MATCHINGS AND COVERINGS FOR GRAPHS

Thesis for the Degree of Ph. D.

MICHIGAN STATE UNIVERSITY

DANIEL HUANG-CHAO MENG

1974

This is to certify that the

thesis entitled

MATCHINGS AND COVERINGS
FOR GRAPHS

presented by

Daniel Huang-Chao Meng

has been accepted towards fulfillment
of the requirements for

Ph.D degree in Mathematics

E. A. Nordhaus
Major professor

Date May 15, 1974

O-7639

ABSTRACT

MATCHINGS AND COVERINGS FOR GRAPHS

By

Daniel Huang-Chau Beck

In graph theory, an extensive amount of research has been devoted to the study of maximum matchings, namely of sets of independent edges or independent vertices which have the maximum cardinality possible. Similarly much attention has been given to covering properties, that is to sets of edges which cover all the vertices of a graph, or to sets of vertices which cover all the edges of a graph and in which the sets have the minimum possible cardinality.

In this thesis the basic notions of matchings and coverings are extended to maximal matchings and minimal coverings, and the interrelations between matchings and coverings are investigated. A well known theorem of Gallai, which relates minimum covers and maximum matchings is extended in various ways.

In particular a ratio called the edge Gallai ratio and one called the vertex Gallai ratio are introduced, and facilitate the study of matchings and coverings. [illegible]

ABSTRACT

MATCHINGS AND COVERINGS FOR GRAPHS

By

Daniel Huang-Chao Meng

In graph theory, an extensive amount of research has been devoted to the study of maximum matchings, namely of sets of independent edges or independent vertices which have the maximum cardinality possible. Similarly much attention has been given to covering properties, that is to sets of edges which cover all the vertices of a graph, or to sets of vertices which cover all the edges of a graph and in which the sets have the minimum possible cardinality.

In this thesis the basic notions of matchings and coverings are extended to maximal matchings and minimal coverings, and the interrelations between matchings and coverings are investigated. A well known theorem of Gallai, which relates minimum covers and maximum matchings is extended in various ways.

In particular a ratio called the edge Gallai ratio and one called the vertex Gallai ratio are introduced, and facilitate the study of matchings and coverings. Precise

Daniel Huang-Chao Meng

upper and lower bounds are obtained for these ratios.

A final section is devoted to generalizations of the concepts of edge dominating numbers and vertex dominating numbers.

There are essentially twelve principal graphical parameters discussed in this thesis, and a useful table is provided listing their values for numerous well known graphs and classes of graphs.

MATCHINGS AND COVERINGS FOR GRAPHS

By

Daniel Huang-Chao Meng

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1974

DEDICATION

To

My Wife

ACKNOWLEDGMENTS

I wish to express my deepest gratitude to Professor E. A. Nordhaus for including me to his "academic family". Special thanks must go to him for his many hours of patient guidance, unfailing interests and timely advice in the development and completion of my program.

My gratitude also goes to Professor B. Grünbaum for suggesting the topic of this thesis and for his many useful comments during its preparation.

I would also like to thank Professors L. M. Kelly, B. M. Stewart, W. E. Kuan and M. J. Winter for serving on my advisory committee. Finally, my thanks go to the Mathematics Department of Michigan State University for their financial support during the writing of this thesis.

TABLE OF CONTENTS

LIST OF FIGURES

Figure		Page

CHAPTER 1

Section 1.1
INTRODUCTION

In graph theory certain concepts have been emphasized and extensively studied. Examples which readily come to mind are the genus of a graph, which might more appropriately be called the minimum genus, the edge matching number associated with maximum matchings, and the edge covering number associated with minimum edge covers. Recently it was found fruitful to consider the concept of the maximum genus of a graph [16].

As Professor Branko Grünbaum [13] has remarked, "People are too frequently preoccupied with maximum matchings and minimum coverings. It is certain that a non-discriminatory approach should lead to a bevy of new results on matchings and coverings and in many other areas of graph theory." This thesis has been motivated by such considerations, and we investigate certain general properties of matchings and coverings and find that it is possible to provide meaningful definitions for minimum matchings and maximum coverings. Some results on minimum matchings have already been noted by Grünbaum. We extend these results to arbitrary matchings and coverings.

In many areas of graph theory, one could see if useful and meaningful results could be obtained by a change in the point of view, such as replacing the word "maximum" by "minimum" or vice-versa. This certainly opens a new door to the interested researcher. Until recently, general matchings and coverings have not been carefully examined, since it appears that almost the entire emphasis has been placed on maximum matchings and minimum coverings.

Definitions of terms as well as some of the notation employed in this thesis are presented in Section 1.2. A survey of known results related to the material of this thesis is presented in Chapter 2.

The first section of Chapter 3 discusses edge matchings, and a condition for a matching to be a minimum. The next section determines non-trivial lower bounds for the number of edges in a minimum matching in terms of the maximum degree $\Delta(G)$ of G. These bounds differ when the minimum degree $\delta(G) \geq 1$ and when $\delta(G) \geq 2$. Section 3.3 deals with the enumeration of minimum matchings for n-th subdivision graphs. Precise formulas are obtained for the number of edges in such matchings.

In Chapter 4, a discussion is made of edge coverings. Certain general properties of maximum coverings are developed, and in Section 4.2 an upper bound for the number of edges in a maximum cover is given in terms of a degree sequence. In Section 4.3 a sufficient condition for a covering to be a maximum is stated and proved.

Chapter 5 deals with the inter-relationship between matchings and coverings. In Section 5.1 certain graphs of Gallai type relative to matchings or coverings are characterized. In Section 5.2 we define an edge Gallai ratio for a graph G and obtain sharp upper and lower bounds for this ratio.

In Chapter 6, we extend some of the ideas developed for edge matchings and edge coverings to independent sets of vertices and to vertex coverings. In Section 6.1 an extension of Gallai's theorem is obtained. This result is that for any graph G of order p, $\alpha_{0U}(G) + \beta_{0L}(G) = p$. In contrast with the case of edge matchings and edge coverings, where all values between the minimum and the maximum values of the parameters are assumed, for vertex coverings and independent set of vertices "gaps" may occur in the parameter values. In Section 6.2 we define a vertex Gallai ratio for a graph G and obtain upper and lower bounds for this ratio.

In Chapter 7, dominating numbers are discussed, and edge dominating sets and vertex dominating sets are introduced. The relation between minimal edge dominating sets and edge matchings and also between minimal vertex dominating sets and independent sets of vertices are studied. Finally, a table is provided giving the values of all of parameters considered in this thesis for certain well known special graphs and special classes of graphs.

Finally, a bibliography lists references which have been useful in the preparation of this thesis.

Section 1.2
BASIC TERMINOLOGY

In this section we present some of the basic definitions and notations which are used in the following chapters. For additional graph theory terminology not explicitly given in this thesis, one may refer to standard texts such as Behzad and Chartrand [4], Berge [3], or Harary [14].

A <u>graph</u> G is a non-empty set V together with a set E of two-element subsets of V. The set V is referred to as the <u>vertex</u> (or point) <u>set</u> of G, and each element of V is called a vertex (or point). The set E is referred to as the <u>edge</u> (or line) <u>set</u> of G. The members of the edge set E are called edges. In general, the vertex set and edge set of a graph G will be denoted by V(G) and E(G) respectively. The graph G is called finite if V(G) and E(G) are both finite. In this thesis all graphs, unless otherwise noted, are assumed to be finite, undirected, and without loops or multiple edges and ordinarily having no isolated vertices.

The <u>order</u> of a graph G, denoted by V(G) or more simply by $|G|$, is the number of elements in V(G). If $|V(G)| = p$ and $|E(G)| = q$, we say that G is a (p,q) graph. A graph G is called <u>empty</u> if E(G) is the empty set. The <u>degree</u> (or valency) of a point of G is the number of edges of G incident with v and is denoted $\deg_G v$, or simply by deg v. In particular, $\delta(G)$ and $\Delta(G)$ are repeatedly used to denote respectively the minimum and the maximum degree of the vertices of G. A graph H is a subgraph of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph <u>induced</u> by a set U of vertices of G, denoted by <U>, is that graph which has U as its vertex set and whose edge set consists of all edges of G which join two vertices of U. Similarly, if F is a non empty subset of E(G), then the subgraph <F> induced by F is the graph whose vertex set consists of those vertices of G incident with at least one edge of F and whose edge set is F. If v is a vertex of G then G-v denotes the graph <V(G) - v>, and in general if S is a proper subset of V(G) then G - S represents the graph <V(G) - S>. Two vertices u and v of a graph G are said to be connected if there exists a u-v path in G, the graph G itself is connected if every two of its vertices are connected.

Given any $X \subseteq V(G)$, there is a largest subgraph H of G such that X = V(H), that is, for u, $v \in X$, $(u,v) \in E(H)$ if and only if $(u,v) \in E(G)$. We call this subgraph H the <u>restriction</u> G/X of G to the vertex set X. There are several special classes of graphs to which we will frequently make reference. A graph of order n which is a path or a cycle is denoted by P_n or C_n respectively, and the number of edges in a path or a cycle is called its <u>length</u>. An acyclic graph is a graph G with no cycles and is a <u>tree</u> if G is also connected. If G is disconnected and acyclic, G is called a <u>forest</u>. The complete graph K_p has every pair of its p vertices adjacent. A <u>bipartite</u> graph G is a graph whose vertex set V(G) can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G is of the form (v_1,v_2) where $v_i \in V_i$, $i = 1, 2$. If V_1 and V_2 have m and n points and G mn edges, we say that G is a <u>complete</u> bipartite graph and write $G = K(m,n)$. A star graph or claw is a complete bipartite graph K(1,n). For $n \geq 4$, the wheel W_n is defined to be the graph $K_1 + C_{n-1}$.

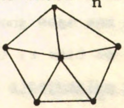


Figure 1.1. The Wheel W_6

A graph $S_n(G)$ referred to as the n-th subdivision graph of G is obtained by replacing every edge of G by a path of length n + 1. When n = 1, this graph is called the subdivision graph of G and is denoted by S(G)

A set M of edges of a graph G is called a matching set provided each vertex of G is incident with at most one edge contained in M. Thus M is a set of independent edges. A matching set M is called a matching of G provided there is no matching set of G which properly contains M.

If M is a matching for G, $|M|$ denotes the number of edges in M. The parameters $\beta_{1L}(G)$ and $\beta_{1U}(G)$ are used to denote respectively the minimum and maximum number of edges in any matching of G. If M is any fixed matching of G, an edge e in M is called a strong edge. Obviously two strong edges are never adjacent. If edge e is not in M, we refer to it as a weak edge. These concepts are of course dependent on the choice of the matching M. A vertex incident only to weak edges relative to M is said to be a weak vertex. A vertex incident with a strong edge and not incident with any weak edge is called a strong vertex (relative to M.) Finally, a vertex incident with a strong edge and also to at least one weak edge (relative to M) is said to be neutral.

Let X be any subset of E(G). An alternative path of (G,X) is one whose successive edges are alternatively in X and not in X. When an orientation of the vertices of a path is made, the first and last edges of a path are called its terminal edges. The terminal vertices of the path consist of the vertex incident to the first edge but not the second, and the vertex incident to the last edge but not to the preceding edge. An augmenting path (G,X) is an alternating path (G,X) whose terminal vertices are incident to no edge of X. If (G,X) has no augmenting path, X is called unaugmentable. The concept of an augmenting path has been used in characterizing maximum matchings [2].

A set C of edges of a graph G is an edge covering set of G provided each vertex of G is incident with at least one edge that belongs to C. An edge covering set C is called an edge covering of G or simply a covering, provided there is no edge covering set of G which is properly contained in C.

If C is a covering of G we denote by $|C|$ the number of edges in C, and by $\alpha_{0L}(G)$ and $\alpha_{0U}(G)$ the minimum and maximum number of edges respectively, in any covering of G. An alternative path of (G,C) is a reducing path if (1) its terminal edges are in C, (2) its terminal vertices are incident to edges of C which are not terminal edges of the path. If (G,C) possesses no reducing path, C is called an irreducible cover.

A graph G having the property that corresponding to an arbitrary edge covering C of G there exists at least one matching M such that $|C| + |M| = |V(G)|$ is called of Gallai type relative to coverings. Similarly, a graph is said to be of Gallai type relative to matchings if it has the property that for every edge matching M of G there is at least one edge covering C of G with $|M| + |C| = |V(G)|$.

CHAPTER 2

HISTORICAL SURVEY

In this chapter we summarize some of the most important known results bearing on matchings and coverings of graphs, and related results.

In 1957, C. Berge [2] used the technique of alternative paths to characterize a maximum matching. He proved that a matching M has maximum cardinality if and only if there exists no argumenting path in (G,M).

According to this theorem, if an edge matching M is given, one can decide if this matching is a maximum matching by searching for all alternating paths starting at a weak vertex. This method has been improved by Edmonds [8] and adapted to a computer search by Witzgall and Zahn [26]. They defined a vertex v to be an outer vertex, rooted at u, if u is a weak vertex which is joined to v by an alternating path of even length. (In particular, all weak vertices are regarded as outer.) The reason for considering outer vertices becomes evident if one examines an augmenting path connecting two weak vertices u_1 and u_2. Let v_1 and v_2 be the neighbors of u_1 and u_2 within the augmenting path.

Then v_2 is an outer vertex rooted at u_1 and v_1 is an outer vertex rooted at u_2. This leads to a reformulation of Berge's result, namely that a matching is maximum if and only if no weak vertex u is adjacent to an outer vertex which is rooted at a weak vertex different from u.

For the purpose of establishing maximality or non-maximality of a matching it is therefore sufficient to search for all outer vertices. This is an improvement over searching for all alternating paths, since there are in general more alternating paths emanating from weak vertices then there are outer vertices.

In 1957, Norman and Rabin [18] presented a method for finding in a graph G a minimum edge cover, employing the concept of a reducing path. They proved the theorem that an edge cover C has minimum cardinality if and only if there exists no reducing path in (G,C).

This theorem gave rise to an algorithm for finding a minimum cover. Norman and Rabin also show that the maximum matching problem and minimum edge cover problem are equivalent. This, of course serves the same purpose as the well known theorem of Gallai, which states that for any graph G of order p we have $\alpha_{0L} + \beta_{0U} = p$, and for any graph having no isolated vertices, $\alpha_{1L} + \beta_{1U} = p$.

J. Weinstein [25] in 1961 found a non-trivial lower bound for $\beta_{1U}(G)$ in terms of the maximum degree $\Delta(G)$ of G, and states two theorems which also involve the minimum degree $\delta(G)$.

(1) For graphs with $\delta \geq 1$, $p \leq (1 + \Delta) \cdot \beta_{1U}(G)$

(2) For graph with $\delta \geq 2$, $2p \leq (2 + \max(4, \Delta) \cdot \beta_{1U}(G))$

Using the technique of "alternating paths" Grünbaum [12] proved the following two intermediate theorems for matchings and coverings.

(1) For every graph G and every integer β_1 satisfying $\beta_{1L} \leq \beta_1 \leq \beta_{1U}$, there exists an edge matching M of G such that $|M| = \beta_1$.

(2) For every graph G and every integer α_1 with $\alpha_{1L} \leq \alpha_1 \leq \alpha_{1U}$ there exists an edge cover C of G such that $|C| = \alpha_1$.

These results are of interest since they show that no gaps are possible in these parameter values. We will show later that gaps can occur in the parameter values for vertex covers and maximal independent sets of vertices.

It is evident that $\beta_{1U} \leq [p/2]$ and that M is an edge matching of G such that $|M| = p/2$ if and only if M is a 1 - factor of G. Grünbaum [13] has also shown that $\beta_{1U} \leq 2\beta_{1L}$ and that there exists a j - connected graph G with arbitrary large order, such that $\beta_{1L}(G) = \lfloor \frac{j+1}{2} \rfloor$.

No general procedure or algorithm has so far been developed to determine the number of edges in an arbitrary matching or in an arbitrary covering. M. J. Stewart [24], has determine the number of edges in a maximum matching for the n-th subdivision graph $S_n(G)$. This number, $\beta_{1U}(S_n(G))$, depends on $q = |E(G)|$, on the parity of n, and sometimes also on the parameter $\beta_{1U}(G)$, namely,

(1) Let G be a connected (p,q) graph. Then

$$\beta_{1U}(S_{2K}(G)) = kq + \beta_{1U}(G).$$

(2) Let G be a connected (p,q) graph. Then

$$\beta_{1U}(S_{2K-1}(G)) = \begin{cases} Kq & \text{if } G \text{ is a tree.} \\ Kq + p - q & \text{otherwise.} \end{cases}$$

Let $K(p_1, p_2, \ldots, p_j)$ denote the complete j - partite graph with sets of vertices containing $p_1, p_2, \ldots, p_j$ elements the notation being such that $p_1 \leq p_2 \leq \ldots \leq p_j$, and let $p = \sum_{i=1}^{j} p_i$. Then we have

$$\beta_{1U}(K(p_1, p_2, \ldots, p_j)) = \min ([{}^{p}/_{2}], p-p_j)$$

and

$$\beta_{1L}(K(p_1, p_2, \ldots, p_j) = \max (p_{j-1}, \{{}^{(p-p_j)}/_{2}\})$$

The first result is due to Chartrand, Geller, and Hedetniemi [6], and the second to B. Grünbaum [13].

Let I_d denote the graph of the d-dimensional cube. Forcade [9] proved that $\beta_{1L}(I_d)/V(I_d)$ is a non-increasing function of d and that $\lim_{d \to \infty} \beta_{1L}(I_d)/V(I_d) = {}^{1}/_{3}$

Finally, let G be a connected (p,q) graph and let LG denote the line graph of G and TG the total graph of G. R. P. Gupta [11] proved the following formulas:

$$\beta_{0U}(LG) = \beta_{1U}(G)$$

$$\alpha_{0L}(LG) = q - \beta_{1U}(G)$$

$$\sigma_{0L}(LG) = \sigma_{1L}(G)$$

$$\beta_{0U}(TG) = p - \sigma_{1L}(G)$$

$$\alpha_{0L}(TG) = q + \sigma_{1L}(G)$$

$$\beta_{1U}(LG) = \lfloor {}^{q}/_{2} \rfloor$$

$$\alpha_{1L}(LG) = \left\{ {}^{q}/_{2} \right\}$$

$$\beta_{1U}(TG) = \left\lfloor \frac{p+q}{2} \right\rfloor$$

$$\alpha_{1L}(TG) = \frac{p+q}{2}$$

$$\alpha_{1L}(G) \leq \beta_{0U}(TG) \leq \lfloor {}^{3}/_{2} \cdot \alpha_{1L}(G) \rfloor$$

Many of these results could be extended to arbitrary matchings and coverings. Line graphs will be considered briefly in chapter 7 of this thesis.

CHAPTER 3

EDGE MATCHINGS

Section 3.1. In 1957, Berge [2] gave a necessary and sufficient condition for determining whether or not a given matching is a maximum, and provided an algorithm for constructing a matching with the maximum number of edges. However, to find a necessary and sufficient condition for a given matching to be a minimum appears to be a difficult question to answer. One reason for this is that a minimum matching is not hereditary, in the sense that if H is a subgraph of G, the inequality $\beta_{1L}(H) \leq \beta_{1L}(G)$ need not be valid. For example, consider the graphs G and H shown in the figure.

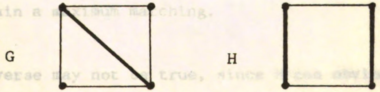


Figure 3.1. A minimum matching which is not hereditary. Here H is a subgraph of G, yet $\beta_{1L}(G) = 1$, and $\beta_{1L}(H) = 2$.

Throughout this chapter we will assume that the word "matching"refers to an edge matching. Furthermore, since an isolated vertex cannot be covered by any edge, we assume that the graphs considered have no isolated vertices.

In this section we prove a sufficient condition for a matching to be minimum. Let M be any matching of a graph G(p,q), and let W, S, and N denote the sets of weak vertices, strong vertices and neutral vertices of G, respectively, relative to M.

First we have the following elementary observation.

<u>Theorem 3.1</u>. If $|W| \leq 1$, then M is a maximum matching.

<u>Proof</u>: If $|W| = 0$, since $p = |N| + |W| + |S|$, then

$$p = |N| + |S|. \quad \text{Now} \quad |N| + |S| = 2|M|,$$

since every vertex incident with an edge in M is a strong or a neutral vertex. Thus $p = 2|M|$ and $|M| = {}^{p}/_{2}$, so M is a maximum matching.

If $|W| = 1$, then $p = 2|M| + 1$, and $|M| = {}^{(p-1)}/_{2} = \lfloor {}^{p}/_{2} \rfloor$. Thus M is again a maximum matching.

The converse may not be true, since M can obviously be a maximum matching even when $|W| \geq 2$. An example is a star graph S of order $p > 3$. Then $\beta_{1L}(S) = \beta_{1U}(S) = 1$ and S has p-2 weak vertices.

