# CONNECTION BLOCKING IN LATTICE QUOTIENTS OF CONNECTED LIE GROUPS

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## ABSTRACT

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Finite blocking is an interesting concept originating as a problem in billiard dynamics and later in the context of Riemannian manifolds. Let $(M, g)$ be a complete connected, infinitely differentiable Riemannian manifold. To *block* a pair of points $m_1, m_2 \in M$ is to find a *finite* set $B \subset M \setminus \{m_1, m_2\}$ such that every geodesic segment joining $m_1$ and $m_2$ intersects $B$. $B$ is called a *blocking set* for the pair $m_1, m_2 \in M$. The manifold $M$ is *secure* if every pair of points in $M$ can be blocked. $M$ is *uniformly secure* if the cardinality of blocking sets for all pairs of points in $M$ has a (finite) upper bound. The main blocking conjecture states that a closed Riemannian manifold is secure if and only if it is flat.

Gutkin [15] initiated a similar study of blocking properties of quotients $G/\Gamma$ of a connected Lie group $G$ by a lattice $\Gamma \subset G$. Here the connection curves are the orbits of one parameter subgroups of $G$. To *block* a pair of points $m_1, m_2 \in M$ is to find a finite set $B \subset M \setminus \{m_1, m_2\}$ such that every connection curve joining $m_1$ and $m_2$ intersects $B$. The lattice quotient $M = G/\Gamma$ is *connection blockable* if every pair of points in $M$ can be blocked, otherwise we call it *non-blockable*. The corresponding main blocking conjecture states that $M = G/\Gamma$ is blockable if and only if its universal cover $\tilde{G} = \mathbb{R}^n$, i.e. $M$ is a torus.

In this dissertation we investigate blocking properties for two classes of lattice quotients, which are lattice quotients of semisimple and solvable Lie groups.

According to the Levi decomposition, every connected Lie group $G$ is a semidirect product of a solvable Lie group $R$, and a semisimple Lie group $S$. A Lie group $G = R \rtimes S$ satisfies *Raghunathan's condition* if the kernel of the action of $S$ on $R$ has no compact factors in its identity component. For a such Lie group $G$, if quotients of $R$ are non-blockable then quotients of $G$ are also non-blockable.

The special linear group $\mathrm{SL}(n, \mathbb{R})$ is a simple Lie group for $n > 1$. Let $M_n = \mathrm{SL}(n, \mathbb{R})/\Gamma$, where $\Gamma = \mathrm{SL}(n, \mathbb{Z})$ is the integer lattice. We focus on $M_2$ and show that the set of blockable pairs is a dense subset of $M_2 \times M_2$, and we use this to conclude manifolds $M_n$ are non-blockable. Next, we review a quaternionic structure of $\mathrm{SL}(2, \mathbb{R})$ and a way for making cocompact lattices in this context. We show that the obtained lattice quotients are not finitely blockable.

In the context of solvable Lie groups, we study lattice quotients of *Sol*. *Sol* is a unimodular solvable Lie group, with the left invariant metric $ds^2 = e^{-2z}dx^2 + e^{2z}dy^2 + dz^2$, and is one of the eight homogeneous Thurston 3-geometries. We prove that all quotients of *Sol* are non-blockable. In particular, we show that for any lattice $\Gamma \subset Sol$, the set of non-blockable pairs is a dense subset of $Sol/\Gamma \times Sol/\Gamma$.

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# KEY TO SYMBOLS

| | | |
|---|---|---|
| $\rtimes$ | — | Sign of the semidirect product of groups (the normal sub group is to the left) |
| $\niplus$ | — | Sign of the semidirect sum of algebras (the ideal is to the left) |
| Ad | — | Adjoint representation of a Lie Group |
| ad | — | Adjoint representation of a Lie Algebra |
| $\mathrm{Sqrt}(X)$ | — | Radical of a set $X$ |
| $G^\circ$ | — | The identity component of a lie group $G$ |
| $C^\infty(G)$ | — | Set of smooth functions defined on $G$ |
| $\mathrm{Rad}\, G$ | — | Radical of a lie group $G$ |
| $\mathrm{Comm}_G(\Lambda)$ | — | Commensurator of a discrete subgroup $\Lambda$ in $G$ |
| $\mathbb{F}\text{-rank}(G)$ | — | Rank of a Lie group $G$ over a field $\mathbb{F}$ |
| $\mathfrak{der}\,\mathfrak{g}$ | — | Lie algebra of derivations of $\mathfrak{g}$ |
| $\mathfrak{rad}\,\mathfrak{g}$ | — | Radical of a Lie algebra $\mathfrak{g}$ |
| $I_{m,n}$ | — | Diagonal $n \times n$ matrix where its diagonal consists of $m$ 1's followed by $n$ (-1)'s |
| $\mathrm{GL}(n, \mathbb{F})$ | — | Group of invertible $n \times n$ matrices over a field $\mathbb{F}$ |
| $\mathrm{SL}(n, \mathbb{F})$ | — | Linear special group (subgroup of $\mathrm{GL}(n, \mathbb{F})$ whose elements have determinant 1) |
| $\mathrm{O}(n)$ | — | Orthogonal group of $n \times n$ matrices (over $\mathbb{R}$) |
| $\mathrm{SO}(n)$ | — | Special orthogonal group of $n \times n$ matrices (over $\mathbb{R}$) |
| $\mathrm{SO}(m, n)$ | — | Special orthogonal group of $n \times n$ matrices $g$ with $g^T I_{m,n} g = I_{m,n}$ |
| $\mathbb{H}_{\mathbb{F}}^{a,b}$ | — | Algebra of quaternions over a field $\mathbb{F}$ |
| $\mathrm{N}_{\mathrm{red}}$ | — | Reduced norm |
| $\mathrm{SL}(\mathbb{H}_{\mathbb{F}}^{a,b})$ | — | Subalgebra of quaternions with norm one, over a field $\mathbb{F}$ |
| $n_T(x, y)$ | — | Number of geodesic segments joining $x, y$ of length $\leq T$ |
| $m_T(x, y)$ | — | Number of geodesic segments connecting $x, y$ of length $\leq T$ |

# CHAPTER 1

# INTRODUCTION

The theme of finite blocking originates from a problem in the Leningrad Mathematical Olympiad worded as follows. The president and a terrorist are moving in a rectangular room. The terrorist intends to shoot the president with his 'magic gun' whose bullets bounce off the walls perfectly elastically: the angles of incidence and reflection are equal. Presidential protection detail consists of superhuman body guards. They are not allowed to be where the president or the terrorist are located, but they can be anywhere else, changing their locations instantaneously, as the president and the terrorist are moving about the room. Their task is to put themselves in the way of terrorist's bullets shielding the president. The problem asks how many body guards suffice.

To translate this problem into a mathematical setting, let $\Omega$ be a bounded plane domain. For arbitrary points $p, q \in \Omega$ let $\Gamma(p, q)$ be the family of billiard orbits in $\Omega$ connecting these points. Body guards correspond to $b_1, ..., b_N \in \Omega \setminus \{p, q\}$ such that every $\gamma \in \Gamma(p, q)$ passes through one of these points. If for any $p, q \in \Omega$ there is a blocking set $B = B(p, q) = b_1, ..., b_N$ then the domain is uniformly secure. The minimal possible $N$ is then the blocking number of $\Omega$. The Olympiad problem is to show that a rectangle is uniformly secure and to find its blocking number. The solution leads to a problem in plane geometry based on two facts:

1. A rectangle tiles the Euclidean plane under reflections;
2. The torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ is uniformly secure, where the role of billiard orbits is played by the images of straight lines under the projection $\mathbb{R}^2 \to T^2$. A blocking set in the torus is the set of midpoints of all joining segments: It comprises at most 4 points. A blocking set in the rectangle is also the set of midpoints of all joining billiard orbits. It comprises at most $16 = 4{\times}4$ points where the factor 4 is due to the 4 copies of the rectangle needed to tile the torus.

In the context of planar geometry, we may be ask a similar question for polygon billiards. For a solution we need to find out which plane polygons are secure. This problem first appeared in the literature in Hiemer and Snurnikov [22]. A polygon is rational if its corners have $\pi$-rational angles. It is claimed in [22] that all rational polygons are secure. Gutkin studied the security of translation surfaces [16, 17] and proved that the regular $n$-gon is secure if and only if $n = 3, 4, 6$ [12]. Since all regular polygons are rational, this disproves the claim in [22]. The work of Gutkin [13] contains related results on the security of rational polygons, but a solution to the general case remains elusive [14].

The billiard orbits in the rectangle and the straight lines in the torus are examples of geodesics in Riemannian manifolds. The original Olympiad problem expanded into the subject of Riemannian security.

To study the security of Riemannian manifolds, first we need to define the problem in mathematical terms. Let $(M, g)$ be a complete connected, infinitely differentiable Riemannian manifold. For a pair of (not necessarily distinct) points $m_1, m_2 \in M$ let $\Gamma(m_1, m_2)$ be the set of geodesic segments joining these points. A set $B \subset M \setminus \{m_1, m_2\}$ is *blocking* if every $\gamma \in \Gamma(m_1, m_2)$ intersects $B$. The pair $m_1, m_2$ is secure if there is a *finite blocking* set $B = B(m_1, m_2)$. A manifold is secure if all pairs of points are secure. If there is a uniform bound on the cardinalities of blocking sets, the manifold is *uniformly secure* and the best possible bound is the *blocking number*.

Now, the first question naturally arising is which Riemannian manifolds are secure. If we focus on closed Riemannian manifolds, there is the following conjecture as stated by Burns-Gutkin and Lafont-Schmidt [5, 25]:

**Conjecture 1.1.** A closed Riemannian manifold is secure if and only if it is flat.

Conjecture 1.1 says that flat manifolds are the only secure manifolds. This has been verified for several special cases:

- Flat manifolds are uniformly secure, and the blocking number depends only on their dimension (Gutkin-Schroeder [18, 12]). In fact, they are also *midpoint secure*, i.e., the midpoints of connecting geodesics yield a finite blocking set for any pair of points (Gutkin-Schroeder and Bangert-Gutkin)[18, 3, 12].

- A manifold without conjugate points is uniformly secure if and only if it is flat (Burns-Gutkin and Lafont-Schmidt[5, 25]).

- A compact locally symmetric space is secure if and only if it is flat (Gutkin-Schroeder [18]).

- The generic manifold is insecure (Gerber-Ku and Hebda-Ku[8, 20]).

- Conjecture 1.1 holds for compact Riemannian surfaces with genus bigger or equal than one (Bangert-Gutkin [3]).

- Any Riemannian metric has an arbitrarily close, insecure metric in the same conformal class (Hebda-Ku [20]).

To have a better insight of the security concept, we present the proof of Proposition 2 in [18] to show that the flat torus is uniformly (midpoint) secure:

**Proposition 1.2.** *The flat torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ is uniformly secure and the blocking number is $2^n$.*

*Proof.* Let $o \in \mathbb{T}^n$ be the origin. Let $x \in \mathbb{T}^n$ be an arbitrary point. By homogeneity, it suffices to prove that the pair $(o, x)$ is blockable with $2^n$ points. Let $G(o, x)$ be the set of all geodesics connecting the origin $o$, and $x$. There is a one-to-one correspondence between the geodesics $\gamma \in G(o, x)$ and the straight segments $\tilde{\gamma}_{x+z} \in \mathbb{R}^n$ connecting the origin $O \in \mathbb{R}^n$ with the points $x + z,\ z \in \mathbb{Z}^n$. Let $\gamma_{x+z} \in G(o, x)$ be the corresponding connecting geodesic. If $p : \mathbb{R}^n \to T^n$ is the projection, then $\gamma_{x+z} = p(\tilde{\gamma}_{x+z})$. The midpoint of the segment $\tilde{\gamma}_{x+z}$ is $\frac{x}{2} + \frac{z}{2} \in \mathbb{R}^n$. Set $\tilde{F}(x) = \{\frac{x}{2} + \frac{z}{2} : z \in \mathbb{Z}^n\}$. Then the set $F(x) = p(\tilde{F}(x)) \subset T^n$ is finite, and $|F(x) = 2^n|$. Thus, $2^n$ points suffice to block any $\gamma \in G(o, x)$. On the other hand, for a typical $x$, we cannot block $G(o, x)$ with less than $2^n$ points. We leave the verification of this to the reader. □

By the Bieberbach theorem, every closed flat Riemannian manifold $M$ is finitely covered by a flat torus (For statement and proof of the Bieberbach theorem see Wolf [37, p.100]). It is straightforward to see that if the finite cover of a Riemannian manifold is uniformly secure, then the manifold is uniformly secure. Therefore, Proposition 1.2 implies the following corollary:

**Corollary 1.3.** *Every flat closed Riemannian manifold is uniformly secure.*

Gutkin [15] initiated the study of blocking properties of lattice quotients of connected Lie groups. In this context, he speaks of *finite blocking* instead of security. Let $G$ be a connected Lie group, and let $\Gamma \subset G$ be a lattice. Connection curves of the lattice quotient $M = G/\Gamma$ are the orbits of one parameter subgroups of $G$. To *block* a pair of points $m_1, m_2 \in M$ is to find a finite set $B \subset M \setminus \{m_1, m_2\}$ such that every connection curve joining $m_1$ and $m_2$ intersects $B$. If every pair of points in $M$ can be blocked, $M$ is called *connection blockable*, or simply *blockable*, otherwise it is called *non-blockable*. A counterpart of Conjecture 1 for lattice quotients is as follows:

**Conjecture 1.4.** Let $G$ be a connected Lie group with the universal cover $\tilde{G}$, $\Gamma \subset G$ a lattice, and let $M = G/\Gamma$. Then $M$ is blockable if and only if $\tilde{G} = \mathbb{R}^n$, i.e. $M$ is a torus.

To start working on this conjecture, it would be helpful to consider solvable and semisimple Lie groups first. Solvable Lie groups (resp. Lie algebras) and the semisimple Lie groups (resp. Lie algebras) form two large and generally complementary classes. Every connected Lie group $G$ is a semidirect product of a solvable Lie group $R$, and a semisimple Lie group $S$ (Theorem B.4) which is called the Levi decomposition. A connected Lie group $G$ satisfies Raghunathan's condition if the kernel of the action of $S$ on $R$ has no compact factors in its identity component. For such Lie groups $G$, if lattice quotients of $R$ are non-blockable, then lattice quotients of $G$ are also non-blockable. In addition, we will show in Proposition 2.18 that if nilradical of $G$ is not abelian, then lattice quotients of $G$ are non-blockable.

Gutkin in [15] establishes Conjecture 1.4 for lattice quotients of nilpotent Lie groups. Such spaces are called nilmanifolds. He starts with the connection blocking problem in Heisenberg manifolds. He then studies connection blocking in two-step nilmanifolds, and then extends the results to an arbitrary nilmanifold.

The Lie group $H$ is the Lie subgroup of $\mathrm{GL}(3, \mathbb{R})$ defined by

$$H = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \ : \ x, y, z \in \mathbb{R} \right\} .$$

A Heisenberg manifold is a quotient of $H$ by a cocompact lattice. Gutkin's main result can be summarized in the following propositions (See Theorems 1 and 2 in [15]).

**Proposition 1.5.** *Let $M$ be a three-dimensional Heisenberg manifold. Then*

*i) A pair of points in $M$ is blockable if and only if it is midpoint blockable.*

*ii) For every $x \in M$, the pair $(x, x)$ is blockable.*

*iii) The set of blockable pairs of points is a dense countable union of closed submanifolds of positive codimension in $M \times M$.*

*iv) In particular iii) implies that almost all pairs of points $(x, y) \in M \times M$ are non-blockable.*

**Proposition 1.6.** *Let $M$ be a nilmanifold of dimension $n$ . Then the following statements are equivalent:*

*i) $M$ is connection blockable;*

*ii) $M$ is midpoint blockable;*

*iii) $\pi_1(M) = \mathbb{Z}^n$;*

*iv) $M$ is a topological torus;*

*v) $M$ is uniformly blockable and the blocking number depends only on its dimension.*

Every nilpotent Lie groups is solvable; however, solvable Lie groups constitute a much larger class. One the simplest non-nilpotent solvable Lie groups is *Sol*. *Sol* is the Lie group of all vectors

$(x, y, z) \in \mathbb{R}^3$ with the group multiplication $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1+e^{z_1}x_2, y_1+e^{-z_1}y_2, z_1+z_2)$. Details of the definition and properties of *Sol* are presented in Chapter 3.

We prove Conjecture 1.4 for lattice quotients of the Lie group *Sol*. We start with a specific class of lattices in *Sol*: those that are isomorphic to the semidirect product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$, where $A \in \mathrm{SL}(2, \mathbb{Z})$ is a diagonalizable matrix, and $r \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ as $A^r$, so as the multiplication is given by

$$(p_1, q_1, r_1)(p_2, q_2, r_2) = ((p_1, q_1) + A^{r_1}(p_2, q_2), r_1 + r_2)\,.$$

If $P$ is the eigenvector matrix of $A$, by Proposition 3.4 the mapping $(p, q, r) \mapsto (P^{-1}(p, q), sr)$ embeds $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ into *Sol* and the image is a cocompact lattice. We then solve the blocking problem for some of these lattices. In Section 3.3 we prove:

**Theorem 1.7.** *Let $A \in SL(2, \mathbb{Z})$ be a matrix with eigenvalues $\pm\{\lambda, \lambda^{-1}\}$, where $\lambda = e^s$ for some $s \neq 0$. Then there exists $P \in GL(2, \mathbb{R})$ such that $P_{11} = P_{22} = 1$, and*

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

*Let $\Gamma = \Gamma(A) = \{(P(p, q), sr) | p, q, r \in \mathbb{Z}\}$ be the corresponding lattice in $Sol$. Let $m_1 = g_1\Gamma$, $m_2 = g_2\Gamma$ be a pair of points in $Sol/\Gamma$, and assume $g_1^{-1}g_2 = (x_0, y_0, z_0)$. If $x_0 = 0$, $y_0 \neq 0$, or $y_0 = 0$, $x_0 \neq 0$, then $m_2$ is not blockable from $m_1$.*

**Remark.** The above theorem basically shows that if two points are on the planes $x = c$, or $y = c$, (not having the same $y$, or $x$ coordinates, respectively) then their corresponding cosets in the quotient space are not blockable.

Through Proposition 3.6 and Lemma 3.6 in Chapter 3 we show that all lattices of *Sol* are isomorphic to the semidirect product lattices presented in Theorem 1.7, and we then prove non-blockability of all quotients of *Sol*, as stated in the following theorem.

**Theorem 1.8.** *All lattice quotients of Sol are non-blockable. In fact, for every lattice $\Gamma$ in Sol, the set of non-blockable pairs is a dense subset of $Sol/\Gamma \times Sol/\Gamma$.*

In the context of semisimple Lie groups, we investigate blocking problem for quotient lattices of $\mathrm{SL}(n, \mathbb{R})$. Gutkin in [15] proves the lattice quotient $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$ is not *midpoint blockable*. For simplicity, we use the following notation throughout the thesis.

**Notation.** For $n > 1$, $M_n$ denotes the homogeneous space $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$, and when it is clear from the context, $\Gamma$ denotes $\mathrm{SL}(n, \mathbb{Z})$.

We prove that $M_n$ and all quotients of $\mathrm{SL}(n, \mathbb{R})$ whose lattice is commensurable to $\mathrm{SL}(n, \mathbb{Z})$, are non-blockable. Specifically we prove the following theorem.

