EXPANSIONS OF PARABOLIC WAVE AND HARMONIC FUNCTIONS

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Yousef Alavi

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A THESIS

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ABSTRACT

The problem studied in this thesis is the expansions of parabolic wave and harmonic functions. The wave equation separated in the coordinates of the paraboloid of revolution yields solutions referred to as parabolic wave functions. Series expansion for the parabolic wave functions in terms of the spherical wave functions has been obtained, with coefficients of the expansion explicitly determined. These coefficients have been given in terms of certain polynomials due to Pasternack for which the orthogonality relation is known. With this relation then the series expansion has been inverted to express the spherical wave functions, in integral form, in terms of the parabolic wave functions. Two methods have been developed to find the expansion for the parabolic potential functions. Further, the linear generating function for the Pasternack polynomials has been obtained in terms of a hypergeometric function. In addition a new derivation of the bilinear generating function in the continuous case has been given for the parabolic wave functions. Finally a second method for the derivation of the series expansion of the parabolic wave functions has been found.

TABLE OF CONTENTS

I. INTRODUCTION

The parabolic wave functions have received considerable attention in recent years. This has been due, for a large part, to the interest in the physical problem of diffraction of waves both acoustical and electromagnetic, by a paraboloid of revolution. The diffraction problem has been basic in the work of Fock in some recent advances in the general theory of diffraction. Of the many papers of Fock we refer only to reference [8] where other references can be found. The diffraction problem has also been studied by Hochstadt [10] to which we shall refer later.

It is found in the approach to the diffraction problem that the relation of the parabolic wave functions to spherical wave functions is of considerable importance. In this thesis we study the problem of expanding parabolic wave functions in infinite series of spherical wave functions. This assumes as a heuristic principle, that a solution of the wave equation in some coordinate system can be expanded in terms of solutions of some other coordinate system. However the number of cases where this has actually been done and the coefficients explicitly determined, is very small. We shall show that in the present case, the expansion can be done and the coefficients determined.

It is of interest to note that as a by product of the expansion, certain polynomials due to Pasternack [12] will play an important part. These polynomials are orthogonal and the orthogonolity relation is known. Hence the expansion can be inverted and as a result spherical wave functions are then expressed, in integral form, in terms of parabolic wave functions. Further, we are able to find the generating function for the Pasternack polynomials as a hypergeometric function and finally we are able to give a new derivation of the bilinear generating function in the continuous case for the parabolic wave functions themselves.

II. PRELIMINARY NOTIONS AND NOTATIONS

We define the spherical coordinates r, θ, φ, and the coordinates of the paraboloid of revolution ξ, η, φ, by

$$x = r \sin\theta \cos\varphi = \xi\eta \cos\varphi$$
$$y = r \sin\theta \sin\varphi = \xi\eta \sin\varphi$$
$$z = r \cos\theta \qquad = \tfrac{1}{2}(\xi^2 - \eta^2).$$

The wave equation

$$\Delta U + k^2 U = 0$$

transformed to the coordinates ξ, η, φ, is

$$\frac{1}{\xi^2 + \eta^2}\left[\frac{1}{\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial U}{\partial\xi}\right) + \frac{1}{\eta}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial U}{\partial\eta}\right)\right] + \frac{1}{\xi^2\eta^2}\frac{\partial^2 U}{\partial\varphi^2} + k^2 U = 0$$

The method of separation of variables then admits solutions of the form $\psi(\xi,\lambda)\,\psi(\eta,-\lambda)\;e^{-im\varphi}$,
where $\psi(\xi,\lambda)$ satisfies the ordinary differential equation

$$\frac{d^2\psi}{d\xi^2} + \frac{1}{\xi}\frac{d\psi}{d\xi} + \left(k^2\xi^2 - \frac{m^2}{\xi^2} + \lambda\right)\psi = 0, \qquad (1)$$

and $\psi(\eta,-\lambda)$ satisfies a similar equation with the sign of λ reversed. We shall refer to the product of solutions

$\psi_m(\xi,\lambda)\,\psi_m(\eta,-\lambda)e^{-im\varphi}$ or $\psi_m(\xi,\lambda)\;\psi_m(\eta,-\lambda)$ as a parabolic wave function.

From (1), $\psi_m(\xi, \lambda)$ can be given in terms of a confluent hypergeometric function

$$\psi_m(\xi, \lambda) = \xi^m e^{ik\xi^2/2} \, {}_1F_1\left(-\frac{i\lambda}{4k} + \frac{m+1}{2}; m+1; -ik\xi^2\right). \quad (2)$$

For physical reasons we shall expect to have wave functions which are regular and single valued and thus assume m to be integral.

Similarly, the wave equation transformed to the coordinates r, θ, φ, becomes

$$\frac{1}{r^2 \sin\theta}\left[r^2 \sin\theta \frac{\partial^2 U}{\partial r^2} + 2r \sin\theta \frac{\partial U}{\partial r} + \sin\theta \frac{\partial^2 U}{\partial \theta^2} + \cos\theta \frac{\partial U}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial^2 U}{\partial \varphi^2}\right] + k^2 U = 0,$$

and it has solutions of the form $j_n(kr)P_n^{-m}(\cos\theta)e^{-im\varphi}$.

Here $j_n(kr)$ is a spherical Bessel function and satisfies the differential equation

$$\frac{d^2 j_n(kr)}{dr^2} + \frac{2}{r}\frac{d\, j_n(kr)}{dr} + \left[k^2 - \frac{n(n+1)}{r^2}\right] j_n(kr) = 0, \quad (3)$$

and $P_n^{-m}(\cos\theta)$ satisfies

$$\sin^2\theta \frac{d^2 P_n^{-m}(\cos\theta)}{d\theta} + \sin\theta\cos\theta \frac{d\, P_n^{-m}(\cos\theta)}{d\theta} + \left[n(n+1)\sin^2\theta - m^2\right] P_n^{-m}(\cos\theta) = 0.$$

The function $P_n^{-m}(\cos\theta)$ is related to the associated Legendre function $P_n^m(\cos\theta)$ by

$$P_n^{-m}(\cos\theta) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(\cos\theta).$$

Here we choose $P_n^{-m}(\cos\theta)$ so that we can absorb the factor $\frac{(n-m)!}{(n+m)!}$ later in our expansions.

Various notations have been employed to designate confluent hypergeometric functions and the parabolic wave functions, notably those introduced by Whittaker and Bucholz. The Whittaker function, $M_{K,\frac{v}{2}}(x)$, is defined by [3, page 10]

$$M_{K,\frac{v}{2}}(x) = e^{-x/2}\, x^{\frac{v+1}{2}}\, {}_1F_1\left(-K + \frac{v+1}{2}\,;\, v+1;\, x\right)$$

and for v integral, is a regular function of x. It satisfies the differential equation

$$\frac{d^2y}{dx^2} + \left[-\frac{1}{4} + \frac{K}{x} + \frac{1-v^2}{4x^2}\right] y = 0.$$

Our functions $\psi_m(\xi,\lambda)$, $\psi_m(\eta,-\lambda)$ can be expressed in terms of this Whittaker function, as follows:

$$\psi_m(\xi,\lambda) = (-ik)^{-\frac{m+1}{2}} \frac{1}{\xi} M_{\frac{i\lambda}{4k},\frac{m}{2}}(-ik\,\xi^2),$$

$$\psi_m(\eta,-\lambda) = (ik)^{-\frac{m+1}{2}} \frac{1}{\eta} M_{\frac{i\lambda}{4k},\frac{m}{2}}(ik\,\eta^2).$$

If in the above Whittaker function, v is not a negative integer, the function $\mathcal{M}_{K,\frac{v}{2}}(x)$ is defined by Buchholz [3, page 12] as

$$\mathcal{M}_{K,\frac{v}{2}}(x) = \frac{M_{K,\frac{v}{2}}(x)}{\Gamma(1+v)} .$$

In terms of these functions of Buchholz, our functions are

$$\psi_m(\xi, \lambda) = m!\,(-ik)^{-\frac{m+1}{2}} \frac{1}{\xi} \mathcal{M}_{\frac{i\lambda}{4k}, \frac{m}{2}}(-ik\,\xi^2),$$

and

$$\psi_m(\eta, -\lambda) = m!\,(ik)^{-\frac{m+1}{2}} \frac{1}{\eta} \mathcal{M}_{\frac{i\lambda}{4k}, \frac{m}{2}}(ik\,\eta^2),$$

Still a further notation $\mathcal{m}_{\chi}^{(v)}(x)$, useful in applications, has been introduced by Buchholz [3, page 53]. It is a solution of

$$\frac{d}{dx}\left[x\cdot F'(x)\right] - \left(\frac{x}{4} - \chi + \frac{\frac{v^2}{4}}{x}\right) F(x) = 0$$

and is given in terms of the above functions of Whittaker and Buchholz by

$$\mathcal{m}_{\chi}^{(v)}(x) = x^{-\frac{1}{2}}\mathcal{M}_{\chi,\frac{v}{2}}(x) = x^{-\frac{1}{2}} \frac{M_{\chi,\frac{v}{2}}(x)}{\Gamma(1+v)}$$

Hence our functions can be expressed as

$$\psi_m(\xi,\lambda) = m!\,(-ik)^{-\frac{m}{2}}\mathcal{M}^{(m)}_{\frac{i\lambda}{4k}}(-ik\,\xi^2)$$

$$\psi_m(\eta,-\lambda) = m!\,(ik)^{-\frac{m}{2}}\mathcal{M}^{(m)}_{\frac{i\lambda}{4k}}(ik\,\eta^2).$$

Finally we summarize these notations by writing the parabolic wave functions $\psi_m(\xi,\lambda)\,\psi_m(\eta,-\lambda)$ as follows:

$$\psi_m(\xi,\lambda)\;\;\psi_m(\eta,-\lambda) = \frac{1}{\xi\eta}\,k^{-(m+1)}\,M_{\frac{i\lambda}{4k},\frac{m}{2}}(-ik\,\xi^2)$$

$$\cdot\,M_{\frac{i\lambda}{4k},\frac{m}{2}}(ik\,\eta^2),$$

or

$$\psi_m(\xi,\lambda)\;\;\psi_m(\eta,-\lambda) = \frac{(m!)^2}{\xi\eta}\,k^{-(m+1)}\mathcal{M}_{\frac{i\lambda}{4k},\frac{m}{2}}(-ik\,\xi^2)$$

$$\cdot\,\mathcal{M}_{\frac{i\lambda}{4k},\frac{m}{2}}(ik\,\eta^2),$$

or

$$\psi_m(\xi,\lambda)\;\;\psi_m(\eta,-\lambda) = (m!)^2\,k^{-m}\,\mathcal{M}^{(m)}_{\frac{i\lambda}{4k}}(-ik\,\xi^2)$$

$$\cdot\,\mathcal{M}^{(m)}_{\frac{i\lambda}{4k}}(ik\,\eta^2).$$

III. EXPANSIONS

We now attempt to expand the parabolic wave functions $\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda)$ in terms of spherical wave functions $j_n(kr)\,P_n^{-m}(\cos\theta)$ and assume an expansion

$$\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda) = \sum_{n=m}^{\infty} a_n^m\, j_n(kr)\, P_n^{-m}(\cos\theta), \qquad (4)$$

with $r = \frac{1}{2}(\xi^2 + \eta^2)$ and $\cos\theta = \dfrac{\xi^2 - \eta^2}{\xi^2 + \eta^2}$.