If P is a path, then $|P|$ denotes the length of this path, i.e. the number of edges in the path. We need the following three definitions.

Definition 3.1. Let G be a connected graph and M a matching of G. Then two edges e_1 and e_2 in M are said to be near to one another if there exist a path P in G containing e_1 and e_2 such that $|P| \leq 4$.

It is clear that for such a path P, either $|P| = 3$ or $|P| = 4$, since edges in M are disjoint.

Definition 3.2. If G is a connected graph and M is a matching of G, then the matching M is said to have property (P) if for any two near edges e_1, e_2 in M, a shortest path $P(e_1,e_2)$ containing e_1 and e_2 has exactly one weak vertex between e_1 and e_2.

Definition 3.3. If G is a disconnected graph, and if M is a matching of G, then M is said to have property (P) in G, provided the matching induced by M in each component of G has property (P). In the trivial case where $|M| = 1$, we agree that M has property (P).

Theorem 3.2. Let G be a connected graph and M a matching of G. If $|M| \geq 2$, then there exist at least one pair of near edges in M.

Proof: If there exist a path P containing two edges in M such that $|P| \leq 4$, then the theorem follows at once. Hence, we may assume that the length of every path in G containing two arbitrary edges of M is always ≥ 5. Let P be the shortest such path, say $P : v_1, v_2, \ldots, v_K$, where $K \geq 6$. Then we conclude that $v_3, v_4, \ldots, v_{K-2}$ are all weak vertices relative to M. If there were a neutral vertex, say v_i, $3 \leq i \leq K-2$, then by definition v_i is incident with a strong edge (v_i, u). The path $P' : u, v_i, v_{i+1}, \ldots, v_{K-1}, v_K$ contains two edges of M, but $|P'| < |P|$, contradicting the assumption that P is the shortest such path. But if $v_3, v_4, \ldots, v_{K-2}$, are all weak vertices where $K \geq 6$, then M is not a matching, a contradiction. Hence, we conclude there must exist at least one pair of near edges in M.

<u>Theorem 3.3</u>. Let G be a connected graph. A matching M has property (P) if and only if every shortest path $P(e_1, e_2)$ containing any two near edges e_1 and e_2 of M has length four and exactly one weak vertex between e_1 and e_2.

<u>Proof</u>: Since e_1 and e_2 are near edges in M, there exists a path P containing e_1 and e_2 such that $|P| \leq 4$. The shortest path $P(e_1, e_2)$ containing e_1 and e_2 then satisfies $|P(e_1, e_2)| \leq |P| \leq 4$. Since e_1 and e_2 are disjoint edges, $3 \leq |P(e_1, e_2)| \leq 4$. If $|P(e_1, e_2)| = 3$, then $P(e_1, e_2)$ has no weak vertex between e_1 and e_2, contradicting the assumption that M has property (P). Therefore $|P(e_1, e_2)| = 4$ and $P(e_1, e_2)$ contains a vertex v which is

not an end vertex of e_1 or e_2. If v is not a weak vertex, then v must be a neutral vertex and adjacent with an edge e = (u,v) in M. In this case e_1 and e and also e_2 and e are pair of near edges in M, and $|P(e,e_1)| = 3$, $|P(e,e_2)| = 3$, again contradicting the fact that the matching M has property (P). Therefore v is a weak vertex, and the theorem follows.

The converse is clear, since this is merely the definition of property (P)

In order to obtain one of our main results (Theorem 3.4.) we first prove two lemmas.

<u>Lemma 3.1</u>. Let G be a connected graph and M a matching of G having property (P). If v is any weak vertex relative to M, then M has property (P) in G-v.

<u>Proof</u>: We first show that the subgraph G-v has no isolated vertex. Suppose that there exists an isolated vertex v_0 in G-v. Then v_0 is clearly adjacent only to v in G, and since v is a weak vertex then v_0 is also a weak vertex. This contradicts the fact that M is a matching, since no matching permits adjacent weak vertices.

Next, consider the following two cases which arise when v is any weak vertex relative to M.

<u>Case (i)</u>. If v is a cut-vertex of G, then G-v is

disconnected and consists of $L \geq 2$ components $C_1, C_2, \ldots, C_L$. We claim that $M/_{C_i}$, the matching induced by M on the component C_i, has property (P) for $i = 1, 2, \ldots, L$. If $|M/_{C_i}| \geq 2$. Let e_1, e_2 be two near edges in $M/_{C_i}$, therefore these are also two near edges in M. Let $P(e_1,e_2)$ be a shortest path in C_i containing e_1 and e_2. Since $|P(e_1,e_2)| \leq 4$ in C_i, then $|P(e_1,e_2)| = 4$, since the supposition that $|P(e_1,e_2)| = 3$ contradicts the fact that M has property (P). By Theorem 3.3, $M/_{C_i}$ has property (P) in C_i, for each $i = 1, 2, \ldots, L$, so M has property (P) in G-v. For the case $M/_{C_i} = 1$, the result is clear.

<u>Case (ii)</u>. If v is not a cut-vertex of G, then G-v is connected. Let e_1 and e_2 be any two near edges in M, and $P(e_1,e_2)$ a shortest path in G-v containing e_1 and e_2, as in Case (i), $|P(e_1,e_2)| = 4$. Theorem 3.3 again shows that M has property (P) in G-v.

<u>Lemma 3.2</u>. Let G be a connected graph and M any edge matching in G. If v is not a cut-vertex in G, then there exist a matching $\bar{M}$ in G-v such that either $|\bar{M}| = |M|$ or $|\bar{M}| = |M| - 1$.

<u>Proof</u>: We consider two different cases, depending on the degree of vertex v.

<u>Case (i)</u>. v is an end vertex of G. Let v be incident with the edge $e = (u,v)$. (a) If v is a weak vertex, then

M is also a matching in G-v. Setting $\overline{M}$ = M, we have $|\overline{M}| = |M|$. (b) If edge e is in M, and M-e is a matching in G-v, set $\overline{M}$ = M-e, then $|\overline{M}| = |M| - 1$. However, if M-e is not a matching in G-v, then there exist at least one weak vertex w adjacent to u, otherwise if all vertices adjacent to u were neutral, then M-e would be a matching in G-v. Now set $\overline{M}$ = M - e + (u,w). It is clear $\overline{M}$ is a matching in G-v, and $|\overline{M}| = |M| - 1 + 1 = |M|$.

Case (ii). v is not an end vertex of G. (a) Suppose that v is a weak vertex relative to M. Then M is also a matching in G-v. Setting $\overline{M}$ = M, we have $|\overline{M}| = |M|$. (b) Suppose that v is a neutral vertex, so that v is incident with an edge e = (u,v) in M. If M - e is a matching in G-v, set $\overline{M}$ = M - e, so that $|\overline{M}| = |M| - 1$. If M - e is not a matching for G-v. then there exists at least one weak vertex w adjacent to u, otherwise if all vertices adjacent to u were neutral then M - e would be a matching in G-v. Again set $\overline{M}$ = M - e + (u,w). Then $\overline{M}$ is a matching in G-v, because no two adjacent weak vertices exist in G-v. In this case $|\overline{M}| = |M|$.

Remark: This lemma may happen to hold even when v is a cut-vertex, but in general the assumption that v is not a cut-vertex is essential. For example, if v is the center of a star graph, this lemma fails to hold true.

The next theorem gives a sufficient condition for a matching to be a minimum matching. It is assumed that the graph G has no isolated vertices.

<u>Theorem 3.4</u>. Let G be a graph which possesses a matching M having property (P). Then M is a minimum matching.

<u>Proof</u>: We will use induction on the order of G. Assume that for any graph G_0 having order less than that of G then the theorem is true, i.e. if M_0 is a matching of G_0 having property (P) then M_0 is a minimum matching for G.

If G is not connected, then by definition 3.3 the matching induced by M on each component has property (P) provided M has property (P), so by the inductive hypothesis such induced matchings on each component are minimum matchings. Thus M is a minimum matching for G, since the matching for any two components are disjoint.

We henceforth assume that G is connected. If $|M| = 1$, the theorem is trivial. We may therefore assume $|M| \geq 2$. This assumption, together with the fact that M has property (P) implies the existence of weak vertices in G relative to M. Let us assume that M is not a minimum matching, and seek a contradiction. Then there exists a matching $\bar{M}$ in G such that $|\bar{M}| < |M|$. There are two cases to consider.

<u>Case (i)</u>. There exists at least one weak vertex

(relative to M) which is not a cut-vertex of G. Now M is obviously a matching in G-v, and by Lemma 3.1, M has property (P) in G-v. By induction, M is a minimum matching in G-v. Next consider $\overline{M}$ in G. Since v is not a cut-vertex, by Lemma 3.2, there exists a matching M^* in G-v such that either $|M^*| = |M|$ or $|M^*| = |\overline{M}| - 1$. Since M is a minimum matching in G-v, we have $|M^*| \geq |M|$. If $|M^*| = |\overline{M}|$ then $|\overline{M}| \geq |M|$, a contradiction. If $|M^*| = |\overline{M}| - 1$, then $|\overline{M}| - 1 \geq |M|$, and $|\overline{M}| > |M|$, again a contradiction. Thus no such matching $\overline{M}$ can exist, so in case (i), M is a minimum matching.

Case (ii). All weak vertices (relative to M) are cut-vertices of G. Difficulties arise if a cut-vertex of G is removed, since the resulting graph G-v might have isolated vertices, and in this case no matching for G-v is possible. In order to make use of the inductive hypothesis, we resort to a shrinking process applied to suitable subgraph of G. When a subgraph A is shrunk to a vertex v of A, we mean that the entire graph A is replaced by the single vertex v.

Let v be a weak vertex and A_0 any component of G-v. Set $A = \langle A_0 \cup \{v\} \rangle$ and let G_A be the graph constructed from G by shrinking the (block) A into the single vertex v. We show first that the matching M induces matchings on A and on G_A both of which have property (P). Denote by M(A) and $M(G_A)$ the matchings of A and G_A induced by M. If $|M(A)| = 1$

or $|M(G_A)| = 1$, the resulting induced matching is clearly a minimum matching. Therefore we assume that $|M(A)| \geq 2$ and $|M(G_A)| \geq 2$. To show that $M(A)$ has property (P) in A, let e_1, e_2 be any two near edges in $M(A)$, and $P(e_1,e_2)$ a shortest path containing e_1 and e_2. It is clear that $|P(e_1,e_2)| \leq 4$. Also since e_1 and e_2 are two near edges in M, by theorem 3.3, we know that $|P(e_1,e_2)| = 4$ and that $P(e_1,e_2)$ has exactly one weak vertex between e_1 and e_2. Similarly $M(G_A)$ has property (P) in G_A. Therefore by induction $M(A)$ and $M(G_A)$ are minimum matchings of A and G_A, respectively.

Now consider the following two cases:

<u>Case (iia)</u>. The vertex v is a weak vertex with respect to $\overline{M}$, then $\overline{M}(A)$ and $\overline{M}(G_A)$ are matchings for A and G_A respectively. Since $M(A)$ and $M(G_A)$ are minimum matchings of A and G_A respectively, we have $|\overline{M}(A)| \geq |M(A)|$ and $|\overline{M}(G_A)| \geq |M(G_A)|$. Then $|\overline{M}| = |\overline{M}(A)| + |\overline{M}(G_A)| \geq |M(A)| + |M(G_A)| = |M|$, a contradiction.

<u>Case (iib)</u>. The vertex v is not a weak vertex with respect to $\overline{M}$. If $\overline{M}(A)$ and $\overline{M}(G_A)$ are matchings for A and G_A respectively, then by the same argument employed in case (iia) we obtain $|\overline{M}| \geq |M|$, a contradiction. Hence we may assume either $\overline{M}(A)$ is a matching for A, or $\overline{M}(G_A)$ is a matching for G_A, for they cannot both fail to be matchings for A and G_A at the same time, otherwise $\overline{M}$ would not be a

matching. Let us assume $\bar{M}(A)$ is a matching and $\bar{M}(G_A)$ is not a matching, this could happen only when v is a weak vertex relative to $\bar{M}(G_A)$ and there are weak vertices relative to $\bar{M}(G_A)$ adjacent to v. However $\bar{M}(G_A)$ is a matching for G_A-v. Now, consider $M(G_A)$ in G_A. Since v is a weak vertex with respect to M, and also a weak vertex with respect to $M(G_A)$, and $M(G_A)$ has property (P) in G_A, then by lemma 3.1, $M(G_A)$ has property (P) in G_A-v. By induction $M(G_A)$ is a minimum matching in G_a-v. Hence $|\bar{M}(G_A)| \geq |M(G_A)|$. Also since $|\bar{M}(A)| \geq |M(A)|$, then $|\bar{M}| = |\bar{M}(A)| + |\bar{M}(G_A)| \geq |M(A)| + |M(G_A)| = |M|$, again a contradiction. Hence we conclude that M is a minimum matching in G.

<u>Section 3.2</u>. Weinstein [25] in 1961 determined a non-trivial lower bounds for $\beta_{1U}(G)$ in terms of the maximum degree $\Delta(G)$ of G, depending on the value of the minimum degree $\delta(G)$. (1) For $\delta(G) \geq 1$, $|V(G)| \leq (1+\Delta(G))\cdot\beta_{1U}(G)$

(2) For $\delta(G) \geq 2$, $2|V(G)| \leq (2+\max(4,\Delta))\cdot\beta_{1U}(G)$

In this section, we obtain non-trivial lower bounds for $\beta_{1L}(G)$, in terms of $\Delta(G)$ and show that these bounds can be attained, so are sharp.

We first establish a lower bound for $\beta_{1L}(G)$ in the case that G is a tree.

Theorem 3.5. Let T be a tree of order p, and maximum degree $\Delta(T)$. Then $p \le 2\,\Delta(T)\cdot\beta_{1L}(T)$.

Proof: We use induction on the number of edge of T. Suppose T_0 is a tree with $|E(T_0)| < |E(T)|$, we assume that $|V(T_0)| \le 2\,\Delta(T_0)\beta_{1L}(T_0)$, and shall prove that $p \le 2\,\Delta(T)\cdot\beta_{1L}(T)$.

Let M be a minimum matching for T, i.e. $|M| = \beta_{1L}(T)$. If there are no weak edges in T relative to M, then all edges of T are in M. Since T is connected, this is possible only when $T = K_2$. In this trivial case, the theorem follows easily.

Hence, we assume there exist weak edges in T relative to M. Consider the following two cases:

Case (i). There exists at least one weak edge e, which is not an end edge. Consider $T' = T - e$, it is clear M is also a matching in T', with $|V(T')| = |V(T)|$. Since every edge of a tree is a bridge, then T' is a forest. Consider the components $C_1, C_2, \ldots, C_n$ of the forest T'; by induction we have $|V(C_i)| \le 2\,\Delta(C_i)\cdot\beta_{1L}(C_i)$. Hence

$$p = |V(T')| = \sum_{i=1}^{n} |V(C_i)| \le 2\sum_{i=1}^{n} \Delta(C_i)\cdot\beta_{1L}(C_i)$$

$$\le 2\,\Delta(T')\cdot\sum_{i=1}^{n} \beta_{1L}(C_i) \le 2\,\Delta(T)\cdot\beta_{1L}(T)$$

Case (ii). All weak edges of T relative to M are end edges. We claim first that T is a tree with diameter ≤ 3.

Let the path $P(u_1, u_2, \ldots, u_K)$ be a diameter of T if $|P| \geq 4$, then $K \geq 5$, it is clear the edges (u_2, u_3), (u_3, u_4), $\ldots$, (u_{K-2}, u_{K-1}) are not end edges and it is impossible for all these edges to be in M, since M is an independent set of edges, hence there is some edge which is not an end edge. This contradict the assumption. Therefore T is a tree of diameter ≤ 3, i.e. T is a union of two star graphs, joined by an edge between two centers of the stars. It is clear in this case $|M| = 1$. Suppose the degree of these center vertices are d_1 and d_2 respectively, we may assume $1 \leq d_1 \leq d_2$ that is $\Delta(T) = d_2$. Now $|T| = d_1 + d_2$ and $\beta_{1L}(T) = 1$ and $p = d_1 + d_2 \leq 2 \cdot d_2 = 2\Delta(T) \cdot \beta_{1L}(T)$.

<u>Theorem 3.6</u>. Let G be any graph of order p with maximum degree $\Delta(G)$, and having no isolated vertices. Then $p \leq 2\Delta(G) \cdot \beta_{1L}(G)$.

<u>Proof</u>: We use induction on the number of edges. Suppose H is a graph having no isolated vertices and $|E(H)| < |E(G)|$. We assume $|V(H)| \leq 2\Delta(H) \cdot \beta_{1L}(H)$ and we shall show that $p \leq 2\Delta(G) \cdot \beta_{1L}(G)$.

If G is not connected, let $C_1, C_2, \ldots, C_K$ be the components of G. By induction on each component we have $|V(C_i)| \leq 2\Delta(C_i) \cdot \beta_{1L}(C_i)$, $i = 1, 2, \ldots, K$. Then

$$p = \sum_{i=1}^{K} |V(C_i)| \leq 2\sum_{i=1}^{K} \Delta(C_i) \cdot \beta_{1L}(C_i) \leq 2\Delta(G) \cdot \sum_{i=1}^{K} \beta_{1L}(C_i)$$
$$\leq 2\Delta(G) \cdot \beta_{1L}(G)$$

We now assume that G is connected. If G is a tree, our result follows by theorem 3.5. We may therefore assume that G is not a tree. Then there exist a cycle C in G. Let M be a minimum matching for G. Then there exists in C at least one edge e such that e is a weak edge with respect to M. Now set H = G - e. We have $\delta(H) \geq 1$ and $\Delta(H) \leq \Delta(G)$, moreover M is a matching in H. Hence $\beta_{1L}(H) \leq |M|$, now $|V(G)| = |V(H)| = p$ and $|E(H)| < |E(G)|$. By induction, we have $p \leq 2\Delta(H)\cdot\beta_{1L}(H)$ and

$$p \leq 2\Delta(H)\cdot\beta_{1L}(H) \leq 2\Delta(G)\cdot|M| = 2\cdot\Delta(G)\cdot\beta_{1L}(G).$$

Remark: For each $j \geq 1$ there exists a graph G with $\Delta(G) = j$, $\delta(G) = 1$ and $|V(G)| = 2\cdot\Delta(G)\cdot\beta_{1L}(G)$.

For example, let G be a tree of diameter = 3, as in the prove of theorem 3.5. Set $|G| = 2n$, and deg u = deg v = n, $\beta_{1L}(G) = 1$ then $2n = 2\cdot\Delta(G)\cdot\beta_{1L}(G) = 2\cdot n\cdot 1$.

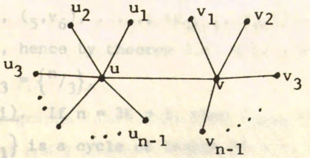


Figure 3.2. A graph illustrating a sharp lower bound when $\delta(G) \geq 1$.

Hence the inequality stated in theorem 3.6. is a best possible result, in the sense that equality can be attained. However, our lower bound for $\beta_{1L}(G)$ in some examples may be poor.

In this section our principal result (Theorem 3.7.) is an analogue of theorem 3.6, but provides a sharper lower bound for $\beta_{1L}(G)$ when the minimum degree $\delta(G) \geq 2$, namely $|V(G)| \leq (1 + \Delta(G)) \cdot \beta_{1L}(G)$. In proving this result we shall use the following lemmas.

<u>Lemma 3.3</u>. Let C_n be a cycle of length n. Then $\beta_{1L}(C_n) = \{^n/_3\}$. In this lemma, $\{x\}$ denotes the smallest integer not less than x.

<u>Proof</u>: Consider the following three cases, where $n = 3K$, $n = 3K + 1$ or $3K + 2$, and $K = 1, 2, 3, \ldots$.

<u>Case (i)</u>. If $n = 3K$, then $C_{3K} = \{v_1, v_2, \ldots, v_{3K}\}$ is a cycle of length 3K, construct a matching M as follows $M = \{(v_2,v_3), (_5,v_6), \ldots, (v_{3K-1}, v_{3K})\}$. M clearly has property (P), hence by theorem 3.4. M is a minimum matching, and $|M| = {^n/_3} = \{^n/_3\}$.

<u>Case (ii)</u>. If $n = 3K + 1$, then $C_{3K+1} = \{v_1, v_2, \ldots, v_{3K}, v_{3K+1}\}$ is a cycle of length 3K + 1. Construct a matching M as follows $M = \{(v_1,v_2), (v_4,v_5), \ldots, (v_{3K-2}, v_{3K-1}), (v_{3K}, v_{3K+1})\}$, $|M| = K + 1$. We cannot employ theorem 3.4, since M does not have property (P). However if M were not a minimum matching, then there exists a

matching M^*, such that $|M^*| < |M| = K + 1$, then $|M^*| \leq K$. Hence there are at least K+1 weak vertices relative to M^*. This implies there exist two consecutive weak vertices in C_{3K+1} relative to M^*, which contradicts the fact that M^* is a matching. Therefore M is a minimum matching and $|M| = K + 1 = \{(3K+1)/3\} = \{n/3\}$.

