KNOT THEORY OF MORSE-BOTT CRITICAL LOCI

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ABSTRACT

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We give an alternative proof of that a critical knot of a Morse-Bott function $f : S^3 \to \mathbb{R}$ is a graph knot where the critical set of f is a link in S^3 [8] [9] [11] [12]. Our proof inducts on the number of index-1 critical knots of f as in [12].

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Chapter 1

Introduction

We will define and study a specific type of Morse-Bott functions [**1**] $f : S^3 \to \mathbb{R}$, called *a k-function* in this thesis, to study knots or links in S^3 where each critical component of f is a knot in S^3. A knot or link in S^3 is called k-mate if it is a sublink of the critical link of some k-function. Our main theorem (Theorem 1) will classify all the k-mate knots in S^3 which are precisely *the graph knots* [**11**] in S^3. Our treatment will be similar to the study of an ordinary Morse function on a compact manifold and we will adopt the standard notions of Morse theory [**13**] in our work .

Unfortunately, there are many different but equivalent definitions of a graph knot or link in a compact 3-manifold M. Our definition in this work will give one equivalent definition for a graph knot in S^3 but it will slightly fall short to capture all the graph links in S^3. A standard definition can be taken as: an irreducible link L in an irreducible, compact, connected 3-manifold M is an irreducible graph link if the JSJ-decomposition [**2**], [**3**] of $M - N(L)$ consists of only Seifert fibered pieces where $N(L)$ is an open tubular neighborhood of L in M [**11**]. Such 3-manifolds are called *graph manifolds* [**4**]. Another characterization of this link L is that *the Gromov volume* of $M - N(L)$ is zero [**5**], [**6**], [**7**]. For compact, connected M with ∂M being equal to a (possibly empty) union of tori or klein bottles, another charactirization is that $L \subseteq M$ is a graph link if L is a subset of the hyperbolic closed orbits of a nonsingular Morse-Smale flow on M [**9**]. There, Morgan's main interest had actually been the existence of nonsingular Morse-Smale flows on M rather than studying links in M but he proved that when M is prime, there exists such a flow on M if and only if M is a graph manifold. For M not necessarily prime, he proved that there exists such a flow on M if and only if each prime summand of M is a graph manifold. Such flows on M are bijectively associatied to *round handle decompositions of* M [**9**], [**10**]. Here, a round handle is either homeomorphic to a solid torus or a solid Klein bottle. We will describe in our work how a k-function induces a round handle decomposition of S^3.

It is shown in [**8**], [**11**], [**12**] that a graph knot K is obtained from an unknot by a finite application of connected sum or cabling operations and we will show that all the k-mate knots in S^3 arise in the same way. We will formally emphasize this elementary perspective of graph knots in Definition 3. In literature, [**9**] seems to be the earliest source to be credited for the classification of graph knots even though the connected sum operation has been overlooked there.

Even though our main theorem will classify the graph knots in a different way, our topological ideas and methods will be close to the ones in [**12**]. The results in [**11**], [**12**] are stronger than ours as they effectively classify all the graph links in S^3. Moreover, the classification in [**11**] studies graph links in a homology 3-sphere M. We will make further remarks on these important sources in the Conclusion section.

Chapter 2

Basic Notions and Results

Definition 1. A real valued smooth function $f : S^3 \to \mathbb{R}$ is *a k-function* if:

(i) The set of critical points of f is a link L in S^3

(ii) The Hessian of f is nondegenerate in the normal direction to L.

(iii) Each knot K in L has a tubular neighborhood U in S^3 with local coordinates (θ, x, y) such that $f(\theta, x, y) = c^2(\pm x^2 \pm y^2) + d$ where $(\theta, 0, 0)$ are the coordinates for K and c, d are scalars $(c \neq 0)$.

The link L is called *the critical link of f* and a component of L is called *a critical knot of f*. The neighborhood U in (iii) is called *a k-model neighborhood of K*. We make an abuse of notation by identifying $U \subseteq S^3$ with $S^1 \times D^2$ where D^2 is the unit disk. The notation (θ, x, y) will refer to such local coordinates of a critical knot throughout this work.

Definition 2. A link L is *k-mate* if L is a subset of the critical link of some k-function.

The basic question is then which knots in S^3 are k-mate. Our main theorem below answers this question. We will provide a proof of it after Theorem 9.

Theorem 1. *A knot is k-mate if and only if it is a graph knot.*

A critical knot K of a k-function f is called a source, sink or saddle respectively if the signs in $f(\theta, x, y) = c^2(\pm x^2 \pm y^2) + d$ are both positive, both negative or opposite respectively. We adopt *the sign conventions in* $f(\theta, x, y) = c^2(y^2 - x^2) + d$ for a saddle.

In the saddle K case, the two circles $S^1 \times (\pm 1, 0)$ are called *stable circles of K in U* and the two circles $S^1 \times (0, \pm 1)$ are called *the unstable circles of K in U*. Similarly, the annulus $S^1 \times [-1,1] \times \{0\}$ is called *the stable annulus of K in U* and the annulus $S^1 \times \{0\} \times [-1,1]$ is called *the unstable annulus of K in U*. Note that neither k-model coordinates (θ, x, y) nor stable or unstable circles of a saddle are unique. The stable and unstable circles of a saddle K can be isotoped to K within the stable or unstable annulus so that they are parallel cable knots of K. They homologically have ± 1 longitude coefficients (and some arbitrary meridian coefficient) in $H_1(\partial U)$. Note that the property (iii) of Definition 1 is not necessary for sources or sinks but it puts a restriction on our saddles. When it is dropped, the stable or unstable regions of a saddle can be a nontrivial line bundle over the saddle (i.e. a Möbius band).

A point in $f(L) \subseteq \mathbb{R}$ where L is the critical link of a k-function f is called *a critical value of f* and a point in $\mathbb{R} - f(L)$ is called *a regular value of f*. We will define *an ordered k-function* later and Lemma 6 shows that the preimage of a regular value of an ordered k-function is a collection of disjoint tori in S^3.

Given a k-function f, there exists *a gradient like vector field X on S^3 for f*. More precisely, $X_p(f)$ is positive if p is not a critical point of f and also, $X(\theta, x, y) = c^2 \cdot (\pm 2x \frac{\partial}{\partial x} \pm 2y \frac{\partial}{\partial y})$ around a critical knot of f (see e.g. **[14]** for the existence of a gradient like vector field for a Morse function). The function f is increasing on the forward flow lines of X. For any point p in S^3 , the flow line $X_t(p)$ converges to a critical point of f as $t \to \pm\infty$. An important application of X is that the flow of X gives an isotopy between the regions $f^{-1}((-\infty, r])$ and $f^{-1}((-\infty, r+\epsilon])$ in S^3 (here, $\epsilon > 0$) when $f^{-1}([r, r+\epsilon])$ does not contain any critical points of f.

A source of a k-function is a sink of $-f$ and vice versa. We may always assume a k-mate knot to be a source of some k-function by the following lemma.

Lemma 2. *A knot K is a source of some k-function if and only if it is a saddle of some k-function.*

Proof. We will prove only one direction as the other one is similar. Suppose that K is a saddle of a k-function f. Let $\tilde{U}$ be a k-model neighborhood of K. Let D be the disk of radius $1/2$ centered at the origin in $\mathbb{R}^2$ and consider the smaller k-model neighborhood $U = S^1 \times D$ of K inside $\tilde{U}$.

Consider the isotopic knots $K_1 := S^1 \times (-2/3, 0)$ and $K_2 := S^1 \times (-5/6, 0)$ and take a small tubular neighborhood V_i of K_i in $\mathrm{Int}(\tilde{U})$ so that V_i intersects each meridian disk $\{\theta\} \times D^2$ of $\tilde{U}$ in a disk (See Fig. 2.1). Moreover, the intersections $V_1 \cap U = \partial V_1 \cap \partial U$ and $V_1 \cap V_2 = \partial V_1 \cap \partial V_2$ are both annuli the core of which are isotopic to K and also, $V_2 \cap U = \varnothing$. We can define a k-function f_1 by modifying f only within $\tilde{U} - U$ so that V_1 contains a k-model neighborhood of the source K_1 of f_1 and V_2 becomes a k-model neighborhood of the saddle K_2 of f_1. Here, K is isotopic to the source K_1.

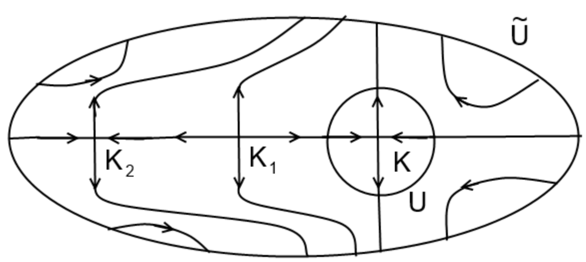

Figure 2.1: The figure shows some of the trajectories of a gradient like vector field for f_1.

□

Lemma 3. *Let f be a k-function without any saddles. Then, f has a single source and a single sink which form a Hopf link in S^3.*

Proof. Since S^3 is closed, f has at least one source K_1 and a sink K_2. Let X be a gradient like vector field for f and X_t be the flow of X. Take a point p in ∂U_1 where U_1 is a k-model neighborhood of K_1. The point $X_t(p)$ will be in a k-model neighborhood U_3 of a sink K_3 of f for large enough t since f has no saddles. Say, $X_{a_p}(p) \in U_3$ for some $a_p > 0$. Since ∂U_1 is compact and connected and f does not have any saddles, we have $X_a(\partial U_1) \subseteq U_3$ for some time $a \geq a_p$. We may assume that $X_a(\partial U_1) = \partial U_3$ after scaling X with a positive smooth function on S^3 if necessary. Then, $X_a(U_1) \cup U_3$ is an embedded, closed and connected 3-manifold in the closed and connected S^3. Therefore, $X_a(U_1) \cup U_3$ is S^3. Hence, the source K_1 and the sink $K_3 = K_2$ are the only critical knots of f. Let P_1 and P_3 denote the core of the solid tori of $X_a(U_1)$ and U_3 respectively. The union $X_a(U_1) \cup U_3$ gives a lens space description of S^3 so that the two solid tori $X_a(U_1)$ and U_3 are two complementary standard solid tori in S^3 by the topological classification of lens spaces. Therefore, $P_1 \cup P_3 \simeq K_1 \cup K_3$ is a Hopf link. $\square$

The above lemma shows that an unknot is k-mate. The next two lemmas describe a way to construct other k-mate knots and as we will show later in Theorem 1, all k-mate knots arise in this way starting with the unknot.

