LOCAL DEFORMATIONS OF WILD GROUP ACTIONS

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ABSTRACT

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In this dissertation, we study deformations of actions of a cyclic group of order p on the formal power series ring $k[[u_1, \ldots, u_n]]$, where k is a field of characteristic $p > 0$. We draw upon work of B. Peskin in [20] to reduce, under certain hypotheses, the task of determining the tangent space of the deformation functor D to a problem in invariant theory. When $n = 2$ and $p = 3$, we use these results to explicitly compute the tangent space of D and then generalize results of Mézard and Bertin for smooth curves to smooth surfaces. In particular, we compute the prorepresentable hull of the equicharacteristic local deformation functor D of a smooth surface with finite, cyclic group action at a point of wild ramification in characteristic 3.

To all those of you who have made this possible: especially my parents, my wife, and my adviser. Thank you!

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1 Introduction

The main idea of deformation theory is to learn how an object can be continuously varied in correspondence with the points of some parameter space. These objects can be chosen to be subschemes of a fixed scheme, line bundles, nonsingular varieties, coherent sheaves, or singularities, among other things. This theory has applications in a wide range of different fields in mathematics. In this dissertation we study the local deformation functor associated to a scheme with certain wild group actions. Along with this functor there often exists a ring, called the (uni)versal deformation ring, that determines what local deformations can arise. The typical goal is to determine this (uni)versal deformation ring for as many group actions as possible.

In the last half century, research in deformation theory has thrived in algebraic geometry. The foundation of this large subject was laid by Kodaira and Spencer, Grothendieck, Schlessinger, Illusie, M. Artin, Deligne, and others. Recently, the deformation theory of Galois representations has found important applications in number theory in the work of Wiles, Taylor-Wiles, and others ([26], [25]). M. Artin and others have studied deformations of singularities. We focus on deformations of schemes with an action by a finite group G over an algebraically closed field of characteristic $p > 0$. Most of the results in this area thus far concern actions on smooth curves. (See [5], [6], and [18] for such results on deformations and local lifting problems). Grothendieck showed that at points of tame ramification, meaning that p does not divide the order $|G|$ of the group G, there are no obstructions to lifting infinitesimal deformations and thus the versal deformation ring is a formal power series ring ([11]). Bertin, Green, and Matignon demonstrate nontrivial obstructions when the ramification is wild, i.e., when p divides $|G|$ ([1], [10]). Progress toward determining the (uni)versal deformation ring at points of wild ramification has been made exclusively in the case of smooth curves. However, even in this case, explicit computations have only been successful in certain special cases. Mézard and Bertin have computed the versal deformation ring when the conductor m of the automorphism σ defining the cyclic group action is $m = 1$ and $p \geq 3$ ([2]). In the case that $m > 1$ and $(m, p) = 1$, they are able to determine a quotient of the hull and compute the hull's Krull dimension. They also prove a local-global principle for deformations. We work to prove local results similar to those of Mézard and Bertin for higher-dimensional schemes. Let us now make the above statements more precise and give an overview of the contents of this dissertation.

In section 2 we introduce some of the preliminary definitions and results that will be required. Particularly, we set up notations for group actions on schemes, give a needed fact for G-torsors, state a result from descent theory that we will use, and recall Schlessinger's theory of functors of Artin rings.

After this we start looking at equivariant deformations in section 3. The focus of our attention will be the local equivariant deformation theory of a smooth scheme X with $\dim(X) = n$ over a field k of prime characteristic $p > 0$ at a point of wild ramification x. Particularly, we consider the completion of the stalk at x, $\widehat{\mathcal{O}}_{X,x} \cong k[[u_1, \ldots, u_n]]$, and fix an action $\rho : G \to \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])$ where $G = \langle \sigma \rangle$ is assumed to be cyclic of order p. We will see that with this data one can define a local deformation functor $D : \mathfrak{C} \to \mathrm{Sets}$ to which Schlessinger's results are applicable. The functor D takes a ring A from the category $\mathfrak{C}$ of local Artinian k-algebras with residue field k to the set of local deformations of the base action ρ to A modulo some equivalence. We end this section by making explicit the fundamental result that shows that the tangent space of this local deformation functor is isomorphic to $H^1(G,\Theta)$, the first group cohomology of the space $\Theta = \oplus_{j=1}^n k[[u_1,\ldots,u_n]]\dfrac{\partial}{\partial u_j}$ of k-derivations of $\widehat{\mathcal{O}}_{X,x}$.

The objective of section 4 is to show that Schlessinger's criterion for the existence of hull are satisfied for the functor D. We then move on to the task of computing the tangent space of this functor in section 5. By the results in section 3 this amounts to computing the first cohomology group $H^1(G,\Theta)$. This is very difficult in general, but in this dissertation, we will exhibit a family of cases in which we can reduce this calculation to a calculation in invariant theory.

These are cases when $n = p-1$ and the action can be put in the following form:

$$\begin{aligned} \sigma u_1 &= u_1 + f(u_1,\ldots,u_n) \\ \sigma u_2 &= u_2 + u_1 \\ &\vdots \\ \sigma u_n &= u_n + u_{n-1}, \end{aligned} \tag{1}$$

where $\operatorname{ord}(f) \geq 2$ and f is invariant. Actually, under the assumption that the linear terms of the action ρ form a single Jordan block when in Jordan form, B. Peskin [20, p. 77] shows that there exists a change of coordinates that puts the action in the form (1). Additionally, we assume that f is invariant under this action and that $n = p-1$.

The quotient $k[[u_0,\ldots,u_n]]/(u_0 - f)$, where $f \in k[[u_1,\ldots,u_n]]$ is the power series given above, can be endowed with a G-action by defining $\sigma u_0 = u_0$ and $\sigma u_i = u_i + u_{i-1}$ for $1 \leq i \leq n$. Then $k[[u_1,\ldots,u_n]] \cong k[[u_0,u_1,\ldots,u_n]]/(u_0 - f)$ as G-modules. Set $R = k[[u_0,\ldots,u_n]]/(u_0 - f)$. Our result that simplifies the calculation of $H^1(G,\Theta)$ is the following:

THEOREM 6. Suppose that the ring $k[[u_1, \ldots, u_n]]$ has G-action given by (1) above with f invariant and $n = p - 1$. Then $H^1(G, \Theta) \cong \widehat{H}^0(G, R)$ and $H^2(G, \Theta) \cong H^1(G, R)$.

Here $\widehat{H}^0(G, R)$ is the Tate cohomology group of G with coefficients in R. The remaining steps to determine the tangent space $H^1(G, \Theta)$ are admittedly still challenging, but Theorem 6 greatly simplifies these computations and allows us to make progress in certain cases. Particularly, since $\widehat{H}^0(G, R) \cong R^G/\mathrm{Im}(Tr)$, the computation of the tangent space of D is reduced to determining the invariant ring R^G and the image of the trace map Tr. Typically, the more challenging part is computing R^G. Even when the action of $G = \langle \sigma \rangle$ on the polynomial ring $k[u_1, \ldots, u_n]$ is linear in the sense that σu_i contains only terms of degree one for all i, this is not trivial. With the aid of computer algebra systems, there is a good deal of current research in computational invariant theory dealing with computing R^G for such wild group actions: for example, see the work of Campbell and Hughes [4], Shank [23], Shank-Wehlau [24], and Campbell-Fodden-Wehlau [3]. Much of this work has been summarized nicely in [7] by Derksen and Kemper.

Using the above and a result of Peskin [20, p. 96] that gives the invariant ring R^G, we then obtain a complete determination of the tangent space when $n = 2$, the characteristic p is 3, and the action is free off the closed point. Also, by the main result of Peskin [20, p. 88], there is an $s \geq 1$ such that after a choice of variables the action (1) above can be written in the form

$$\begin{aligned} \sigma u_1 &= u_1 + y^s \\ \sigma u_2 &= u_2 + u_1, \end{aligned} \tag{2}$$

where $y = Nu_2$ is the norm of u_2 under this action. Note that the norm y and the action itself are defined recursively here. We can then prove the following result.

THEOREM 7. Suppose $\mathrm{char}(k) = 3$ and the action of $G = \langle \sigma \rangle$ on $k[[u_1, u_2]]$ is free off the closed point. Then $H^1(G, \Theta) \cong k[y]/(y^s)$.

With this result in hand we can prove one of the main results of this dissertation. Particularly, we are able to determine the hull of the local equicharacteristic deformation functor D, i.e., the functor restricted to the subcategory $\mathfrak{C}_3$ of $\mathfrak{C}$ consisting of k-algebras of characteristic 3.

THEOREM 8. Suppose $\mathrm{char}(k) = 3$ and the action $\rho : G \to \mathrm{Aut}_k(k[[u_1, u_2]])$ is free off the closed point. Then the hull of the deformation functor $D|_{\mathfrak{C}_3}$ is $k[[x_0, x_1, \ldots, x_{s-1}]]$, where $s \geq 1$ is the integer given in (2) above.

In the final section, we consider a specific example of an action of $\mathbb{Z}/3\mathbb{Z} = \langle \sigma \rangle$ on the Fermat quartic $X \subset \mathbb{P}^3_k$ given by the equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$. Namely, we look at the action given by $\sigma(x_0,\ x_1,\ x_2,\ x_3) = (x_0,\ x_2,\ x_3,\ x_1)$. This example was considered in a paper of Dolgachev and Keum [8, p. 114]. There is a unique fixed point for this action in the case that $\mathrm{char}(k) = 3$. By Peskin's results, we know that the induced local action at this fixed point can be put in the form (2) after some change of coordinates. Applying the above results, we show that the action can be put in this form with $s = 1$. In doing so we demonstrate an example where the local wild action can be lifted to characteristic zero. This also provides a specific example where the local actions we consider in this dissertation arise from a global action.