The coefficients a_n^m, which depend on k, λ, m, can be determined from the condition that the expansion (4) must be a solution of the differential equation (1) for all values of η. Since the particular value of η does not matter, we choose arbitrarily small η, and examine $P_n^{-m}(\cos\theta)$ for η near zero. As $\eta \longrightarrow 0$, $\cos\theta \longrightarrow 1$ and since

$$P_n^{-m}(x) \longrightarrow 2^{-\frac{m}{2}} \frac{(1-x)^{\frac{m}{2}}}{(1+m)}$$

as $x \longrightarrow 1$, we find that

$$P_n^{-m}(\cos\theta) \longrightarrow \frac{1}{m!}\,\frac{\eta^m}{\xi^m}.$$

The expansion (4) thus becomes

$$\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda) = \eta^m \cdot \frac{\xi^{-m}}{m!} \cdot \sum_{n=m}^{\infty} a_n^m\, j_n(kr).$$

It remains here to investigate $\Psi_m(\eta,-\lambda)$ for small η.

Let $iw = \frac{\lambda}{2k}$.

Then

$$\Psi_m(\eta, -\lambda) = \eta^m e^{ik\eta^2/2} {}_1F_1(-\frac{w}{2} + \frac{m+1}{2}; m+1; -ik\eta^2)$$

$$= \eta^m e^{ik\eta^2/2} \left[1 + \frac{-\frac{w}{2} + \frac{m+1}{2}}{m+1}(-ik\eta^2) + O(\eta^4) \cdots \right].$$

But the ${}_1F_1$ above approaches 1 for small η, so that we have

$$\Psi_m(\eta, -\lambda) \approx \eta^m$$

and the expansion (4) reduces to

$$\Psi_m(\xi, \lambda) = \frac{1}{m!} \sum_{n=m}^{\infty} a_n^m \xi^{-m} j_n(k\nu)$$

with $\nu = \xi^2/2$.

We now determine the condition that

$$\sum_{n=m}^{\infty} a_n^m \xi^{-m} j_n(k\xi^2/2)$$

satisfies the differential equation in (1). Let

$$\psi = \sum_{n=m}^{\infty} a_n^m \xi^{-m} j_n(k\xi^2/2).$$

Substituting ψ in the differential equation (1) leads to

$$\sum_{n=m}^{\infty} a_n^m \left[\frac{k\xi^2}{2} j_n''(k\xi^2/2) + (1-m)\, j_n'(k\xi^2/2) + \left(\frac{k}{2}\xi^2 + iw\right) j_n(k\xi^2/2) \right] = 0.$$

If we put $v = (k/2)\xi^2$ the last equation becomes

$$\sum_{n=m}^{\infty} a_n^m \left[v \cdot j_n''(v) + (1-m) j_n'(v) + (v + iw) j_n(v) \right] = 0, \qquad (5)$$

in which we can eliminate the $j_n''(v)$ and $j_n'(v)$ by using the differential equation for $j_n(v)$ and the recursion formulas

$$\frac{(2n+1)}{v} j_n(v) = j_{n+1}(v) + j_{n-1}(v), \qquad (6)$$

$$(2n+1)\, j_n'(v) = n \cdot j_{n-1}(v) - (n+1)\, j_{n+1}(v)\,. \qquad (7)$$

The differential equation for $j_n(v)$ is

$$v \cdot j_n''(v) + 2j_n'(v) + \left[v - \frac{n(n+1)}{v} \right] j_n(v) = 0,$$

or

$$v \cdot j_n''(v) = \left[\frac{n(n+1)}{v} - v \right] j_n(v) - 2j_n'(v).$$

If we now introduce $v \cdot j_n''(v)$ into (5), it becomes, after simplifying

$$\sum_{n=m}^{\infty} a_n^m \left\{ \left[\frac{(n+1)n}{v} + iw \right] j_n(v) - (1+m)\, j_n'(v) \right\} = 0.$$

Using the recursion formulas, (6), and (7), the last equation reduces to

$$\sum_{n=m}^{\infty} a_n^m \left\{ iw \cdot j_n(v) + \frac{n(n+1)}{2n+1} \left[j_{n+1}(v) + j_{n-1}(v) \right] - \frac{(m+1)}{2n+1} \left[n\, j_{n-1}(v) - (n+1)\, j_{n+1}(v) \right] \right\} = 0,$$

or

$$\sum_{n=m}^{\infty} a_n^m \frac{n(n-m)}{2n+1} j_{n-1}(v) + \sum_{n=m}^{\infty} a_n^m\, iw\, j_n(v) + \sum_{n=m}^{\infty} a_n^m \frac{(n+1)(n+m+1)}{(2n+1)} j_{n+1}(v) = 0.$$

This can be rewritten as

$$\sum_{n=m}^{\infty} j_n(v) \left[\frac{(n+1)(n-m+1)}{(2n+3)} a_{n+1}^m + iw\, a_n^m + \frac{n(n+m)}{(2n-1)} a_{n-1}^m \right] = 0,$$

which can be reduced to a simpler form if we let

$$a_n^m = (2n+1)\, b_n^m .$$

This leads to

$$\sum_{n=m}^{\infty} j_n(v) \left[(n+1)(n-m+1)\, b_{n+1}^m + iw(2n+1)\, b_n^m + n(n+m)\, b_{n-1}^m \right] = 0. \tag{8}$$

Hence for (8) to hold, the coefficient of $j_n(v)$ must vanish, and we therefore have

$$(n+1)(n-m+1)\, b_{n+1}^m + iw(2n+1)\, b_n^m + n(n+m)\, b_{n-1}^m = 0, \tag{9}$$

which is a three-term recursion formula for the coefficients

$b_n^m = \dfrac{a_n^m}{2n+1}$ in our expansion (4).

The problem is now to determine the coefficients a_n^m. This can be done by comparing the recursion formula (9) for the b_n^m with a recursion formula for certain polynomials studied by Pasternack [12]. In order to do this, in (5) let

$$b_n^m = \frac{(n+m)!}{(n-m)!} B_n^m .$$

This leads to

$$(n+1)(n+1+m)\, B_{n+1}^m + iw(2n+1)\, B_n^m + n(n-m)\, B_{n-1}^m = 0, \qquad (10)$$

which is a recursion formula for the B_n^m. Now let $B_n^m = i^n\, C_n^m$. Then (10) reduces to

$$(n+1)(n+1+m)\, C_{n+1}^m + w(2n+1)\, C_n^m - n(n-m)\, C_{n-1}^m = 0$$

This is the recursion formula for the set of polynomials $F_n^m(w)$ given by Pasternack [12]. They are defined as

$$F_n^m(z) = {}_3F_2(-n\, , n+1\, ,\ \tfrac{1}{2} + \tfrac{m}{2} + \tfrac{z}{2}\, ; 1, m+1; 1)$$

for all m, real or complex, except m a negative integer. Therefore we have

$$a_n^m = (2n+1)\, b_n^m = (2n+1)\, \frac{(n+m)!}{(n-m)!} B_n^m$$

$$= (2n+1)\, \frac{(n+m)!}{(n-m)!}\, i^n\, C_n^m$$

where the C_n^m are proportional to $F_n^m(w)$. Hence

$$\frac{a_n^m}{a_m^m} = \frac{(2n+1)}{(2m+1)} \frac{(n+m)!}{(2m)!(n-m)!} i^{n-m} \frac{F_n^m(w)}{F_m^m(w)} , \tag{11}$$

with $iw = \lambda/2k$. The a_n^m can now be given if we evaluate a_m^m from the expansion (4),

$$\Psi_m(\xi, \lambda)\Psi_m(\eta, -\lambda) = \sum_{n=m}^{\infty} a_n^m \, j_n(kr) \, P_n^{-m}(\cos\theta),$$

by dividing by $\xi^m \eta^m = r^m \sin^m\theta$, and taking the limit as $\xi, \eta, r \rightarrow 0$. Now on the right-hand-side the dominant term for small r is the first term of the series for

$$j_m(kr) = \sqrt{\frac{\pi}{2kr}} \; J_{m+\frac{1}{2}}(kr),$$

which, after dividing by r^m, becomes $\frac{2^m \, m!}{(2m+1)!} k^m$.

Also, $P_m^{-m}(\cos\theta) = \frac{\sin^m\theta}{2^m \, m!}$ which, after dividing by $\sin^m\theta$, becomes $\frac{1}{2^m \, m!}$. So that we have now for the right-hand-side

$$a_m^m \frac{2^m k^m m!}{(2m+1)!} \cdot \frac{1}{2^m \, m!} = a_m^m \frac{k^m}{(2m+1)!} .$$

But the left-hand-side

$$\frac{\Psi_m(\xi, \lambda)\Psi_m(\eta, -\lambda)}{\xi^m \eta^m}$$

approaches 1 in the limit, and therefore we have for the a_m^m

$$a_m^m = \frac{(2m+1)!}{k^m} .$$

Therefore we have for the coefficients a_n^m, substituting

a_m^m into (11),

$$a_n^m = \frac{(2n+1)}{k^m} \cdot \frac{(n+m)!}{(n-m)!} \, i^{n-m} \frac{F_n^m(w)}{F_m^m(w)} .$$

We have finally for our expansion (4)

$$\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda) = \sum_{n=m}^{\infty} a_n^m \, j_n(kr)\, P_n^{-m}(\cos\theta),$$

$$a_n^m = (2n+1)\,\frac{(n+m)!}{(n-m)!} \cdot \frac{i^{n-m}}{k^m} \, \frac{F_n^m(w)}{F_m^m(w)} , \text{ with } iw = \frac{\lambda}{2k} . \qquad (12)$$

