

SPECIALIZATIONS OF RESIDUAL
CENTRALITY IN GROUPS

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Roger Dale Konyndyk

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A handwritten signature in blue ink, appearing to read "R. D. Phelms".

Major professor

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ABSTRACT

SPECIALIZATIONS OF RESIDUAL CENTRALITY IN GROUPS

By

Roger D. Konyndyk

A group G is residually central if each non-identity element $g \in G$ is not an element of $[g, G]$. An unsolved problem is whether such groups must have a central series. Some partial solutions to this problem are obtained. Residually central groups in which each element has only finitely many conjugates are locally nilpotent and therefore have a central series. Using group ring techniques, it is shown that a finitely generated residually central abelian by nilpotent group is residually nilpotent. The question of when the standard wreath product $W = A \wr G$ of groups A and G is residually central is also taken up. If A and G are locally nilpotent, then W is residually central if and only if either G is torsion-free or there exists a prime p such that all elements of W of finite order have p -power order.

If $g \in G$, define $R_0(g) = g^G$ and $R_n(g) = [R_{n-1}(g), G]$ for positive integers n . If for each $g \in G$, $\bigcap_{n=0}^{\infty} R_n(g) = 1$, then G is called a $(*)$ -group. $(*)$ -groups are ZD-groups whose lower central series has length at most $\omega + 1$, where ω is the first limit ordinal. Many classes of $(*)$ -groups must be residually nilpotent: wreath products of non-trivial groups,



nilpotent by cyclic groups, and cyclic by nilpotent groups. Counterexamples show that property (*) is not equivalent to residual nilpotence.

Let $\mathfrak{X}$ be one of the following classes of groups: residually nilpotent groups, (*)-groups, residually solvable groups, residually finite groups. If G is a $\mathfrak{X}$ -group, then G/H is an $\mathfrak{X}$ group if H meets any of the following conditions:

- a) H is maximal with respect to $H \triangleleft G$ and $\gamma_n(H) = 1$, where $n \geq 2$.
- b) H is maximal with respect to $H \triangleleft G$ and $H^{(n)} = 1$, where $n \geq 1$.
- c) $H \triangleleft G$ and $H = C_G(K)$ for some subset K of G .



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By

Roger D. Konyndyk

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CHAPTER I

BASIC DEFINITIONS AND RESULTS

Let G be a group, $g_1, \dots, g_n \in G$. Then the commutator of g_1 and g_2 is $[g_1, g_2] = g_1^{-1}g_2^{-1}g_1g_2$, and the n -fold commutator is $[g_1, \dots, g_{n-1}, g_n] = [[g_1, \dots, g_{n-1}], g_n]$ for $n > 2$. If A and B are subsets of G , then $[A, B] = \langle [a, b] : a \in A \text{ and } b \in B \rangle$. If α is an ordinal greater than 2, then

$$[A, {}_{\alpha}B] = \begin{cases} [[A, {}_{\beta}B], B] & \text{if } \alpha \text{ has a predecessor } \beta \\ \cap \{[A, {}_{\beta}B] : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

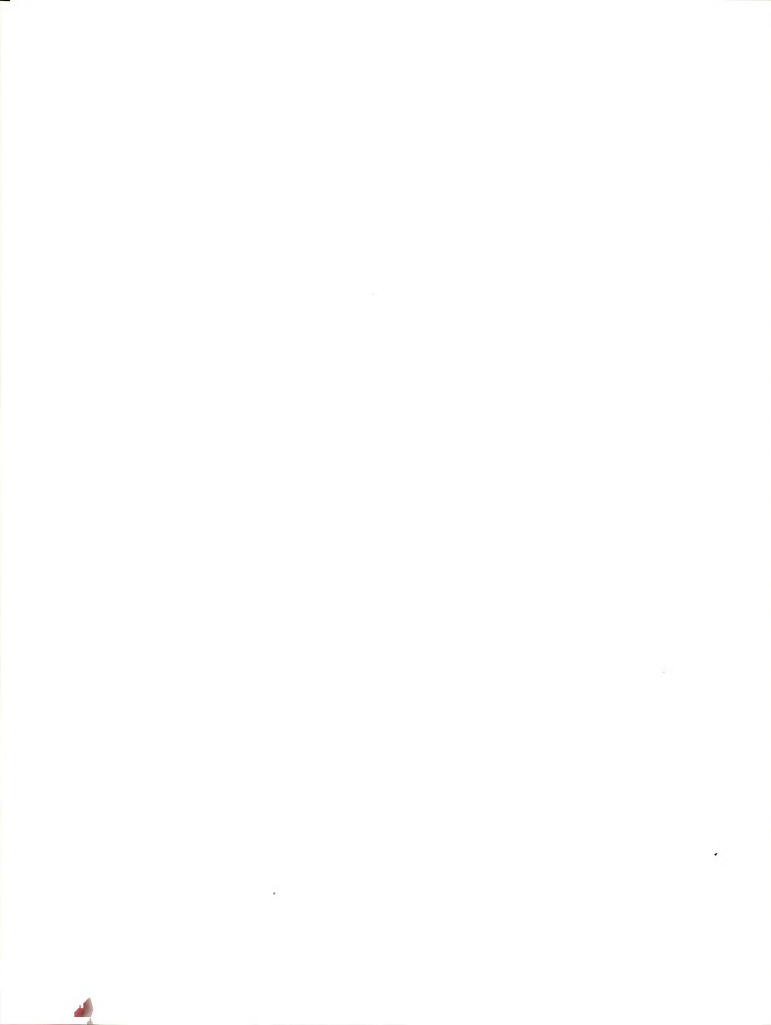
In the case $A = \{g\}$, it will be convenient to write $[g, {}_{\alpha}B]$ for $[\{g\}, {}_{\alpha}B]$. If in addition $B = G$, we set $R_0(g) = g^G$, the normal closure of g in G , and $R_{\alpha}(g) = [g, {}_{\alpha}G]$ for ordinals $\alpha > 0$.

The following identities are routine to verify and will be used repeatedly.

Lemma 1.1. Let x, y, z be elements of a group G . Then

$$(1) \quad [x, yz] = [x, z][x, y]^z$$

$$(2) \quad [xy, z] = [x, z]^y[y, z].$$



The next few results are corollaries of 1.1. Proofs may be found in [17, pp. 43-44].

Lemma 1.2. Let A, B be non-empty subsets of a group such that B is a subgroup. Then $[A, B, B] \leq [A, B]$ and $[A, B] \triangleleft \langle A, B \rangle$.

Lemma 1.3 (The three subgroup lemma). Let A, B, C be subgroups of a group G . If any two of the subgroups $[A, B, C]$, $[B, C, A]$, $[C, A, B]$ are contained in a normal subgroup of G , then so is the third.

The following definitions are taken from [17] and [18]. For a group G and ordinal α , define

$$\gamma_1(G) = G$$

$$\gamma_\alpha(G) = \begin{cases} [\gamma_\beta(G), G] & \text{if } \alpha \text{ has a predecessor } \beta \\ \cap \{\gamma_\beta(G) : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

G is said to be nilpotent of class $\leq c$ if there is an integer c such that $\gamma_{c+1}(G) = 1$. The subgroups $\gamma_\alpha(G)$ are characteristic in G and form the lower central series of G . There must be a first ordinal α_0 such that $\gamma_{\alpha_0}(G) = \gamma_{\alpha_0+1}(G)$; this subgroup is called the hypocenter of G . If for some α_0 , $\gamma_{\alpha_0}(G) = 1$, then G is called a ZD-group (hypo-central).

The elements of the upper central series of G are defined by

$$\zeta_0(G) = 1$$

$$\zeta_1(G) = \text{the center of } G$$

$$\zeta_\alpha(G)/\zeta_\beta(G) = \zeta_1\left(\frac{G}{\zeta_\beta(G)}\right) \quad \text{if } \alpha \text{ has a predecessor } \beta.$$

$$\zeta_\alpha(G) = \bigcup\{\zeta_\beta(G) : \beta < \alpha\} \quad \text{if } \alpha \text{ is a limit ordinal.}$$

There is a first ordinal α_0 such that $\zeta_{\alpha_0}(G) = \zeta_{\alpha_0+1}(G)$; this subgroup is called the hypercenter of G . If for some ordinal α_0 , $\zeta_{\alpha_0}(G) = G$, then G is called a ZA-group (hypercentral). Another characterization of ZA-groups may be found in [12, p. 219]:

Theorem 1.4. G is a ZA-group if and only if for every sequence $x_1, x_2, \dots$ of elements of G , there is an integer n such that $[x_1, x_2, \dots, x_n] = 1$.

A central series is a collection Φ of subgroups $\{G_\alpha : \alpha \in I\}$ of G such that

(i) I is a fully ordered set such that if $\alpha, \beta \in I$ and $\alpha < \beta$, then $G_\alpha \leq G_\beta$.

(ii) If $G_\alpha < G_\beta$, $\alpha, \beta \in I$, and no element of Φ is properly between G_α and G_β , then $[G_\beta, G] \leq G_\alpha$; that is, $G_\alpha/G_\beta \leq \zeta_1(G/G_\beta)$.

(iii) For any subset A of I , $\bigcup\{G_\alpha : \alpha \in A\} \in \Phi$ and $\bigcap\{G_\alpha : \alpha \in A\} \in \Phi$.

(iv) $\{1\} \in \Phi$ and $G \in \Phi$.

A group which has a central series is called a Z-group; thus ZA- and ZD-groups are Z-groups.

If P is a group-theoretic property, then the group G has P residually if for every $1 \neq g \in G$, there is a normal subgroup N_g of G such that $g \notin N_g$ and G/N_g has P . It is an easy exercise to show that G is residually nilpotent if and only if $\gamma_\omega(G) = \bigcap_{n=1}^{\infty} \gamma_n(G) = 1$. (ω denotes the first infinite ordinal.) Following Kurosh [12], a group G is said to have P locally if every finitely generated subgroup of G has the property P . P is called a local property if every group which has P locally itself has P . Mal'cev has shown that the property of being a Z-group is a local property. (For a proof, see [18, pp. 93-99].)

A group G is residually central if for all $1 \neq g \in G$, $g \notin [g, G]$. This is equivalent to the condition that for all $1 \neq g \in G$, there is a normal subgroup N_g of G such that $g \notin N_g$ and $gN_g \in \zeta(G/N_g)$; i.e. every non-trivial element of G is residually in the center of G . Residually central groups were first studied by Durbin in [3] and [4]. They have also been discussed by Ayoub [1], Slotterbeck [21], and Stanley [22] and [23].

The following results may be found in [3] and [18, pp. 6-8].

Proposition 1.5. If G is residually central and satisfies Min- n (the minimal condition on normal subgroups), then G satisfies Min and is hypercentral.

Proposition 1.6. G is residually central if any of the following conditions holds:

- (1) G is a subgroup of a residually central group.
- (2) G is locally a residually central group.
- (3) G is residually a residually central group.

Proposition 1.7. If G is finite, then G is residually central if and only if G is nilpotent.

Proposition 1.8. Let G be a residually central group, and let N be a normal subgroup of G contained in the hypercenter of G . Then G/N is residually central.

Proposition 1.9. Elements of a residually central group which have relatively prime, finite orders commute.

Stanley in [23] has obtained more information concerning when homomorphic images of residually central groups are residually central. In general, however, the question appears quite difficult and little progress has been made beyond Proposition 1.8.