<u>Case (iii)</u>. If $n = 3K+2$, then $C_{3K+2} = \{v_1, v_2, \ldots, v_{3K+1}, v_{3K+2}\}$ is a cycle of length 3K+2. Construct a matching M as in case (ii). If M were not a minimum matching then there exists a matching M^*, such that $|M^*| < |M|$. This will imply that there exists a matching $\bar{M}$ in C_{3K+1} such that $|\bar{M}| \leq |M| = K + 1$, which contradicts the fact that a minimum matching of C_{3K+1} is of order K+1. Then $|M| = K + 1 = \{(3K+2)/3\} = \{n/3\}$.

Hence we conclude that in all three cases $\beta_{1L}(C_n) = \{n/3\}$.

<u>Lemma 3.4</u>. Let G be a connected graph, and M a matching of G. Let X and Y be subsets of V(G), such that they form a partition of V(G), i.e. $X \cup Y = V(G)$ and $X \cap Y = \Phi$. Let H and K be any graphs (not necessarily subgraphs of G) having vertex sets X and Y respectively, such that $\Delta(H) \leq \Delta(G)$ and $\Delta(K) \leq \Delta(G)$. If M_1 and M_2 are matchings for H and K respectively with $|M_1| + |M_2| \leq |M|$, and in H and K we have the inequalities $|X| = |V(H)| \leq (1 + \Delta(H))\cdot M_1$. $|Y| = |V(K)| \leq (1 + \Delta(K))\, M_2$, then in G we have $|V(G)| \leq (1 + \Delta(G))\cdot|M|$.

<u>Proof</u>: The proof is trivial, since

$$|V(G)| = |V(H)| + |V(K)| \leq (1 + \Delta(H)) \cdot |M_1| + (1 + \Delta(K)) \cdot |M_2|$$
$$\leq (1 + \Delta(G)) \cdot (|M_1| + |M_2|) = (1 + \Delta(G)) \cdot |M|.$$

<u>Theorem 3.7</u>. Let G be a graph of order p with $\delta(G) \geq 2$. Then $p \leq (1 + \Delta(G)) \cdot \beta_{1L}(G)$.

<u>Proof</u>: We use induction on the edge number $|E(G)|$ of G, and assume that whenever H is a graph with $\delta(H) \geq 2$ and $|E(H)| < |E(G)|$ then $|V(H)| \leq (1 + \Delta(H)) \cdot \beta_{1L}(H)$. We shall show that $p \leq (1 + \Delta(G)) \cdot \beta_{1L}(G)$. Throughout the entire proof of this theorem, we will let M be a minimum matching for G. If G is not connected, let $C_1, C_2, \ldots, C_K$ be the components of G and M/C_i, $i = 1, 2, \ldots, K$ the matching in C_i induced by the matching M. Then $|M/C_i| = \beta_{1L}(C_i)$. By using induction on each component, we have $|V(C_i)| \leq (1 + \Delta(C_i)) \cdot \beta_{1L}(C_i)$, $i = 1, 2, \ldots, K$. Then

$$p = \sum_{i=1}^{K} |V(C_i)| \leq \sum_{i=1}^{K} (1 + \Delta(C_i)) \cdot \beta_{1L}(C_i)$$
$$\leq (1 + \Delta(G)) \sum_{i=1}^{K} \beta_{1L}(C_i) = (1 + \Delta(G)) \cdot \beta_{1L}(G)$$

We next assume that G is connected, and define $S = \{v \in V(G) \mid \deg v \geq 3\}$. $T = \{v \in V(G) \mid \deg v = 2\}$. we first note that we may assume that $S \neq \phi$, for if $S = \phi$, since G is connected then G is a cycle and by lemma 3.3.

$\beta_{1L}(G) = \{P/3\}$. Now $(1 + \Delta(G))\cdot\beta_{1L}(G) = 3\{P/3\} \geq \{3\cdot P/3\} = p$, so the theorem holds in this case.

We may also assume that $T \neq \phi$, for if $T = \phi$, then there exist two S - vertices joined by a weak edge $e = (u,v)$. Setting $H = G - \{e\}$, we have $\delta(H) \geq 2$ and $|E(H)| < |E(G)|$. Hence by induction $|V(H)| \leq (1 + \Delta(H))\cdot\beta_{1L}(H)$. But since M is also a matching of H, we have $|M| \geq \beta_{1L}(H)$. Also since $p = |V(H)|$, we obtain

$$p \leq (1 + \Delta(H))\cdot\beta_{1L}(H) \leq (1 + \Delta(G))\cdot\beta_{1L}(G),$$

and again the theorem follows.

Note in the above discussion of the case where $T \neq \phi$ we have also shown that the graph $G/_S$ is either discrete (i.e. totally disconnected) or joined by edges in M, between S vertices.

Now we are going to prove the theorem in the following five steps in the case where $S \neq \phi$ and $T \neq \phi$.

(1) Let P be a path between two S - vertices. If the terminal edges of P are weak edges the theorem follows if $|P| > 2$, so we may assume $|P| \leq 2$.

(2) The theorem follows when $G/_S$ is not discrete (i.e. totally disconnected), so we may assume that $G/_S$ is discrete.

(3) If there exists an edge in M which is not incident with an S - vertex, the theorem follows. Therefore we may assume that all edges in M are incident with S - vertices.

(4) For any path P between two S - vertices, the

theorem follows when $|P| \leq 4$.

(5) If the assumptions made in steps 1-4 hold. Then $|S| = |M|$ and $\Delta(G)\cdot|S| \geq |T|$.

After proving these five steps, the proof of the theorem will be immediate, since $|S| + |T| = p$.
Now $\Delta(G)\cdot|S| \geq |T| = p - |S|$.

$(\Delta(G) + 1)\cdot|S| \geq p$, but since $|S| = |M| = \beta_{1L}(G)$.
Therefore we conclude $(1 + \Delta(G))\cdot\beta_{1L}(G) \geq p$.

(1) Let P be a path between two S vertices x and y say, and the terminal edges of P are weak and $|P| > 2$.

<u>Case (a)</u>. $x = y$

<u>(a-1)</u>. Deg x = deg $y \geq 4$.

Set $H = P(u, u_1, \ldots, v)$, and $K = G - V(H)$. Since $\delta(H) \geq 1$, by theorem 3.5. we have $|V(H)| \leq 2\Delta(H)\cdot\beta_{1L}(H) = 4\cdot\beta_{1L}(H) \leq (1 + \Delta(G))\cdot\beta_{1L}(H)$, and in K we have $\delta(K) \geq 2$ and $|E(K)| < |E(G)|$. By induction we have $|V(K)| \leq (1+\Delta(K))\cdot\beta_{1L}(K)$. Let M(H) and M(K) be the matchings on H and K induced by M, then we have $|M(H)| \geq \beta_{1L}(H)$ and $|M(K)| \geq \beta_{1L}(K)$. Thus by lemma 3.4. the theorem follows.

<u>(a-2)</u>. Deg x = deg $y = 3$, let us trace the path $P(x, x_1, \ldots, x_N)$ to the nearest S vertex x_N, since (u,x) and (v,x) are weak edges, the first neutral vertex on the path P (start from x) must be either $x = y$, or x_1. Also we note that there must exist a T vertex w adjacent to x_N,

for if all vertices adjacent to x_N are S vertices, this implies that there are weak edges joining two S vertices, which contradicts our assumption.

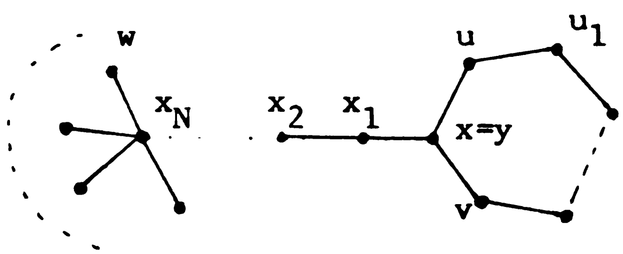


Figure 3.3. Existence of a T vertex w adjacent to x_N.

If x = y were the first neutral vertex, we can set $H = P(u, u_1, \ldots, v)$, and $K = G - V(H) + (x,w)$. In H by theorem 3.5. we have $|V(H)| \leq 2\cdot\Delta(H)\cdot\beta_{1L}(H) = 4\cdot\beta_{1L}(H)$

$$\leq (1 + \Delta(G))\cdot\beta_{1L}(H).$$

In K we have $\delta(K) \geq 2$, $|E(K)| < |E(G)|$ and $\Delta(K) \leq \Delta(G)$. By induction we have $|V(K)| \leq (1 + \Delta(K))\cdot\beta_{1L}(K)$. Let M(H) and M(K) be the matchings on H and K induced by M, then we have $|M(H)| \geq \beta_{1L}(H)$ and $|M(K)| \geq \beta_{1L}(K)$. Thus by lemma 3.4. the theorem follows. If x_1 were the first neutral vertex we can set $H = C(x, u, \ldots, v, x)$ and $K = G - V(H) + (w,x_1)$. Similarly, by induction and lemma 3.4. the theorem follows.

<u>Case (b)</u>. $x \neq y$. Set $H = P(u, u_1, \ldots, v)$ and $K = G - V(H)$. In H, theorem 3.5. implies $|V(H)| \leq 2\cdot\Delta(H)\cdot\beta_{1L}(H) \leq (1 + \Delta(G))\cdot\beta_{1L}(H)$

In K, we have $\delta(K) \geq 2$, $|E(K)| < |E(G)|$ and $\Delta(K) \leq \Delta(G)$. Again by lemma 3.4. the theorem follows. Therefore we may assume that $|P| = 2$.

(2) $^{G}/_{S}$ is not discrete. We have shown that if $^{G}/_{S}$ is not discrete, then there are only strong edges joined between vertices in S. If there exist two S vertices x and y, joined by a strong edge (x,y), then by step (1) any path between x and y has length 2.

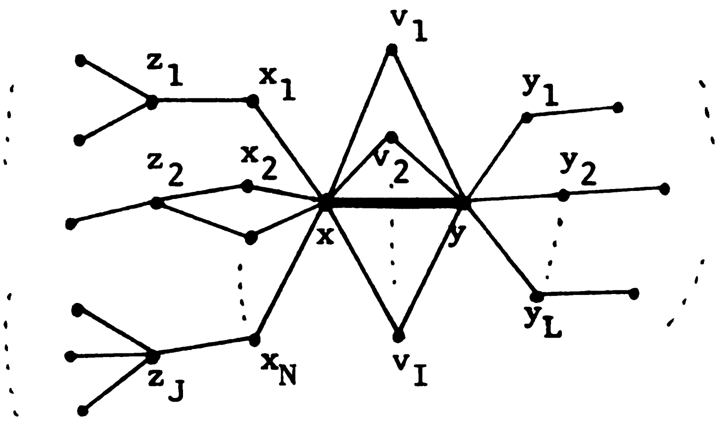


Figure 3.4. Two S vertices x and y joined by a strong edge.

Set deg x = N + I, deg y = L + I. We may assume without loss of generality that $N \geq L$. Now, let us consider the following various of cases.

<u>Case 2-a</u>. N = L = 0. Then $I \geq 2$, we have $\Delta(G) = I + 1$, $p = I + 2$ and $\beta_{1L}(G) = 1$, then $p = I + 2 = (\Delta(G) + 1)\cdot\beta_{1L}(G)$, thus the theorem follows.

<u>Case 2-b</u>. $N = 1$, $L = 0$, and $I \geq 2$. Let us trace the path $P(x, x_1, x_2, \ldots, x_R)$ to the nearest S vertex x_R, we have $R \geq 2$, by the same argument as in step (1) case (a-2) there exists a T vertex w adjacent to x_R. Now consider the following two possibilities, start from x in the path P. Either x_1 is the first neutral vertex or x_2 is the first neutral vertex. If x_1 were the first neutral vertex, set $H = {}^{G}/_{<x, y, v_1, \ldots, v_I>}$ and $K = G - V(H) + (x_1,w)$. In H we have $|V(H)| = I + 2 = (1 + \Delta(H)) \cdot \beta_{1L}(H)$. In K we have $\delta(K) \geq 2$, $|E(K)| < |E(G)|$, by induction $|V(K)| \leq (1 + \Delta(K)) \cdot \beta_{1L}(K)$. Similar argument as before the theorem follows by lemma 3.4. If x_2 were the first neutral vertex, we can set $H = {}^{G}/_{<x_1, x, y, v_1, \ldots, v_I>}$ and $K = G - V(H)$ if $R = 2$, $K = G - V(H) + (x_2,w)$ if $R > 2$. In H we have $|V(H)| = I + 3$ and $I + 3 = (1 + (I + 2)) \cdot 1 = (1 + \Delta(H)) \cdot \beta_{1L}(H)$. Again in K by induction we have $|V(K)| \leq (1 + \Delta(K)) \cdot \beta_{1L}(K)$. From lemma 4.3. the theorem follows.

<u>Case 2-c</u>. $N \geq 2$, $L = 0$ and $I \geq 2$. We divide the set of vertices $\{x_1, x_2, \ldots, x_N\}$ into two sets, namely M(x) of those vertices incident with edges in M, and M'(x) of those vertices not incident with edges in M. Set $H = {}^{G}/_{<y, v_1, \ldots, v_I, x \text{ and } x_i \in M'(x)>}$ and $\bar{K} = G - V(H)$. In H we have $|V(H)| = I + 2 + |M'(x)|$, $\Delta(H) = I + |M'(x)| + 1$, $\beta_{1L}(H) = 1$. Therefore $|V(H)| = I + 2 + |M'(x)| = ((I + |M'(x)| + 1) + 1) \cdot 1 = (1 + \Delta(H)) \cdot \beta_{1L}(H)$.

As for $\overline{K}$, we are going to construct a graph K from $\overline{K}$, such that $|V(K)| = |V(\overline{K})|$, $\delta(K) \geq 2$ and $\Delta(K) \leq \Delta(G)$. Moreover there exists a matching M^* in K such that $|M^*| = |M| - 1$. If in $\overline{K}$ we set $M^* = M/_{\overline{K}}$ (i.e. M^* is the matching induced by M on $\overline{K}$), then for $x_i \in M(x)$, the x_i are strong vertices relative to M^* and are of degree one in $\overline{K}$. As for the vertices z's joined to $x_i \in M'(x)$, they are either neutral or strong vertices relative to M^* depending upon whether their degree > 1 or = 1. Now, we can construct K from $\overline{K}$ by joining the x_i's in M(x) and the z's of degree 1, among themselves in pairs if they are even in number. If the number is odd the extra vertex can be joined to a T - vertex, so that $\Delta(K) \leq \Delta(G)$, $\delta(K) \geq 2$ and $|V(K)| = |V(\overline{K})|$. It is clear by the construction that M^* is a matching of K and $|M^*| = |M| - 1$, hence by induction we have $|V(K)| \leq (1 + \Delta(K)) \cdot \beta_{1L}(K)$. Thus by lemma 3.4. the theorem follows.

<u>Case 2-c</u>. $N \geq L \geq 1$. We first establish that we may assume there exists a T vertex incident with an edge in M. For if there were no T vertex incident with an edge in M then $|M| = |S|/_2$. To verify this statement, it is clear that since M is incident only with S vertices (as will be proved in step 3 on page 39), then $|M| \leq |S|/_2$. Conversely if there exists an S vertex, say s_0, not incident with edges in M, then one of the vertices joined to s_0, say s, must be a neutral vertex. If deg s ≥ 3, then we have two S vertices joined by

a weak edge. If deg s = 2, this contradicts our assumption. Hence $2|M| \geq |S|$, i.e. $|M| = {}^{|S|}/_{2}$. Now count the number of edges in G. Since the path between two adjacent S vertices is either an edge in M or two weak edges we have

$$q = \sum_{s \in S} \deg s - {}^{|S|}/_{2} \geq 2|T| + {}^{|S|}/_{2}.$$

and $\Delta \cdot |S| - {}^{|S|}/_{2} \geq 2|T| + {}^{|S|}/_{2}$.

$$\Delta \cdot |S| \geq 2|T| + |S| = 2p - |S|.$$

$$|S| \cdot (\Delta + 1) \geq 2p. \quad {}^{|S|}/_{2} \cdot (\Delta + 1) = |M| \cdot (\Delta + 1) \geq p.$$

Therefore the theorem follows. Thus we may assume that there exist T vertices incident with edges in M.

N = L = 1 and $z_1 \neq u_1$, set $H = {}^{G}/_{<x_1,\, x,\, y,\, v_1,\, \ldots,\, v_I>}$

or $= {}^{G}/_{<x,\, y,\, v_1,\, \ldots,\, v_I>}$

depending on whether z_1 or x_1 is a neutral vertex. Then $K = G - V(H) + (z_1,y_1)$ or $K = G - V(H) + (x_1,y_1)$, and the theorem follows by lemma 3.4. If N = L = 1 and $z_1 = u_1$, if (x_1,z_1) in M or similarly (y_1,z_1) in M.

Set $H = {}^{G}/_{<x,\, y,\, v_1,\, \ldots,\, v_I>}$ and $K = G - V(H) + (x_1,y_1)$. The result is clear. If both x_1 and y_1 are weak vertices set $H = {}^{G}/_{<x_1,\, x,\, y,\, v_1,\, \ldots,\, v_I>}$ and $K = G - V(H) + (y_1,w)$, where w is a neutral T vertex, we have established the existence of such a vertex at the beginning of case 2-d, therefore the theorem follows by lemma 3.4.

$N \geq L > 1$. We may assume all y_i's are weak vertices, for if there exists one vertex, say y_0 which is a neutral vertex, we can join all y_i's to y_0, and treat the rest as in case 2-c. i.e. Set $H = {}^{G}/_{\langle y,x,v_1,\ .\ .\ .,\ v_I \text{ and } x_i \in M'(x)\rangle}$ and $\overline{K} = G - V(H)$. Construct K from $\overline{K}$ by joining all y_i's to y_0, then from case 2-c, the theorem follows. Thus all y_i's are weak vertices. Now set $H = {}^{G}/_{\langle y,x,v_1,\ .\ .\ .,\ v_I, \text{ and } x \in M'(x)\rangle}$, ($M'(x)$ as in case 2-c) and $\overline{K} = G - V(H)$. Assume there are J vertices (z_1, . . ., z_J) adjacent to x_i's. After removing H from G we obtain J strong or neutral vertices, and the total degree decrease arising from these J vertices is N. Also there are L weak vertices of degree 1 in $\overline{K}$. Since $N \geq L$, we can construct K from $\overline{K}$ by joining these two sets of vertices in an appropriate way without increasing the degree of $\overline{K}$. From such a construction $\Delta(K) \geq 2$, $\Delta(K) \leq \Delta(G)$ and $|\overline{K}| = |K|$. Moreover set $M^* = {}^{M}/_{K}$ the matching induced by M on K. It is clear from the construction of K that $|M^*| = |M| - 1$, and the theorem follows by lemma 3.4.

<u>Note</u>: Case 2-d also shows that no strong edge can join an S vertex to a T vertex and then be followed by a weak S vertex. If it does, we can set $H = {}^{G}/_{\langle y,x,x_i \in M'(x)\rangle}$ and $K = G - V(H)$.

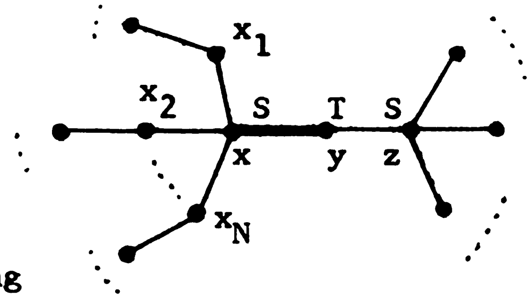


Figure 3.5. A T vertex joining two S vertices.

The same argument as case 2-c, will prove the theorem.

(3) All edges in M are incident with S vertices. For if there exist an edge e = (x,y) not incident with an S vertex, then deg x = deg y = 2.

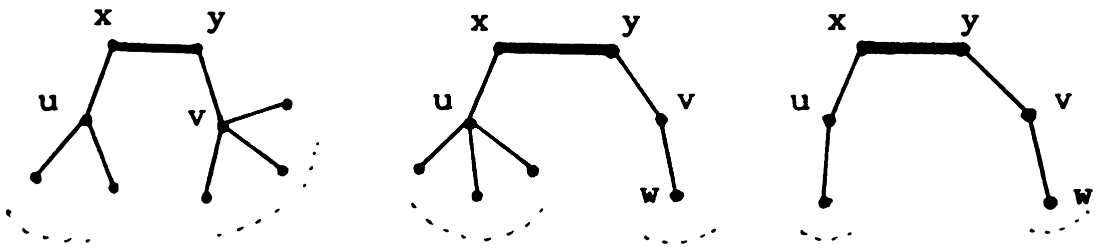


Figure 3.6. An edge (x,y) not incident with S vertices.