2 Preliminaries

2.1 Schemes with Group Action

We now work in the category Sch$/k$ of schemes over a fixed algebraically closed field k. Let G be a group. If X is a scheme in Sch$/k$, an *action of G on X* is given by a group homomorphism $\rho : G \to \mathrm{Aut}_k(X)$. The image of $g \in G$ under ρ will be denoted by ρ_g, or by g when no confusion could arise. If T is a set on which G acts, we will denote the set of fixed points by T^G. The functor of fixed points X^G of $X \in$ Sch$/k$ with G-action is given by

$$X^G : \mathrm{Sch}/k \to \mathrm{Sets}, \ \ (T \to k) \mapsto X(T)^G,$$

where $X(T) = \{T \to X\}$ denotes the set of T-valued points of X. It can be shown that X^G is represented by a subscheme of X [9, p. 293]. The action of G on X is *free* if the set of P-valued points $X^G(P)$ is empty for all $P \in$ Sch$/k$. If S is a local ring with maximal ideal m_S, we say that the action of G on $X =$ Spec S is *free off the closed point* if X^G is supported on $\{m_S\} \in$ Spec $S = X$.

One result related to actions of a group G on a scheme X that we will require in section 4 is the following result, which can be found in [13, p. 216].

THEOREM. Let R be a ring, A an R-algebra, and G a finite group acting on A by R-algebra automorphisms. Suppose that G acts freely on A in the sense that (Spec $A)^G = \emptyset$. Then A is a finite étale G-torsor over A^G, and the natural map

$$A \otimes_{A^G} A \to \prod_{g \in G} A, \ \ x \otimes y \mapsto (\cdots, x \otimes g(y), \cdots)$$

is an isomorphism of left A-algebras.

In the above theorem, we say A is a *G-torsor* over A^G when the map $\mathrm{Spec}(A) \times_{\mathrm{Spec}(A^G)} G \to \mathrm{Spec}(A) \times_{\mathrm{Spec}(A^G)} \mathrm{Spec}(A)$ given by $(a, g) \mapsto (a, ga)$ is an isomorphism. We will also say that Spec A is a *G-principal homogeneous space* over $\mathrm{Spec}(A^G)$. Note here that $\mathrm{Spec}(A^G)$ is the quotient $\mathrm{Spec}(A)/G$.

In the sections that follow we will also require a well-known fact from descent theory. If A is a ring, M is an A-module, and the group G acts on both A and M, we say that M has compatible G-action if $g \cdot (am) = (ga) \cdot (gm)$. As in [16, p. 18], one can prove that when G acts freely on A the following map gives an equivalence of categories:

$$\left\{ \begin{array}{l} A\text{-modules with} \\ \text{compatible } G\text{-action} \end{array} \right\} \longleftrightarrow \left\{ A^G\text{-modules} \right\},$$

$$M \longmapsto M^G$$

$$A \otimes_{A^G} N \longleftarrow N.$$

Under this correspondence, we note that M is finitely generated if and only if M^G is finitely generated.

2.2 Functors of Artin Rings

Let Λ be a local Noetherian ring with maximal ideal m_Λ with residue field $k = \Lambda/m_\Lambda$. Denote by $\mathfrak{C}$ the category of Artinian local Λ-algebras having residue field k. Let $\widehat{\mathfrak{C}}$ be the category consisting of complete local Noetherian Λ-algebras A for which $A/m_A^n \in \mathfrak{C}$ for all n and having local Λ-algebra homomorphisms as morphisms. Note that $\mathfrak{C}$ is a full subcategory of $\widehat{\mathfrak{C}}$. Suppose $F : \mathfrak{C} \to \mathrm{Sets}$ is a functor such that $F(k)$ is a singleton set. A *couple* (A, ε) is a pair such that $A \in \mathfrak{C}$ and $\varepsilon \in F(A)$. We extend the definition of F from $\mathfrak{C}$ to $\widehat{\mathfrak{C}}$ by setting $\widehat{F}(A) = \varprojlim_n F(A/m_A^n)$ for $A \in \widehat{\mathfrak{C}}$. With this definition, we can consider *pro-couples* (A, ε), where $A \in \widehat{\mathfrak{C}}$ and $\varepsilon \in \widehat{F}(A)$.

Let us now sketch an argument showing that $\widehat{F}(A) \cong \mathrm{Hom}(h_A, F)$, where $h_A(R) = \mathrm{Hom}(A, R)$ for $R \in \mathfrak{C}$. In order to set up this isomorphism, start with $\xi = (\xi_j) \in \widehat{F}(A)$. Let $u : A \to R$ be a homomorphism in $\mathfrak{C}$. Since R is Artinian, there is some l such that $m_R^l = 0$. Since $u(m_A) = m_R$, it follows that $m_A^l \subset \mathrm{Ker}(u)$. Thus u factors through $u_l : A/m_A^l \to R$. To build the desired homomorphism in $\mathrm{Hom}(h_A, F)$ from ξ we send $u : A \to R$ to $F(u_l)(\xi_l)$. It is easily checked that this gives an isomorphism. A pro-couple (A, ε) therefore naturally induces a morphism of functors $h_A \to F$.

DEFINITION. A morphism $F \to H$ of functors is said to be *smooth* if $F(S) \to F(R) \times_{H(R)} H(S)$ is surjective for any surjection $S \to R$ in $\mathfrak{C}$.

As discussed in Schlessinger, it suffices to check the surjectivity of this map for all small extensions $S \to R$ in $\mathfrak{C}$ in order to show that $F \to H$ is smooth. A map $S \to R$ in $\mathfrak{C}$ is called a *small extension* if its kernel is a nonzero principal ideal (t) such that $t \cdot m_S = 0$. We define the *tangent space* of a functor F by $F(k[\varepsilon])$, and denote this by t_F. We will use the special notation t_A to denote the tangent space of the functor h_A.

DEFINITION. A pro-couple (A, ε) is a *prorepresentable hull* if the induced map $h_A \to F$ is smooth and the map $t_A \to t_F$ of tangent spaces is bijective. The pair (A, ε) is a *universal deformation ring* if the induced map $h_A(R) \to F(R)$ is an isomorphism for all R in $\mathfrak{C}$.

In [21, p. 212], Schlessinger provides necessary and sufficient conditions for a functor $F : \mathfrak{C} \to \mathrm{Sets}$, with $F(k)$ a singleton set, to have a prorepresentable hull and a universal deformation ring.

THEOREM. Suppose $\phi' : A' \to A$ and $\phi'' : A'' \to A$ are maps in $\mathfrak{C}$ and consider the natural map

$$f : F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'').$$

Then F has a prorepresentable hull if and only if the conditions (H_1), (H_2), and (H_3) below are satisfied:

(H_1) If ϕ'' is a small extension, then f is surjective.

(H_2) If $A = k$ and $A'' = k[\varepsilon]$, then f is a bijection.

(H_3) $F(k[\varepsilon])$ is a finite-dimensional vector space over k.

F has a universal deformation ring if and only if F satisfies conditions $(H_1) - (H_3)$ above and the condition:

(H_4) For any small extension $A' \to A$, $f : F(A' \times_A A') \xrightarrow{\sim} F(A') \times_{F(A)} F(A')$ is a bijection.

For more information concerning functors of Artin rings, one can consult Schlessinger's famous paper [21] where these notions were first introduced.

3 The Deformation Theory of Schemes with Group Action

Suppose X is a connected n-dimensional, finite type separated scheme over a fixed algebraically closed field k of characteristic $p > 0$. We will assume that X is smooth over k. Let $G \subset \mathrm{Aut}_k(X)$ be a finite subgroup and denote this inclusion by $\rho : G \hookrightarrow \mathrm{Aut}_k(X)$. A *deformation of the pair* (X, ρ) to an object R in the category $\mathfrak{C}$ is a triple $(\widetilde{X}, \widetilde{\rho}, \widetilde{\phi})$ consisting of a smooth scheme $\widetilde{X}$ of finite type over R, an injective morphism $\widetilde{\rho} : G \hookrightarrow \mathrm{Aut}_R(\widetilde{X})$, $\sigma \mapsto \widetilde{\rho\sigma}$, and an isomorphism $\widetilde{\phi} : \widetilde{X} \otimes_{\mathrm{Spec}\ R} \mathrm{Spec}\ k \to X$ of schemes over k such that $\overline{\widetilde{\rho}} = \rho$ via this map. Two deformations (X_1, ρ_1, ϕ_1) and (X_2, ρ_2, ϕ_2) are said to be isomorphic if there exists an isomorphism $\psi : X_1 \to X_2$ of schemes over R such that $\phi_1 \circ (\psi \otimes_R k) = \phi_2$

and $\psi \circ (\rho_2)_\sigma = (\rho_1)_\sigma \circ \psi$ for all $\sigma \in G$. We say that a closed point $x \in X(k)$ is *a point of wild ramification* when the stabilizer subgroup $G_x \subset \mathrm{Aut}_k(X)$ is of order divisible by p. Further, $\widehat{\mathcal{O}}_{X,x} \cong k[[u_1, \ldots, u_n]]$ and it follows that there is an injective morphism $G_x \hookrightarrow \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])$. A representative of a deformation of (X, ρ) to R gives a similar morphism $G_x \hookrightarrow \mathrm{Aut}_R(R[[u_1, \ldots, u_n]])$, which reduces by $R \to R/m_R \cong k$ to the initial representation of G_x over $k[[u_1, \ldots, u_n]]$. Henceforth, fix a representation $\rho : G \hookrightarrow \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])$, and denote the image of σ under ρ by ρ_σ. One is led to define the local deformation functor

$$D = D_{G,\rho} : \mathfrak{C} \to \text{Sets}, \qquad R \mapsto \left\{ \begin{array}{l} \text{liftings } G \to \mathrm{Aut}_R(R[[u_1, \ldots, u_n]]) \text{ of } \rho \\ \text{modulo conjugation by an element} \\ \text{of } \ker\left(\mathrm{Aut}_R(R[[u_1, \ldots, u_n]]) \to \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])\right) \end{array} \right\}.$$

Progress toward determining the (uni)versal deformation ring has been made by Mézard and Bertin [2] for the case of smooth projective curves, which corresponds to $n = 1$. We work to obtain similar results for schemes of higher dimension.