Residually central groups are discussed in chapters II and III. In chapter II, relationships between residual centrality and other group-theoretical properties are discussed. The main result is Theorem 2.7: Let G be a finitely generated residually central group with a normal abelian subgroup A . If G/A is nilpotent, then G is residually nilpotent. In chapter III we take up the question of when the standard restricted wreath product $W = A \wr G$ is residually

central. The question is completely answered in the case where G is orderable and in the case where A and G are locally nilpotent.

If G is a Z -group with central series $\Phi = \{G_\alpha | \alpha \in I\}$, and $1 \neq g \in G$, then $\cap \{G_\alpha | g \in G_\alpha\}$ is an element G_{α_1} of Φ , and $\cup \{G_\alpha | g \notin G_\alpha\}$ is an element G_{α_2} of Φ . No element G_α of Φ can satisfy $G_{\alpha_2} < G_\alpha < G_{\alpha_1}$. Thus $[G_{\alpha_1}, G] \leq G_{\alpha_2}$, and $[g, G] \leq [G_{\alpha_1}, G] \leq G_{\alpha_2}$. Since $g \notin G_{\alpha_2}$, $g \notin [g, G]$. Thus any Z -group is residually central. Whether or not the converse is true is unknown. The close relationship between residually central groups and Z -groups may be seen in the following theorem due to Hickin and Phillips [10].

Proposition 1.10. G is a Z -group if and only if for every finitely generated subgroup $1 \neq K$ of G , $K \not\leq [K, G]$.

In chapter IV a specialization of residual centrality is examined: A group G is a $(*)$ -group if for all $g \in G$, $\bigcap_{n=0}^{\infty} R_n(g) = 1$. It turns out that property $(*)$ is very close to residual nilpotence: A $(*)$ -group is a ZD -group whose lower central series has length at most $\omega + 1$ (Proposition 4.4). Property $(*)$ turns out to be equivalent to residual nilpotence for wreath products of groups (Theorem 4.13), nilpotent by cyclic groups (Proposition 4.15), and groups with trivial center (Corollary 4.6).

Chapter V looks at homomorphic images of $(*)$ -groups and other classes of groups, and looks at descendance in $(*)$ -groups. We obtain

Corollary 5.4. Let $\mathfrak{X}$ be one of the following classes of groups: Residually nilpotent groups, residually solvable groups, residually finite groups. If G is an $\mathfrak{X}$ -group, then G/H is an $\mathfrak{X}$ -group if H satisfies any of the following conditions:

- (1) H is maximal with respect to $H \triangleleft G$ and $H^{(n)} = 1$, where $H^{(n)}$ denotes the n^{th} derived subgroup of H , $n \geq 1$.
- (2) H is maximal with respect to $H \triangleleft G$ and $\gamma_{n+1}(H) = 1$, where $n \geq 1$.
- (3) $H \triangleleft G$ and $H = C_G(K)$ for some subset K of G .

Corollary 5.9. Let G be a $(*)$ -group, H as in (1), (2), or (3) of 5.4. Then G/H is residually nilpotent.

Results on descendance are found in 5.10: If G is a $(*)$ -group, and A is an abelian subgroup of G or if $A \leq \zeta_\omega(G)$, then A is descendant in G of order type at most $\omega + 1$.

We close this section by stating two lemmas for future use.

Lemma 1.11. If G is any group and $g \in G$, then $[g, G] = [g^G, G]$. Hence if $n \geq 0$, $R_{n+1}(g) = [R_n(g), G]$.

Proof: Since $g^G = \langle g \rangle [g, G]$,

$$\begin{aligned} [g^G, G] &= [\langle g \rangle [g, G], G] \\ &\cong [\langle g \rangle, G]^{[g, G]} [[g, G], G] \quad (\text{by 1.1}) \end{aligned}$$



$$\begin{aligned}
&= [\langle g \rangle, G][g, G, G] && (\text{since } [\langle g \rangle, G] \triangleleft G) \\
&\leq [\langle g \rangle, G][g, G] && (\text{by 1.2})
\end{aligned}$$

Thus it suffices to show that for any positive integer n and element $h \in G$, $[g^n, h] \in [g, G]$ and $[g^{-n}, h] \in [g, G]$. But $[g^n, h] = [gg^{n-1}, h] = [g, h]^{g^{n-1}} [g^{n-1}, h] \in [g, G]$ by induction on n . Similarly, $[g^{-n}, h] \in [g^{-1}, G]$. Now $[g^{-1}, h]^g = g^{-1} g h^{-1} g^{-1} h g = [h, g] \in [g, G]$. Since $[g, G] \triangleleft G$, $[g^{-1}, G] \leq [g, G]$.

Lemma 1.12. If $g_1, \dots, g_n \in G$, then $\prod_{i=1}^n [g_i, G] = [\langle g_1, \dots, g_n \rangle, G] \triangleleft G$.

Proof: Since each $[g_i, G] \leq [\langle g_1, \dots, g_n \rangle, G]$, it is clear that $\prod_{i=1}^n [g_i, G] \leq [\langle g_1, \dots, g_n \rangle, G]$. Any element of $[\langle g_1, \dots, g_n \rangle, G]$ is the product of finitely many elements of the form $[g_{i_1} \dots g_{i_m}, h]^{\pm 1}$, where $h \in G$, and $1 \leq i_j \leq n$. But $[g_{i_1} (g_{i_2} \dots g_{i_m}), h] = [g_{i_1}, h]^{g_{i_2} \dots g_{i_m}} [g_{i_2} \dots g_{i_m}, h] = [g_{i_1}, h]^{g_{i_2} \dots g_{i_m}} \dots [g_{i_{m-1}}, h]^{g_{i_m}} [g_{i_m}, h] \in \prod_{i=1}^n [g_i, G]$, since each $[g_i, G] \triangleleft G$. Finally $[\langle g_1, \dots, g_n \rangle, G]$ is the product of finitely many normal subgroups of G and therefore is normal.



CHAPTER II

RESIDUALLY CENTRAL GROUPS

In this chapter relationships between residually central groups and other classes of groups are examined.

An FC-group is a group in which each element has only finitely many conjugates.

Proposition 2.1. A residually central FC-group is locally nilpotent.

Proof: Let G be a residually central FC-group, and $H = \langle h_1, \dots, h_n \rangle$ a finitely generated subgroup. It suffices to show that H must be nilpotent. By 1.8, $H/\zeta_1(H)$ is residually central. Since each h_i has only finitely many conjugates in H , each $[H:C_H(h_i)] < \infty$. $\zeta_1(H) = \bigcap_{i=1}^{\infty} C_H(h_i)$ is the intersection of finitely many subgroups each of finite index in H and therefore has finite index in H . Thus $H/\zeta_1(H)$ is a finite residually central group, which must be nilpotent, by 1.7. Then H itself is nilpotent.

Proposition 2.2. Let G be a residually central group with a normal subgroup N such that G/N is nilpotent and N satisfies the minimal condition on subgroups. Then G is a ZA-group.

Proof: Let $x_1, x_2, \dots$ be any set of elements of G , and let $y_k = [x_1, x_2, \dots, x_k]$. By 1.4, it suffices to show that some $y_k = 1$. Since G/N is nilpotent, there is an integer n such that $\gamma_n(G) \leq N$, and so $y_n \in N$. Let i be a non-negative integer. If $y_{n+i} = 1$, there is nothing to prove. If $y_{n+i} \neq 1$, then $y_{n+i} \notin [y_{n+i}, G]$, since G is residually central. $y_{n+i+1} = [y_{n+i}, x_{n+i+1}] \in [y_{n+i}, G] \triangleleft G$, and so $y_{n+i+1}^G \leq [y_{n+i}, G] < y_{n+i}^G \leq N$. Thus there is a descending sequence of subgroups $y_n^G > y_{n+1}^G > \dots$ of F . Since F satisfies Min, for some j we must have $y_{n+j}^G = y_{n+j+1}^G$. This can happen only if $y_{n+j} = 1$.

Corollary 2.3. Let G be a finitely generated residually central group. If G has a finite normal subgroup N such that G/N is nilpotent, then G is nilpotent.

Proof: Since N is finite, N satisfies Min, and 2.2 applies. Mal'cev [17, p. 50] has shown that ZA-groups are locally nilpotent, completing the proof.

The necessity of the condition in 2.2 that the subgroup N has Min is shown by the example of the infinite dihedral group. This group has two generators, is metabelian, and is residually nilpotent (hence residually central), but is not nilpotent. If the group were a ZA-group, it would be locally nilpotent and hence nilpotent.

If G is a group with a normal abelian subgroup A , and $\Gamma = G/A$, then A may be viewed as a right $Z\Gamma$ -module,



where $Z\Gamma$ is the integral group ring. Whenever this is done, A will be written additively and Γ will be written multiplicatively. Thus if $a \in A$ and $\sum_{i=1}^n z_i(g_i A) \in Z\Gamma$, where $z_i \in Z$, $g_i \in G$, $1 \leq i \leq n$, then the module action $a(\sum_{i=1}^n z_i(g_i A))$ is defined to be (in multiplicative notation) $\prod_{i=1}^n (a^{z_i})^{g_i}$; the module multiplication is basically that of group conjugation. It is routine to check that this is well-defined. Note that submodules are subgroups of A which are normal in G . Note also that for $a \in A$, $g \in G$, $[a, g] = a^{-1}a^g$ may be written in module notation as $-a + a(gA) = a(gA - 1)$. The augmentation ideal Δ of $Z\Gamma$ is defined to be the (two-sided) ideal generated by $\{\gamma - 1 \mid \gamma \in \Gamma\}$. If B is any subset of A , then $[B, G]$ may be written as $B\Delta$ in module notation; repeating this m times shows that $[B, {}_m G] = B\Delta^m$ for positive integers m . Δ may be characterized by the well-known

Lemma 2.4. If Γ is a group and $Z\Gamma$ the integral group ring, then the augmentation ideal $\Delta = \{ \sum_{i=1}^n z_i \gamma_i \mid z_i \in Z, \gamma_i \in \Gamma, \text{ and } \sum_{i=1}^n z_i = 0 \}$.

Proof: The containment $\subseteq$ is clear. Conversely, suppose that $\sum_{i=1}^n z_i \gamma_i \in Z\Gamma$ with $\sum_{i=1}^n z_i = 0$. Then $\sum_{i=1}^n z_i \gamma_i - 0 = \sum_{i=1}^n z_i \gamma_i - \sum_{i=1}^n z_i = \sum_{i=1}^n z_i (\gamma_i - 1) \in \Delta$.

Let R be a ring and I an ideal in R . Following Roseblade [20], I is said to be a polycentral ideal of R if there is a chain $0 = I_0 \leq I_1 \leq \dots \leq I_c = I$ of ideals of

R such that I_{i+1}/I_i is generated by one element in the center of R/I_i , $0 \leq i < c$. Roseblade has proved [20]

Theorem 2.5. Every ideal of the integral group ring $Z\Gamma$ is polycentral if and only if Γ is a finitely generated nilpotent group.

Robinson in [19] uses the term polycentral to mean that each I_{i+1}/I_i is generated by some subset of the center of R . He shows, in Theorem 5 of a preprint of [19],

Theorem 2.6. Let R be a ring, I a polycentral ideal of R , and M a noetherian R -module. Then $a \in \bigcap_{n=1}^{\infty} MI^n$ if and only if $a = ai$ for some $i \in I$.

These two results give

Theorem 2.7. Let G be a finitely generated residually central group with a normal abelian subgroup A . If G/A is nilpotent, then G is residually nilpotent.