<u>Case 3-a</u>. Deg u $\geq$ 3 and deg v $\geq$ 3. Set H = (x,y) and K = G - V(H).

<u>Case 3-b</u>. Deg u $\geq$ 2, and deg v = 2. If v is a weak vertex and w a neutral vertex. Set H = P(x,y,v) and K = G - V(H) + (u,w). If v is a neutral vertex, set H = (x,y) and K = G - V(H) + (u,w). The theorem follows again by induction and lemma 3.4.

(4) Let P be any path between two S vertices then $|P| \leq 4$. For if there exists a path between two S vertices of length ≥ 5, then such a path must contains a strong edge disjoint from S vertices which contradicts (3).

(5) $^{G}/_{S}$ is discrete, and all the paths P between S vertices satisfy $|P| \leq 4$. In particular, if the terminal edges of P are weak edges, we have $|P| = 2$. Claim $|S| = |M|$

and $\Delta \cdot |S| \geq |T|$. We have shown in (3) that every edge of M is incident with S vertices, hence $|M| \leq |S|$. If there exists an S vertex, say s, which is not incident with any edge in M, then s is a weak vertex. Let s_1 be a vertex joined to s. Then deg $s_1 = 2$ (otherwise we have two S vertices joined by a weak edge), also (s_1, s_2) is in M. If deg $s_2 = 2$, we contradict (3). If deg $s_2 \geq 3$, again we contradict the remark stated before (3). Therefore every S vertex is incident with an edge of M, so $|M| = |S|$.

Finally, we show that $\Delta \cdot |S| \geq |T|$, let us divide the set of vertices T into three classes of subsets. Let α be the set of T vertices, such that a vertex is in α if it is adjacent to two T vertices. Let β be the set of T vertices, such that a vertex is in β if it is adjacent to one T vertex and one S vertex. Let γ be the set of T vertices, such that a vertex is in γ, if it is adjacent to two S vertices. Now consider the number of edges q in G. We have

$$q = \sum_{s \in S} \deg s + |\alpha| + |\beta|/2$$

also $q = 2|\gamma| + |\alpha| + 3|\beta|/2$.

Hence $\Delta \cdot |S| + |\alpha| + |\beta|/2 \geq 2|\gamma| + |\alpha| + 3|\beta|/2$.

and $\Delta \cdot |S| \geq 2|\gamma| + |\beta|$.

We claim that $|\gamma| \geq |\alpha|$. It is clear that $|\alpha| \leq |S|/2$, for a vertex v in α, v must be on a path P of length 4, and the terminal edges of P are therefore strong edges. There will be no other α vertex on any path joining these two terminal S vertices. Also we have $|\gamma| \geq |S|/2$, for if $|\gamma| < |S|/2$

then $\quad 3|S| + |\alpha| + |\beta|/2 \leq q = 2|\gamma| + |\alpha| + 3|\beta|/2.$

$$3|S| + |\beta|/2 \leq 2|\gamma| + 3|\beta|/2 < |S| + 3|\beta|/2.$$

i.e. $2|S| < |\beta|$.

Since one edge in M can contribute at most two β vertices then $|\beta| \leq 2|M| = 2|S|$, which yields a contradiction. Therefore $|\gamma| \geq |S|/2 \geq |\alpha|$ and $\Delta \cdot |S| \geq 2|\gamma| + |\beta| \geq |\alpha| + |\beta| + |\gamma| = |T|$. Thus we conclude that $\Delta \cdot |S| \geq |T|$ and the proof of theorem 3.7. is completed.

Section 3.3. M. J. Stewart [24], has determined the number of edges in a maximum matching for the n-th subdivision graph $S_n(G)$. This number, $\beta_{1U}(S_n(G))$, depends on $q = |E(G)|$, on the parity of n, and sometimes also on the parameter $\beta_{1U}(S_1(G))$. Similar precise results are obtained here for the minimum matching number $\beta_{1L}(S_n(G)$. Various cases arise depending on the value of n modulo 3. We first prove the following theorem for the case when $n = 3K-1$, where $K = 1, 2, 3, \ldots$. It will be observed that in this case the value of $\beta_{1L}(S_n(G))$ depends only on K and q.

Theorem 3.8. Let G be a connected (p,q) graph. Then $\beta_{1L}(S_{3K-1}(G)) = K \cdot q$.

Proof: Let both the vertices of G and the vertices of

$S_{3K-1}(G)$ which correspond to the vertices of G be labeled as $v_1, v_2, \ldots, v_p$. To each edge (v_i, v_j) present in G, denote the corresponding v_i-v_j path of length 3K in $S_{3K-1}(G)$ by P_{ij}: ($v_i, u_1, u_2, \ldots, u_{3K-1}, v_j$). Next construct a matching M in $S_{3K-1}(G)$ as follows: to each path P_{ij} in $S_{3K-1}(G)$ choose the edges $(u_1,u_2), (u_4,u_5), \ldots, (u_{3K-2}, u_{3K-1})$ to be the edges in M. We are choosing for M the middle edge in each set of three consecutive edges of P_{ij}. It is obvious from the construction that M is an edge matching in $S_{3K-1}(G)$, and that each path P_{ij} contributes exactly K edges to M. Hence $|M| = K \cdot q$. The theorem then follows if we can show that M has property (P), since by theorem 3.4. M is then a minimum matching. Let e_0 be any edge in M and consider in M all the near edges of e_0. Suppose that e_1 is any one of these near edges. Two possibilities arise: Case 1. The near edges e_0 and e_1 lie on the same path P_{ij}. It is clear from the construction of M that $|P(e_0,e_1)| = 4$, and that there exists a weak vertex between e_0 and e_1. Case 2. The near edges e_0 and e_1 lie on two different paths P_{ij} and P_{ik}. Since $|P(e_0,e_1)| \leq 4$, edges e_0 and e_1 are two terminal edges of the paths P_{ij} and P_{ik} which are in M. Again, by the construction of M, we conclude that $|P(e_0,e_1)| = 4$, and that the vertex v_i is the weak vertex between e_0 and e_1. (See figure 3.7.) In each case M has property (P), so $|M| = \beta_{1L}(S_{3K-1}(G)) = K \cdot q$.

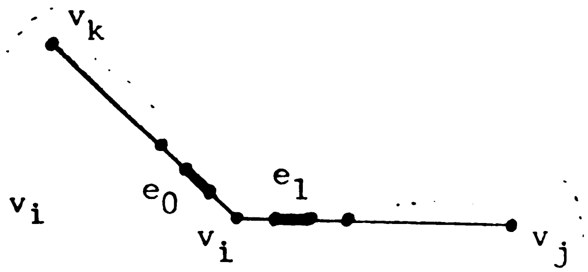


Figure 3.7. A weak vertex v_i between e_0 and e_1.

<u>Corollary</u>: Let G be an arbitrary (p,q) graph. Then

$\beta_{1L}(S_{3K-1}(G)) = K\cdot q$. $K = 1, 2, 3, \ldots$

<u>Proof</u>: The connected case has already been considered in theorem 3.8. If G is not connected, let $G = C_1 \cup C_2 \cup \ldots \cup C_N$, where $C_1, \ldots, C_N$ are the non-empty components of G, and $N \geq 2$. Then

$$\beta_{1L}(S_{3K-1}(G)) = \sum_{i=1}^{N} \beta_{1L}(S_{3K-1}(C_i))$$

$$= \sum_{i=1}^{N} K\cdot |E(C_i)| = K\cdot q.$$

In the remaining cases when n = 3K+1 or when n = 3K+3, where K = 0, 1, 2, . . . , the formulas for $\beta_{1L}(S_n(G))$ are somewhat more complicated than that given in theorem 2.8. since they depend also on the values of $\beta_{1L}(S_1(G))$ and $\beta_{1L}(S_3(G))$. Nevertheless these formulas are useful, especially when K is large.

When an edge e of G and its end vertices are replaced by a path P of length n+1 in $S_n(G)$, we find it convenient to say that the edge e "contains" the n+1 edges of the path P. To each edge (v_i,v_j) in G, denote the corresponding path of length n+1 in $S_n(G)$ by P_{ij}: $(v_i,u_1,u_2, \ldots, u_n,v_j)$. For instance the corresponding path of length 2 in $S_1(G)$ is P_{ij}: (v_i, u_1, v_j).

<u>Theorem 3.9</u>. Let G be a connected (p,q) graph. Then

$$\beta_{1L}(S_{3K+1}(G)) = \beta_{1L}(S_1(G)) + K \cdot q. \quad K = 0, 1, 2, \ldots$$

<u>Proof</u>: Let $|M_0| = m$ denote the number of edges in a minimum matching of $S_1(G)$, so $\beta_{1L}(S_1(G)) = m$. Then there are exactly m edges of G which contain an edge of M_0, and q-m edges of G which do not contain an edge of M_0.

Let both the vertices of G and the vertices of $S_{3K+1}(G)$ which correspond to the vertices of G be labeled $v_1, v_2, \ldots, v_p$. To each edge (v_i,v_j) in G denote the corresponding v_i-v_j path of length 3K+2 in $S_{3K+1}(G)$ by P_{ij}: $(v_i, u_1, u_2, \ldots, u_{3K+1}, v_j)$. Now, construct a matching M in $S_{3K+1}(G)$ in a manner we now describe. Corresponding to a path P_{ij}:(v_i,u_1,v_j) in $S_1(G)$ containing an edge of M_0, one of the vertices v_i and v_j must be incident to an edge in M_0. If v_i does, we choose the available K+1 edges in P_{ij}, namely $(v_i,u_1), (u_3,u_4), \ldots, (u_{3K},u_{3K+1})$ to be in M. Corresponding to a path P_{ij} in $S_1(G)$ which does not contain an edge in M_0, we choose the available K edges in P_{ij} as follows $(u_2,u_3), (u_5,u_6), \ldots, (u_{3K-1},u_{3K})$. Thus $|M| = m(K+1) + (q-m)K = q \cdot K + m$.

Now claim (1) M is a matching and (2) M is a minimum matching. It is clear that M is an independent set of edges in $S_{3K+1}(G)$. If M is not a matching, then on some path in $S_{3K+1}(G)$ there are at least two consecutive vertices which are not incident with edges in M. This can possibly occur only on two ends of path P_{ij} of $S_{3K+1}(G)$ which contains K edges in M. If v_i, u_1 are two such vertices, then v_i must be a weak vertex relative to M_0 in $S_1(G)$. However in $S_1(G)$, P_{ij} contains no edges in M_0. Therefore v_i, u_1 in $S_1(G)$ are two consecutive weak vertices, contradicting the fact that M_0 is a matching in $S_1(G)$. Thus v_i must incident with an edge in M. Thus M is a matching. We next show that M is a minimum matching in $S_{3K+1}(G)$. If false, let M_1 be a minimum matching of $S_{3K+1}(G)$, and $|M_1| < |M|$. We can obtain another matching M_2 from M_1 having the same cardinality as M_1 in the following way. The edges of M_1 are distributed among the q-path of $S_{3K+1}(G)$. It is clear that no path P_{ij} contains K+3 edges in M_1, since M_1 is a minimum matching. If there exists a path P_{ij} containing K+2 edges of M_1, then there must exist a path P_{i1} or P_{jk} containing K edges of M_1, since if all paths adjacent to P_{ij} contain K+1 edges of M_1, we can construct a new matching with fewer edges than M_1. This contradicts the assumption that M_1 is a minimum matching. We can therefore rearrange the edges in M_1 so that both P_{ij} and P_{i1} (or P_{jk}) contain K+1 edges of M_1. We may then assume that among these paths there are s which contain K+1 edges in M_1, while the other q-s paths each contain only K edges of M_1. Whenever a path P_{ij}, contain K edges of M_1, both v_i and v_j must be incident with edges of M_1 not in P_{ij}. For a path P_{ij} containing K+1 edges of M_1, at most one of v_i and v_j can be incident with an edge of M_1 not in P_{ij}, since otherwise K edges could be used in M_1 for that path P_{ij}, contradicting the minimality of M_1. This set of K+1 edges in M_1 can be replaced by a set of K+1 edges which form a matching for P_{ij}, such that exactly one of v_i and v_j is incident with an edge of M_1 in P_{ij}. If this latter set is distinct from the former, we obtain a minimal matching having the same cardinality as M_1. Repeating this process for every path P_{ij} containing K+1 edges of M_1, we obtain a matching M_2, where $|M_2| = |M_1|$. We observe that M_2 has the property that a path P_{ij} contain K edges of M_2 if and only if both v_i amd v_j are incident with edges of M_2 not in P_{ij}, and a path P_{ij} contains K+1 edges of M_2 if and only if exactly one of v_i and v_j is incident with an edge of M_2 in P_{ij}. Now we choose a new independent set of edges M^* in $S_1(G)$ in the following way: corresponding to a s-path P_{ij} containing K+1 edges of M_2, we choose the edge (v_i,u_1) or (u_1,v_j) to be in M^* depending upon whether v_i or v_j is incident with an edge of M_2 in P_{ij}. It is clear that M^* is an independent set of edges, and we claim it is a matching for $S_1(G)$. If not then there exists at least two consecutive vertices in $S_1(G)$ which are not incident with edges in M^*, say v_i and u_1. Then the path P_{ij} contains only K edges in M_2, otherwise u_1 would be incident with an edge in M^*. This implies that both v_i and v_j must be incident with edges of M_2 not in P_{ij}. If v_i is incident with an edge of M_2 in path P_{ik}, then the path P_{ik} contains K+1 edges of M_2. By the construction of M^*, v_i would be incident with an edge in M^*, contradicting the assumption. Hence we conclude that M^* is a matching of $S_1(G)$, and $|M^*| = s$. Since M_0 is a minimum matching of $S_1(G)$, we have $m \leq s$. However $|M_1| = s(K+1) + (q-s)\cdot K = s + q\cdot K \geq m + q\cdot K = |M|$. Hence we have $|M_1| = |M|$, i.e. M is a minimum matching.

<u>Corollary</u>: For any graph G of order p having q edges,

$$\beta_{1L}(S_{3K+1}(G)) = \beta_{1L}(S_1(G)) + K\cdot q. \quad K = 0, 1, 2, \ldots$$

<u>Proof</u>: The proof is similar to that given in the corollary of theorem 3.8. and is omitted.

The next theorem completes our study of the minimum matching number $\beta_{1L}(S_n(G))$ by treating the remaining case when $n = 3K+3$, $K = 0, 1, 2, \ldots$

<u>Theorem 3.10</u>. Let G be a connected (p,q) graph. Then

$$\beta_{1L}(S_{3K+3}(G)) = \beta_{1L}(S_3(G)) + K\cdot q.$$

<u>Proof</u>: In $S_3(G)$, for each edge (v_i,v_j) in G, the corresponding path P_{ij}: (v_i,u_1,u_2,v_j) is of length 4. A

matching for $S_3(G)$ then involves either one or two of the edges in each such path. Let M_0 be a minimum matching for $S_3(G)$, and let r be the number of edges of G which contain two edges in M_0. Then q-r is the number of edges of G which contain only one edge of M_0, and $|M_0| = 2r + (q-r) = r+q$. Let both the vertices of G and the vertices of $S_{3K+3}(G)$ corresponding to the vertices of G be labeled as $v_1, v_2, \ldots, v_p$. For each edge (v_i, v_j) occuring in G, denote the corresponding path of length 3K+4 in $S_{3K+3}(G)$ by P_{ij}: $(v_i, u_1, u_2, \ldots, u_{3K+3}, v_j)$. Again the vertices between v_i and v_j on P_{ij} have been labeled in a way not showing their dependence on i and j. Now construct a matching M for $S_{3K+3}(G)$ in the following manner. Corresponding to an edge (v_i, v_j) in G which contains two edges of M_0 in $S_3(G)$, at least one of v_i and v_j or both must be incident with edges of M_0 in the path P_{ij} of $S_3(G)$. In the former case, if v_i has this incidence property, we choose the edges (v_i, u_1), $(u_3, u_4), \ldots, (u_{3K+2}, u_{3K+3})$ to be in M. It is to be noted that the last edge selected was not (u_{3K+3}, v_j) but the preceeding edge. In the latter case, we choose the edges (v_i, u_1), $(u_3, u_4), \ldots, (u_{3K+3}, v_j)$ to be in M. Corresponding to an edge (v_i, v_j) in G which contains only one edge of M_0 in $S_3(G)$, we observe that at least one of v_i and v_j must be incident with an edge of M_0 not in the path P_{ij} of $S_3(G)$, otherwise M_0 would not be a matching. If v_i has this incidence property, so that v_i is a neutral vertex in $S_3(G)$, we can choose (u_2,u_3), (u_5,u_6), ..., (u_{3K+2},u_{3K+3}). Hence we have $|M| = r(K+2) + (q-r)(K+1) = (r+q) + q\cdot K$

$$= \beta_{1L}(S_3(G)) + K\cdot q.$$

We next show (1) M is a matching, and (2) M is a minimum matching. For (1), it is clear that from the choice of the edges of M that M is an independent set of edges in $S_{3K+3}(G)$. If M is not a matching, then there must exists at least two consecutive vertices in $S_{3K+3}(G)$ which are not incident with edge in M. By the construction of M, this is impossible for any path P_{ij} containing K+2 edges of M. For those paths P_{ij} which contain only K+1 edges in M, this could possibly occur only at an end vertex of the path P_{ij}. If v_i and u_1 are two consecutive such vertices on path P_{ij}, then v_i must be a weak vertex relative to M_0 in $S_3(G)$, for if not then v_i must be incident with an edge in M_0. By construction of M, this v_i must then be incident with an edge in M in $S_{3K+3}(G)$, constradicting the fact that v_i is a weak vertex relative to M. Hence v_i is a weak vertex relative to M_0. In this case v_j must be incident with an edge of M_0 not in P_{ij}, again by the construction of M_0. Since the vertex on P_{ij} adjacent to v_j will then be a neutral vertex relative to M, we find another contradiction. Thus M is a matching.

To prove (2), suppose that M is not a minimum matching for $S_{3K+3}(G)$. Let M_1 be any minimum matching of $S_{3K+3}(G)$ so $|M_1| < |M|$. We can obtain another matching M_2 from M_1 having the same cardinality as M_1 in the following way. The edges of M_1 are distributed among the q paths P_{ij} of $S_{3K+3}(G)$. Some of these paths - say s in number - contain K+2 edges of M_1, while the remaining q-s paths each contains only K+1 edges of M_1. We will refer to the former as s-edges and to the latter as q-s edges in G. Whenever a path P_{ij} in $S_{3K+3}(G)$ contains K+1 edges of M_1, at least one of v_i and v_j must be incident with an edge of M_1 not in path P_{ij}. Whenever a path P_{ij} contain K+2 edges of M_1, we may replace this set of K+2 edges by a set of K+2 edges of P_{ij} which form a matching in the subgraph P_{ij} itself. Moreover, one of v_i and v_j, or both, is incident with edges of M_1 in P_{ij}, if this latter set is distinct from the former. It is impossible for both v_i and v_j to be incident with edges of M_1 not in P_{ij}, otherwise K+1 edges could have been used in M_1 for the path P_{ij}, contradicting the minimality of $|M_1|$. This replacement yields a matching having the same cardinality as M_1. Repeating this process for every path P_{ij} containing K+2 edges of M_1, we obtain a matching M_2, where $|M_2| = |M_1|$. Next, we choose a new independent set M^* in $S_3(G)$ with the assistance of M_2. For a path P_{ij} containing K+2 edges in M_2, we are going to choose from P_{ij} in $S_3(G)$ two edges in M^*, these two edges will be incident with v_i or

v_j depending on whether or not v_i and v_j are incident with edges of M in P_{ij}, and choose from P_{ij} in $S_3(G)$ one edge in M^*, if the path P_{ij} contains K+1 edges in M_2, and this edge is distance two away from the vertex v_i (i.e. from v_i to the near end vertex of this edge is distance two) or v_j, if v_i or v_j is incident with an edge of M^* not in P_{ij}. It is clear that v_i and v_j cannot both be weak vertices relative to M_2 and cannot both be incident with edges in P_{ij} in M_2. In either case, K+2 edges are needed for P_{ij} in order that M_2 be a matching. In case both v_i and v_j are incident with edges of M^* not in P_{ij}, we can choose an edge of P_{ij} in $S_3(G)$ distance two away from either vertex v_i or v_j. Now, we show that M^* is a matching in $S_3(G)$. If not, there exists at least two consecutive weak vertices on P_{ij} relative to M^* in $S_3(G)$. By the construction of M^*, one of these vertices must be a vertex v_i of G. Then the corresponding path, say P_{ij}, in $S_{3K+3}(G)$ is either an s-edge or a q-s edge. If (v_i,v_j) were a s-edge, then v_j must be incident with an edge in M^* in P_{ij}, and there would be two strong edges relative to M^* between v_i and v_j. Hence consecutive weak vertices on P_{ij} are not possible. If (v_i,v_j) were a q-s edge, then v_j must be incident with an edge of M^* not in P_{ij}. Thus by the construction above, the edge chosen in M^* is distance two away from v_j. This makes the vertex in P_{ij} adjacent to v_i a neutral vertex and not a weak vertex relative to M^*. Hence M^* is a matching and $|M^*| = s + q$. Since M_0 is a minimum matching in $S_3(G)$ we have $r+q \leq s+q$, i.e. $r \leq s$. Now

$$|M_1| = |M_2| = s(K+2) + (q-s)(K+1)$$
$$= (s+q) + q \cdot K \leq |M|$$
$$= (r+q) + K \cdot q.$$

Hence $s = r$, i.e. $|M| = |M_1|$.

