lifting property" (Chapter I, property IV, p.7), A a projective R-module and S a separable R-algebra imply that A is a projective S-module. Since S is connected, $\operatorname{rank}_S(A)$ is defined. But then by Chapter I, property X, p.8, rank is multiplicative and $3 = \operatorname{rank}_R(A) = \operatorname{rank}_R(S)\cdot\operatorname{rank}_S(A) = 2\cdot\operatorname{rank}_S(A)$. Thus 2 divides 3, a contradiction.

To do the case when S is not connected, we need the following lemmas:

Lemma 2.3. Let R be a connected ring and $A = \dfrac{R[x]}{\langle x^3-x^2+px\rangle}$ for $p \in \operatorname{rad} R$. If A has an idempotent e, $e \neq 0,1$, then e or $1 - e$ is of the form $a_1x + a_2x^2$ where $a_1 \in \operatorname{rad} R$ and $\overline{a_2}$ is an idempotent in $R/\operatorname{rad} R$.

Proof: Let $e = a_0 + a_1x + a_2x^2$, $a_i \in R$, represent e in the free R-basis for A, $\{1,x,x^2\}$.

$$0 = e^2 - e = (a_0^2-a_0) + (2a_0a_1-a_1)x + (a_1^2-a_2+2a_0a_2)x^2 + (2a_1a_2)x^3 + (a_2^2)x^4.$$

Now applying the relations, $x^3 = x^2 - px$ and $x^4 = (1-p)x^2 - px$, we get

$$0 = (a_0^2-a_0) + (2a_0a_1-a_1-2a_1a_2p-a_2^2p)x + (a_1^2-a_2+2a_0a_2+2a_1a_2+a_2^2(1-p))x^2.$$

$\{1,x,x^2\}$ being a free basis of A over R implies the following relations:

$$(-1-)\quad a_0^2 = a_0$$

$$(-2-)\quad 2a_0a_1 - a_1 - 2a_1a_2p - a_0^2p = 0$$

$$(-3-)\quad a_1^2 - a_2 + 2a_0a_2 + 2a_1a_2 + a_2^2(1-p) = 0.$$

Since R is connected, equation (-1-) gives $a_0 = 0$ or $a_0 = 1$. The conclusion follows by examining equations (-2-) and (-3-) when $a_0 = 0$ and when $a_0 = 1$.

Lemma 2.4. Let R be a connected ring. If $A = \dfrac{R[x]}{\langle x^3-x^2+px\rangle}$, where $p \in \operatorname{rad} R$, has an inertial subalgebra S such that S is not a connected ring, then $S \cong Re \oplus R(1-e)$ (as rings) for some idempotent $e \in A$.

Proof: Since R is connected, $\mu_m(A)$ is well defined, and $\mu_m(A) = 2$ for any maximal ideal m of R. Suppose A has an inertial subalgebra S. Then by Proposition 1.7, $2 = \mu_m(A) = \operatorname{rank}_R(S)$.

If S is not connected then $S = Se \oplus S(1-e)$ for some idempotent $e \in A$, $e \neq 0,1$. If $\operatorname{rank}_R(Se) = 0$ then $(Se)_p = 0$ for all prime ideals p of R and so $Se = 0$ and $e = 0$. Similarly $\operatorname{rank}_R(S(1-e)) \neq 0$. Since rank of direct sums is additive (Chapter I, property IX, p.8), $\operatorname{rank}_R(Se) = \operatorname{rank}_R(S(1-e)) = 1$. Since Se and $S(1-e)$ are projective modules over a connected ring, they are faithful R-modules.

Se is thus a finitely generated, projective, faithful Re-module. By [6, Corollary 1.11, p.8 and Corollary 4.2, p.56] Re is an Re-direct summand of Se. Therefore there exists a finitely generated Re-module U such that $Se \cong Re \oplus U$ as Re-modules. Localizing at each prime ideal p of $R \cong Re$ gives $U_p = 0$, and thus $U = 0$. Thus $Se \cong Re$ as rings and similarly $S(1-e) \cong R(1-e)$, and therefore $S \cong Re \oplus R(1-e)$ (as rings).

<u>Lemma 2.5</u>. Let $A = \dfrac{R[x]}{\langle x^3-x^2+px\rangle}$ for $p \in \text{rad } R$. Let S be a subring of A, $S = Re \oplus R(1-e)$ for e an idempotent of A of the form $e = a_1x + a_2x^2$ where $a_1 \in \text{rad } R$ and $\bar{a}_2$ is an idempotent of $R/\text{rad } R$. Then $A = S + N$ implies $\bar{e} = \bar{x}^2$ in $\dfrac{A}{\text{rad}(R)\cdot A}$ and a_2 is a unit of R.