We henceforth assume that $G = \langle \sigma \rangle$ is cyclic of order p. So σ is an automorphism of $k[[u_1, \ldots, u_n]]$ of order p. The proof of a result given by Cornelissen and Kato for curves [6, p. 442] applies also in the case of higher-dimensional schemes because their argument is purely formal. Namely, letting $\Theta_A = \left(\Omega^1_{A/k}\right)^*$, we have the following fundamental result. Note that when $A = k[[u_1, \ldots, u_n]]$, we will denote $\Theta_{k[[u_1,\ldots,u_n]]}$ by Θ and we have $\Theta = \oplus_{j=1}^n k[[u_1, \ldots, u_n]] \dfrac{\partial}{\partial u_j}$.

PROPOSITION 1. The map $D(k[\varepsilon]) \to H^1(G, \Theta)$ given by $\widetilde{\rho} \mapsto d_{\widetilde{\rho}}$, where $d_{\widetilde{\rho}}$ is the 1-cocycle given by

$$d_{\widetilde{\rho}}(\sigma) = \frac{\widetilde{\rho}_\sigma \circ \rho_\sigma^{-1} - \mathrm{Id}}{\varepsilon} \text{ for } \sigma \in G,$$

is a bijection. Here we identify the k-derivation $d_{\widetilde{\rho}}(\sigma)$ with $\displaystyle\sum_{j=1}^{n} d_{\widetilde{\rho}}(\sigma)(u_j)\frac{\partial}{\partial u_j} \in \Theta$.

Proof. The G-action on Θ is the adjoint action. Namely, for a k-derivation $\delta \in \Theta$ and $\sigma \in G$,

$$\sigma \cdot \delta = \rho_\sigma \circ \delta \circ \rho_\sigma^{-1}.$$

For a lifting $\widetilde{\rho}$ of ρ, write $\widetilde{\rho}_\sigma(x) = \rho_\sigma(x) + \rho'_\sigma(x)\varepsilon$ for $\sigma \in G$ and $x \in k[[u_1, \dots, u_n]]$. We first show that the cocycle $d_{\widetilde{\rho}}$ is determined by ρ'_σ. First note that

$$\widetilde{\rho}_\sigma(x + y\varepsilon) = \rho_\sigma(x) + (\rho'_\sigma(x) + \rho_\sigma(y))\varepsilon.$$

Therefore, $d_{\widetilde{\rho}}(\sigma) = \frac{\widetilde{\rho}_\sigma \circ \rho_\sigma^{-1} - \mathrm{Id}}{\varepsilon} = \rho'_\sigma \circ \rho_\sigma^{-1}$. Next note that for $x, y \in k[[u_1, \dots, u_n]]$,

$$\begin{aligned}
\rho_\sigma(x)\rho_\sigma(y) + \rho'_\sigma(xy)\varepsilon &= \widetilde{\rho}_\sigma(xy) \\
&= \widetilde{\rho}_\sigma(x)\widetilde{\rho}_\sigma(y) \\
&= (\rho_\sigma(x) + \rho'_\sigma(x)\varepsilon)(\rho_\sigma(y) + \rho'_\sigma(y)\varepsilon) \\
&= \rho_\sigma(x)\rho_\sigma(y) + (\rho_\sigma(x)\rho'_\sigma(y) + \rho_\sigma(y)\rho'_\sigma(x))\varepsilon,
\end{aligned}$$

which implies $\rho'_\sigma(xy) = \rho_\sigma(x)\rho'_\sigma(y) + \rho_\sigma(y)\rho'_\sigma(x)$. Thus for $a, b \in k[[u_1, \dots, u_n]]$, we have

$$
\begin{aligned}
d_{\widetilde{\rho}}(\sigma)(ab) &= \rho'_\sigma(\rho_\sigma^{-1}(ab)) \\
&= \rho'_\sigma(\rho_\sigma^{-1}(a)\rho_\sigma^{-1}(b)) \\
&= \rho_\sigma(\rho_\sigma^{-1}(a))\rho'_\sigma(\rho_\sigma^{-1}(b)) + \rho'_\sigma(\rho_\sigma^{-1}(a))\rho_\sigma(\rho_\sigma^{-1}(b)) \\
&= a(d_{\widetilde{\rho}}(b)) + (d_{\widetilde{\rho}}(a))b
\end{aligned}
$$

and so $d_{\widetilde{\rho}}(\sigma)$ is a k-derivation. Next, for $\sigma, \tau \in G$,

$$
\begin{aligned}
\rho_{\sigma\tau}(x) + (\rho'_{\sigma\tau}(x) + \rho_{\sigma\tau}(y))\varepsilon &= \widetilde{\rho}_{\sigma\tau}(x + y\varepsilon) \\
&= \widetilde{\rho}_\sigma(\widetilde{\rho}_\tau(x + y\varepsilon)) \\
&= \widetilde{\rho}_\sigma(\rho_\tau(x) + (\rho'_\tau(x) + \rho_\tau(y))\varepsilon) \\
&= \rho_{\sigma\tau}(x) + (\rho'_\sigma(\rho_\tau(x)) + \rho_\sigma(\rho'_\tau(x)) + \rho_\sigma(\rho_\tau(y)))\varepsilon.
\end{aligned}
$$

This implies that $\rho'_{\sigma\tau} = \rho'_\sigma \circ \rho_\tau + \rho_\sigma \circ \rho'_\tau$, and so

$$
\begin{aligned}
d_{\widetilde{\rho}}(\sigma\tau)(x) &= \rho'_{\sigma\tau}(\rho_{\sigma\tau}^{-1}(x)) \\
&= (\rho'_\sigma \circ \rho_\tau + \rho_\sigma \circ \rho'_\tau)(\rho_{\sigma\tau}^{-1}(x)) \\
&= \rho'_\sigma(\rho_\sigma^{-1}(x)) + \rho_\sigma(\rho'_\tau(\rho_\tau^{-1}(\rho_\sigma^{-1}(x)))) \\
&= d_{\widetilde{\rho}}(\sigma)(x) + \sigma \cdot (d_{\widetilde{\rho}}(\tau))(x).
\end{aligned}
$$

Thus $d_{\widetilde{\rho}}$ is a cocycle.

To show the map d is well-defined, consider two isomorphic liftings $\widetilde{\rho}$ and $\widetilde{\widetilde{\rho}}$ of ρ. That is, there exists $\psi \in \mathrm{Aut}_{k[\varepsilon]}(k[\varepsilon][[u_1, \ldots, u_n]])$ such that $\overline{\psi} = \mathrm{Id}_{k[[u_1,\ldots,u_n]]}$ and $\psi \circ \widetilde{\widetilde{\rho}}_\sigma = \widetilde{\rho}_\sigma \circ \psi$. Since $\overline{\psi} = \mathrm{Id}$, we can write $\psi(x) = x + \delta(x)\varepsilon$ for $x \in k[[u_1, \ldots, u_n]]$ where $\delta \in \Theta$. We then have

$$
\begin{aligned}
&\psi(\widetilde{\widetilde{\rho}}_\sigma(x + y\varepsilon)) = \widetilde{\rho}_\sigma(\psi(x + y\varepsilon)) \\
\Rightarrow &\rho_\sigma(x) + (\rho''_\sigma(x) + \delta(\rho_\sigma(x)) + \rho_\sigma(y))\varepsilon = \rho_\sigma(x) + (\rho'_\sigma(x) + \rho_\sigma(\delta(x)) + \rho_\sigma(y))\varepsilon \\
\Rightarrow &\rho''_\sigma(x) + \rho_\sigma(y) + \delta(\rho_\sigma(x)) = \rho'_\sigma(x) + \rho_\sigma(\delta(x)) + \rho_\sigma(y) \\
\Rightarrow &(\rho''_\sigma - \rho'_\sigma)(x) = \rho_\sigma(\delta(x)) - \delta(\rho_\sigma(x)).
\end{aligned}
$$

Thus

$$
\begin{aligned}
(d_{\widetilde{\widetilde{\rho}}}\sigma - d_{\widetilde{\rho}}\sigma)(x) &= (\rho''_\sigma \circ \rho_\sigma^{-1} - \rho'_\sigma \circ \rho_\sigma^{-1})(x) \\
&= (\rho''_\sigma - \rho'_\sigma)(\rho_\sigma^{-1}(x)) \\
&= (\rho_\sigma \circ \delta - \delta \circ \rho_\sigma)(\rho_\sigma^{-1}(x)) \\
&= \rho_\sigma(\delta(\rho_\sigma^{-1}(x))) - \delta(x) \\
&= (\sigma \cdot \delta - \delta)(x) \\
&= (\sigma - \mathrm{Id})(\delta)(x).
\end{aligned}
$$

Hence $(d_{\widetilde{\widetilde{\rho}}}) - (d_{\widetilde{\rho}})$ is a coboundary.