**Theorem 1.9.** *Two elements $m_1 = g_1\Gamma$ and $m_2 = g_2\Gamma \in M_2$ are not finitely blockable from each other if $g_1^{-1}g_2 \in SL(2, \mathbb{Q})$. In fact, the set of non-blockable pairs is a dense subset of $M_2 \times M_2$.*

This easily implies the following:

**Theorem 1.10.** *Any lattice quotient $M_n$, $n > 2$, has infinitely many pairs of non-blockable points.*

**Remark.** Lattices $\Gamma_1$ and $\Gamma_2$ of a Lie group $G$ are commensurable if there exists $g \in G$ such that the group $\Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index in both $\Gamma_1$ and $g\Gamma_2 g^{-1}$. By Corollary 2.13, quotient spaces of a Lie group mod two commensurable lattices carry the same blocking property. The Margulis Arithmeticity Theorem (See Morris and Margulis [31, p.92], [29, p.298]), implies every lattice of $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$ is arithmetic. As a result, a large class of lattices in $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$ are commensurable to $\mathrm{SL}(n, \mathbb{Z})$. In particular, if $\Gamma$ is a lattice and the subgroup $\Gamma \cap \mathrm{SL}(n, \mathbb{Z})$ is of finite index in $\Gamma$, then $\Gamma$ and $\mathrm{SL}(n, \mathbb{Z})$ are commensurable. Hence, all lattice quotients $\mathrm{SL}(n, \mathbb{R})/\Gamma$, for such lattices $\Gamma$ are non-blockable. Moreover, for every lattice $\Gamma \subset \mathrm{SL}(n, \mathbb{Q})$, $\mathrm{SL}(n, \mathbb{R})/\Gamma$ is non-blockable [29, p.319].

A lattice $\Gamma \subset G$ is called *cocompact* if the quotient space $G/\Gamma$ is compact. $\mathrm{SL}(2, \mathbb{Z})$ is the most basic example of a non-cocompact lattice in $\mathrm{SL}(2, \mathbb{R})$. Up to commensurability and conjugates, this is the only one that is not cocompact, Morris [31, p.115]. Since lattice quotients of the same Lie group mod conjugate or commensurable lattices have identical blocking property (See Corollary 2.13), we conclude following corollary.

**Corollary 1.11.** *For every non-cocompact lattice* $\Gamma \subset SL(2,\mathbb{R})$*, the quotient space* $SL(2,\mathbb{R})/\Gamma$ *is non-blockable.*

For $\mathrm{SL}(2,\mathbb{R})$, we additionally study the blocking problem for a class of compact lattice quotients. We show that for a large class of cocompact lattices $\Gamma \subset \mathrm{SL}(2,\mathbb{R})$, defined using a quaternionic structure, $\mathrm{SL}(2,\mathbb{R})/\Gamma$ is non-blockable. For any field $\mathbb{F}$, and any nonzero $a, b \in \mathbb{F}$, $\mathbb{H}_{\mathbb{F}}^{a,b}$ is a quaternion algebra, in which the multiplication and the norm function depend on $a, b$, and $\mathrm{SL}(1,\mathbb{H}_{\mathbb{F}}^{a,b})$ is the subgroup of elements with norm one. See Section 4.3 for the formal mathematical definitions. We specifically prove the following theorem.

**Theorem 1.12.** *Let* $a, b$ *be positive integers such that* $\Gamma = SL(1,\mathbb{H}_{\mathbb{Z}}^{a,b})$ *is a cocompact lattice of* $G = SL(1,\mathbb{H}_{\mathbb{R}}^{a,b})$*. If* $g = x + yi \in SL(1,\mathbb{H}_{\mathbb{Q}}^{a,b})$*, then* $g\Gamma \subset G/\Gamma$ *is not finitely blockable from* $m_0 = \Gamma$*. Therefore the lattice quotient* $G/\Gamma$ *is not finitely blockable.*

The organization of the dissertation is as follows. In Chapter 2, we briefly review the requirements needed to study the blocking problem. Section 2.1 includes Lie theory basic concepts and theorems, as presented in the standard textbooks. In Section 2.2 we review general blocking properties of lattice quotients of a connected Lie group. In Chapter 3 we study the blocking problem in lattice quotients of *Sol*. In Section 3.1 we derive an explicit formula for *Sol*'s one parameter subgroups. Section 3.2 introduces the semidirect product lattices in *Sol*. We describe a group presentation for all lattices in *Sol* according to Molnár [30] and then prove that all lattices in *Sol* are conjugate to semidirect product lattices. In Section 3.3, we first prove a few technical lemmas, then proceed to prove Theorems 1.7 and 1.8. In Chapter 4 we study the blocking problem in lattice quotients of $\mathrm{SL}(n,\mathbb{R})$, mainly focusing on $\mathrm{SL}(2,\mathbb{R})$. In Section 4.1 we describe one parameter subgroups in the Lie group $\mathrm{SL}(2,\mathbb{R})$. In Section 4.2, we first prove a technical proposition, then state and prove Theorems 1.9 and 1.10. Section 4.3 presents a quaternionic structure of $\mathrm{SL}(2,\mathbb{R})$ and a way for making cocompact lattices in this context. The section concludes with the proof of Theorem 1.12. Chapter 5 includes concluding remarks and problems for further research. Section 5.1 discusses the blocking problem for some other solvable and semisimple Lie groups. In Section

5.2 we discuss how the blocking problem in manifolds with conjugate points relates to the behavior of the exponential map near singularities. We then state a conjecture about the number of geodesic segments of certain length, and propose a sketch for a potential proof.

# CHAPTER 2

# PRELIMINARIES

This chapter is a brief review of the background needed to study the blocking problem. The first section includes Lie Theory basic concepts and theorems, as presented in standard textbooks. In the second section we introduce the concept of connection blocking in lattice quotients of connected Lie groups and review some general blocking properties for these spaces.

## 2.1 Lie Theory Background

In this section we briefly review the basic concepts of Lie Theory. Proofs of the theorems and detailed discussions can be found in many Lie theory textbooks, for example Abbaspour [1], Hilgert [23], and Hall [19]. See Morris [31] for a detailed discussion about lattices and arithmetic subgroups.

### 2.1.1 Lie Groups and Lie Algebras

By a Lie group $G$ we mean a topological group with a differentiable structure such that the mapping $G \times G \to G$ given by $(x, y) \to xy^{-1}, x, y \in G$, is differentiable. It follows that *left translations* $L_g : G \to G, L_g(h) = gh$, and *right translations* $R_g : G \to G, R_g(h) = hg$, are diffeomorphisms. We say that a Riemannian metric on $G$ is *left invariant* if $< U, V >_h =< d(L_g)_h U, d(L_g)_h V >$ for all $g, h \in G, U, V \in T_h G$, that is, if $L_g$ is an isometry. Analogously, we can define a *right invariant* Riemannian metric. A Riemannian metric on $G$ which is both right and left invariant is said to be *bi-invariant*. We say that a differentaible vector field $X$ on a Lie group $G$ is *left invariant* if $dL_g(X) = X$ for all $g \in G$. From The left translation $L_g$ one can, for any vector $X_e \in T_e G$, define a left invariant vector field $X$ on $G$ by

$$X_g = (dL_g)(X_e) .$$

$X$ is left invariant since $dL_g(X_h) = dL_g \circ dL_h(X_e) = dL_{gh}(X_e) = X_{gh}$ . Let $\mathfrak{g}$ represent the set of left invariant vector fields on $G$. For $a, b \in \mathbb{R}$ and $X, Y \in \mathfrak{g}$, it is easy to see $aX + bY \in \mathfrak{g}$, which means $\mathfrak{g}$ is a $\mathbb{R}$-vector space. Recall that the Lie bracket of two vector fields $X, Y$ on $G$ is defined as $[X, Y] := XY - YX$, which is the same as Lie derivative of $Y$ with respect to $X$. It's not difficult to see if $X, Y \in \mathfrak{g}$, so is their Lie bracket $[X, Y]$.

It follows that for a Lie group $G$ as an $n$-dimensional smooth manifold, $\mathfrak{g}$ is an $n$-dimensional vector space, and together with Lie bracket operation $[\cdot, \cdot]$ it's called the *Lie Algebra* of $G$. Since every left invariant vector field $X$ is identified by $X_e$, its value at identity, $\mathfrak{g}$ can also be defined as the tangent space at identity, $T_eG$. Lie bracket of $X_e, Y_e \in T_eG$ is then defined by $[X_e, Y_e] := [X, Y]_e$. A *Lie Algebra homomorphism* between two Lie algebras is a linear map that preserves Lie algebra structure. Suppose $\phi : G \to H$ is a Lie group homomorphism, then its differential at $e$, $d\phi : T_eG \to T_eH$ gives a linear map from $T_eG$ to $T_eH$. Considering the identification of $T_eG$ with $\mathfrak{g}$ and $T_eH$ with $\mathfrak{h}$, we state the following theorem without proof.

**Theorem 2.1.** *If $\phi : G \to H$ is a Lie group homomorphism, then the induced map $d\phi : \mathfrak{g} \to \mathfrak{h}$ is a Lie algebra homomorphism.*

**Remark.** A Lie algebra in abstract sense is defined as a vector space $\mathfrak{g}$ together with a bilinear, antisymmetric map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(X, Y) \mapsto [X, Y]$, called *the Lie bracket*, satisfying Jacobi identity, i.e. for all $X, Y, Z \in \mathfrak{g}$, $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. A Lie algebra $\mathfrak{g}$ is abelian if the Lie bracket vanishes identically, that is for all $X, Y \in \mathfrak{g}$, $[X, Y] = 0$; otherwise it's called non-abelian. As we saw earlier, every Lie group gives rise to a Lie algebra. Conversely, to any finite-dimensional Lie algebra over real or complex numbers, there is a corresponding connected Lie group unique up to covering (Lie's third theorem, see Theorem B.4). This correspondence between Lie groups and Lie algebras allows one to study Lie groups in terms of Lie algebras.

### 2.1.2 One Parameter Subgroups

Let G be a Lie group, $X_e \in T_eG$ be a tangent vector at the identity element, and $X \in \mathfrak{g}$ be the left invariant vector field generated by $X_e$. As an exercise one can show that any left invariant vector field on $G$ is complete. So for any $g \in G$ there is a unique integral curve of $X$ defined on the whole real line $\mathbb{R}$,

$$\gamma_g : \mathbb{R} \to G,$$

so that $\gamma_g(0) = g$. We are interested in the special map $\phi := \gamma_e$, the integral curve of $X$ that starts at $e$. It is easy to see that the map $\phi = \gamma_e$ is a Lie group homomorphism from $\mathbb{R}$ to $G$, i.e. $\phi(s+t) = \phi(s)\phi(t)$ holds for all $s, t \in \mathbb{R}$.

**Definition 2.2.** A one-parameter subgroup of a Lie group $G$ is a Lie group homomorphism $\phi : \mathbb{R} \to G$, that is $\phi$ is smooth and, $\phi(s+t) = \phi(s)\phi(t)$ holds for all $s, t \in \mathbb{R}$.

So the argument above shows that for any $X \in \mathfrak{g}$ (or for any $X_e \in T_eG$), one can construct a one-parameter subgroup $\phi$ of $G$. Conversely, for any one-parameter subgroup $\phi : \mathbb{R} \to G$, we must have $\phi(0) = e$, and thus construct a left-invariant vector field $X$ on $G$ via the vector

$$X_e = \dot{\phi}(0) = (d\phi)_0 \left(\frac{d}{dt}\right) \in T_eG\,.$$

It is not hard to see that different vectors in $T_eG$ give rise to different one-parameter subgroups, and different one-parameter subgroups give rise to different vectors in $T_eG$. As a consequence, we get one-to-one correspondence between

1. One-parameter subgroups of $G$,

2. Left invariant vector fields on $G$,

3. Tangent vectors at $e \in G$.

So we have three different descriptions of the Lie algebra $\mathfrak{g}$.

### 2.1.3 The Exponential Map

For any $X \in \mathfrak{g}$, let $\phi_X$ be the one-parameter subgroup $G$ corresponding to $X$.

**Definition 2.3.** The exponential map of $G$ is the map

$$\exp : \mathfrak{g} \to G, X \mapsto \phi_X(1)\,.$$

Since $\tilde{\phi}(s) = \phi_X(ts)$ is the one parameter subgroup corresponding to $tX$, we have

$$\exp(tX) = \phi_X(t)\,.$$

Note that the zero vector $0 \in T_eG$ generates the zero vector field on $G$, whose integral curve through constant $e$ is the constant curve. So $\exp(0) = e$. The exponential map $\exp : \mathfrak{g} \to G$ is a local diffeomorphism near 0 and it's differential at 0 is the identity map. i.e. $(d\exp)_0 = \mathrm{Id}$ .

The exponential map is natural, which means for any Lie group homomorphism $\phi : G \to H$, the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{d}\phi"] \arrow[d, "\exp_{\mathfrak{g}}"'] & \mathfrak{h} \arrow[d, "\exp_{\mathfrak{h}}"] \\ G \arrow[r, "\phi"] & H \end{tikzcd} \tag{2.1.1}$$

is commutative, i.e. $\phi \circ \exp_{\mathfrak{g}} = \exp_{\mathfrak{h}} \circ d\phi$.

### 2.1.4 Linear Lie Groups

Recall that $M(n, \mathbb{R})$, the set of all $n \times n$ real matrices, is diffeomorphic to $\mathbb{R}^{n^2}$.

**Definition 2.4.** A *linear Lie group*, or *matrix Lie group*, is a submanifold of $M(n, \mathbb{R})$ which is also a Lie group, with group structure the matrix multiplication.

Let's begin with the "largest" linear Lie group, the general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \{X \in M(n, \mathbb{R}) \mid \det X \neq 0\}\,.$$

Since the determinant map is continuous, $\mathrm{GL}(n,\mathbb{R})$ is open in $M(n,\mathbb{R})$ and thus a submanifold. Moreover, $\mathrm{GL}(n,\mathbb{R})$ is closed under the group multiplication and inversion operations, so it is a Lie group. Obviously $\mathrm{GL}(n,\mathbb{R})$ is an $n^2$-dimensional non-compact Lie group, and it is not connected. In fact, it consists of exactly two connected components which are

$$\mathrm{GL}_+(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \,|\, \det X > 0\},$$

and

$$\mathrm{GL}_-(n,\mathbb{R}) = \{X \in M(n,\mathbb{R}) \,|\, \det X < 0\}.$$

The fact that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R}) \cong \mathbb{R}^{n^2}$ also implies that the Lie algebra of $\mathrm{GL}(n,\mathbb{R})$, as the tangent space at $e = I_n$, is the set $M(n,\mathbb{R})$ itself, i.e.

$$\mathfrak{gl}(n,\mathbb{R}) = \{A \,|\, A \text{ is an } n \times n \text{ real matrix}\}.$$

Using the coordinate system computations, it turns out that the Lie bracket operation on $\mathfrak{g}$ is the matrix commutator, that is for all $A, B \in \mathfrak{g}$

$$[A, B] = AB - BA.$$

Given any $A \in \mathfrak{gl}(n,\mathbb{R})$, we can define the matrix exponential

$$e^A = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots .$$

It is easy to check that the series converges, and

$$e^{sA}e^{tA} = e^{(s+t)A}.$$

Notice that $e^{0A} = I_n$, and $(e^{tA})^{-1} = e^{-tA}$. In particular, $e^{tA} \in \mathrm{GL}(n,\mathbb{R})$. So $e^{tA}$ is a one-parameter subgroup of $\mathrm{GL}(n,\mathbb{R})$. Since $\left.\frac{d}{dt}\right|_{t=0} e^{tA} = A$, we conclude that the exponential map $\mathfrak{gl}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is

$$\exp(A) = I_n + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots + \frac{A^n}{n!} + \cdots .$$

The exponential map is not surjective, not even to $\mathrm{GL}_+(n, \mathbb{R})$.

A subgroup $H$ of a Lie group $G$ is called a *Lie subgroup* if it is a Lie group (with respect to the induced operation), and the inclusion map $\iota_H : H \hookrightarrow G$ is an immersion (and therefore a Lie group homomorphism). Suppose $H$ is a Lie subgroup of $G$, and let $\mathfrak{h}$ be the Lie algebra of $H$. Since $\iota : H \hookrightarrow G$ is an immersion and is a Lie group homomorphism, $d\iota_H : \mathfrak{h} \to \mathfrak{g}$ is injective and is a Lie algebra homomorphism. So we think of $\mathfrak{h}$ as a *Lie subalgebra*, i.e. a linear subspace that is closed under the Lie bracket of $\mathfrak{g}$. Note that a one-parameter subgroup of $H$ is automatically a one-parameter subgroup of $G$ (with initial vector in $T_eH$), so the exponential map $\exp_H : \mathfrak{h} \to H$ is exactly the restriction of $\exp_G : \mathfrak{g} \to G$ onto $\mathfrak{h}$. The following well known theorem is very useful to determine the Lie algebra of a lie subgroup.

**Theorem 2.5.** *Suppose $H$ is a Lie subgroup of $G$. Then as a Lie subalgebra of $\mathfrak{g}$,*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \textit{ for all } t \in \mathbb{R}\} .$$

**The special linear group.** The special linear group is defined as

$$\mathrm{SL}(n, \mathbb{R}) = \{X \in \mathrm{GL}(n, \mathbb{R}) \mid \det X = 1\} .$$

It is easy to see that $\mathrm{SL}(n, \mathbb{R})$ is a subgroup, and a $n^2 - 1$ dimensional submanifold of $\mathrm{GL}(n, \mathbb{R})$. It follows that $\mathrm{SL}(n, \mathbb{R})$ is a (connected non-compact) Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$. To determine its Lie algebra $\mathfrak{sl}(n, \mathbb{R})$, first notice that $\det e^A = e^{\mathrm{Tr}(A)}$. So for an $n \times n$ matrix $A$, $e^A \in \mathrm{SL}(n, \mathbb{R})$ if and only if $\mathrm{Tr}(A) = 0$. We conclude that

$$\mathfrak{sl}(n, \mathbb{R}) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid \mathrm{Tr}(A) = 0\} .$$

### 2.1.5 Lattices

By a discrete subgroup $\Lambda$ of a Lie group $G$ we mean a Lie subgroup of $G$ such that it has discrete topology as a topological subspace.

**Commensurability.** When studying discrete subgroups and lattices, we usually wish to ignore minor differences that come from passing to a finite-index subgroup. This leads us to the concept of commensurability.

**Definition 2.6.** Two subgroups $H_1$ and $H_2$ of the same group $G$ are said to be (strictly) *commensurable* if $H_1 \cap H_2$ is a finite index subgroup of both $H_1$ and $H_2$.

A basic exercise in group theory shows that the intersection of two finite-index subgroups is again a finite index subgroup. It follows that being commensurable is an equivalence relation on the set of subgroups of a given group. If one wishes to go from geometry to algebra, there are some modifications that we need to consider. In the context of blocking problem the following definition is the appropriate modification we need to make.

**Definition 2.7.** Two subgroups $H_1$ and $H_2$ of a group $G$ are said to be *weakly commensurable* if there is an element of $g \in G$ such that $H_1$ and $gH_2g^{-1}$ are strictly commensurable.

It is straightforward to see that being weakly commensurable is again an equivalence relation on the subgroups of a given group $G$. Two discrete subgroups of a Lie group which are weakly commensurable have very similar geometric structure; this leads us to the following definition:

**Definition 2.8.** An element $g \in G$ *commensurates* $\Lambda$ if $g\Lambda g^{-1}$ is commensurable to $\Lambda$. Let

$$\mathrm{Comm}_G(\Lambda) := \{g \in G \mid g \text{ commensurates } \Lambda\} .$$

This is called the *commensurator* of $\Lambda$.