The coefficients a_n^m can also be given in terms of $F_n^{-m}(w)$, if, for the moment, m is not an integer. This can be done if we let $b_n^m = i^n B_n^m$. Then the recursion formula for the b_n^m becomes, with $iw = \lambda/2k$,

$$(n+1)(n-m+1)\, i^{n+1} B_{n+1}^m + (2n+1)(iw)\, i^n B_n^m$$

$$+ n(n+m)\, i^{n-1} B_{n-1}^m = 0,$$

or

$$(n+1)(n-m+1)\, B_{n+1}^m + (2n+1)\, w\, B_n^m - n(n+m)\, B_{n-1}^m = 0,$$

which is the same as the recursion formula

$$(n+1)(n-m+1)\, F_{n+1}^{-m}(z) + (2n+1)\, z\, F_n^{-m}(z)$$

$$- n(n+m)\, F_n^{-m}(z) = 0.$$

So that now we have

$$a_n^m = \frac{(2n+1)}{k^m} (2m)!\ i^{n-m} \frac{F_n^{-m}(w)}{F_m^{-m}(w)}$$. But $\frac{F_n^{-m}(w)}{F_n^{-m}(w)}$ is defined as m approaches an integer, and thus our expansion (12) can be given as

$$\Psi_m(\xi,\lambda)\Psi_m(\eta,-\lambda) = \sum_{n=m}^{\infty} \frac{(2n+1)}{k^m} (2m)!\ i^{n-m} \cdot \frac{F_n^{-m}(w)}{F_m^{-m}(w)} \cdot j_n(kr)\ P_n^{-m}(\cos\theta). \tag{13}$$

Now, in order to establish the convergence of our series expansions for the parabolic wave functions, we write (12) as

$$\Psi_m(\xi,\lambda)\Psi_m(\eta,-\lambda) = \sum_{n=m}^{\infty} (2n+1) \frac{i^{n+m}}{k^m} \cdot \frac{F_n^m(w)}{F_m^m(w)} j_n(kr)\ P_n^m(\cos\theta).$$

To investigate the behavior of $F_n^m(w)$ for large n, we can use the asymptotic expansion of a certain polynomial given by Rice [13]. He defines these polynomials as

$$H_n(\zeta,p,v) = {}_3F_2(-n,n+1,\zeta;\ 1,p;v),$$

and with $\zeta = \frac{1+m+w}{2}$ and $p = m+1$ and $v = 1$ these reduce to Pasternack polynomials. $H_n(\zeta,p,1)$ however, behave as a power of n for large n. Hence the coefficients a_n^m in the expansion behave as a power of n. This is also the case for the function $P_n^m(\cos\theta)$. But $j_n(kr)$ behaves as $O(\frac{1}{n!})$ for large n, so that the series converges everywhere.

IV. INVERSION OF THE SERIES EXPANSION FOR PARABOLIC WAVE FUNCTIONS

It is now proposed to invert the series expansion for parabolic wave functions and express the spherical wave functions in terms of the parabolic wave functions. This can be done using the known orthogonality relation for Pasternack polynomials. The orthogonality relation is given by Bateman [1] as

$$\frac{\pi}{2}\cdot\frac{1}{\Gamma(1+m)\Gamma(1-m)}\int_{-\infty}^{\infty}\frac{F_n^m(ix)\,F_p^{-m}(-ix)}{\cosh\pi x+\cos m\pi}\,dx=\begin{cases}0, & p\neq n,\\ 1/2n+1, & p=n.\end{cases}\tag{1}$$

Our series expansion for parabolic wave functions is

$$\Psi_m(\xi,\lambda)\Psi_m(\eta,-\lambda)=\sum_{n=m}^{\infty}\frac{i^{n-m}}{k^m}(2n+1)\,\frac{(n+m)!}{(n-m)!}\cdot\frac{F_n^m(-\frac{i\lambda}{2k})}{F_m^m(-\frac{i\lambda}{2k})}\cdot j_n(kr)\,P_n^{-m}(\cos\theta),$$

which can be rewritten as

$$\Psi_m(\xi,\lambda)\Psi_m(\eta,-\lambda)=\sum_{n=m}^{\infty}\frac{i^{n-m}}{k^m}(2n+1)\,\frac{F_n^m(-\frac{i\lambda}{2k})}{F_m^m(-\frac{i\lambda}{2k})}\cdot j_n(kr)\,P_n^m(\cos\theta)\tag{2}$$

For convenience, we let $t = \frac{-\lambda}{2k}$, and restrict this substitution for this section only. Now, multiplying (2) by

$$\frac{F_p^m(-it)}{\cosh \pi t + \cos m\pi}$$

and integrating with respect to t from $-\infty$ to ∞, and as it is permissible here, interchanging the summation and the integration, leads to

$$(-i)^m k^m \int_{-\infty}^{\infty} \frac{F_m^m(it)\, F_p^m(-it)}{\cos \pi t + \cos m\pi} \Psi_m(\xi, \lambda) \Psi_m(\eta, -\lambda) dt$$

$$= \sum_{n=m}^{\infty} i^n j_n(kr) P_n^m(\cos \theta)(2n+1) \int_{-\infty}^{\infty} \frac{F_n^m(it)\, F_p^m(-it)}{\cosh \pi t + \cos m\pi} dt \quad (3)$$

From Pasternack [12] we have, if m is not an integer,

$$F_n^{-m}(z) = \frac{(1+m)_n}{(1-m)_n} F_n^m(z) = \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \cdot \frac{\Gamma(1-m)}{\Gamma(1+m)} F_n^m(z) \quad (4)$$

Under this condition, introducing (4) into right-hand-side of (3) for $F_p^m(-it)$, and then applying the orthogonality condition (1), leads to

$$i^n j_n(kr) P_n^m(\cos \theta) = \frac{\pi}{2}(-i)^m k^m \frac{\Gamma(n+m+1)}{\Gamma(n-m+1)} \cdot \frac{1}{[\Gamma(m+1)]^2}$$

$$\cdot \int_{-\infty}^{\infty} \frac{F_m^m(it)\, F_n^m(-it)}{\cosh \pi t + \cos m\pi} \cdot \Psi_m(\xi, \lambda) \Psi_m(\eta, -\lambda) dt. \quad (5)$$

The factor in the integrand involving Pasternack's polynomials can be expressed in terms of hypergeometric and Gamma functions. We have

$$F_n^m(-it) = {}_3F_2(-n,n+1, \tfrac{1}{2} + \tfrac{m}{2} - \tfrac{it}{2}; m+1,1;1).$$

Using the transformation (9)

$${}_3F_2\begin{bmatrix}\alpha_1,\alpha_2,\alpha_3 ; 1\\ \beta_1,\beta_2\end{bmatrix} = \frac{\Gamma(\beta_2)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\,\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2)}$$

$$\cdot\,{}_3F_2\begin{bmatrix}\beta_1-\alpha_1,\beta_1-\alpha_2, \alpha_3;1\\ \beta_1,\beta_1+\beta_2-\alpha_1-\alpha_2\end{bmatrix}$$

leads to

$$F_n^m(-it) = \frac{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}m + \tfrac{it}{2})}{\Gamma(m+1)\,\Gamma(\tfrac{1}{2} - \tfrac{1}{2}m + \tfrac{it}{2})}$$

$$\cdot\,{}_3F_2(-n+m,n+m+1, \tfrac{1}{2} + \tfrac{1}{2}m + \tfrac{it}{2}; m+1,m+1;1) \qquad (6)$$

Also, we have

$$F_m^m(it) = {}_2F_1(-m, \tfrac{1}{2} + \tfrac{1}{2}m + \tfrac{it}{2};1;1),$$

which, with the relation [14, page 282]

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},$$

leads to

$$F_m^m(it) = \frac{\Gamma(\tfrac{1}{2} + \tfrac{1}{2}m - \tfrac{it}{2})}{\Gamma(m+1)\,\Gamma(\tfrac{1}{2} - \tfrac{1}{2}m - \tfrac{it}{2})}. \qquad (7)$$

Hence, from (6) and (7) we get

$$F_n^m(-it)\, F_m^m(it) = \frac{1}{[\Gamma(m+1)]^2} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{it}{2})\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{it}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m+\frac{it}{2})\Gamma(\frac{1}{2}-\frac{1}{2}m-\frac{it}{2})}$$

$$\cdot\ {}_3F_2(-n+m, n+m+1, \tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2}; m+1, m+1; 1) \qquad (8)$$

If we now use the functional equation

$$\Gamma(\tfrac{1}{2}+z)\,\Gamma(\tfrac{1}{2}-z) = \frac{\pi}{\cos \pi z} \quad ,$$

we can write for

$$\frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{it}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m-\frac{it}{2})} \cdot \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{it}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m+\frac{it}{2})} = \frac{1}{\pi}\left[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{it}{2})\right]^2$$

$$\cdot \cos \pi(\tfrac{m}{2}+\tfrac{it}{2})\ \frac{1}{\pi}\left[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2})\right]^2 \cos \pi(\tfrac{m}{2}-\tfrac{it}{2})$$

$$= \frac{1}{2\pi^2}\left[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{it}{2})\ \Gamma(\tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2})\right]^2 (\cosh \pi t + \cos m\pi) \qquad (9)$$

Substitution of (9) into (8) leads to

$$F_n^m(-it)F_m^m(it) = \frac{1}{2\pi^2[\Gamma(1+m)]^2}\left[\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{it}{2})\ \Gamma(\tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2})\right]^2$$

$$\cdot\ (\cosh \pi t + \cos m\pi)\ {}_3F_2(-n+m, n+m+1, \tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2}; m+1, m+1; 1) \qquad (10)$$

Equation (10) can now be introduced into (5), and thus we get

$$i^n\, j_n(kr)\, P_n^m(\cos\theta) = \frac{1}{4\pi}\,(-i)^m\, k^m\, \frac{1}{[\Gamma(m+1)]^4}\cdot\frac{\Gamma(n+m+1)}{\Gamma(n-m+1)}$$

$$\cdot\int_{-\infty}^{\infty}\left[\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{it}{2}\right)\,\Gamma\left(\tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2}\right)\right]^2$$

$$\cdot\,{}_3F_2\left(-n+m,n+m+1,\tfrac{1}{2}+\tfrac{1}{2}m-\tfrac{it}{2};m+1,m+1;1\right)\,\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda)\,dt \qquad (11)$$

This result, which now holds for integral values of m, can be reduced to a representation due to Buchholz [2, page 202] if we use his parabolic functions. Buchholz redefines the functions $\mathcal{M}_K^{(m)}(z)$ [2, page 198] as

$$\mathcal{M}_K^{(m)}(z) = \left(\frac{\pi}{2z}\right)^{1/2} M_{K,\frac{m}{2}}(z).$$

Using this definition and the expression relating our parabolic wave functions and the Whittaker functions, section II, we get

$$\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda) = \frac{2}{\pi}\,k^{-m}\,\mathcal{M}_{\frac{i\lambda}{4k}}^{(m)}(-ik\,\xi^2)$$

$$\cdot\,\mathcal{M}_{\frac{i\lambda}{4k}}^{(m)}(ik\,\eta^2). \qquad (12)$$

Therefore, for integers m, with $-\frac{it}{2} = s$, and the relation (12), equation (11) reduces to the result obtained by Buchholz, and we get

$$i^n\, j_n(kr)\, P_n^m(\cos\theta) = \frac{2}{\pi}(-i)^m \frac{1}{(m!)^4}\frac{(n+m)!}{(n-m)!}$$

$$\cdot \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} [\Gamma(\tfrac{1}{2} + \tfrac{1}{2}m + s)\Gamma(\tfrac{1}{2} + \tfrac{1}{2}m - s)]^2$$

$$\cdot\, {}_3F_2(-n+m, n+m+1, \tfrac{1}{2} + \tfrac{1}{2}m + s;\, m+1, m+1; 1)$$

$$\cdot\, \mathcal{M}_s^{(m)}(-ik\xi^2)\, \mathcal{M}_s^{(m)}(ik\eta^2)\, ds.$$

V. SPECIAL CASES

1. A SERIES EXPANSION FOR THE FUNCTION $\Psi_m(\xi,\lambda)$. By the use of the expansion (12), section III, we can now derive a series expansion for the parabolic function $\Psi_m(\xi,\lambda)$ in terms of the spherical Bessel function $j_n(kr)$. It is then possible to give an expansion in terms of the function $j_n(kr)$ for the Whittaker function $M_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2)$.