If $d_{\widetilde{\widetilde{\rho}}}\sigma - d_{\widetilde{\rho}}\sigma = (\sigma - \mathrm{Id})(\delta)$ for some $\delta \in \Theta$, then we can define a map $\psi(x + y\varepsilon) = x + (y + \delta(x))\varepsilon$. ψ is clearly additive since δ is additive, and

$$
\begin{aligned}
\psi\left((x_1+y_1\varepsilon)(x_2+y_2\varepsilon)\right) &= \psi\left(x_1x_2+(x_1y_2+x_2y_1)\,\varepsilon\right)\\
&= x_1x_2+(x_1y_2+x_2y_1+\delta(x_1x_2))\varepsilon\\
&= x_1x_2+(x_1y_2+x_2y_1+x_1\delta(x_2)+x_2\delta(x_1))\varepsilon\\
&= \left(x_1+(y_1+\delta(x_1))\right)\left(x_2+(y_2+\delta(x_2))\right)\\
&= \psi(x_1+y_1\varepsilon)\psi(x_2+y_2\varepsilon).
\end{aligned}
$$

Further, $\psi(x+y\varepsilon)=0 \implies x+(y+\delta(x))\varepsilon=0 \implies x=y=0$, so ψ is injective. ψ is also obviously surjective. Noting that $d_{\widetilde{\widetilde{\rho}}}\sigma - d_{\widetilde{\rho}}\sigma = (\sigma-\mathrm{Id})(\delta)$ implies that $\rho''_\sigma(x)-\rho'_\sigma(x) = \rho_\sigma(\delta(x))-\delta(\rho_\sigma(x))$, we have

$$
\begin{aligned}
\psi\circ\widetilde{\widetilde{\rho}}(x+y\varepsilon) &= \psi\left(\rho_\sigma(x)+(\rho''_\sigma(x)+\rho_\sigma(y))\varepsilon\right)\\
&= \rho_\sigma(x)+\left(\rho''_\sigma(x)+\rho_\sigma(y)+\delta(\rho_\sigma(x))\right)\varepsilon\\
&= \rho_\sigma(x)+\left(\rho_\sigma(\delta(x))-\delta(\rho_\sigma(x))+\rho'_\sigma(x)+\rho_\sigma(y)+\delta(\rho_\sigma(x))\right)\varepsilon\\
&= \rho_\sigma(x)+\left(\rho'_\sigma(x)+\rho_\sigma(y+\delta(x))\right)\varepsilon\\
&= \widetilde{\rho}\,(x+(y+\delta(x))\varepsilon)\\
&= \widetilde{\rho}\circ\psi(x+y\varepsilon).
\end{aligned}
$$

So ψ is an automorphism that shows $\widetilde{\rho}$ and $\widetilde{\widetilde{\rho}}$ are isomorphic. Thus d is injective.

Lastly, we show that d is surjective. Starting with a 1-cocycle $f : G \to \Theta$, we get an automorphism $\widetilde{\rho}_\sigma$ by defining $\widetilde{\rho}_\sigma(x) = \rho_\sigma(x)+(f\sigma)(\rho_\sigma(x))\varepsilon$ for each $\sigma\in G$. This gives the desired lifting of ρ corresponding to the cocycle f. ■

4 The Existence of a Hull

We will now show that conditions $(H_1) - (H_3)$ are satisfied for the functor D defined in the previous section.

PROPOSITION 2. Suppose the action of G on $S = k[[u_1, \ldots, u_n]]$ is free off the closed point of Spec$S$. Then the local deformation functor $D : \mathfrak{C} \to$ Sets defined in section 3 has a prorepresentable hull.

Proof. Let $\phi' : A' \to A$ and $\phi'' : A'' \to A$ be morphisms in the category $\mathfrak{C}$.

(H_1) : Suppose that the map $\phi'' : A'' \to A$ is a small extension with kernel (t). In order to verify the (H_1) property, we must show that $f : D(A' \times_A A'') \to D(A') \times_{D(A)} D(A'')$ is surjective. Let $(\varepsilon', \varepsilon'') \in D(A') \times_{D(A)} D(A'')$, where $D(\phi')(\varepsilon') = D(\phi'')(\varepsilon'') = \varepsilon$. Suppose that $\varepsilon' = [\rho']$ and $\varepsilon = [\rho]$ are representatives such that $\rho' : G \to \text{Aut } A'[[u_1, \ldots, u_n]]$ is a lift of $\rho : G \to \text{Aut } A[[u_1, \ldots, u_n]]$ via ϕ'. Let ρ'' be a representative of ε''. Since $D(\phi'')(\varepsilon'') = \varepsilon$, there is an automorphism $\psi \in \text{Aut}_A (A[[u_1, \ldots, u_n]])$ showing that $D(\phi'')(\rho'')$ and ρ are equivalent, i.e., $\overline{\psi} = \text{Id}$ and $\psi \circ D(\phi'')(\rho'') \circ \psi^{-1} = \rho$. Suppose that ψ is the automorphism given by

$$u_i \mapsto \textstyle\sum_{j=1}^{n} c_{i,j} u_j + d_i$$

for $1 \leq i \leq n$, where $c_{i,j} \in A$ and $d_i \in A[[u_1, \ldots, u_n]]$.

Since ϕ'' is surjective, there are lifts of the elements $c_{i,j} \in A$ and the coefficients of d_i to A'': call these lifts $c''_{i,j}$ and d''_i, respectively. Next we show that the map $\widetilde{\psi}$ given by

$$u_i \mapsto \textstyle\sum_{j=1}^{n} c''_{i,j} u_j + d''_i$$

is an A''-automorphism. Since the $c''_{i,j}$ and d''_i are lifts under the small extension $A'' \to A$, we have that $c''_{i,j} = c_{i,j} + t\widetilde{c}_{i,j}$ and $d''_i = d_i + t\widetilde{d}_i$ for some $\widetilde{c}_{i,j} \in A''$ and $\widetilde{d}_i \in A''[[u_1, \ldots, u_n]]$. Let a_i denote the constant term of d_i and a''_i the constant term of d''_i. Since ψ is an automorphism, $a_i \in m_A$. Thus $a''_i \in \phi''^{-1}(a_i) \subseteq \phi''^{-1}(m_A) \subseteq m_{A''}$. Also, we know that the determinant of the matrix of coefficients of the linear terms for ψ is a unit in A. Then

$$\begin{aligned}
\det\left(c''_{i,j}\right)_{i,j} &= \sum_{\tau\in S_n} \operatorname{sgn}(\tau) c''_{1,\tau(1)} \cdots c''_{n,\tau(n)} \\
&= \sum_{\tau\in S_n} \operatorname{sgn}(\tau)(c_{1,\tau(1)} + t\widetilde{c}_{1,\tau(1)}) \cdots (c_{n,\tau(n)} + t\widetilde{c}_{n,\tau(n)}) \\
&= \sum_{\tau\in S_n} \operatorname{sgn}(\tau) c_{1,\tau(1)} \cdots c_{n,\tau(n)} + t \sum_{j=1}^{n} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) c_{1,\tau(1)} \cdots \widetilde{c}_{j,\tau(j)} \cdots c_{n,\tau(n)} \\
&= \det\left(c_{i,j}\right)_{i,j} + t \sum_{j=1}^{n} \sum_{\tau\in S_n} \operatorname{sgn}(\tau) c_{1,\tau(1)} \cdots \widetilde{c}_{j,\tau(j)} \cdots c_{n,\tau(n)}
\end{aligned}$$

is a unit in $A'' = A \oplus At$, since $\det\left(c_{i,j}\right)_{i,j}$ is a unit in A. Hence $\widetilde{\psi}$ is an automorphism. Thus the action given by $\widetilde{\psi} \circ \rho'' \circ \widetilde{\psi}^{-1} : G \to \operatorname{Aut} A''[[u_1, \ldots, u_n]]$ is equivalent to that of ρ'' and, by construction, we have that $D(\phi'')(\widetilde{\psi} \circ \rho'' \circ \widetilde{\psi}^{-1}) = \rho$. We now use the representative $\widetilde{\psi} \circ \rho'' \circ \widetilde{\psi}^{-1}$ for ε''.

Lastly, suppose that the actions $\widetilde{\psi} \circ \rho'' \circ \widetilde{\psi}^{-1}$, ρ', and ρ are determined by the following automorphisms, respectively:

$$u_i \mapsto \left(\textstyle\sum_{j=1}^{n} r''_{i,j} u_j\right) + s''_i, \quad u_i \mapsto \left(\textstyle\sum_{j=1}^{n} r'_{i,j} u_j\right) + s'_i, \quad u_i \mapsto \left(\textstyle\sum_{j=1}^{n} r_{i,j} u_j\right) + s_i.$$

with $r''_{i,j} \in A''$, $s''_i \in A''[[u_1, \ldots, u_n]]$, $r'_{i,j} \in A'$, $s'_i \in A'[[u_1, \ldots, u_n]]$, $r_{i,j} \in A$, and $s_i \in A[[u_1, \ldots, u_n]]$. Define $\widetilde{\rho} : G \to \operatorname{Aut} A' \times_A A''[[u_1, \ldots, u_n]]$ by sending σ to the automorphism

$$u_i \mapsto \left(\textstyle\sum_{j=1}^{n} \left(r'_{i,j}, r''_{i,j}\right) u_j\right) + \left(s'_i, s''_i\right).$$

Note that the determinant of interest for this map will be a unit because the corresponding determinants in A' and A'' are units and since multiplication/addition is done component-wise in fiber products. The equivalence class of this automorphism will map to $(\varepsilon', \varepsilon'')$ under

f and so the (H_1) condition is verified.