$\mathrm{Comm}_G(\Lambda)$ contains the normalizer $\mathcal{N}_G(\Lambda)$ of $G$, and is sort of generalized normalizer of $G$. Indeed, $\mathrm{Comm}_G(\Lambda)$ can be thought of as the stabilizer of the commensurability class of $\Lambda$ for the conjugacy action of $G$ on the set of commensurability classes of its subgroups. For example, if $G = \mathrm{SL}(n, \mathbb{R})$ and $\Lambda = \mathrm{SL}(n, \mathbb{Z})$, then $\mathcal{N}_G(\Lambda)$ is commensurable to $\Lambda$, but $\mathrm{Comm}_G(\Lambda)$ contains $\mathrm{SL}(n, \mathbb{Q})$; so $\mathrm{Comm}_G(\Lambda)$ is dense in $G$, even though $\mathcal{N}_G(\Lambda)$ is discrete (see Morris [31, p.49]).

**Remark.** In the context of blocking problem, when we refer to commensurability, we mean weak commensurability. Therefore two discrete subgroups $\Gamma_1$ and $\Gamma_2$ of a Lie group $G$ are commensurable if there exists $g \in G$ such that the group $\Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index in both $\Gamma_1$ and $g\Gamma_2 g^{-1}$. We will see that quotients of a given Lie group mod two commensurable lattices, carry the same blocking property.

The definition of a lattice in a Lie group requires the introduction of Haar measure. To see a brief summary of Haar measure definition and properties see Appendix C. Let $\mu$ be Haar measure on $G$ and $\Lambda \subset G$ be a discrete subgroup. Then there exists a unique (up to a scalar multiple) $\sigma$-finite, $G$-invariant Borel measure $\nu$ on $G/\Lambda$ which can be defined via natural quotient map. (See Proposition 4.1.3 in [31] for proof). The Haar measure $\mu$ on $G$ is given by a smooth volume form. so the associated measure $\nu$ on $G/\Lambda$ is also given by a volume form, Therefore we say that $G/\Lambda$ has *finite volume* if $\nu(G/\Lambda) < \infty$.

**Definition 2.9.** A subgroup $\Gamma$ of $G$ is a lattice in $G$ if

- $\Gamma$ is a discrete subgroup of $G$, and
- $G/\Gamma$ has finite volume.

*Examples.* If $\Gamma$ is discrete and $G/\Gamma$ is compact, then $\Gamma$ is a lattice in $G$. Such a lattice is called cocompact. In addition, every finite index subgroup of a lattice is also a lattice [31, p.15, p.46].

$\mathrm{SL}(2,\mathbb{Z})$ is a lattice in $\mathrm{SL}(2,\mathbb{R})$. First, discreteness is obvious. To see the second condition we note that $\mathbb{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ (See [31, p.9]) where $\mathbb{H}^2$ is the hyperbolic plane as a Lie group with multiplication. Let

$$\mathcal{F} = \{z \in \mathbb{H}^2 \mid |z| \geq 1 \text{ and } -1/2 \leq \mathrm{Re} z \leq 1/2\}\,.$$

It is well known that $\mathcal{F}$ is a fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{H}^2$; it therefore suffices to show that $\mathcal{F}$ has finite volume, or, more precisely, finite hyperbolic area. The hyperbolic area $dA$ of an infinitesimal rectangle is the product of its hyperbolic length and its hyperbolic width.

If the Euclidean length is $dx$ and the Euclidean width is $dy$, and the rectangle is located at the point $x + iy$, then, by definition of the hyperbolic metric, the hyperbolic length is $(dx)/(2y)$ and the hyperbolic width is $(dy)/(2y)$. Therefore,

$$dA = \frac{dxdy}{4y^2}.$$

Since $\mathrm{Im} z \geq \sqrt{3}/2$ for all $z \in \mathcal{F}$, we have

$$\mathrm{vol}(\mathcal{F}) = \int_{x+iy\in\mathcal{F}} dA \leq \int_{\sqrt{3}/2}^{\infty} \int_{-1/2}^{1/2} \frac{dxdy}{4y^2} = \frac{1}{4} \int_{\sqrt{3}/2}^{\infty} \frac{1}{y^2}\, dy < \infty.$$

Similar but more complicated calculations show that $\mathrm{SL}(n, \mathbb{Z})$ is a lattice in $\mathrm{SL}(n, \mathbb{R})$. As in the above example, the hard part is to find a fundamental domain for the action of $\Gamma$ on $G$ (or an appropriate approximation of a fundamental domain); then it is not difficult to see its volume is finite. These are special cases of the following general theorem which implies that every simple Lie group has a lattice (Theorem 1.3.9 in [31]).

**Theorem 2.10.** *Assume $G \subset SL(n, \mathbb{R})$ and there exist simple Lie groups $G_1, \cdots, G_m$ such that $G = G_1 \times \cdots \times G_m$. Moreover, assume $G \cap SL(n, \mathbb{Q})$ is dense in $G$. Then $G_{\mathbb{Z}} = G \cap SL(n, \mathbb{Z})$ is a lattice in $G$.*

Lattices constructed by taking the integer points of $G$ in this way are said to be *arithmetic*. When $n$ is large, there is more than one way to embed $G$ in $\mathrm{SL}(n, \mathbb{R})$, and different embeddings can lead to quite different intersections with $\mathrm{SL}(n, \mathbb{Z})$. In particular, for a non-compact simple Lie group $G$, we can take an appropriate embedding of $G$ in some $\mathrm{SL}(n, \mathbb{R})$, and construct a non cocompact lattice $\Gamma$ in $G$; we also can take a different embedding, and construct a cocompact lattice $\Gamma'$ in $G$ [31, p.16].

## 2.2 Connection Blocking in Lattice Quotients of Connected Lie Groups

A differentiable manifold $M$ with a transitive action of a Lie group $G$ on it is called a homogeneous space of $G$. It can be shown that any homogeneous space of G is isomorphic to $G/H$, where $H \subset G$

is a Lie subgroup, with the canonical $G$-action. Homogeneous spaces are the most important and interesting objects of geometry.

Let $G$ be a connected Lie group and $\Gamma \subset G$ be a lattice. $\Gamma$ acts on $G$ through left (or right) multiplication. Let $\Gamma \subset G$ be a lattice in $G$. The lattice quotient space $M = G/\Gamma$ is a homogeneous space of $G$. In this section we study connection blocking properties of $M = G/\Gamma$, following the notation and text in Gutkin [15].

For $g \in G, m \in M$, $g \cdot m$ denotes the action of $G$ on $M$. Let $\mathfrak{g}$ be the Lie Algebra of $G$ and let $\exp : \mathfrak{g} \to G$ be the exponential map. For $m_1, m_2 \in M$ let $C_{m_1,m_2}$ be the set of parametrized curves $c(t) = \exp(tx) \cdot m, 0 \leq t \leq 1$, such that $c(0) = m_1, c(1) = m_2$. We say that $C_{m_1,m_2}$ is the collection of *connecting curves* for the pair $m_1, m_2$. Let $I \subset \mathbb{R}$ be any interval. If $c(t), t \in I$, is a curve, we denote by $c(I) \subset M$ the set $\{c(t) : t \in I\}$. A *finite set* $B \subset M \setminus \{m_1, m_2\}$ is a *blocking set* for the pair $m_1, m_2$ if for any curve $c$ in $C_{m_1,m_2}$ we have $c([0, 1]) \cap B \neq \emptyset$. If a blocking set exists, the pair $m_1, m_2$ is *connection blockable*, or simply *blockable*. We also say that $m_1$ is *blockable* (resp. *not blockable*) *away* from $m_2$. The analogy with Riemannian security [12, 25, 21, 4] suggests the following:

**Definition 2.11.** Let $M = G/\Gamma$ be a lattice quotient.

i) $M$ is *connection blockable* if every pair of its points is blockable. If there exists at least one non-blockable pair of points in $M$, then $M$ is non-blockable.

ii) $M$ is *uniformly blockable* if there exists $N \in \mathbb{N}$ such that every pair of its points can be blocked with a set $B$ of cardinality at most $N$. The smallest such $N$ is the blocking number for $M$.

iii) A pair $m_1, m_2 \in M$ is *midpoint blockable* if the set $\{c(1/2) : c \in C_{m_1,m_2}\}$ is finite. A lattice quotient is midpoint blockable if all pairs of its points are midpoints blockable.

iv) A lattice quotient is *totally non-blockable* if no pair of its points is blockable.

Blocking property of lattice quotients carries some straightforward and expected properties which can be summarized in the following proposition.

**Proposition 2.12.** *Let $M = G/\Gamma$ where $\Gamma \subset G$ is a lattice, and let $m_0 = \Gamma$ be the identity element of $M$. Then the following holds:*

- *i) The lattice quotient $M$ is blockable (resp. uniformly blockable, midpoint blockable) if and only if all pairs $m_0, m$ are blockable (resp. uniformly blockable,midpoint blockable). The space $M$ is totally non-blockable if and only if no pair $m_0, m$ is blockable;*
- *ii) Let $\tilde{\Gamma} \subset \Gamma$ be lattices in G, let $M = G/\Gamma, \tilde{M} = G/\tilde{\Gamma}$, and let $p : \tilde{M} \to M$ be the covering. Let $m_1, m_2 \in M$ and let $\tilde{m_1}, \tilde{m_2} \in \tilde{M}$ be such that $m_1 = p(\tilde{m_1}), m_2 = p(\tilde{m_2})$. If $B \subset M$ is a blocking set for $m_1, m_2$ (resp. $\tilde{B} \subset \tilde{M}$ is a blocking set for $\tilde{m_1}, \tilde{m_2}$) then $p^{-1}(B)$ (resp. $p(\tilde{B})$ is a blocking set for $\tilde{m_1}, \tilde{m_2}$ (resp. $m_1, m_2$).*
- *iii) Let $G', G''$ be connected Lie groups with lattices $\Gamma' \subset G', \Gamma'' \subset G''$, and let $M' = G'/\Gamma', M'' = G''/\Gamma''$. Set $G = G' \times G''$, $M = M' \times M''$. Then a pair $(m_1', m_1''), (m_2', m_2'') \in M$ is connection blockable if and only if both pairs $m_1', m_2' \in M'$ and $m_1'', m_2'' \in M''$ are connection blockable.*

*Proof.* Claim *i)* is immediate from the definitions. The proofs of claim *ii)* and claim *iii)* are analogous to the proof of their counterparts for riemannian security. See Proposition 1 in Gutkin [18] for claim *ii)*, and Lemma 5.1 and Proposition 5.2 in Burns [5] for claim *iii)*. □

We say lattice quotients $M_1, M_2$ have *identical blocking property* if both are blockable (or not), midpoint blockable (or not), totally non-blockable (or not), etc. Recall that two subgroups $\Gamma_1, \Gamma_2 \subset G$ are *commensurable*, $\Gamma_1 \sim \Gamma_2$, if there exists $g \in G$ such that the group $\Gamma_1 \cap g\Gamma_2 g^{-1}$ has finite index in both $\Gamma_1$ and $g\Gamma_2 g^{-1}$. Commensurability yields an equivalence relation in the set of lattices in $G$. We will use the following immediate Corollary of Proposition 2.12.

**Corollary 2.13.** *i) If lattices $\Gamma_1, \Gamma_2 \subset G$ are commensurable, then the lattice quotients $M_i = G/\Gamma_i$, $i = 1, 2$ have identical blocking properties.*

*ii) Let $M_1 = G_1/\Gamma_1$, $M_2 = G_2/\Gamma_2$ be lattice quotients. Then $M_1 \times M_2 \cong (G_1 \times G_2)/(\Gamma_1 \times \Gamma_2)$ is blockable (resp. midpoint blockable, uniformly blockable) if and only if both $M_1$ and $M_2$ are blockable (resp. midpoint blockable, uniformly blockable).*

Let $\exp : \mathfrak{g} \to G$ be the exponential map. For $\Gamma \subset G$ denote by $p_\Gamma : G \to G/\Gamma$ the projection, and set $\exp_\Gamma = p_\Gamma \circ \exp : \mathfrak{g} \to G/\Gamma$. We will say that a pair $(G, \Gamma)$ is of *exponential type* if the map $\exp_\Gamma$ is surjective. Let $M = G/\Gamma$. For $m \in M$ set $\mathrm{Log}(m) = \exp_\Gamma^{-1}(m)$. Note, $\mathrm{Log}(m)$ may have more than one element. We will use the following basic fact to prove a point is not blockable from identity.

**Proposition 2.14.** *Let $G$ be a Lie group, $\Gamma \subset G$ a lattice such that $(G, \Gamma)$ is of exponential type, and let $M = G/\Gamma$. Then $m \in M$ is blockable away from $m_0$ if and only if there is a map $x \mapsto t_x$ of $Log(m)$ to $(0, 1)$ such that the set $\{\exp(t_x x) : x \in Log(m)\}$ is contained in a finite union of $\Gamma$-cosets.*

*Proof.* Connecting curves are $c_x(t) = \exp(tx)\Gamma/\Gamma$ for some $x \in \mathrm{Log}(m)$. Since $c(1) = m$, there is $\gamma \in \Gamma$ such that $\exp(x) = g\gamma$. Thus

$$c(t) = \exp(t \log(g\gamma)) \cdot m_0 \tag{2.2.1}$$

for some $\gamma \in \Gamma$, and every such curve is connecting $m_0$ with $m$.

Suppose $m$ is blockable away from $m_0$, and let $B \subset G/\Gamma$ be a blocking set. Let $t_x \in (0, 1)$ be such that $c_x(t_x) \in B$. Set $A = \{\exp(t_x x) : x \in \mathrm{Log}(m)\} \subset G$. Then $(A\Gamma/\Gamma) \subset B$, hence finite. Thus $A$ is contained in a finite union of $\Gamma$-cosets.

On the other hand, if for any collection $\{t_x \in (0, 1) : x \in \mathrm{Log}(m)\}$ the set $A = \{\exp(t_x x) : x \in \mathrm{Log}(m)\}$ is contained in a finite union of $\Gamma$-cosets, then $(A\Gamma/\Gamma) \subset M$ is a finite blocking set. □

If $A \subset G$ is any subset, we will say that

$$\mathrm{Sqrt}(A) = \{g \in G : g^2 \in A\} \tag{2.2.2}$$

is the *square root* of $A$. We will say that a pair $(G, \Gamma)$ is of *virtually exponential type* if there exists $\tilde{\Gamma} \sim \Gamma$ such that $(G, \tilde{\Gamma})$ is of exponential type.

**Corollary 2.15.** *Let $\Gamma \subset G$ be a lattice such that $(G, \Gamma)$ is of virtually exponential type. Then:*

*i) The lattice quotient $M = G/\Gamma$ is midpoint blockable if and only if the square root of any coset $g\Gamma$ is contained in a finite union of $\Gamma$-cosets.*

*ii) Any point in $M$ is midpoint blockable away from itself if and only if the square root of $\Gamma$ is contained in a finite union of $\Gamma$-cosets.*

*Proof.* By Corollary 2.13, we can assume that $(G, \Gamma)$ is of exponential type. Set $t_x \equiv 1/2$ in Proposition 2.14. □

The following lemma relates blocking property of a lattice quotient and its closed subspaces. The proof is straightforward and is left to the reader.

**Lemma 2.16.** *Let $G$ be a Lie group, and let $\Gamma \subset G$ be a lattice. Let $H \subset G$ be a closed subgroup such that $\Gamma \cap H$ is a lattice in $H$. Let $X = G/\Gamma, Y = H/(\Gamma \cap H)$ be the lattice quotients, and let $Y \subset X$ be the natural inclusion.*

*i) If $Y$ is not blockable (resp. not midpoint blockable, etc) then $X$ is not blockable (resp. not midpoint blockable, etc).*

*ii) If $Y$ contains a point which is not blockable (resp. not midpoint blockable) away from itself, then no point in $X$ is blockable (resp. not midpoint blockable) away from itself.*

Solvable Lie groups (resp. Lie algebras) and the semisimple Lie groups (resp. Lie algebras) form two large and generally complementary classes. Every connected Lie group $G$ is a semidirect product of a solvable Lie group $R$, and a semisimple Lie group $S$ (Theorem B.4) which is called the Levi decomposition. Connection blocking in lattice quotients of connected Lie groups satisfying *Raghunathan's condition* defined in the following definition, is related to connection blocking of its components quotients.

**Definition 2.17.** Let $G$ be a connected Lie group. The maximum connected closed nilpotent subgroup of $G$ is called the *nilradical* of $G$. Let $G = R \rtimes S$ be the Levi decomposition, where

$R = \operatorname{Rad} G$ is the radical of $G$ and $S$ is semisimple. Let $\sigma$ denote the action of $S$ on $R$. We say $G$ satisfies Raghunathan's condition if the kernel of $\sigma$ has no compact factors in its identity component.

**Proposition 2.18.** *Let $G$ be a connected Lie group which is not semisimple and satisfies Raghunathan's condition and let $\Gamma \subset G$ be a lattice. Let $G = R \rtimes S$ be the Levi decomposition, and $N \leq R$ be the nilradical of $G$. Then*

*i)* *$\Gamma \cap R$ and $\Gamma \cap N$ are lattices in $R$ and $N$, respectively.*

*ii)* *If $R/(\Gamma \cap R)$ is non-blockable, then $G/\Gamma$ is non-blockable.*

*iii)* *If $N$ is not abelian, then $G/\Gamma$ is non-blockable.*

*Proof.* See Raghunathan [34, Corollary 8.28] for the proof of *i)*. Since $R$ is a closed (nontrivial) subgroup of $G$, Lemma 2.16 implies *ii)*. Note that $N/(\Gamma \cap N)$ is a nilmanifold. If $N$ is not abelian, by Proposition 1.6 $N/(\Gamma \cap N)$ is non-blockable. Now *iii)* follows from Lemma 2.16. $\square$

Proposition 2.18 is particularly interesting since it relates connection blocking in lattice quotients of $G$ to its algebraic structure, the structure of its nilradical. In addition, Proposition 2.18 implies that proving Conjecture 1.4 for lattice quotients of solvable and semisimple Lie groups, also proves the conjecture for lattice quotients of all connected Lie groups satisfying Raghunathan's condition.

# CHAPTER 3

# CONNECTION BLOCKING IN QUOTIENTS OF *SOL*

In this chapter we investigate blocking properties in lattice quotients of *Sol*, an important non-nilpotent solvable Lie group and one of the eight homogeneous Thurston 3-geometries. We prove that all lattice quotients of *Sol* are non-blockable.

## 3.1 *Sol* and One Parameter Subgroups

In this section we derive an explicit formula for one parameter subgroups in $Sol$, which is essential to study its blocking properties.

**Definition 3.1.** By *Sol* we mean the Lie group $\mathbb{R}^2 \rtimes \mathbb{R}$ where $z \in \mathbb{R}$ acts on $\mathbb{R}^2$ as

$$\begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix},$$

so as multiplication is given by $(x_1, y_1, z_1)(x_2, y_2, z_2) = (x_1 + e^{z_1} x_2, y_1 + e^{-z_1} y_2, z_1 + z_2)$, together with a left invariant Riemannian metric $ds^2 = e^{-2z} dx^2 + e^{2z} dy^2 + dz^2$.