<u>Proof</u>: $N = \langle \text{rad } R, x(x-1)\rangle$ and $\frac{A}{N} \cong \dfrac{(R/\text{rad}R)[x]}{\langle x^2-x\rangle}$.
$A = S + N$ implies there exist $r_1, r_2 \in R$ and $n \in N$ such that $x = r_1e + r_2(1-e) + n = r_2 + (r_1-r_2)e + n$. Since $\bar{e} = \bar{a}_2\bar{x}^2$ in $A/(\text{rad}(R)\cdot A)$, $\bar{x} = \bar{r}_2 + (\bar{r}_1-\bar{r}_2)\bar{a}_2\bar{x}^2 + \bar{n}$ in $A/(\text{rad}(R)\cdot A)$. Therefore in A/N, $\bar{\bar{x}} = \bar{\bar{r}}_2 + (\bar{\bar{r}}_1-\bar{\bar{r}}_2)\bar{\bar{a}}_2\bar{\bar{x}}$. Thus $\bar{\bar{r}}_2 = \bar{\bar{0}}$ and $\bar{\bar{x}} = \bar{\bar{r}}_1\bar{\bar{a}}_2\bar{\bar{x}}$. Therefore $\bar{\bar{1}} = \bar{\bar{r}}_1\bar{\bar{a}}_2$. Since $\bar{\bar{a}}_2$ is both a unit and an idempotent of A/N, $\bar{\bar{a}}_2 = \bar{\bar{1}}$. But then by [7, Lemma 1.2, p.46], $\bar{a}_2 = \bar{1}$ in $A/(\text{rad}(R)\cdot A)$. Thus $\bar{e} = \bar{x}^2$ and a_2 is a unit of R.

The following lemma is a generalization by Greco of a result for local rings due to Nakayama [2, Lemma 3, p.134].

Lemma 2.6. [7, see proof of Theorem 3.1, p.54] Let $f(x) \in R[x]$ be a monic polynomial. Suppose $A = \frac{R[x]}{\langle f(x)\rangle} = \mathfrak{A} \oplus \mathfrak{B}$ for ideals $\mathfrak{A}, \mathfrak{B}$ of A. Suppose further that $A/\mathrm{rad}(R)\cdot A \cong \langle g_0(x)\rangle \oplus \langle h_0(x)\rangle$ for monic, coprime polynomials $g_0(x), h_0(x) \in (R/\mathrm{rad}R)[x]$ such that $\mathfrak{A}/(\mathrm{rad}\ R\cdot\mathfrak{A}) = \langle g_0(x)\rangle$ and $\mathfrak{B}/(\mathrm{rad}\ R\cdot\mathfrak{B}) = \langle h_0(x)\rangle$. Then there exist monic, coprime polynomials $g(x), h(x) \in R[x]$ such that $f(x) = g(x)\cdot h(x)$, and $\overline{g(x)} = g_0(x)$, $\overline{h(x)} = h_0(x)$ in $\frac{R}{\mathrm{rad}\ R}[x]$.

We can now complete the proof of Theorem 2.2:

Case 2. Assume S is not connected. By Lemmas 2.3, 2.4, and 2.5 S must be of the form $S = Re \oplus R(1-e)$, for e an idempotent in A of the form $e = a_1x + a_2x^2$ where $a_1 \in \mathrm{rad}\ R$, $\bar{a}_2 = \bar{1}$ in $R/\mathrm{rad}\ R$. Thus

$$\begin{aligned} A = Ae \oplus A(1-e) &= \langle a_2x^2+a_1x\rangle \oplus \langle a_2x^2+a_1x-1\rangle \\ &= \langle x^2+a_2^{-1}a_1x\rangle \oplus \langle x^2+a_2^{-1}a_1x-a_2^{-1}\rangle. \end{aligned}$$

Furthermore $x^2 + a_2^{-1}a_1x \equiv x^2$ modulo $\mathrm{rad}(R)\cdot A$ and $x^2 + a_2^{-1}a_1x - a_2^{-1} \equiv x^2 - 1$ modulo $\mathrm{rad}(R)\cdot A$. Finally $\langle x^2-1\rangle = \langle x-1\rangle$ in $A/(\mathrm{rad}(R)\cdot A)$, since $\langle x^2-1\rangle \subseteq \langle x-1\rangle$ and $(x-1) = (1-x)(x^2-1)$ in $A/(\mathrm{rad}(R)\cdot A)$ implies $\langle x-1\rangle \subseteq \langle x^2-1\rangle$. Thus by Lemma 2.6, there exist monic polynomials $g(x), h(x)$ in $R[x]$ such that $x^3 - x^2 + px = g(x)\cdot h(x)$ with $\overline{g(x)} = x^2$ and $\overline{h(x)} = x - 1$ in $(R/\mathrm{rad}R)[x]$. This contradicts the choice of p.

Theorem 2.2 states that if R is a connected, inertial coefficient ring then $R/\text{rad } R$ is connected and so $(R, \text{rad } R)$ trivially is an L.I. pair. We next use the decomposition space to extend Theorem 2.2 to an arbitrary inertial coefficient ring.

Proposition 2.7. Let A be a finitely generated R-algebra. (A,N) is an L.I. pair if (A_x, N_x) is an L.I. pair for all $x \in X(R)$.

Proof: Let $u \in A$ be such that $u^2 - u \in N$. We must find an idempotent $e \in A$ such that $u - e \in N$.

Since R_x is a flat R-module [17, Proposition II.18, p.33] without ambiguity we can let N_x denote the image of N under the canonical homomorphism $A \to A/I(x)\cdot A = A_x$. Now $u_x = u + I(x)\cdot A$ is an element of A_x such that $\bar{u}_x$ is an idempotent element of A_x/N_x. Since (A_x, N_x) is an L.I. pair and since an idempotent in A_x can be lifted to an idempotent in A [17, Proposition II.20, p.34], there exists an idempotent $f(x) \in A$ such that $u_x = [f(x)]_x + [n(x)]_x$ for some $n(x) \in N$. By Proposition 1.16 for each $x \in X(R)$ there exists an idempotent $e(x) \in R$ such that $u\cdot e(x) = f(x)\cdot e(x) + n(x)\cdot e(x)$ and $u_y = [f(x)]_y + [n(x)]_y$ for all $y \in N(e(x))$.