(H_2) : A sketch of the proof of this criterion can be found at [15, p. 390]. For each $\rho'' \in E'' = \mathrm{Hom}(G, \mathrm{Aut}\ A''[[u_1,\ldots,u_n]])$, define

$$C(\rho'') = \{\psi \in \mathrm{Ker}(\mathrm{Aut}\ A''[[u_1,\ldots,u_n]] \to \mathrm{Aut}\ k[[u_1,\ldots,u_n]]) \mid \psi\rho''(\sigma) = \rho''(\sigma)\psi\}.$$

We can similarly define $C(\rho)$ for $\rho \in E = \mathrm{Hom}(G, \mathrm{Aut}\ A[[u_1,\ldots,u_n]])$, where ρ is the image of ρ'' under ϕ''. For ease of reference, set $G_0 = \mathrm{Ker}(\mathrm{Aut}\ A[[u_1,\ldots,u_n]] \to \mathrm{Aut}\ k[[u_1,\ldots,u_n]])$, $G_1 = \mathrm{Ker}(\mathrm{Aut}\ A'[[u_1,\ldots,u_n]] \to \mathrm{Aut}\ k[[u_1,\ldots,u_n]])$, $G_2 = \mathrm{Ker}(\mathrm{Aut}\ A''[[u_1,\ldots,u_n]] \to \mathrm{Aut}\ k[[u_1,\ldots,u_n]])$, and $G_3 = \mathrm{Ker}(\mathrm{Aut}\ A' \times_A A''[[u_1,\ldots,u_n]] \to \mathrm{Aut}\ k[[u_1,\ldots,u_n]])$. We will need the following lemma in order to verify (H_2).

LEMMA 3. If $C(\rho'') \to C(\rho)$ is surjective for all $\rho'' \in E''$, then the map f is injective.

Proof. Suppose that $\rho_1, \rho_2 \in \widetilde{E} = \mathrm{Hom}(G, \mathrm{Aut}\ A' \times_A A''[[u_1,\ldots,u_n]])$ are such that $f([\rho_1]) = f([\rho_2])$. Let $f([\rho_1]) = [\phi_1] \times_{[\phi_0]} [\phi_2]$ and $f([\rho_2]) = [\widetilde{\phi_1}] \times_{[\widetilde{\phi_0}]} [\widetilde{\phi_2}]$. Since $[\phi_2] = [\widetilde{\phi_2}]$, by definition it follows that there exists $\psi_2 \in G_2$ such that $\psi_2\phi_2\psi_2^{-1} = \widetilde{\phi_2}$. Applying the map ϕ'' to this equation gives a $\psi_0 \in G_0$ such that $\psi_0\phi_0\psi_0^{-1} = \widetilde{\phi_0}$, i.e, ψ_0 is the image of ψ_2 under the natural map $G_2 \to G_0$. Next $[\phi_1] = [\widetilde{\phi_1}]$ implies that there exists $\psi_1 \in G_1$ such that $\psi_1\phi_1\psi_1^{-1} = \widetilde{\phi_1}$. Applying the map ϕ' to this last equation gives a $\psi_0' \in G_0$ such that $\psi_0'\phi_0\psi_0'^{-1} = \widetilde{\phi_0} \Rightarrow \phi_0 = \psi_0'^{-1}\widetilde{\phi_0}\psi_0'$. Thus $\psi_0'^{-1}\psi_0\phi_0(\psi_0'^{-1}\psi_0)^{-1} = \phi_0$ and so $\psi_0'^{-1}\psi_0 \in C(\phi_0)$. So, by assumption, there exists $\beta_2 \in C(\phi_2)$ such that $\beta_2 \mapsto \psi_0'^{-1}\psi_0$ via $C(\phi_2) \to C(\phi_0)$. Then note that $\widetilde{\psi_2} = \psi_2\beta_2^{-1} \in G_2$ is such that $\widetilde{\psi_2}\phi_2\widetilde{\psi_2}^{-1} = \psi_2\beta_2^{-1}\phi_2\beta_2\psi_2^{-1} = \psi_2\phi_2\psi_2^{-1} = \widetilde{\phi_2}$, using that $\beta_2 \in C(\phi_2)$. Also, when we reduce $\widetilde{\psi_2}$ via $A'' \to A$, we get $\psi_0(\psi_0'^{-1}\psi_0)^{-1} = \psi_0'$. Therefore, $\widetilde{\psi_2}\phi_2\widetilde{\psi_2}^{-1}$ reduces to $\psi_0'\phi_0\psi_0'^{-1} = \widetilde{\phi_0}$ on the level of A. Next we define the map $g = \psi_1 \times_{\psi_0'} \widetilde{\psi_2}$, which can be viewed as an element of G_3. This map is well-defined on the fiber product and will have unit determinant of its linear coefficients because this is also true for both ψ_1 and $\widetilde{\psi_2}$. Then, by construction, we have that $g\rho_1 g^{-1} = \rho_2$. Thus $[\rho_1] = [\rho_2]$ and so f is injective. ■

The (H_2) condition is that f is a bijective when $A'' = k[\epsilon]$ and $A = k$. By the already proved (H_1), we have surjectivity of f and so we only have to show injectivity. However, since $C(\rho'') \to C(\rho)$ is clearly surjective when $A'' = k[\epsilon]$ and $A = k$, this follows from Lemma 3.

$(H_3):$ Since $D(k[\epsilon]) \cong H^1(G, \Theta)$ by Proposition 1, it suffices to show that $H^1(G, \Theta)$ is a finite k-vector space. As in [19, p. 622], start by noting that $H^1(G, \Theta)$ is finitely generated as an $R = S^G$-module. Thus $\widetilde{H^1(G, \Theta)}$ is a coherent sheaf on Spec$R$. We now show that this sheaf is supported at the closed point of Spec$R$. Namely, that $H^1(G, \Theta)_P = H^1(G, \Theta_P) = 0$ for all $P \in \mathrm{Spec}R$ with $P \neq m_R$. Consider the diagram

$$\begin{array}{ccc} S \otimes_R R_P & \longleftarrow & S \\ \uparrow & & \uparrow \\ R_P & \longleftarrow & R \end{array}$$

Here Θ_P is an $S \otimes_R R_P$-module. Since the action of G on S is free off the closed point, by the descent result stated in section 2.1, $\Theta_P \cong S \otimes_R R_P^d$ for some d and $S \otimes_R R_P$ is a G-torsor. Thus $(S \otimes_R R_P) \otimes_{R_P} (S \otimes_R R_P) \cong S \otimes_R R_P[G]$ and so

$$
\begin{aligned}
H^1(G, S \otimes_R R_P) \otimes_{R_P} (S \otimes_R R_P) &\cong H^1(G, (S \otimes_R R_P) \otimes_{R_P} (S \otimes_R R_P)) \\
&\cong H^1(G, S \otimes_R R_P[G]) \\
&\cong (0).
\end{aligned}
$$

Since $S \otimes_R R_P$ is faithfully flat over R_P, this implies that $H^1(G, S \otimes_R R_P) = 0$ by [14, p. 47]. Therefore, $H^1(G, \Theta_P) = H^1(G, S \otimes_R R_P^d) \cong H^1(G, S \otimes_R R_P)^d \cong (0)$, as desired. Therefore, $H^1(G, \Theta)$ is a vector space over k. It contains a copy of k, since $R \to S$ is totally ramified at m_S and so $\sigma - \mathrm{id} \equiv 0$ modulo m_S. Hence $H^1(G, \Theta)$ is also finite-dimensional.

Since $(H1) - (H3)$ are satisfied, D has a prorepresentable hull. ■.

5 Computation of the Hull

As stated earlier, we have fixed a base action $\rho : G \to \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])$. Peskin shows in [20, p. 77] that such an action can be transformed by a change of coordinates so that it consists of blocks of the form

$$
\begin{aligned}
\sigma u_i &= u_i + f_i(u_1, \ldots, u_n) \\
\sigma u_{i+1} &= u_{i+1} + u_i \\
&\vdots \\
\sigma u_{i+j} &= u_{i+j} + u_{i+j-1},
\end{aligned}
\tag{3}
$$

where f_i has order ≥ 2 and $j + 1 \leq p$.

5.1 The Single Jordan Block Case

We will focus on the case where the linear terms of the G-action consist of a single Jordan block in its Jordan form and f_i is fixed by the action. Namely, we assume that ρ is given by

$$\begin{aligned} \sigma u_1 &= u_1 + f(u_1, \ldots, u_n) \\ \sigma u_2 &= u_2 + u_1 \\ &\vdots \\ \sigma u_n &= u_n + u_{n-1}, \end{aligned} \tag{4}$$

where $\mathrm{ord}(f) \geq 2$ and f is invariant. Under the further assumption $n = p - 1$, Peskin shows [20, p. 88] that there is a coordinate change so that f can be realized as $f = (Nu_{p-1})^s$, a power of the norm of u_{p-1} for some $s \geq 1$. In this case, note that this means that the norm Nu_{p-1} and the action will be defined recursively. To simplify notation, we will denote the norm Nu_{p-1} by y.