Consider the three curves $\mathbb{R} \to Sol$ given by $\gamma_1 : t \mapsto (t, 0, 0)$, $\gamma_2 : t \mapsto (0, t, 0)$ and $\gamma_3 : t \mapsto (0, 0, t)$. These have tangent vectors

$$\frac{\partial \gamma_1}{\partial t} = \frac{\partial}{\partial x}, \quad \frac{\partial \gamma_2}{\partial t} = \frac{\partial}{\partial y}, \quad \frac{\partial \gamma_3}{\partial t} = \frac{\partial}{\partial z}$$

at $(0, 0, 0)$, respectively, and these vectors span the tangent space at that point. The left action of the group on these vectors gives a collection of three invariant vector fields $X_1, X_2$ and $X_3$ which form a basis for the tangent space at each point. Since $(x, y, z)\gamma_1 \mapsto (x + e^z t, y, z)$, $(x, y, z)\gamma_2 \mapsto (x, y + e^{-z} t, z)$, and $(x, y, z)\gamma_3 \mapsto (x, y, z + t)$, it follows that

$$X_1(x, y, z) = \frac{\partial}{\partial t}(x, y, z)\gamma_1 \bigg|_{t=0} = e^z \frac{\partial}{\partial x}, \quad X_2(x, y, z) = \frac{\partial}{\partial t}(x, y, z)\gamma_2 \bigg|_{t=0} = -e^{-z} \frac{\partial}{\partial y},$$

and

$$X_3(x,y,z) = \frac{\partial}{\partial t}(x,y,z)\gamma_3\bigg|_{t=0} = \frac{\partial}{\partial z}\,.$$

We construct the metric to be orthogonal at every point with respect to these vector fields. Thus

$$\begin{aligned}
(\frac{\partial}{\partial x}\bigg|_{(x,y,z)}, \frac{\partial}{\partial x}\bigg|_{(x,y,z)}) &= (e^{-z}X_1(x,y,z), e^{-z}X_1(x,y,z)) = e^{-2z}\,,\\
(\frac{\partial}{\partial y}\bigg|_{(x,y,z)}, \frac{\partial}{\partial y}\bigg|_{(x,y,z)}) &= (-e^{z}X_2(x,y,z), -e^{z}X_2(x,y,z)) = e^{2z}\,,\\
(\frac{\partial}{\partial z}\bigg|_{(x,y,z)}, \frac{\partial}{\partial z}\bigg|_{(x,y,z)}) &= (X_3(x,y,z), X_3(x,y,z)) = 1\,,
\end{aligned}$$

and so we obtain the metric given above.

Let $\mathfrak{sol}$ denote the Lie algebra of left invariant vector fields in $Sol$, together with the basis $X_1, X_2, X_3$ as above. We have the following proposition:

**Proposition 3.2.** *The exponential map of Sol is given as the following: Given any vector* $X = a_1X_1 + a_2X_2 + a_3X_3 \in \mathfrak{sol}$,

$$\exp(tX) = \left(\frac{a_1}{a_3}(e^{a_3t}-1), \frac{a_2}{a_3}(e^{-a_3t}-1), a_3t\right),$$

*if* $a_3 \neq 0$*. If* $a_3 = 0$, $\exp(tX) = (a_1t, -a_2t, 0)$.

*Proof.* Let $\gamma(t) = (x(t), y(t), z(t))$ be the integral curve to $X$ so that

$$\gamma'(t) = X(t) = a_1e^{z}\frac{\partial}{\partial x} - a_2e^{-z}\frac{\partial}{\partial y} + a_3\frac{\partial}{\partial z}\,.$$

This leads to the first order system $x'(t) = a_1e^{z(t)}$, $y'(t) = -a_2e^{-z(t)}$, $z'(t) = a_3$, $\gamma(0) = (0,0,0)$ which can be easily solved giving the exponential formula. □

**Remark.** For every $g \in Sol$, the exponential map formula shows that the equation $\exp(X) = g$ has a unique solution. Let $g^t = \exp(t\log g)$ be the unique one parameter subgroup joining identity and $g$. A direct computation gives us the following corollary.

**Corollary 3.3.** *If $g = (x, y, z) \in Sol$ and $z \neq 0$,*

$$g^t = \left(\frac{x}{e^z - 1}(e^{tz} - 1), \frac{y}{e^{-z} - 1}(e^{-tz} - 1), tz\right) .$$

*If $g = (x, y, 0)$, $g^t = (tx, -ty, 0)$.*

## 3.2 Lattices in $Sol$

A complete classification of lattices in $Sol$ is presented in Molnár [30]. In this paper, $Sol$ lattices are classified in an algorithmic way into 17 different types, but infinitely many $Sol$ affine equavalence classes, in each type. For the purpose of the blocking problem, we consider a class of lattices constructed by the following proposition. We then prove, every lattice in $Sol$ is conjugate to a lattice in this class.

**Proposition 3.4.** *Let $A \in SL(2, \mathbb{Z})$. Suppose that $A$ is conjugate in $GL(2, \mathbb{R})$ to a matrix of the form*

$$\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$$

*for some positive $\lambda \neq 1$. Then there is a monomorphism $\mathbb{Z}^2 \rtimes_A \mathbb{Z} \hookrightarrow Sol$ and the image is a lattice.*

Note that by $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ we mean the semidirect product where $r \in \mathbb{Z}$ acts on $\mathbb{Z}^2$ as $A^r$ so as the multiplication is given by $(p_1, q_1, r_1)(p_2, q_2, r_2) = ((p_1, q_1) + A^{r_1}(p_2, q_2), r_1 + r_2)$.

*Proof.* By assumption there exists $P \in \mathrm{GL}(2, \mathbb{R})$, and $s \in \mathbb{R} \setminus \{0\}$ such that $\lambda = e^s$, and

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} .$$

Define the embedding by $(p,q,r) \mapsto (P(p,q), sr)$ and note that since $s \neq 0$, and $P$ is nonsingular this is an injection. The following calculation demonstrate that this gives a homomorphism:

$$\begin{aligned}(p_1,q_1,r_1)(p_2,q_2,r_2) &= ((p_1,q_1) + A^{r_1}(p_2,q_2), r_1 + r_2) \\ &\mapsto \big(P(p_1,q_1) + PA^{r_1}(p_2,q_2), s(r_1+r_2)\big) \\ &= (P(p_1,q_1) + \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix}^{r_1} P(p_2,q_2), sr_1 + sr_2) \\ &= (P(p_1,q_1) + \begin{pmatrix} e^{sr_1} & 0 \\ 0 & e^{-sr_1} \end{pmatrix} P(p_2,q_2), sr_1 + sr_2) \\ &= (P(p_1,q_1), sr_1)\,(P(p_2,q_2), sr_2)\ .\end{aligned}$$

The quotient of $Sol$ by $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is a $\mathbb{T}^2$ bundle over $S^1$ so is compact. Thus $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ is indeed a lattice in $Sol$. We now show that the action of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ on $Sol$ is proper. Let $g = (X,Y,Z) \in Sol$ and let $\gamma = (p,q,r) \in \mathbb{Z}^2 \rtimes_A \mathbb{Z} \setminus \{1\}$. Then $\gamma g = (P(p,q) + (e^r X, e^{-r} Y), sr + Z)$. If $r \neq 0$ then $d(g,\gamma g) \geq |s| \geq 0$. If $r = 0$ then $\gamma g = (P(p,q) + (X,Y), Z)$ and both $g$ and $\gamma g$ lie in the same horizontal plane $z = Z$ on which the metric restricts to $ds^2 = e^{-2Z}dx^2 + e^{2Z}dy^2 + dz^2$. In this case let $\mu = \min\{e^{-2Z}, e^{2Z}\} > 0$ and let $K = \inf_{\|(x,y)\|_2 = 1} \|P(x,y)\|_2 > 0$. Then $d(g,\gamma g) \geq \mu K \|(p,q)\|_2$ and since $\gamma \neq 1$, $(p,q) \neq (0,0)$ so $d(g,\gamma g) \geq \mu K$. We have thus shown that for all $\gamma \in \mathbb{Z}^2 \rtimes_A \mathbb{Z}$ with $\gamma \neq 1$, $d(g,\gamma g) \geq \min\{s, \mu K\} > 0$. Hence the action of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ on $Sol$ is proper. □

$Sol$ multiplication can be projectively interpreted by "left translations" on its points as $L_\tau : (x,y,z) \mapsto \tau(x,y,z)$, $\tau \in Sol$. Let $L(T)$ denote the set of left translations on $Sol$ and assume $\Gamma < L(T)$ is a subgroup, generated by three independent translations $\tau_1 = (x_1,y_1,z_1), \tau_2 = (x_2,y_2,z_2), \tau_3 = (x_3,y_3,z_3)$ with non-commutative addition, or in this case $\mathbb{Z}$ linear combinations.

**Notation.** Let $\tau_1$ and $\tau_2$ be left translations. $[\tau_1,\tau_2] := \tau_1^{-1}\tau_2^{-1}\tau_1\tau_2$ denotes the commutator. For matrices $A, P \in \mathrm{GL}(2,\mathbb{R})$, $A^P := P^{-1}AP$.

The concept of a lattice can be rephrased as a subgroup of left translations. The theorem below clarifies the algebraic structure of lattices in $Sol$ (see [30]).

**Theorem 3.5.** *For each lattice* $\Gamma$ *of Sol there exists* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ *with* $tr(A) > 2$*, and* $\tau_i = (x_i, y_i, z_i)$, $i = 1, 2, 3$ *such that:*

*i)* $\Gamma$ *has a group presentation*

$$\Gamma = \Gamma(A) = \langle \tau_1, \tau_2, \tau_3 : [\tau_1, \tau_2] = 1, \tau_3^{-1}\tau_1\tau_3 = \tau_1 A^P, \tau_3^{-1}\tau_2\tau_3 = \tau_2 A^P \rangle,$$

*ii)* $\tau_1 = (x_1, y_1, z_1), \tau_2 = (x_2, y_2, z_2)$ *satisfy the equalities* $z_1 = z_2 = 0$*, and the matrix* $P = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \in GL(2, \mathbb{R})$ *satisfies:*

$$A^P = P^{-1}AP = \begin{pmatrix} e^{z_3} & 0 \\ 0 & e^{-z_3} \end{pmatrix}.$$

**Remark.** Molnár's definition of $Sol$ multiplication is slightly different from our definition. In his paper he defines the lattices as a subgroup of right translations of $Sol$. As a result, the statement of Theorem 3.5 has been readjusted accordingly.

Using notations of Proposition 3.4 and Theorem 3.5, it's easy to see that lattices of Proposition 3.4, correspond to $\Gamma(A) = \langle \tau_1, \tau_2, \tau_3 \rangle$ where, $x_3 = y_3 = 0$. Then $e^{z_3}, e^{-z_3}$ are eigenvalues of $A$ and $P$ is the eigenvector matrix of $A$. Now pairing Proposition 3.4 and Theorem 3.5, we conclude the following proposition which will be used later to study blocking property of all quotients of $Sol$.

**Proposition 3.6.** *Every lattice of Sol is conjugate to a semidirect product lattice presented by Proposition 3.4.*

*Proof.* Given a lattice $\Gamma = \Gamma(A) = \langle \tau_1, \tau_2, \tau_3 \rangle$ as in Theorem 3.5, let $\Gamma_0 = \Gamma_0(A) = \langle \tau_1, \tau_2, \tau_3' = (0, 0, z_3) \rangle$, $g = \left( \frac{x_3}{e^{z_3}-1}, \frac{y_3}{e^{-z_3}-1}, 0 \right)$, $\phi_g \in \text{Aut}(Sol) : (x, y, z) \mapsto g^{-1}(x, y, z)g$. Since $g$ commutes with $\tau_1, \tau_2$, $\phi_g(\tau_1) = \tau_1, \phi_g(\tau_2) = \tau_2$. In addition, we compute:

$$\begin{aligned}
\phi_g(\tau_3') = g^{-1}\tau_3' g &= \left(-\frac{x_3}{e^{z_3}-1}, -\frac{y_3}{e^{-z_3}-1}, 0\right)(0,0,z_3)\left(\frac{x_3}{e^{z_3}-1}, \frac{y_3}{e^{-z_3}-1}, 0\right) \\
&= \left(-\frac{x_3}{e^{z_3}-1}, -\frac{y_3}{e^{-z_3}-1}, z_3\right)\left(\frac{x_3}{e^{z_3}-1}, \frac{y_3}{e^{-z_3}-1}, 0\right) \\
&= (x_3, y_3, z_3) = \tau_3\,.
\end{aligned}$$

Hence $\phi_g(\Gamma_0) = \Gamma$. □

## 3.3 Blocking Property of *Sol* Quotient Spaces

This section concludes with the proof of the main theorems. We first need a few technical lemmas that will be applied in the body of the proofs.

**Lemma 3.7.** *For an integer $n > 2$, $n^2 - 4$ is never a perfect square.*

*Proof.* By contrary suppose there exist a positive integer $k$ such that $n^2 - 4 = k^2$, so that $(n-k)(n+k) = 4$. Noting $0 < n - k < n + k$, it follows that $n - k = 1$ and $n + k = 4$, and thus $n = 5/2$ contradicting the assumption. □

**Lemma 3.8.** *Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ with eigenvalues $\lambda = e^s \neq 1, \lambda^{-1}$. Then $\lambda \notin \mathbb{Q}$. The matrix*

$$P = \begin{pmatrix} 1 & -\frac{1}{c}(e^{-s} - d) \\ -\frac{1}{b}(e^s - a) & 1 \end{pmatrix}$$

*is invertible, and*

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

*Given such a matrix P,*

$$(P(p,q), sr) = (p - \frac{1}{c}(e^{-s} - d)q, q - \frac{1}{b}(e^s - a)p, sr)\,.$$

*Proof.* Since $\lambda + \lambda^{-1} = \mathrm{tr}(A)$, solving the quadratic equation for $\lambda$ it follows that $\lambda = \mathrm{tr}(A)/2 \pm \sqrt{\mathrm{tr}(A)^2 - 4}/2$. Note that $\mathrm{tr}(A) \in \mathbb{Z}$ and $\mathrm{tr}(A) = (e^s + e^{-s}) > 2$. Now Lemma 3.7 implies $\mathrm{tr}(A)^2 - 4$ is not a perfect square, so it's irrational. Let $\boldsymbol{v_1}, \boldsymbol{v_2}$ be the eigenvectors associated to $\lambda, \lambda^{-1}$, so that the first component of $\boldsymbol{v_1}$ and the second component of $\boldsymbol{v_2}$ are equal to 1, respectively, and assume $\tilde{P} = [v_1, v_2]$. A direct computation shows that:

$$\boldsymbol{v_1} = \begin{pmatrix} 1 \\ \frac{1}{b}(e^s - a) \end{pmatrix}, \quad \boldsymbol{v_2} = \begin{pmatrix} \frac{1}{c}(e^{-s} - d) \\ 1 \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} 1 & \frac{1}{c}(e^{-s} - d) \\ \frac{1}{b}(e^s - a) & 1 \end{pmatrix}.$$

$\tilde{P}$ is the eigenvector matrix, so it's invertible and

$$\tilde{P}^{-1} A \tilde{P} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}. \tag{3.3.1}$$

Let $P = (\tilde{P}/\det(\tilde{P}))^{-1}$. Following (3.3.1) it's easy to see that

$$PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Using the common formula to find inverse of the $(2\times2)$ matrix $\tilde{P}/\det(\tilde{P})$ and noting $\det(\tilde{P}/\det(\tilde{P})) = (\det(\tilde{P}))^{-1}$, if follows that:

$$P = \frac{1}{\det(\tilde{P})^{-1}} \begin{pmatrix} \det(\tilde{P})^{-1} & -\det(\tilde{P})^{-1}\frac{1}{c}(e^{-s} - d) \\ -\det(\tilde{P})^{-1}\frac{1}{b}(e^s - a) & \det(\tilde{P})^{-1} \end{pmatrix}.$$

The last statement of the Lemma follows from direct computation. □

Next we prove the following lemma.

**Lemma 3.9.** *Let $A$ be conjugate to its eigenvalue matrix via matrices $P_1, P_2 \in GL(2, \mathbb{R})$ as in the statement of Proposition 3.4, and $\Gamma_i$, $i = 1, 2$ be the two associated lattices in Sol, i.e. images of the embeddings $(p, q, r) \mapsto (P_i(p, q), sr)$. Then $B = P_2 P_1^{-1}$ is diagonal and the mapping $\phi : Sol \to Sol$, $(x, y, z) \mapsto (B(x, y), z)$ is a Lie group automorphism. In addition $\phi(\Gamma_1) = \Gamma_2$, hence quotient spaces $Sol/\Gamma_1$ and $Sol/\Gamma_2$ have identical blocking property, i.e. $m = g\Gamma_1$ is blockable from the identity $m_1^0 = \Gamma_1$ if and only if $\phi(g)\Gamma_2$ is blockable from $m_2^0 = \Gamma_2$.*

*Proof.* Since both $P_1^{-1}, P_2^{-1}$ are eigenvector matrices of $A$, there exists a diagonal and invertible matrix $B$ such that $P_1^{-1} = P_2^{-1}B$, and thus $B = P_2 P_1^{-1}$. Since $B$ is nonsingular, $\phi$ is a diffeomorphism on $Sol$, and it's clear from the definition $\phi(\Gamma_1) = \Gamma_2$. The following calculation demonstrates that the mapping is a homomorphism:

$$\begin{aligned}
(x_1, y_1, z_1)(x_2, y_2, z_2) &= ((x_1, y_1) + \begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{pmatrix}(x_2, y_2), z_1 + z_2) \\
&\mapsto (B(x_1, y_1) + B\begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{pmatrix}(x_2, y_2), z_1 + z_2) \\
&= (B(x_1, y_1) + \begin{pmatrix} e^{z_1} & 0 \\ 0 & e^{-z_1} \end{pmatrix} B(x_2, y_2), z_1 + z_2) \\
&= (B(x_1, y_1), z_1)(B(x_2, y_2), z_2)\,.
\end{aligned}$$

Since Lie group isomorphisms map one parameter subgroups to one parameter subgroups, the last statement follows immediately. $\square$

Now we are ready to prove the main theorems.

*Proof of Theorem 1.7.* Assume that matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{Z})$ with eigenvalues $\lambda \neq 1, \lambda^{-1}$ is given, and $P \in \mathrm{GL}(2, \mathbb{R})$ is such that $PAP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ and $P_{11} = P_{22} = 1$ (Note that since switching $A \leftrightarrow -A$ doesn't change $P$ we may assume $\lambda > 0$ and since $\lambda \neq 1$, $\mathrm{tr}(A) > 2$). Let $\lambda = e^s, s \neq 0$, $\Gamma$ be the image lattice of $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ through the embedding in Proposition 3.4, $g = (0, y, z) \in Sol$, $y \neq 0$, and $m = g\Gamma$. We prove that $m$ is not blockable from the identity $m_0 = \Gamma$. Changing the representative $g$ for $m = g\Gamma$ if necessary, we may assume $z \neq 0$. To the contrary, assume that $m$ is blockable from identity $m_0$. Let $r_i$ be a sequence of integers, so that $sr_i$ is strictly increasing and, $sr_i \to \infty$, as $i \to \infty$, and let $\gamma_i = (0, 0, sr_i) \in \Gamma$. By Proposition 2.14, for a suitable choice of $t_i$'s where $0 < t_i < 1$, there exist $\tilde{g_1}, \cdots, \tilde{g_n} \in Sol$ such that $\{(g\gamma_i)^{t_i}\} \subset \cup_{n=1}^{N} \tilde{g_n}\Gamma$; passing to a subsequence if necessary, we may assume there exists a fixed $\tilde{g} \in Sol$ such that for each $i$, $(g\gamma_i)^{t_i} \in \tilde{g}\Gamma$. In addition we may assume $\tilde{g} = (g\gamma_1)^{t_1}$. In particular, there exist $\tilde{y}, \tilde{z} \in \mathbb{R}$ such that $\tilde{g} = (g\gamma_1)^{t_1} = (0, \tilde{y}, \tilde{z})$.