$\{N(e(x))\}_{x \in X(R)}$ is an open cover of $X(R)$ and thus by the partition property, Proposition 1.14, there exists a finite refinement of disjoint open and closed sets

$U_1, \ldots, U_m$. By Proposition 1.15 (iv), $U_i = N(e_i)$ for some idempotent $e_i \in R$. Since $\{N(e_i)\}_{i=1}^m$ covers $X(R)$ and since $N(e_i) \cap N(e_j) = \emptyset$, by Proposition 1.15 (i), (ii), and (iii) $1 = \sum_{i=1}^m e_i$ and $e_i \cdot e_j = 0$ for each $i \neq j$. Furthermore $N(e_i) \subseteq N(e(x_i))$ for some $x_i \in X$, and thus $N(e_i) \cap N(e(x_i)) = N(e_i)$ implies $e_i \cdot e(x_i) = e_i$; hence $u \cdot e_i = f_i \cdot e_i + n_i \cdot e_i$ where $f_i = f(x_i)$ and $n_i = n(x_i)$. Let $e = \sum_{i=1}^m (f_i \cdot e_i)$. Since the e_i are pairwise orthogonal and each f_i is an idempotent, e is an idempotent. Now $u = \sum_{i=1}^m ue_i = \sum_{i=1}^m (f_i e_i + n_i e_i) = e + \sum_{i=1}^m n_i e_i$, and therefore $u - e \in N$.

<u>Corollary 2.8</u>. If R is an inertial coefficient ring, $(R, \text{rad } R)$ is an L.I. pair.

<u>Proof</u>: Since a homomorphic image of an inertial coefficient ring is an inertial coefficient ring (Chapter I, property VIII, p.8) $R_x = R/I(x)$ is a connected inertial coefficient ring for every $x \in X(R)$. By Theorem 2.2 $(R_x, \text{rad}(R_x))$ is an L.I. pair for every $x \in X(R)$. Since $(\text{rad } R)_x \subseteq \text{rad}(R_x)$, by Lemma 1.10 (ii) $(R_x, (\text{rad } R)_x)$ is an L.I. pair for every $x \in X(R)$; hence by Proposition 2.7 $(R, \text{rad } R)$ is an L.I. pair.

We are now able to prove the general case:

<u>Theorem 2.9</u>. Let R be an inertial coefficient ring and A be a finitely generated R-algebra. Then (A,N) is an L.I. pair.

<u>Proof</u>: $R/\mathrm{annih}_R A$, being a homomorphic image of R, is an inertial coefficient ring and A is a faithful $R/\mathrm{annih}_R A$-algebra. Thus replacing $R/\mathrm{annih}_R A$ by R we may assume A is a faithful R-algebra.

Let $c \in A$ be such that $c^2 - c \in N$. We must find an idempotent $e \in A$ such that $c - e \in N$. Let $B = R[c]$ denote the R-subalgebra of A generated by c. B is a finitely generated, commutative R-algebra. By Proposition 1.3 $N \cap B \subseteq \mathrm{rad}\, B$. Let $(R/\mathrm{rad}R)[\bar{c}]$, where $\bar{c} = c + N$, denote the $R/\mathrm{rad}\, R$-subalgebra of A/N generated by $\bar{c}$. Define a homomorphism $\psi : B \to (R/\mathrm{rad}\, R)[\bar{c}]$ by $\psi(\sum_{i=0}^{n} r_i c^i) = \sum_{i=0}^{n} \bar{r}_i (\bar{c})^i$. ψ is surjective and $\ker \psi \subseteq N \cap B \subseteq \mathrm{rad}\, B$. Now $B/\ker \psi \cong (R/\mathrm{rad}R)[\bar{c}]$ is a homomorphic image of $\dfrac{(R/\mathrm{rad}R)[x]}{\langle x^2 - x\rangle} \cong R/\mathrm{rad}\, R \oplus R/\mathrm{rad}\, R$ and therefore is a separable R-algebra. Since $B/\mathrm{rad}\, B \cong \dfrac{B/\ker \psi}{\mathrm{rad}\, B/\ker \psi}$, $B/\mathrm{rad}\, B$ is a separable R-algebra. By Chapter I, property VII, p.8, B is an inertial coefficient ring, and thus by Corollary 2.8 $(B, \mathrm{rad}\, B)$ is an L.I. pair. By Lemma 1.10 (ii), $(B, N \cap B)$ is an L.I. pair. Then $c^2 - c \in N \cap B$ implies that there exists an idempotent $e \in B$ such that $c - e \in N \cap B$. But then $e \in A$ and $c - e \in N$.

Notice that in the preceding proof we showed that if R is an inertial coefficient ring and if $\bar{c} \in A/N$ is idempotent, then there exists an idempotent $e \in B = R[c]$ lifting $\bar{c}$. Thus e is a polynomial over R in c. Furthermore, the following result is a consequence of Corollary 1.12 and the proof of Theorem 2.9.

<u>Corollary 2.10</u>. The following are equivalent properties of a commutative ring R:

(i) All finitely generated R-algebras have the idempotent lifting property.

(ii) All finitely generated, commutative R-algebras are Hensel rings.

Any algebra A having the idempotent lifting property must satisfy the two properties below. A consequence of Theorem 2.9 is that these results hold for any finitely generated algebra over an inertial coefficient ring.

1) Any countable sequence of pairwise orthogonal idempotents in A/N can be lifted to a sequence of pairwise orthogonal idempotents in A [16, Proposition 2, p.73].