We have already shown, page 9, that

$$\Psi_m(\xi,\lambda) = \frac{1}{m!\,\xi^m} \sum_{n=m}^{\infty} a_n^m\, j_n(v), \tag{1}$$

with $v=k\xi^2/2$, and where

$$a_n^m = (2n+1)\,\frac{(n+m)!}{(n-m)!}\cdot\frac{i^{n-m}}{k^m}\cdot\frac{F_n^m(w)}{F_m^m(w)},$$

with $iw = \lambda/2k$. Now with

$$\Psi_m(\xi,\lambda) = (-ik)^{-(m+1)/2}\,\frac{1}{\xi}\,M_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2)$$

(1) becomes

$$M_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2) = \frac{(-ik)^{(m+1)/2}}{m!\,\xi^{m-1}} \sum_{n=m}^{\infty} a_n^m\, j_n(v),$$

or

$$M_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2) = \sum_{n=m}^{\infty} \frac{(-1)^m(-ik)}{m!\,\xi^{m-1}}^{-(m-1)/2} (2n+1)i^n$$

$$\cdot\frac{(n+m)!}{(n-m)!}\,\frac{F_n^m(w)}{F_m^m(w)}\, j_n(v), \tag{2}$$

where we have substituted for the coefficients a_n^m , page 14.

The series expansion (2) can now be compared with an expansion for the Whittaker function given by Buchholz [6, page 128]. In this case, however, essentially the same Whittaker function has been expanded in an infinite series whose terms are a finite product of a sum of two Bessel functions of the first kind whose orders are half an integer.

2. THE DERIVATION OF THE SERIES EXPANSION OF HOCHSTADT

In [10] Hochstadt makes use of an expansion of parabolic wave functions in terms of spherical wave functions and we shall show that his result is a special case of ours.

Hochstadt gives, for s and m integral,

$$\Omega_s^m(P) \equiv \frac{\Gamma(1+s+m)}{s!} \mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(-2i\,k\xi)$$

$$\cdot \mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(2i\,k\eta)e^{-im\varphi} = \sum_{n=m}^{\infty} \frac{i^{n+m}}{m!}(2n+1)$$

$$\cdot \sum_{r=0}^{s} \frac{(-1)^r(m-n)_r(m+n+1)_r(r+m+1)_{(s-r)}}{(m+1)_r(s-r)!\;r!}$$

$$\cdot j_n(kr)\;P_n^m(\cos\theta)e^{-im\varphi}\;. \tag{3}$$

Our expansion is

$$\Psi_m(\xi,\lambda)\Psi_m(\eta,-\lambda) = \sum_{n=m}^{\infty}(2n+1)\frac{(n+m)!}{(n-m)!}\frac{i^{n-m}}{k^m}\frac{F_n^m(w)}{F_m^m(w)}$$

$$\cdot j_n(kr)\;P_n^{-m}(\cos\theta), \tag{4}$$

with $iw = \lambda/2k$.

Using the notation of Buchholz for our parabolic wave functions (4) becomes

$$(m!)^2\,k^{-m}\mathcal{M}^{(m)}_{-\frac{w}{2}}(-ik\xi^2)\cdot\mathcal{M}^{(m)}_{-\frac{w}{2}}(ik\eta^2) = \sum_{n=m}^{\infty}(2n+1)$$

$$\cdot(-1)^m\frac{i^{n-m}}{k^m}\;\frac{F_n^m(w)}{F_m^m(w)}\;j_n(kr)\;P_n^m(\cos\theta),$$

or

$$\mathcal{M}^{(m)}_{-\frac{w}{2}}(-ik\xi^2)\cdot\mathcal{M}^{(m)}_{-\frac{w}{2}}(ik\eta^2) = \sum_{n=m}^{\infty}\frac{(2n+1)}{(m!)^2}\, i^{n+m}$$

$$\cdot\frac{F^m_n(w)}{F^m_m(w)}\, j_n(kr)\, P^m_n(\cos\theta). \tag{5}$$

Now let $-w = 2s + 1 + m$, where s and m are integers, then (5) becomes

$$\mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(-ik\xi^2)\cdot\mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(ik\eta^2) = \sum_{n=m}^{\infty}\frac{(2n+1)}{(m!)^2}\, i^{n+m}$$

$$\cdot\frac{F^m_n(-2s-1-m)}{F^m_m(-2s-1-m)}\, j_n(kr)\, P^m_n(\cos\theta). \tag{6}$$

We have

$$F^m_m(-2s-1-m) = {}_2F_1(-m,-s;1;1)$$

$$= \frac{(1+s)_m}{(1)_m}$$

$$= \frac{(s+m)!}{m!\; s!} = \frac{\Gamma(s+m+1)}{m!\; s!}\,. \tag{7}$$

Also

$$F^m_n(-2s-1-m) = {}_3F_2(-n,n+1,-s;m+1,1;1)$$

$$= \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+s+\frac{1}{2}m+\frac{1}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m+s+\frac{1}{2}m+\frac{1}{2})\,\Gamma(m+1)}$$

$$\cdot{}_3F_2(-n+m,n+m+1,-s;m+1,m+1;1)$$

$$= \frac{\Gamma(s+m+1)}{s!\ m!} \sum_{r=0}^{s} \frac{(-n+m)_r (n+m+1)_r (-s)_r}{(m+1)_r (m+1)_r \cdot r!}$$

$$= \frac{\Gamma(s+m+1)}{s!\ m!} \sum_{r=0}^{s} \frac{(m-n)_r (n+m+1)_r (-1)^r\ s!}{(m+1)_r (m+1)_r\ r!\ (s-r)!} .$$

Here we have used the transformation [9, page 499]

$$ {}_3F_2\left[\begin{matrix}\alpha_1, \alpha_2, \alpha_3; 1\\ \beta_1, \beta_2\end{matrix}\right] = \frac{\Gamma(\beta_1)\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)}{\Gamma(\beta_2-\alpha_3)\Gamma(\beta_1-\beta_2-\alpha_1-\alpha_2)}$$

$$\cdot\ {}_3F_2\left[\begin{matrix}\beta_1-\alpha_1, \beta_1-\alpha_2, \alpha_3; 1\\ \beta_1, \beta_1+\beta_2-\alpha_1-\alpha_2\end{matrix}\right].$$

Hence we have with, some simplifying,

$$F_n^m(-2s-1-m) = \sum_{r=0}^{s} \frac{(-1)^r (n-m)_r (n+m+1)_r (m+1+r)_{(s-r)}}{(m+1)_r\ r!\ (s-r)!} . \tag{8}$$

Introducing (7) and (8) into (6) leads to

$$\mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(-ik\xi^2) \cdot \mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(ik\eta^2)$$

$$= \sum_{n=m}^{\infty} \frac{(2n+1)}{(m!)^2}\ i^{n+m}\ \frac{m!\ s!}{(s+m+1)}$$

$$\cdot \sum_{n=m}^{\infty} \frac{(-1)^r (n-m)_r (n+m+1)_r (m+1+r)_{(s-r)}}{(m+1)_r\ r!\ (s-r)!}$$

$$\cdot\ j_n(kr)\ P_n^m(\cos\theta),$$

or

$$\frac{\Gamma(s+m+1)}{s!}\,\mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(-ik\xi^2)\cdot\mathcal{M}^{(m)}_{s+\frac{1+m}{2}}(ik\eta^2)$$

$$=\sum_{n=m}^{\infty}\frac{i^{n-m}(2n+1)}{m!}\sum_{r=0}^{s}\frac{(-1)^r(n-m)_r(r+m+1)_{(s-r)}}{(m+1)_r(s-r)!\;r!}$$

$$\cdot\; j_n(kr)\; P_n^m\,(\cos\theta),$$

which is the series expansion given by Hochstadt with the parabolic coordinates $X=2\sqrt{\xi\eta}\;\cos\varphi$, $Y=2\sqrt{\xi\eta}\,\sin\varphi$, $Z=\xi-\eta$.

VI. THE EXPANSION FOR PARABOLIC POTENTIAL FUNCTIONS

The parabolic potential functions are the solutions of LaPlace's equation

$$\Delta U = 0$$

in parabolic coordinates. The expansion for these functions can be derived in two ways. Directly, by repeating the process for LaPlace's equation which we did for the wave equation, and this will be shown later in this section. Secondly, the expansion for these functions can be derived by taking the limit of the expansion for the parabolic wave functions, (12), section III, as $k \longrightarrow 0$. Then the parabolic wave functions become Bessel functions in the limit, as is the case if we separate the LaPlace's equation in the parabolic coordinates and find its solution. To show this, the differential equation satisfied by $\Psi_m(\xi, \lambda)$ is

$$\frac{d^2 \Psi_m(\xi, \lambda)}{d\xi^2} + \frac{1}{\xi}\frac{d\Psi_m(\xi, \lambda)}{d\xi} + \left[k^2 \xi^2 - \frac{m^2}{\xi^2} + \lambda\right] \cdot \Psi_m(\xi, \lambda) = 0,$$

which becomes, as $k \longrightarrow 0$,

$$\xi^2 \frac{d^2 \Psi_m(\xi, \lambda)}{d\xi^2} + \xi \frac{d\Psi_m(\xi, \lambda)}{d\xi} + \left[-m^2 + (\lambda^{1/2}\xi)^2\right] \cdot \Psi_m(\xi, \lambda) = 0. \qquad (1)$$

This is Bessel's differential equation with a solution regular at $\xi = 0$, $J_m(\lambda^{1/2}\xi)$. For the equation involving

η, the corresponding solution is $J_m(i\,\lambda^{1/2}\eta)$.