Thus we conclude M is a minimum matching.

<u>Corollary</u>: Let G be any (p,q) graph. Then

$$\beta_{1L}(S_{3K+3}(G)) = \beta_{1L}(S_3(G)) + K \cdot q. \quad K = 0, 1, 2, \ldots$$

<u>Proof</u>: The proof is similar to that given in the corollary to theorem 3.8. and is omitted.

CHAPTER 4

EDGE COVERINGS

Section 4.1. The graphs considered in this chapter are understood to have no isolated vertices, since no edge can cover an isolated vertex. A set C of edges of a graph G is called an edge covering set of G, provided each vertex of G is incident with at least one edge that belongs to C. An edge covering set C is called an edge covering (or simply a covering) of G provided there is no edge covering set of G which is properly contained in C. The set of all edges of G, for example, is a covering set but ordinarily is not a covering. The usual edge covering number $\alpha_1(G)$ denotes the cardinal number of a covering having the minimum number of edges. We will designate this number by $\alpha_{1L}(G)$. In addition to the minimum covering number, we also define the maximum covering number $\alpha_{1U}(G)$ as the cardinal number of a covering having the largest possible number of edges.

Let G be a (p,q) graph and C an arbitrary covering of G. We denote by $\alpha = |C|$ the number of edges in C. It is clear that $p/2 \leq \alpha_{1L}(G) \leq \alpha(G) \leq \alpha_{1U}(G) \leq p-1$.
Since $p \leq 2\alpha_{1L}(G)$, we also have

$$\alpha_{1L}(G) \le \alpha_{1U}(G) \le 2\alpha_{1L}(G)-1.$$

If $G=K_p$ denotes a complete graph of order p, we have $\alpha_{1U}(K_p) = p-1$. If G is a cycle C_n or a path P_n, then

$$\alpha_{1U}(C_n) = [^{2n}/_3] \text{ and } \alpha_{1U}(P_n) = [^{2(n+1)}/_3].$$

__Theorem 4.1__. Let C be any covering of a graph G of order p. Then the number of components of <C> in G is p-|C|.

__Proof:__ For any covering C of G, no three edges in C can be the edges of a path of length three in G, since the middle edge could then be omitted from C, violating the definition of a covering. Thus the induced graph <C> must be a union of λ star subgraphs. Let v_j be the center of the j-th star subgraph, $j = 1, 2, \ldots, \lambda$. In case the star subgraph is K_2, it is immaterial which end vertex is called the center. If v_j is adjacent to α_j vertices in the j-th star graph then α_j is less than or equal to the degree of v_j in G. Since C is a covering of G and covers all vertices of G, then

$$\lambda + \sum_{j=1}^{\lambda} \alpha_j = p.$$

If α is the edge covering number of C, then

$$\alpha = \sum_{j=1}^{\lambda} \alpha_j = |C|.$$

and $\lambda + \alpha = p$, or $\lambda = p-\alpha = p-|C|$.

If we choose one edge from each components of C, this set of edges constitutes a matching set M for G, but is not necessary a matching. Then $\lambda=|M|$, and we at once have the following result, which is a variation of Gallai's theorem.

Corollary: Corresponding to every covering C of G, there exists a matching set M of G such that $|C| + |M| = |V(G)|$.

Let $K(p_1, p_2, \ldots, p_j)$ denote a complete j-partite graph with j-pairwise disjoint sets of vertices containing $p_1, p_2, \ldots, p_j$ vertices respectively, the notation being chosen so that $p_1 \leq p_2 \leq \ldots \leq p_j$. Set $p = \sum_{i=1}^{j} p_i$. Then by Gallai's formula and from [6].

$$\beta_{1U}(K(p_1, p_2, \ldots, p_j)) = \min\{[p/2],\ p-p_j\}.$$

We have $\alpha_{1L}(K(p_1, p_2, \ldots, p_j)) = p-\min\{[p/2],\ p-p_j\}$.

Also it is easy to show that

$\alpha_{1U}(K(p_1, p_2, \ldots, p_j)) = \sum_{i=2}^{j} p_i = p-p_1$, a result which we now prove.

It is obvious from theorem 4.1. that $|C|$ depends on λ and that $|C|$ attains its maximum when λ attains its minimum value. When $G = K(p_1, \ldots, p_j)$, the minimum number of components a covering C can have is p_1. Hence from $|C| = p-\lambda$, we have $\alpha_{1U}(K(p_1, \ldots, p_j)) = p-p_1 = \sum_{i=2}^{j} p_i$.

Section 4.2. It is not hard to see that an edge covering of G and the degree sequence of the vertices of a graph of G are closely related. Consider, for example, the following illustrations. Let G_1 and G_2 be graphs of order 7 with degree sequences (5, 3, 2, 2, 2, 1, 1) and (6, 3, 3, 2, 2, 1, 1) respectively (see figure 4.1.)

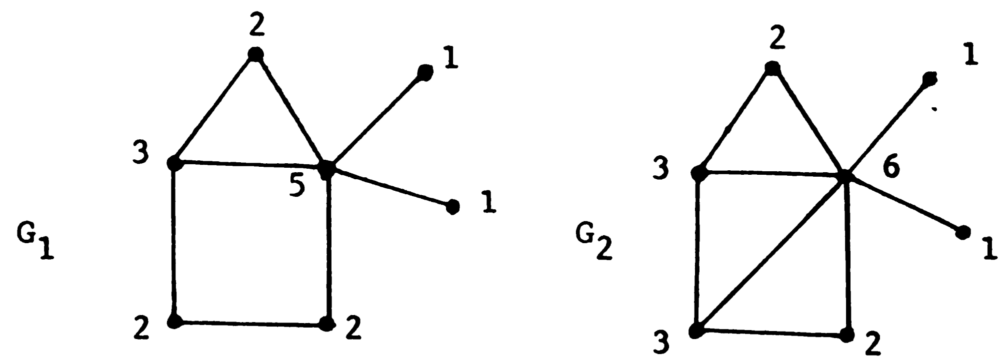


Figure 4.1. The degree sequences of G_1 and G_2.

Here $\alpha_{1U}(G_1) = 5$ and $\alpha_{1U}(G_2) = 6$. It will therefore come as no surprise that an upper bound for $|C|$ can be derived in terms of the degree sequence of G.

Let G be a (p,q) graph with the degree sequence d_1, d_2, ..., d_p, where $d_1 \geq d_2 \geq . . . . . \geq d_p$. Then the degree sequence has the properties $d_1 + 1 \leq p$

and

$$p \leq \sum_{i=1}^{p} d_i .$$

Hence it is possible to determine a unique integer k with

$1 \le k < p$ such that

$$\sum_{i=1}^{k} (d_i + 1) \le p < \sum_{i=1}^{k+1} (d_i + 1).$$

Define $\hat{\alpha} = p - k$ (This definition is due to Prof. B. M. Stewart), we have the following.

<u>Theorem 4.2</u>. Given any graph $G(p,q)$ with degree sequence $d_1, \ge d_2 \ge . . . . \ge d_p$, then for any covering C of G we have $|C| \le \hat{\alpha}$. ($\hat{\alpha}$ defined as above.)

<u>Proof</u>: From theorem 4.1. if C is any covering of G, then

$$p = \lambda + |C| = \sum_{j=1}^{\lambda} (\alpha_j + 1).$$

If we suppose that the star subgraphs of <C> to have been lebeled so that $\alpha_1 \ge \alpha_2 \ge . . . . \ge \alpha_\lambda$ it follows that $\alpha_j \le d_j$ for $1 \le j \le \lambda$.

We have $p = \lambda + |C| = k + \hat{\alpha}$.

Suppose $\lambda < k$, then

$$p = \sum_{j=1}^{\lambda} (\alpha_j + 1) \le \sum_{j=1}^{\lambda} (d_j + 1) < \sum_{j=1}^{k} (d_j + 1) \le p,$$

a contradiction. Hence $k \le \lambda$ and $|C| \le \hat{\alpha}$.

Since in theorem 4.2. C is any covering of G, thus we have derived an upper bound for any covering C of G. Also the equality of the upper bound can be attained. Consider, for example, the case when G consists of n copies of K_2. Then there are n components, with degree sequence $d_1 = d_2 = d_3 = \ldots = d_{2n} = 1$ we have $\hat{\alpha} = \sum_{i=1}^{n} d_i = n = P/_2$.

Section 4.3. In 1957 Norman and Rabin [18] presented a necessary and sufficient condition for determining whether or not a given edge covering is a minimum, and also provided an algorithm for finding a minimum cover. In the case of maximum covers, however, we have been unable to obtain similar necessary and sufficient conditions. But, we were able to find a sufficient condition for a covering to be a maximum. To develop this result, we first need several definitions.

Definition 4.1. Let X be any subset of the edge set E(G). If P(G,X) is a path in G with the property that as one traverses the path from one of the end vertices to the other the successive edges are alternatively two in X and one not in X or vice versa, then P(G,X) is called a bi-alternative path. In the special cases where $|P| < 3$, we agree to regard paths having two consecutive edges in X or a single edge not in X as bi-alternative paths.

Let G be a given graph and C a covering of G. The vertices of G can be partitioned into two sets:

$$C_1 = \{v \mid \deg_C v = 1\}, \quad C_2 = \{v \mid \deg_C v > 1\}.$$

Here $\deg_C v$ (the degree of v relative to C) denotes the number of edges of C incident with v. Evidently $\sum_{v \in C_2} \deg_C v \leq |C_1|$, and equality folds if and only if the induced graph <C> has no component isomorphic to K_2.

<u>Definition 4.2</u>. Let C be a covering of G. We say that C has <u>property (P^*)</u>, if every path joining two C_1 vertices is bi-alternative.

Our main result in this section is to show that a covering C which has property (P^*) must be a maximum covering. The converse is false. For example, consider a graph G consisting of two star subgraphs and an edge joining the two centers of the stars, as shown in the figure 4.2. The edges shown shaded clearly form a maximum cover, but the path P: (v_1, v_2, v_3, v_4) joining two C_1 vertices is not bi-alternative.

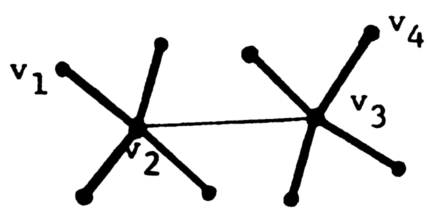


Figure 4.2. A maximum covering failing to have property (P*).

Before proving the main theorem (Theorem 4.3.), we need to establish four lemmas.

<u>Lemma 4.1</u>. If a covering C of graph G has property (P^*), then no edge of G joins two C_2 vertices.

<u>Proof</u>: Let $v_1, v_2 \in C_2$, so v_1 and v_2 are centers of star subgraphs of G. There exist C_1 vertices, say u_1, u_2 joined to v_1, v_2 respectively. If there exists an edge (v_1, v_2) in E(G), it is clear that (v_1, v_2) is not in C. Then P: (u_1, v_1, v_2, u_2) would be a path joining two C_1 vertices which is not bi-alternative. This contradicts the assumption that the covering C has property (P^*).

<u>Lemma 4.2</u>. If a covering C has property (P^*), then no component of <C> is K_2.

<u>Proof</u>: Suppose there exists a component of <C> which is a single edge (u_1, u_2). Then P: (u_1, u_2) is a path between two C_1 vertices which is not bi-alternative.

<u>Lemme 4.3</u>. If a covering C of G has property (P^*), then every C_1 vertex v has $\deg_G v \leq 2$.

<u>Proof</u>: Let $v \in C_1$, so there is exactly one edge of C incident to v. If $\deg_G v > 2$, then there are at least two edges in G which are not in C and are incident to v, let u_1 and u_2 be the other end vertices of two such edges.

Three cases arise. <u>Case (i)</u>. Both u_1, u_2, in C_1. Then P: (u_1, v, u_2) is a path joining two C_1 vertices, which is not bi-alternative, a contradiction.

<u>Case (ii)</u>. Suppose u_1 in C_1 and u_2 in C_2. Then there exists at least one C_1 vertex, say w, adjacent to u_2. If $u_1 \neq w$, then the path P: (w, u_2, v, u_1) between two C_1 vertices is not bi-alternative. If $u_1 = w$, then P: (u_1, u_2, v) between two C_1 vertices is not bi-alternative, again a contradiction.

<u>Case (iii)</u>. Both u_1, u_2 in C_2, then there exists a vertex $w \in C_1$ joined to u_1 (or u_2). The path P: (w, u_1, v) between two C_1 vertices is again not bi-alternative. Hence we conclude that $\deg_G v \leq 2$.

<u>Lemma 4.4</u>. If C is any covering in G which has property (P^*), then for every vertex v in C_2 we have $\deg_C v = \deg_G v$.

<u>Proof</u>: Let v be any vertex in C_2, so $\deg_C v > 1$. If every edge of G incident with v is in C, then $\deg_C v = \deg_G v$ and the lemma follows. Thus we may assume that there exists an edge e incident with v which is not in C. Let u be the other end vertex of edge e. Then u cannot be a C_2 vertex by lemma 4.1. Hence u in C_1. Since v is in C_2, there exists at least one C_1 vertex, say $w \neq u$, joined to v. Then the path P: (w, v, u) is not bi-alternative. But P is a path joining two C_1 vertices, contradicting the assumption

that C has property P^*. Therefore we conclude that $\deg_G v = \deg_C v$.

<u>Theorem 4.3</u>. If C is a covering in G having property (P^*), then C is a maximum covering.

<u>Proof</u>: By lemma 4.2. no component of <C> is K_2, so if λ is the number of components for <C>, then $\lambda = |C_2|$. Also $|C| = |C_1| = \sum_{v \in C_2} \deg_C v$. Let us label the vertices of C_2 as $v_1, v_2, \ldots, v_\lambda$ and those vertices in C_1 as $v_\lambda+1, v_\lambda+2, \ldots, v_p$. Consider the corresponding degree sequence (in G): $d_1, d_2, \ldots, d_\lambda, d_\lambda+1, \ldots, d_p$. From lemma 4.4. we know that $\deg_C v_i = \deg_G v_i = d_i$, for $1 \leq i \leq \lambda$, and from lemma 4.3. we have $\deg_G v_j = d_j \leq 2$, for $\lambda+1 \leq j \leq p$. By the definition of $\hat{\alpha}$ and theorem 4.2. we have

$$\hat{\alpha} = \sum_{i=1}^{\lambda} d_i = \sum_{v \in C_2} \deg_C v = |C|.$$

Since $\hat{\alpha}$ is an upper bound for every covering C of G, in this case $|C| = \hat{\alpha}$, C is then a maximum covering for G.

CHAPTER 5

EDGE MATCHINGS AND COVERINGS

Section 5.1. In [18], Norman and Rabin discussed relations between minimum edge coverings and maximum edge matchings. They proved that if one begins with a minimum covering C, a maximum matching M can be produced from it, and conversely that from a maximum matching M one can construct a minimum cover C. In this section we develop analogous results for arbitrary matchings and coverings. These results generalize Gallai's Theorem in various ways. The graphs discussed are assumed to have no isolated vertices, so $\delta(G) \geq 1$ always holds.

In the corollary to theorem 4.1. we have proved that corresponding to any covering C of G there exists a matching set M of G such that $|C| + |M| = |V(G)|$. The first result developed in this section is that if G is a tree a valid result is obtained when we replace "a matching set M" by "a matching M".

This modified result is not true in general. As a counter example, consider the graph G of order 6 having a covering C shown shaded in the figure 5.1.

Here $|C| = 4$, but $\beta_{1L}(G) = \beta_{1U}(G) = 3$, so far this choice of C, there is no matching M for which $|C| + |M| = 6$.

G

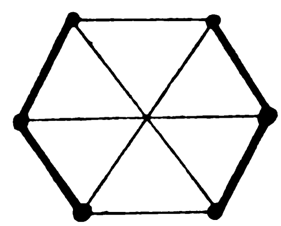


Figure 5.1. A counter example.

This example shows that it is reasonable to make the following definitions.

Definition 5.1. A graph G having the property that corresponding to an arbitrary edge covering C of G, there exists a matching M such that $|C| + |M| = |V(G)|$ is called of Gallai type relative to coverings.

An analogous definition is useful for matchings.

Definition 5.2. A graph G having the property that corresponding to an arbitrary edge matching M of G, there exists a covering C such that $|C| + |M| = |V(G)|$ is called of Gallai type relative to matchings.

Before we prove our next theorem 5.1. we need a preliminary result.

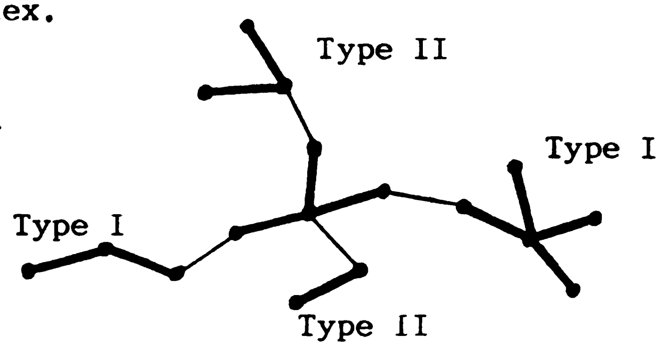
Lemma 5.1. Let T be a tree and C any covering of T. If <C> has more than one component, then there is at least one component C_i of <C> which is joined to the subgraph $T-V(C_i)$ by exactly one edge.

Proof: Let the components of C be the star graphs $C_1, C_2, \ldots C_k$, where $k>1$. If C_1 is joined to $T-V(C_1)$ by exactly one edge, then the lemma follows. If not, then C_1 is joined to $T-V(C_1)$ by more than one edge. Let e_1 be one of these edges. Then e_1 is adjacent to some component, say C_2. If e_1 is the only edge joining C_2 to $T-V(C_2)$, then the lemma follows. Otherwise there exists another edge $e_2 \neq e_1$, and e_2 is adjacent to a component, say C_3, etc. In this process we never encounter a component which has already appeared in our list, since paths in a tree are unique. Hence for $i \neq j$, $C_i \neq C_j$. Since the number of components in <C> is finite, we must terminate our list with a component C_i which is joined to $T-V(C_i)$ by exactly one edge.

Theorem 5.1. Let T be a non-trivial tree and C any covering of T. Then there exists a matching M of T such that $|C| + |M| = |V(T)|$.

Proof: From the corollary of theorem 4.1. we know the existence of a matching set M' such that $|C| + |M'| = |T|$. Of course M' may not be a matching, but only an independent set of edges. In the case of a tree, our problem is to choose an appropriate edge from each component of <C> so that the resulting set of edges constitutes a matching M. The theorem follows at once if <C> has only one component. In this case C = E(T), the edge set of T. Henceforth assume that <C> has more than one component. There are then two kind of components which may occur, namely a component which is connected to the tree by exactly one edge (the existence of at least one such component is proved in lemma 5.1.) and those components of <C> which are connected to the tree by more than one edge. Let us called the former outer components of <C>, and the latter inner components of <C>. Each outer component is adjacent to some component of <C> by a single edge in T. In some cases this single edge is not joined to the center of the star graph (type I), and in the other case the edge is joined to the center of the star graph (type II). If an outer component is K_2, we agree that the center is not an end vertex.

Figure 5.2. The types of outer components.



The adjoining figure shows outer components of each type. When an outer component is removed from T, the edges of C remaining define a cover for the resulting subtree.