2) If $A/N \cong B_{n \times n}$, the full $n \times n$ matrix ring over a finitely generated $R/\text{rad } R$-algebra B, then there exists a finitely generated R-algebra C such that $\frac{C}{\text{rad } C} \cong B$ and $A \cong C_{n \times n}$, the full $n \times n$ matrix ring over C [14, see proof of Theorem 1, p.55].

<u>Corollary 2.11</u>. If R is an inertial coefficient ring then $R/\mathfrak{A}$ is a Hensel ring for any ideal $\mathfrak{A}$ of R.

<u>Proof</u>: By Chapter I, Property VIII, p.8, $R/\mathfrak{A}$ is an inertial coefficient ring and by Theorem 2.9 all finitely generated $R/\mathfrak{A}$-algebras have the idempotent lifting property. Thus by Corollary 2.10 $R/\mathfrak{A}$ is a Hensel ring.

A consequence of Corollary 2.11 is that all homomorphic images of an inertial coefficient ring have the following properties of a Hensel ring R:

1) For every projective $\frac{R}{\text{rad } R}$ - module $\bar{P}$ of rank n there exists a unique (up to isomorphism) projective R-module P of rank n such that $\bar{P} = \frac{P}{\text{rad } R \cdot P}$ [7, Corollary 5.4, p.58].
2) The homomorphism $\mathfrak{B}(R) \to \mathfrak{B}(R/\text{rad } R)$ is an isomorphism, where $\mathfrak{B}(R)$ denotes the Brauer group of R [22].

<u>Corollary 2.12</u>. The following are equivalent properties of a commutative ring R:

(i) For all finitely generated, commutative R-algebras A such that A/N is R-separable, there exists a separable R-subalgebra S of A such that $A = S + N$.

(ii) All finitely generated, commutative R-algebras have the idempotent lifting property.

<u>Proof</u>: The fact that (ii) implies (i) follows from Theorem 1.5. The proof of Theorem 2.9 shows that (i) implies (ii).

If Ingraham's conjecture is true then it is unnecessary to restrict the algebras A in Corollary 2.12 to commutative R-algebras, for then the lifting of idempotents from A/N to A in all finitely generated R-algebras A is equivalent to the lifting of the separability of A/N to a separable R-subalgebra S of A in all finitely generated R-algebras.

CHAPTER III

NEW INERTIAL COEFFICIENT RINGS

In this chapter we shall show that a ring R is an inertial coefficient ring if and only if for every $x \in X(R)$ each connected ring R_x is an inertial coefficient ring. We shall use this criterion to produce new inertial coefficient rings.

W.C. Brown [4, Theorem 1, p.370] used the decomposition space $X(R)$ to show von Neumann regular rings are inertial coefficient rings. Our result is a generalization of his result, for when R is a von Neumann regular ring each R_x is a field and therefore each R_x is an inertial coefficient ring. Our proof is closely patterned after Brown's proof. The technique is to show that an inertial subalgebra exists if and only if a particular finite collection of equations holds. To find an inertial subalgebra of an R-algebra A we use the fact that certain equations hold in each R_x-algebra A_x and then using the topology on $X(R)$ we patch together elements of A to obtain equations in A which hold in every A_x and therefore hold in A.

We shall use the following criterion for the separability of a finitely generated R-algebra S.

Lemma 3.1. Let S be an R-algebra generated as an R-module by $s_1, \ldots, s_n$. Then S is separable if and only if there exists $b_i, b_i' \in S$ $i = 1, \ldots, m$ such that

(i) $\sum_{i=1}^{m} b_i b_i' = 1$, and

(ii) $\sum_{i=1}^{m} s_j b_i \otimes_R b_i' = \sum_{i=1}^{m} b_i \otimes_R b_i' s_j$ holds in $S \otimes_R S^O$ for $j = 1, \ldots, n$.

Proof: Proof follows from [6, Proposition 1.1 (iii), p.40].

The main result of this chapter is the following theorem.

Theorem 3.2. R is an inertial coefficient ring if and only if R_x is an inertial coefficient ring for all $x \in X(R)$.

Proof: If R is an inertial coefficient ring then $R_x = R/I(x)$ is an inertial coefficient ring by Chapter I, property VIII, p.8.

Conversely, suppose R_x is an inertial coefficient ring for all $x \in X(R)$, and let A be a finitely generated R-algebra such that A/N is R-separable. Then $A_x = A/(I(x)\cdot A)$ is a finitely generated R_x-algebra. Since R_x is a flat R-module [17, Proposition II.18, p.33] without ambiguity we can let N_x denote the image of N under the canonical homomorphism $A \to A/(I(x)\cdot A) = A_x$. Furthermore, since $0 \to N \to A \to A/N \to 0$

is an exact sequence of R-modules and R_x is R-flat, $\frac{A_x}{N_x} \cong (A/N)_x$ is a separable R_x-algebra. Since each R_x is an inertial coefficient ring and $N_x \subseteq \mathrm{rad}(A_x)$, for every $x \in X(R)$ there exists a separable R_x-algebra $\hat{S}^x$ such that $\hat{S}^x + N_x = A_x$ [12, Corollary, p.3].

By Chapter I, property V, p.8, each $\hat{S}^x$ is a finitely generated R_x-algebra; for each $x \in X(R)$ let $s_1(x),\ldots,s_{n(x)}(x) \in A$ be such that $(s_1(x))_x,\ldots,(s_{n(x)}(x))_x$ are R_x-module generators of $\hat{S}^x$. Let $S^x = \sum_{i=1}^{n(x)} R\cdot s_i(x)$ be the R-submodule of A generated by $\{s_i(x)\}_{i=1}^{n(x)}$. Then $(S^x)_x = \hat{S}^x$.