A knot K is a cable knot of J if K can be isotoped into ∂U where U is a closed tubular neighborhood of J in S^3. Here, K is allowed to bound a disk in ∂U so that an unknot is a trivial cable knot of any knot J. Even when K does not bound a disk in ∂U so that K is not a trivial cable knot of J, the cable knot K can be a meridian of J or a longitude of an unknot J so that K is still a trivial knot. We will use the notation $K \simeq J_{p,q}$ which says that the cable knot K of J is homologically p longitudes plus q meridians of J. We will sometimes conveniently suppress the coefficients p and q and use the notation $K \succ J$ instead.

Lemma 4. *A cable knot K of a k-mate knot J is k-mate.*

Proof. If K is trivial, then it is k-mate by Lemma 1. Otherwise, K can be isotoped to be transverse to the meridian disks of k-model neighborhood U of J. We may assume that J is a source of a k-function f by Lemma 2. The rest of the proof will follow exactly as in that lemma where f gets modified only within U but still preserving a smaller tubular neighborhood of J. The knot K becomes another source and a saddle isotopic to K gets inserted between K and J. □

A connected sum $K_1 \# K_2$ of two knots K_1 and K_2 is not well defined in general unless both K_1 and K_2 and their ambient spaces S^3's are all oriented. One can regard K_1 and K_2 as a split link in the same ambient space S^3 and an orientation of this single S^3 can be fixed easily. However, a k-function does not induce a natural orientation on a critical knot of it. While we study k-functions, we will strictly work with *unoriented knots.* The notation $K_1 \# K_2$ will then denote a knot in the set $\{K_1^+ \# K_2^+, K_1^+ \# K_2^-\}$ of knots where $K_i^{\pm}$ specifies an orientation of K_i.

Lemma 5. *A connected sum $K_1 \# K_2$ of k-mate knots K_1 and K_2 is k-mate.*

Proof. The k-mate knots K_1 and K_2 are sources of some k-functions f_1 and f_2 by Lemma 2 respectively. Let S be a sphere in S^3 which yields the connected summands K_1 and K_2 of $K_1 \# K_2$. Let $\tilde{S}$ be a small closed tubular neighborhood of S in S^3 so that $\tilde{S} \cap K_1 \# K_2$ is two unknotted arcs in $\tilde{S} \simeq S^2 \times [0,1]$. Let V be a small closed tubular neighborhood of $K_1 \# K_2$ in S^3 such that $V \cap \partial \tilde{S}$ is four disjoint disks and also, $V \cup \tilde{S}$ is smoothly embedded in S^3. Let J denote the core of the annulus $S - V$. The region $S^3 - \mathrm{Int}(V \cup \tilde{S})$ has two connected components each of which is diffeomorphic to the complement of K_1 or K_2 in S^3. Let R_i denote the component of $S^3 - \mathrm{Int}(V \cup \tilde{S})$ that is diffeomorphic to the complement of K_i.

Let U_i be a small closed k-model neighborhood of the source K_i of f_i. We may assume that $S^3 - \mathrm{Int}(U_i) = R_i$ by isotoping U_i in S^3. See Figure 2.2. By using the flow of the gradient like vector fields for f_1 and f_2, we may scale f_1 and f_2 and add some constants so that they agree in a small tubular neighborhood of ∂U_1 or ∂U_2 with $f_1(\partial U_1) = f_2(\partial U_2) = 1$ (see e.g. **[14]** for such scaling of f_i).

We can now define a k-function f such that:

(i) V is a k-model neighborhood of the source $K_1 \# K_2$ of f with $f(K_1 \# K_2) = 0$

(ii) $\tilde{S} - V$ contains a k-model neighborhood of the saddle J of f with $f(J) = 0.5$ and also, $S - V$ is a stable annulus of J in $\tilde{S} - V$.

(iii) $f_{|U_i} = f_{i|U_i}$

Therefore, $K_1 \# K_2$ is k-mate.

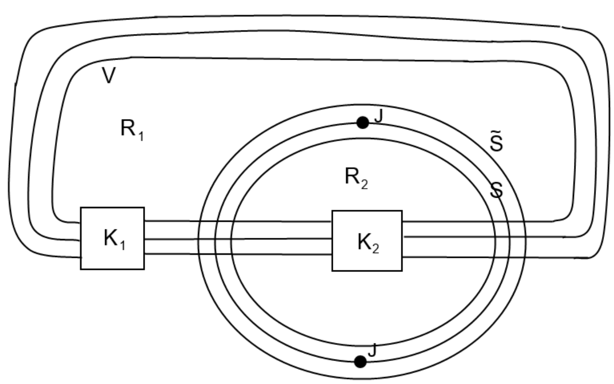

Figure 2.2: The situation for $K_1 \# K_2$

□

Chapter 3

Round Handle Decomposition of an ordered k-function

Before studying the preimage of a regular value of f and how it changes when we pass a critical level, we first introduce *ordered k-functions* where we make local modifications near the critical link L of f without changing the critical set of f or the type of each component of L. The number $\epsilon > 0$ will denote a sufficiently small positive number throughout the text.

For a source K of f with local k-model coordinates (θ, x, y), we can use an increasing smooth function $h : [0, 1] \to (-\infty, 0]$ with $h(z) = 0$ near $z = 1$ and linear near $z = 0$ to change f locally by redefining $\tilde{f}(\theta, x, y) := f(\theta, x, y) + h(x^2 + y^2) \leq f(\theta, x, y)$ so that f can have arbitrarily small values on the source K. Similarly, f can be redefined near a sink to have an arbitrarily large value on it. For a saddle K of f, we can use a decreasing (or increasing) smooth $h : [0, 1] \to [0, \epsilon]$ (or $[-\epsilon, 0]$) and $h(z) = \pm\epsilon$ near $z = 0$ and $h(z) = 0$ near $z = \pm\epsilon$. We can then redefine f near K as $\tilde{f}(\theta, x, y) = f(\theta, x, y) + h(x^2 + y^2)$ which changes the saddle value $f(K)$ by $\pm\epsilon$. Here, $|h'(z)|$ is also small enough so that K remains to be a saddle of $\tilde{f}$ without creating any other critical points.

An ordered k-function f has then the following properties:

(i) The critical values of the critical knots of f are all distinct.

(ii) The critical values of f are ordered as: source values $\leq$ saddle values $\leq$ sink values.

Say, $a_1 < \cdots < a_j < b_1 \cdots < b_k < c_1 < \cdots < c_m$ where a_i, b_i and c_i correspond to a source, a saddle and a sink of a ordered k-function f respectively. Recall the smallness of ϵ: if z_0 is a critical value of f, then z_0 is the only critical value of f in $[z_0 - \epsilon, z_0 + \epsilon]$. We now describe a round handle decomposition **[9]** of S^3 by analyzing the preimages of an ordered k-function f. Such an analysis will be repeatedly used in our proofs.

Start with $r < a_1$ having $f^{-1}((-\infty, r]) = \varnothing$. When we increase r, each time r passes a source value of f, the preimage $f^{-1}([a_1, r])$ will have one more solid torus in S^3; *a round 0-handle* is attached to the empty set. The region $f^{-1}([a_1, a_i + \epsilon])$ will consist of i disjoint solid tori.

When we pass b_1, the preimage $\tilde{V} := f^{-1}([a_1, b_1 + \epsilon])$ is the union of $V := f^{-1}([a_1, b_1 - \epsilon])$ which consists of j disjoint solid tori and a region $f^{-1}([b_1 - \epsilon, b_1 + \epsilon])$. One connected component R_1 of this latter region is a solid torus that contains a k-model neighborhood of the saddle K_1 where K_i is the saddle of f with $f(K_i) = b_i$. The component R_1 contains a tubular neighborhood $\tilde{A}_1$ of the stable annulus of K_1 in $f^{-1}([b_1 - \epsilon, b_1 + \epsilon])$. For an appropriate choice of $\tilde{A}_1$, one can show that $f^{-1}([a_1, b_1 + \epsilon])$ is isotopic to $V \cup \tilde{A}_1$ in S^3 (see e.g. **[13]** for a Morse analogue of this fact). This tubular neighborhood $\tilde{A}_1$ is attached to V along two disjoint annuli in $\partial V \cap \partial \tilde{A}_1$ the cores of which are isotopic to the unstable circles of K. The region $f^{-1}([a_1, b_1 + \epsilon])$ is topologically equivalent to the union of $f^{-1}([a_1, b_1 - \epsilon])$ and a solid torus $\tilde{A}_1$ in S^3 that intersect each other along two parallel annuli in $\partial \tilde{A}_1 \cap V$ the cores of which have ± 1 longitude coefficients in $H_1(\partial \tilde{A}_1)$. In this case, $\tilde{A}_1$ is *a round 1-handle* that is attached to V along two annuli that are tubular neighborhoods of stable circles in ∂V.

The consecutive passes of saddle values of f look similar. Each time we pass a saddle value b_i, the region $f^{-1}([a_1, b_i + \epsilon])$ is isotopic to the union $V := f^{-1}([a_1, b_i - \epsilon])$ and a

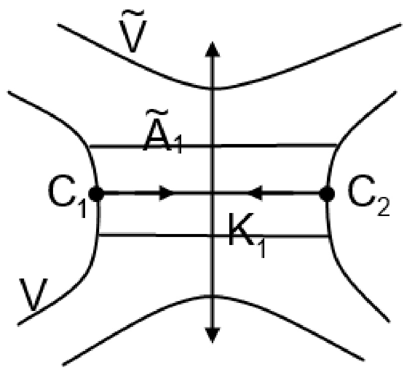

Figure 3.1: A $\{\theta\} \times D^2$ cross section of a k-model neighborhood of K_1

solid torus $\tilde{A}_i$ in S^3 where $V \cap \tilde{A}_i = \partial V \cap \partial \tilde{A}_i$ consists of two parallel annuli the cores of which have ± 1 longitude coefficients in $H_1(\partial \tilde{A}_i)$. Here, the solid torus $\tilde{A}_i$ is a tubular neighborhood of a stable annulus of the saddle K_i. When we pass b_i, the boundary of the preimage changes from ∂V to $\partial(V \cup \tilde{A}_i)$ by a surgery on the two stable circles C_1 and C_2 of K_i in $\partial V \cap \partial \tilde{A}_i$. Specifically, a tubular neighborhood $\simeq S^1 \times \partial I \times I$ of C_1 and C_2 in ∂V gets replaced by another two disjoint annuli $\simeq S^1 \times I \times \partial I$ which is now a tubular neighborhood of the unstable circles of K_i in $\partial(V \cup \tilde{A}_1)$. They have the following identification where $I = [-1, 1]$:

$$S^1 \times \partial I \times (I - \{0\}) \xrightarrow{\simeq} S^1 \times (I - \{0\}) \times \partial I$$

$(\theta, x, ty) \longrightarrow (\theta, tx, y)$ where $t \in (0, 1]$ and $x, y = \pm 1$

We will use the notation $s(\cdot)$ to denote a surgered surface in S^3 coming from the pass of a saddle value of f so that in the above situation, the surface $s(\partial V)$ is isotopic to $\partial(V \cup \tilde{A}_i)$ in S^3.