As in Peskin's paper, we will use the following clever trick to simplify computations. By introducing an extra variable u_0 and taking an appropriate "slice", we can produce a linear model for our action. Particularly, begin with an action $G \to \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])$ in the form (4) above. Consider $f \in k[[u_1, \ldots, u_n]]$ as an element of $k[[u_0, u_1, \ldots, u_n]]$. The power series ring $k[[u_0, u_1, \ldots, u_n]]$ has a G-action given by $\sigma u_0 = u_0$, $\sigma u_i = u_i + u_{i-1}$ for $1 \leq i \leq n$. We can then define a surjective G-equivariant map ψ by

$$\begin{aligned} k[[u_0, u_1, \ldots, u_n]] &\to k[[u_1, \ldots, u_n]] \\ u_0 &\mapsto f \\ u_i &\mapsto u_i \qquad \text{for } i \geq 1. \end{aligned}$$

The kernel of ψ is generated by the element $u_0 - f$. Thus there is a G-isomorphism $k[[u_0, u_1, \ldots, u_n]]/(u_0 - f) \cong k[[u_1, \ldots, u_n]]$, where the action on the quotient is given by the same formulas in (5) above. We will use $k[[u_0, u_1, \ldots, u_n]]/(u_0 - f)$ to do our future computations. Henceforth, set $S = k[[u_0, u_1, \ldots, u_n]]$ and $R = k[[u_0, u_1, \ldots, u_n]]/(u_0 - f)$, with the actions of G given above. Our first goal is to compute $H^1(G, \Theta_R)$ and $H^2(G, \Theta_R)$: since these groups give the tangent space and the obstructions to infinitesimal local deformations, respectively.

Begin by noting that $\Theta_S = \left(\Omega^1_{S/k}\right)^* = \mathrm{Hom}_S\left(\Omega^1_{S/k}, S\right)$ and $\Theta_R = \left(\Omega^1_{R/k}\right)^*$. By basic algebra, we have the following short exact sequence

$$I/I^2 \xrightarrow{\gamma} \Omega^1_{S/k} \otimes_S R \to \Omega^1_{R/k} \to 0,$$

where $I = (u_0 - f)$. In [20, p. 82], it's explained that when $n = p - 1$ the ring S^G is factorial and there exists a unit $s \in S$ such that I is generated by the invariant element $(u_0 - f)s \in S^G$. Thus I/I^2 is isomorphic to R as G-modules. The map $\gamma : R \to \Omega^1_{S/k} \otimes_S R$ is given by $r \mapsto rd(u_0 - f)$ and is clearly injective. Thus the above sequence of G-modules is also left exact. Taking the dual of this short exact sequence gives the short exact sequence

$$0 \to \Theta_R \hookrightarrow \Theta_S \otimes_S R \to R \to 0.$$

We will need the following results.

LEMMA 4. $H^i(G, \Theta_S) = 0$ for $i > 0$.

Proof. The key to the proof of this fact is that $\Theta_S \cong k[G] \otimes_k S$ as G-modules, so let's first prove this. First, we have $\Theta_S = \bigoplus_{i=0}^{n} S \frac{\partial}{\partial u_i} \cong \left(\bigoplus_{i=0}^{n} k \frac{\partial}{\partial u_i}\right) \otimes_k S$. Set $V = \bigoplus_{i=0}^{n} k \frac{\partial}{\partial u_i}$. Note that $V = \left(m/m^2\right)^*$, where $m = (u_0, \ldots, u_n)$ is the maximal ideal of $k[[u_0, u_1, \ldots, u_n]]$. The action of G on m/m^2 is given by $\sigma u_0 = u_0$, $\sigma u_i = u_i + u_{i-1}$ for $1 \leq i \leq n$. Define a k-linear map $\psi : m/m^2 \to k[G]$ by $u_i \mapsto (\sigma - 1)^{n-i}$ and note that this is clearly bijective. ψ is G-equivariant because

$$\begin{aligned}
\psi\left(\sigma u_i\right) &= \psi\left(u_i + u_{i-1}\right) \\
&= (\sigma - 1)^{n-i} + (\sigma - 1)^{n-i+1} \\
&= (\sigma - 1)^{n-i}(1 + \sigma - 1) \\
&= \sigma(\sigma - 1)^{n-i} = \sigma\psi\left(u_i\right)
\end{aligned}$$

for $0 \leq i \leq n$. Thus ψ is an isomorphism of G-modules. Since $k[G]$ is self-dual as a G-module, it follows that $V = (m/m^2)^* \cong (k[G])^* \cong k[G]$. Hence, we get that $\Theta_S \cong k[G] \otimes_k S$ where the action on this latter module is the diagonal action. We can change this diagonal action to an action on the left factor only using Frobenius reciprocity. Particularly, we have $k[G] \otimes_k S \xrightarrow{\sim} k[G] \otimes_k S$ by $g \otimes s \mapsto g \otimes g^{-1}s$. Hence we have $\Theta_S \cong k[G] \otimes_k S$ with G acting on the left factor of this tensor product. As a result, for $i > 0$, we get

$$H^i\left(G, \Theta_S\right) \cong H^i\left(G, k[G] \otimes_k S\right) \cong H^i\left(G, k[G]\right) \otimes_k S = 0. \quad \blacksquare$$

As a result of this, we can now prove:

LEMMMA 5. $H^i(G, \Theta_S \otimes_S R) = 0$ for $i > 0$.

Proof. Since $\Theta_S \otimes_S R \cong \Theta_S/(u_0 - f)$, we can form the short exact sequence

$$0 \to \Theta_S \to \Theta_S \to \Theta_S \otimes_S R \to 0.$$

The map $\Theta_S \to \Theta_S$ is given by multiplication by $u_0 - f$. Now if we consider the corresponding long exact sequence of cohomology (as explained in [22, p. 111])

$$\begin{aligned} 0 &\to (\Theta_S)^G \to (\Theta_S)^G \to (\Theta_S \otimes_S R)^G \\ &\to H^1(G, \Theta_S) \to H^1(G, \Theta_S) \to H^1(G, \Theta_S \otimes_S R) \\ &\to H^2(G, \Theta_S) \to H^2(G, \Theta_S) \to H^2(G, \Theta_S \otimes_S R) \\ &\to H^3(G, \Theta_S) \to \cdots \end{aligned}$$

and use that $H^i(G, \Theta_S) = 0$ for $i > 0$ by Lemma 4, it follows that $H^i(G, \Theta_S \otimes_S R) = 0$ for $i > 0$. ■

Using these results, we can now say something about $H^i(G, \Theta_R)$ for $i = 1, 2$.

THEOREM 6. Suppose that the G-action $\rho : G \hookrightarrow \mathrm{Aut}_k(k[[u_1, \ldots, u_n]])$ with $n = p - 1$ is such that the linear terms form a single Jordan block when in its Jordan form and f is invariant when the action is put in the form (4) by [20, p. 77]. Then $H^1(G, \Theta_R) \cong \widehat{H}^0(G, R)$ and $H^2(G, \Theta_R) \cong H^1(G, R)$.

Proof. The short exact sequence $0 \to \Theta_R \hookrightarrow \Theta_S \otimes_S R \to R \to 0$ gives us the following long exact sequence (as in [22, p. 128])

$$\begin{aligned} \cdots &\to \widehat{H}^0(G, \Theta_R) \to \widehat{H}^0(G, \Theta_S \otimes_S R) \to \widehat{H}^0(G, R) \\ &\to H^1(G, \Theta_R) \to H^1(G, \Theta_S \otimes_S R) \to H^1(G, R) \\ &\to H^2(G, \Theta_R) \to H^2(G, \Theta_S \otimes_S R) \to H^2(G, R) \\ &\to H^3(G, \Theta_R) \to H^3(G, \Theta_S \otimes_S R) \to \cdots . \end{aligned}$$

Applying Lemma 5 to this sequence, it follows that $H^3(G, \Theta_R) \cong H^2(G, R)$ and $H^2(G, \Theta_R) \cong H^1(G, R)$. Since G is cyclic, it is a standard fact that the group cohomology will be 2-periodic. Thus $H^1(G, \Theta_R) \cong H^3(G, \Theta_R) \cong H^2(G, R) \cong \widehat{H}^0(G, R)$. So the desired result holds. ■

5.2 The Case $p = 3$

We now assume that the characteristic of k is $p = 3$ and the action $\rho : G \to \mathrm{Aut}_k(k[[u_1, u_2]])$ is free off the closed point. By our comments in the last section and [20, p. 88], there is an $s \geq 1$ such that after a change of coordinates the action ρ can be put in the form

$$\begin{aligned} \sigma u_1 &= u_1 + y^s \\ \sigma u_2 &= u_2 + u_1 \end{aligned} \tag{5}$$

with $y = Nu_2$ and this action defined recursively. Further, we can take advantage of a result of Peskin in [20, p. 96] that states that the invariant ring for this G-action on $R = k[[u_0, u_1, u_2]]/(u_0 - y^s)$ is

$$R^G = k[[x, y, z]]/(z^3 + y^{2s}z^2 - y^{3s+1} - x^2),$$

where $x = Nu_1$, $y = Nu_2$, and $z = u_1^2 - y^s u_1 + y^s u_2$. In general positive characteristics, it is very difficult to compute this invariant ring or to do the other computations necessary to explicitly compute the cohomology groups $H^1(G, \Theta_R)$ and $H^2(G, \Theta_R)$. However, with this result in hand, we can prove the following.

THEOREM 7. Suppose $\mathrm{char}(k) = 3$ and the action of G on $k[[u_1, u_2]]$ is free off the closed point. Then $H^1(G, \Theta_R) \cong k[y]/(y^s)$.