Let $a_1,\ldots,a_p$ be R-module generators of A. Since for each $x \in X(R)$ $\hat{S}^x = (S^x)_x$ is a separable R_x-algebra such that $N_x + (S^x)_x = A_x$, there exist elements $r_{ijk}(x)$, $r_i(x), t_{\ell j}(x), r_{hj}(x), r'_{hj}(x) \in R$, elements $z_\ell(x) \in N$, and elements $b_h(x), b'_h(x) \in S^x$ for $i,j,k = 1,\ldots,n(x)$, $\ell = 1,\ldots,p$, and $h = 1,\ldots,m(x)$ such that:

(-1-) $\quad (s_i(x))_x(s_j(x))_x = \sum_{k=1}^{n(x)} (r_{ijk}(x))_x(s_k(x))_x$

for $i,j = 1,\ldots,n(x)$.

(-2-) $\quad 1_x = \sum_{i=1}^{n(x)} (r_i(x))_x(s_i(x))_x.$

(-3-) $\quad (a_\ell)_x = (z_\ell(x))_x + \sum_{j=1}^{n(x)} (t_{\ell j}(x))_x(s_j(x))_x$

for $\ell = 1,\ldots,p$.

(-4-) $\quad 1_x = \sum_{h=1}^{m(x)} (b_h(x))_x(b'_h(x))_x.$

(-5-) $$\sum_{h=1}^{m(x)} [(s_j(x))_x (b_h(x))_x \otimes_{R_x} (b_h'(x))_x] = \sum_{h=1}^{m(x)} [(b_h(x))_x \otimes_{R_x} (b_h'(x))_x (s_j(x))_x]$$
in $(S^x)_x \otimes_{R_x} (S^x)_x^o$ for $j = 1,\ldots,n(x)$.

(-6-) $$b_h(x) = \sum_{j=1}^{n(x)} r_{hj}(x) s_j(x) \text{ and } b_h'(x) = \sum_{j=1}^{n(x)} r_{hj}'(x) s_j(x)$$
for $h = 1,\ldots,m(x)$.

Using Proposition 1.16 and by intersecting the appropriate neighborhoods of x if necessary, for each $x \in X(R)$ there exists an idempotent $e(x) \in R$ such that $[e(x)]_x = 1_x$ and equations 1-5 hold for all $y \in N(e(x))$ when we replace the subscript x with the subscript y (for example, $\sum_{i=1}^{m(x)} ([s_j(x)]_y [b_i(x)]_y \otimes_{R_y} [b_i'(x)]_y) = \sum_{i=1}^{m(x)} ([b_i(x)]_y \otimes_{R_y} [b_i'(x)]_y [s_j(x)]_y)$ holds in $(S^x)_y \otimes_{R_y} (S^x)_y^o$).

The neighborhoods $\{N(e(x))\}_{x \in X(R)}$, where each $e(x)$ is chosen as above, form an open cover of $X(R)$. By Proposition 1.15 and the partition property (Proposition 1.14), there exist pairwise orthogonal idempotents $\{e_i\}_{i=1}^t$ contained in R such that $\{N(e_i)\}_{i=1}^t$ is a disjoint open cover of $X(R)$ refining $\{N(e(x))\}_{x \in X(R)}$. Let x_i denote a point of $X(R)$ such that $N(e_i) \subseteq N(e(x_i))$, $i = 1,\ldots,t$. Let $n = \underset{i=1,\ldots,t}{\text{maximum}} \{n(x_i)\}$ and $m = \underset{i=1,\ldots,t}{\text{maximum}} \{m(x_i)\}$. For each k define $s_j(x_k) = 0$, $r_j(x_k) = 0$, $r_{ij\ell}(x_k) = 0$ for all i, j, ℓ, $n(x_k) < i \le n$, $n(x_k) < j \le n$, or $n(x_k) < \ell \le n$, define

$t_{i,j}(x_k) = 0$ for all i,j, $0 \leq i \leq p$ and $n(x_k) < j \leq n$, and define $b_j(x_k) = b_j'(x_k) = 0$ for all j, $m(x_k) < j \leq m$.

We patch together the data from the stalks R_x by defining the following:

$$s_j = \sum_{k=1}^{t} s_j(x_k)e_k \qquad j = 1,\ldots,n$$

$$r_j = \sum_{k=1}^{t} r_j(x_k)e_k \qquad j = 1,\ldots,n$$

$$r_{ij\ell} = \sum_{k=1}^{t} r_{ij\ell}(x_k)e_k \qquad i,j,\ell = 1,\ldots,n$$

$$z_j = \sum_{k=1}^{t} z_j(x_k)e_k \qquad j = 1,\ldots,p$$

$$t_{ij} = \sum_{k=1}^{t} t_{ij}(x_k)e_k \qquad \begin{matrix} i = 1,\ldots,p \\ j = 1,\ldots,n \end{matrix}$$

$$b_j = \sum_{k=1}^{t} b_j(x_k)e_k \qquad j = 1,\ldots,m$$

$$b_j' = \sum_{k=1}^{t} b_j(x_k)e_k \qquad j = 1,\ldots,m.$$

Any $x \in X(R)$ is contained in $N(e_i)$ for some i and is not contained in $N(e_j)$ for $j \neq i$. Thus for all prime ideals $p \in x$, $1 - e_i \in p$ and therefore $(e_i)_x = 1_x$. For all $p \in x$ and $j \neq i$, $1 - e_j \notin p$ and $e_j(1-e_j) = 0 \in p$ imply that $e_j \in p$, and therefore $(e_j)_x = 0_x$. Thus when $x \in N(e_k)$ $(s_j)_x = (s_j(x_k))_x$, $(r_j)_x = (r_j(x_k))_x$, $(z_j)_x = (z_j(x_k))_x$, $(r_{ij\ell})_x = (r_{ij\ell}(x_k))_x$, $(t_{ij})_x = (t_{ij}(x_k))_x$, $(b_j)_x = (b_j(x_k))_x$, and $(b_j')_x = (b_j'(x_k))_x$.