When we pass the first sink value c_1, a k-model neighborhood U of the sink in $f^{-1}(c_1)$ fills in $V := f^{-1}([a_1, c_1 - \epsilon])$ in S^3. The region $f^{-1}([a_1, c_1 + \epsilon])$ is isotopic to $V \cup U$ in S^3 where $V \cap U = \partial V \cap \partial U$ is a torus. Here, U is *a round 2-handle* that is attached to V along a torus. So, the boundary of $f^{-1}([a_1, c_1 - \epsilon])$ is m disjoint tori and each time we pass a sink value, one of those m tori is filled in by a solid torus coming from a k-model neighborhood of a sink. This process ends with $f^{-1}([a_1, c_m]) = S^3$.

As we have explained above, an ordered k-function on S^3 induces a round handle decomposition of S^3. The converse induction almsot holds. The only exceptions where this induction fails depend on the round 1-handles. In our definition of a saddle, the stable and unstable annuli of a saddle are trivial line bundles over S^1. In the round 1-handles defined in **[9]**, such unstable and stable line bundles over S^1 may be non-trivial; i.e. they can be Möbius bands. Our restriction on saddles implies that every k-mate link is a graph link but not necessarily the converse. However, we will still be able to provide an alternative proof to Theorem 1 which states that a knot is k-mate if and only if it is a graph knot. Lemma 2 shows that a knot can be realized as a saddle of a k-function if and only if it can be realized as a source of a k-function and our definition of a source is general without any restrictions.

We briefly remark that each ordered k-function naturally defines a Morse function on S^3 with the following indices of critical points: A pair of 0 and 1 from a source, a pair of 1 and 2 from a saddle and a pair of 2 and 3 from sink. For each such a $\{j, j+1\}$ critical index pair, *the attaching sphere* of the $j+1$-handle intersects *the belt sphere* of the j-handle geometrically twice.

The converse also holds. If $f_0 : S^3 \to \mathbb{R}$ is a Morse function such that:

(i) The critical values of f_0 are all distinct.

(ii) The indices of the critical points of f_0 come in adjacent pairs and their critical values are ordered on the real line $\mathbb{R}$ as: First $\{0, 1\}$ pairs, then $\{1, 2\}$ pairs and then $\{2, 3\}$ pairs

(iii) For each $\{j, j+1\}$ index pair of critical points p_j and p_{j+1} of f_0 respectively, the attaching sphere of the $j+1$-handle of p_{j+1} intersects the belt sphere of the j-handle of p_j geometrically twice (also, with the same signs when $\{j, j+1\} = \{1, 2\}$).

Then, f_0 induces a ordered k-function f on S^3. Each $\{0,1\}, \{1,2\}$ and $\{2,3\}$ index pair of paired critical points of f_0 gives rise to a source, saddle and a sink of f respectively.

An alternative proof of the below lemma is in **[9]**.

Lemma 6. *If f is an ordered k-function and r is a regular value of f, then each connected component of $f^{-1}(r)$ is an embedded torus in S^3.*

Proof. Let $a_1 < \cdots < a_j < b_1 \cdots < b_k < c_1 < \cdots < c_m$ denote the critical values of f where a_i, b_i and c_i correspond to a source, a saddle and a sink of f respectively. The lemma is clear for $r < b_1$ or $r > b_k$. Assume now that for some $b_1 < r < b_k$, the surface $f^{-1}(r)$ has a non-torus component. Since a torus has Euler characteristic 0 and the Euler characteristic of $f^{-1}(b_i - \epsilon)$ does not change after a surgery during the pass of b_i, there exists a sphere $\hat{S}_1$ in some $f^{-1}(w_1)$ where w_1 is a regular value of f.

The sphere $\hat{S}_1$ bounds a 3-ball on each side in S^3 and let B_1 denote the one of them such that $B_1 \cap f^{-1}(w_1 - \epsilon) = \varnothing$. As $f^{-1}(z)$ is a union of tori for a regular value z with $z > b_k$, there must be a surgery on a sphere S_1 isotopic to $\hat{S}_1$ in B_1 during the pass of a saddle value but it may happen that the surface $s(S_1)$ produced by surgery contains a sphere in B_1. Take a regular value $w_2 \geq w_1$ large enough such that $f^{-1}(w_2)$ contains a sphere S_2 in B_1 but the produced surface $s(S_2)$ does not contain a sphere after the pass of a saddle value β_2. Moreover, we can find such w_2 and S_2 such that the 3-ball B_2 bounded by S_2 in S^3 with $B_2 \cap f^{-1}(w_2 - \epsilon) = \varnothing$ satisfies $B_2 \subseteq B_1$.

Let K_2 be the saddle with $f(K_2) = \beta_2$. Let A_2 denote the stable annulus of K_2 and

$\{C_1, C_2\} = \partial A_2$ denote the stable circles of K_2 with $C_1 \subseteq S_2$. Then, C_2 is not in S_2 but in another component Σ of $f^{-1}(\beta_2 - \epsilon)$ with genus greater than 1 because $s(S_2)$ does not contain a sphere.

The existence of such Σ with big genus implies that B_2 contains at least one source K and $B_2 \cap f^{-1}(b_1 - \epsilon) \neq \varnothing$. Moreover, a surgery must happen on (not necessarily distinct) tori T_a and T_b in B_2 containing the stable circles C_a and C_b of a saddle respectively, such that both C_a and C_b bound disjoint disks in T_a and T_b respectively. This surgery then produces also a sphere S_3. Say, $S_3 \subseteq f^{-1}(w_3)$. Let B_3 denote the 3-ball bounded by S_3 in S^3 such that $B_3 \cap f^{-1}(w_3 - \epsilon) = \varnothing$. We can find such S_3 and B_3 such that $B_3 \subseteq B_2$ and $K \subseteq B_2 - B_3$. We can now apply our last argument to $S_3 = \partial B_3$ instead of $S_1 = \partial B_1$ to conclude that B_3 contains at least one source J with $J \neq K$, $B_4 \subseteq B_3$ and $J \subseteq B_3 - B_4$ where B_4 is a 3-ball in S^3 and the sphere ∂B_4 is in the preimage of a regular value of f. Therefore, B_1 contains infinitely many sources of f and we have reached the desired contradiction. $\square$

Chapter 4

Characterization of k-mate knots

Theorem 9 will essentially state the equivalence between k-mate knots and graph knots. Its proof will be divided into several cases most of which will be handled locally. These local arguments are not difficult when one studies each possible situation with a careful thought. There is an exceptional case (the Subcase 3 of Case 3) though where a local analysis does not suffice as in the previous cases and we will employ various technical methods to tackle this difficult case. We first prove the below technical lemma which will provide us a nonlocal picture in S^3 in certain situations. We will later define *a graph kit of a graph knot* (Definition 4) and strengthen the statement of Theorem 9 with parts (i), (ii) and (iii) there to obtain a more global picture in this exceptional case. We will be able to complete the proof of this case with these extra technical details at hand.

Suppose that K_1 and K_2 are two unknots such that K_1 is a cable knot of K_2. Since both K_1 and K_2 are unknots, K_2 is then a cable knot of K_1 as well. If it is a trivial cabling or K_2 is a longitude of K_1, then $K_1 \cup K_2$ is a split link of two unknots. If $lk(K_1, K_2) = \pm 1$, then $K_1 \cup K_2$ is a Hopf link. The below lemma exposes such cabled two unknots K_1 and K_2 in generality but we will encounter many split links of two unknots or Hopf links in its proof.

Lemma 7. *Suppose that K is a critical unknot of an ordered k-function f and K_R is the unknot core of an unknotted solid torus R in S^3 such that $K \cap R = \varnothing$ and also, ∂R is in* *the preimage of a regular value of f. Then, the unknots K and K_R are cable knots of each other.*

Proof. If f has no saddles, then Lemma 3 shows that $K \cup K_R$ is a Hopf link. Assume now that f has a saddle and let $a_1 < \cdots < a_j < b_1 < \cdots < b_k < c_1 < \cdots < c_m$ denote the critical values of f where a_i, b_i and c_i correspond to a source, a saddle and a sink of f respectively. Let K_1 be the saddle of f with $f(K_1) = b_1$. We will induct on the number of saddles k by analyzing the stable circles C_1 and C_2 of K_1 in $f^{-1}(b_1 - \epsilon)$ and also a stable annulus A of K_1 in $f^{-1}([b_1 - \epsilon, b_1])$. We may assume $r \notin [b_1 - \epsilon, b_1 + \epsilon]$ where $\partial R \subseteq f^{-1}(r)$. The circles C_1 and C_2 are cable knots of (not necessarily distinct) sources P_1 and P_2 respectively. Let E_i denote the solid torus component of $f^{-1}([a_1, b_1 - \epsilon])$ containing P_i. Let E denote the component of $f^{-1}([a_1, b_1 + \epsilon])$ which contains $P_1 \cup P_2 \cup K_1$. In each case below, we will define an ordered k-function f_1 with at most $k - 1$ saddles.

Case 1. *Only one of C_1 and C_2 bounds a disk in $f^{-1}(b_1 - \epsilon)$.*

Say, C_1 bounds a disk D in $f^{-1}(b_1 - \epsilon)$ so that $K_1 \simeq C_1$ is an unknot. Then, P_2 cannot be nontrivial. Otherwise, C_2 must be a meridian of P_2 and P_2 will intersect the sphere, which is the union of D, A and a meridian disk of P_2, geometrically once. As $H_2(S^3) = 0$, such a single geometric intersection of a 1-cycle and a 2-cycle of S^3 is not possible. So, P_2 is an unknot and C_2 is a longitude of P_2 as it bounds the disk $D \cup A$ in the complement of P_2.