Proof: For some integer n , $\gamma_n(G) \leq A$ since G/A is nilpotent. Let $A_1 = \gamma_n(G)$. By [2, Theorem 3.6], if $G = \langle g_1, \dots, g_k \rangle$, then $A_1 = \langle [g_{i_1}, g_{i_2}, \dots, g_{i_n}] \mid 1 \leq i_j \leq k \rangle^G$. G/A_1 is nilpotent, and A_1 is finitely generated as a $Z\Gamma$ -module, where $\Gamma = G/A_1$. Now G is residually nilpotent if and only if $\bigcap_{m=1}^{\infty} [A_1, {}_m G] = 1$. If Δ is the augmentation ideal of $Z\Gamma$ this condition is equivalent to $\bigcap_{m=1}^{\infty} A_1 \Delta^m = 0$ in module notation. By 2.5, Δ is a polycentral ideal of

$Z\Gamma$ in the senses of both Roseblade and Robinson. P. Hall has shown [7, Theorem 1] that $Z\Gamma$ is a right-noetherian ring. Since A_1 is a finitely generated right $Z\Gamma$ -module, A_1 is a noetherian module. By 2.6, $a \in \bigcap_{m=1}^{\infty} A_1 \Delta^m$ if and only if $a = a\delta$ for some $\delta \in \Delta$. This means that $a \in a\Delta$, which says that $a \in [a, G]$. Thus a must be the identity, since G is residually central.

Corollary 2.8. Suppose G is a semi-direct product of A by F , where A is abelian and F is finite. If G is finitely generated and residually central, then G is residually nilpotent.

Proof: Since F is finite and residually central, F is nilpotent, and 2.7 applies.

Phillips and Roseblade recently have shown [15] that if in 2.8 G is merely an extension of A by F instead of a semi-direct product, then G is a ZD-group, and $\gamma_{w+d}(G) = 1$ for some integer d .



CHAPTER III

RESIDUALLY CENTRAL WREATH PRODUCTS

Let A and G be non-trivial groups. For each $g \in G$, let $A_g \simeq A$, and set $\bar{A} = \bigcup \{A_g \mid g \in G\}$. Any element α of $\bar{A}$ can be thought of a function $\alpha: G \rightarrow A$ such that $\alpha(g) = 1$ for all but finitely many $g \in G$. Map G into $\text{Aut}(\bar{A})$ by $\alpha^h(g) = \alpha(gh^{-1})$, for $g, h \in G, \alpha \in \bar{A}$. The resulting semi-direct product $W = \bar{A} \rtimes G$ is called the (standard restricted) wreath product of A by G , written $W = A \wr G$. The subgroup $\bar{A}$ is called the base group. If $\alpha \in \bar{A}$, we shall usually write $\alpha = \sum_{i=1}^n a_i^{g_i}$ to mean that $\alpha(g_i) = a_i$, $1 \leq i \leq n$, and $\alpha(g) = 1$ if $g \notin \{g_1, \dots, g_n\}$. In this notation, if $g \in G$, then $\alpha^g = \sum_{i=1}^n a_i^{g_i g}$. If $B \triangleleft A$, then $\frac{A}{B} \wr G$ is a homomorphic image of $A \wr G$ in the obvious way; the kernel of the homomorphism is $\bar{B} = B^G = \bigcup \{B_g \mid g \in G\}$. More information about the structure of W may be found in P. Neumann's paper [13].

Hartley [9] has determined which wreath products are residually nilpotent. Motivated by this, we now turn to the question of which wreath products are residually central. In the sequel, W will always denote the wreath product $A \wr G$, and $\bar{A}$ will denote the base group. Note that A and

G can be embedded in W . Since subgroups of residually central groups are residually central, A and G must be residually central if W is to be.

Proposition 3.1. Suppose that $W = A \wr G$ is residually central. If G is infinite, then A is a Z -group.

Proof: Let $a_1, \dots, a_n \in A$, $K = \langle a_1, \dots, a_n \rangle$. By 1.10, it suffices to show that $K \not\leq [K, A]$. Let $g_1, \dots, g_n$ be distinct elements of G , and let $\alpha = \sum_{i=1}^n a_i^{g_i} \in \bar{A}$. Since W is residually central, $\alpha \notin [\alpha, W] \geq [\alpha, \bar{A}] = \sum_{i=1}^n [a_i, A]^{g_i}$ as a direct sum. Fix $g \in G$, and let $b_i \in [a_i, A]$, $1 \leq i \leq n$.

Then $b_i^{g_i} \in [a_i, A]^{g_i} \leq [\alpha, W] \triangleleft W$; thus $b_i^g = (b_i^{g_i})^{g_i^{-1}g} \in [\alpha, W]$. Hence $(\prod_{i=1}^n [a_i, A])^g = [K, A]^g \leq [\alpha, W]$ by 1.11. Since g was arbitrary, $\sum \{[K, A]^g : g \in G\} \leq [\alpha, W]$. If $K \leq [K, A]$, then $a_i \in [K, A]$, $1 \leq i \leq n$, and $\alpha = \sum_{i=1}^n a_i^{g_i} \in \sum_{i=1}^n [K, A]^{g_i} \leq [\alpha, W]$, a contradiction.

A group G is ordered if there is a total order $<$ on G such that if $a < b$ in G and $c, d \in G$, then $cad < cbd$; that is, the order on G is preserved by right and left multiplication. A group on which it is possible to impose such an order is called orderable. Every orderable group G must be torsion-free: If $1 \neq g \in G$, then either $g > 1$ or $g^{-1} > 1$. An easy induction then shows that $g^n > 1$ or $(g^{-1})^n > 1$ for every positive integer n . It is also clear that subgroups of orderable groups are orderable. For further

information, the reader may consult [11]. The following facts are proved there.

Proposition 3.2.[11, pp. 4,5] The cartesian product of orderable groups is an orderable group, and any group which is residually an orderable group is orderable.

Proposition 3.3.[11, p. 10] A group is orderable if and only if every finitely generated subgroup is orderable; that is, orderability is a local property.

Proposition 3.4.[11, p. 16] A locally nilpotent torsion-free group is orderable.

Proposition 3.5.[11, p. 17] All free groups are orderable.

If g is an element of the ordered group G , let $|g| = \max\{g, g^{-1}\}$. Then g is said to be infinitely small relative to h , written $g \ll h$, if $|g|^n < |h|$ for all positive integers n . A subset K of G is convex if for all $g \in G$, $h \in K$, $|g| < |h|$ implies that $g \in K$. The next result is in [11, p. 14,15].

Proposition 3.6. Let G be an ordered group, $g, h \in G$. Then

- (1) $g \ll h$ if and only if there is a convex subgroup containing g but not h .
- (2) $|[g, h]| \ll \max\{|g|, |h|\}$.
- (3) If G is nilpotent, then $|[g, h]| \ll |g|$.



Lemma 3.7. If G is a Z -group with central series $\{G_\alpha | \alpha \in I, I \text{ a totally ordered set}\}$, then G is orderable if G/G_α is torsion-free, for all $\alpha \in I$.

Proof: This is really a corollary of Theorem 2.2.3 in [11, p. 16]. Each $G_\alpha \triangleleft G$, and if $\alpha, \beta \in I$ such that $G_\alpha < G_\beta$ and there is no $\gamma \in I$ for which $G_\alpha < G_\gamma < G_\beta$, then $[G_\beta, G] \leq G_\alpha$, and G_β/G_α is a torsion-free abelian group, and hence is orderable, by 3.4. Since $G_\beta/G_\alpha \leq \zeta_1(G/G_\alpha)$, elements of G_β/G_α are fixed under conjugation by elements of G/G_α . Thus $\{G_\alpha | \alpha \in I\}$ meets the conditions of Theorem 2.2.3 of [11].

Lemma 3.8. Let A and G be residually central groups. Then $W = A \wr G$ is residually central if for all $1 \neq \alpha \in \bar{A}$, $\alpha \notin [\alpha, G][\alpha, \bar{A}]^G$.

Proof: Since W is the semi-direct product $\bar{A}G$, any element of W can be written uniquely in the form αg , where $\alpha \in \bar{A}$ and $g \in G$. If $g \neq 1$, then $\alpha g \notin \bar{A}[g, G]$, since $g \notin [g, G]$. Now $[\alpha g, W] \leq [\alpha, W][g, W] \leq \bar{A}[g, \bar{A}G] \leq \bar{A}[g, G][g, A]^G \leq \bar{A}[g, G]$. Thus $\alpha g \notin [\alpha g, W]$.

If $g = 1$, then $[\alpha, W] = [\alpha, \bar{A}G] \leq [\alpha, G][\alpha, \bar{A}]^G$. Thus W is residually central if $\alpha \notin [\alpha, G][\alpha, \bar{A}]^G$.

Theorem 3.9. If G is a residually central ordered group, and A is a Z -group, then $W = A \wr G$ is residually central.

Proof: Let $\alpha = \sum_{i=1}^m a_i^{g_i} \in \bar{A}$, where $g_i \in G$, $a_i \in A$, $1 \leq i \leq m$. By 3.8, it is enough to assume that $\alpha \in [\alpha, G][\alpha, \bar{A}]^G$

and reach a contradiction. Let $L = [\langle a_1, \dots, a_m \rangle, A]$. Since A is a Z -group, some $a_i \notin L$, by 1.10. If $\bar{L} = \sum\{L^g : g \in G\} \triangleleft W$, then $\alpha \notin \bar{L}$; however, $\alpha \in \zeta_1(\bar{A}/\bar{L})$, by definition of L and $\bar{L}$. If $A_1 = A/L$, then $\alpha \in [\alpha, W]$ implies that $\alpha\bar{L} \in [\alpha\bar{L}, W]$ in $A_1 \text{ wr } G$. Because $\alpha\bar{L} \in \zeta_1(\bar{A}_1)$, a characteristic subgroup of $\bar{A}$, $[\alpha\bar{L}, G] \leq \zeta_1(\bar{A}_1)$. Let $A_2 = \zeta_1(\bar{A}_1)$; then $W_2 = A_2 \text{ wr } G$ is not residually central; we may assume that the base group is abelian.

Again, let $\alpha \in \bar{A}$, where A is abelian. View $\bar{A}$ as ZG -module. Then $\alpha \in [\alpha, G]$ means, in module notation that $\alpha \in \alpha\Delta$, where Δ is the augmentation ideal of ZG . Then there is an element $\delta \in \Delta$ such that $\alpha\delta = \alpha$. α and δ may be expressed in module notation as

$$\alpha = \sum_{i=1}^m a_i g_i$$

$$\delta = \sum_{j=1}^n z_j h_j, \text{ where } a_i \in A, g_i \in G, 1 \leq i \leq m,$$

$z_j \in Z$, $\sum_{j=1}^n z_j = 0$, $h_j \in G$, $1 \leq j \leq n$, and $g_1 < g_2 < \dots < g_m$ and $h_1 < h_2 < \dots < h_n$ in the ordered group G . Now,

$$\alpha\delta = \sum_{i=1}^m \sum_{j=1}^n (z_j a_i) g_i h_j = (z_1 a_1) g_1 h_1 + \dots + (z_n a_m) g_m h_n.$$

Either $h_n > 1$ or $h_1 < 1$, or both. (Note that $n \geq 2$, since $\sum_{j=1}^n z_j = 0$.) We treat the case where $h_n > 1$; the case $h_1 < 1$ is almost identical.