Let S be the R-submodule of A generated by $\{s_j\}_{j=1}^n$. For each $x \in X(R)$, $x \in N(e(x_k))$ for some k, so for each $i,j = 1,\ldots,n$

$$(s_i)_x(s_j)_x = (s_i(x_k))_x(s_j(x_k))_x = \sum_{\ell=1}^n (r_{ij\ell}(x_k))_x(s_\ell(x_k))_x$$
$$= \sum_{\ell=1}^n (r_{ij\ell})_x(s_\ell)_x$$

and

$$1_x = \sum_{i=1}^n (r_i(x_k))_x(s_i(x_k))_x = \sum_{i=1}^n (r_i)_x(s_i)_x.$$

Thus $s_is_j - \sum_{\ell=1}^n r_{ij\ell}s_\ell$ $i,j = 1,\ldots,n$ and $1 - \sum_{i=1}^n r_is_i$ are elements of A which equal 0_x in A_x for all $x \in X(R)$. By Proposition 1.17 $s_is_j - \sum_{\ell=1}^n r_{ij\ell}s_\ell = 0$ for $i,j = 1,\ldots,n$ and $1 - \sum_{i=1}^n r_is_i = 0$, so S is an R-algebra and $S_x = S_x^{x_k}$, where $x \in N(e_k) \subseteq N(e(x_k))$.

Setting previously undefined $r_{ij}(x_k) = 0$ and using equation (-6-) and the fact that $\{e_i\}_{i=1}^t$ are pairwise orthogonal idempotents the following equalities show that $b_i \in S$:

$$b_i = \sum_{k=1}^t b_i(x_k)e_k = \sum_{k=1}^t \sum_{j=1}^{n(x_k)} r_{ij}(x_k)s_j(x_k)e_k$$
$$= \sum_{k=1}^t \sum_{j=1}^n r_{ij}(x_k)s_j(x_k)e_k$$
$$= \sum_{j=1}^n \sum_{k=1}^t r_{ij}(x_k)e_k\ s_j(x_k)e_k$$
$$= \sum_{j=1}^n (\sum_{k=1}^t r_{ij}(x_k)e_k)(\sum_{k=1}^t s_j(x_k)e_k)$$
$$= \sum_{j=1}^n (\sum_{k=1}^t r_{ij}(x_k)e_k)s_j.$$

Similarly $b_i' \in S$.

Now $1 - \sum_{i=1}^{m} b_i b_i'$ is an element of S which is 0_x in S_x for all $x \in X(R)$. Therefore $1 = \sum_{i=1}^{m} b_i b_i'$ by Proposition 1.17. $\sum_{i=1}^{m} (s_j b_i \otimes_R b_i') - \sum_{i=1}^{m} (b_i \otimes_R b_i' s_j)$ is an element of $S \otimes_R S^o$ which is 0_x in $(S \otimes_R S^o) \otimes_R R_x \cong S_x \otimes_{R_x} S_x^o = S_x^{x_i} \otimes_{R_x} (S_x^{x_i})^o$ where $x \in N(e_i) \subseteq N(e(x_i))$, and therefore by Proposition 1.17 $\sum_{i=1}^{m} (s_j b_i \otimes_R b_i') = \sum_{i=1}^{m} (b_i \otimes_R b_i' s_j)$ in $S \otimes_R S^o$. Hence S is a separable R-algebra.

Finally since equation (-3-) holds in A_x for all $x \in X(R)$, $a_i = z_i + \sum_{j=1}^{n} t_{ij} s_j$ and $A = N + S$.

We have produced an inertial subalgebra of A and have shown, therefore, that R is an inertial coefficient ring.

We shall next use Theorem 3.2 to produce three new types of inertial coefficient rings. Our method will be to compute $X(R)$ and to show the R_x are inertial coefficient rings. In the second and third examples we shall view $X(R)$ as the collection of maximal Boolean ideals.

Our first new example of an inertial coefficient ring is the ring $S = C(X,R)$ of continuous functions from a profinite space X to a connected, inertial coefficient ring R endowed with the discrete topology.

Proposition 3.3. Let $S = C(X,R)$ be the ring of continuous functions from a profinite space X to a connected, inertial coefficient ring R having the discrete topology. Then S is an inertial coefficient ring.

Proof: By [16, pp.40-1] for every $x \in X(S)$, S_x is a homomorphic image of R and so is an inertial coefficient ring. By Theorem 3.2 S is an inertial coefficient ring.

A particular example of the previous proposition is the following:

Example 3.4. Let $S = C(X,C)$ where X is the subspace $\{1,\frac{1}{2},\frac{1}{3},\ldots,0\}$ of the real numbers under the usual topology and C is the complex numbers under the discrete topology. S is isomorphic to the ring of eventually constant sequences in C. Since X is a profinite space, by Proposition 3.3 S is an inertial coefficient ring.