We consider the situation $P_1 \neq P_2$ first. The region E is then isotopic to E_1 in S^3. If K is equal to K_1 or P_2, then K is contained in a small 3-ball B containing the disk $A \cup D$ such that $B \cap K_R = \varnothing$. The link $K \cup K_R$ is then a split link of two unknots.

Assume now that K is distinct from K_1 and P_2. We define a k-function f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source P_1 of f_1. Then, f_1 is ordered because the region $f^{-1}((-\infty, b_1+\epsilon])$ contains just a single saddle of f. If $E \cap R = \varnothing$, then r is a regular value of f_1 with $\partial R \subseteq f_1^{-1}(r)$. An application of the induction hypothesis to the critical unknot K of f_1 and the unknotted solid torus R proves the lemma.

If $R \subseteq E$, then R is a tubular neighborhood of P_2 or an unknot P_1. If R is a tubular neighborhood of P_2, then $K_R \simeq P_2$ is contained in a 3-ball not containing K and $K \cup K_R$ is a split link of two unknots. If R is a tubular neighborhood of an unknot P_1, then $K \neq P_1$. Also, R is isotopic to E in S^3 and $K \nsubseteq E$ as K is distinct from K_1 and P_2. So, the induction hypothesis applies to the critical unknot K of f_1 and the unknotted solid torus E to prove the lemma in this situation.

We now prove the lemma for the situation $P_1 = P_2$. Then, E is a solid torus and $P_1 \cup K_1$ is a split link of two unknots. If K is equal to P_1 or K_1, then K is inside a 3-ball B in E such that $B \cap K_R = \varnothing$ so that $K \cup K_R$ is a split link of two unknots. Assume now $K \nsubseteq E$. Let K_E denote the core of E. We define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source K_E of f_1. If $E \cap R = \varnothing$, we can apply the induction hypothesis as before. If $R \subseteq E$, then R is either a k-model neighborhood of P_1 or K_E is an unknot and both R and E are tubular neighborhoods of K_E. In the former case, K_R is contained in a 3-ball within E not containing K so that $K \cup K_R$ is a split link of two unknots. In the latter case, we apply the induction hypothesis just as before where $K \cup K_E \simeq K \cup K_R$.

Case 2. *Both C_1 and C_2 bound disks D_1 and D_2 in $f^{-1}(b_1 - \epsilon)$ respectively.*

Then, K_1 is an unknot saddle. We first consider the case that the sources P_1 and P_2 are distinct. In this situation, there are two possible ways to attach the round handle corresponding to K_1 to the solid tori E_1 and E_2. One way leads to a sphere component of

∂E which is not possible by Lemma 6. We must have the other possibility where ∂E is a disjoint union of two tori. The sphere $D_1 \cup D_2 \cup A$ bounds 3-ball B_1 and B_2 on either side in S^3 and the cut of E along this sphere produces two punctured solid tori in S^3.

If the interior of some B_i contains only one of the knots K and K_R, then $K \cup K_R$ is a split link of two unknots. Otherwise, say $K \cup K_R \subseteq \mathrm{Int}(B_1)$. The 3-ball B_2 contains only one of P_1 and P_2. Say, $P_2 \subseteq B_2$ and $P_1 \cap B_2 = \varnothing$. The region $E \cup B_2$ is then isotopic to E_1 in S^3. We define f_1 by $f_1(p) := f(p)$ for $p \notin E \cup B_2$ so that $E \cup B_2$ becomes a k-model neighborhood of the source P_1 of f_1. An application of the induction hypothesis finishes the proof when $R \nsubseteq E$. When R is a subset of $E - B_2$, it is then a tubular neighborhood of P_1 and it is isotopic to $E \cup B_2$ in S^3. In this situation, we apply the induction hypothesis for the unknotted solid torus $E \cup B_2$ and the critical knot K of f_1.

Assume now that $P_1 = P_2$. The disks D_1 and D_2 cannot be disjoint since otherwise, $f^{-1}(b_1 + \epsilon)$ would contain a sphere contradicting Lemma 6. Say, $D_2 \subseteq D_1$. Let A_1 denote the annulus $D_1 - \mathrm{Int}(D_2)$. The torus $T_0 = A_1 \cup A$ separates S^3 into two closed regions and let R_0 denote the one of them such that $\mathrm{Int}(R_0) \cap E_1 = \varnothing$. Similarly, let $\tilde{R}_0$ denote the the component of $S^3 - \mathrm{Int}(E)$ such that $\tilde{R}_0$ is isotopic to R_0 in S^3.

As R_0 in S^3 is bounded by a torus, it is diffeomorphic to the complement of a knot K_0 in S^3 (after smoothing the corners of T_0). Take a small 3-ball identified with $D_2 \times [0, 1]$ coming from the push off of the disk D_2 into the exterior of E_1 in a normal direction so that $D_2 \times \{0\} := D_2 \subseteq \partial E_1$ and $\partial D_2 \times [0, 1] \subseteq A$. We can first take K_0 to be the union of a properly embedded arc in A_1 and another properly embedded arc in A. We can then slightly push off this union of two arcs into the exterior of R_0 in a normal direction to achieve $K_0 \cap R_0 = \varnothing$. Then, $S^3 - R_0$ is a tubular neighborhood of K_0 and $D_2 \times \{0\}$ is a meridian disk of K_0. Therefore, $B_0 := R_0 \cup D_2 \times [0, 1]$ is diffeomorphic to a 3-ball which intersects

E_1 in the disk D_1. Hence, the region $E_1 \cup R_0 \cup D_2 \times [0,1]$ is isotopic to E_1 in S^3 and similarly, so is the region $E \cup \tilde{R}_0$. So, the surface $s(\partial E_1)$, which has two components, has one component isotopic to ∂E_1 and another component isotopic to T_0 in S^3.

If $K = K_1$, then K can be isotoped along A into the 3-ball $D_2 \times [0,1]$ and we can easily isotope K_R out of this 3-ball if necessary without removing K from that 3-ball. So, $K \cup K_R$ is a split link of two unknots. We will assume $K \neq K_1$ from now on.

If $(K \cup R) \cap (E \cup \tilde{R}_0) = \varnothing$, we define f_1 by $f_1(p) := f(p)$ for $p \notin E \cup \tilde{R}_0$ so that $E \cup \tilde{R}_0$ becomes a k-model neighborhood of the source P_1 of f_1. An application of the induction hypothesis to the critical unknot K of f_1 and the solid torus R proves the lemma.

If $(K \cup R) \subseteq \tilde{R}_0$, we define f_1 by $f_1(p) := f(p)$ for $p \in \tilde{R}_0$ so that $S^3 - \tilde{R}_0$ becomes a k-model neighborhood of the source K_0 of f_1. We can then apply the induction hypothesis just as before.

Assume now that only one of R and K is inside $\tilde{R}_0$ and the other is outside $\tilde{R}_0$. Say, $K_a \subseteq \tilde{R}_0$ where $\{K_a, K_b\} = \{K_R, K\}$. Then, K_a is contained in the 3-ball B_0 but K_b is not so that $K \cup K_R$ is a split link of two unknots.

The final possible situation is that only one of R and K is inside E but none of them are inside $\tilde{R}_0$. Then, P_1 is either equal to K or isotopic to K_R within R and in the latter case, we may assume $P_1 = K_R$. Say, $K_a = P_1$ where $\{K_a, K_b\} = \{K_R, K\}$. Then, $K_b \nsubseteq E \cup \tilde{R}_0$. We define f_1 by $f_1(p) := f(p)$ for $p \notin E \cup \tilde{R}_0$ so that $E \cup \tilde{R}_0$ becomes a k-model neighborhood of the source P_1 of f_1. An application of the induction hypothesis proves the lemma.

Case 3. *None of C_1 and C_2 bounds a disk in $f^{-1}(b_1 - \epsilon)$.*

Subcase 1. *Both C_1 and C_2 bound meridian disks D_1 and D_2 of P_1 and P_2 respectively.*

Then, K_1 is an unknot saddle. The sources P_1 and P_2 are equal since otherwise, P_1

would intersect the sphere $S_1 := D_1 \cup D_2 \cup A$ geometrically once in S^3. The sphere S_1 yields $P_1 \simeq P_a \# P_b$ and the region $S^3 - E$ has two components R_a and R_b which are isotopic to the complement of P_a and P_b in S^3 respectively.

We consider the situation $K = K_1$ first. If $R \subseteq E$, then P_1 is an unknot and R is a tubular neighborhood of it. We see that $K \cup K_R$ is a Hopf link in this setting. If $R \nsubseteq E$, then say $R \subseteq R_b$. Since K is in the 3-ball B_a bounded by S_1 and containing the region R_a but K_R is not in B_a, the link $K \cup K_R$ is a split link of two unknots. We will assume $K \neq K_1$ from now on.

Assume $R \subseteq E$ so that R is a tubular neighborhood of the unknot P_1 and P_a and P_b are unknots as well. As $K \neq K_1$, the unknot K is either in R_a or R_b. Say, $K \subseteq R_a$. We define f_1 by $f_1(p) := f(p)$ for $p \in R_a$ so that $S^3 - R_a$ becomes a k-model neighborhood of the unknot source P_a of f_1. We can now apply the induction hypothesis to the critical unknot K of f_1 and a k-model neighborhood of the source P_a of f_1 to conclude that P_a and K are cable knots of each other. The lemma then follows because P_a can be isotoped to K_R within $S^3 - R_a$ so that $P_a \cup K \simeq K_R \cup K$.

If $R \nsubseteq E$, then say $R \subseteq R_b$. If $K \subseteq R_b$, we define f_1 by $f_1(p) := f(p)$ for $p \in R_b$ so that $S^3 - R_b$ becomes a k-model neighborhood of the source P_b of f_1. An application of the induction hypothesis proves the lemma. If $K \subseteq R_a$, then K is in the 3-ball B_a not containing R so that $K \cup K_R$ is a split link of two unknots. If $K \nsubseteq R_a \cup R_b$, then $K = P_1$ because $K \neq K_1$ as well. This final situation is similar to the previous situation where $R \subseteq E$ and $K \subseteq R_a \cup R_b$.