First, suppose that some a_i has infinite order. Let $g_{i_0} = \max\{g_i \mid a_i \text{ has infinite order}\}$. Then $g_{i_0} h_n > g_{i_0}$ in G .

Since $\alpha = \alpha\delta$, the element in the $g_{i_0} h_n^{\text{th}}$ "slot" of α , say b , must be equal to $\sum_{g_i h_j = g_{i_0} h_n} z_j a_i g_i h_j$. By definition, b has finite order. However, $z_n a_{i_0}$ has infinite order, and no other summand of b has infinite order. Thus b has infinite order, a contradiction.

Next, suppose that $A_3 = \langle a_1, \dots, a_m \rangle$ is a finite p -group, for some prime p . If for some j_0 , $z_{j_0} a_i = 0$, $1 \leq i \leq m$, let $\delta' = \sum_{j \neq j_0} z_j h_j$. Then $\alpha_{\delta'} = \alpha\delta = \alpha$; we may therefore assume that for each j there is an i such that $z_j a_i \neq 0$. Since $g_m > g_i$ if $i < m$ and $h_n > h_j$ if $j < n$, $g_i h_j < g_m h_n$ if $i < m$ or $j < n$. Thus the $g_m h_n^{\text{th}}$ position in $\alpha\delta$ is $z_n a_m$. Because $g_m h_n > g_i$, $1 \leq i \leq m$, and $\alpha = \alpha\delta$, $z_n a_m = 0$. For some i , $z_n a_i \neq 0$; not all the elements of $A_3 = \langle a_1, \dots, a_m \rangle$ can have the same order, and hence A_3 has exponent greater than p . Suppose the exponent of A_3 is p^k . Let $A_4 = A_3 / p^{k-1} A_3$, and let ϕ be the natural map. $0 \neq \phi(\alpha)$, and

$$\phi(\alpha) = \phi(\alpha\delta) = \phi(\alpha)\delta.$$

Since A_4 has exponent p , this is impossible.

Finally, suppose that A_3 is a finite abelian group. Then there are primes $p_1, \dots, p_r$ such that $A_3 = B_1 \oplus \dots \oplus B_r$, where B_k is a p_k -group, $1 \leq k \leq r$. As before, let $A_5 = A_3 / (B_2 \oplus \dots \oplus B_r)$ with ψ the natural map. Then

$$0 \neq \psi(\alpha) = \psi(\alpha\delta) = \psi(\alpha)\delta$$

in the p_1 -group A_5 , which is impossible.

This shows that if G is a residually central orderable group, then W is residually central if and only if A is a Z-group. Free groups are orderable (3.5) and are residually nilpotent [18, Theorem 9.11], and hence are Z-groups. Thus the wreath product of two free groups is residually central.

Lemma 3.10. Suppose that $W = A \wr G$ is residually central, and G has an element g of prime order p . Then every element of A and of G of finite order has p -power order.

Proof: Suppose $a \in A$ has prime order $q \neq p$. Identify a with the element of the base group $\bar{A}$ defined by $a(1_G) = a$ and $a(h) = 1_A$ if $h \neq 1_G$. By 1.9, a and g commute, which is impossible.

Suppose $h \in G$ has prime order $q \neq p$. By 1.9, h and g commute, so that $\langle h, g \rangle$ is cyclic of order pq . Let $1 \neq a \in A$. $W_1 = \langle a \rangle \wr \langle g, h \rangle$ is a finitely generated, metabelian, residually central group. By 2.7, W_1 is residually nilpotent. However, Hartley [9] has shown that this is impossible.

Theorem 3.11. Suppose that A and G are locally nilpotent. Then $W = A \wr G$ is residually central if and only if either

- (1) G is torsion-free, or
- (2) All elements of G finite order have p -power order, where p is a prime, and all elements of A of finite order also have p -power order.

Proof: The necessity of (1) or (2) is clear from Lemma 3.10.

If (1) holds, then by 3.4, G is orderable, and Theorem 3.9 applies.

Suppose (2) holds. Since residual centrality is a local property, it suffices to consider a finitely generated subgroup $\langle w_1, \dots, w_m \rangle$ of W . Each $w_i = \alpha_i g_i$, where $\alpha_i \in \bar{A}$ and $g_i \in G$. Each $\alpha_i = \sum_{j=1}^n a_{ij}^{g_{ij}}$. Hence $\langle w_1, \dots, w_m \rangle \leq \langle a_{ij}, g_i, g_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n \rangle \leq \langle a_{ij} \rangle \text{ wr } \langle g_i, g_{ij} \rangle$. Thus we may assume that A and G are finitely generated and therefore are nilpotent. By [2, Theorem 2.1], A can be embedded in $P_A \oplus T_A$, and G can be embedded in $P_G \oplus T_G$, where P_A, P_G are finite p -groups, and T_A, T_G are torsion-free finitely generated nilpotent groups.

By Lemma 3.8, it suffices to show that if $\alpha = \sum_{k=1}^l a_k^{g_k} \in \bar{A}$, then $\alpha \notin [\alpha, G][\alpha, \bar{A}]^G$. Suppose that there is an α such that $\alpha \in [\alpha, G][\alpha, \bar{A}]^G$. Since A is nilpotent, there is an integer r such that each $a_i \in \zeta_r(A)$ and some $a_i \notin \zeta_{r-1}(A)$. Then

$$\begin{aligned} [\alpha, \bar{A}]^G &\leq [\langle a_1, \dots, a_l \rangle, A]^G \\ &\leq [\zeta_r(A), A]^G \\ &\leq \zeta_{r-1}(A)^G. \end{aligned}$$

$W_1 = (A/\zeta_{r-1}(A)) \text{ wr } G$ is a homomorphic image of W in the obvious way. Let $\bar{\alpha}$ denote the image of α in W_1 . Because $\alpha \in [\alpha, G][\alpha, \bar{A}]^G$, $\bar{\alpha} \in [\bar{\alpha}, G][\bar{\alpha}, \overline{A/\zeta_{r-1}(A)}]^G = [\bar{\alpha}, G]$ in W_1 . Let



$A_1 = \zeta_r(A)/\zeta_{r-1}(A)$. Then $A_1 \text{ wr } G$ is a subgroup of W_1 which contains $\bar{\alpha}$. Also, $[\bar{\alpha}, G] \leq \bar{A}_1$, since $\bar{\alpha} \in \zeta_1(A/\zeta_{r-1}(A))$, a characteristic subgroup of $A/\zeta_{r-1}(A)$. By [2, Corollary 2.11], every element of A_1 of finite order has p -power order. By [2, Theorem 2.2], A_1 and G are residually finite p -groups, and hence are residually nilpotent p -groups of finite exponent. Since $\bar{\alpha} \in [\bar{\alpha}, G]$, $A_1 \text{ wr } G$ is not residually central, and therefore not residually nilpotent. However, Hartley [9] has shown that $A_1 \text{ wr } G$ is residually nilpotent, a contradiction.

Corollary 3.12. If A is abelian and G is locally nilpotent, then $W = A \text{ wr } G$ is residually central if and only if W is locally a residually nilpotent group.

Proof: The sufficiency of the condition is clear. Theorem 3.11 and Theorems B1 and B2 of [9] combine to prove the necessity.

Thus we have succeeded in classifying those restricted wreath products $W = A \text{ wr } G$ which are residually central in the case where G is orderable and in the case where A and G are locally nilpotent. In addition to this, Hartley's paper [9] gives conditions for W to be residually nilpotent; his conditions clearly are sufficient conditions for W to be residually central. Necessary conditions are that A must be a Z -group if G is infinite, G can have at most one relevant prime, and if G has an element of prime order p , then every element of A and of G of finite order has p -power order.

To expand our results to the case where A is not locally nilpotent appears to be difficult. In order to use



group ring techniques, as was done above, and as Hartley did in [9], it is necessary to work within an abelian "slice" of A and to know something about the orders of elements of that slice. This is much more difficult if A is not locally nilpotent. Thus it seems likely that different techniques will be required to expand 3.11 significantly.

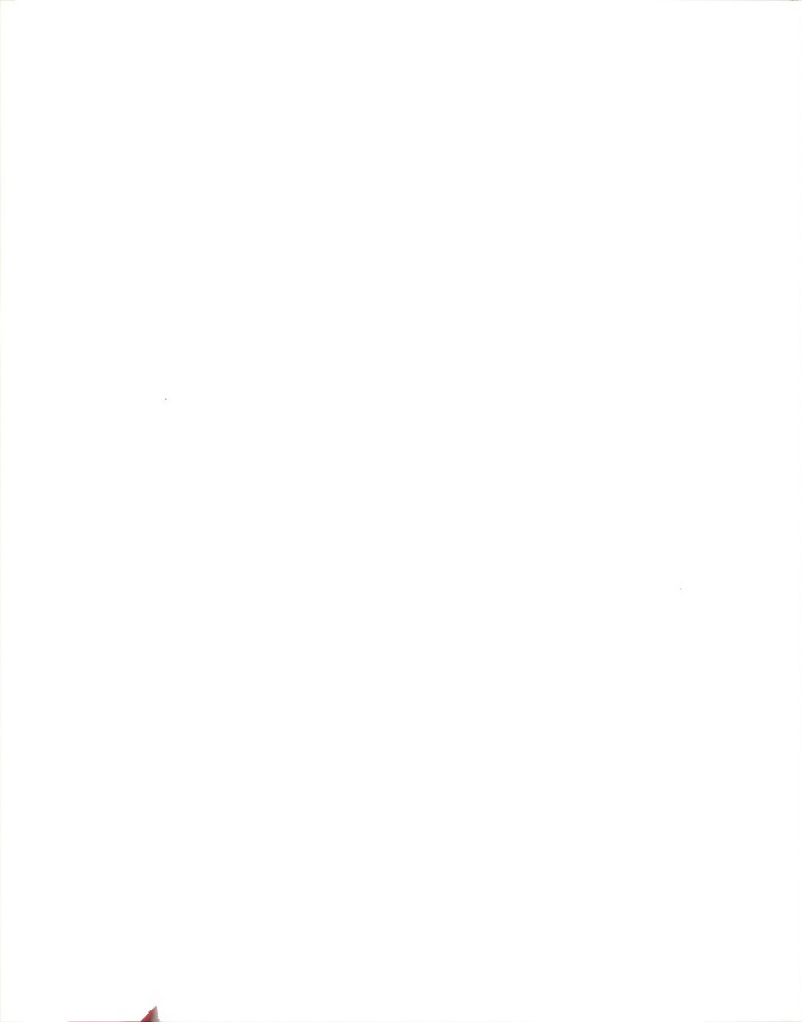
If the usual base group $\Sigma\{A_g | g \in G\}$ is replaced by $\pi\{A_g | g \in G\}$, the resulting group is called the unrestricted wreath product, denoted by $A \text{Wr} G$. Because this is a much "larger" group, one would expect that far more restrictive conditions would be necessary to make $A \text{Wr} G$ residually central. That this is indeed the case is illustrated by the following result.

Proposition 3.13. Let $W = A \text{Wr} G$. If G contains an element g of infinite order, and A contains an element a_1 of finite order or a 2-divisible subgroup, then W is not residually central.

Proof: Suppose A contains a 2-divisible subgroup A_1 . It is enough to show that $A_1 \text{Wr} \langle g \rangle$ is not residually central.