A suitable matching M can now be constructed for T. In constructing a matching, one must select one edge from each star graph. From each outer component of type I, the edge adjacent to the single edge joining the adjacent component is selected. From each outer component of type II, we choose an arbitrary edge. Next, we remove all these outer components from T and designate these components as belonging to level I. A new tree is obtained. The argument is now repeated. After a finite number of steps, there will be one or two components of <C> left. In case there is only one component remaining, we choose any edge from it to be an element of M. In case there are two components left, only one edge join these components, since otherwise there would be a cycle in T, a contradiction. Thus either component may be regarded as an outer component. Depending on whether this outer component is of type I or of type II, we choose a suitable edge, and from the remaining component select an arbitrary edge, completing the construction of M. We next show that M is a matching for T. It is already clear that M is a set of independent edges. If M is not a matching, then there exists at least two adjacent weak vertices relative to M in G, say v_i and v_j. These vertices cannot belong to the same component of <C>, so must belong to two distinct adjacent components, say C_i and C_j. The vertices v_i and v_j cannot have degree in C greater than one, since such vertices are neutral with respect to M. Therefore $\deg_C v_i = 1$ and $\deg_C v_j = 1$. Moreover the components C_i and C_j cannot belong to the same level of outer components, because components on the same level are not adjacent. If C_i and C_j were adjacent, this would imply the existence of a cycle in T, except in the trivial case where C_i and C_j are the only components in C. However, in this case, the theorem follows easily by an appropriate choice of one edge from each component. Therefore at some level one of C_i and C_j must be an outer component, and the other an inner component. Let C_i be the outer component, and C_j the inner. Since $\deg_C v_i = 1$, then v_i is a neutral vertex relative to M, by the manner in which M was constructed. This contradicts the assumption that v_i was a weak vertex. The proof that M is a matching in T, and that $|C| + |M| = |V(T)|$ is now complete.

We conclude from definition 5.1. and from theorem 5.1. that all trees are of Gallai type relative to coverings. A number of sufficient conditions for a graph to be of Gallai type relative to matchings (see definition 5.2.) will next be developed.

<u>Theorem 5.2</u>. Let G be a graph of order p with maximum degree $\Delta(G) = p-1$. Then G is of Gallai type relative to matchings.

<u>Proof</u>: Let M be an arbitrary matching for G, and v_0 be a vertex having maximum degree p-1 in G. Suppose that v_0 is a weak vertex relative to M. Then v_0 is the only weak vertex relative to M, for if there exists another weak vertex v in G, then the edge $(v,v_0) \in E(G)$, and $M \cup (v,v_0)$ is a matching set in G containing M, contradicting the assumption that M is a matching. Thus all other vertices of G are then neutral vertices. If u is any neutral vertex, the edge (v_0,u) is in G and the set of edges $C = M \cup (v_0,u)$ is clearly a covering for G. Then $|C| = |M| + 1$, so $|M| + |C| = 2|M| + 1 = p$, proving the theorem when v_0 is a weak vertex relative to M.

Next let v_0 be a neutral vertex relative to M, and W the set of edges joining v_0 to all the weak vertices (relative to M) in G, and set $C = M \cup W$. Then $|C| = |M| + |W|$, and $|C| + |M| = 2|M| + |W| = p$.

The restriction that $\Delta(G) = p-1$, cannot be weakened, as the following example shows. Consider $G = P_4$, so $p = |G| = 4$ and $\Delta(G) = 2$. There exists a matching M with $|M| = 1$, and only one possible covering C, with $|C| = 2$. Then $|M| + |C| < 4$, so G is not of Gallai type relative to matchings.

<u>Theorem 5.3</u>. Let G be a graph of degree p and maximum degree Δ such that $\alpha_{1U}(G) - \alpha_{1L}(G) \geq \frac{p(\Delta-1)}{2}$. Then G is a graph of Gallai type relative to matchings.

<u>Proof</u>: Let M be a matching of order m. If $m = \beta_{1U}(G)$, then by Gallai's theorem there exists a covering C of order $\alpha_{1L}(G)$ such that $\beta_{1U}(G) + \alpha_{1L}(G) = p$.

We may therefore assume that $m < \beta_{1U}(G)$, or $m = \beta_{1U}(G) + r$, for some positive integer r. One must show that there exists a covering C, such that $|C| = \alpha_{1L}(G) + r$, since then $|M| + |C| = p$. To show the existence of such a covering C, by the intermediate theorem of covering [12] we only have to show that $\alpha_{1L}(G) \leq \alpha_{1L}(G) + r \leq \alpha_{1U}(G)$. By theorem 3.6. $\beta_{1L}(G) \geq p/2$ and $\beta_{1U}(G) - \beta_{1L}(G) \leq p/2 - p/2\Delta$

$$= \frac{p(\Delta-1)}{2\Delta}. \quad \text{i.e. } r \text{ is always bounded by } 1 \leq r \leq \frac{p(\Delta-1)}{2\Delta}.$$

Hence $\alpha_{1L}(G) \leq \alpha_{1L}(G) + r \leq \alpha_{1L}(G) + \frac{p(\Delta-1)}{2\Delta} \leq \alpha_{1U}(G)$. and there exists a covering C, such that $|C| = \alpha_{1L}(G) + r$.

Similarly we employ theorem 3.7. to prove a slightly different version of theorem 5.3.

Theorem 5.4. Let G be a graph of order p with maximum degree Δ, and minimum degree $\delta(G) \geq 2$, and having $\alpha_{1U}(G) - \alpha_{1L}(G) \geq \frac{p(\Delta-1)}{2(\Delta+1)}$. Then G is of Gallai type relative to matchings.

Proof: Let M be a matching of order m. If $m = \beta_{1U}(G)$ then $|M| + |C| = p$ is an immediate consequence of Gallai's theorem, since a cover C with $|C| = \alpha_{1L}(G)$ always exists. We henceforth assume that $m < \beta_{1U}(G)$, so $m = \beta_{1U}(G) - r$, for some positive integer r. We must demonstrate the existence of a covering C such that $|C| = \alpha_{1L}(G) + r$. By theorem 3.7. $\beta_{1L}(G) \geq p/(1+\Delta)$ and $\beta_{1U}(G) - \beta_{1L}(G) \leq p/2 - p/(1+\Delta) = \frac{p(\Delta-1)}{2(1+\Delta)}$ i.e. r is bounded by $1 \leq r \leq \frac{p(\Delta-1)}{2(1+\Delta)}$.

Then $\alpha_{1L}(G) \leq \alpha_{1L}(G) + r \leq \alpha_{1L}(G) + \frac{p(\Delta-1)}{2(1+\Delta)} \leq \alpha_{1U}(G)$. By the intermediate theorem of covering [12] there exists a covering C, such that $|C| = \alpha_{1L}(G) + r$, and hence $|M| + |C| = p$.

It is not true in general that a graph which is of Gallai type relative to coverings is also of Gallai type relative to matchings. Neither is it true in general that a graph which is of Gallai type relative to matchings is also of Gallai type relative to coverings, we present some examples to illustrate these assertions.

Example 5.1. Let T be a tree of order 12, as shown in the figure 5.3.

T

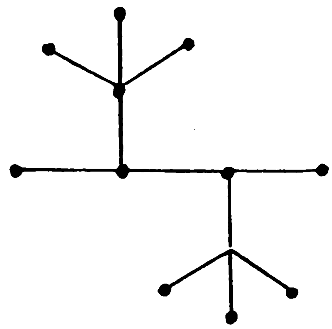


Figure 5.3. A graph not of Gallai type relative to matchings.

By theorem 5.1. T is of Gallai type relative to coverings. There exists a matching M of order 2, but no covering C such that $|C| + |M| = 12$. since $|C| = 8$. Thus T is not of Gallai type relative to matchings.

<u>Example 5.2</u>. Let G be the graph of order 6 shown in figure 5.4.

G

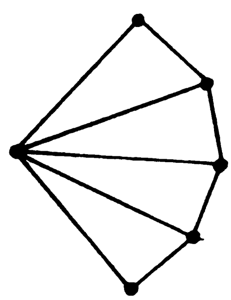


Figure 5.4. A graph not of Gallai type relative to coverings.

Since $\Delta(G) = 5$, by theorem 5.2. G is of Gallai type relative to matchings. Consider a maximum cover C, for which $|C| = 5$. Since $\beta_{1L}(G) = 2$. $|M| \geq 2$ for any matching M, and $|C| + |M| \geq 7$. Thus G is not of Gallai type relative to coverings.

One might wonder if some connectivity requirement might force a graph which is of Gallai type relative to coverings to be of Gallai type relative to matchings, or vice versa. We show the following counter examples, using graphs which are blocks, that is, are 2-connected.

Example 5.3. Let $G = K_p$ with $P \geq 4$. Then G is a 2-connected graph and $\Delta(G) = p-1$, so by theorem 5.2. G is of Gallai type relative to matchings. However G is not of Gallai type relative to coverings. As we now show, consider a covering C of K_p consisting of all p-1 edges incident with a fixed vertex. Since $\beta_{1L}(G) \geq 2$, then for any matching M, $|M| + |C| \geq p+1$.

Example 5.4. Let G be a graph of order 14, having 6 vertices of degree 5, and 8 vertices of degree 3, as shown in figure 5.5. We notice that G is 2-connected, it is connected and has no cut vertices. We readily calculate the values $\alpha_{1L}(G) = 8$, $\alpha_{1U}(G) = 10$ and $\beta_{1L}(G) = 3$, $\beta_{1U}(G) = 6$. Thus by the intermediate theorem for any covering C of G there exists a matching M of G such that $|M| + |C| = 14$. For a matching M with $|M| = 3$, there is no covering C such that $|C| + |M| = 14$, since $8 \leq |C| \leq 10$ for any covering C. We conclude that G is Gallai type relative to coverings but not of Gallai type relative to matchings.

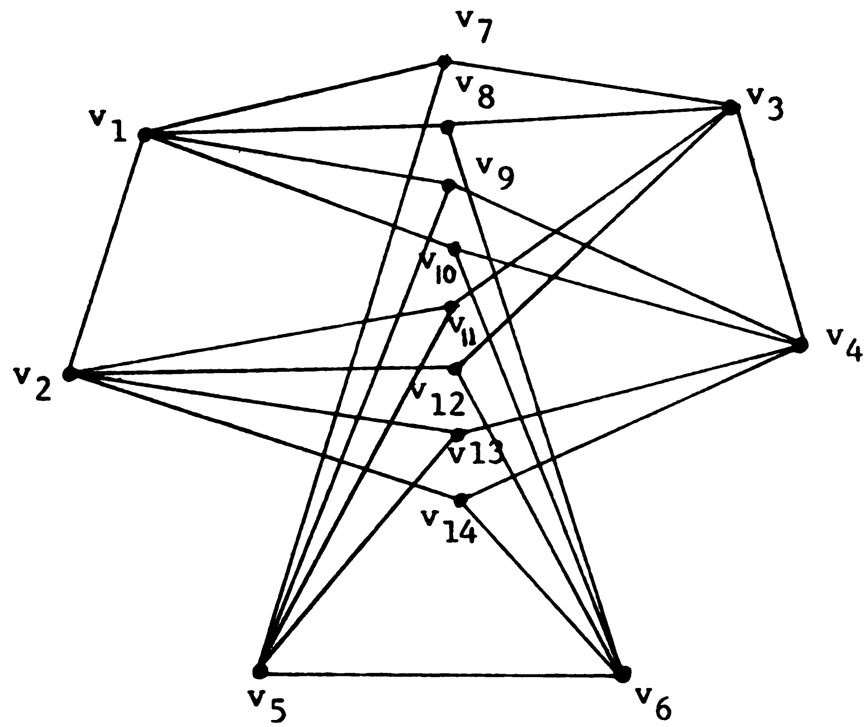


Figure 5.5. A block not of Gallai type relative to matchings.

However, we do have a characterization of graphs which are of Gallai type relative to both matchings and coverings.

<u>Theorem 5.5</u>. A necessary and sufficient condition for a graph G or order p to be of Gallai type relative to both matchings and coverings is that $\alpha_{1U}(G) + \beta_{1L}(G) = p$. We observe that this equation is similar to Gallai's equation $\alpha_{1L}(G) + \beta_{1U}(G) = p$.

<u>Proof</u>: Let G be a graph which is of Gallai type relative both to matchings and to coverings. Consider a minimum matching M of G for which $|M| = \beta_{1L}(G)$. By

hypothesis there exists a covering C of G such that $|M| + |C| = p$. Similarly for a maximum covering C' of G, for which $|C'| = \alpha_{1U}(G)$, there exists a matching M' of G, such that $|M'| + |C'| = p$. Now $|M| \leq |M'|$ and $|C| \leq |C'|$. But inequality is impossible, because $|M| + |C| = |M'| + |C'| = p$. Thus $|M| = |M'|$ and $|C| = |C'|$, so $|M| + |C'| = p$.

Conversely, suppose that $\alpha_{1U}(G) + \beta_{1L}(G) = p$. We seek to prove that G is of Gallai type relative to both matchings and coverings. By Gallai's formula, $\alpha_{1L}(G) + \beta_{1U}(G) = p$. Let M be any matching for G, so $\beta_{1L}(G) \leq |M| \leq \beta_{1U}(G)$.

Then $\quad p - \alpha_{1U}(G) \leq |M| \leq p - \alpha_{1L}(G)$

or $\quad \alpha_{1L}(G) \leq p - |M| \leq \alpha_{1U}(G)$.

By intermediate value theorem for covering, there exists a covering C, such that $|C| = p - |M|$. i.e. $|M| + |C| = p$. Similarly, let C be any covering for G, so $\alpha_{1L}(G) \leq |C| \leq \alpha_{1U}(G)$. Then $\quad p - \beta_{1U}(G) \leq |C| \leq p - \beta_{1L}(G)$.

or $\quad \beta_{1L}(G) \leq p - |C| \leq \beta_{1U}(G)$.

By intermediate value theorem for matchings, there exists a matching M, such that $|M| = p - |C|$. Then $|M| + |C| = p$. Thus G is of Gallai type relative to both coverings and matchings.

Section 5.2. We again consider graphs G of order p having no isolated vertices. For any matching M of G:

$$1 \leq \beta_{1L}(G) \leq |M| \leq \beta_{1U}(G) \leq p/2.$$

The following inequalities hold for an arbitrary covering C of G:

$$p/2 \leq \alpha_{1L}(G) \leq |C| \leq \alpha_{1U}(G) \leq p-1.$$

It should be observed that every number in the second sequence of inequalities is greater or equal to each number in the first sequence. Clearly a covering C is also a matching if and only if $|C| = p/2$, so G has a 1-factor. From these inequalities we readily obtain

$$1/2 < 1/2 + 1/p \leq \frac{|M| + |C|}{p} \leq 3/2 - 1/p < 3/2.$$

Definition 5.3. For an arbitrary matching M and an arbitrary covering C of a graph G, we define the edge Gallai ratio.

$$\lambda_1 = \lambda_1(G,M,C) = \frac{|M| + |C|}{p}.$$

We first prove that there exist graphs G and suitable natchings and coverings of G, such that the Gallai ratio is arbitrary close to the upper bound $3/2$. To show this, let G be a complete graph K_p.

Case (i). If $p = 2n$, then $\alpha_{1U}(K_p) = p-1$, and $\beta_{1L}(K_p) = n$. thus $\alpha_{1U}(K_p) + \beta_{1U}(K_p) = 2n - 1 + n = 3n - 1$, and

$$\frac{\alpha_{1U}(K_p) + \beta_{1U}(K_p)}{p} = \frac{3}{2} - \frac{1}{2n}.$$

Clearly the edge Gallai ratio is arbitrary near $3/2$ for p sufficiently large.

Case (ii). If $p = 2n + 1$, then $\alpha_{1U}(K_p) = 2n$ and

$\beta_{1U}(K_p) = \beta_{1L}(K_p) = n$. Thus $\alpha_{1U}(K_p) + \beta_{1U}(K_p) = 3n$ and $\frac{\alpha_{1U}(K_p) + \beta_{1U}(K_p)}{p} = \frac{3n}{2n+1} = \frac{3}{2 + {}^{1}/_{n}}$

Again, it is evident that edge Gallai ratio is arbitrary near ${}^{3}/_{2}$ when p is sufficiently large.

Surprisingly, it turns out that the lower bound for the Gallai ratio can be substantially improved. We prove the following result.

Theorem 5.6. Let G be a graph of order p. Then for arbitrary matchings M and coverings C of G, $\lambda_1(G,M,C) \geq {}^{3}/_{4}$, and there exist graphs for which this lower bound is attained.

Proof: Suppose that there exists a graph G, a matching M and a covering C for G, such that $\lambda_1(G,M,C) < {}^{3}/_{4}$. We seek a contradiction. In particular, then $\alpha_{1L}(G) + \beta_{1L}(G) < {}^{3p}/_{4}$, so we may as well assume that $|M| = \beta_{1L}(G)$ and $|C| = \alpha_{1L}(G)$. Thus $|M| + |C| < {}^{3p}/_{4}$ and so ${}^{p}/_{2} \leq |C| < {}^{3p}/_{4}$.

Now, the edges in matching M cover exactly $2|M|$ vertices, so there exists $p-2|M|$ weak vertices in G, relative to M. No two of these weak vertices can be adjacent, otherwise M could be enlarged by including the edge joining these two weak vertices, contradicting the fact that M is a matching. Therefore there exists at least $p-2|M|$ independent vertices in G. Since $|M| < \frac{3}{4}p - |C|$, then $p - 2|M| > 2|C| - {}^{p}/_{2}$.

If p is even, there exists at least $1 + 2|C| - {}^{p}/_{2}$ independent vertices in G. Since $2|C| \geq p$ then $|C| + 1 \geq {}^{p}/_{2}$, and $2|C| - {}^{p}/_{2} + 1 > |C|$. If p is odd, there exists at least $2|C| - [{}^{p}/_{2}]$independent vertices in G. Since $2|C| \geq p$ and p is odd, then $2|C| > p$, $|C| > [{}^{p}/_{2}]$, and $2|C| - [{}^{p}/_{2}] > |C|$. But C is an edge covering this implies that G has at most $|C|$ independent vertices. In any case a contradiction. This completes the proof of the theorem.

The equality $\alpha_{1L}(G) + \beta_{1L}(G) = {}^{3p}/_{4}$ can be attained for certain graphs G whose order is a multiple of four. A very simple example is afforded by $G = P_4$, a path of length 3. Here $\lambda_1(P_4, M, C)$ assumes only two values, namely ${}^{3}/_{4}$ and 1. A more general example is given by taking for G the union of n vertex disjoint copies of P_4. By adjoining suitable edges one can easily obtain a connected graph for which equality holds.

We next consider properties of graphs which attain this minimum possible value of the Gallai ratio, and employ an argument similar to that used in theorem 5.6.

<u>Theorem 5.7</u>. Let G be a graph of order $p = 4n$. Then $\alpha_{1L}(G) + \beta_{1L}(G) = {}^{3p}/_{4}$ if and only if $\alpha_{1L}(G) = {}^{p}/_{2}$ and $\beta_{1L}(G) = {}^{p}/_{4}$.

<u>Proof</u>: Suppose that $\alpha_{1L}(G) + \beta_{1L}(G) = {}^{3p}/_{4}$, so p is a multiple of four, and ${}^{p}/_{2} \leq \alpha_{1L}(G) < {}^{3p}/_{4}$. Let M be a

matching with $|M| = \beta_{1L}(G)$ and C a covering with $|C| = \alpha_{1L}(G)$. Now the matching M covers exactly $2|M|$ vertices of G, so there are $p-2|M|$ weak vertices with respect to M, no two of which can be adjacent. Hence there are at least $p-2|M|$ independent vertices in G. We next show that there are at least two such weak vertices. Since $|M| = {}^{3p}/_{4} - |C|$, then $p - 2|M| = 2|C| - {}^{p}/_{2}$, and since ${}^{p}/_{2} \leq |C| < {}^{3p}/_{4}$, we have $2|C| \geq p$, and $p - 2|M| \geq {}^{p}/_{2} \geq 2$. There are $p - |C|$ components in C , hence G has at most $|C|$ independent vertices in G. Then ${}^{p}/_{2} \leq 2|C| - {}^{p}/_{2} \leq |C|$, so $|C| \leq {}^{p}/_{2}$. Thus $|C| = {}^{p}/_{2}$ and $|M| = {}^{p}/_{4}$. The converse is trivial.

In the remainder of this section, we prove two theorems which relate matchings and coverings to the Gallai ratio of a graph. They provide a characterization of graphs for which every covering (or every matching) contains the same number of edges.

<u>Theorem 5.8</u>. Let G be a graph of order p. Then $\alpha_{1L}(G) = \alpha_{1U}(G)$ if and only if $\lambda_1(G,M,C) \leq 1$ for all matchings M and coverings C of G.

<u>Proof</u>: If $\alpha_{1L}(G) = \alpha_{1U}(G)$, then for any matching M and covering C, we have
$|C| + |M| \leq \alpha_{1U}(G) + \beta_{1U}(G) = \alpha_{1L}(G) + \beta_{1U}(G) = p$, by Gallai's theorem. Hence $\lambda_1(G,M,C) \leq 1$.

Conversely, if $\lambda_1(G,M,C) \leq 1$ for all matchings M and coverings C of G, then by Gallai's formula

$$p = \alpha_{1L}(G) + \beta_{1U}(G) \leq \alpha_{1U}(G) + \beta_{1U}(G) \leq p.$$

Thus $\alpha_{1L}(G) = \alpha_{1U}(G)$.