Subcase 2. *Only C_1 bounds a meridian disk D_1 of P_1.*

Then, K_1 is an unknot saddle. The sources P_1 and P_2 are distinct since C_2 is a non-meridian, nontrivial cable knot of P_2. Since C_2 bounds the disk $D_1 \cup A_1$, both C_2 and P_2 are unknots and C_2 is a longitude of P_2. Also, P_2 is a meridian of P_1 and E is isotopic to E_1 in S^3.

We first consider the situation where K is equal to K_1 or P_2. If $R \subseteq E$, then R is a tubular neighborhood of either P_1 or P_2. If $R \supseteq P_2$, then $K \cup K_R \simeq C_2 \cup P_2$ is a split link of two unknots. If $R \supseteq P_1$, then P_1 is an unknot and $K \cup K_R \simeq C_1 \cup P_1$ is a Hopf link. If $R \nsubseteq E$, then K is contained in a small 3-ball B inside E with $K_R \cap B = \varnothing$ so that $K \cup K_R$ is a split link of two unknots. We will assume that K is distinct from K_1 and P_2 from now on.

Assume $R \nsubseteq E$. We define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source P_1 of f_1. The induction hypothesis can then be applied as before.

Assume now $R \subseteq E$ so that R is a tubular neighborhood of either P_1 or P_2. If $K = P_1$, then K_R is isotopic to P_2 within R and $K \cup K_R \simeq P_1 \cup P_2$ is a Hopf link. Assume now $K \cap E = \varnothing$. If $R \supseteq P_2$, then K_R is isotopic to P_2 within R where P_2 is in the 3-ball B not containing K so that $K \cup K_R \simeq K \cup P_2$ is a split link of two unknots. If $R \supseteq P_1$, we define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the unknot source P_1 of f_1. The induction hypothesis can be applied as before.

Subcase 3. *None of C_1 and C_2 is a meridian of P_1 and P_2 respectively.*

The isotopic cable knots C_1 and C_2 are, say, $C_1 \simeq (P_1)_{p,q}$ and $C_2 \simeq (P_2)_{r,s}$ where $p, r \neq 0$ as C_i is not a meridian of P_i. We will first consider the situation $P_1 \neq P_2$. Let $\tilde{A}$ be a closed tubular neighborhood of the stable annulus A so that the annulus $\tilde{C}_i := \tilde{A} \cap E_i$ becomes a tubular neighborhood of C_i in ∂E_i and also, E is isotopic to $E_1 \cup E_2 \cup \tilde{A}$ in S^3. The boundary of $E_1 \cup E_2 \cup \tilde{A}$ is a torus which comes from the union of the two annuli $\partial E_i - \mathrm{Int}(\tilde{C}_i)$ $(i = 1, 2)$

and the two annuli that are parallel copies of A in $\tilde{A}$. Let $R_1 := S^3 - \mathrm{Int}(E_1 \cup E_2 \cup \tilde{A})$.

If $p = \pm 1$, then P_1 is isotopic to C_1 in E_1. As C_1 and C_2 are isotopic, we get $P_1 \succ P_2$ (i.e., P_1 is a cable knot of P_2). Similarly, if $r = \pm 1$, then $P_2 \simeq K_1$ and $P_2 \succ P_1$. In these situations, E is isotopic to E_1 (when $P_2 \succ P_1$) or E_2 (when $P_1 \succ P_2$) in S^3.

We will now prove that $P_1 \cup P_2$ is a Hopf link when $p, r \neq 0, \pm 1$. The solid torus E_1 admits a Seifert fibration with a single singular fiber P_1 of multiplicity $|p|$ so that the annulus $\tilde{C_1}$ becomes a union of regular fibers because C_1 is not a meridian of P_1. This fibration on $\tilde{C_1}$ extends to a regular Seifert fibration on $\tilde{A}$ because C_1 is a cable knot of K_1 with $C_1 \simeq (K_1)_{\pm 1, \beta}$. We may assume that $\tilde{C_2} \subseteq \partial \tilde{A}$ is a union of regular fibers since the cable knots C_1 and C_2 of K_1 have the same slope. This fibration on $\tilde{C_2}$ can then be extended to a Seifert fibration of E_2 with a single singular fiber P_2 of multiplicity $|r|$ because C_2 is not a meridian of P_2. So, the region $E_1 \cup E_2 \cup \tilde{A}$ becomes a Seifert fibered manifold over a disk with two singular fibers of multiplicities $|p|$ and $|r|$. The torus $\partial(E_1 \cup E_2 \cup \tilde{A})$ bounds $E_1 \cup E_2 \cup \tilde{A}$ and R_1 in S^3 at least one of which must be a solid torus. Since $\pi_1(E_1 \cup E_2 \cup \tilde{A}, *) \simeq < z, w; z^p = w^r > \not\simeq \mathbb{Z}$, the region R_1 must be a solid torus. A regular fiber in $\partial(E_1 \cup E_2 \cup \tilde{A}) = \partial R_1$ is nontrivial there. It cannot bound a meridian disk in R_1 since otherwise $\pi_1(S^3, *) = \pi_1(E_1 \cup E_2 \cup \tilde{A} \cup R_1, *) \simeq < z, w; z^p = w^r = 1 > \not\simeq 1$. Therefore, the Seifert fibration on ∂R_1 can be extended into R_1 with at most one singular fiber so that we obtain a Seifert fibration of $S^3 = E_1 \cup E_2 \cup \tilde{A} \cup R_1$ over a sphere with two or three singular fibers. It follows now from the classification of Seifert fibered manifolds that S^3 cannot have a Seifert fibration with three singular fibers over a sphere but only two so that R_1 must have a regular Seifert fibration (see e.g. **[15]** or **[16]**). Moreover, if one takes the base sphere as the union of two disks each of which contains a point corresponding to a singular fiber P_1 or P_2, then those two disks will correspond to two complementary solid tori in S^3 so that the cores P_1 and P_2 of those two complementary solid tori form a Hopf link in S^3.

We first consider the cases $P_1 \succ P_2$ or $P_2 \succ P_1$ where E is isotopic to E_1 or E_2 in S^3. Say, $P_a \succ P_b$ where $\{P_a, P_b\} = \{P_1, P_2\}$. Assume $R \nsubseteq E$. We define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source P_b of f_1. If K is distinct from P_a and K_1, then an application of the induction hypothesis proves the lemma. If K is equal to P_a or K_1, then P_b is also an unknot because $P_a \simeq K_1$ is a nontrivial, non-meridian cable knot of P_b. Moreover, P_a is isotopic to P_b within E. An application of the induction hypothesis to the critical unknot P_b of f_1 and R proves the lemma because $K \cup K_R \simeq P_b \cup K_R$.

Assume now $R \subseteq E$. Then, R is a tubular neighborhood of (possibly both) P_a or P_b. In either case, P_b is an unknot because P_a is a nontrivial, non-meridian cable knot of P_b. Also, K_R is isotopic to P_b within E. If $K \subseteq E$, then K is equal to P_a, P_b or K_1 and also, $K \cup K_R \simeq P_a \cup P_b$ which proves the lemma. If $K \nsubseteq E$, we define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source P_b of f_1. We apply the induction hypothesis just as before.

We consider the Hopf link $P_1 \cup P_2$ case now. We still have both $P_1 \succ P_2$ and $P_2 \succ P_1$ but E is no longer isotopic to E_1 or E_2 in S^3. The torus knot $C_1 \simeq (P_1)_{p,q}$ is nontrivial since $p \neq 0, \pm 1$. The region $V := S^3 - \mathrm{Int}(E)$ is a solid torus the core of which is isotopic to $C_1 \simeq K_1$.

Assume $K \cup R \subseteq V$. Since the core of V is nontrivial, each of the unknots K and K_R is contained in some 3-balls B_K and B_R inside V respectively. Let $g : V \to S^3$ be an embedding such that $g(V)$ is a standard, unknotted solid torus in S^3. Then, both $g(K)$ and $g(K_R)$ are unknots since each of K and K_R is contained in a 3-ball inside V. We define f_1 by $f_1(p) := f(g^{-1}(p))$ for $p \in g(V)$ so that $S^3 - g(V)$ becomes a k-model neighborhood of an unknot source K_g of f_1. The unknots $g(K_R)$ and $g(K)$ are then cable knots of each other by the induction hypothesis. Therefore, so is the link $g^{-1}(g(K) \cup g(K_R)) = K \cup K_R$.

Assume now that only one of K and R is inside V. The one inside V is then contained in a 3-ball not containing the other one so that $K \cup K_R$ is a split link of two unknots. The final remaining case is $K \cup R \subseteq E$ where $K \cup K_R \simeq P_1 \cup P_2$ is a Hopf link.

We will now prove this subcase of the lemma for the situation $P_1 = P_2$. Let $\tilde{A}$, $\tilde{C}_i$ and $C_1 \simeq (P_1)_{p,q}$ $(p \neq 0)$ be just as before where C_1 is now isotopic to C_2 in ∂E_1. The nontrivial circles C_1 and C_2 separates ∂E_1 into two closed annuli A_1 and A_2 and the components of $s(\partial E_1)$ are isotopic to the tori $\Sigma_1 := A_1 \cup A$ and $\Sigma_2 := A_2 \cup A$ in S^3. Let H_i denote the closed region bounded by Σ_i in S^3 such that $\mathrm{Int}(H_i) \cap E_1 = \varnothing$. Similarly, let $\tilde{H}_i$ denote the component of $S^3 - \mathrm{Int}(E)$ that is isotopic to H_i in S^3.

Assume that H_1 is not a solid torus. Then, $E_1 \cup H_2$ bounded by Σ_1 is a solid torus. If C_1 bounds a disk in $E_1 \cup H_2$, then C_1 is a meridian of the core of $E_1 \cup H_2$ and also a longitude of the unknot P_1 because C_1 is a nontrivial, non-meridian cable knot of P_1. The region $E_1 \cup H_2$ is then isotopic to H_2 in S^3 so that H_2 is a solid torus. When C_1 does not bound a disk in $E_1 \cup H_2$, the region $E_1 \cup H_2$ admits a Seifert fibration with at most one single singular fiber where the annuli A_1 and A in its boundary become a union of regular fibers. As $\partial A_2 = \partial A_1$ consists of two regular fibers, the annulus A_2 can then be isotoped into $\partial(E_1 \cup H_2)$ relative to its boundary in the Seifert fibered solid torus $E_1 \cup H_2$ so that H_2 is again isotopic to $E_1 \cup H_2$ in S^3.