Since $g\gamma_i = (0, y, z + sr_i)$, Corollary 3.3 implies

$$(g\gamma_i)^{t_i} = \left(0, \frac{y}{e^{-z-sr_i} - 1}(e^{-t_i(z+sr_i)} - 1), t_i(z + sr_i)\right) . \tag{3.3.2}$$

In particular, for each $i, j$ the first component of $[(g\gamma_i)^{t_i}]^{-1} \cdot [(g\gamma_j)^{t_j}]$ is 0. As $[(g\gamma_i)^{t_i}]^{-1} \cdot [(g\gamma_j)^{t_j}] \in \Gamma$, there exist $\tilde{p}, \tilde{q}, \tilde{r} \in \mathbb{Z}$ so that $[(g\gamma_i)^{t_i}]^{-1} \cdot [(g\gamma_j)^{t_j}] = (\tilde{p} - \frac{1}{c}(e^{-s} - d)\tilde{q}, \tilde{q} - \frac{1}{b}(e^s - a)\tilde{p}, s\tilde{r})$. Hence $\tilde{p} - \frac{1}{c}(e^{-s} - d)\tilde{q} = 0$, and since $e^s = \lambda \notin \mathbb{Q}$, we must have $\tilde{p} = \tilde{q} = 0$. Therefore it follows that

$$[(g\gamma_i)^{t_i}]^{-1} \cdot [(g\gamma_j)^{t_j}] = (0, 0, s\tilde{r_{ij}}),\ \tilde{r_{ij}} \in \mathbb{Z}, \tag{3.3.3}$$

letting $i = 1$, we conclude that

$$(g\gamma_j)^{t_j} = (g\gamma_1)^{t_1}(0, 0, s\tilde{r_j}) = \tilde{g}(0, 0, s\tilde{r_j}) = (0, \tilde{y}, \tilde{z} + s\tilde{r_j}) \tag{3.3.4}$$

which means $\{(g\gamma_i)^{t_i}\} \cap \tilde{g}\Gamma$ lies on a vertical line in $y$-$z$ plane. In addition, comparing the third component of both sides in equation 3.3.3 implies

$$t_j(z + sr_j) - t_i(z + sr_i) = s\tilde{r_{ij}},\ r_{ij} \in \mathbb{Z}\,. \tag{3.3.5}$$

Solving for $t_i$ using the second components of equations 3.3.2 and 3.3.4 it follows that

$$t_i = -\frac{1}{z + sr_i} \ln\left(\frac{\tilde{y}}{y}(e^{-z-sr_i} - 1) + 1\right); \tag{3.3.6}$$

Since for each $i$, $0 < t_i < 1$, letting $i \to \infty$ shows that $\tilde{y}/y$ has to be positive. Now plugging the formula for $t_i$ and $t_j$ in equation 3.3.5 gives us:

$$\frac{e^{-z-sr_i} - 1 + y/\tilde{y}}{e^{-z-sr_j} - 1 + y/\tilde{y}} = e^{s\tilde{r_{ij}}} \tag{3.3.7}$$

Setting, $j = i + 1$, $i \to \infty$, the left side of the above equation goes to 1. So, for large enough $i, j = i + 1$, $\tilde{r_{ij}} = 0$, and so $r_i = r_{i+1}$ which is a contradiction.

Now, if $g = (x, 0, z)$, $x \neq 0$, replace $(0, 0, sr_i)$ with $(0, 0, -sr_i)$; repeating a similar argument on the first component of $(g\gamma_i)^{t_i}$, proves that $m = (x, 0, z)\Gamma$ is also not blockable from $m_0$. $\square$

Knowing all lattices of $Sol$ are conjugate to semidirect products, we are ready to prove the second theorem.

*Proof of Theorem 1.8.* By Proposition 3.6 and Lemma 3.9 it suffices to prove the theorem for a lattice $\Gamma$ presented in Theorem 1.7. From Theorem 1.7, all cosets in $X = \{(0, y, z)\Gamma \mid y, z \in \mathbb{R}, y \neq 0\}$ are non-blockable from the identity. We show that the group elements in $X\Gamma$ is dense in $Sol$, which implies $X$ is dense in $Sol/\Gamma$.

Fix $g = (x_0, y_0, z_0) \in Sol$, and assume $\epsilon > 0$ is given. Since $e^s$ is not rational,

$$\{(e^{z_0}(p - \frac{1}{c}(e^{-s} - d)q) \mid p, q \in \mathbb{Z}\}$$

is dense in $\mathbb{R}$. Let $p_1, q_1$ be such that

$$|(e^{z_0}(p_1 - \frac{1}{c}(e^{-s} - d)q_1) - x_0| < \epsilon/2,$$

moreover choose a non-zero real number $y_1$ such that

$$|y_1 + e^{-z_0}(q_1 - \frac{1}{b}(e^s - a)p_1) - y_0| < \epsilon/2\,.$$

Let $g_1 = (0, y_1, z_0) \in Sol$ and $\gamma_1 = (P(p_1, q_1), 0) \in \Gamma$. Then

$$g_1\gamma_1 = ((e^{z_0}(p_1 - \frac{1}{c}(e^{-s} - d)q_1), y_1 + e^{-z_0}(q_1 - \frac{1}{b}(e^s - a)p_1), z_0) \in X\Gamma,$$

and the above argument shows the Euclidean distance $d(g_1\gamma_1, g) < \epsilon$, and hence $X\Gamma$ is dense in $Sol$. Thus $X$ is a dense subset of $Sol/\Gamma$, not blockable from the identity $m_0$, which implies the statement of Theorem 1.8. $\square$

## CHAPTER 4

## CONNECTION BLOCKING IN SEMISIMPLE LATTICE QUOTIENTS

In this chapter we study connection blocking in lattice quotients of $\mathrm{SL}(n,\mathbb{R})$. We start with quotients of $\mathrm{SL}(2,\mathbb{R})$, and then extend the result to quotients of $\mathrm{SL}(n,\mathbb{R})$, $n > 2$. Since by Corollary 2.13 direct product of lattice quotients carries the same blocking property as its components, the results of the current chapter can be extended to lattice quotients of a large class of semisimple Lie groups.

### 4.1 One Parameter Families of $\mathrm{SL}(2,\mathbb{R})$ and Modified Times

In this section we derive an explicit formula for one parameter families in $\mathrm{SL}(2,\mathbb{R})$, which is essential to study its blocking properties.

The exponential map for $\mathrm{SL}(2,\mathbb{R})$ can be formulated in terms of trigonometric functions. The formula is directly derived from the exponential power series $\exp(X) = \sum_{k=0}^{\infty} X^k/k!$, doing some matrix algebra. For details see Rossmann [35, pp. 17-19]. We have the following proposition:

**Proposition 4.1.** *Let $g_0$ denote the identity element of $SL(2,\mathbb{R})$. For a given matrix*

$$X = \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{R}), \tag{4.1.1}$$

*if $a^2+b > 0$, then let $\omega(X) = \sqrt{a^2+bc} > 0$ and if $a^2+b < 0$, then let $\omega(X) = \sqrt{-(a^2+bc)} > 0$. In the first case we have*

$$exp(X) = (\cosh\omega)\, g_0 + \left(\frac{\sinh\omega}{\omega}\right) X, \tag{4.1.2}$$

*and in the second case $(a^2+bc < 0)$, we have*

$$exp(X) = (\cos\omega)\, g_0 + \left(\frac{\sin\omega}{\omega}\right) X. \tag{4.1.3}$$

*If $a^2+bc = 0$, then $exp(X) = g_0 + X$. Furthermore, every matrix $g \in SL(2,\mathbb{R})$ whose trace satisfies $tr(g) \geq -2$ is in the image of the exponential map. Consequently, for any $g \in SL(2,\mathbb{R})$, either $g$ or*

*$-g$ is of the form exp$(X)$, for some $X \in \mathfrak{sl}(2,\mathbb{R})$. Therefore, $(SL(2,\mathbb{R}), SL(2,\mathbb{Z}))$ is of exponential type.*

For $g \in \mathrm{SL}(2,\mathbb{R})$ with $\mathrm{tr}(g) \geq 2$, $\log(g)$ is unique and we use the notations $\omega_g = \omega(\log(g))$, $g^t = \exp(t \log g)$, $0 \leq t \leq 1$. We have the following lemma:

**Lemma 4.2.** *If $tr(g) \geq 2$, we have*

$$g^t = \left(\cosh(t\omega_g) - \frac{\sinh t\omega_g}{\sinh \omega_g} \cosh \omega_g\right) g_0 + \frac{\sinh t\omega_g}{\sinh \omega_g} g, \tag{4.1.4}$$

*where $g_0$ is the identity element of $SL(2,\mathbb{R})$.*

*Proof.* From (4.1.2) it follows that

$$\log g = \frac{\omega_g}{\sinh \omega_g} \left(g - \cosh(\omega_g) g_0\right) \tag{4.1.5}$$

Noting that $\omega(t\log(g)) = t\omega(\log(g))$, substituting (4.1.5) in the equation

$$g^t = \exp(t \log g) = \cosh(t\omega_g) g_0 + \frac{\sinh t\omega_g}{t\omega_g}(t \log g)$$

gives the desired formula.

□

**Definition 4.3.** For a fixed $g \in \mathrm{SL}(2,\mathbb{R})$ and an arbitrary $\gamma \in \Gamma$ . We use the notation

$$\lambda_\gamma = \frac{\sinh(t\omega_{g\gamma})}{\sinh(\omega_{g\gamma})},\ 0 \leq \lambda_\gamma \leq 1\,.$$

We call $\lambda_\gamma$ the modified time associated with $\gamma$.

Let

$$a(\lambda_\gamma) = \left(1 + \left(\mathrm{tr}(g\gamma)^2/4 - 1\right) \lambda_\gamma^2\right)^{1/2}\,. \tag{4.1.6}$$

From (4.1.2) we have $\cosh \omega_\gamma = \mathrm{tr}(g\gamma)/2$; a direct computation from (4.1.5) gives the following formula

$$(g\gamma)^t = \left[a(\lambda_\gamma) - 1/2\mathrm{tr}(g\gamma)\lambda_\gamma\right] g_0 + \lambda_\gamma g\gamma \,. \tag{4.1.7}$$

Modified time as defined in above, will be pivotal for the proof of the main theorem.

**Notation.** While working with a sequence $\{\gamma_i\} \in \Gamma$, by $\lambda_i, a(\lambda_i)$ we mean $\lambda_{\gamma_i}, a(\lambda_{\gamma_i})$.

## 4.2 Blocking Properties of $M_n$

This section concludes with the proof of Theorem 1.9. The proof will be based on the technical Proposition 4.8, which is the main body of this section.

Throughout the section, $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, $M_2 = \mathrm{SL}(2,\mathbb{R})/\Gamma$. We assume:

$$g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Q}),\ \{\gamma_i\} \subset \Gamma,\ g\gamma_i = \begin{pmatrix} x_i & y_i \\ z_i & w_i \end{pmatrix}.$$

Moreover, since $g$ and $-g$ have identical blocking properties, we may assume $x > 0$.

In order to prove Proposition 4.8, we first need a few Lemmas.

**Lemma 4.4.** *Suppose that $R(\alpha,\beta) \in \mathbb{R}[\alpha,\beta]$ has the form $R(\alpha,\beta) = c\alpha^n + P(\alpha,\beta)$ for some $n > 0$, $c \neq 0$, and polynomial $P(\alpha,\beta)$ of degree of at most $n-1$ in $\alpha$. Then given any sequence of positive real numbers $\{\beta_i\}$ such that $\beta_i \to \infty$, as $i \to \infty$, there exists an increasing function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that $R(\beta_{f(i)}, \beta_{f(j)}) \neq 0$, $\forall i > j$.*

*Proof.* Define $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ inductively as follows. Set $f(1) = 1$, and assuming $f(k)$ is defined, define $f(k+1)$ in the following way. The $k$-polynomials $R_1(\alpha) = R(\alpha, \beta_{f(1)}), \cdots, R_k(\alpha) = R(\alpha, \beta_{f(k)})$ are all degree $n$ in $\alpha$. Choose $l$ large enough so that $R_1(\beta_l), \cdots, R_k(\beta_l) \neq 0$, and define $f(k+1) = l$. □

**Lemma 4.5.** *For a given element $g \in SL(2,\mathbb{R})$:*

*i) Every five elements of coset $g\Gamma$ are $\mathbb{Z}$ linearly dependent.*

*ii) Let $g\gamma_1, \cdots, g\gamma_n$, $n \leq 4$, be $\mathbb{Z}$(or $\mathbb{Q}$)-linearly independent elements of $g\Gamma$. Then there exists a non-zero integer $m_0$ such that for every $g\gamma \in span_{\mathbb{Q}} < g\gamma_1, \cdots, g\gamma_n >$, there exists $(m_1, \cdots, m_n) \in \mathbb{Z}^n$ so that $\sum_{i=1}^n m_i(g\gamma_i) = m_0(g\gamma)$.*

*Proof.* To prove the first part note that $g\gamma \in \mathrm{span}_{\mathbb{Q}} < g\gamma_1, \cdots, g\gamma_n >$ if and only if $\gamma \in \mathrm{span}_{\mathbb{Q}} < \gamma_1, \cdots, \gamma_n >$; therefore we may assume $g\Gamma = \Gamma$. Considering $\Gamma$ as a subset of $\mathbb{Q}$-vector space $\mathbb{Q}^4$, immediately implies every five elements of it are $\mathbb{Q}$ (and therefore $\mathbb{Z}$)-linearly dependent.

Now we prove part *ii)* of the Lemma. First a conventional notation; For

$$\gamma = \begin{pmatrix} \gamma^1 & \gamma^2 \\ \gamma^3 & \gamma^4 \end{pmatrix} \in \Gamma,$$

define $[\gamma] = (\gamma^1, \gamma^2, \gamma^3, \gamma^4)^T$; moreover we use the notation $[a] = (a_1, \cdots, a_n)^T$, to denote an arbitrary element of $\mathbb{R}^n$ as a $n \times 1$ matrix. Let $A = ([\gamma_1] \cdots [\gamma_n])$. Note that $A$ is a $4 \times n$ matrix of rank $n$, thus there exists an invertible $n \times n$ submatrix $\tilde{A}$ consisting of rows, say, $i_1, \cdots, i_n$. Take an arbitrary element $\gamma \in \mathrm{span}_{\mathbb{Q}} < \gamma_1, \cdots, \gamma_n >$. Since $\tilde{A}^{-1}$ has rational entries, we can choose a fixed integer $m_0$ so that $m_0\tilde{A}^{-1}$ has integer entries. Hence the linear equation $\tilde{A}[m] = m_0(\gamma^{i_1}, \cdots, \gamma^{i_n})^T$ has a solution $[m] = (m_1, \cdots, m_n)^T \in \mathbb{Z}^n$. For $1 \leq j \leq 4$, $j \neq i_1, \cdots, i_n$, let $A_j$ denote the $j$-th row of $A$, and assume $A_j = \sum_{k=1}^n \alpha_{jk} A_{i_k}$. Since $\gamma \in \mathrm{span}_{\mathbb{Q}} < \gamma_1, \cdots, \gamma_n >$, there exists $[r] \in \mathbb{Q}^n$ such that $[\gamma] = A[r]$. It follows that:

$$\gamma^j = A_j[r] = (\sum_{k=1}^n \alpha_{jk} A_{i_k})[r] = \sum_{k=1}^n \alpha_{jk}(A_{i_k}[r]) = \sum_{k=1}^n \alpha_{jk}\gamma^{i_k} .$$

Hence we have:

$$A_j[m] = (\sum_{k=1}^n \alpha_{jk} A_{i_k})[m] = \sum_{k=1}^n \alpha_{jk}(A_{i_k}[m]) = \sum_{k=1}^n \alpha_{jk} m_0 \gamma^{i_k} = m_0\gamma^j .$$

Therefore we conclude $A[m] = m_0[\gamma]$, that implies $m_0\gamma = \sum_{i=1}^n m_i\gamma_i$. □

**Lemma 4.6.** *Every coset $m \in SL(2, \mathbb{Q})/\Gamma$ has a representative of the form*

$$g = \begin{pmatrix} x & 0 \\ z & 1/x \end{pmatrix};$$

*that is* $m = g\Gamma$*, where* $x, z \in \mathbb{Q}$*.*

*Proof.* Let

$$g_1 = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

be an arbitrary representative of coset $m$. If $y \neq 0$, let $s/q = -x/y$, $\gcd(s, q) = 1$ and choose $p, r \in \mathbb{Z}$ so that $ps - rq = 1$, and let

$$\gamma = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

It is clear that $g = g_1\gamma \in m$ and $g$ has the desired form. □

**Lemma 4.7.** *Let* $g \in SL(2, \mathbb{R})$*, and* $\gamma_1, \cdots, \gamma_n \in \Gamma$ *have forms*

$$g = \begin{pmatrix} x & 0 \\ z & 1/x \end{pmatrix}, \ x, z \in \mathbb{R} \quad and \quad \gamma_i = \begin{pmatrix} p_i & 1 \\ r_i & s_i \end{pmatrix}.$$

*Let* $(g\gamma_1)^{t_1}, \cdots, (g\gamma_n)^{t_n}$ *be* $\mathbb{Z}$*-linearly dependent, that is*

$$\sum_{i=1}^{n} m_i (g\gamma_i)^{t_i} = 0, \ m_i \in \mathbb{Z}. \tag{4.2.1}$$

*Then we have* $\sum_{i=1}^{n} m_i\lambda_i = 0$ *and* $\sum_{i=1}^{n} m_i a(\lambda_i) = 0$*.*

*Proof.* By (4.1.7) we have

$$(g\gamma_i)^{t_i} = \begin{pmatrix} a(\lambda_i) + 1/2\lambda_i(x_i - w_i) & \lambda_i y_i \\ \lambda_i z_i & a(\lambda_i) + 1/2\lambda_i(w_i - x_i) \end{pmatrix}.$$

Now (4.2.1) implies

$$\sum_{i=1}^{n} m_i\lambda_i y_i = 0, \tag{4.2.2}$$

$$\sum_{i=1}^{n} m_i a(\lambda_i) + 1/2\lambda_i(x_i - w_i) = 0, \tag{4.2.3}$$

$$\sum_{i=1}^{n} m_i a(\lambda_i) + 1/2\lambda_i(w_i - x_i) = 0\,. \tag{4.2.4}$$

Since $y_1 = \cdots = y_n = x$, (4.2.2) immediately implies $\sum_{i=1}^n m_i\lambda_i = 0$. To obtain $\sum_{i=1}^n m_i a(\lambda_i) = 0$, add (4.2.3) and (4.2.4).