Now for the right-hand-side of the expansion III (12) we must find $F_n^m(w)$, $F_m^m(w)$, and $j_n(kr)$, as $k \longrightarrow 0$, while $P_n^{-m}(\cos\theta)$ is independent of k.

We have

$$F_n^m(w) = {}_3F_2(-n,n+1,\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{1}{2}w;1,m+1;1)$$

$$= \sum_{r=0}^{n} \frac{(-n)_r(n+1)_r(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}w)_r}{(1)_r\;(m+1)_r} \cdot \frac{1}{r!}$$

The last term in series (r=n) contains $(\frac{1}{2}+\frac{1}{2}m+\frac{1}{2}w)_n$.

This factor is of degree n in w. All other terms of r < n are of degree less than n in w. Therefore, near k = 0, with r = n and $w = -i\lambda/2k$, we have

$$F_n^m\left(-\frac{i\lambda}{2k}\right) \approx \frac{1}{n!}\cdot(-1)^n\, n!\cdot\frac{(2n)!}{n!}\cdot\frac{1}{n!}\cdot\frac{m!}{(n+m)!}$$

$$\cdot\frac{1}{2^{2n}}(-1)^n\, i^n\,\frac{\lambda^n}{k^n} = \frac{m!(2n)!\; i^n\lambda^n}{(n+m)!(n!)^2\; 2^{2n}}\cdot\frac{1}{k^n}$$

Similarly, near k = 0, we have

$$F_m^m\left(-\frac{i\lambda}{2k}\right) \approx \frac{m!(2m)!\; i^m\lambda^m}{(2m)!(m!)^2\; 2^{2m}}\cdot\frac{1}{k^m}$$

$$= \frac{i^m\,\lambda^m}{m!\;2^{2m}}\cdot\frac{1}{k^m}$$

Then, near k = 0, for the coefficients a_n^m we have

$$a_n^m \approx \frac{(2n+1)(n+m)!}{k^m(n-m)!}\, i^{n-m}\, \frac{m!(2n)!\,i^n \lambda^n}{(n+m)!(n!)^2\, 2^{2n}}\, \frac{1}{k^n}$$

$$\cdot \frac{m!\; 2^{2m}\, k^m}{i^m\, \lambda^m} ,$$

or

$$a_n^m \approx \frac{(2n+1)}{(n-m)!}\, \frac{(m!)^2}{(n!)^2}\, (-\tfrac{1}{4})^{n-m}\, \lambda^{n-m}\, (2n)!\, \frac{1}{k^n} .$$

The dominant term in the expansion of $j_n(kr) = \sqrt{\frac{\pi}{2kr}}\, J_{n+\frac{1}{2}}(kr)$

near $k = 0$ is the leading term and hence for small k

$$j_n(kr) \approx \frac{2^n (kr)^n\, n!}{(2n+1)!}$$

Thus the right-hand-side of our expansion III (12) becomes, in the limit as $k \longrightarrow 0$,

$$\sum_{n=m}^{\infty} \frac{(m!)^2}{(n-m)!\,n!}\, (-\tfrac{1}{4})^{n-m}\, 2^n r^n\, \lambda^{n-m}\, P_n^{-m}\,(\cos\theta).$$

It remains now to investigate the left-hand-side of our expansion as $k \longrightarrow 0$. We have

$$\Psi_m(\xi, \lambda) = \xi^m\, e^{ik\,\xi^2/2}\; {}_1F_1(-\tfrac{i\lambda}{4k} + \tfrac{m+1}{2}; m+1; -ik\,\xi^2).$$

Then

$$\lim_{k \longrightarrow 0} \Psi_m(\xi, \lambda) = \xi^m \left[1 - \frac{1}{(m+1)} \cdot \frac{\left(\frac{\sqrt{\lambda}}{2}\,\xi\right)^2}{1!} + \frac{1}{(m+1)(m+2)}\, \frac{\left(\frac{\sqrt{\lambda}}{2}\,\xi\right)^4}{2!} - \cdots\cdots \right]$$

$$= \frac{m!2^m}{(\sqrt{\lambda})^m}\left[\frac{\left(\frac{\sqrt{\lambda}}{2}\xi\right)^m}{m!} - \frac{\left(\frac{\sqrt{\lambda}}{2}\xi\right)^{m+2}}{(m+1)!\ 1!} + \frac{\left(\frac{\sqrt{\lambda}}{2}\xi\right)^{m+4}}{(m+1)!\ 2!} - \cdots\cdots\right]$$

$$= \frac{m!\ 2^m}{m/2}\sum_{p=0}^{\infty}\frac{(-1)^p\left(\frac{\sqrt{\lambda}}{2}\xi\right)^{m+2p}}{\Gamma(m+p+1)\ p!}$$

$$= \frac{m!\ 2^m}{\lambda^{m/2}}\ J_m(\sqrt{\lambda}\,\xi).$$

Similarly

$$\Psi_m(\eta,-\lambda) = \eta^m e^{-ik\eta^2/2}\ {}_1F_1\left(+\frac{i\lambda}{4k}+\frac{m+1}{2};\ m+1;\ -ik\eta^2\right)$$

approaches

$$\frac{m!\ 2^m}{i^m\ \lambda^{m/2}}\cdot J_m(i\sqrt{\lambda}\,\eta).$$

Therefore the expansion III (12) now becomes

$$\frac{(m!)^2\ 2^{2m}}{i^m\ \lambda^m} J_m(\lambda^{1/2}\xi)\cdot J_m(i\lambda^{1/2}\eta) = \sum_{n=m}^{\infty}\frac{(m!)^2\ r^n}{(n-m)!\ n!}$$

$$\cdot\left(-\frac{1}{4}\right)^{n-m}\frac{2^n\lambda^n}{\lambda^m}P_n^{-m}(\cos\theta),$$

or

$$J_m(\lambda^{1/2}\xi)\, J_m(i\,\lambda^{1/2}\eta) = \sum_{n=m}^{\infty} \frac{i^m}{n!}(2\lambda)^n \frac{(-1)^m}{(n-m)!}\left(-\frac{1}{4}\right)^n$$

$$\cdot\, r^n\, P_n^{-m}(\cos\theta)$$

$$= \sum_{n=m}^{\infty} i^m \frac{\left(-\frac{1}{4}\right)^n (2\lambda)^n}{n!\,(n+m)!}\, r^n$$

$$\cdot\, P_n^m(\cos\theta). \qquad (2)$$

which is the expansion for the parabolic potential functions.

As mentioned above a different way to get this expansion for the potential functions would be to repeat the process for LaPlace's equation that we did for the wave equation. We assume the expansion

$$J_m(\lambda^{1/2}\xi)\, J_m(i\,\lambda^{1/2}\eta) = \sum_{n=m}^{\infty} b_n^m\, r^n\, P_n^m(\cos\theta), \qquad (3)$$

with $r = \dfrac{\xi^2 + \eta^2}{2}$, $\cos\theta = \dfrac{\xi^2 - \eta^2}{\xi^2 + \eta^2}$. Again we attempt to determine the coefficients b_n^m by substituting (3) in the differential equation for $J_m(\lambda^{1/2}\xi)$ for all η.

Since the particular value of η does not matter, we choose arbitrarily small η. Then in the right-hand-side of the expansion (3), for η near zero, we have

$$P_n^m\left(\frac{\xi^2-\eta^2}{\xi^2+\eta^2}\right)\approx(-1)^m\ \frac{(n+m)!}{(n-m)!}\cdot\frac{1}{m!}\frac{\eta^m}{\xi^m}\ ,$$

and $\quad r^n\approx\dfrac{\xi^{2n}}{2^n}$.

For the function $J_m(i\,\lambda^{1/2}\eta)$, near $\eta = 0$, we have

$$J_m(i\lambda^{1/2}\eta)\approx\frac{i^m\ \lambda^{m/2}\eta^m}{m!}\ .$$

Thus the expansion (3) for $\eta \longrightarrow 0$ reduces to

$$J_m(\lambda^{1/2}\xi)=\sum_{n=m}^{\infty} b_n^m\ i^{-m}\ (-1)^m\ \frac{(n+m)!}{(n-m)!}\ \frac{\lambda^{-m/2}}{2^n}\ \xi^{2n-m}\ .$$

We now determine the condition that

$$\sum_{n=m}^{\infty} b_n^m\ i^{-m}\ (-1)^m\ \frac{(n+m)!}{(n-m)!}\ \frac{\lambda^{-m/2}}{2^n}\ \xi^{2n-m}$$

satisfies the differential equation (1). This leads to

$$\sum_{n=m}^{\infty} b_n^m\frac{(n+m)!}{(n-m)!}\ \frac{1}{2^n}\ (4n^2-4nm)\ \xi^{2n-m}$$

$$+\sum_{n=m}^{\infty} b_n^m\frac{(n+m)!}{(n-m)!}\ \frac{1}{2^n}\ \lambda\ \xi^{2n-m+2}=0,$$

which can be simplified to

$$\sum_{n=m+1}^{\infty} \left[b_n^m \frac{(n+m)!}{(n-m)!} \frac{1}{2^n} (4n^2 - 4nm) + b_{n-1}^m \frac{(n-1+m)!}{(n-1-m)!} \frac{1}{2^{n-1}} \lambda \right]$$

$$\cdot\ \xi^{2n-m} = 0 \tag{4}$$

Hence for (4) to hold the coefficient of ξ^{2n-m} must vanish, and thus we have the two-term recursion formula

$$b_n^m = - \frac{(n+m-1)!}{(n-m-1)!} \frac{1}{2^{n-1}} \lambda \cdot \frac{(n-m)!}{(n+m)!} \frac{2^n\, b_{n-1}^m}{4n(n-m)} ,$$

or

$$b_n^m = - \frac{(2\lambda)}{4n(n+m)} b_{n-1}^m \ . \tag{5}$$

The solution of (5) gives, for b_n^m in terms of b_m^m,

$$b_n^m = \frac{(-1)^{n-m}\ (2\lambda)^{n-m}\ m!\ (2m)!}{4^{n-m}\ \ n!\ \ (m+n)!}\, b_m^m \ .$$

Now the coefficients b_n^m can be given if we evaluate b_m^m from expansion (3). This can be done by dividing (3) by $\xi^m \eta^m = r^m \sin^m\theta$, and taking the limit when $\xi, \eta \longrightarrow 0$. This gives for the right-hand-side

$$\frac{(-1)^m\ (2m)!}{2^m\ m!}\, b_m^m \ .$$

The left-hand-side is

$$\frac{J_m(\lambda^{1/2}\xi)\ J_m(i\,\lambda^{1/2}\eta)}{\xi^m\ \eta^m} ,$$

and this becomes

$$\frac{i^m \lambda^m}{2^{2m} (m!)^2}$$

as $\xi, \eta \longrightarrow 0$, and therefore we have for the b_m^m

$$b_m^m = \frac{(-1)^m i^m \lambda^m}{2^m \, m! \, (2m)!}$$

Finally, we have for the coefficients b_n^m

$$b_n^m = \frac{i^m (-\frac{1}{4})^n (2\lambda)^n}{n! \, (m+n)!} \, .$$

The expansion for the parabolic potential functions is therefore

$$J_m(\lambda^{1/2} \xi) \, J_m(i \lambda^{1/2} \eta) = \sum_{n=m}^{\infty} \frac{i^m (-\frac{1}{4})^n (2\lambda)^n}{n! \, (n+m)!} r^n \cdot P_n^m (\cos \theta).$$

This is exactly the same expansion as in (2), and it can be rewritten as

$$J_m(\lambda^{1/2} \xi) \, J_m(i \lambda^{1/2} \eta) = \sum_{n=m}^{\infty} \frac{i^m (-\frac{1}{4})^n \lambda^n}{n! \, (n+m)!} \cdot (\xi^2 + \eta^2)^n \, P_n^m \left(\frac{\xi^2 - \eta^2}{\xi^2 + \eta^2} \right).$$

It follows immediately that the series is convergent for all ξ and η.