Therefore, at least one of H_1 and H_2, say H_1, is a solid torus. Let K_H denote the core of both H_1 and $\tilde{H}_1$. The union $E_1 \cup H_1$ of two solid tori intersecting each other at an annulus in their boundaries is then similar to the union $E_1 \cup E_2 \cup \tilde{A}$ in our previous situation $P_1 \neq P_2$. Therefore, either $P_1 \cup K_H$ is a Hopf link with C_1 being a nontrivial torus knot or one of the knots P_1 and K_H is a cable knot of the other one. In the latter case, $E_1 \cup H_1$ is isotopic to E_1 or H_1 in S^3. In the former Hopf link case, the region H_2 is also a solid torus the core of which is isotopic to C_1.

We have $S^3 = E \cup \tilde{H}_1 \cup \tilde{H}_2$ where the interiors of those three regions are disjoint. There are various possibilities about where K and R might be. We start with the assumption $R \cup K \subseteq \tilde{H}_2$. The case where $P_1 \cup K_H$ is a Hopf link and C_1 is a nontrivial torus knot has already been analyzed in the "Hopf link $P_1 \cup P_2$" situation before and the lemma holds in this case. If $P_1 \succ K_H$ or $K_H \succ P_1$ and also, $E_1 \cup H_1$ is isotopic to E_1 or H_1 in S^3, we define f_1 by $f_1(p) := f(p)$ for $p \notin E \cup \tilde{H}_1$ so that $E \cup \tilde{H}_1$ becomes a k-model neighborhood of the source P_1 or K_H of f_1. An application of the induction hypothesis proves the lemma.

Assume now $R \cup K \subseteq \tilde{H}_1$. The case where K_H is nontrivial has already been analyzed in the "Hopf link $P_1 \cup P_2$" situation before and the lemma holds in this case. If K_H is an unknot, we apply the induction hypothesis more directly by simply defining f_1 by $f_1(p) := f(p)$ for $p \in \tilde{H}_1$ so that $S^3 - \tilde{H}_1$ becomes a k-model neighborhood of an unknot source of f_1.

Assume now that only one of R and K is in $\tilde{H}_1$ and the other one is in $\tilde{H}_2$. Say, $K_a \subseteq \tilde{H}_1$ where $\{K_a, K_b\} = \{K_R, K\}$. If K_H is nontrivial so that $\tilde{H}_1$ is a knotted solid torus, then $K \cup K_R$ is a split link of two unknots. If $K_H \cup P_1$ is a Hopf link and C_1 is a nontrivial torus knot, then $\tilde{H}_2$ is a knotted solid torus and $K \cup K_R$ is again a split link of two unknots. The remaining situation is that K_H is trivial and $E \cup \tilde{H}_1$ is isotopic to $\tilde{H}_1$ in S^3. Also, $\tilde{H}_2$ is an unknotted solid torus in S^3. Here, each of the unknots K and K_R are in one of the two complementary unknotted solid tori $E \cup \tilde{H}_1$ and $\tilde{H}_2$ but not in the same one. Let J_H denote the unknot core of $\tilde{H}_2$ so that $K_H \cup J_H$ is a Hopf link. We define f_1 by $f_1(p) := f(p)$ for $p \in \tilde{H}_1$ so that $E \cup \tilde{H}_2$ becomes a k-model neighborhood of the source J_H of f_1. An application of the induction hypothesis shows that K_a is a cable knot of J_H. Since $K_H \cup J_H$

is a Hopf link, K_a is a cable knot of K_H as well by redifining the core K_H of H_1 if necessary. A similar argument shows that the other unknot K_b is a cable knot of K_H as well. Since $K_a \subseteq \tilde{H}_1$ but $K_b \not\subseteq \tilde{H}_1$, the unknot K_a can be isotoped into an arbitrarily small tubular neighborhood of K_H without affecting K_b. Therefore, the unknots in $K_a \cup K_b = K_R \cup K$ are cable knots of each other as well.

The only possibility we haven't considered so far is that R or K is inside E. If $K \cup R \subseteq E$, then R is a tubular neighborhood of P_1 and $K = K_1$. The lemma follows from $K \cup K_R \simeq C_1 \cup P_1$ and $C_1 \succ P_1$. Assume now that only one of K and R is inside E. If $R \subseteq E$, we may assume $K_R = P_1$. Say, $K_a \subseteq E$ where $\{K_a, K_b\} = \{K, K_R\}$. Then K_a is equal to P_1 or K_1 and in the latter case, P_1 is also an unknot because the unknot K_1 is a nontrivial, non-meridian cable knot of P_1.

First assume $K_b \subseteq \tilde{H}_2$. If $P_1 \cup K_H$ is a Hopf link and C_1 is a nontrivial torus knot, then $\tilde{H}_2$ is a knotted solid torus and $K \cup K_R$ is a split link of two unknots. Otherwise, $E \cup \tilde{H}_1$ is isotopic to E_1 in S^3 and we define f_1 by $f_1(p) := f(p)$ for $p \in \tilde{H}_2$ so that $E \cup \tilde{H}_1$ becomes a k-model neighborhood of the unknot source P_1 of f_1. The unknots in $P_1 \cup K_b$ are then cable knots of each other by the induction hypothesis. So are the unknots in $K_1 \cup K_b$ because K_1 can be isotoped into an arbitrarily small neighborhood of P_1 without affecting K_b. Since $K_R \cup K$ is either equal to $P_1 \cup K_b$ or $K_1 \cup K_b$, the lemma is proven in this situation.

Assume now $K_b \subseteq \tilde{H}_1$. If K_H is nontrivial, then $K \cup K_R$ is a split link of two unknots. If $P_1 \cup K_H$ is a Hopf link, the unknots K and K_R are in two complementary unknotted solid tori but not in the same one and the lemma has been proven in this situation above. The remaining case is that K_H is trivial and $E \cup \tilde{H}_1$ is isotopic to both E_1 and $\tilde{H}_1$ in S^3 where the unknots K_H and P_1 are cable knots of each other. In this case, $\tilde{H}_2$ is a standard solid torus and $P_1 \cup J_H \simeq K_H \cup J_H$ is a Hopf link where J_H is the core of $\tilde{H}_2$. We define f_1 by

$f_1(p) := f(p)$ for $p \in E \cup \tilde{H}_1$ so that $\tilde{H}_2$ becomes a k-model neighborhood of the unknot source J_H of f_1. An application of the induction hypothesis shows $K_b \succ J_H$ so that K_b can be isotoped into $\partial\tilde{H}_1$ within $\tilde{H}_1$ and hence, $K_b \succ P_1$. We now see $K_b \succ K_a$ as well even when $K_a \neq P_1$ since in this case $K_a = K_1$ and, $K_b = K_R$ can be isotoped into an arbitrarily small tubular neighborhood of P_1 without affecting K_1. This completes the proof of the lemma. $\square$

We will take the following elementary description as a definition of a graph knot [**8**], [**11**].

Definition 3. Let $\mathbb{S}_0 := \{\text{unknot}\}$. To define $\mathbb{S}_n$ inductively ($n \in \mathbb{N}$), assume that $\mathbb{S}_k$ is defined for $0 \leq k < n$ and let $\mathbb{S}_n$ denote the set of cable knots of P where P is a connected sum of knots in $\mathbb{S}_{n-1}$. A knot in $\mathbb{S} := \cup_{k\in\mathbb{N}} \mathbb{S}_k$ is called *a graph knot.*

We have the following facts about the set of graph knots $\mathbb{S}$: The set $\mathbb{S}_1$ is the set of (trivial or nontrivial) torus knots. Since the cable knot $(K)_{1,r}$ of any knot K is isotopic to K, we have $\mathbb{S}_{n+1} \supseteq \mathbb{S}_n$ for all $n \in \mathbb{N}$. If $\{K_1, \ldots, K_m\} \subseteq \mathbb{S}_n$, then $K_1\#\cdots\#K_m \simeq (K_1\#\cdots\#K_m)_{1,r} \in \mathbb{S}_{n+1}$. If we have a sequence of cable knots $U \prec K_1 \prec K_2 \prec \cdots \prec K_m$ where U is an unknot, then $K_i \in \mathbb{S}_i$ for $1 \leq i \leq m$.

Definition 4. Suppose that K is a graph knot in $\mathbb{S}_n$. If $n = 0$, we define *the graph knot kit* or shortly *the graph kit of* K to be the empty set. For $n > 0$, fix a (not necessarily unique) expression of K with $K \simeq (P)_{q,r}$ and $P \simeq P_{K,1}\#\cdots\#P_{K,m}$ where $P_{K,i} \in \mathbb{S}_{n-1}$ for $1 \leq i \leq m$. We define $\Gamma(K)$ corresponding to this fixed expression of K by $\Gamma(K) := \{P_{K,1}, \ldots, P_{K,m}\}$. Let $\Phi_1 := \Gamma(K)$. For $1 \leq j < n$, we define Φ_{j+1} inductively by $\Phi_{j+1} := \Phi_j \cup \{J :\ J \in \Gamma(H) \text{ where } H \in \Phi_j\}$ where the elements $P_{K,i} \in \Gamma(K)$ and $P_{J,s} \in \Gamma(J)$ are distinct elements of Φ_{j+1} whenever K and J are distinct in Φ_j. Then, we say that Φ_n is *a graph kit of* K and also, K is *woven from the graph knots in* Φ_n.

Note that even when the expression $K \simeq (P)_{q,r}$ and $P \simeq P_{K,1} \# \cdots \# P_{K,m}$ of $K \in \mathbb{S}_n$ is unique, we can consider K in $\mathbb{S}_{n+1} \supseteq \mathbb{S}_n$ and may produce a different graph kit of K. Also, distinct graph knots in a graph kit can be isotopic. For example, for the torus knots $T_{2,3}$ and $T_{2,5}$ in $\mathbb{S}_1$, the set $\Phi := \{T_{2,3}, T_{2,5}, \text{unknot}_{2,3}, \text{unknot}_{2,5}\}$ is a graph kit of $T_{2,3} \# T_{2,5}$ (where Φ is valid for any orientations of $T_{2,3}$ and $T_{2,5}$ defining $T_{2,3} \# T_{2,5}$).