<u>Corollary</u>: Let G be a graph of order p. Then $\alpha_{1L}(G)=\alpha_{1U}(G)$ if and only if $\lambda_1(G,M,C) \leq 1$ for a maximum matching M and a maximum covering C of G.

<u>Theorem 5.9</u>. Let G be a graph of order p. Then $\beta_{1L}(G) = \beta_{1U}(G)$ if and only if $\lambda_1(G,M,C) \geq 1$ for all matchings M and coverings C of G.

<u>Proof</u>: If $\beta_{1L}(G) = \beta_{1U}(G)$, then for any matching M of G and any covering C of G we have

$$|C| + |M| \geq \alpha_{1L}(G) + \beta_{1L}(G) = \alpha_{1L}(G) + \beta_{1U}(G) = p.$$

Hence $\lambda_1(G,M,C) \geq 1$.

Conversely, if $\lambda_1(G,M,C) \geq 1$ for all matchings M and coverings C of G, by Gallai's formula

$$p = \alpha_{1L}(G) + \beta_{1U}(G) \geq \alpha_{1L}(G) + \beta_{1L}(G) \geq p.$$

Hence $\beta_{1U}(G) = \beta_{1L}(G)$.

<u>Corollary</u>: Let G be a graph of order p. Then $\beta_{1L}(G)=\beta_{1U}(G)$ if and only if $\lambda_1(G,M,C) \geq 1$ for a minimum matching M and a minimum covering C of G.

CHAPTER 6

VERTEX COVERINGS AND INDEPENDENT SETS OF VERTICES

Section 6.1. Many of the concepts which we have developed in our study of edge matchings and edge coverings can be extended to maximal independent vertex sets and to vertex coverings. As was the case for maximum edge matchings and minimum edge coverings, earlier investigations have been devoted almost exclusively to the study of maximum independent sets of vertices and minimum vertex covers. Little or no attention has been given to a general study of independent vertex sets and to vertex covers. We find it possible to define vertex coverings and maximal independent vertex sets in a more general way. We again assume that the graphs considered have no isolated vertices.

Definition 6.1. A vertex covering set C_0 is any subset of the vertex set V(G) of G, which covers all the edges of G. In particular, V(G) itself is a vertex covering set for G.

Definition 6.2. A vertex covering is a vertex covering set which is minimal, in the sense that it contains no proper subset which is also a vertex covering set.

Let $\alpha_{0U} = \alpha_{0U}(G)$ denote the number of vertices in a vertex covering having maximum cardinality and $\alpha_{0L} = \alpha_{0L}(G)$ the number of vertices in a vertex covering with minimum cardinality. Let G be any (p.q) graph with no isolated vertices, and C_0 any vertex covering of G of order α_0. Then the following inequalities are obvious:

$$1 \leq \alpha_{0L} \leq \alpha_0 \leq \alpha_{0U} \leq p-1.$$

Furthermore, there exist graph G such that $\alpha_{0U}(G) = p-1$ and $\alpha_{0L}(G) = 1$, so both bounds are attainable. In fact, both bounds can be attained with a single graph, as shown by the example of a star graph S of order p. Clearly $\alpha_{0U}(S) = p-1$, and $\alpha_{0L}(S) = 1$. It is readily seen that there exist only two vertex coverings for S. This example also shows that one cannot obtain an intermediate value theorem as was the case for edge coverings, since for any integer K such that $1 < K < p-1$ there is no vertex covering C_0 such that $|C_0|=K$.

We next define a maximal independent set of vertices for a graph G.

<u>Definition 6.3</u>. <u>An independent vertex set M_0</u> is any subset of the vertex set V(G) of G with the property that no two vertices in M_0 are adjacent.

Definition 6.4. A maximal independent vertex set is an independent vertex set M_0 which is maximal. This means that M_0 ceases to be an independent vertex set if any vertex of G not in M_0 is adjoined to M_0.

Let $\beta_{0U} = \beta_{0U}(G)$ denote the number of vertices in a maximal independent vertex set having maximum cardinality, and $\beta_{0L} = \beta_{0L}(G)$ in one having minimum cardinality. Let G be a (p,q) graph and M_0 any maximal independent vertex set with order β_0. Then it is clear we have

$$1 \leq \beta_{0L} \leq \beta_0 \leq \beta_{0U} \leq p-1.$$

We again consider the example of a star graph S of order $p \geq 3$, for which $\beta_{0L}(S) = 1$ and $\beta_{0U}(S) = p-1$. There are clearly only two maximal independent vertex sets for S, one consisting of the center of the star graph S and the other consisting of the remaining vertices. We again see that no intermediate value theorem is possible. If K is any integer such that $\beta_{0L}(G) < K < \beta_{0U}(G)$, there may or may not be a maximal independent vertex set having order K.

Gallai [10] has shown that for any graph G, of order p: $\alpha_{0L}(G) + \beta_{0U}(G) = p$.

We will prove in this section a rather surprising variation of Gallai's result, namely that

$$\alpha_{0U}(G) + \beta_{0L}(G) = p.$$

Thus Gallai's theorem also holds when the subscripts L and

U are interchanged. First we need to establish the following.

Lemma 6.1. Let G be a graph of order p having no isolated vertices. Then C_0 is a vertex covering for G if and only if the complementary vertex set $M_0 = V(G) - C_0$ is a maximal independent set in G.

Proof: Let C_0 be a vertex covering for G. Now $M_0 = V(G) - C_0$ is an independent set of vertices, for if there exist two vertices v_1 and v_2 in M_0 which are not independent, then v_1 is adjacent to v_2 and the edge $e = (v_1, v_2)$ is not covered by any vertex in C_0. This contradicts the assumption that C_0 is a vertex covering.

We next prove that M_0 is maximal. Suppose that M_0 is not maximal, so that there exists a set $U \subset C_0$ such that $U \cup M_0$ is an independent set of vertices. Let $u \in U$, so u is joined only to vertices in C_0 and not to vertices in M_0. Then $C_0 - u$ is also a vertex covering, contradicting the minimality of C_0.

Conversely, let M_0 be a maximal independent vertex set for G, and set $C_0 = V(G) - M_0$. Then C_0 must be a vertex covering set, for if there exists an edge not covered by C_0 this edge must join two vertices in M_0, contradicting the assumption that the vertices of M_0 are independent. The set C_0 is also a vertex covering. If the contrary is assumed, then C_0 has a proper subset R which is a vertex covering set. Let $v \in C_0 - R$. Then v is joined only to vertices in C_0, for if v is joined to some vertex in M_0 by an edge e, this edge e clearly is not covered by R. But if v joins only vertices in C_0, then $M_0 \cup \{v\}$ is an independent set of vertices, contradicting the maximality of M_0. Thus C_0 is a vertex covering.

We can now prove our principal result of this section, an extension of Gallai's well known result for minimum vertex covers and maximum sets of independent vertices.

__Theorem 6.1__. Let G be any graph of order p with no isolated vertices. Then $\alpha_{0U}(G) + \beta_{0L}(G) = p$.

__Proof__: Let C_0 be a vertex covering of maximum order $\alpha_{0U}(G)$, and let $M_0 = V(G) - C_0$. By lemma 6.1. M_0 is a maximal independent vertex set, we seek to show that $|M_0| = \beta_{0L}(G)$, that is that M_0 has the minimum number of vertices possible. If M_0 is not a minimum, then there exists a maximal independent vertex set M_0^* such that $|M_0^*| < |M_0|$. Let $C_0^* = V(G) - M_0^*$. By lemma 6.1. C_0^* is a vertex covering and $|C_0^*| > |C_0|$, contradicting the assumption that C_0 had maximum order. Hence M_0 is a minimum maximal independent vertex set, and $|M_0| = \beta_{0L}(G)$. Since $|C_0| + |M_0| = p$, the theorem follows.

It is obvious that the assumption made in theorem 6.1. that G has no isolated vertices is not essential. Suppose there exists a set N of isolated vertices in G. Then we can set $\bar{G} = G - N$, and $|\bar{G}| = p - n$, where $|N| = n$. By theorem 6.1. $\alpha_{0U}(\bar{G}) + \beta_{0L}(\bar{G}) = p\text{-}n$. Since $\alpha_{0U}(\bar{G}) = \alpha_{0U}(G)$ and $\beta_{0L}(\bar{G}) = \beta_{0L}(G) - n$, then $\alpha_{0U}(G) + \beta_{0L}(G) = p$.

Section 6.2. We again consider graphs G of order p having no isolated vertices. We have observed in section 6.1. that for any maximal independent vertex set M_0 of G:

$$1 \leq \beta_{0L}(G) \leq |M_0| \leq \beta_{0U}(G) \leq p\text{-}1.$$

and for any vertex covering C_0 of G, we have:

$$1 \leq \alpha_{0L}(G) \leq |C_0| \leq \alpha_{0U}(G) \leq p\text{-}1.$$

From these inequalities we readily obtain:

$$0 < \frac{2}{p} \leq \frac{|M_0| + |C_0|}{p} \leq 2 - \frac{2}{p} < 2.$$

Definition 6.5. For an arbitrary maximal independent set of vertex M_0 and an arbitrary vertex covering C_0 of a graph G, we define the vertex Gallai ratio

$$\lambda_0 = \lambda_0(G, M_0, C_0) = \frac{|M_0| + |C_0|}{p}.$$

We first show that there exist graphs G and suitable maximal independent vertex sets and vertex coverings of G such that the Gallai ratio is arbitrary close to the lower bound 0 and to the upper bound 2. To show this, let G be a star graph of order p, then $\alpha_{0L}(G) = \beta_{0L}(G) = 1$ and $\lambda_0 = \frac{\alpha_{0L} + \beta_{0L}}{p} = \frac{2}{p}$. Clearly λ_0 is arbitrary near 0 when p is sufficiently large. Also $\alpha_{0U}(G) = \beta_{0U}(G) = p-1$ and $\lambda_0 = \frac{\alpha_{0U} + \beta_{0U}}{p} = 2 - \frac{2}{p}$. It is evident that λ_0 will be arbitrary near 2, when p is sufficiently large.

Next we prove a theorem which relates independent vertex sets and vertex coverings to the vertex Gallai ratio of a graph. It also provides a characterization of graphs for which all vertex coverings contain the same number of vertices and all maximal independent vertex sets also have the same number of vertices.

Theorem 6.2. Let G be a graph of order p. Then the following three statements are equivalent.

(1) $\alpha_{0L}(G) = \alpha_{0U}(G)$.

(2) $\beta_{0L}(G) = \beta_{0U}(G)$.

(3) $\lambda_0(G, M_0, C_0) = 1$ for all maximal independent vertex sets M_0 and vertex coverings C_0 of G.

Proof: (1) implies (2). By Gallai's theorem $p = \alpha_{0L}(G) + \beta_{0U}(G) = \alpha_{0U}(G) + \beta_{0U}(G)$, and from theorem 6.1. $p = \alpha_{0U}(G) + \beta_{0L}(G)$. Hence $\beta_{0L}(G) = \beta_{0U}(G)$.

(2) implies (3). For any maximal independent vertex set M_0 and vertex covering C_0, we have:

$|C_0| + |M_0| \leq \alpha_{0U}(G) + \beta_{0U}(G) = \alpha_{0U}(G) + \beta_{0L}(G) = p$, by theorem 6.1. Hence $\lambda_0(G, M_0, C_0) \leq 1$. Furthermore, $|C_0| + |M_0| = |C_0| + \beta_{0U}(G) \geq \alpha_{0L}(G) + \beta_{0U}(G) = p$, by Gallai's formula. Hence $\lambda_0(G, M_0, C_0) \geq 1$, and we conclude that $\lambda_0(G, M_0, C_0) = 1$.

(3) implies (1). Since $p = \alpha_{0L}(G) + \beta_{0U}(G) = \alpha_{0U}(G) + \beta_{0U}(G)$ then $\alpha_{0L}(G) = \alpha_{0U}(G)$.

It is interesting to compare theorem 6.2. with the analogous results obtained for edge matchings and coverings in theorem 5.8. and theorem 5.9.

<u>Remark</u>: If for some graph G, $\alpha_{0L}(G) \neq \alpha_{0U}(G)$, (or if $\beta_{0L}(G) \neq \beta_{0U}(G)$), then the vertex Gallai ratio $\lambda_0(G)$ assumes values less than unity and also greater then unity. By theorem 6.2. $\alpha_{0L}(G) \neq \alpha_{0U}(G)$ implies $\beta_{0L}(G) \neq \beta_{0U}(G)$, and vice-versa. From theorem 6.1. $\alpha_{0U}(G) + \beta_{0L}(G) = p$ and this implies that $\alpha_{0U}(G) + \beta_{0U}(G) > p$ and also that $\alpha_{0L}(G) + \beta_{0L}(G) < p$. The last two inequalities show that $\lambda_0 > 1$ and $\lambda_0 < 1$ both occur for such graphs.

CHAPTER 7

DOMINATING NUMBERS

Section 7.1. Dominating numbers have been discussed by Ore [22] and also by Berge [1], who refers to them as coefficients of external stability. An application of dominating numbers which readily comes to mind is the problem of the five queens. In the game of chess, what is the fewest number of queens which can be placed on a standard chessboard so that every square is guarded (dominated) by at least one of the queens? It is easy to show that five queens can be placed so that this condition is satisfied, and that no fewer suffice.

In this chapter it is assumed that G is a (p,q) graph which has no isolated vertices.

Definition 7.1. A subset D_0 of V(G) is called a vertex dominating set, if every vertex of G not in D_0 is adjacent to at least one vertex in D_0.

Definition 7.2. A vertex dominating set D_0 is called minimal if no proper subset of D_0 is a vertex dominating set.

Let us denote the minimum and maximum number of vertices in any minimal vertex dominating set of graph G by $\sigma_{0L} = \sigma_{0L}(G)$ and $\sigma_{0U} = \sigma_{0U}(G)$ respectively, and refer to these parameters as the vertex dominating numbers. If D_0 is any minimal vertex dominating set of order σ_0, then it is clear that $1 \leq \sigma_{0L} \leq \sigma_0 \leq \sigma_{0U} \leq p-1$.
A star graph S of order p serves as an example to show that the upper and lower bounds for σ_0 can both be attained, since $\sigma_{0L}(S) = 1$ and $\sigma_{0U}(S) = p-1$. From this example it is also evident that there exist only two minimal vertex dominating sets for S, and hence in general there is no possible intermediate value theorem as in the case of edge coverings and edge matchings. The range of values of $\sigma_0(G)$ may therefore be expected to contain gaps.

Ore [22] proved that an independent vertex set is maximal if and only if it is a vertex dominating set. We shall prove the following generalization:

Theorem 7.1. An independent vertex set is maximal if and only if it is a minimal vertex dominating set.

Proof: Let C_0 be a maximal independent vertex set. If C_0 failed to be a vertex dominating set, then there exists some vertex v in $V(G) - C_0$, such that v is not adjacent to any of the vertices in C_0. Then $C_0 \cup \{v\}$ forms a larger independent set of vertices, which contradicts the maximality of C_0. Hence C_0 is a vertex dominating set. If C_0 is not minimal (as a dominating set), then there exists a vertex u in C_0 such that $C_0 - \{u\}$ also forms a vertex dominating set. But vertex u fails to be dominated by $C_0 - \{u\}$, a contradiction.

Conversely, let D_0 be any independent vertex set which is also a minimal vertex dominating set. If D_0 were not a maximal independent vertex set, then there exists some vertex w in $V(G) - D_0$ such that $\{w\} \cup D_0$ is an independent vertex set. This implies that w is not dominated by D_0, again a contradiction.

The vertex independence numbers are bounded above and below by the vertex dominating numbers.

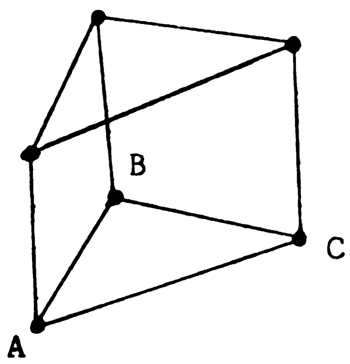
Corollary 7.1. For any graph G of order p.

$$1 \leq \sigma_{0L}(G) \leq \beta_{0L}(G) \leq \beta_{0U}(G) \leq \sigma_{0U}(G) \leq p-1.$$

Proof: Since from theorem 7.1. every maximal independent set of vertices is a minimal vertex dominating set, the result follows immediately.

There exist graphs such that $\beta_{0U} < \sigma_{0U}$ and there are graphs such that $\sigma_{0L} < \beta_{0L}$. For example, consider the following graphs shown in the figure 7.1. and figure 7.2.

G_1



G_2

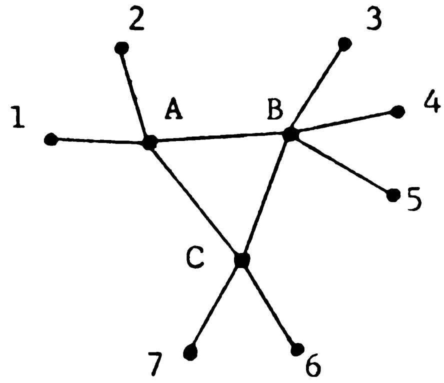


Figure 7.1. A graph illustrating $\beta_{0U}(G_1) < \sigma_{0U}(G_1)$.

Figure 7.2. A graph illustrating $\sigma_{0L}(G_2) < \beta_{0L}(G_2)$.

Then $\beta_{0U}(G_1) = 2$ and $\sigma_{0U}(G_1) = 3$, since the vertices A, B, and C form a minimal vertex dominating set. Where $\sigma_{0L}(G) = 3$ and $\beta_{0L}(G_2) = 5$. Here the vertex set $\{A,B,C\}$ and $\{1,2,B,6,7\}$ can be used.

Definitions can be made for minimal edge dominating sets analogous to those made for minimal vertex dominating sets.

Definition 7.3. A subset D_1 of E(G) is called an edge dominating set if every edge of G not in D_1 is adjacent to at least one edge in D_1.

Definition 7.4. An edge dominating set D_1 is called minimal if no proper subset of D_1 is an edge dominating set.

We denote by $\sigma_{1L} = \sigma_{1L}(G)$ and $\sigma_{1U} = \sigma_{1U}(G)$ respectively the minimum and maximum number of edges in any minimal edge dominating set of G and refer to these parameters as the edge dominating numbers. If D_1 is any minimal edge dominating set having cardinality σ_1, then

$$1 \le \sigma_{1L} \le \sigma_1 \le \sigma_{1U} \le p-2.$$

The fact that p-2 is an upper bound for σ_1 will be shown later. A star graph S of order p is an example showing that the lower bound can be attained, and the following figure 7.3. shows that the upper bound p-2 is also attainable.

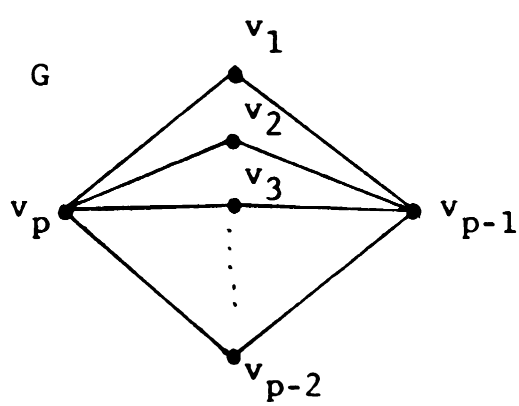


Figure 7.3. $\sigma_{1U}(G) = p-2$.

Then $\sigma_{1U}(G) = p-2$, as can be seen by considering the edge set $\{(v_i,v_p) \mid i = 1, 2, \ldots, p-2\}$. Also $\sigma_{1L}(G) = 2$, and there are no minimal edge dominating sets whose cardinality lies between 2 and p-2. Thus no intermediate value theorem is possible, and the range of values of the edge dominating numbers may contain gaps.

We next show that p-2 is indeed an upper bound for σ_{1U}.

Suppose a graph G of order p has a minimal edge dominating set D_1 such that $|D_1| = p-1$. Each component of the edge induced graph $<D_1>$ is a star graph, so $<D_1>$ is a forest. Supposing that $<D_1>$ has p' vertices, then $<D_1>$ has $p' - \lambda$ edges, where λ is the number of components in $<D_1>$. Then by hypothesis $p' - \lambda = p - 1$ where $\lambda \geq 1$, so $p' \geq p$. Hence $p = p'$ and $\lambda = 1$, i.e. $<D_1>$ is connected and is a spanning star graph. If we consider any edge of G not in D_1 dominated by an edge (v_1,v_2) of D_1, it is clear that it is already dominated by other edges of D_1. Then $D_1 - (v_1,v_2)$ also forms an edge dominating set, contradicting the minimality of D_1. Therefore $|D_1| \leq p-2$.

There is a relation between edge matchings and minimal edge dominating sets, as shown in the following

<u>Theorem 7.2</u>. An independent set of edges of a graph G is a matching for G if and only if it is a minimal edge dominating set.