VII. GENERATING FUNCTIONS

1. Linear Generating Function for Pasternack Polynomials.

In order to find this generating function we first consider the representation

$$B(\beta, \gamma - \beta)\ {}_2F_1\ (\alpha,\beta;\gamma;x) = \int_0^1 u^{\beta-1}\ (1-u)^{\gamma-\beta-1}\ (1-xu)^{-\alpha}\ du, \qquad (1)$$

which is an integral representation for the hypergeometric function [11, page 12]. Let $\alpha = m + 1/2$, $\beta = (m+1+w)/2$, $\gamma = m+1$, and $x = -4t/(1-t)^2$, then (1) becomes

$$B(\tfrac{m+1+w}{2}, \tfrac{m+1-w}{2})\,{}_2F_1(m+\tfrac{1}{2}, \tfrac{m+1+w}{2}\ ;\ m+1;\ -\frac{4t}{(1-t)^2}\ )$$

$$= \int_0^1 u^{\frac{m-1+w}{2}}\ (1-u)^{\frac{m-1-w}{2}} \left[1 + \frac{4tu}{(1-t)^2}\right]^{-m-\frac{1}{2}} du, \qquad (2)$$

and

$$\frac{1}{(1-t)^{2m+1}}\ B(\tfrac{m+1+w}{2}\ ,\ \tfrac{m+1-w}{2})\ {}_2F_1\ (m+\tfrac{1}{2}\ ,\ \tfrac{m+1+w}{2};m+1;-\frac{4t}{(1-t)^2})$$

$$= \int_0^1 u^{\frac{m-1+w}{2}}\ (1-u)^{\frac{m-1-w}{2}} \left[1+t^2-2t(1-2u)\right]^{-m-\frac{1}{2}} du. \qquad (3)$$

Now the generating function for the Gegenbauer polynomial $C_n^{v+1/2}(z)$ is $(1-2hz + h^2)^{-v-1/2}$, and we have [5, page 175]

$$(1-2hz+h^2)^{-v-\frac{1}{2}} = \sum_{n=m}^{\infty} C_n^{v+\frac{1}{2}}(z)h^n =$$

$$2^v \frac{\Gamma(v+1)}{\Gamma(2v+1)} \sum_{n=m}^{\infty} \frac{\Gamma(n+2v+1)}{\Gamma(n+1)} (1-z^2)^{-\frac{v}{2}} P_{n+v}^{-v}(z)h^n. \quad (4)$$

which is convergent for $|h| < \left|z \pm (z^2-1)^{1/2}\right|$.

Substitution of (4) into (3) leads to, with v=m, h=t, and z=1-2u,

$$\frac{1}{(1-t)^{2m+1}} B\left(\frac{m+1+w}{2}, \frac{m+1-w}{2}\right) {}_2F_1\left(m+\frac{1}{2}, \frac{m+1+w}{2}; m+1; -\frac{4t}{(1-t)^2}\right)$$

$$= (-1)^m \frac{m!}{(2m)!} \sum_{n=m}^{\infty} t^{n-m} 2^{m-1} \int_{-1}^{1} (1-z^2)^{-\frac{m}{2}} P_n^m(z)$$

$$\cdot \left[\frac{1-z}{2}\right]^{\frac{m+w-1}{2}} \left[\frac{1+z}{2}\right]^{\frac{m-w-1}{2}} dz. \quad (5)$$

where we have applied the relation $P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} \cdot P_n^m(x)$, started the summation from n=m, and, as it is permissible here, changed its order with the integration.

Now from Buchholz [2, page 202] we have

$$\int_0^{\pi} \left(\tan\frac{\varphi}{2}\right)^w P_n^m(\cos\varphi)\, d\varphi = \frac{(-1)^m}{(m!)^2} \frac{(n+m)!}{(n-m)!} \Gamma\left(\frac{1+m-w}{2}\right)$$

$$\cdot \Gamma\left(\frac{1+m+w}{2}\right) {}_3F_2\left(-n+m, n+m+1, \frac{m+1+w}{2} ; m+1, m+1; 1\right). \tag{6}$$

If we put $\tan \varphi/2 = \sqrt{\frac{1-z}{1+z}}$, $z = \cos \varphi$ in (6), we get

$$\int_{-1}^{1} \left[\frac{1-z}{1+z}\right]^{\frac{w}{2}} P_n^m(z) \frac{1}{\sqrt{1-z^2}} dz = \int_{-1}^{1} \frac{(1-z)^{\frac{m+w-1}{2}}}{(1-z)^{\frac{m}{2}}} \frac{(1+z)^{\frac{m-w-1}{2}}}{(1+z)^{\frac{m}{2}}}$$

$$\cdot P_n^m(z) dz$$

$$= 2^{m-1} \int_{-1}^{1} (1-z^2)^{-\frac{m}{2}} \left[\frac{1-z}{2}\right]^{\frac{m+w-1}{2}} \left[\frac{1+z}{2}\right]^{\frac{m-w-1}{2}} P_n^m(z) dz, \tag{7}$$

which except for a constant factor is the integral in (5). Introducing (7) into (5) leads to, from (6),

$$\frac{1}{(1-t)^{2m+1}} B\left(\frac{m+1+w}{2}, \frac{m+1-w}{2}\right) {}_2F_1\left(m + \frac{1}{2}, \frac{m+1+w}{2} ; m+1; -\frac{4t}{(1-t)^2}\right)$$

$$= (-1)^m \frac{m!}{(2m)!} \sum_{n=m}^{\infty} t^{n-m} \frac{(-1)^m}{(m!)^2} \frac{(n+m)!}{(n-m)!} \Gamma\left(\frac{m+1-w}{2}\right)$$

$$\cdot \Gamma\left(\frac{m+1+w}{2}\right) {}_3F_2\left(-n+m, n+m+1, \frac{m+1+w}{2}; m+1, m+1; 1\right),$$

or

$$\frac{t^m}{(1-t)^{2m+1}} {}_2F_1\left(m + \frac{1}{2}, \frac{m+1+w}{2} ; m+1; -\frac{4t}{(1-t)^2}\right) =$$

$$\frac{1}{2m!}\sum_{n=m}^{\infty}\frac{(n+m)!}{(n-m)!}\,{}_3F_2(-n+m, n+m+1, \frac{m+1+w}{2}; m+1, m+1; 1)\cdot t^n$$

Using -w for w, we get

$$\frac{t^m}{(1-t)^{2m+1}}\;{}_2F_1(m+\tfrac{1}{2}, \frac{m+1-w}{2}; m+1; -\frac{4t}{(1-t)^2}) = \frac{1}{2m!}$$

$$\cdot\sum_{n=m}^{\infty}\frac{(n+m)!}{(n-m)!}\,{}_3F_2(-n+m, n+m+1; \frac{m+1-w}{2}; m+1, m+1; 1)\cdot t^n \quad (8)$$

But we have

$${}_3F_2(-n+m, n+m+1, \frac{m+1-w}{2}; m+1, m+1; 1) = \frac{m!\,\Gamma(\frac{1}{2}-\frac{1}{2}m+\frac{w}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{w}{2})}$$

$$\cdot\; F_n^m(-w), \quad (9)$$

and

$$F_m^m(w) = \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{w}{2})}{m!\,\Gamma(\frac{1}{2}-\frac{1}{2}m+\frac{w}{2})}\,. \quad (10)$$

Introducing (9) and (10) into (8) leads to

$$\frac{t^m}{(1-t)^{2m+1}}\;{}_2F_1(m+\tfrac{1}{2}, \frac{m+1-w}{2}; m+1; -\frac{4t}{(1-t)^2}) = \frac{1}{(2m)!}$$

$$\cdot\sum_{n=m}^{\infty}\frac{(n+m)!}{(n-m)!}\,\frac{F_n^m(-w)}{F_m^m(w)}\cdot t^n\;. \quad (11)$$

But from Pasternack [12, page 212] we have

$$F_n^m(-z) = (-1)^n F_n^m(z),$$

so that (11) becomes

$$\frac{t^m}{(1-t)^{2m+1}}\ {}_2F_1\left(m+\tfrac{1}{2},\ \frac{m+1-w}{2};\ m+1;\ -\frac{4t}{(1-t)^2}\right) = \frac{1}{(2m)!}$$

$$\cdot\sum_{n=m}^{\infty} (-1)^n \frac{(n+m)!}{(n-m)!}\cdot\frac{F_n^m(w)}{F_m^m(w)}\ t^n,$$

or, multiplying by $(ik)^m$,

$$\frac{(2m)!}{(ik)^m}\cdot\frac{t^m}{(1-t)^{2m+1}}\ {}_2F_1\left(m+\tfrac{1}{2},\ \frac{m+1-w}{2};\ m+1;-\frac{4t}{(1-t)^2}\right)$$

$$=\sum_{n=m}^{\infty}\frac{(-1)^n}{(ik)^m}\ \frac{(n+m)!}{(n-m)!}\ \frac{F_n^m(w)}{F_m^m(w)}\ t^n. \tag{12}$$