Proof. It follows from Theorem 6 and standard results of group cohomology that

$$H^1(G, \Theta_R) \cong \widehat{H}^0(G, R) = R^G/\mathrm{Im}(Tr).$$

Therefore, to prove the desired result we need to compute the image of the trace map $Tr : R \to R^G$. As explained in [20, p. 94], R is a free $R' = k[[x, y]]$-module of rank 9 with basis $\left\{u_1^i u_2^j\right\}_{0 \leq i,j \leq 2}$. Thus to compute $\mathrm{Im}(Tr)$ we compute the image of these basis elements under Tr as power series in x, y, and z. Particularly, $Tr(1) = Tr(u_1) = 0$, $Tr(u_2) = y^s$, $Tr(u_1^2) = Tr(u_1 u_2) = -y^{2s}$, $Tr(u_2^2) = y^{2s} - z$, $Tr(u_1^2 u_2) = y^{3s} - y^s z$, $Tr(u_1 u_2^2) = -y^{3s} - x + y^s z$, and $Tr(u_1^2 u_2^2) = y^{4s} - z^2 - y^s x - y^{2s} z$. Thus $\mathrm{Im}(Tr)$ is generated by x, y^s, z, and z^2 as an R'-module. Since R^G is of rank 3 as an R'-module, generated by 1, z, and z^2, it follows that $H^1(G, \Theta_R) \cong k[y]/(y^s)$. ■

Consider the following action of G on $k[\varepsilon][[u_1, \ldots, u_n]]$ that lifts the base action ρ:

$$\widetilde{\sigma} u_1 = u_1 + \widetilde{y}^s + \varepsilon \widetilde{y}^k$$

$$\widetilde{\sigma} u_2 = u_2 + u_1,$$

where both this action and the norm $\widetilde{y} = N u_2$ are defined recursively here. A simple computation shows that $\widetilde{y} = y + \varepsilon y'$ for some power series $y' \in k[[u_1, u_2]]$ and that $\widetilde{\sigma} y = \widetilde{y}$. Under the bijection $D(k[\varepsilon]) \xrightarrow{\sim} H^1(G, \Theta)$ from Proposition 1 we have

$$
\begin{aligned}
\left(d_{\widetilde{\rho}}\sigma\right)(u_1) &= \frac{\widetilde{\sigma}(\sigma^{-1}(u_1)) - u_1}{\varepsilon} \\
&= \frac{\widetilde{\sigma}(u_1 - y^s) - u_1}{\varepsilon} \\
&= \frac{u_1 + \widetilde{y}^s + \varepsilon\widetilde{y}^k - \widetilde{y}^s - u_1}{\varepsilon} \\
&= y^k
\end{aligned}
$$

and

$$
\begin{aligned}
\left(d_{\widetilde{\rho}}\sigma\right)(u_2) &= \frac{\widetilde{\sigma}(\sigma^{-1}(u_2)) - u_2}{\varepsilon} \\
&= \frac{\widetilde{\sigma}(u_2 - u_1 + y^s) - u_2}{\varepsilon} \\
&= \frac{u_2 + u_1 - u_1 - \widetilde{y}^s - \varepsilon\widetilde{y}^k + \widetilde{y}^s - u_2}{\varepsilon} \\
&= -y^k.
\end{aligned}
$$

So $d_{\widetilde{\rho}}\sigma = y^k \frac{\partial}{\partial u_1} - y^k \frac{\partial}{\partial u_2}$. Thus the following correspondence holds:

$$
\left\{ \begin{array}{l} \sigma u_1 = u_1 + \widetilde{y}^s + \varepsilon f(y) \\ \\ \sigma u_2 = u_2 + u_1 \end{array} \right\} \longleftrightarrow \left\{ f(y)\frac{\partial}{\partial u_1} - f(y)\frac{\partial}{\partial u_2} \right\},
$$

where $f(y) \in k[y]$ is a polynomial of degree $< s$. Since $\frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_2}$, $y\frac{\partial}{\partial u_1} - y\frac{\partial}{\partial u_2}$, $\ldots$, $y^{s-1}\frac{\partial}{\partial u_1} - y^{s-1}\frac{\partial}{\partial u_2}$ are linearly independent over k and $H^1(G, \Theta_R) \cong k[y]/(y^s)$ is s-dimensional by Theorem 6, we're led to consider deformations of ρ to $A \in \mathfrak{C}$ given by

$$\sigma u_1 = u_1 + \widetilde{y}^s + a_0 + a_1\widetilde{y} + \cdots + a_{s-1}\widetilde{y}^{s-1} \tag{6}$$

$$\sigma u_2 = u_2 + u_1$$

where the a_i lie in the maximal ideal m_A of A and this action and the element $\widetilde{y} = Nu_2$ are defined recursively. As long as A is of characteristic 3, we note that such a map does indeed give a local deformation because it is assumed to be an A-linear map and $\widetilde{y}$ is invariant by definition: thus a simple calculation shows that $\sigma^3 = \mathrm{id}$. Further, we know that the deformations of this form will cover the tangent space. We can now get some information about the prorepresentable hull of the local deformation functor $D = D_\rho : \mathfrak{C} \to \mathrm{Sets}$. In fact, if we restrict D to the subcategory $\mathfrak{C}_3 = \{A \in \mathfrak{C} \mid \mathrm{char}(A) = 3\}$ of $\mathfrak{C}$, then we can compute the prorepresentable hull of the restriction $D|_{\mathfrak{C}_3}$.

THEOREM 8. Suppose $\mathrm{char}(k) = 3$ and the action $\rho : G \to \mathrm{Aut}_k(k[[u_1, u_2]])$ is free off the closed point. Then the hull of the deformation functor $D|_{\mathfrak{C}_3}$ is $k[[x_0, x_1, \ldots, x_{s-1}]]$, where $s \geq 1$ is the integer given by [20, p. 88] as explained above.

Proof. By Propositions 1 and Theorem 6, $t_R \cong t_D$. So it remains to show that $h_R \to D$ is smooth, i.e., $f : \mathrm{Hom}(R, A') \to \mathrm{Hom}(R, A) \times_{D(A)} D(A')$ is surjective for any small extension $\phi : A' \to A$ in $\mathfrak{C}_3$. Begin with a homomorphism $g : R \to A$ that induces a deformation $[\xi] \in D(A)$. ξ is given by (7), where $a_0,\ a_1,\ \ldots,\ a_{s-1}$ are the images of $x_0,\ x_1,\ \ldots,\ x_{s-1}$ under g. Suppose that $[\widetilde{\xi}] \in D(A')$ is a lifting of the deformation $[\xi]$ to A'. Take any lifts $a'_0,\ a'_1,\ \ldots,\ a'_{s-1} \in A'$ of $a_0,\ a_1,\ \ldots,\ a_{s-1}$ via ϕ. Thus $a'_0 = a_0 + t\widetilde{a}_0$, $a'_1 = a_1 + t\widetilde{a}_1, \ldots a'_{s-1} = a_{s-1} + t\widetilde{a}_{s-1}$, where $J = (t)$ is the kernel of ϕ. We lift g to a homomorphism $g' \in \mathrm{Hom}(R, A')$ by sending $x_i \mapsto a'_i$. It remains to show that g gets sent to the desired element under f.

As explained in [21, p. 213], by Schlessinger's (H_2) condition for D and the isomorphism $A' \times_{A'/J} A' = A' \times_A A' \xrightarrow{\sim} A' \times_k k[J], (a,b) \mapsto (a, a_o + b - a)$ where $a, b \in A'$ and a_o is the k-residue of a, we obtain a map

$$D(A') \times (t_D \otimes J) \to D(A') \times_{D(A)} D(A')$$

This map determines an action of $t_D \otimes J$ on the fibers $D(\phi)^{-1}(\eta)$ for each $\eta \in D(A)$. This action is transitive by (H_1). Similarly, we have another transitive action of $t_R \otimes J$ on each fiber $h_R(\phi)^{-1}(b)$, for $b \in h_R(A)$, that is compatible with the previous action via $G : \mathrm{Hom}(R, A') \to D(A')$. Since the image of g' under G is in $D(\phi)^{-1}([\xi])$, there exists an element $w \in t_D \otimes J$ such that $wG(g') = [\widetilde{\xi}]$. Thus the element $w' \in t_R \otimes J$ corresponding to w under the natural isomorphism $t_R \otimes J \xrightarrow{\sim} t_D \otimes J$ is such that the image of $w'g'$ under $\mathrm{Hom}(R, A') \to D(A')$ is $[\widetilde{\xi}]$. However, the image of $w'g'$ under $h_R(\phi)$ is still g: since $g' \in h_R(\phi)^{-1}(g)$ and $t_R \otimes J$ acts on $h_R(\phi)^{-1}(g)$. Therefore, $w'g'$ maps to the desired element $(g, [\widetilde{\xi}])$ and so f is surjective. ■

6 An Example

One problem in the area of deformation theory that has garnered a lot of interest is that of lifting certain wild actions to characteristic zero. In the local setting where our results hold, the question is whether or not the base action $\rho : G \hookrightarrow \mathrm{Aut}_k(k[[u,v]])$ can be lifted from k to a ring of characteristic zero. One may also want to explore in what ways local actions can be realized from global actions on smooth projective varieties.