Our second new example of an inertial coefficient ring is the polynomial ring $R[y_1,\ldots,y_m]$, the formal power series ring $R[[y_1,\ldots,y_m]]$, and the convergent power series ring $R\langle\langle y_1,\ldots,y_m\rangle\rangle$ over a von Neumann regular ring R. We first compute the stalks of these rings.

Lemma 3.5. Let $f(y) = \sum_{j=0}^{\infty} a_j y^j$ be an idempotent in $R[[y]]$. Then $f(y) = a_0$ where a_0 is an idempotent of R.

<u>Proof</u>: Equating constant terms and coefficients of y in the equation $(\sum_{j=0}^{\infty} a_j y^j)^2 = \sum_{j=0}^{\infty} a_j y^j$ gives the relations:

$$(-1-) \quad a_0^2 = a_0 \quad \text{and} \quad (-2-) \quad 2a_0a_1 = a_1.$$

By equation (-1-) a_0 is an idempotent of R. We shall show by induction that $a_j = 0$ for $j \geq 1$. Multiplying equation (-2-) by a_0 and using equation (-1-) gives $2a_0a_1 = a_0a_1$ or $a_0a_1 = 0$; thus $2a_0a_1 = a_1 = 0$. Now suppose $a_j = 0$ for $j \leq i$. As before $(a_0 + \sum_{j=i+1}^{\infty} a_j y^j)^2 = a_0 + \sum_{j=i+1}^{\infty} a_j y^j$ implies that $2a_{i+1}a_0 = a_{i+1}$ and so $2a_{i+1}a_0 = a_{i+1}a_0$ giving $a_{i+1}a_0 = 0$ and hence $a_{i+1} = 2a_{i+1}a_0 = 0$.

<u>Proposition 3.6</u>. If R is a ring such that $R_x[y_1,\ldots,y_m]$ (respectively $R_x[[y_1,\ldots,y_m]]$, $R_x\langle\langle y_1,\ldots,y_m\rangle\rangle$) is an inertial coefficient ring for all $x \in X(R)$ then $R[y_1,\ldots,y_m]$ (respectively $R[[y_1,\ldots,y_m]]$, $R\langle\langle y_1,\ldots,y_m\rangle\rangle$) is an inertial coefficient ring.

<u>Proof</u>: Let $S = R[y_1,\ldots,y_m]$, $T = R[[y_1,\ldots,y_m]]$, and $U = R\langle\langle y_1,\ldots,y_m\rangle\rangle$. Using Lemma 3.5 and induction on m one can show that any idempotent in S, T, or U is an idempotent of R. Since the decomposition space of a commutative ring A is the collection of maximal Boolean ideals of A, it follows immediately that $X(S) = X(T) = X(U) = X(R)$. Since for each $x \in X(S)$, $I(x)$ is the ideal of S generated by the idempotents of x, and since idempotents of S are contained

in R, then $I(x) = I \cdot S$ where I the ideal of R generated by the idempotents of x, and $S_x = S/I(x) = R/I[y_1, \ldots, y_m] = R_x[y_1, \ldots, y_m]$. Similarly $T_x = R_x[[y_1, \ldots, y_m]]$ and $U_x = R_x\langle\langle y_1, \ldots, y_m\rangle\rangle$. The result now follows from Theorem 3.2.

Corollary 3.7. If R is a von Neumann regular ring, $R[y_1, \ldots, y_m]$, $R[[y_1, \ldots, y_m]]$, and $R\langle\langle y_1, \ldots, y_m\rangle\rangle$ are inertial coefficient rings.

Proof: For each $x \in X(R)$, R_x is a field and hence $R_x[y_1, \ldots, y_m]$ (13, Corollary 2, p.553), $R_x[[y_1, \ldots, y_m]]$ (18, Theorem 30.3, p.104), and $R_x\langle\langle y_1, \ldots, y_m\rangle\rangle$ (18, Theorem 45.5, p.193) are known to be inertial coefficient rings.

If $S = \oplus \sum_{i=1}^{n} R_i$ is a finite direct sum of rings R_i then for every $x \in X(S)$ $S_x = (R_i)_{x_i}$ for some $x_i \in X(R_i)$. This fact suggests that Theorem 3.2 might be of value in studying infinite direct sums (with 1 adjoined) and direct products. As our final example of a new inertial coefficient ring we shall see that infinite direct sums (with 1 adjoined) and a few very special direct products can be shown to be inertial coefficient rings using the decomposition space.

Let R be a "ring" perhaps without an identity. R can be embedded in a ring which has an identity element in the usual manner: Let $R^* = R \oplus Z$ where Z denotes the integers. Define addition in R^* coordinatewise and multiplication by $(a,i)\cdot(b,j) = (ab+ib+ja, ij)$ for $a,b \in R$ and $i,j \in Z$. The element $(0,1)$ is the identity element of R^*. The following lemma follows easily from the definition of multiplication in R^*.

<u>Lemma 3.8</u>. All idempotents of R^* are of the form $(e,0)$ or $(-e,1)$ where e is an idempotent of R.

Let $R = \oplus \sum R_\alpha$ be the direct sum of a collection of commutative rings $\{R_\alpha\}$. We next compute the points of $X(R^*)$ and the stalks $(R^*)_x$ by finding the maximal Boolean ideals of R^* for this particular R. Any idempotent e of R has only finitely many nonzero coordinates each of which must be an idempotent. Let e_α denote the α^{th} coordinate of e.