<u>Proof</u>: Let M be an edge matching. If M fails to be an edge dominating set, then there exists an edge e in E(G) - M, which is not adjacent to any edge of M. This implies that $M \cup \{e\}$ is an independent set of edges contradicts the assumption that M is a matching. Hence M is an edge dominating set. If M is not minimal, then there exists an edge e' in M such that $M - \{e'\}$ also forms an edge dominating

set, but then edge e' is not dominated by M - $\{e'\}$, a contradiction. Thus M is a minimal edge dominating set.

Conversely, let D_1 be an independent set of edges which is a minimal edge dominating set. If D_1 were not a matching, then there exists an edge e" in E(G) - D_1 such that $D_1 \cup \{e"\}$ is an independent set of edges. However, this implies that edge e" is not dominated by D_1, a contradiction.

The following corollary is the analog of corollary 7.1. shows that the edge independence numbers are bounded above and below by the edge dominating numbers.

<u>Corollary 7.2</u>. For any graph G of order p we have

$$1 \leq \sigma_{1L}(G) \leq \beta_{1L}(G) \leq \beta_{1U}(G) \leq \sigma_{1U}(G) \leq p-2.$$

<u>Proof</u>: Since from theorem 7.2. every edge matching is a minimal edge dominating set, the result follows immediately.

There exist graph having $\beta_{1U} < \sigma_{1U}$. For example from the adjoining figure 7.4.

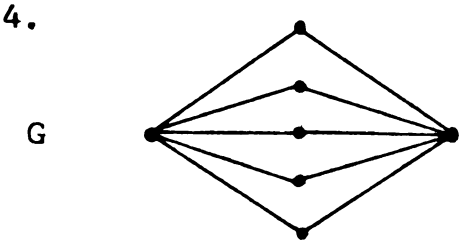


Figure 7.4. A graph illustrating $\beta_{1U}(G) < \sigma_{1U}(G)$.

We have $\beta_{1U} = 2$ and $\sigma_{1U} = 5$.

The situation is quite different for the parameters $\beta_{1L}(G)$ and $\sigma_{1L}(G)$. We have the following.

Theorem 7.3. For any graph G with no isolated vertices,

$$\sigma_{1L}(G) = \beta_{1L}(G).$$

In order to establish this equality, we need the following.

Lemma 7.1. Let D_1 be a minimal edge dominating set of G, having minimum cardinality, so $|D_1| = \sigma_{1L}(G)$. Suppose that the edge induced subgraph $\langle D_1 \rangle$ has a component C which is a star graph different from K_2, so $|C| \geq 3$. Then there exists a non-empty set of at least $|C| - 2$ vertices in $G - \langle D_1 \rangle$, which are joined to at least $|C| - 2$ end vertices of the component C.

Proof: Define $W = G - \langle D_1 \rangle$. The graph W is a set of independent vertices, for if two vertices of W were joined by an edge e, then e is not dominated by D_1, a contradiction. Let $|C| = n+1$, where $n \geq 2$. Since D_1 is an edge dominating set in G of minimum cardinality, each edge of C dominates at least one edge of G not in C, since otherwise such an edge of C could be omitted from D_1, a contradiction.

We maintain it is always possible to choose a set of distinct vertices $u_1, u_2, \ldots, u_m$ in W, such that (1) no three end vertices of C can be joined to a single vertex of W and to no other vertices of W and (2) no two (or more) pairs of end vertices of C can be joined to two (or more) distinct vertices of W and to no other vertices of W.

Suppose there exists three end vertices v_1, v_2, v_3 of C which are joined to a single vertex u_1 of W and to no other vertices of W, (see figure 7.5.)

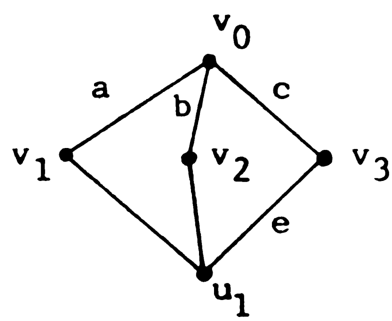


Figure 7.5. Three end vertices of C, joined to a single vertex of W.

Then we can delete edges a and b from D_1 and add edge e to D_1, thereby obtaining a new edge dominating set with smaller cardinality than $|D_1|$, a contradiction.

Next suppose that there exists two pairs of end vertices of C which are joined to two distinct vertices u_1 and u_2 of W and to no other vertices of W, (see figure 7.6.)

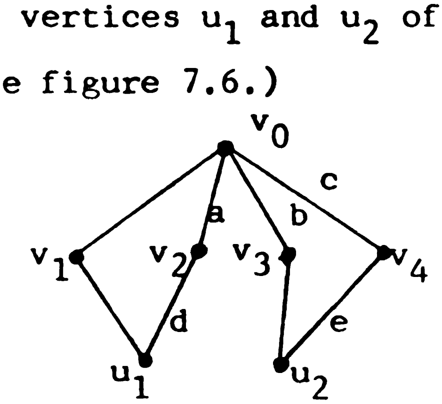


Figure 7.6. Two pairs of end vertices of C, joined to two distinct vertices of W.

Then we can delete edges a, b, c from D_1 and add edges d, e to D_1. We obtain a new edge dominating set with cardinality less than $|D_1|$, a contradiction.

From (1) and (2) we conclude that at most two end vertices of C can be joined to a single vertex of W, therefore $m \geq n-1$. Hence there exists a non-empty set of at least $|C| - 2 = n-1$ vertices in W, which are joined to at least n-1 end vertices of the component C.

We are now able to complete the proof of theorem 7.3. Let D_1 be a minimal edge dominating set having $|D_1| = \sigma_{1L}(G)$. It is clear from corollary 7.2. that $\sigma_{1L}(G) \leq \beta_{1L}(G)$. If D_1 is also an independent set of edges, then D_1 is an edge matching and $\sigma_{1L}(G) \geq \beta_{1L}(G)$, so $\sigma_{1L}(G) = \beta_{1L}(G)$, and the theorem is proved in this case.

If D_1 is not an independent set of edges, then at least one of the components of $\langle D_1 \rangle$ is a star graph different from K_2. Let $\langle D_1 \rangle = \bigcup_{i=1}^{\lambda} C_i$, where C_i denotes the i-th component of $\langle D_1 \rangle$, $i = 1, 2, \ldots, \lambda$. Consider a component C_1 of order n_1+1, where $n_1 \geq 2$. By lemma 7.1. there exists at least n_1-1 vertices in $G - \langle D_1 \rangle$ which are joined to n_1-1 end vertices, say $v_{12}, v_{13}, \ldots, v_{1n_1}$ of the graph C_1. Now we are going to construct a new edge dominating set by replacing the edges $(v_{10}, v_{12}), (v_{10}, v_{13}), \ldots$

. ., (v_{10}, v_{1n_1}) by the set of edges (v_{12}, u_{12}), (v_{13}, u_{13}), . . ., (v_{1n_1}, u_{1n_1}). Let us denote this edge dominating set by D_{11}. It is clear $|D_1| = |D_{11}|$. Suppose that a component C_2 in $\langle D_1 \rangle$ has order n_2+1, where $n_2 \geq 2$. We claim there is no end vertices of C_2 can be joined to vertices u_{1i}'s only, and to no other vertices of W. For example figure 7.7.

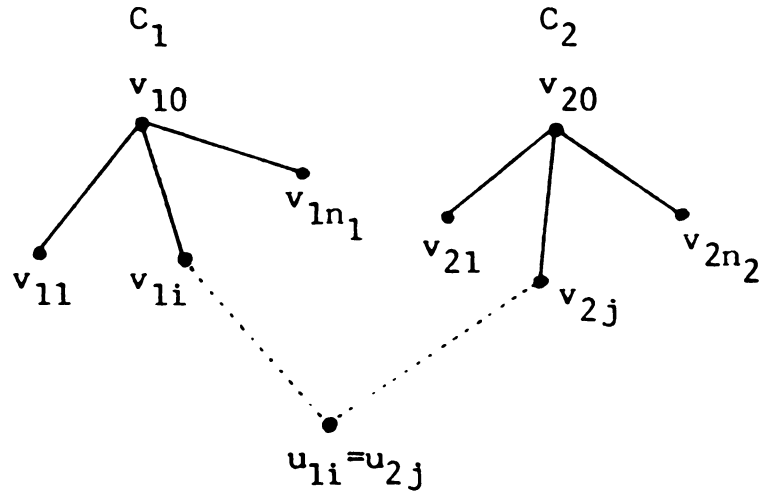


Figure 7.7. An end vertex of C_2 joined to vertex u_{1i}.

If, v_{2j} is joined to $u_{2j} = u_{1i}$, then $D_{11} - (v_{20}, v_{2j})$ is an edge dominating set, which contradicts the minimality of D_1. From this assumption and lemma 7.1. there exists at least n_2-1 vertices in G - $\langle D_1 \rangle$ namely $u_{22}, u_{23}, \ldots, u_{2n_2}$ different from $u_{11}, u_{12}, \ldots, u_{1n1}$ which are joined to n_2-1 end vertices of C_2, say $v_{22}, v_{23}, \ldots, v_{2n_2}$. Now, we may construct a new edge dominating set D_{12} by replacing the set of edges (v_{20}, v_{22}), (v_{20}, v_{23}), . . ., (v_{20}, v_{2n_2}). by the set of edges (v_{22}, u_{22}), (v_{23}, u_{23}), . ., (v_{2n2}, u_{2n2}).

After a finite number of such steps, all components in the edge dominating set are reduced to single edges, and no star component different from K_2 exists. The remaining edges in these components must then be independent, and therefore form an edge matching. Thus $\sigma_{1L}(G) \geq \beta_{1L}(G)$, and equality follows.

An application of theorem 7.3. is the following.

Theorem 7.4. Let G be a graph of order p, and having no isolated vertices. Then $\sigma_{0L}(G) + \sigma_{1L}(G) \leq p$.

Proof: Let D_0 be a minimal vertex dominating set such that $|D_0| = \sigma_{0L}(G)$. Define $W = V(G) - D_0$. If $|D_0| \leq p/2$, then $\sigma_{0L}(G) + \sigma_{1L}(G) = |D_0| + \beta_{1L}(G) \leq p$, since no independent edge set has more than $p/2$ elements.

Suppose next that $|D_0| > p/2$. We first note that if v_1 is in D_0 and is incident with a vertex v_2 in D_0, then there exist at least one vertex u_1 in W such that u_1 is dominated only by v_1 and not by any other vertices in D_0. For if the set of vertices in W which are dominated by v_1 are also dominated by some other vertices in D_0, then since v_1 itself is dominated by v_2, this implies that $D_0 - \{v_1\}$ will be a vertex dominating set. This contradicts the minimality of D_0.

Now, let M_1 be a minimum edge matching in G, so

$|M_1| = \beta_{1L}(G) = \sigma_{1L}(G)$. M_1 is a set of independent edges, which join pairs of vertices in D_0, or in W, or join a vertex in D_0 to one in W. Suppose there exist r edges in M_1 which join two vertices in D_0, then there are at least 2r vertices in W which cannot join vertices in D_0 to form edges in M_1, by the previous argument. Now, we may form an upper bound for $|M_1|$ by joining these 2r vertices in W by r edges and join the remaining $|W| - 2r$ vertices of W to vertices in D_0. This is possible since $|D_0| > |W|$. Thus $|M_1| \leq r + r + |W| - 2r = |W|$. This shows that $|M_1|$ is bounded above by $|W|$, and is independent of the number of edges chosen from the set $\langle D_0 \rangle$. Hence we conclude that $\sigma_{0L}(G) + \sigma_{1L}(G) = |D_0| + |M_1| \leq |D_0| + |W| = p$.

We have considered in this chapter sets of edges which dominate all edges of a graph G, and also sets of vertices which dominate all vertices of G. One might wonder if it would be useful to consider sets of edges which dominate all vertices of a graph, or sets of vertices which dominate all edges.

A moment's reflection convinces one that under the usual interpretation of domination, the parameters arising are exactly the edge and vertex covering numbers discussed in chapters 4 and 5.

Section 7.2. This final section contains a few miscellaneous results on line graphs. The line graph LG of G by definition has a vertex set V(LG) which is in one-to-one correspondence with the edge set E(G) of G. Two vertices of LG are adjacent if and only if the corresponding edges of G are adjacent. The line graph of G is sometimes called an interchange graph or derivative of G. Gupta [11] mentioned without proof a few results concerning the relationship between minimum covers and maximum matchings for line graphs. We present a few new ones. We assume G is a (p,q) graph and has no isolated vertices. It is clear that $\beta_{0U}(LG) = \beta_{1U}(G)$, $\beta_{0L}(LG) = \beta_{1L}(G)$.
and $\sigma_{0U}(LG) = \sigma_{1U}(G)$, $\sigma_{0L}(LG) = \sigma_{1L}(G)$.
From Gallai's formula we readily have

$$\alpha_{0L}(LG) + \beta_{0U}(LG) = q.$$

Hence $\alpha_{0L}(LG) = q - \beta_{0U}(LG) = q - \beta_{1U}(G)$.
From theorem 6.1. we have

$$\alpha_{0U}(LG) + \beta_{0L}(LG) = q.$$

Hence $\alpha_{0U}(LG) = q - \beta_{0L}(LG) = q - \beta_{1L}(G)$

We also have the following.

Theorem 7.5. Let G be a graph with no isolated vertices. Then $\beta_{0L}(LG) = \sigma_{0L}(LG)$

Proof: Now $\beta_{0L}(LG) = \beta_{1L}(G)$ and $\sigma_{0L}(LG) = \sigma_{1L}(G)$.

By theorem 7.3, $\beta_{1L}(G) = \sigma_{1L}(G)$. Hence we conclude that $\beta_{0L}(LG) = \sigma_{0L}(LG)$.

__Corollary 7.3__. Let G be a (p,q) graph which has no isolated vertices. Then $\sigma_{0L}(LG) + \alpha_{0U}LG) = q$.

The proof is trivial and is omitted.

Table of Parameters for Certain Classes of Graphs

Graph	α_{1U}	α_{1L}	β_{1U}	β_{1L}	α_{0U}	α_{0L}	β_{0U}	β_{0L}	σ_{1U}	σ_{1L}	σ_{0U}	σ_{0L}
Complete graph K_p	p-1	$\{\frac{p}{2}\}$	$[\frac{p}{2}]$	$[\frac{p}{2}]$	p-1	p-1	1	1	$[\frac{p}{2}]$	$[\frac{p}{2}]$	1	1
Path P_n	$[\frac{2n}{3}]$	$\{\frac{n}{2}\}$	$[\frac{n}{2}]$	$\{\frac{n-1}{3}\}$	$\{\frac{n}{2}\}$	$[\frac{n}{2}]$	$\{\frac{n}{2}\}$	$[\frac{n}{2}]$	$[\frac{n}{2}]$	$\{\frac{n-1}{3}\}$	$\{\frac{n}{2}\}$	$\{\frac{n}{3}\}$
Cycle C_n	$[\frac{2n}{3}]$	$\{\frac{n}{2}\}$	$[\frac{n}{2}]$	$\{\frac{n}{3}\}$	$\{\frac{n}{2}\}$	$\{\frac{n}{2}\}$	$[\frac{n}{2}]$	$[\frac{n}{2}]$	$[\frac{n}{2}]$	$\{\frac{n}{3}\}$	$\{\frac{n}{2}\}$	$\{\frac{n}{3}\}$
Star graph K(1,p-1)	p-1	p-1	1	1	p-1	1	p-1	1	1	1	p-1	1
Bipartite graph K(p,p)	p	p	p	p	p	p	p	p	p	p	p	p
K(m,n) $m \leq n$	n	n	m	m	n	m	n	m	m	m	n	m
Wheel $W_n = K_1 + C_{n-1}$	n-1	$\{\frac{n}{2}\}$	$[\frac{n}{2}]$	$\{\frac{n}{3}\}$	n-1	$\{\frac{n+1}{2}\}$	$[\frac{n-1}{2}]$	1	$[\frac{n}{2}]$	$\{\frac{n}{3}\}$	$[\frac{n-1}{2}]$	1

Graph	α_{1U}	α_{1L}	β_{1U}	β_{1L}	α_{0U}	α_{0L}	β_{0U}	β_{0L}	σ_{1U}	σ_{1L}	σ_{0U}	σ_{0L}
Complete j-partite graph $K(p_1, p_2, \ldots p_j)$ $p_1 \le p_2 \le \ldots \le p_j$ $p = \sum_{i=1}^{j} p_i$	$p-p_1$	$p-\min([\frac{p}{2}], p-p_j)$	$\min([\frac{p}{2}], p-p_j)$	$\max(p_{j-1}, \{\frac{p-p_j}{2}\})$	$p-p_1$	$p-p_j$	p_j	p_1	$\min([\frac{p}{2}], p-p_j)$	$\max(p_{j-1}, \{\frac{p-p_j}{2}\})$	p_j	p_1
Comb	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\{\frac{p}{4}\}$	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\frac{p}{2}$	$\{\frac{p}{4}\}$	$\frac{p}{2}$	$\frac{p}{2}$
The Petersen graph p=10	7	5	5	3	7	6	4	3	5	3	5	3
Tetrahedron p=4	3	2	2	2	3	3	1	1	2	2	1	1
Cube p=8	6	4	4	3	4	4	4	4	4	3	4	2
Octahedron p=6	4	3	3	2	4	4	2	2	3	2	2	2
Dodecahedron p=20	14	10	10	6	14	13	7	6	10	6	7	6
Isosahedron p=12	10	6	6	5	9	9	3	3	6	5	3	2

BIBLIOGRAPHY

1. C. Berge. The Theory of Graphs. Methuen, London (1962).

2. C. Berge. Two Theorems in Graph Theory. Proc. Nat. Acad. Sci. USA. 43 (1957), 842-844.

3. C. Berge. Graphs and Hypergraphs. North-Holland, London (1973).

4. M. Behzad and G. Chartrand. An Introduction to the Theory of Graphs. Allyn and Bacon, Boston (1972).

5. L. M. Beineke and M. D. Plummer. On the 1-Factors of a Non-separable Graph. J. Combinatorial Theory 2 (1967), 285-289.

6. G. Chartrand, D. Geller and S. Hedetniemi. Graphs with Forbidden Subgraphs. J. Combinat. Theory 10B (1971), 12-41.

7. J. Edmonds. Covers and Packings in a Family of Sets. Bull. Am. Math. Soc. 68 (1962), 494-499.

8. J. Edmonds. Paths, Trees and Flowers. Canad. J. Math. 17 (1965), 449-467.

9. R. Forcade. Smallest Maximal Matchings in the Graph of the d-dimensional Cube. J. of Combinatorial Theory. Series B. 14 no. 2 (1973), 157-162.

10. T. Gallai. Über extreme Punkt-und Kantenmengen. Ann. Univ. Sci. Budapest, Eötvös Sect. Math. 2 (1959), 133-138.

11. R. Gupta. Independence and Covering Numbers of Line Graphs and Total Graphs. Proof Techniques in Graph Theory. Academic Press, New York (1969), 61-62.

12. B. Grünbaum. Some Problems Concerning Graphs. Univ. of Washington (1972).

13. B. Grünbaum. Networks. (To appear.)

14. F. Harary. Graph Theory. Addison-Wesley, Reading (1969).

15. S. Hedetniemi. On Hereditary Properties of Graphs. Studia Sci. Math. Hungar. (1972).

16. E. A. Nordhaus, B. M. Stewart and A. T. White. On the Maximum Genus of a Graph. J. Combinatorial Theory. Series B. 11 (1971), 258-267.

17. E. A. Nordhaus. On the Density and Chromatic Numbers of Graphs. The Many Facets of Graph Theory. Springer-Verlag, Berlin (1969).

18. R. Norman and M. Rabin. An Algorithm for a Minimum Cover of a Graph. Proc. Amer. Math. Soc. 10 (1959), 315-319.

19. O. Ore. Arc Coverings of Graphs. Ann. Mat. Pura. Appl. 55 (1961), 315-321.

20. O. Ore. Incidence Matchings in Graphs. J. Math. Pura. Appl. 40 (9) (1961), 123-127.

21. O. Ore. Graphs and Matchings Theorems. Duke Math J. 22 (1955), 625-639.

22. O. Ore. Theory of Graphs. Amer. Math. Soc. Colloq. Publ. 38, Providence (1962).

23. D. Ray and Chaudhudi. An Algorithm for a Minimum Cover of an Abstract Complex Chains. Canad. J. Math. 15 (1963), 11-24.

24. M. J. Stewart. Graphs and Their Subdivisions. Ph.D. Thesis, Michigan State University (1972).

25. J. H. Weinstein. On the Number of Disjoint Edges in a Graph. Canad. J. Math. 15 (1963), 106-111.

26. C. Witzgall and C. Zahn. Modification of Edmonds' Maximum Matching Algorithm. J. of Res. N. B. S. 69 B. (1965), 91-98.