Let $1 \neq a \in A_1$. For each positive integer i , there is an element b_i such that $b_i^{2^i} = a$ and $b_i = b_{i+1}^2$. Define $\alpha \in \pi\{A_{1_g^n} | n \in \mathbb{Z}\}$ by

$$\alpha(g^i) = \begin{cases} a^{2^i} & \text{if } i \geq 0 \\ b_i & \text{if } i < 0. \end{cases}$$



$$\begin{aligned}
\text{Then } [\alpha, g^{-1}](g^i) &= (\alpha^{-1} \alpha g^{-1})(g^i) \\
&= (\alpha(g^i))^{-1} \alpha(g^{i+1}) \\
&= (\alpha(g^i))^{-1} \alpha(g^{i+1}) \\
&= (a^{2^i})^{-1} a^{2^{i+1}} \quad \text{if we denote}
\end{aligned}$$

$$\begin{aligned}
b_i \text{ by } a^{2^{-1}} \\
&= (a^{2^i})^{-1} a^{2^i} a^{2^i} \\
&= a^{2^i} \\
&= \alpha(g^i).
\end{aligned}$$

Hence $[\alpha, g^{-1}] = \alpha$, and W is not residually central.

Suppose A has an element a_1 of finite order. If the order of a_1 is odd, then A contains a subgroup A_1 isomorphic to a cyclic group odd prime order. Such a group is 2-divisible, and the above argument applies.

Suppose A contains an element a_1 of order two. Again, it suffices to show that $\langle a_1 \rangle \text{Wr} \langle g \rangle$ is not residually central. Define α in the base group by

$$\alpha(g^i) = \begin{cases} a & \text{if 3 does not divide } i \\ 1 & \text{if } i \text{ is divisible by 3.} \end{cases}$$

$$\begin{aligned}
\text{Then } [\alpha, g^{-1}](g^i) &= (\alpha(g^i))^{-1} \alpha(g^{i+1}) \\
&= \alpha(g^i) \alpha(g^{i+1}), \text{ since } a = a^{-1} \\
&= \begin{cases} a & \text{if } i \text{ or } i+1 \text{ is divisible by 3} \\ 1 & \text{otherwise.} \end{cases}
\end{aligned}$$



Similarly, $[\alpha, g^{-2}](g^i) = \alpha(g^i)\alpha(g^{i+2})$

$$= \begin{cases} a & \text{if } i \text{ or } i+2 \text{ is divisible by } 3 \\ 1 & \text{otherwise.} \end{cases}$$

Then $([\alpha, g^{-1}][\alpha, g^{[2]}](g^i) = \begin{cases} 1 & \text{if } i \text{ is divisible by } 3 \\ a & \text{otherwise.} \end{cases}$

Therefore $[\alpha, g^{-1}][\alpha, g^{-2}] = \alpha$, $\alpha \in [\alpha, G]$, and w is not residually central.

CHAPTER IV

A SPECIALIZATION OF RESIDUAL CENTRALITY

Recall that, for $g \in G$, $R_0(g) = g^G$ and $R_{n+1}(g) = [R_n(g), G]$ for non-negative integers n . If for each element g of a group G , $\bigcap_{n=0}^{\infty} R_n(g) = 1$, then G is said to be a $(*)$ -group. Because each $R_n(g) \leq \gamma_{n+1}(G)$, every residually nilpotent group is a $(*)$ -group. Note also that if $g \in [g, G]$, then $R_1(g) = [g, G] = [g, G, G] = R_2(g)$. Thus if $\bigcap_{n=0}^{\infty} R_n(g) = 1$, then $[g, G] > [g, G, G]$, and therefore every $(*)$ -group is residually central. By 1.7, property $(*)$ is equivalent to nilpotence for finite groups.

Proposition 4.1. The class of $(*)$ -groups is closed under the taking of subgroups and Cartesian products; hence a residually $(*)$ -group is a $(*)$ -group.

Proof: Let H be a subgroup of G , and let $h \in H$. Since for each n $[h, {}_n G] \geq [h, {}_n H]$, $1 = \bigcap_{n=1}^{\infty} [h, {}_n G] \geq \bigcap_{n=1}^{\infty} [h, {}_n H]$.

Let $\{G_i \mid i \in I\}$ be a collection of $(*)$ -group, and $g = (g_i)_{i \in I} \in \prod_{i \in I} G_i$. Then $[g, \prod_{i \in I} G_i] \leq \prod_{i \in I} [g_i, G_i]$, and for each n , $R_n(g) \leq \prod_{i \in I} R_n(g_i)$.

Since $\bigcap_{n=0}^{\infty} R_n(g_i) = 1_{G_i}$ for each $i \in I$, $\bigcap_{n=0}^{\infty} R_n(g) = 1$.



Suppose a group G is residually a $(*)$ -group.

Then for each $g \in G$, there is a normal subgroup H_g of G such that $g \notin H_g$ and H_g is a $(*)$ -group. G can be embedded in the Cartesian product $\prod \{G/H_g : g \in G\}$ by the map $g \mapsto (gH_g)_{g \in G}$. The Cartesian product of the $(*)$ -groups $\{G/H_g : g \in G\}$ is a $(*)$ -group, and so G is a $(*)$ -group.

Free groups are residually nilpotent [18, p. 117] and thus are $(*)$ -groups. Since every group is a homomorphic image of a free group, and, e.g., finite non-nilpotent groups are not $(*)$ -groups, a homomorphic image of a $(*)$ -group need not be a $(*)$ -group. The symmetric group on three symbols is metabelian but not a $(*)$ -group; this shows that a $(*)$ -group extended by a $(*)$ -group need not be a $(*)$ -group.

The following lemma is a fairly well-known extension of the three-subgroup Lemma 1.3.

Lemma 4.2. Let H and K be subgroups of a group G such that $[H, {}_n K] < G$ for all positive integers n . Then for any such n , $[\gamma_n(K), H] \leq [H, {}_n K]$.

Proof: Induct on n . The case $n = 1$ is clear. Suppose $n \geq 2$. Then

$$\begin{aligned} [H, {}_n K] &= [[H, {}_{(n-1)} K], K] \geq [[\gamma_{(n-1)}(K), H], K] \text{ by} \\ &\text{induction} \\ &= [\gamma_{(n-1)}(K), H, K]. \end{aligned}$$

Similarly,



$$[H, {}_n K] = [[H, K], (n-1)K] \geq [\gamma_{(n-1)}(K), [H, K]] \text{ by}$$

induction

$$= [H, K, \gamma_{(n-1)}(K)].$$

By the three subgroup Lemma 1.3,

$$[H, {}_n K] \geq [K, \gamma_{(n-1)}(K), H] = [\gamma_n(K), H].$$

We begin by exploring the relationships between (*)-groups and some other classes of groups.

Proposition 4.3. A group G is hypercentral with the upper central series having length at most ω if and only if for each $1 \neq g \in G$, there is an integer n_g such that $R_{n_g}(g) = 1$. In particular, such a group is a (*)-group.

Proof: Necessity. Suppose that $G = \zeta_\omega(G) = \bigcup_{n=1}^{\infty} \zeta_n(G)$, and let $1 \neq g \in G$. Let n_g be the least integer such that $g \in \zeta_{n_g}(G)$. For each positive integer n , $[\zeta_n(G), G] \leq \zeta_{n-1}(G)$, and so

$$R_{n_g}(g) = [g, {}_{n_g} G] \leq [\zeta_{n_g}(G), {}_{n_g} G] \leq \zeta_0(G) = 1.$$

Sufficiency. Suppose that for all $g \in G$, there is an integer n_g such that $R_{n_g}(g) = 1$. Since $\zeta_{n_g}(G) = \{x \in G : [x, {}_{n_g} G] = 1\}$, $g \in \zeta_{n_g}(G)$. Thus $G = \bigcup_{n=1}^{\infty} \zeta_n(G)$.

However, a group can be hypercentral of length $\omega + 1$ and yet contain an element g such that $R_\alpha(g) \neq 1$ for all ordinals α .



Example 4.1. Let P be a 2^ω -group, and let $a \in \text{Aut}(P)$ be defined by $p^a = p^{-1}$, where $p \in P$. Let $G = P] \langle a \rangle \simeq \text{Dih}(Z_{(2^\omega)})$. Let $p \in P$. $[a, p] = (p^{-1})^a p = p^2$. Because P is divisible, $P^2 = P$; i.e., every element of P is the square of some other element of P . Hence $[a, P] = P^2 = P$, and $R_\alpha(a) = P$ for every ordinal $\alpha \geq 1$.

To see that G has ZA-length $\omega + 1$, view P as the abelian group generated by $\{p_i : i = 1, 2, 3, \dots\}$, where $p_1^2 = 1$ and $p_{i+1}^2 = p_i$ for $i \geq 1$. Suppose that $p_i a^j \in \zeta_1(G)$, where $i \geq 1$, $j = 0$ or 1 . $(p_i a^j)^a = p_i^{-1} a^j = p_i a^j$ if and only if $i = 1$, for only $p_1 = p_1^{-1}$. Also, $p_2 p_1^a = p_2^a = p_2^{-1}$. Thus $\zeta_1(G) = \{1, p_1\}$. Note that $G/\zeta_1(G) \simeq G$, and so $\zeta_2(G) = \langle p_2 \rangle$. Similarly, $\zeta_n(G) = \langle p_n \rangle$ for each integer n . Then

$$\zeta_\omega(G) = \bigcup_{n=1}^{\infty} \zeta_n(G) = \bigcup_{n=1}^{\infty} \langle p_n \rangle = P.$$

Since G/P is abelian, $\zeta_{\omega+1}(G) = G$.

Proposition 4.4. A $(*)$ -group is a ZD-group whose lower central series has length at most $\omega + 1$.

Proof: Let G be a $(*)$ -group, and let $g \in G$. By Lemma 3.2, $\bigcap_{n=1}^{\infty} [g, \gamma_n(G)] \leq \bigcap_{n=1}^{\infty} R_n(g) = 1$. Thus

$$[g, \gamma_\omega(G)] = [g, \bigcap_{n=1}^{\infty} \gamma_n(G)] \leq \bigcap_{n=1}^{\infty} R_n(g) = 1.$$

Thus $\gamma_\omega(G)$ centralizes every $g \in G$. Then $\gamma_{\omega+1}(G) = [\gamma_\omega(G), G] \leq [\zeta_1(G), G] = 1$.



Combining this with 4.3 gives a result proved by Hartley [8].

Corollary 4.5. A ZA-group of length at most ω is also a ZD-group of length at most $\omega + 1$.

Corollary 4.6. If G is a $(*)$ -group, and $\zeta_1(G) = 1$, then G is residually nilpotent.

Corollary 4.7. A $(*)$ -group satisfying the minimal condition on normal subgroups is nilpotent.

Corollary 4.8. A group G is residually nilpotent if and only if G is a $(*)$ -group and for all $1 \neq g \in \zeta_1(G)$, there is a normal subgroup M_g of G such that $g \notin M_g$ and G/M_g is nilpotent.

Proof: The necessity of the condition is clear. Let $1 \neq g \in G$. To show that G is residually nilpotent, it is necessary to find a normal subgroup M_g of G such that $g \notin M_g$ and G/M_g is nilpotent. If $g \in \zeta_1(G)$, this is true by hypothesis. Suppose $g \notin \zeta_1(G)$. By 4.4, $\bigcap_{n=1}^{\infty} \gamma_n(G) \leq \zeta_1(G)$, and so there is an integer n such that $g \notin \gamma_n(G)$. Since $G/\gamma_n(G)$ is nilpotent, the result follows.