□

**Proposition 4.8.** *Let* $g = \begin{pmatrix} x & 0 \\ z & 1/x \end{pmatrix} \in SL(2,\mathbb{Q})$*, and* $m = g\Gamma$ *be finitely blockable from identity* $m_0$*. Then there exists a sequence*

$$\gamma_i = \begin{pmatrix} p_i & 1 \\ p_i s_i - 1 & s_i \end{pmatrix} \in \Gamma$$

*and a sequence of times* $\{t_i\} \subset (0,1)$ *such that*

*i) all elements of* $\{(g\gamma_i)^{t_i}\}$ *belong to the same coset, and all modified times are the same, i.e.,* $\lambda_i = \lambda = const,$

*ii)* $\lambda_i^2, \lambda_i a(\lambda_i) \in \mathbb{Q}$*,*

*iii)* $C_i = tr(g\gamma_i)$ *is an increasing sequence of positive rational numbers with the same denominator,* $C_i \to \infty$*, as* $i \to \infty$*, and*

*iv)* $\{p_i\}$ *is an increasing sequence of positive integers.*

*Proof.* Let $m = g\Gamma$, $g \in \mathrm{SL}(2,\mathbb{Q})$, be blockable from identity $m_0$. Suppose $x = a/b,\ a, b \in \mathbb{Z}^+$. Let $p_i = 2ib^2$, $s_i = (a - 2a^2)i$, then a direct computation shows $C_i = \mathrm{tr}(g\gamma_i) = z + ib$. It is clear that passing to a subsequence if necessary, we may assume $2 < C_1 < C_2 < \cdots$. Note that $C_i$'s are rational numbers with the same denominator. By Proposition 2.14 for a suitable choice of $t_i$'s where $0 < t_i < 1$ we have $\{(g\gamma_i)^{t_i}\} \subset \cup_{n=1}^N \tilde{g_n}\Gamma$; passing to a subsequence if necessary it follows that there exists a sequence

$$\gamma_i = \begin{pmatrix} p_i & 1 \\ p_i s_i - 1 & s_i \end{pmatrix},\ \gamma_i \in \Gamma$$

such that $tr(g\gamma_i) = C_i = z + n_i b$, $p_i = 2n_i b^2$ where $n_i \in \mathbb{Z}^+$, $n_1 < n_2 < \cdots$ and $(g\gamma_i)^{t_i} \in \tilde{g}\Gamma$ for some fixed $\tilde{g} \in G$. Now let $\lambda_i = \dfrac{\sinh(t_i \omega_{\gamma_i})}{\sinh(\omega_{\gamma_i})}$ be modified times $\lambda_i \in (0,1)$. We show that for every pair of indexes $(i, j)$, $\lambda_i \lambda_j \in \mathbb{Q}$. By (4.1.7) we have

$$(g\gamma_i)^{t_i} = \begin{pmatrix} a(\lambda_i) + 1/2\lambda_i(x_i - w_i) & \lambda_i y_i \\ \lambda_i z_i & a(\lambda_i) + 1/2\lambda_i(w_i - x_i) \end{pmatrix},$$

where

$$a(\lambda_i) = a(\lambda_{\gamma_i}) = \left[1 + \left(1/4\mathrm{tr}(g\gamma_i)^2 - 1\right)\lambda_i^2\right]^{1/2}.$$

Since $\left[(g\gamma_i)^{t_i}\right]^{-1} \cdot \left[(g\gamma_j)^{t_j}\right] \in \Gamma$ it follows that

$$\begin{pmatrix} a(\lambda_i) + 1/2\lambda_i(w_i - x_i) & -\lambda_i y_i \\ -\lambda_i z_i & a(\lambda_i) + 1/2\lambda_i(x_i - w_i) \end{pmatrix} \cdot \begin{pmatrix} a(\lambda_j) + 1/2\lambda_j(x_j - w_j) & \lambda_j y_j \\ \lambda_j z_j & a(\lambda_j) + 1/2\lambda_j(w_j - x_j) \end{pmatrix} \in \Gamma$$

which can be written as

$$B(i, j) \begin{pmatrix} a(\lambda_i)a(\lambda_j) \\ \lambda_i a(\lambda_j) \\ a(\lambda_i)\lambda_j \\ \lambda_i \lambda_j \end{pmatrix} \in \mathbb{Z}^4 \tag{4.2.5}$$

where

$$B(i, j) = \begin{pmatrix} 1 & 1/2(w_i - x_i) & 1/2(x_j - w_j) & 1/4(w_i - x_i)(x_j - w_j) - y_i z_j \\ 0 & -y_i & y_j & 1/2y_j(w_i - x_i) - 1/2y_i(w_j - x_j) \\ 0 & -z_i & z_j & 1/2z_j(x_i - w_i) - 1/2z_i(x_j - w_j) \\ 1 & 1/2(x_i - w_i) & 1/2(w_j - x_j) & 1/4(x_i - w_i)(w_j - x_j) - z_i y_j \end{pmatrix}$$

We claim that passing to a subsequence of $\{(g\gamma_i)^{t_i}\}$ if necessary, we may assume $\det(B(i, j)) \neq 0$. Let $u_i = x_i - w_i$, then a direct but lengthy computation shows that

$$\det(B(i, j)) = x\left(u_i^2 z_j + u_j^2 z_i - u_i u_j(z_i + z_j) - x(z_j - z_i)^2\right)$$

Noting that $u_i = 2xp_i - C_i = (4ab-b)n_i - z$, $z_i = -xp_i^2 + C_ip_i - 1/x = b^3(2-4a)n_i^2 + 2zb^2n_i - b/a$, we see that $\det(B(i,j)) = -a^2b^4(2-4a)^2n_j^4 + P(n_i, n_j)$ where $P$ is a third degree polynomial in $n_j$. Now Lemma 4.4 proves the claim.

Now, from (4.2.5) $\lambda_i^2, \lambda_i\lambda_j, \lambda_i a(\lambda_j) \in \mathbb{Q}$. Let $1 \leq n_0 \leq 4$ be the biggest integer such that there are $n_0$ $\mathbb{Q}$ (or $\mathbb{Z}$)-linearly independent elements of $(g\gamma_i)^{t_i} \in \tilde{g}\Gamma$. Then it is clear that for all $i$, $(g\gamma_i)^{t_i} \in \mathrm{span}_{\mathbb{Q}} < (g\gamma_1)^{t_1}, \cdots, (g\gamma_{n_0})^{t_{n_0}} >$. Lemma 4.7 implies that for every $i$, there exist integers $m_{i,0}, m_{i,1} \cdots, m_{i,n_0}$ such that

$$m_{i,0}\lambda_i = m_{i,1}\lambda_1 + \cdots + m_{i,n_0}\lambda_{n_0}$$

and since $(g\gamma_i)^{t_i}, (g\gamma_1)^{t_1}, \cdots, (g\gamma_{n_0})^{t_{n_0}}$ all belong to the same coset, by Lemma 4.5 we can assume $m_{i,0} = m_0$ is nonzero and fixed. Now, from previous step and the equation

$$m_0^2\lambda_i^2 = (m_{i,1}\lambda_1 + \cdots + m_{i,n_0}\lambda_{n_0})^2$$

we conclude $\{\lambda_i\}$ does not have any accumulation point and since $\{\lambda_i\} \subset (0,1)$ it follows that it's finite. Passing to a subsequence again, we may assume $\lambda_i = \lambda = const$. $\square$

Now we are ready to prove Theorem 1.9.

*Proof of Theorem 1.9.* By contrary suppose $m = g\Gamma \in \mathrm{SL}(2,\mathbb{Q})/\Gamma$ is blockable from identity $m_0$. By Lemma 4.6 we may assume:

$$g = \begin{pmatrix} x & 0 \\ z & 1/x \end{pmatrix}, \; x, z \in \mathbb{Q}\,.$$

Let $\{(g\gamma_i)^{t_i}\}$ be a sequence as in Proposition 4.8, and suppose $\mathrm{tr}(g\gamma_i) = C_i = x_i/y$, $x_i, y \in \mathbb{Z}^+$, and $\lambda_i^2 = \lambda^2 = k/l < 1$, $k, l \in \mathbb{Z}^+$. Substituting theses into (4.1.6) it follows that

$$(\lambda_i a(\lambda_i))^2 = \frac{1}{4y^2l^2}\left(4kly^2 - 4k^2y^2 + k^2x_i^2\right)\,.$$

By Proposition 4.8, *ii*), we have $\lambda_i a(\lambda_i) = \lambda a(\lambda_i) \in \mathbb{Q}$, so $(\lambda_i a(\lambda_i))^2 = a_i^2/b_i^2$, for some $a_i, b_i \in \mathbb{Z}^+$. Thus there exists $\tilde{a}_i \in \mathbb{Z}^+$ so that

$$\left(4kl - 4k^2\right)y^2 + k^2x_i^2 = \tilde{a}_i^2\,,$$

which can be rewritten as

$$\left(4kl - 4k^2\right) y^2 = (\tilde{a}_i + kx_i)(\tilde{a}_i - kx_i)\,.$$

Since $k < l$, left side is a constant positive integer. Letting $x_i \to \infty$, the above equation yields a contradiction. $\square$

From above theorem it immediately follows:

**Corollary 4.9.** *Two elements $m_1 = g_1\Gamma$ and $m_2 = g_2\Gamma \in M_2$ are not blockable from each other if $g_1^{-1}g_2 \in SL(2,\mathbb{Q})$, therefore the set of non-blackable pairs is a dense subset of $M_2 \times M_2$.*

Following the proof of Proposition 9 in Gutkin [15], we prove Theorem 1.10:

*Proof of Theorem 1.10.* For $1 \le i \le n-1$ let $G_i \subset \mathrm{SL}(n,\mathbb{R})$ be the group $\mathrm{SL}(2,\mathbb{R})$ embedded in $\mathrm{SL}(n,\mathbb{R})$ via the rows and columns $i, i+1$. Then $G_i \cap \mathrm{SL}(n,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z})$, and hence $G_i\mathrm{SL}(n,\mathbb{Z})/\mathrm{SL}(n,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$. Set $M_n^{(i)} = G_i\mathrm{SL}(n,\mathbb{Z})/\mathrm{SL}(n,\mathbb{Z}) \subset M_n$. By Theorem 1.9, each $M_n^{(i)}$ has infinitely many non-blockable pairs $m_1, m_2$, yielding the the claim. $\square$

## 4.3 Blocking Property and Cocompact Lattices of SL$(2,\mathbb{R})$

In the previous section we dealt with non-cocompact lattice qutients of $\mathrm{SL}(2,\mathbb{R})$. As stated in Corollary 1.11, all non-cocompact quotients of $\mathrm{SL}(2,\mathbb{R})$ are non-blockable.

To address the cocomapct lattices, we need to know more about the structure of these lattices. There are several ways to construct cocompact lattices of $\mathrm{SL}(2,\mathbb{R})$. In this section we study blocking properties for a class of cocompact lattices, in $\mathrm{SL}(2,\mathbb{R})$, derived from quaternion algebras. We follow the notation and discussion used in Morris [31, p.118]. First we need a few preliminaries.

**Definition 4.10.** 1. For any field $F$, and any nonzero $a, b \in F$, the corresponding **quaternion algebra** over $F$ is the ring

$$\mathbb{H}_F^{a,b} = \{x + yi + zj + wk \mid x, y, z, w \in F\},$$

where

a) addition is defined in the obvious way, and

b) multiplication is determined by the relations

$$i^2 = a,\ j^2 = b,\ ij = k = -ji,$$

together with the requirement that every element of $F$ is in the center of $\mathbb{H}_F^{a,b}$. (Note that $k^2 = k \cdot k = (-ji)(ij) = -ab$.)

2. The **reduced norm** of $g = x + yi + zj + wk \in \mathbb{H}_F^{a,b}$ is

$$\mathrm{N}_{\mathrm{red}}(g) = g\overline{g} = x^2 - ay^2 - bz^2 + abw^2 \in F,$$

where $\overline{g} = x - yi - zj - wk$ is the **conjugate** of $g$. (Note that $\overline{gh} = \overline{g}\overline{h}$.)

There are a few straightforward facts left to the reader to verify, for example: $\mathbb{H}_F^{a^2,b} \cong \mathrm{Mat}_{2\times 2}(F)$ for any nonzero $a, b \in F$, $\mathbb{H}_{\mathbb{C}}^{a,b} \cong \mathrm{Mat}_{2\times 2}(\mathbb{C})$.

We need the following proposition:

**Proposition 4.11.** *Fix positive integers a and b, and let*

$$G = SL(1, \mathbb{H}_{\mathbb{R}}^{a,b}) = \{g \in \mathbb{H}_{\mathbb{R}}^{a,b} \mid N_{red}(g) = 1\}.$$

*Then:*

*i)* $G \cong SL(2, \mathbb{R})$*,*

*ii)* $G_{\mathbb{Z}} = SL(1, \mathbb{H}_{\mathbb{Z}}^{a,b})$ *is an arithmetic subgroup of G, and*

*iii) the following are equivalent:*

*a)* $G_{\mathbb{Z}}$ *is cocompact in G.*

*b)* $(0, 0, 0, 0)$ *is the only integer solution* $(p, q, r, s)$ *of the Diophantine equation*

$$w^2 - ax^2 - by^2 + abz^2 = 0.$$

*c) Every nonzero element of* $\mathbb{H}_{\mathbb{Q}}^{a,b}$ *has a multiplicative inverse (so* $\mathbb{H}_{\mathbb{Q}}^{a,b}$ *is a "division algebra").*

**Remark.** It is well known that the Diophantine equation $w^2 - ax^2 - by^2 + abz^2 = 0$ has only trivial integer solution if and only if the equation $ax^2 + by^2 = z^2$ has only trivial integer solution [31, p.121]. This can happen if $a, b$ are prime, or if $a$ is not a square mod $b$, and $b$ is not a square mod $a$. Throughout the section we assume $a$ and $b$ are such integers, so the norm equation has only trivial solution (and thus $G_{\mathbb{Z}}$ is cocompact). In particular, $a$ and $b$ can not be perfect squares.

We refer the reader to [31, p.119] for a proof. We will use the fact that the isomorphism in *i*) is given by:

$$\phi(x + yi + zj + wk) = \begin{pmatrix} x + y\sqrt{a} & z + w\sqrt{a} \\ b(z - w\sqrt{a}) & x - y\sqrt{a} \end{pmatrix}. \tag{4.3.1}$$

Next, we discuss the exponential mapping. Let $\mathfrak{g} \cong T_1(\mathrm{SL}(1, \mathbb{H}_{\mathbb{R}}^{a,b}))$ and $\mathfrak{sl}(2, \mathbb{R}) \cong T_{\mathrm{Id}}\mathrm{SL}(2, \mathbb{R})$ be the lie algebras of $G$ and $\mathrm{SL}(2, \mathbb{R})$ respectively. Since $\phi$ in equation (4.3.1) is an isomorphism of lie groups, $\mathrm{d}\phi_1 : T_1(\mathrm{SL}(1, \mathbb{H}_{\mathbb{R}}^{a,b})) \to T_{\mathrm{Id}}\mathrm{SL}(2, \mathbb{R})$ is a Lie algebra isomorphism. Moreover, since $\mathrm{SL}(1, \mathbb{H}_{\mathbb{R}}^{a,b})$ and $\mathrm{SL}(2, \mathbb{R})$ are embedded manifolds in $\mathbb{R}^4$, $\mathrm{d}\phi_1$ is the restriction of the corresponding differential when $\phi$ is regarded as a function from $\mathbb{R}^4$ to $\mathbb{R}^4$. Note that $T_1(\mathrm{SL}(1, \mathbb{H}_{\mathbb{R}}^{a,b})) = \{(0, u_1, u_2, u_2) \mid u_1, u_2, u_3 \in \mathbb{R}\}$, computing $\mathrm{d}\phi_1$ it follows that:

$$\begin{pmatrix} 1 & \sqrt{a} & 0 & 0 \\ 0 & 0 & 1 & \sqrt{a} \\ 0 & 0 & b & -b\sqrt{a} \\ 1 & -\sqrt{a} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} u_1\sqrt{a} \\ u_2 + u_3\sqrt{a} \\ bu_2 - b\sqrt{a}u_3 \\ -u_1\sqrt{a} \end{pmatrix} \tag{4.3.2}$$

Since the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{d}\phi_1"] \arrow[d, "\exp"'] & \mathfrak{sl}(2, \mathbb{R}) \arrow[d, "\exp"] \\ G \arrow[r, "\phi"] & \mathrm{SL}(2, \mathbb{R}) \end{tikzcd} \tag{4.3.3}$$

commutes Proposition 4.1 easily implies the following:

**Proposition 4.12.** *Let $G = SL(1, \mathbb{H}_{\mathbb{R}}^{a,b})$ and $\mathfrak{g} \cong \mathbb{R}^3$ be its Lie algebra. Given $U = (u_1, u_2, u_3) \in \mathfrak{g}$, let $\omega = \sqrt{|u_1^2 a + u_2^2 b - u_3^2 ab|}$. Then we have the following:*

*i)* $\exp(U) = \cosh\omega + \dfrac{\sinh\omega}{\omega}u_1 i + \dfrac{\sinh\omega}{\omega}u_2 j + \dfrac{\sinh\omega}{\omega}u_3 k$, *if* $u_1^2 a + u_2^2 b - u_3^2 ab > 0$,

*ii)* $\exp(U) = 1 + u_1 i + u_2 j + u_3 k$, *if* $u_1^2 a + u_2^2 b - u_3^2 ab = 0$, *and*

*iii)* $\exp(U) = \cos\omega + \dfrac{\sin\omega}{\omega}u_1 i + \dfrac{\sin\omega}{\omega}u_2 j + \dfrac{\sin\omega}{\omega}u_3 k$, *if* $u_1^2 a + u_2^2 b - u_3^2 ab < 0$.

For $g = x + yi + zj + wk \in G$ with $x > 1$, $\log(g)$ is unique; let $\omega_g = \omega(\log(g))$, $g^t = \exp(t \log g)$, $0 \leq t \leq 1$. The following Lemma is the counterpart to Lemma 4.2 and is stated as follows:

**Lemma 4.13.** *Let $g = x + yi + zj + wk \in G$ with $x > 1$, we have:*

$$g^t = \left(\cosh(t\omega_g) - \frac{\sinh t\omega_g}{\sinh \omega_g}\cosh\omega_g\right)1 + \frac{\sinh t\omega_g}{\sinh\omega_g}g. \tag{4.3.4}$$

*Proof.* Follow the steps of Lemma 4.2. □

Let $\Gamma = \mathrm{SL}(1, \mathbb{H}_{\mathbb{Z}}^{a,b})$ be a cocompact lattice. Following notations of Section 2, for a fixed $g$ and an arbitrary $\gamma \in \Gamma$, $\lambda_\gamma = \dfrac{\sinh(t\omega_{g\gamma})}{\sinh(\omega_{g\gamma})}$, $0 \leq \lambda_\gamma \leq 1$ is the modified time. Through similar step we can easily conclude:

$$(g\gamma)^t = \left[a(\lambda_\gamma) - x\lambda_\gamma\right]1 + \lambda_\gamma g\gamma, \tag{4.3.5}$$

where

$$a(\lambda_\gamma) = \left(1 + \left(x^2 - 1\right)\lambda_\gamma^2\right)^{1/2}. \tag{4.3.6}$$

To follow through the proof of Proposition 4.8 for cocompact lattices we only consider elements $g = x + yi \in \mathrm{SL}(1, \mathbb{H}_{\mathbb{Q}}^{a,b})$. For a sequence $\{\gamma_i\} \subset \Gamma$ let $g\gamma_i = x_i + y_i i + z_i j + w_i k$. We need the following lemma.

**Lemma 4.14.** *Let $g = x + yi \in SL(1, \mathbb{H}_{\mathbb{Q}}^{a,b})$. There exists a sequence $\gamma_i = p_i + q_i i + r_i j + s_i k \in \Gamma$, such that $z_i$ and $w_i$ in $g\gamma_i$, are nonzero and fixed for all i, $z_i^2 - aw_i^2 \neq 0$, and $x_i \to \infty$, as $i \to \infty$.*

*Proof.* Fix an element $\gamma_1 = p_1 + q_1 i + r_1 j + s_1 k \in \Gamma$ such that $xr_1 + ays_1 \neq 0$, and $xs_1 + yr_1 \neq 0$. Since $a$ is not a perfect square, $r_1^2 - as_1^2 \neq 0$. Let $n = p_1^2 - aq_1^2$. It is well known that if the Pell's equation $p^2 - aq^2 = n$ has one solution (and $a$ is not a perfect square), it has infinitely many solutions. Let $(p_i, q_i) \in \mathbb{Z}^2$ be an infinite set of distinct solutions such that $xp_i,\ yq_i > 0$, and let $\gamma_i = p_i + q_i i + r_1 j + s_1 k$. Then it is easily seen $z_i = xr_1 + ays_1, w_i = xs_1 + yr_1$ are fixed, $z_i^2 - aw_i^2 = (x^2 - ay^2)(r_1^2 - as_1^2) \neq 0$, and $x_i = xp_i + ayq_i \to \infty$ as $i \to \infty$. □

It can be easily seen Lemma 4.5 is valid for the cocompact lattices $\Gamma$, if we think of elements of $\Gamma$ as two by two matrices with integer entries. The following proposition is the counterpart to Proposition 4.8 for cocompact lattices.