If Φ is a finite collection of unoriented knots, then $\underset{J \in \Phi}{\#} J$ denotes a connected sum of the knots in Φ which are assigned an arbitrary orientation before their connected sum is taken. If Φ is a graph kit of some graph knot and P is a nontrivial graph knot in Φ, then $\Gamma_\Phi(P)$ will denote the finite set of graph knots such that P is a cable knot of a connected sum of the graph knots in $\Gamma_\Phi(P)$ and also, $\Gamma_\Phi(P) \subseteq \Phi$. In this case, the orientations of the knots in the connected sum $\underset{J \in \Gamma_\Phi(P)}{\#} J$ are not arbitrary but in such a way so that the graph knot P becomes a cable knot of the graph knot $\underset{J \in \Gamma_\Phi(P)}{\#} J$.

Lemma 8. *A graph knot K is k-mate.*

Proof. Say, $K \in \mathbb{S}_n$. If $n = 0$, then K is an unknot which is k-mate by Lemma 3. Assume now that K is nontrivial and $n > 0$. We induct on n. Each graph knot in $\Gamma(K)$ is k-mate by the induction hypothesis. Applications of Lemma 4 and Lemma 5 to the graph knots in $\Gamma(K)$ show that K is k-mate. $\square$

Theorem 9. *Suppose that f is an ordered k-function. Then, every critical knot K of f is a graph knot. Moreover, there exist a graph kit Φ of K such that:*

(i) Each graph knot P in Φ is isotopic to the core of a solid torus R_P where $\partial R_P \subseteq f^{-1}(r)$ for some regular value r of f.

(ii) For each nontrivial graph knot P in Φ, there exists a solid torus $R_{\Gamma(P)}$ such that the

core of $R_{\Gamma(P)}$ *is isotopic to* $\#_{J \in \Gamma_\Phi(P)} J$ *and also,* $\partial R_{\Gamma(P)} \subseteq f^{-1}(r)$ *for some regular value* r *of* f*. Moreover, the core of* R_P *can be isotoped into* $\partial R_{\Gamma(P)}$ *in* S^3.

(iii) There exists a solid torus $R_{\Gamma(K)}$ *such that the core of* $R_{\Gamma(K)}$ *is isotopic to* $\#_{J \in \Gamma_\Phi(K)} J$ *and also,* $\partial R_{\Gamma(K)} \subseteq f^{-1}(r)$ *for some regular value* r *of* f*. Moreover,* K *can be isotoped into* $\partial R_{\Gamma(K)}$.

Remark. Part *(i)* of Theorem 9 does not say that a saddle K of an ordered k-function f is the core of a solid torus R where $\partial R \subseteq f^{-1}(r)$ for some regular value r of f but only that there exists a graph kit ϕ of K such that this is true for every graph knot in Φ. However, K is not in Φ.

Proof. There is nothing to prove if K is trivial because we can take the empty set as a graph kit of K. If f has no saddles, then f has a single unknot source and a single unknot sink which form a Hopf link by Lemma 3 and the theorem holds in this case. Assume now that K is nontrivial so that f has saddles. Let $a_1 < \cdots < a_j < b_1 < \cdots < b_k < c_1 < \cdots < c_m$ denote the critical values of f where a_i, b_i and c_i correspond to a source, a saddle and a sink of f respectively. We will apply the proof technique in Lemma 7 to induct on the number k of saddles of f. As we will cover the similar cases or subcases, we will omit some details which can be found in that proof.

Let K_1 be the saddle of f with $f(K_1) = b_1$. Let A be the stable annulus of K_1 in $f^{-1}([b_1 - \epsilon, b_1])$ and C_1 and C_2 be the stable circles of K_1 in $f^{-1}(b_1 - \epsilon)$. The circles C_1 and C_2 are cable knots of (not necessarily distinct) sources P_1 and P_2 respectively. Let E_i denote the solid torus component of $f^{-1}([a_1, b_1 - \epsilon])$ containing P_i. Let E denote the component of $f^{-1}([a_1, b_1 + \epsilon])$ which contains $P_1 \cup P_2 \cup K_1$.

Case 1. *Only one of* C_1 *and* C_2 *bounds a disk in* $f^{-1}(b_1 - \epsilon)$.

Say, C_1 bounds a disk D in $f^{-1}(b_1 - \epsilon)$ Then, K_1 and P_2 are unknots so that K is distinct from K_1 and P_2 and also, C_2 is a longitude of P_2.

If $P_1 \neq P_2$, then E is isotopic to E_1 in S^3. We define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source P_1 of f_1. We apply the induction hypothesis to the critical knot K of f_1 to conclude that K is a graph knot and also, there exists a graph kit Φ of K satisfying the properties stated in the theorem for f_1 where these properties include a collection of various solid tori in S^3. The boundary $\partial\Omega$ of one of these solid tori Ω is in the preimage of a regular value of f except possibly when $\partial\Omega \subseteq E$ but then, the solid torus Ω is a tubular neighborhood of P_1 and we can find another appropriate tubular neighborhood $\tilde{\Omega}$ of P_1 in E such that $\partial\tilde{\Omega} \subseteq f^{-1}(a_j + \epsilon)$. Therefore, the graph kit Φ together with a slightly modified collection of solid tori (if necessary) works for the k-function f as well.

If $P_1 = P_2$, then E is a solid torus not containing K. Let K_E denote the core of E. We define f_1 by $f_1(p) := f(p)$ for $p \notin E$ so that E becomes a k-model neighborhood of the source K_E of f_1. The rest of the proof continues just as in the previous $P_1 \neq P_2$ situation.

Case 2. *Both C_1 and C_2 bound disks D_1 and D_2 in $f^{-1}(b_1 - \epsilon)$ respectively.*

Then, K_1 is an unknot so that $K \neq K_1$. First assume that $P_1 \neq P_2$ so that E is a connected sum of two solid tori. Let B_1 and B_2 denote the 3-balls bounded by the sphere $D_1 \cup D_2 \cup A$ in S^3 where the cut of E along this sphere produces two punctured solid tori in S^3. Say, $K \subseteq \mathrm{Int}(B_1)$ and also, say $P_2 \subseteq B_2$. The region $E \cup B_2$ is then isotopic to E_1 in S^3. We define f_1 by $f_1(p) := f(p)$ for $p \notin E \cup B_2$ so that $E \cup B_2$ becomes a k-model neighborhood of the source P_1 of f_1. We apply the induction hypothesis to the critical knot K of f_1 to prove that K is a graph knot and also, it has a graph kit Φ together with various collection of solid tori satisfying the properties of the theorem for f_1. This collection of solid tori will work for f after modifying the ones in $E \cup B_2$ if necessary.

We now consider the situation $P_1 = P_2$. The disks D_1 and D_2 are not disjoint and say, $D_2 \subseteq D_1$. Let A_1 denote the annulus $D_1 - \mathrm{Int}(D_2)$. The torus $T_0 = A_1 \cup A$ separates S^3 into two closed regions and let R_0 denote the one of them such that $\mathrm{Int}(R_0) \cap E_1 = \varnothing$. Similarly, let $\tilde{R}_0$ denote the component of $S^3 - \mathrm{Int}(E)$ such that $\tilde{R}_0$ is isotopic to R_0 in S^3. The region R_0 is diffeomorphic to the complement of a knot K_0 in T_0. Since $E_1 \cup R_0$ is isotopic to E_1 in S^3, we can define an ordered k-function $\tilde{f}$ by modifying f within a small neighborhood U of $R_0 \cup D_1$ such that $\tilde{f}(p) = f(p)$ for $p \notin U$ and also, $\tilde{f}$ does not have any critical points in U. So, the saddle K_1 of f in U is removed.

If $K \nsubseteq R_0$, we define f_1 by $f_1(p) := \tilde{f}(p)$. The induction hypothesis applies to the critical knot K of f_1 so that K is a graph knot and also, there exists a graph kit Φ of K such that the solid tori corresponding to the graph knots in Φ satisfy the properties stated in the theorem for f_1. For the boundary of one of those solid tori lying in $f_1^{-1}(r)$ for a regular value r of f_1, we will have $f_1^{-1}(r) \subseteq f^{-1}(r)$ if $r > b_1 + \epsilon$. We may have $f_1^{-1}(r) \nsubseteq f^{-1}(r)$ for some regular value r with $r \leq b_1 + \epsilon$ but then we can use another appropriate choice of regular value $r_0 \leq b_1 + \epsilon$ instead of r and we can achieve $f_1^{-1}(r_0) = f^{-1}(r_0)$. Therefore, the graph kit Φ of K satisfies the properties stated in the theorem for f as well.

If $K \subseteq R_0$, we define f_1 by $f_1(p) := f(p)$ for $p \in \tilde{R}_0$ so that $S^3 - \tilde{R}_0$ becomes a k-model neighborhood of the source K_0 of f_1. We apply the induction hypothesis just as before to conclude that K is a graph knot and also, there exists a graph kit Φ of K satisfying the properties stated in the theorem for f_1. That graph kit will satisfy those properties for f as well except that there may be a tubular neighborhood of K_0 associated to a graph knot in Φ. This tubular neighborhood may not exactly work for f but it is isotopic to an appropriate tubular neighborhood $S^3 - \mathrm{Int}(\tilde{R}_0)$ coming from f.

Case 3. *None of C_1 and C_2 bounds a disk in $f^{-1}(b_1 - \epsilon)$.*

Subcase 1. *Both C_1 and C_2 bound meridian disks D_1 and D_2 of P_1 and P_2 respectively.*

Then, K_1 is an unknot with $K \neq K_1$. We have $P_1 = P_2$ and the sphere $S_1 := D_1 \cup D_2 \cup A$ yields $P_1 \simeq P_a \# P_b$. The region $S^3 - E$ has two components R_a and R_b which are isotopic to the complement of P_a and P_b in S^3 respectively.

Assume $K \neq P_1$. Say, $K \subseteq R_a$. We define f_1 by $f_1(p) := f(p)$ for $p \in R_a$ so that $S^3 - R_a$ becomes a k-model neighborhood of the source P_a of f_1. We then apply the induction hypothesis just as before.

Assume now $K = P_1$. We can use f_1 above and similarly define f_2 for the regions R_b and $S^3 - R_b$ to conclude that P_a and P_b are graph knots and also, there exist graph kits Φ_a and Φ_b of P_a and P_b respectively such that Φ_a and Φ_b satisfy the properties stated in the theorem for f_1 and f_2 respectively. Since $P_1 \simeq P_a \# P_b$, the source P_1 is also a graph knot and $\Phi := \Phi_a \cup \Phi_b \cup \{P_a, P_b\}$ is a graph kit of P_1.