<u>Lemma 3.9</u>. All $x \in X(R^*)$ are of the form

$x_0 = \{(e,0) : e$ an idempotent of $R\}$ or

$x_\alpha^\beta = \{(e,0) : e$ an idempotent of R and $e_\alpha \in x_\beta \in X(R_\alpha)\}$
$\cup \{(-e,1) : e$ an idempotent of R and $e_\alpha \notin x_\beta\}$.

$(R^*)_{x_0} \cong Z$, the integers and $(R^*)_{x_\alpha^\beta} \cong (R_\alpha)_{x_\beta}$.

<u>Proof</u>: One can easily show that x_0 and x_α^β are maximal Boolean ideals of R^*. We shall show that any maximal Boolean ideal x of R^* is one of these ideals.

Let $x(\alpha) = \{e_\alpha \mid (e,0) \in x\}$. If for every α, $1_\alpha \in x(\alpha)$, then $x_0 \subseteq x$ implies $x_0 = x$.

Suppose there exists an α such that $1_\alpha \notin x(\alpha)$. It is easily checked that $x(\alpha)$ is now a maximal Boolean ideal of R_α and so $x(\alpha) \in X(R_\alpha)$, say $x(\alpha) = x_\beta$. If $(e,0) \in x$ then by definition of x_β, $e_\alpha \in x_\beta$, whence $(e,0) \in x_\alpha^\beta$. To show $x \subseteq x_\alpha^\beta$ we shall show that $(-f,1) \in x$ implies $f_\alpha \notin x_\beta$, or equivalently that $1_\alpha - f_\alpha \in x_\beta$. Let $e = (0,\ldots,0,1_\alpha,0,\ldots,0) \in R$. Since $(e,0)\cdot(-f,1) = (e-ef,0) \in x$ and $e - ef = (0,\ldots,0,1_\alpha - f_\alpha,0,\ldots,0)$, by the definition of x_β we have $1_\alpha - f_\alpha \in x_\beta$. Thus $x_\alpha^\beta \subseteq x$ and so $x_\alpha^\beta = x$.

It is clear that $R^*_{x_0} = Z$. One can check that the ring homomorphism $\varphi : R \oplus Z \to (R_\alpha)_{x_\beta}$ given by $\varphi(r,j) = \tau(r_\alpha + j_\alpha)$, where $j_\alpha = j\cdot 1_\alpha$ and $\tau : R_\alpha \to (R_\alpha)_{x_\beta}$ is the canonical homomorphism, induces an isomorphism between $R^*_{x_\alpha^\beta} = (R \oplus Z)_{x_\alpha^\beta}$ and $(R_\alpha)_{x_\beta}$.

<u>Proposition 3.10</u>. If $\{R_\alpha\}_{\alpha \in I}$ is a collection of inertial coefficient rings, then $(\oplus \sum_{\alpha \in I} R_\alpha)^*$ is an inertial coefficient ring.

<u>Proof</u>: The result follows from Theorem 3.2, Lemma 3.9, and the fact that the integers form an inertial coefficient ring.

<u>Example 3.11</u>. $(\oplus \sum_{n=1}^{\infty} Z/p^n Z)^*$ where Z denotes the integers is an inertial coefficient ring which has radical which is nil but not nilpotent.

<u>Proposition 3.12</u>. Let $\prod_I R$ denote the direct product of $\{R_\alpha\}_{\alpha\in I}$, where each R_α is isomorphic to a fixed finite, connected ring R. $\prod_I R$ is an inertial coefficient ring.

<u>Proof</u>: If I and R are given the discrete topology, $\prod_I R \cong C(I,R)$. Let $\beta(I)$ denote the Stone-Čech compactification of I. It is not hard to show that $\beta(I)$ is totally disconnected, and hence a profinite space. By Proposition 3.3 $C(\beta(I),R)$ is an inertial coefficient ring, since any finite ring is an inertial coefficient ring. The natural ring homomorphism $\varphi: C(\beta(I),R) \to C(I,R)$ given by restriction is surjective since R is compact. Then $C(I,R)$, being a homomorphic image of an inertial coefficient ring, is itself an inertial coefficient ring.

<u>Corollary 3.13</u>. Let $\prod_I R_\alpha$ denote the direct product of $\{R_\alpha\}_{\alpha\in I}$, where each R_α is isomorphic to a finite ring of cardinality less than some fixed integer n. $\prod_I R_\alpha$ is an inertial coefficient ring.

<u>Proof</u>: Since there are only a finite number of distinct isomorphism classes of connected rings of cardinality less than n and since $\prod_I R_\alpha \cong \oplus \sum_{i=1}^{M} (\prod_{I_i} R_i)$, a finite direct sum of rings $\prod_{I_i} R_i$, where each $\prod_{I_i} R_i$ is a direct product of a collection of rings each isomorphic to a fixed finite, connected ring R_i, then $\prod_I R_\alpha$ is an inertial coefficient ring.

We have thus far been unable to use the decomposition space to determine whether more general direct products, e.g. ΠZ, are inertial coefficient rings.

The main result of this chapter, that R is an inertial coefficient ring if and only if each R_x is an inertial coefficient ring, is parallel to a result which follows from Proposition 2.7 in Chapter II, that all finitely generated R-algebras A have (A,N) an L.I. pair if and only if all finitely generated R_x-algebras B have $(B,\text{rad } B)$ an L.I. pair for all $x \in X(R)$. This result further suggests the equivalence suggested by Ingraham of inertial coefficient rings and rings R all of whose finitely generated R-algebras A have the idempotent lifting property, since both these properties can be determined from the connected stalks R_x of the ring R.

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