The coefficients of t^n in the expansion (11) are proportional to the coefficients of the expansion in section III. In fact if we rewrite (12) as

$$\frac{(2m)!}{(ik)^m}\cdot\frac{t^m}{(1-t)^{2m+1}}\ {}_2F_1\left(m+\tfrac{1}{2},\ \frac{m+1-w}{2};\ m+1;\ -\frac{4t}{(1-t)^2}\right)$$

$$=\sum_{n=m}^{\infty}\frac{i^{n-m}}{k^m}\ \frac{(n+m)!}{(n-m)!}\ \frac{F_n^m(w)}{F_m^m(w)}\ i^n\ t^n\ ,$$

it is seen at once that the coefficients of $i^n t^n$ here are exactly the b_n^m section III, page 12 and 14. That the coefficients in (12) are proportional to the coefficients b_n^m can also be seen by considering the differential equation

$$(t^2-1)f'' + \left[(m+3)t+2w+\frac{m-1}{t}\right] f' + (m+1+\frac{w}{t})f = 0, \tag{13}$$

satisfied by the function on the left-hand-side of (12). However, as t=0 is a regular singlular point of (13), if we assume a solution of the form

$$f = \sum_{n=m}^{\infty} c_n \, i^n t^n,$$

upon introducing this series into the differential equation, we get

$$\sum_{n=m}^{\infty} \Big[n(n+m+2)+(n+1) \Big] c_n \, i^n t^n + \sum_{n=m}^{\infty} w(2n+1)c_n \, i^n t^{n-1}$$

$$+ \sum_{n=m}^{\infty} n(m-n)c_n \, i^n t^{n-2} = 0,$$

which after simplifying becomes

$$\sum_{n=m}^{\infty} i^{n-1} s^{n-1} \Big[(n+1)(n-m+1)c_{n+1} + iw(2n+1)c_n + n(n+m)c_{n-1} \Big] = 0.$$

Hence for (14) to hold, the coefficient of t^{n-1} must vanish, and thus we are lead to the following three-term recursion formula for the c_n

$$(n+1)(n-m+1)c_{n+1} + iw(2n+1)c_n + n(n+m)c_{n-1} = 0. \qquad (14)$$

With $iw = \lambda/2k$, this is exactly the same recursion formula, equation (9), page 11, satisfied by the coefficients b_n^m. Thus the c_n must be proportional to the coefficients b_n^m, and hence the function on the left-hand-side of the equation (12) is a generating function for the coefficients b_n^m.

2. Bilinear Continuous Generating Function for Parabolic Wave Functions and Whittaker Confluent Hypergeometric Functions.

The bilinear continuous generating function for the Whittaker functions has been given by Erdélyi [4, page 66] as

$$\frac{(txy)^{1/2}}{1+t} e^{\left(-\frac{x+y}{2}\cdot\frac{1-t}{1+t}\right)} J_{2\mu}\left[\frac{2(txy)^{1/2}}{1+t}\right] = \frac{1}{2\pi i}$$

$$\cdot\int_L t^K \frac{\Gamma(\frac{1}{2}-K+\mu)\,\Gamma(\frac{1}{2}+K+\mu)}{[\Gamma(2\mu+1)]^2} M_{K,\mu}(x)$$

$$\cdot M_{K,\mu}(y)\,dK, \qquad (1)$$

where L is a path from $-i\infty$ to $+i\infty$, separating the poles of $\Gamma(\frac{1}{2}+K+\mu)$ from those of $\Gamma(\frac{1}{2}-K+\mu)$.

The purpose of this section is to derive the bilinear continuous generating function for our parabolic wave functions using our series expansion for these functions and some properties of Pasternack polynomials. Then with the relation between our parabolic wave functions and Whittaker functions already established, the above formula, a proof of which has been given by Erdélyi [4], can at once be given.

We start with (Appendix, equation 4)

$$J_m(z \sin\alpha \sin\beta)e^{iz\cos\alpha \cos\beta} = \sum_{n=m}^{\infty} i^{n-m}(2n+1)$$

$$\cdot \frac{(n+m)!}{(n-m)!} j_n(z)\, P_n^{-m}(\cos\alpha)\, P_n^{-m}(\cos\beta).$$

Now let $z = kr = \dfrac{\xi^2+\eta^2}{2}k$, and $\cos\beta = \dfrac{\xi^2-\eta^2}{\xi^2+\eta^2}$, then

we have

$$J_m(k\,\xi\eta \sin\alpha)e^{ik\frac{\xi^2-\eta^2}{2}\cos\alpha} = \sum_{n=m}^{\infty} i^{n-m}(2n+1)$$

$$\frac{(n+m)!}{(n-m)!} j_n(kr)\, P_n^{-m}(\cos\theta)\, P_n^{-m}(\cos\alpha), \qquad (2)$$

with $r = \dfrac{\xi^2+\eta^2}{2}$, and $\cos\theta = \dfrac{\xi^2-\eta^2}{\xi^2+\eta^2}$.

If we now substitute $\cos\alpha = \dfrac{1-t}{1+t}$, $\sin\alpha = \dfrac{2t^{1/2}}{1+t}$

in (2), we get

$$J_m\left(2k\frac{t^{1/2}\xi\eta}{1+t}\right) e^{ik\frac{\xi^2-\eta^2}{2}\cdot\frac{1-t}{1+t}} = \sum_{n=m}^{\infty} i^{n-m}$$

$$\cdot(2n+1)\frac{(n+m)!}{(n-m)!} j_n(kr)\, P_n^{-m}(\cos\theta)\, P_n^{-m}\left(\frac{1-t}{1+t}\right). \qquad (3)$$

Now we make use of a certain integral involving the Pasternack polynomial $F_n^m(x)$. Pasternack gives [12, page 216]

$$\Gamma(m+1)\, e^{mx} \operatorname{sech} x\, P_n^{-m}(\tanh x) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{ixz} F_n^m(iz-m)$$

$$\cdot \operatorname{sech}(\tfrac{1}{2}\pi z)\, dz . \qquad (4)$$

If we substitute $\tanh x = \frac{1-t}{1+t}$, so that $\operatorname{sech} x = \frac{2t^{1/2}}{1+t}$,

and $x = \ln t^{-1/2}$, and replace iz for z, (4) becomes

$$\Gamma(m+1) \frac{2t^{1/2}}{1+t} P_n^{-m}\left(\frac{1-t}{1+t}\right) = \frac{i}{2} \int_{i\infty}^{-i\infty} t^{\frac{m+z}{2}} F_n^m(-z-m)$$

$$\cdot \operatorname{sech}(\tfrac{1}{2}\pi i z)\, dz ,$$

or

$$\Gamma(m+1) \frac{2t^{1/2}}{1+t} P_n^{-m}\left(\frac{1-t}{1+t}\right) = \frac{1}{2i} \int_{-i\infty}^{i\infty} t^{\frac{m+z}{2}} F_n^m(-z-m)$$

$$\cdot \sec(\tfrac{1}{2}\pi z)\, dz . \qquad (5)$$

Let $w = \frac{-i\lambda}{2k} = -(m+z)$, then (5) becomes

$$\Gamma(m+1) \frac{t^{1/2}}{1+t} P_n^{-m}\left(\frac{1-t}{1+t}\right) = -\frac{1}{4i} \int_{m-i\infty}^{m+i\infty} t^{-\frac{w}{2}} F_n^m(w)$$

$$\cdot \sec \tfrac{1}{2}\pi(-m-w)\, dw,$$

which can be written as

$$\frac{t^{1/2}}{1+t} P_n^{-m}\left(\frac{1-t}{1+t}\right) = -\frac{1}{4i}\frac{1}{(m!)^2}\int_{m-i\infty}^{m+i\infty} t^{-\frac{w}{2}} \frac{F_n^m(w)}{F_m^m(w)}$$

$$\cdot \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{w}{2})}{\Gamma(\frac{1}{2}-\frac{1}{2}m-\frac{w}{2})} \cdot \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{w}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{w}{2})} \sec \frac{1}{2}\pi(-m-w)\, dw. \quad (6)$$

But we have

$$\Gamma(\tfrac{1}{2}-\tfrac{1}{2}m-\tfrac{w}{2})\cdot\Gamma(\tfrac{1}{2}+\tfrac{1}{2}m+\tfrac{w}{2}) = \pi \sec \tfrac{1}{2}\pi(m+w),$$

so that (6) becomes

$$t^{1/2} P_n^{-m}\left(\frac{1-t}{1+t}\right) = -\frac{1}{4\pi i}\int_{m-i\infty}^{m+i\infty} t^{-\frac{w}{2}}$$

$$\cdot \frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{w}{2})\,\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{w}{2})}{[\Gamma(m+1)]^2} \cdot \frac{F_n^m(w)}{F_m^m(w)}\, dw. \quad (7)$$

Returning to equation (3) above, if we multiply it by $\frac{t^{1/2}}{1+t}$ and introduce (7) there, we get

$$\frac{t^{1/2}}{1+t} J_m\left(2k\frac{t^{1/2}\xi\eta}{1+t}\right) e^{ik\frac{\xi^2-\eta^2}{2}} \cdot \frac{1-t}{1+t} = \sum_{n=m}^{\infty} i^{n-m}(2n+1)$$

$$\cdot \frac{(n+m)!}{(n-m)!} j_n(kr)\, P_n^{-m}(\cos\theta) \cdot \frac{-1}{4\pi i}$$

$$\cdot \int_{m-i\infty}^{m+i\infty} t^{-\frac{w}{2}} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}m + \frac{w}{2})\, \Gamma(\frac{1}{2} + \frac{1}{2}m - \frac{w}{2})}{[\Gamma(m+1)]^2} \frac{F_n^m(w)}{F_m^m(w)}\, dw. \qquad (8)$$

We can interchange the summation and integration here, and then use our expansion III(12) for the parabolic wave functions, thus (8) becomes

$$\frac{t^{1/2}}{1+t} J_m \left(2k\xi\eta \frac{t^{1/2}}{1+t}\right) e^{ik\frac{\xi^2-\eta^2}{2}\cdot\frac{1-t}{1+t}}$$

$$= -\frac{k^m}{4\pi i} \int_{m-i\infty}^{m+i\infty} t^{-\frac{w}{2}}$$

$$\cdot \frac{\Gamma(\frac{1}{2} + \frac{1}{2}m + \frac{w}{2})\, \Gamma(\frac{1}{2} + \frac{1}{2}m - \frac{w}{2})}{[\Gamma(m+1)]^2} \Psi_m(\xi, \lambda)$$

$$\cdot \Psi_m(\eta, -\lambda)\, dw. \qquad (9)$$

This is the bilinear generating function for our parabolic wave functions. We have already shown that

$$\Psi_m(\xi, \lambda)\, \Psi_m(\eta, -\lambda) = \frac{1}{\xi\eta} k^{-(m+1)} M_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2)$$

$$\cdot M_{-\frac{w}{2},\frac{m}{2}}(ik\eta^2),$$

with $w = -\frac{i\lambda}{2k}$,

so that for the Whittaker confluent hypergeometric functions (9) becomes, after multiplying by $\xi\eta$,

$$k\,\xi\eta\,\frac{t^{1/2}}{1+t}\,J_m\!\left(2k\,\xi\eta\,\frac{t^{1/2}}{1+t}\right)\,e^{ik\frac{\xi^2-\eta^2}{2}\cdot\frac{1-t}{1+t}}$$

$$= -\frac{1}{4\pi i}\int_{m-i\infty}^{m+i\infty} t^{-\frac{w}{2}}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}m+\frac{w}{2})\,\Gamma(\frac{1}{2}+\frac{1}{2}m-\frac{w}{2})}{[\Gamma(m+1)]^2}$$

$$\cdot\,M_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2)\cdot M_{-\frac{w}{2},\frac{m}{2}}(ik\eta^2)\,dw\ ,$$

which is essentially formula (1) with $x = -ik\xi^2$, $y = ik\eta^2$, $K = -\frac{w}{2} = i\lambda/4k$, and $2\mu = m$.