An obvious question now is whether property $(*)$ is equivalent to residual nilpotence (ZD-length at most ω) or to having ZD-length at most $\omega + 1$. The next two examples show that it is equivalent to neither.



Example 4.2. Let $D_n = \langle a_n, b_n : a_n^2 = 1, b_n^{2^{n-1}} = 1, b_n a_n = a_n b_n^{-1} \rangle$, the dihedral group of order 2^n . It is easy to check that $\zeta_1(D_n) = \langle b_n^{2^{n-2}} \rangle$ has order two and that $D_n / \zeta_1(D_n) \simeq D_{n-1}$ if $n \geq 3$. Since the nilpotency class of D_3 is 2 (i.e., $\zeta_2(D_3) = D_3$ and $\zeta_{2-1}(D_3) \neq D_3$), the nilpotency class of D_n is $n-1$ if $n \geq 3$.

Let $G = \sum_{n=3}^{\infty} D_n$, a group which has ZA-length ω and ZD-length ω . Let $\phi_{m,n} : \zeta_1(D_m) \rightarrow \zeta_1(D_n)$ by $\phi_{mn}(b_m^{2^{m-2}}) = b_n^{2^{n-2}}$, for $m, n \geq 3$. Let $H = \langle z^{-1} \phi_{mn}(z) \mid z \in \zeta_1(D_m); m, n \geq 3 \rangle$, and set $\Gamma = G/H$, called the central product of the D_n 's.

Let $L = \zeta_1(D_3)$. If $m \neq n$, then

$$HD_n/H \cap HD_m/H = LH/H \simeq L/(H \cap L) = L.$$

Thus $\zeta_1(\Gamma) = LH/H$ has order two.

Since G has ZA-length ω , Γ has ZA-length $\leq \omega$ and hence is a (*)-group. To show that Γ is not residually nilpotent, the following well-known lemma is needed.

Lemma 4.9. If Γ is a ZA-group, then every non-trivial normal subgroup of Γ intersects $\zeta_1(\Gamma)$ non-trivially.

Proof: Suppose $1 \neq N \triangleleft \Gamma$ and $N \cap \zeta_1(\Gamma) = 1$. Since Γ is a ZA-group, there is a least ordinal α such that $N \cap \zeta_\alpha(\Gamma) > 1$. Clearly, α cannot be a limit ordinal. Then α has a predecessor β , and $N \cap \zeta_\beta(\Gamma) = 1$. Since $N \cap \zeta_1(\Gamma) = 1$, $1 \neq [N \cap \zeta_\alpha(\Gamma), \Gamma] \leq N \cap [\zeta_\alpha(\Gamma), \Gamma] \leq N \cap \zeta_\beta(\Gamma)$, a contradiction.

Since $\zeta_1(\Gamma)$ has order two, every nontrivial normal subgroup of Γ contains $\zeta_1(\Gamma)$. Since Γ contains copies of each D_n , Γ cannot have finite nilpotency class; no $\gamma_n(\Gamma)$ can be the identity. Thus for each n , $\zeta_1(\Gamma) \leq \gamma_n(\Gamma)$, and

$$1 \neq \zeta_1(\Gamma) \leq \bigcap_{n=1}^{\infty} \gamma_n(\Gamma).$$

Gluskov [6] has constructed a very similar class of examples. He takes the central product of upper $n \times n$ unitriangular matrices over a finite field K . For $n \geq 2$, the center of the group M_n of upper unitriangular $n \times n$ matrices turns out to be

$$\left\{ \begin{bmatrix} 1 & 0 & \dots & 0 & a \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \\ \vdots & & & & 0 \\ 0 & \dots & \dots & \dots & 1 \end{bmatrix} \cdot a \in K \right\},$$

which is isomorphic to the additive group K . Each M_n is a nilpotent group, and so the central product has ZA-length $\leq \omega$.

Let p be the characteristic of K , and p^m the order of K . If the additive group K is generated by k , then every non-trivial subgroup of K must contain $\langle k^{p^{m-1}} \rangle$, which is cyclic of order p . If $\mathcal{M}$ is the central product of the M_n 's, by 3.8 every non-trivial normal subgroup of $\mathcal{M}$ intersects $\zeta_1(\mathcal{M}) \simeq K$ non-trivially, and therefore must contain



$$\left\langle \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & k^{p^{m-1}} \\ 0 & 1 & 0 & \dots & 0 & \\ \vdots & & & & & \vdots \\ 0 & \dots & \dots & 0 & 1 & \end{bmatrix} \right\rangle.$$

This subgroup is contained by $\gamma_n(\mathcal{M})$ for $n \geq 1$, and is therefore contained in $\bigcap_{n=1}^{\infty} \gamma_n(\mathcal{M})$. Thus $\mathcal{M}$ is not residually nilpotent.

The next example shows that property (*) is not equivalent to having ZD-length at most $\omega + 1$.

Example 4.3. Let H be the abelian group with presentation $\langle x, y_1, y_2, \dots : x^2 = 1, y_i^{2^i} = x \rangle$. (For a note on the existence of such a group see [5, p. 118 and Theorem 36.1, p. 121].) Define $a \in \text{Aut}(H)$ by $h^a = h^{-1}$, $h \in H$, and set $G = H \langle a \rangle$. $\gamma_2(G) = [H \langle a \rangle, H \langle a \rangle] = [H, \langle a \rangle]$. If $h \in H$, then $[h, a] = h^{-1}h^a = h^{-2}$, and thus $\gamma_2(G) = H^2$. Now suppose that $\gamma_i(G) = H^{2^{i-1}}$. Then

$$\gamma_{i+1}(G) = [H^{2^{i-1}}, H \langle a \rangle] = [H^{2^{i-1}}, \langle a \rangle] = (H^{2^{i-1}})^2 = H^{2^i}.$$

Therefore

$$\gamma_{\omega}(G) = \bigcap_{i=2}^{\infty} \gamma_i(G) = \bigcap_{i=2}^{\infty} H^{2^{i-1}} = \langle x \rangle,$$

and

$$\gamma_{\omega+1}(G) = [\gamma_{\omega}(G), H \langle a \rangle] = [\langle x \rangle, H \langle a \rangle] = [\langle x \rangle, \langle a \rangle] = 1.$$

However, $\gamma_2(G) = [H, \langle a \rangle] = R_1(a)$, and so



$$\gamma_{i+1}(G) = R_i(a) \quad \text{for } i \geq 1.$$

Thus $\bigcap_{n=1}^{\infty} R_n(a) = \gamma_{\omega}(G) = \langle x \rangle \neq 1$; G is not a $(*)$ -group even though G is a ZD-group of length $\omega + 1$.

All $(*)$ -groups have ZD-length at most $\omega + 1$. The interesting question now is when $(*)$ -groups have ZD-length ω ; that is, which $(*)$ -groups are residually nilpotent. We first investigate (restricted) wreath products.

Lemma 4.10. Let $W = A \wr G$, where G is infinite. If W has property $(*)$, then W is residually nilpotent.

Proof: By 4.6 it is sufficient to show that $\zeta_1(W) = 1$. However, this is already well-known; see [13, p. 34].

Now it suffices to consider $W = A \wr G$ for finite groups G . If W is to be a $(*)$ -group, both A and G must be $(*)$ -groups, because both A and G are embedded in W . Then the finite group G must be nilpotent, since G is a ZD-group. If W is to be a $(*)$ -group, W must be residually central; Lemma 3.10 now shows that G must be a p -group for some prime p . Since for some finite c , $\gamma_c(G) = 1$, $\gamma_c(W)$ is contained in the base group $\bar{A}$; thus W is residually nilpotent if and only if $\bigcap_{n=1}^{\infty} [\bar{A}, {}_n W] = 1$. If A is abelian, this reduces to the condition that $\bigcap_{n=1}^{\infty} [\bar{A}, {}_n G] = 1$. As in the discussion of residually central wreath products, $\bar{A}$ will be viewed as a module of the integral group ring $\mathbb{Z}G$ in this case.

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Let $g \in G$ have order p , and suppose that A is abelian. Let $W^* = A \text{ wr } \langle g \rangle$, and denote its base group by $\overline{A^*}$. If $n \geq 1$, then

$$[\overline{A^*}, {}_n \langle g \rangle] = [\overline{A^*}, g], {}_{(n-1)} \langle g \rangle] \leq R_n(g).$$

If W is a $(*)$ -group, so is W^* , and thus

$$\bigcap_{n=1}^{\infty} [\overline{A^*}, {}_n \langle g \rangle] \leq \bigcap_{n=1}^{\infty} R_n(g) = 1,$$

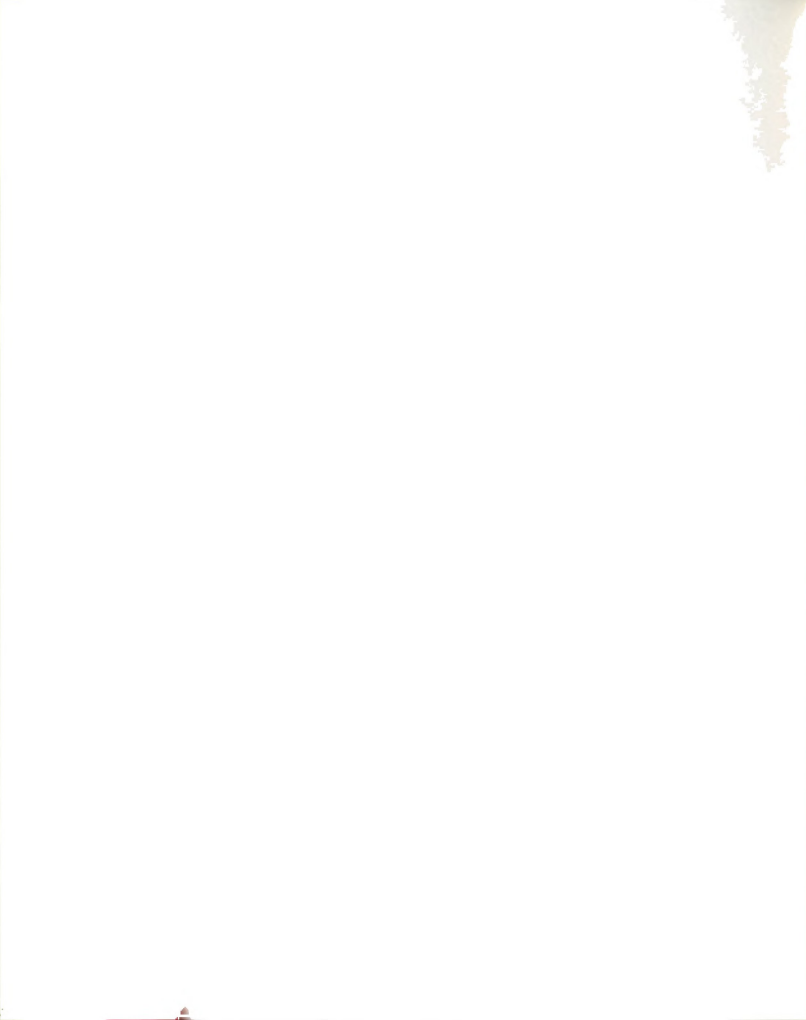
and W^* is residually nilpotent. By Lemma 8 of [9], A is residually a p -group of finite exponent. By [9, Theorem B1], W is residually nilpotent. We have proved

Lemma 4.11. If A is abelian and G is finite, then $W = A \text{ wr } G$ is a $(*)$ -group if and only if W is residually nilpotent. In this case, there is a prime p such that G is a p -group, and A is residually a p -group of finite exponent.