**Proposition 4.15.** *Let $g = x + yi \in SL(1, \mathbb{H}_{\mathbb{Q}}^{a,b})$. $m = g\Gamma$, is finitely blockable from identity $m_0$, then there exists a sequence $\gamma_i = p_i + q_i i + r_i j + s_i k$ and a sequence of times $\{t_i\} \subset (0, 1)$ such that*

- *i) all elements of $\{(g\gamma_i)^{t_i}\}$ belong to the same coset, and all modified times are the same, i.e., $\lambda_i = \lambda = const$,*
- *ii) $\lambda_i^2, \lambda_i a(\lambda_i) \in \mathbb{Q}$,*
- *iii) $x_i = Re(g\gamma_i)$ is an increasing sequence of positive rational numbers with the same denominator, $x_i \to \infty$, as $i \to \infty$, and*
- *iv) $\{p_i\}$ is an increasing sequence of positive integers.*

*Proof.* Let $m = g\Gamma$, be blockable from identity $m_0$. Let $\{\gamma_i\}$ be a sequence as in Lemma 4.14. Then $x_i = \mathrm{Re}(x_i)$ is an increasing sequence of rational numbers with the same denominator, and $x_i \to \infty$, as $i \to \infty$. By Proposition 2.14 for a suitable choice of $t_i$'s where $0 < t_i < 1$ we should have $\{(g\gamma_i)^{t_i}\} \subset \cup_{n=1}^{N} \tilde{g_n}\Gamma$; passing to a subsequence if necessary, we may assume $(g\gamma_i)^{t_i} \in \tilde{g}\Gamma$ for some fixed $\tilde{g} \in G$.

Now let $\lambda_i = \dfrac{\sinh(t_i \omega_{\gamma_i})}{\sinh(\omega_{\gamma_i})}$ be modified times $\lambda_i \in (0,1)$. We show that for every pair of indexes $(i,j)$, $\lambda_i \lambda_j \in \mathbb{Q}$.

By (4.3.5) and (4.3.6) we have

$$(g\gamma_i)^{t_i} = a(\lambda_i) + \lambda_i(y_i i + z_i j + w_i k)$$

where

$$a(\lambda_i) = \left(1 + \left(x^2 - 1\right)\lambda_i^2\right)^{1/2}.$$

Since $\left[(g\gamma_i)^{t_i}\right]^{-1} \cdot \left[(g\gamma_j)^{t_j}\right] \in \Gamma$ it follows that

$$(a(\lambda_i) - \lambda_i(y_i i + z_i j + w_i k)) \cdot \left(a(\lambda_j) + \lambda_j(y_j i + z_j j + w_j k)\right) \in \Gamma$$

which can be written as

$$B(i,j)\begin{pmatrix} a(\lambda_i)a(\lambda_j) \\ \lambda_i a(\lambda_j) \\ a(\lambda_i)\lambda_j \\ \lambda_i\lambda_j \end{pmatrix} \in \mathbb{Z}^4 \tag{4.3.7}$$

where

$$B(i,j) = \begin{pmatrix} 1 & 0 & 0 & w_i w_j ab - y_i y_j a - z_i z_j b \\ 0 & -y_i & y_j & (z_i w_j - w_i z_j)b \\ 0 & -z_i & z_j & (w_i y_j - y_i w_j)a \\ 0 & -w_i & w_j & z_i y_j - y_i z_j \end{pmatrix}$$

We claim that passing to a subsequence of $\{g\gamma_i^{t_i}\}$ if necessary, we may assume $\det(B(i,j)) \neq 0$. A direct computation shows that

$$\det(B(i,j)) = -(w_i y_j - y_i w_j)^2 a - (w_i z_j - z_i w_j)^2 b + (z_i y_j - y_i z_j)^2$$

Note that $\det(B(i,j))$ is a second degree polynomial in $y_j$ (the coefficient of $y_j^2$ is $z_i^2 - a w_i^2 \neq 0$); so by Lemma 4.4 and passing to a subsequence if necessary, we may assume $\det(B(i,j)) \neq 0$.

Now, from (4.3.7) $\lambda_i^2, \lambda_i\lambda_j, \lambda_i a(\lambda_j) \in \mathbb{Q}$. Let $1 \leq n_0 \leq 4$ be the biggest integer such that there are $n_0$ $\mathbb{Q}$ (or $\mathbb{Z}$)-linearly independent elements of $(g\gamma_i)^{t_i} \in \tilde{g}\Gamma$. Then it is clear that $(g\gamma_i)^{t_i} \in \text{span}_{\mathbb{Q}} < (g\gamma_1)^{t_1}, \cdots, (g\gamma_{n_0})^{t_{n_0}} >$, for arbitrary $i$ which implies, considering the $z$-component,

$$m_{i,0}\lambda_i z_i = m_{i,1}\lambda_1 z_1 + \cdots + m_{i,n_0}\lambda_{n_0} z_{n_0}.$$

By Lemma 4.14 $z_i = z$ is nonzero and fixed, and since $(g\gamma_i)^{t_i}, (g\gamma_1)^{t_1}, \cdots, (g\gamma_{n_0})^{t_{n_0}}$ all belong to the same coset, by Lemma 4.5 we can assume $m_{i,0} = m_0 \neq 0$ is also fixed and does not depend on $i$. Now, from previous step and the equation

$$m_0^2\lambda_i^2 = (m_{i,1}\lambda_1 + \cdots + m_{i,n_0}\lambda_{n_0})^2$$

we conclude $\{\lambda_i\}$ does not have any accumulation point and since $\{\lambda_i\} \subset (0, 1)$ it follows that it's finite. Passing to a subsequence again, we may assume $\lambda_i = \lambda = const$. □

*Proof of Theorem 1.12.* The proof is quite similar to proof of Theorem 1.9, just replace $C_i$ with $2x_i$, and Proposition 4.8 with Proposition 4.15. □

## CHAPTER 5

## CONCLUDING REMARKS AND PROPOSED PROBLEMS FOR FURTHER RESEARCH

### 5.1 Connection Blocking Problems in Other Lattice Quotients

In the context of solvable Lie groups, one can study the blocking problem in higher dimensional versions of *Sol*. Let $A$ be a positive definite (symmetric) $n \times n$ matrix with no eigenvalues equal to one. By $Sol_A := \mathbb{R}^{n-1} \rtimes_A \mathbb{R}$ we mean semi direct product of $\mathbb{R}^{n-1}$ and $\mathbb{R}$, where $t \in \mathbb{R}$ acts on $\mathbb{R}^{n-1}$ as $A^t$, i.e. $(\boldsymbol{x}_1, t_1)(\boldsymbol{x}_2, t_2) = (\boldsymbol{x}_1 + A^{t_1}\boldsymbol{x}_2, t_1 + t_2)$, $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{R}^{n-1}$, $t_1, t_2 \in \mathbb{R}$. $Sol_A$ is an $n$-dimensional solvable Lie group which is a generalization of *Sol*. An argument similar to the proof of Proposition 3.4 shows that there is a monomorphism $\mathbb{Z}^{n-1} \rtimes_A \mathbb{Z} \hookrightarrow Sol_A$ and the image $\Gamma \subset Sol_A$ is a lattice. It would be then natural to ask the following questions:

Q1: How other lattices in $Sol_A$ are related to $\Gamma$?

Q2: Are all lattice quotients of $Sol(n)$ non-blockable?

Delving into the method of proof for non-blockability of *Sol* quotients reveals that the method might be applicable to $Sol_A/\Gamma$. It would also be interesting to investigate the connection between $Sol_A$ and $n-1$-dimensional hyperbolic space $\mathbb{H}^{n-1}$.

In the context of lattice quotients of semisimple Lie groups there is a lot of room to work on the connection blocking problem. By Margulis Arithmeticity Theorem (see Theorem D.5), every lattice of $\mathrm{SL}(n, \mathbb{R})$, $n \geq 3$ is arithmetic, that is its algebraic structure looks very similar to $\mathrm{SL}(n, \mathbb{Z})$. In fact modding out normal compact subgroups, arithmetic subgroups are commensurable to an integer points lattice, i.e. a lattice of the form $G_{\mathbb{Z}} = G \cap \mathrm{SL}(n, \mathbb{Z})$, where $G$ is a subgroup of $\mathrm{SL}(n, \mathbb{R})$ (see appendix D). Since connection blocking is invariant through modding out normal compact subgroups and commensurability, it suffices to consider integer points lattice of the form $G_{\mathbb{Z}}$. It would be then worthwhile to investigate the possibility of applying *modified times* method to a lattice $G_{\mathbb{Z}}$ and prove the following quotient spaces are non-blockable. The other interesting problem is studying finite blocking for quotients of *Special orthogonal group*, $\mathrm{SO}(m,n) := \{g \in \mathrm{SL}(m+n,\mathbb{R}) | g^T I_{m,n} g = I_{m,n}\}$ ($I_{m,n} = \mathrm{diag}(1,1,\cdots,1,-1,-1,\cdots,-1) \in \mathrm{Mat}_{(m+n)\times(m+n)}(\mathbb{R})$, where the number of 1's is $m$ and the number of -1's is $n$). This paves the way to prove lattice quotients of every linear, semisimple Lie group $G$ are non-blockable. Indeed, without losing any main ideas, it may be assumed that $G$ is either $\mathrm{SL}(n,\mathbb{R})$ or $\mathrm{SO}(m,n)$, or a prodcut of these, Morris [31, p.43].

The security problem in Riemannian manifolds and connection blocking in lattice quotient are closely related problems. Given a connected Lie group $G$ with a left invariant Riemannian metric $dg$, in general one parameter subgroups passing through the identity are not geodesics. However if certain conditions are met these two classes of paths coincide. In particular, if $G$ admits a bi-invariant Riemannian metric, the one parameter subgroups are also geodesics. A Lie group $G$ admits a bi-invariant metric if the adjoint group $\mathrm{Ad}(G) := \{\mathrm{Ad}(g) \,|\, g \in G\}$ is relatively compact, i.e. it is included in a compact set (See Theorem 2, Pennec [33]). For compact Lie groups, the adjoint group is the image of a compact set by a continuous mapping and is thus also compact. Thus, bi-invariant metrics exist in such a case.

Let $G$ be a connected Lie group with a bi-invariant Riemannian metric $dg$, and let $M = G/\Gamma$ be a lattice quotient of $G$. Then $dg$ induces a Riemannian metric on the lattice quotient $M$ through the projection map. In addition, geodesics passing through the identity and one parameter subgroups of $M$ coincide. Thus for such a lattice quotient, investigating security of $M$ as a Riemannian manifold and its connection blocking property are the same problem.

## 5.2 Behavior of Exponential Map Near Singularities

Security of a closed Riemannian manifold has been verified for various classes of manifolds without conjugate points. Schmidt and Lafont have shown secure compact non-positively curved Riemannian manifolds are flat, Lafont [25]. Burns and Gutkin [5] prove that compact Riemannian manifolds with no conjugate points and positive topological entropy are totally insecure. They also prove that uniformly secure closed Riemannian manifolds without conjugate points are flat. Both of these result deal with manifolds without conjugate points. It seems for dealing with manifolds with conjugate points, understanding the behavior of the exponential map near singularities is essential.

A *configuration* in $M$ is an ordered pair of points in $M$. Let $(x, y)$ be a configuration in $M$. A geodesic $\gamma$ *joins* $x$ to $y$, if $x$ is its initial point and $y$ is its final point. A geodesic $\gamma$ *connects* $x$ and $y$ if it joins $x$ and $y$ and does not pass through either $x$ or $y$. Let $G_T(x, y)$ and $\Gamma_T(x, y)$ denote the set of geodesics joining, and connecting $x$ and $y$ with length $\leq T$, respectively; $n_T(x, y) := |G_T(x, y)|$ and $m_T(x, y) := |\Gamma_T(x, y)|$. Moreover, let $s_T(x, y)$ be the minimal cardinality of a blocking set for $\Gamma_T(x, y)$.

Burns and Gutkin's method of proof uses a famous identity involving topological entropy due to Mañé [28]:

$$h_{\text{top}} = \lim_{T\to\infty} \frac{1}{T} \log \int_{M\times M} n_T(x, y)\, d\mu(x) d\mu(y) \tag{5.2.1}$$

If a Riemannian manifold has no conjugate point this identity simply implies the following stronger identity:

$$\forall x, y \in M, \quad h_{\text{top}} = \lim_{T\to\infty} \frac{1}{T} \log n_T(x, y) \tag{5.2.2}$$

which is then applied to obtain the aforementioned results. Therefore, a main approach for proving the main conjecture concerns studying a weaker version of Equation 5.2.2, and a modified technique to obtain similar results for manifolds with conjugate points. A big step in this regard is to understand behavior of the exponential map near singularities, which is mainly unknown. A famous paper by Warner [36], gives a very good insight about the conjugate locus and the type of conjugate points. Regarding $m_T(x, y)$ (or $n_T(x, y)$) functions, I believe the following conjecture to be true which is an interesting problem by itself:

**Conjecture 5.1.** Let $(M, g)$ be a closed connected Riemannian manifold. For $T > 0$, $x, y \in M$, let $m_T(x, y) \leq \infty$ be the cardinality of the set of geodesic segments connecting $x$ to $y$ of length $\leq T$. Then there exists a pair of points $x^*, y^* \in M$ such that

$$m_T(x^*, y^*) = \sup_{x,y \in M} m_T(x, y) .$$

Since $M$ is compact and $m_T(x, y)$ is a discrete function, the statement of Conjecture 5.1 is trivial if $\sup_{x,y\in M} m_T(x, y) < \infty$. If $\sup_{x,y\in M} m_T(x, y) = \infty$, let $(x_n, y_n)$ be a sequence in $M \times M$ such that $m_T(x_n, y_n) \to \infty$, as $n \to \infty$. Since $M \times M$ is compact, passing to a subsequence if necessary, we may assume there exists $(x^*, y^*) \in M \times M$ such that $(x_n, y_n) \to (x^*, y^*)$, as $n \to \infty$. One approach for the Conjecture is to assume $m_T(x^*, y^*) < \infty$ and that there exists a sequence $y_n \to y^*$, where $m_T(x^*, y_n) \to \infty$, as $n \to \infty$ and derive a contradiction. Let $\exp_T$ be the exponential map at $x^*$, restricted to a closed ball of radius $T$ in the tangent space $T_{x^*}M$. Assume $\exp_T(p) = y^*$ and $\exp_T(p_n^j) = y_n$, $j = 1, \cdots, j_n$, where $j_n \to \infty$, as $n \to \infty$. Interestingly, it can be proved (passing to a subsequence if necessary) that we can interpolate all $p_n^j$'s through a smooth path $v : [0, a] \to T_x M$ starting at $p$, so the parametrized surface $f(s, t) = \exp_T(tv(s))$, $0 \leq s \leq a$, $0 \leq t \leq 1$, would be a strange surface in $M$ and its boundary $c(s) = \exp_T(v(s))$, winds around itself a lot of times starting at $y^*$. It would be interesting to study the Jacobi field $J_s(t) = \partial_s f(s, t)$ of this surface, where a different geometric or analytic technique may be applied to possibly refute the existence of such surface.

If proved, Conjecture 5.1 together with Mañé identity 5.2.1, has interesting implications regarding $m_T(x, y)$ estimates which may also be advantageous for proving general Conjecture 1.1 for a certain class of manifolds with conjugate points. Conjecture 5.1, basically states that if $\exp_T$ is *finite to one*, that is the preimage of every point has finite cardinality) then it is a *finite* map at every point of its domain (See Golubitsky [10, pp. 167-169]). For a Riemnnian manifold $M$ and a point $x \in M$, a conjugate point $p \in T_x M$ is called *regular* if there exists a neighborhood $U$ of $p$ such that each ray of $T_x M$ contains at most one point in $U$ which is a conjugate point. A conjugate point which is not regular is called a *singular* or *intersection* point [36]. Warner [36, Theorem

3.3] proves for almost every regular point, the map $\exp_T$ can be formulated in local coordinates, which would imply the finiteness of $\exp_T$ at these points. However, for a class of regular conjugate points, and subsequently singular conjugate points proving the finiteness would be tough. There is also much room to investigate behavior of the exponential map near such points using singularity theory, and derive properties beyond just the finiteness.

# APPENDICES

# APPENDIX A

## Semidirect Product and Semidirect Sum

In many cases it is convenient to describe the structure of Lie groups in terms of semidirect products. The *semidirect product* of abstract groups $G_1$ and $G_2$ is the direct product of sets $G_1$ and $G_2$ endowed with the group structure via

$$(g_1, g_2)(h_1, h_2) = (g_1 \cdot b(g_2)h_1, g_2h_2),$$

where $b$ is a homomorphism of $G_2$ into the group Aut$G_1$ of automorphisms of the group $G_1$. We will denote the semidirect product by $G_1 \rtimes G_2$, or more precisely $G_1 \rtimes_b G_2$. The elements of the form $(g_1, e)$ (resp. $(e, g_2)$) form a subgroup in $G_1 \rtimes G_2$ isomorphic to $G_1$ (resp. $G_2$). This subgroup is usually identified with $G_1$ (resp. $G_2$). The subgroup $G_1$ is normal and

$$g_2 g_1 g_2^{-1} = b(g_2)g_1,\ g_1 \in G_1, g_2 \in G_2\,. \tag{A.0.1}$$

The subgroup $G_2$ is normal if and only if $b$ is trivial, i.e. $b(G_2) = e$; in this case the semidirect product coincides with the direct product $G_1 \times G_2$ .

One says that a group $G$ *splits into a semidirect product of subgroups* $G_1$ and $G_2$ if

1. $G_1$ is normal;
2. $G1G2 = G$;
3. $G1 \cap G_2 = \{e\}$.

In this case we have the isomorphism

$$G_1 \rtimes_b G_2 \cong G,\ (g_1, g_2) \mapsto g_1 g_2\,, \tag{A.0.2}$$

where $b : G_2 \to \operatorname{Aut} G_1$ is the homomorphism defined by A.0.1 and we will write $G = G_1 \rtimes G_2$.

A *semidirect product of Lie groups* $G_1$ and $G_2$ is defined as a semidirect product of abstract groups endowed with a differentiable structure as the direct product of differentiable manifolds. It is additionally required that $b$ define differentiable $G_2$-action on $G_1$ , i.e. that the map

$$G_1 \times G_2 \to G_1,\ (g_1, g_2) \mapsto b(g_2)g_1$$

be differentiable. (In particular, the automorphism $b(g_2)$ of $G_1$ must be differentiable for any $g_2 \in G_2$). This ensures the differentiability of group actions in the semidirect product. One says that a *Lie group G splits into a semidirect product of Lie subgroups* $G_1$ and $G_2$ if it splits into their semidirect product as an abstract group. In this case the action $b$ of $G_2$ on $G_1$ defined by A.0.1 is differentiable and the abstract isomorphism A.0.2 is a Lie group isomorphism.

To semidirect products of Lie groups there correspond *semidirect sums* of Lie algebras (which could as well have been called *semidirect products*). The tangent algebra of $\mathrm{Aut}\,\mathfrak{g}$ is the Lie algebra $\mathfrak{der}\,\mathfrak{g}$ of derivations of $\mathfrak{g}$, i.e. linear transformation of $\mathfrak{g}$ satisfying the product rule (see Onishchick [32, p.23]). Let $\beta$ be a Lie algebra homomorphism $\mathfrak{g}_2 \longrightarrow \mathfrak{der}\,\mathfrak{g}_1$. A semidirect sum of Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ is the direct sum of vector spaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ endowed with the bracket

$$[(\xi_1, \xi_2), (\eta_1, \eta_2)] = ([\xi_1, \eta_1] + \beta(\xi_2)\eta_1 - \beta(\eta_2)\xi_1, [\xi_2, \eta_2])\,.$$

We denote the semidirect sum by $\mathfrak{g}_1 \oplus\!\!\!\!+ \mathfrak{g}_2$, or more prudently $\mathfrak{g}_1 \oplus\!\!\!\!+_\beta \mathfrak{g}_2$. It is not difficult to verify a semidirect sum of Lie algebras is a Lie algebra. The elements of the form $(\xi_1, 0)$ (resp. $(0, \xi_2)$) constitute a subalgebra of $\mathfrak{g}_1 \oplus\!\!\!\!+ \mathfrak{g}_2$ isomorphic to $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$), usually identified with $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$). The subalgebra $\mathfrak{g}_1$ is an ideal and

$$[\xi_2, \xi_1] = \beta(\xi_2)\xi_1,\ (\xi_1 \in \mathfrak{g}_1, \xi_2 \in \mathfrak{g}_2)\,. \tag{A.0.3}$$

The subalgebra $\mathfrak{g}_2$ is an ideal if and only if $\beta = 0$. In this case the semidirect sum is isomorphic to the direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.