APPENDIX

It is possible to derive the series expansion for the parabolic wave functions in terms of the spherical wave functions with a different method. This can be done by using an equation due to Buchholz that expresses the product of the parabolic functions as denoted by him, in terms of an integral of a Bessel function of the first kind. Buchholz gives [3, page 85]

$$\mathcal{M}_{x,\mu/2}(a_1 t)\cdot\mathcal{M}_{x,\mu/2}(a_2 t) = \frac{(a_1 a_2)^{1/2}\cdot t}{\Gamma(\frac{1+\mu}{2}+x)\,\Gamma(\frac{1+\mu}{2}-x)}$$

$$\cdot\int_0^{\pi} e^{-\frac{1}{2}(a_1+a_2)t\cos\varphi}\, J_\mu(t\sqrt{a_1 a_2}\,\sin\varphi)$$

$$\cdot\,(\cot\tfrac{\varphi}{2})^{2x}\,d\varphi\,. \tag{1}$$

Also from Erdelyi [6 , Page 102]

$$(\sin\alpha\,\sin\beta)^{1/2-\nu}\,J_{\nu-\frac{1}{2}}(z\sin\alpha\,\sin\beta)e^{i\,z\cos\alpha\,\cos\beta}$$

$$= 2^{2\nu-\frac{1}{2}}(\pi z)^{-\frac{1}{2}}[\Gamma(\nu)]^2\cdot\sum_{n=0}^{\infty}\frac{i^n\,n!(n+\nu)}{(2\nu+n)}\,J_{n+\nu}(z)$$

$$\cdot\,C_n^\nu(\cos\alpha)C_n^\nu(\cos\beta). \tag{2}$$

here $J_\nu(z)$ is the Bessel function of the first kind and $C_n^\nu(x)$ is the Gegenbauer polynomial. Equation (2), with $\nu = m + \frac{1}{2}$, becomes

$$(\sin\alpha \sin\beta)^{-m} J_m(z \sin\alpha \sin\beta)\, e^{iz \cos\alpha \cos\beta} = 2^{2m + \frac{1}{2}} \cdot [\Gamma(m + \tfrac{1}{2})]^2 (\pi z)^{-\frac{1}{2}} \cdot \sum_{n=0}^{\infty} \frac{i^n\, n!(m+n+\frac{1}{2})}{\Gamma(2m+n+1)} J_{m+n+\frac{1}{2}}(z) \cdot C_n^{m+\frac{1}{2}}(\cos\alpha)\, C_n^{m+\frac{1}{2}}(\cos\beta). \qquad (3)$$

Substituting for Gegenbauer functions in terms of Legendre functions [5, page 175]

$$C_n^{m+\frac{1}{2}}(x) = \frac{2^m\, \Gamma(m+1)\, \Gamma(n+2m+1)}{\Gamma(2m+1)\, \Gamma(n+1)} (1-x^2)^{-\frac{1}{2}m} P_{n+m}^{-m}(x),$$

in (3) leads to

$$(\sin\alpha \sin\beta)^{-m} J_m(z \sin\alpha \sin\beta)\, e^{iz \cos\alpha \cos\beta} = 2^{2m+\frac{1}{2}} \cdot [\Gamma(m+\tfrac{1}{2})]^2 (\pi z)^{-\frac{1}{2}} \cdot \sum_{n=0}^{\infty} \frac{i^n\, n!(m+n+\frac{1}{2})}{\Gamma(2m+n+1)} J_{m+n+\frac{1}{2}}(z) \cdot 2^{2m} \cdot \left[\frac{\Gamma(m+1)}{\Gamma(2m+1)} \cdot \frac{\Gamma(n+2m+1)}{\Gamma(n+1)}\right]^2 (\sin\alpha \sin\beta)^{-m} P_{n+m}^{-m}(\cos\alpha) \cdot P_{n+m}^{-m}(\cos\beta).$$

This simplifies with

$$\Gamma(m+\tfrac{1}{2}) = \frac{\sqrt{\pi}}{2^{2m}} \cdot \frac{(2m)!}{m!} ,$$

to

$$J_m(z \sin\alpha \sin\beta)\, e^{i z \cos\alpha \cos\beta} = \sqrt{\frac{\pi}{2z}} \sum_{n=0}^{\infty} i^n$$

$$\cdot \frac{(2m+2n+1)(n+2m)!}{n!} J_{m+n+\frac{1}{2}}(z)\, P_{n+m}^{-m}(\cos\alpha)$$

$$\cdot P_{n+m}^{-m}(\cos\beta) = \sum_{n=m}^{\infty} i^{n-m} \frac{(2n+1)(n+m)!}{(n-m)!} j_n(z)$$

$$\cdot P_n^{-m}(\cos\alpha)\, P_n^{-m}(\cos\beta). \qquad (4)$$

Now, comparing (1) and (4), we can substitute (4) into (1) if we put $\beta = \varphi$, $iz \cos\alpha = -\frac{1}{2}(a_1 + a_2)t$, and $z \sin\alpha = t\sqrt{a_1 a_2}$. Thus

$$z = \frac{it}{2}(a_1 - a_2)$$

$$\cos\alpha = \frac{a_1 + a_2}{a_1 - a_2} .$$

Hence (1) becomes, with $\mu = m$, and we can change the order of integration and summation here,

$$\mathcal{M}_{x,\frac{m}{2}}(a_1 t) \cdot \mathcal{M}_{x,\frac{m}{2}}(a_2 t) = \frac{(a_1 a_2)^{1/2}\, t}{\Gamma(\frac{1+m}{2} + x)\, \Gamma(\frac{1+m}{2} - x)}$$

$$\cdot \sum_{n=m}^{\infty} i^{n-m} \frac{(2n+1)(n+m)!}{(n-m)!} j_n \left[\frac{it}{2} (a_1 - a_2)\right]$$

$$\cdot P_n^{-m} \left(\frac{a_1+a_2}{a_1-a_2} \right) \cdot \int_0^{\pi} P_n^{-m}(\cos \varphi)(\tan \frac{\varphi}{2})^{-2x} \, d\varphi$$

$$= \frac{(a_1 a_2)^{1/2} \, t}{\Gamma(\frac{1+m}{2} + x) \, \Gamma(\frac{1+m}{2} - x)} \cdot \sum_{n=m}^{\infty} i^{n-m} (2n+1)(-1)^m$$

$$\cdot j_n \left[\frac{it}{2}(a_1-a_2)\right] P_n^{-m} \left(\frac{a_1+a_2}{a_1-a_2} \right) \cdot \int_0^{\pi} P_n^{m}(\cos \varphi)$$

$$\cdot (\tan \varphi)^{-2x} \, d\varphi \, . \tag{5}$$

Using equation (6) section VII. 1, (5) becomes

$$\frac{1}{(a_1 a_2)^{1/2} \, t} \mathcal{M}_{x,\frac{m}{2}}(a_1 t) \cdot \mathcal{M}_{x,\frac{m}{2}}(a_2 t) = \sum_{n=m}^{\infty} \frac{i^{n-m}}{(m!)^2} (2n+1)$$

$$\cdot \frac{(n+m)!}{(n-m)!} \, {}_3F_2 \left(-n+m, n+m+1, -x + \frac{m+1}{2} ; m+1, m+1; 1\right)$$

$$\cdot j_n\left[\frac{it}{2}(a_1-a_2)\right] P_n^{-m} \left(\frac{a_1+a_2}{a_1-a_2} \right) . \tag{6}$$

Now, as before, we have

$$ {}_3F_2\left(-n, n+m+1, -x+\frac{m+1}{2};\ m+1, m+1; 1\right) $$

$$ = \frac{m!\ \Gamma(\frac{1}{2}-\frac{1}{2}m+x)}{\Gamma(\frac{1}{2}+\frac{1}{2}m+x)} \cdot F_n^m(-2x) $$

$$ = \frac{F_n^m(-2x)}{F_m^m(-2x)}, $$

so that (6) becomes

$$ \frac{1}{(a_1a_2)^{1/2}\,t}\ \mathcal{M}_{x,\frac{m}{2}}(a_1t)\cdot\ \mathcal{M}_{x,\frac{m}{2}}(a_2t) = \sum_{n=m}^{\infty} i^{n-m}\,(2n+1) $$

$$ \cdot\frac{(n+m)!}{(n-m)!}\ \frac{F_n^m(-2x)}{F_m^m(-2x)}\ j_n\left[\frac{it}{2}(a_1-a_2)\right] P_n^{-m}\left(\frac{a_1+a_2}{a_1-a_2}\right). \tag{7} $$

Now in (7) we put $X = -\frac{W}{2}$, $a_1t = -ik\,\xi^2$, $a_2t = ik\,\eta^2$,

and

$$ i\,\frac{a_1-a_2}{2}\,t = k\,\frac{\xi^2+\eta^2}{2} = kr $$

$$ \frac{a_1+a_2}{a_1-a_2} = \frac{\xi^2-\eta^2}{\xi^2+\eta^2} = \cos\theta, $$

and get

$$\frac{(m!)^2}{k^{m+1}\xi\eta}\mathcal{M}_{-\frac{w}{2},\frac{m}{2}}(-ik\xi^2)\cdot\mathcal{M}_{-\frac{w}{2},\frac{m}{2}}(ik\eta^2)$$

$$=\sum_{n=m}^{\infty}\frac{i^{n-m}}{k^m}\frac{(2n+1)(n+m)!}{(n-m)!}\frac{F_n^m(w)}{F_m^m(w)}$$

$$\cdot\, j_n(kr)\,P_n^{-m}(\cos\theta)\,. \qquad (8)$$

We have for our parabolic wave functions

$$\Psi_m(\xi,\lambda)\,\Psi_m(\eta,-\lambda)=\frac{(m!)^2}{\xi\eta}\,k^{-(m+1)}$$

$$\cdot\,\mathcal{M}_{-\frac{w}{2},\frac{m}{2}}(ik\xi^2)\cdot\mathcal{M}_{-\frac{w}{2},\frac{m}{2}}(ik\eta^2)\,.$$

Introducing this into (8) gives at once our expansion (12) for the parabolic wave functions in terms of the spherical wave functions in section III.

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