Let $X \subset \mathbb{P}^3_k$ be the Fermat quartic given by the equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$. Suppose that the group $G = \langle \sigma \rangle \cong \mathbb{Z}/3\mathbb{Z}$ acts on X via the automorphism $\sigma(x_0,\ x_1,\ x_2,\ x_3) = (x_0,\ x_2,\ x_3,\ x_1)$. This action was considered in a different context in [8, p. 114]. Note that $\sigma(x_0,\ x_1,\ x_2,\ x_3) = (x_0,\ x_1,\ x_2,\ x_3)$ implies that $(x_0,\ x_1,\ x_2,\ x_3) = (x_0,\ x_2,\ x_3,\ x_1)$ in $\mathbb{P}^3_k$. If $x_0 \neq 0$, we can scale these tuples by $1/x_0$ and conclude that $x_1 = x_2 = x_3$ since we are then working in an open affine subset of $\mathbb{P}^3_k$. So $x_0^4 + 3x_1^4 = 0 \Longrightarrow 3\left(\frac{x_1}{x_0}\right)^4 = -1$. If $\mathrm{char}(k) = 3$, we arrive at a contradiction. If $\mathrm{char}(k) = 0$, we find four solutions for $\frac{x_1}{x_0}$ and hence four fixed points of the action. Now suppose that $x_0 = 0$. There must exist $c \in k^*$ such that $x_1 = cx_2$, $x_2 = cx_3$, and $x_3 = cx_1$. We can further assume that $x_1 \neq 0$, since $x_1 = 0$ would imply that $x_0 = x_1 = x_2 = x_3 = 0$. The equation $x_0^4 + x_1^4 + x_2^4 + x_3^4 = 0$ then becomes $(c^8 + c^4 + 1)x_1^4 = 0$, which gives that $c^8 + c^4 + 1 = 0$. The relations $x_1 = cx_2$, $x_2 = cx_3$, and $x_3 = cx_1$ also yield that $x_1 = c^3 x_1 \Longrightarrow (c^3 - 1)x_1 = 0 \Longrightarrow c^3 = 1$. If $\mathrm{char}(k) = 3$, this implies that $c = 1$ and thus $(0, 1, 1, 1)$ is the only fixed point. However, if $\mathrm{char}(k) = 0$, then $c^8 + c^4 + 1 = 0$ and $c^3 = 1$ imply that $c^2 + c + 1 = 0$. Thus, we obtain two more fixed points in the characteristic zero case, giving us a total of six fixed points.

We henceforth assume that $\mathrm{char}(k) = 3$. If we consider the induced action on $\widehat{\mathcal{O}}_{X,x} \cong k[[u, v]]$, the completion of the stalk at the fixed point $x = (x_0, x_1 - x_2, x_2 - x_3)$, we get an action of the type we have thus far been studying. By the results of Peskin [20, p. 88], this action must be of the form

$$\sigma u = u + (Nv)^s$$

$$\sigma v = v + u$$

where $s \geq 1$ and both the action and Nv are defined recursively. Our goal is to determine the value of s and thus obtain a specific example where the actions we have been studying can be lifted to characteristic zero: and also an example where our local actions arise from a global action on a surface. To get a handle on the action, let's first consider the action on the open affine set $U = \{x_1 \neq 0\} \bigcap \{x_2 \neq 0\} \bigcap \{x_3 \neq 0\} = \text{Spec } \frac{k[x,y,z,y^{-1},z^{-1}]}{\left(1+x^4+y^4+z^4\right)}$ where $x = \frac{x_0}{x_1}$, $y = \frac{x_2}{x_1}$, and $z = \frac{x_3}{x_1}$. The action here is given by

$$\begin{aligned} \sigma: \; x = \frac{x_0}{x_1} &\mapsto \frac{x_0}{x_2} = xy^{-1} \\ y = \frac{x_2}{x_1} &\mapsto \frac{x_3}{x_2} = zy^{-1} \\ z = \frac{x_3}{x_1} &\mapsto \frac{x_1}{x_2} = y^{-1} \\ y^{-1} = \frac{x_1}{x_2} &\mapsto \frac{x_2}{x_3} = yz^{-1} \\ z^{-1} = \frac{x_1}{x_3} &\mapsto \frac{x_2}{x_1} = y. \end{aligned}$$

Next consider the stalk at $m = (x, y-1, z-1)$ and take the completion. In the completion we note that the ideal m is generated by just the elements $u = y - 1$ and $v = z - 1$: since $1 + x^4 + y^4 + z^4 = 0 \Longrightarrow x^4 = -y^4 - z^4 - 1$ and we can solve this equation for x in terms of u and v using the Binomial Theorem for rational exponents, which is allowed since we are working in a power series ring once we take the completion. Therefore, the completed stalk with the induced action is isomorphic to $k[[u, v]]$ with the action

$$\begin{aligned} \sigma: \; u = y - 1 &\mapsto zy^{-1} - 1 = \frac{v-u}{1+u} = (v-u)\sum_{i=0}^{\infty}(-u)^i \\ v = z - 1 &\mapsto y^{-1} - 1 = \frac{-u}{1+u} = \sum_{i=0}^{\infty}(-u)^{i+1}. \end{aligned}$$

Proceeding as suggested by Peskin's work, we make the change of coordinates $\widehat{u} = u + v$, $\widehat{v} = u$ to put the linear terms of this action in Jordan form:

$$\sigma : \ \widehat{u} \mapsto \widehat{u} - \widehat{u}\widehat{v} - \widehat{v}^2 + (\widehat{u} + \widehat{v} + 1)\sum_{i=2}^{\infty}(-\widehat{v})^i = \frac{\widehat{u}}{1+\widehat{v}}$$

$$\widehat{v} \mapsto \widehat{u} + \widehat{v} + (\widehat{u} + \widehat{v})\sum_{i=1}^{\infty}(-\widehat{v})^i = \frac{\widehat{u} + \widehat{v}}{1+\widehat{v}}.$$

Lastly, make a final change of coordinates $\widetilde{v} = \widehat{v}$, $\widetilde{u} = (\sigma - 1)\widehat{v}$ to put the action in the form:

$$\sigma\widetilde{u} = \widetilde{u} + f$$

$$\sigma\widetilde{v} = \widetilde{u} + \widetilde{v},$$

where $f = (\sigma - 1)^2\widehat{v} = Tr(\widehat{v})$ and so is clearly invariant.

Using that $\widehat{v} = \widetilde{v}$ and $\widetilde{u} = (\sigma - 1)\widehat{v} = \frac{\widehat{u}+\widehat{v}}{1+\widehat{v}} - \widehat{v} \Longrightarrow \widehat{u} = (\widetilde{u} + \widetilde{v})(1 + \widetilde{v}) - \widetilde{v}$, we can find a closed form for f:

$$\begin{aligned} f &= (\sigma - 1)^2\widehat{v} \\ &= (\sigma^2 + \sigma + 1)\widehat{v} \\ &= \frac{\widehat{v} - \widehat{u}}{1 + \widehat{u} - \widehat{v}} + \frac{\widehat{u} + \widehat{v}}{1 + \widehat{v}} + \widehat{v} \\ &= \frac{\widetilde{v} - (\widetilde{u} + \widetilde{v})(1 + \widetilde{v}) + \widetilde{v}}{1 + (\widetilde{u} + \widetilde{v})(1 + \widetilde{v}) - \widetilde{v} - \widetilde{v}} + \frac{(\widetilde{u} + \widetilde{v})(1 + \widetilde{v})}{1 + \widetilde{v}} + \widetilde{v} \\ &= \frac{\widetilde{u}^2 + \widetilde{u}^2\widetilde{v} - \widetilde{v}^3}{(1 + \widetilde{v})(1 + \widetilde{u} + \widetilde{v})}. \end{aligned}$$

By Theorem 7, we know that the dimension of $H^1(G,\Theta)$ as a k-vector space is s. Thus we can achieve our goal by computing $H^1(G,\Theta)$. By Theorem 6, $H^1(G,\Theta) \cong R^G/\mathrm{Im}(Tr)$. The results in [20, p. 96] show that

$$R^G = \frac{k[[\widetilde{x},\widetilde{y},\widetilde{z},f]]}{(\widetilde{z}^3 + \widetilde{y}^{2s}\widetilde{z}^2 - \widetilde{y}^{3s+1} - \widetilde{x}^2)}$$

where $\widetilde{x} = \mathrm{Norm}(\widetilde{u})$, $\widetilde{y} = \mathrm{Norm}(\widetilde{v})$, and $\widetilde{z} = \widetilde{u}^2 - \widetilde{u}f + \widetilde{v}f$. The same calculations done in Theorem 7 to compute $\mathrm{Im}(Tr)$ show that $\mathrm{Im}(Tr) = fR^G + \widetilde{z}R^G + \widetilde{x}R^G$. Next

$$\begin{aligned}
f &= \frac{\widetilde{u}^2 + \widetilde{u}^2\widetilde{v} - \widetilde{v}^3}{(1+\widetilde{v})(1+\widetilde{u}+\widetilde{v})} \\
&= (\widetilde{u}^2 + \widetilde{u}^2\widetilde{v} - \widetilde{v}^3)(1 - \widetilde{u} + \widetilde{v} + \cdots) \\
&= \widetilde{u}^2 - \widetilde{u}^2\widetilde{v} - \widetilde{v}^3 - \widetilde{u}^3 - \widetilde{u}^3\widetilde{v} - \widetilde{u}\widetilde{v}^3 + \widetilde{u}^2\widetilde{v}^2 - \widetilde{v}^4 + \cdots \\
&= \widetilde{z} - \widetilde{y} + h(\widetilde{x},\widetilde{y},\widetilde{z},f)
\end{aligned}$$

where $\mathrm{ord}(h) \geq 2$. Thus, since $f \equiv \widetilde{z} \equiv 0$ in the quotient $R^G/\mathrm{Im}(Tr)$, we have $0 \equiv f = \widetilde{z} - \widetilde{y} + h(\widetilde{x},\widetilde{y},\widetilde{z},f) \equiv -\widetilde{y} + h(\widetilde{x},\widetilde{y},\widetilde{z},f)$. Hence $\widetilde{y} \equiv h(\widetilde{x},\widetilde{y},\widetilde{z},f) \equiv g(\widetilde{y})$ for a power series g with $\mathrm{ord}(g) \geq 2$, since all the terms of h containing $\widetilde{x}$, $\widetilde{z}$, or f are $\equiv 0$ in the quotient. Iterating this equivalence, we have that $\widetilde{y} \equiv (\underbrace{g \circ g \circ \cdots \circ g}_{l})(\widetilde{y})$ for any l and so it follows that $\widetilde{y} \equiv 0$. Thus $\mathrm{Im}(Tr)$ also contains $\widetilde{y}$. Therefore, $H^1(G,\Theta) \cong k$ and so $s = 1$.

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