Lemma 4.12. Suppose $W = A \text{ wr } G$ is a $(*)$ -group, where G is finite. Then for some prime p , G is a p -group, and A is residually a nilpotent p -group of finite exponent.

Since by Theorem B1 of [9] such a group is residually nilpotent, this lemma and Lemma 4.10 complete the proof of

Theorem 4.13. The standard wreath product $W = A \text{ wr } G$ of non-trivial groups has property $(*)$ if and only if W is residually nilpotent.



Proof of 4.12. Suppose that W is a $(*)$ -group, and let $1 \neq a \in A$. Then $\langle a \rangle \text{wr} G$ is also a $(*)$ -group. By 4.11 G is a p -group, and if a has finite order, then the order of a is a power of p . Let $g \in G$ have order p , and let Δ be the augmentation ideal of the group ring $\mathbb{Z}\langle g \rangle$. What follows is an adaptation of Lemma 8 of [9].

Claim. If n is a positive integer, then there is an element r_n of $\mathbb{Z}\langle g \rangle$ such that

$$p^n(1-g) = r_n(1-g)^{n(p-1)+1} \quad (1).$$

Proof of claim: By [9, Lemma 6], $p(1-g) \in \Delta^p$. For $n = 1$, then, there is an element $r_1 \in \mathbb{Z}\langle g \rangle$ such that $p(1-g) = r_1(1-g)^p$, since Δ^p is generated by $(1-g)^p$. Now suppose that (1) holds for some $n \geq 1$.

$$\begin{aligned} p^{n+1}(1-g) &= p(p^n(1-g)) = p \cdot r_n(1-g)^{n(p-1)+1} \\ &= r_n(1-g)^{n(p-1)} p(1-g) \\ &= r_n(1-g)^{n(p-1)} r_1(1-g)^p \quad \text{by the case } n=1 \\ &= (r_n r_1)(1-g)^{n(p-1)+p} \\ &= (r_n r_1)(1-g)^{(n+1)(p-1)+1}, \quad \text{as desired.} \end{aligned}$$

Identify A with the subgroup of the base group given by $\{f: G \rightarrow A \mid f(g) = 1_A \text{ if } g \neq 1_G\}$. Write $\langle a \rangle$ additively, and view the base group of $\langle a \rangle \text{wr} \langle g \rangle$ as a $\mathbb{Z}\langle g \rangle$ -module. $[a^{p^n}, g]$ may be written in module notation as $p^n a - (p^n a)g = a \cdot p^n(1-g)$. By (1), there is an element $r_n \in \mathbb{Z}\langle g \rangle$ such that $ap^n(1-g) = ar_n(1-g)^{n(p-1)+1} \in a\Delta^{n(p-1)+1}$. Thus



$$\begin{aligned}
[a^{p^n}, g] &\in [a, \gamma_{n(p-1)+1}(g)] \\
&\leq [\langle g \rangle, A, \gamma_{n(p-1)}(g)] \\
&\leq R_{n(p-1)+1}(g) \quad \text{in } W. \quad (2).
\end{aligned}$$

Define $A^m = \langle a^m : a \in A \rangle$ for positive integers m . Let

$A_n = A^{p^n} (A \cap \gamma_{n(p-1)+1}(W))$. Then

$$\begin{aligned}
[A_n, g] &\leq [A^{p^n}, g][A \cap \gamma_{n(p-1)+1}(W), g] \\
&\leq R_{n(p-1)+1}(g)[\gamma_{n(p-1)+1}(W), g] \quad \text{by (2)} \\
&\leq R_{n(p-1)+1}(g) \quad \text{by 4.2. Thus}
\end{aligned}$$

$$[\bigcap_{n=1}^{\infty} A_n, g] \leq \bigcap_{n=1}^{\infty} [A_n, g] \leq \bigcap_{n=1}^{\infty} R_{n(p-1)+1}(g) = 1$$

since W is a $(*)$ -group. However, for any $1 \neq b \in A$,

$[b, g] \neq 1$. Thus $[\bigcap_{n=1}^{\infty} A_n, g] = 1$ implies that $\bigcap_{n=1}^{\infty} A_n = 1$.

Since A_n contains the subgroup generated by all $p^{n\text{th}}$ powers of elements of A , A/A_n has exponent dividing p^n .

Because A_n contains $\gamma_{n(p-1)+1}(A)$, each A/A_n is also nilpotent. Finally, if $1 \neq b \in A$, there is an integer n such that $b \notin A_n$, since $\bigcap_{n=1}^{\infty} A_n = 1$, and A/A_n is a nilpotent p -group of finite exponent. Thus the lemma is established.

Lemma 4.14. Suppose that the group G has a normal subgroup B such that $G/B = \langle aB \rangle$ for some $a \in G$. Then for each positive integer n , $\gamma_n(G) = \gamma_n(B)R_{n-1}(a)$.



Proof: Induct on n . For $n = 1$,

$$\gamma_1(B)R_0(a) = Ba^G = G = \gamma_1(G).$$

Suppose that the lemma is true for some $n \geq 1$. Then

$$\begin{aligned}\gamma_{n+1}(G) &= [\gamma_n(G), G] \\ &= [\gamma_n(B)R_{(n-1)}(a), G] \\ &\leq [\gamma_n(B), G][R_{(n-1)}(a), G] \\ &= [\gamma_n(B), B\langle a \rangle]R_n(a) \\ &\cong [\gamma_n(B), \langle a \rangle][\gamma_n(B), B]^{\langle a \rangle}R_n(a) \\ &\leq \gamma_{n+1}(B)R_n(a) \quad \text{by 4.2.}\end{aligned}$$

The opposite inclusion is trivial.

Proposition 4.15. A $(*)$ -group G which is nilpotent by cyclic is residually nilpotent.

Proof: There is a subgroup $B \triangleleft G$ such that $G/B = \langle aG \rangle$ for some $a \in G$, and $\gamma_c(B) = 1$ for some integer c . For $n \geq c$, by Lemma 4.14.

$$\gamma_n(G) = \gamma_n(B)R_{n-1}(a) = R_{n-1}(a).$$

Thus $\bigcap_{n=1}^{\infty} \gamma_n(G) = \bigcap_{n=1}^{\infty} R_{n-1}(a) = 1$.

Proposition 4.16. A $(*)$ -group G which is cyclic by nilpotent is residually nilpotent.

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Proof: There is an element $a \in G$ such that $\langle a \rangle \triangleleft G$ and $G/\langle a \rangle$ is nilpotent. Thus for some integer c , $\gamma_c(G) \leq \langle a \rangle$, and so for $n \geq 0$, $\gamma_{c+n}(G) \leq [\langle a \rangle, {}_n G] = R_n(a)$.
$$\bigcap_{n=0}^{\infty} \gamma_{c+n}(G) \leq \bigcap_{n=0}^{\infty} R_n(a) = 1.$$
 Note that if $\langle a \rangle$ is finite, then G is nilpotent.

Proposition 4.17. Suppose that G has a normal subgroup B such that G/B is nilpotent, and B satisfies the minimal condition on normal subgroups. If G is a ZD-group, then G is nilpotent.

Proof: For some integer c , $\gamma_c(G) \leq B$. The subgroups $\gamma_c(G) \geq \gamma_{c+1}(G) \geq \dots$ form a decreasing chain in B . Since B satisfies Min- n , there is an integer n such that $\gamma_{c+n}(G) = \gamma_{c+n+1}(G)$. Since G is a ZD-group, $\gamma_{c+n}(G) = 1$.

In example 4.2, $\zeta_1(\Gamma)$ is cyclic and $\Gamma/\zeta_1(\Gamma) \simeq \sum_{n=3}^{\infty} D_n$, a residually nilpotent group. Since Γ is not residually nilpotent, this shows that the hypotheses of the last two propositions cannot be weakened to read, " $G/\langle a \rangle$ of G/B is residually nilpotent".

Proposition 4.18. Let G be a $(*)$ -group with a normal subgroup H such that $\zeta_1(G) \cap H = 1$ and G/H is residually nilpotent. Then G is residually nilpotent.

Proof: Since G/H is residually nilpotent, $\gamma_w(G) = \bigcap_{n=1}^{\infty} \gamma_n(G) \leq H$. By 4.4, $1 = \gamma_{w+1}(G) = [\gamma_w(G), G]$. Thus $\gamma_w(G) \leq \zeta_1(G)$. Since $\zeta_1(G) \cap H = 1$, $\gamma_w(G) = 1$.

CHAPTER V
DESCENDANCE AND HOMOMORPHIC IMAGES

Robinson in [16] has given a construction which can easily be adapted to yield information about the factors $R_n(g)/R_{n+1}(g)$. Let N be a normal subgroup of G , and let $a \in [N, (n-1), G]$ and $b \in G$. Let

$$\bar{a} = a[N, {}_n G] \in [N, (n-1)G]/[N, {}_n G]$$

$$b^* = b\gamma_2(G) \in G/\gamma_2(G).$$

We first show that the function given by

$$(\bar{a}, b^*) \mapsto [a, b][N, (n-1)G] \in [N, {}_n G]/[N, (n+1)G]$$

is well-defined. Let $r \in [N, {}_n G]$, $s \in \gamma_2(G)$.

$$\begin{aligned} (\bar{a}, b^*) &= (\overline{ar}, (bs)^*) \mapsto [ar, bs][N, (n+1)G] \\ &= [ar, s][ar, b]^s[N, (n+1)G] \\ &= [a, s]^r[r, s][a, b]^{rs}[r, b]^s[N, (n+1)G]. \end{aligned}$$

Because $r \in [N, {}_n G]$, $[r, s] \in [N, (n+1)G]$ and $[r, b]^s \in [N, (n+1)G]$. Also,

$$[a, s]^r \in [[N, (n-1)G], \gamma_2(G)] \leq [N, (n+1)G]$$

by 4.2. Finally, $[a, b] \in [N, {}_n G]$ implies that $[a, b][N, (n+1)G]$



is in the center of $G/[N, (n+1)G]$. Therefore

$$(\overline{a}r, (bs)^*) \mapsto [a, b][N, (n+1)G].$$

Next, we show that this function is bilinear. Let $a_1, a_2 \in [N, (n-1)G]$, and $b \in G$.

$$\begin{aligned} (\overline{a_1 a_2}, b^*) &\mapsto [a_1 a_2, b][N, (n+1)G] \\ &= [a_1, b][a_1, b, a_2][a_2, b][N, (n+1)G] \\ &= [a_1, b][a_2, b][N, (n+1)G]. \end{aligned}$$

Now let $a \in [N, (n-1)G]$ and $b_1 b_2 \in G$.

$$\begin{aligned} (\overline{a}, (b_1 b_2)^*) &\mapsto [a, b_1 b_2][N, (n+1)G] \\ &= [a, b_2][a, b_1][a, b_1, b_2][N, (n+1)G] \\ &= [a, b_1][a, b_2][N, (n+1)G], \end{aligned}$$

since $[N, {}_n G]/[N, (n+1)G]$ is abelian. Setting $N = g^G$ gives

Theorem 5.1. Let $g \in G$, n a positive integer. Then there is an onto homomorphism

$$\theta_n : \frac{R_{n-1}(g)}{R_n(g)} \otimes \frac{G}{[G, G]} \rightarrow \frac{R_n(g)}{R_{n+1}(g)}.$$

Proof: The preceding work shows the existence of the homomorphism θ_n . Because each $R_n(g)$ is generated by the elements $[a, b]$, where $a \in R_{n-1}(g)$ and $b \in G$, the image of θ_n contains all the generators of $R_n(g)/R_{n+1}(g)$. Thus θ_n is onto.