One says that a Lie algebra $\mathfrak{g}$ splits into a *semidirect sum of Lie subalgebras* $\mathfrak{g}_1$ and $\mathfrak{g}_2$ if

1. $\mathfrak{g}_1$ is an ideal;

2. $\mathfrak{g}$ is the direct sum of subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ as a vector space.

In this case we have an isomorphism

$$\mathfrak{g}_1 \dotplus_\beta \mathfrak{g}_2 \cong \mathfrak{g}, \ (\xi_1, \xi_2) \mapsto \xi_1 + \xi_2,$$

where $\beta : \mathfrak{g}_2 \longrightarrow \mathfrak{der}\,\mathfrak{g}_1$ is the homomorphism defined by formula A.0.3. In this situation we will write $\mathfrak{g} = \mathfrak{g}_1 \dotplus_\beta \mathfrak{g}_2$. The following theorem relates semidirect product of two Lie groups to semidirect sum of their corresponding Lie algebra, [32, p.37].

**Theorem A.1.** *The tangent Lie algebra of the semidirect product $G_1 \rtimes_b G_2$ of Lie groups $G_1$ and $G_2$ is the semidirect sum $\mathfrak{g}_1 \dotplus_\beta \mathfrak{g}_2$ of their tangent algebras and $\beta = dB$, where $B : G_2 \longrightarrow Aut(\mathfrak{g}_1)$ is a Lie group homomorphism defined by the formula $B(g_2) = d(b(g_2))$, for any $g2 \in G_2$.*

# APPENDIX B

## Levi Decomposition

Let $G$ be a connected Lie group and $\mathfrak{g} = T_e G$ its Lie algebra. Recall that the *iterated commutator groups* $G^{(k)}$ $(k = 0, 1, 2, \cdots)$ of $G$ are defined by induction:

$$G^0 = G, \quad G^{(1)} = G', \quad G^{(k+1)} = \left(G^{(k)}\right)' .$$

A Lie group $G$ is called *solvabale* if there exists an integer $m$ such that $G^{(m)} = \{e\}$.

The *derived algebra* of a Lie algebra $\mathfrak{g}$ is the subalgebra $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}'$ generated by the brackets $[\xi, \eta]$, where $\xi, \eta \in \mathfrak{g}$. It is the smallest ideal such that the corresponding quotient algebra is commutative. The *iterated derived algebras* $\mathfrak{g}^{(k)}$ $(k = 0, 1, 2, \cdots)$ of a Lie aglebra $\mathfrak{g}$ are defined by induction as:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = \mathfrak{g}', \quad \mathfrak{g}^{(k+1)} = \left(\mathfrak{g}^{(k)}\right)' .$$

A Lie algebra is called *solvable* if there exists an $m$ such that $\mathfrak{g}^{(m)} = \{0\}$. A connected Lie group $G$ is solvable if and only if so is its Lie algebra. More precisely, $G^{(m)} = \{e\}$ if and only if $\mathfrak{g}^{(m)} = \{0\}$, Onishchick [32, p.54]. The sum of solvable ideals of a Lie algebra is a solvable ideal [32, p.55]. It follows that in any Lie algebra $\mathfrak{g}$ there exists the largest solvable ideal. It is called the *radical* of $\mathfrak{g}$. We denote it by $\mathfrak{rad}\,\mathfrak{g}$. Similarly, the largest connected solvable normal Lie subgroup of $G$ is called *radical* of the Lie group $G$ and is denoted by $\mathrm{Rad}\,G$. The following theorem guarantees existence of such subgroup (see [32, p.55] for proof).

**Theorem B.1.** *In any Lie group G there is the largest connected solvable normal Lie subgroup. Its tangent Lie algebra coincides with* $\mathfrak{rad}\,\mathfrak{g}$.

*Examples.*

1. The group $B_n(K)$ the set of invertible (upper) triangular $n \times n$ matrices over the field $K$ is a solvable Lie group, and its Lie algebra $\mathfrak{b}_n(K)$, the set of all upper triangular $n \times n$ matrices over $K$ is a solvable Lie algebra [32, p.53].

2. It is well known that semidirect product of two solvable abstract group is solvable. Thus *Sol* is solvable, so is its Lie algebra $\mathfrak{sol}$.

3. Nilpotent Lie groups (resp. Lie algebras) are, a fortiori solvable but the converse is not true, Hall [19, p.54].

A Lie group $G$ (resp. a Lie algebra $\mathfrak{g}$) is called *semisimple* if $\mathrm{Rad}\, G = \{e\}$ (resp. $\mathfrak{rad}\,\mathfrak{g} = \{0\}$). Obviously, a Lie group is semisimple if and only if its tangent Lie algebra is semisimple. For any Lie group $G$ (resp. Lie algebra $\mathfrak{g}$) the quotient group $G/\mathrm{Rad}G$ (resp. the quotient algebra $\mathfrak{g}/\mathfrak{rad}\,\mathfrak{g}$) is semisimple. Equivalently, a Lie algebra is semisimple if it is a direct sum of simple Lie algebras, i.e., there exist non-abelian Lie algebras $\mathfrak{g}_\mathfrak{i}$, $i = 1, \cdots, n$ whose only ideals of $\mathfrak{g}_\mathfrak{i}$ are 0 and $\mathfrak{g}_\mathfrak{i}$ itself, and $\mathfrak{g} = \mathfrak{g}_g \oplus \cdots \oplus \mathfrak{g}_\mathfrak{n}$ [19, p.173].

**Remark.** A connected non-Abelian Lie group $G$ is simple if it has no nontrivial, connected, closed, proper, normal subgroup. A non-abelian Lie algebra $\mathfrak{g}$ is simple if its only ideals are 0 and itself (or equivalently, a Lie algebra of dimension 2 or more, whose only ideals are 0 and itself). It can be shown that $G = \mathrm{SL}(n, \mathbb{R})$, $n > 1$ (resp. $\mathfrak{so}(n, \mathbb{R})$, $n > 1$) is a simple Lie group (resp. Lie algebra). A direct product of simple Lie groups (ex. $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(3, \mathbb{R})$) is semisimple. But in general, semisimple Lie groups are a much larger class of Lie groups. see Morris [31, p.428] for a definition based on direct product of simple Lie groups.

The solvable Lie algebras and the semisimple Lie algebras form two large and generally complementary classes, as is shown by Levi decomposition.

**Definition B.2.** Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $K = \mathbb{C}$ or $\mathbb{R}$. A subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is called a *Levi subalgebra* if $\mathfrak{g}$ splits into the semidirect sum

$$\mathfrak{g} = \mathfrak{rad}\,\mathfrak{g} \niplus \mathfrak{l}\,. \tag{B.0.1}$$

Decomposition B.0.1 is called the *Levi decomposition* of $\mathfrak{g}$.

**Theorem B.3** (Levi)**.** *Any finite dimensional Lie algebra $\mathfrak{g}$ over $K = \mathbb{C}$ or $\mathbb{R}$ contains a Levi subalgebra.*

Analogous statements hold for simply connected Lie groups. The following theorem states one of the fundamental facts of Lie theory which sometimes is called Lie's third theorem (See Onishchick [32, p.284] for proof).

**Theorem B.4.** *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra (over $\mathbb{C}$ or $\mathbb{R}$), $\mathfrak{l}$ its Levi subalgebra. Then there exists a simply connected Lie group $G$ (either complex or real respectively) whose tangent algebra is isomorphic to $\mathfrak{g}$. Moreover,*

$$G = A \rtimes L,$$

*where $A = Rad\, G$, and $L$ is a simply connected Lie subgroup with the tangent Lie algebra $\mathfrak{l}$.*

**Remark.** $\mathrm{Rad}\, G$ (resp. $\mathfrak{rad}\,\mathfrak{g}$) is solvable. In addition, for any Lie group $G$ (resp. Lie algebra $\mathfrak{g}$) the quotient group $G/\mathrm{Rad}G$ (resp. the quotient algebra $\mathfrak{g}/\mathfrak{rad}\,\mathfrak{g}$) is semisimple. Therefore Levi decompostion implies every simply connected Lie group (resp. finite-dimensional Lie algebra) is a semidirect product (resp. semidirect sum) of a solvable Lie subgroup (resp. Lie subalgebra) and a semisimple Lie subgroup (resp. Lie subaglebra).

# APPENDIX C

## Haar Measure

Standard texts in real analysis construct a translation-invariant measure on $\mathbb{R}^n$ which is called Lebesgue measure, but the analogue for the Lie groups is called *Haar* measure. We state the following proposition without proof.

**Proposition C.1** (Existence and uniqueness of Haar measure)**.** *If $G$ is any Lie group, then there exists a unique (up to scalar multiple) $\sigma$-finite borel measure $\mu$ on $G$, such that*

1. *$\mu(C)$ is finite for every compact subset $C$ of $G$, and*

2. *$\mu(gA) = \mu(A)$, for every Borel subset $A$ of $G$, and every $g \in G$.*

**Definition C.2.** The measure $\mu$ of Proposition C.1 is called the *left Haar measure* on $H$. Analogously, there exists a unique *right Haar measure* with $\mu(Ag) = \mu(A)$. $G$ is *unimodular* if the left Haar measure is also a right Haar measure, i.e. $\mu(Ag) = \mu(gA) = \mu(A)$.

Haar measure is always *inner regular*. This means $\mu(A)$ is the supremum of the measures of the compact subsets of $A$.

**Proposition C.3.** *There is a continuous homomorphism $\Delta : G \to \mathbb{R}^+$, such that, if $\mu$ is any (left or right) Haar measure on $G$, then $\mu(gAg^{-1}) = \Delta(h)\mu(A)$, for all $g \in G$ and any Borel set $A \subset G$.*

*Proof.* Let $\mu$ be a left Haar measure. For each $g \in G$, define $\phi_h : G \to G$ by $\phi_g(x) = gxg^{-1}$. Then $\phi_g$ is an automorphism of $G$, so $(\phi_g)_*\mu$ is a left Haar measure. By uniqueness, we conclude that there exists $\Delta(g) \in \mathbb{R}^+$, such that $(\phi_g)_*\mu = \Delta(g)\mu$. It is easy to see that $\Delta$ is a continuous homomorphism. If $\mu$ is a left Haar measure it is easy to see that $\tilde{\mu}(A) := \mu(A^{-1})$ is a right Haar measure and to verify the same formula also applies to it. □

**Definition C.4.** The function $\Delta$ defined in Proposition C.3 is called the *modular function* of $G$.

**Corollary C.5.** *Let $\Delta$ be the modular function of $G$, and let $A$ be a Borel subset of $G$.*

*i) If $\mu$ is a right Haar measure on $G$, then $\mu(gA) = \Delta(g)\mu(A)$, for all $g \in G$.*

*ii) If $\mu$ is a left Haar measure on $G$, then $\mu(Ag) = \Delta(g^{-1})\mu(A)$, for all $g \in G$.*

*iii) $G$ is unimodular if and only if $\Delta(g) = 1$, for all $g \in G$.*

*iv) $\Delta(g) = |\det(Ad_g)|$, for all $g \in G$.*

**Proposition C.6.** *Let $\mu$ be a left Haar measure on a Lie group $G$. Then $\mu(G) < \infty$ if and only if $G$ is compact.*

*Proof.* ($\Leftarrow$): See Proposition C.1

($\Rightarrow$): Since $\mu(G) < \infty$ (and the measure $\mu$ is inner regular), there exists a compact subset $C$ of $G$, such that $\mu(C) > \mu(G)/2$. Then, for any $g \in G$, we have

$$\mu(gC) + \mu(C) = \mu(C) + \mu(C) = 2\mu(C) > \mu(G),$$

so $gC$ can not be disjoint from $C$. This implies that $g$ belongs to the set $C \cdot C^{-1}$, which is compact. Since $g$ is an arbitrary element of $G$, we conclude that $G = C \cdot C^{-1}$ is compact. □

# APPENDIX D

## Arithmetic Subgroups

A lattice of the form $G_{\mathbb{Z}} = G \cap \mathrm{SL}(n, \mathbb{Z})$ is said to be *arithmetic*. However, for the following reasons, a somewhat more general class of lattices is also said to be arithmetic. The reason is that there are some obvious modifications of $G_{\mathbb{Z}}$ that are also lattices, and they should also be regarded as arithmetic subgroups. We want to modify the definition so that:

1. If $\phi : G_1 \to G_2$ is an isomorphism, and $\Gamma_1$ is an arithmetic subgroup of $G_1$, then we wish to be able to say that $\phi(\Gamma_1)$ is an arithmetic subgroup of $G_2$.
2. We wish to ignore compact groups, that is, modding out a compact subgroup should not affect arithmeticity. So we wish to be able that if $K$ is a compact normal subgroup of $G$, and $\Gamma$ is a lattice in $G$, then $\Gamma$ is arithmetic if and only if $\Gamma K/K$ is an arithmetic subgroup of $G/K$.
3. And finally, arithmeticity should be independent of commensurability.

First we need the following definition.

**Definition D.1.** For a subset $\mathcal{Q}$ of $\mathbb{Q}[x_{1,1}, \cdots, x_{n,n}]$, we define

$$\mathrm{Var}(\mathcal{Q}) := \{g \in \mathrm{SL}(n, \mathbb{R}) \mid Q(g) = 0, \text{ for all } Q \in \mathcal{Q}\},$$

which is a subgroup of $\mathrm{SL}(n, \mathbb{R})$. Let $H$ be a closed subgroup of $\mathrm{SL}(n, \mathbb{R})$. We say that $H$ is *defined over* $\mathbb{Q}$ (or that $H$ is a $\mathbb{Q}$-*subgroup*) if there exists a subset $\mathcal{Q} \subset \mathbb{Q}[x_{1,1}, \cdots, x_{n,n}]$ such that $H^\circ = \mathrm{Var}(\mathcal{Q})^\circ$, and $H$ has only finitely many components. In other words, $H$ is commensurable to the variety $\mathrm{Var}(\mathcal{Q})$, for some set $\mathcal{Q}$ of $\mathbb{Q}$-polynomials.

*Examples.*

1. $\mathrm{SL}(n, \mathbb{R})$ is defined over $\mathbb{Q}$, just let $\mathrm{Var}(\mathcal{Q}) = \emptyset$.

2. If $m < n$, we may embed $\mathrm{SL}(m, \mathbb{R})$ in the top left corner of $\mathrm{SL}(n, \mathbb{R})$. This copy of $\mathrm{SL}(m, \mathbb{R})$ is defined over $\mathbb{Q}$: Let $Q = \{x_{i,j} - \delta_i^j \mid \max\{i, j\} > n\}$.

We state the following important theorem from Morris [31, p.88] without proof.

**Theorem D.2.** *If $G$ is defined over $\mathbb{Q}$, then $G_{\mathbb{Z}}$ is lattice in $G$.*

This theorem immediately implies that $\mathrm{SL}(n, \mathbb{Z})$ is a lattice in $\mathrm{SL}(n, \mathbb{R})$. These considerations lead us to the following definition:

**Definition D.3.** $\Gamma$ is an *arithmetic* subgroup of $G$ if and only if there exist

i) a closed, connected, semisimple subgroup $G'$ of some $\mathrm{SL}(n, \mathbb{R})$, such that $G'$ is defined over $\mathbb{Q}$,

ii) compact normal subgroups $K$ and $K'$ of $G^\circ$ and $G'$, respectively, and

iii) an isomorphism $\phi : G^\circ/K \to G'/K'$,

such that $\phi(\overline{\Gamma})$ is commensurable to $\overline{G'_{\mathbb{Z}}}$, where $\overline{\Gamma}$ and $\overline{G'_{\mathbb{Z}}}$ are the images of $\Gamma \cap G^\circ$ and $G'_{\mathbb{Z}}$ in $G^\circ/K$ and $G'/K'$, respectively.

$\mathrm{SL}(n, \mathbb{Z})$ is the most basic example of an arithmetic group. In Section 4.3 we present a quaternionic structure of $\mathrm{SL}(2, \mathbb{R})$ and we make arithmetic subgroups $\mathrm{SL}(1, \mathbb{H}_{\mathbb{Z}}^{a,b})$ which are cocompact lattices of $\mathrm{SL}(2, \mathbb{R})$. Note that up to conjugacy, there are only countably many arithmetic lattices in $G$, because there are only countably many finite subsets of the polynomial ring $\mathbb{Q}[x_{1,1}, \cdots, x_{n,n}]$. The following theorem is a very helpful criterion for arithmeticity. Recall that the subgroup

$$\mathrm{Comm}_G(\Gamma) = \{g \in G \mid g\Gamma g^{-1} \text{ is commensuarble to } \Gamma\}$$

is called the *commensurator* of $\Gamma$ in $G$. It is easy to see that if $G$ is defined over $\mathbb{Q}$, then $G_{\mathbb{Q}} \subset \mathrm{Comm}_G(G_{\mathbb{Z}})$.

For a connected semisimple non-compact Lie group $G$, a lattice $\Gamma \subset G$ is called *irreducible* if for every non-compact closed normal subgroup $N$ of $G$, $\Gamma N$ is dense in $G$. In particular, lattices of the form $\Gamma_1 \times \Gamma_2 \subset G_1 \times G_2$ are excluded from this definition and are called *reducible*. Note that $\mathrm{SL}(n, \mathbb{R})$ is a simple Lie group, thus all lattices $\Gamma \subset \mathrm{SL}(n, \mathbb{R})$ are irreducible.

**Theorem D.4** (Commensurability Criterion for Arithmeticity)**.** *Let $G$ be a connected semisimple Lie group with no compact factors, and $\Gamma \subset G$ an irreducible lattice. Then $\Gamma$ is arithmetic if and only if $Comm_G(\Gamma)$ of $\Gamma$ is dense in $G$.*

The astonishing theorem due to Gregory Margulis shows that taking the integer points is usually the only way to make a lattice. If $G$ is a semisimple algebraic group defined over the field $\mathbb{F}$, then $\mathbb{F}$-rank$(G)$ is defined to be the maximal dimension of an abelian $\mathbb{F}$-subgroup of $G$ which is $\mathbb{F}$-split, i.e. which can be diagonalized over $\mathbb{F}$. If $G$ is a connected semisimple Lie group then we can realize $\mathrm{Ad}(G)$ as a subgroup of finite index in the $\mathbb{R}$-points of an $\mathbb{R}$-group (See Zimmer [38, Proposition 3.1.6]). We then define $\mathbb{R}$-rank$(G)$ to be the $\mathbb{R}$-rank of this algebraic group. Thus $\mathbb{R}$-rank$(\mathrm{SL}(n, \mathbb{R})) = n - 1$, the $\mathbb{R}$-split abelian subgroup of maximal dimension being diagonal matrices of dimension one.

**Theorem D.5** (Margulis Arithmeticity)**.** *Let $G$ be a connected semisimple Lie group with trivial center and no compact factors. Let $\Gamma \subset G$ be an irreducible lattice. Assume $\mathbb{R}$-rank$(G) \geq 2$. Then $\Gamma$ is arithmetic.*

Since $\mathrm{rank}(\mathrm{SL}(n, \mathbb{R})) = n - 1$, it follows that:

**Corollary D.6.** *Every lattice $\Gamma \subset SL(n, \mathbb{R})$, $n \geq 3$, is arithmetic.*

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