Corollary 5.2. (a) If for some positive integer m , $g^m = 1$ or $G/[G, G]$ has exponent m , then $R_n(g)/R_{n+1}(g)$ is an abelian group of exponent dividing m , for $n = 0, 1, 2, \dots$.

(b) If G is finitely generated and g^G is finitely generated, then $R_n(g)/R_{n+1}(g)$ is finitely generated, for $n = 0, 1, 2, \dots$. If in addition g has finite order m and $\bigcap_{n=0}^{\infty} R_n(g) = 1$, then $g^G = R_0(g)$ is residually a finite group of order dividing a power of m .

(c) If $\bigcap_{n=0}^{\infty} R_n(g) = 1$ and g^G and G are finitely generated, then g^G is residually finite.

Proof: (c) is the only part requiring comment. Let $h \in g^G$. There is an integer k such that $h \notin R_k(g)$. Each factor $R_n(g)/R_{n+1}(g)$ is a finitely generated abelian group. Thus there exist subgroups $H_0 \triangleleft H_1 \triangleleft \dots \triangleleft H_\ell$ of g^G such that $H_0 = R_k(g)$, $H_\ell = g^G$, and each factor H_{i+1}/H_i is cyclic, that is, $g^G/R_k(g)$ is polycyclic. Such groups are residually finite [2, Corollary 1.21]. Hence there is a normal subgroup $K/R_k(g)$ of $g^G/R_k(g)$ not containing $hR_k(g)$ and such that $[g^G/R_k(g) : K/R_k(g)]$ is finite. Then $g \notin K$ and g^G/K is finite.

Theorem 5.3. Let G be a group, n a positive integer, and $\{N_\alpha : \alpha \in I\}$ a collection of normal subgroups of G such that $\bigcap \{N_\alpha : \alpha \in I\} = 1$.

(a) If H is a subgroup maximal with respect to $H \triangleleft G$ and $H^{(n)} = 1$, where $H^{(n)}$ denotes the n^{th} derived subgroup of H , then $\bigcap \{HN_\alpha : \alpha \in I\} = H$.

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(b) If H is maximal with respect to $H \triangleleft G$ and $\gamma_{n+1}(H) = 1$, then $\cap\{HN_\alpha : \alpha \in I\} = H$.

(c) If $H = C_G(K)$ for some subset K of G , then $\cap\{HN_\alpha : \alpha \in I\} = H$.

Proof: In each case, let $R = \cap\{HN_\alpha : \alpha \in I\}$.

(a) For each $\alpha \in I$, $R^{(n)} \leq (HN_\alpha)^{(n)}$. For $0 < k \leq n$, $(HN_\alpha)^{(k)}$ is the product of normal subgroups $H^{(k)}$, $N_\alpha^{(k)}$, and commutators involving both $H^{(i)}$'s and $N_\alpha^{(j)}$'s, where $0 \leq i, j \leq k-1$. Any such commutator is contained in the normal subgroup N_α . Thus $R^{(n)} \leq (HN_\alpha)^{(n)} \leq H^{(n)}N_\alpha = N_\alpha$ for each $\alpha \in I$. Therefore $R^{(n)} \leq \cap\{N_\alpha : \alpha \in I\} = 1$. R is a normal subgroup containing H ; by the maximality of H , $R = H$.

(b) An argument similar to that in (a) shows that for each α ,

$$\gamma_{n+1}(R) \leq \gamma_{n+1}(HN_\alpha) \leq \gamma_{n+1}(H)N_\alpha = N_\alpha.$$

Again $\gamma_{n+1}(R) \leq \cap\{N_\alpha : \alpha \in I\} = 1$. By the maximality of H , $R = H$.

(c) Let $x \in R = \cap\{C_G(K)N_\alpha : \alpha \in I\}$. For each $\alpha \in I$, $x = c_\alpha n_\alpha$, where $c_\alpha \in C_G(K)$ and $n_\alpha \in N_\alpha$. Let $k \in K$. For any $\alpha \in I$,

$$\begin{aligned} [x, k] &= [c_\alpha n_\alpha, k] = [c_\alpha, k]^{n_\alpha} [n_\alpha, k] \\ &= [n_\alpha, k] \in N_\alpha. \end{aligned}$$

Thus $[x, k] \in \cap\{N_\alpha : \alpha \in I\} = 1$, so that $x \in C_G(K)$.

Corollary 5.4. Let H be a normal subgroup as in (a), (b), or (c) of 5.3, and let $\mathfrak{x}$ be one of the following classes

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of groups: residually nilpotent groups, (*)-groups, residually solvable groups, residually finite groups. If G is an $\mathfrak{X}$ -group, then G/H is an $\mathfrak{X}$ -group.

Proposition 5.5. Let H be a subgroup of G such that H satisfies the minimal condition on subgroups normal in H . If $\{N_i | i = 1, 2, 3, \dots\}$ is a nested sequence of normal subgroups of G such that $\bigcap_{i=1}^{\infty} N_i = 1$, then $\bigcap_{i=1}^{\infty} HN_i = H$.

Proof: Because H satisfies Min-n, and $H \cap N_1 \geq H \cap N_2 \geq \dots$, there is an integer K such that

$$H \cap N_K = H \cap N_{K+1} = \dots = \bigcap_{i=1}^{\infty} (H \cap N_i) \leq \bigcap_{i=1}^{\infty} N_i = 1.$$

Let $x \in \bigcap_{i=1}^{\infty} HN_i$. For each integer i , x can be written $x = h_i n_i$, where $h_i \in H$ and $n_i \in N_i$. Let $j > k$. Then $h_k n_k = x = h_j n_j$, implying that $h_j^{-1} h_k = n_j n_k^{-1} \in H \cap N_k = 1$. Thus if $j > k$, $n_j = n_k$; $n_k \in N_j$ for all $j > k$. But then $n_k \in \bigcap_{i=1}^{\infty} N_i = 1$. Thus $x = h_k \in H$.

Corollary 5.6. Let H be a normal subgroup of G such that H satisfies Min-n, and let γ be one of the following classes of groups: residually nilpotent groups, (*)-groups, residually solvable groups, countable residually finite groups. If G is a γ -group, then G/H is a γ -group.

Corollary 5.4 can be improved in the case that $\mathfrak{X}$ is the class of (*)-groups.

Proposition 5.7. Let G be a (*)-group, K any subset of G . Then $\bigcap_{n=1}^{\infty} C_G(K) \gamma_n(G) = C_G(K)$.

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Proof: Let $k \in K$, $x \in \bigcap_{n=1}^{\infty} C_G(K) \gamma_n(G)$. For each n , x can be written as $x = c_n g_n$, where $c_n \in C_G(K)$ and $g_n \in \gamma_n(G)$. Furthermore,

$$[x, k] = [c_n g_n, k] = [g_n, k] \in [\gamma_n(G), k]$$

$$\leq R_n(k) \text{ by 4.2.}$$

Hence $[x, k] \in \bigcap_{n=1}^{\infty} R_n(k) = 1$, forcing x to centralize k .

Corollary 5.8. Let G be a $(*)$ -group. For any positive integer n , $G/\zeta_n(G)$ is residually nilpotent.

Proof: Set $K = G$ in 5.7 and induct on n .

Corollary 5.9. Let G be a $(*)$ -group, n a positive integer. Then G/H is residually nilpotent if any one of the following conditions holds:

- (a) H is maximal with respect to $H \triangleleft G$ and $H^{(n)} = 1$.
- (b) H is maximal with respect to $H \triangleleft G$ and $\gamma_{n+1}(H) = 1$.
- (c) $H = C_G(K)$, where $K \triangleleft G$.

Proof: (c) follows immediately from 5.7. For (a) and (b), note that $\zeta_1(G)$ will be contained in any such maximal subgroup H . Thus G/H is a homomorphic image of the residually nilpotent group $G/\zeta_1(G)$. Now 5.4 applies.

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Petty [14] has shown that corollary 5.8 cannot be extended to $G/\zeta_\alpha(G)$ for limit ordinals α . Let G be as defined in example 4.3, and let $\bar{\zeta}(G)$ denote the hypercenter of G . He shows that $G/\bar{\zeta}(G)$ is not a ZD-group, although it is a Z-group.

Beyond what is presented in chapter one, the only information about homomorphic images of residually central groups seems to be Stanley's result [23, Corollary 8] that if G is a residually central group satisfying the minimal condition on normal subgroups, then every homomorphic image of G is residually central.

Let H be a subgroup of a group G , and $\{\beta: \beta \leq \alpha_0\}$ a set of ordinal numbers. If there is a collection $\{K_\beta: \beta \leq \alpha_0\}$ of subgroups of G such that $K_0 = G$, $K_{\alpha_0} = H$, $K_{\beta+1} \triangleleft K_\beta$ for $\beta + 1 \leq \alpha_0$, and $K_\alpha = \cap \{K_\beta: \beta < \alpha\}$ for limit ordinals α , then H is said to be descendant in G with order type α_0 . For example, if α_0 is finite, then the definition is equivalent to saying that H is a subnormal subgroup of G .

Concerning descendance in $(*)$ -groups, we have

Theorem 5.10. Let G be a $(*)$ -group.

- (1) If A is an abelian subgroup of G , then A is descendant of order type at most $\omega + 1$.
- (2) If $g \in G$ has finite order, then $\langle g \rangle$ is descendant of order type at most ω .
- (3) If $H \leq \zeta_\omega(G)$, then H is descendant of order type at most $\omega + 1$.

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Proof: (1) Consider the descending series

$$G = \gamma_1(G)C_G(A) \geq \gamma_2(G)C_G(A) \geq \dots$$

It is routine to verify that each $\gamma_{i+1}(G)C_G(A) \triangleleft \gamma_i(G)C_G(A)$, and by Theorem 5.3(c), $\bigcap_{i=1}^{\infty} \gamma_i(G)C_G(A) = C_G(A)$. Since A is abelian, $A \leq C_G(A)$, and the result follows.

(2) The series

$$G \geq g^G = \langle g \rangle R_1(g) \geq \langle g \rangle R_2(g) \geq \dots$$

is a normal series, and $\bigcap_{n=1}^{\infty} \langle g \rangle R_n(g) = \langle g \rangle$ by Proposition 5.5.

(3) Consider the series

$$G = \gamma_1(G)H \geq \gamma_2(G)H \geq \dots \bigcap_{n=1}^{\infty} \gamma_n(G)H \geq H.$$

As in (1), each $\gamma_{n+1}(G)H < \gamma_n(G)H$. It remains to show that $h \triangleleft \bigcap_{n=1}^{\infty} \gamma_n(G)H$. Let $h \in H$. Since $H \leq \bigcup_{i=1}^{\infty} \zeta_i(G)$, $h \in \zeta_m(G)$ for some integer m . Thus $R_m(h) = 1$. Let $y \in \bigcap_{n=1}^{\infty} \gamma_n(G)H$. Then y can be written as $y = g'h'$, where $g' \in \gamma_n(G)$ and $h \in H$. Thus

$$h^y = h[h, y] = h[h, g'h'] = h[h, h'] [h, g'] [[h, g'], h'].$$

Now $[h, g'] \in [h, \gamma_m(G)] \leq R_m(h) = 1$ by 4.2. Hence $h^y = h[h, h'] \in H$ as desired.

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