For each graph knot P in $\Phi_a \cup \Phi_b$, we already have a solid torus R_P or $R_{\Gamma(P)}$ associated to it. The boundaries ∂R_P or $\partial R_{\Gamma(P)}$ are then in $f^{-1}(r)$ for some regular value r of f except possibly when R_P or $R_{\Gamma(P)}$ is a tubular neighborhood of P_a or P_b but this problem in those exceptional cases can be easily resolved by just picking a more appropriate tubular neighborhood R_P or $R_{\Gamma(P)}$ of P_a or P_b in the beginning. For the graph knots P_a and P_b in Φ, we associate the solid tori $E \cup R_b$ and $E \cup R_a$ to R_{P_a} and R_{P_b} respectively. Finally, we regard K as the cable knot $(K)_{1,q}$ of itself and define $R_{\Gamma(K)} := E_1$. The collection of all these solid tori associated to the graph knots in the graph kit Φ of K satisfies then the properties stated in the theorem for f.

Subcase 2. *Only C_1 bounds a meridian disk D_1 of P_1.*

Then, K_1 is an unknot so that $K \neq K_1$. The sources P_1 and P_2 are distinct since C_2 is a non-meridian, nontrivial cable knot of P_2. Also, P_2 is an unknot and C_2 is a longitude of P_2. The region E is isotopic to E_1 in S^3. This subcase is then similar to Case 1 and the proof in this case can be completed similarly.

Subcase 3. *None of C_1 and C_2 is a meridian of P_1 and P_2 respectively.*

We will first consider the situation $P_1 \neq P_2$. The isotopic cable knots C_1 and C_2 are, say, $C_1 \simeq (P_1)_{p,q}$ and $C_2 \simeq (P_2)_{r,s}$ where $p, r \neq 0$ as C_i is not a meridian of P_i.

If p or r is equal to 1, then $P_1 \succ P_2$ or $P_2 \succ P_1$ and E is a tubular neighborhood of P_1 or P_2. Say, $P_a \succ P_b$ where $\{P_a, P_b\} = \{P_1, P_2\}$. Let $\{E_a, E_b\} := \{E_1, E_2\}$ be such that E_a and E_b are tubular neighborhoods of P_a and P_b respectively. We define f_1 by $f_1(p) := f(p)$ for $p \in (S^3 - E) \cup E_b$ so that E becomes a k-model neighborhood of the source P_b of f_1. If K is distinct from K_1 and P_a, we can then apply the induction hypothesis to critical knot K of f_1 to prove the theorem. If K is equal to P_a or K_1, then $K \succ P_b$. We apply the induction hypothesis to the source P_b of f_1 to conclude that P_b is a graph knot and there exists a graph kit Φ_b of P_b satisfying the properties stated in the theorem for f_1. Since $K \succ P_b$, the knot K is also a graph knot and also, $\Phi_b \cup \{P_b\}$ is a graph kit of K. The solid tori R_P or $R_{\Gamma(P)}$ is already defined for a graph knot P in Φ_b. We define $R_{P_b} := E_b$ and $R_{\Gamma(K)} := E_b$. The collection of these solid tori satisfies then the properties stated in the theorem for f.

If $p, r \neq 0, \pm 1$, then $P_1 \cup P_2$ is a Hopf link so that nontrivial K is distinct from P_1 and P_2. If $K = K_1$, the saddle K is a torus knot. The graph kit $\{P_1\}$ of K_1 together with the solid tori $R_{P_1} := E_1$ and $R_{\Gamma(K)} := E_1$ proves the theorem. Assume now that K is inside the solid torus $V := S^3 - \mathrm{Int}(E)$ where the core K_V of V is a nontrivial torus knot $\simeq (P_1)_{p,q}$.

This is the situation where we will need the fact from Lemma 7 and also, the utility of a graph kit satisfying the properties stated in the theorem as we now embed V into S^3 by $g : V \to S^3$ such that $g(V)$ becomes a standard, unknotted solid torus in S^3. We define f_1 by $f_1(p) := f(g^{-1}(p))$ for $p \in g(V)$ so that $S^3 - g(V)$ becomes a k-model neighborhood of an unknot source K_g of f_1.

If $g(K)$ is trivial, then K_g and $g(K)$ are cable knots of each other by Lemma 7. Therefore, K is a nontrivial, non-meridian cable knot of K_V. The graph kit $\{K_V, P_1\}$ of K together with the solid tori $R_{K_V} := V$, $R_{\Gamma(K_V)} := E_1$, $R_{P_1} := E_1$ and $R_{\Gamma(K)} := V$ proves the theorem.

Assume now that $g(K)$ is nontrivial. An application of the induction hypothesis to the critical knot $g(K)$ of f_1 shows that $g(K)$ is a graph knot and also, it produces a graph kit Φ_g of $g(K)$ satisfying the properties stated in the theorem for f_1. For P in Φ_g, let R_P and $R_{\Gamma(P)}$ (when P is nontrivial) be the solid tori as stated in the theorem. Let $R_{\Gamma(g(K))}$ be the solid torus for $\Gamma_{\Phi_g}(g(K))$ as stated in the theorem. If R_P is not a tubular neighborhood of K_g, we can assume $R_P \subseteq g(V)$. If R_P is a tubular neighborhood of K_g, then the standard solid torus R_P can be replaced by the standard solid torus $g(V)$ because the unknotted solid tori R_P and $g(V)$ are isotopic in S^3 and also, a cable knot of the unknot core of R_P is a cable knot of the unknot core of $g(V)$ as well. Hence, we can assume that $R_P \subseteq g(V)$ for each $P \in \Phi_g$. Similarly, we can assume that $R_{\Gamma(P)} \subseteq g(V)$ for each nontrivial $P \in \Phi_g$ and also, $R_{\Gamma(g(K))} \subseteq g(V)$.

Let K_P denote the core of R_P for $P \in \Phi_g$. Since the graph kit Φ_g is a collection of isotopy classes of knots, the knot $g^{-1}(P)$ is *not defined*. However, the knot $g^{-1}(K_P)$ is *well defined* and it will do the job. If K_P is an unknot, then Lemma 7 asserts that K_P is a cable knot of K_g. Therefore, the knot $g^{-1}(K_P)$, which is the core of the solid torus $g^{-1}(R_P)$, is a cable knot of K_V so that it is a graph knot. Let $\Phi_1 := \{g^{-1}(K_P) : P \in \Phi_g\}$. The properties of Φ_g stated in the theorem imply that every knot in Φ_1 is a graph knot because $g^{-1}(K_P)$ is a graph knot for every unknot P in Φ_g.

Let $\Phi_0 := \{P \in \Phi_g : K_P$ is trivial but $g^{-1}(K_P)$ is not trivial$\}$. Let $\Phi_2 := \Phi_1 \cup \{(K_V)_P : P \in \Phi_0\} \cup \{(P_1)_P : P \in \Phi_0\}$, where the latter two sets contain just distinct copies of the same knots K_V and P_1. For each nontrivial graph knot $g^{-1}(K_P)$ in Φ_1, we take $\Gamma(g^{-1}(K_P))$ as $\{g^{-1}(K_J) : J \in \Gamma_{\Phi_g}(P)\}$ (when P is nontrivial) or $\{(K_V)_P\}$ (when P is trivial). We also take $\Gamma((K_V)_P)$ as $\{(P_1)_P\}$. So, we have $\Gamma(J) \subseteq \Phi_2$ when J is a nontrivial graph knot in Φ_2. To each graph knot $g^{-1}(K_P)$ in Φ_1, we associate the solid torus $g^{-1}(R_P)$. To each graph knot $(K_V)_P$ or $(P_1)_P$ in Φ_2, we associate the solid tori V or E_1 respectively. When $g^{-1}(K_P)$ in Φ_2 is nontrivial, we take the solid torus $g^{-1}(R_{\Gamma(P)})$ (when P is nontrivial) or V (when P is trivial) for $R_{\Gamma(g^{-1}(K_P))}$. Finally, we define $R_{\Gamma((K_V)_P)} := E_1$ for $(K_V)_P \in \Phi_2$ and $R_{\Gamma(K)} := g^{-1}(R_{\Gamma(g(K))})$. The collection of these solid tori associated to the trivial or nontrivial graph knots in Φ_2 satisfies the properties stated in the theorem for the ordered k-function f. These properties of the solid tori imply now that the knot $g^{-1}(g(K)) = K$ is a graph knot that is woven from the graph knots in Φ_2.

We will now prove the theorem for the situation $P_1 = P_2$. The nontrivial circles C_1 and C_2 separates ∂E_1 into two closed annuli A_1 and A_2 and the components of $s(\partial E_1)$ are isotopic to the tori $\Sigma_1 := A_1 \cup A$ and $\Sigma_2 := A_2 \cup A$ in S^3. Let H_i denote the closed region bounded by Σ_i in S^3 such that $\mathrm{Int}(H_i) \cap E_1 = \varnothing$. Similarly, let $\tilde{H}_i$ denote the component of $S^3 - \mathrm{Int}(E)$ that is isotopic to H_i in S^3. Then, at least one of H_1 and H_2, say H_1, is a solid torus. Let K_H denote the core of H_1. The link $P_1 \cup K_H$ is either a Hopf link with C_1 being a nontrivial torus knot or one of P_1 and K_H is a cable knot of the other one. In the latter cable knot cases, the region $H_1 \cup E_1$ is isotopic to H_1 or E_1 in S^3.

There are now several possibilities for the location of K in regard of $\tilde{H}_1 \cup E \subseteq S^3$. We start with the assumption $K \not\subseteq \tilde{H}_1 \cup E$. If $P_1 \succ K_H$ or $K_H \succ P_1$, let $\{K_a, K_b\} := \{P_1, K_H\}$ be such that $K_a \succ K_b$. We define f_1 by $f_1(p) := f(p)$ for $p \notin \tilde{H}_1 \cup E$ so that $\tilde{H}_1 \cup E$ becomes a k-model neighborhood of the source K_b of f_1. An application of the induction hypothesis proves the theorem. If $P_1 \cup K_H$ is a Hopf link, the region $S^3 - \tilde{H}_1 \cup E$ is a tubular neighborhood of a nontrivial torus knot isotopic to K_1 and we have already proven the theorem in this setting which was analyzed in the situation $P_1 \neq P_2$.

Assume now $K \subseteq E$ so that K is equal to P_1 or K_1. We first consider $K = P_1$. As K is nontrivial, $P_1 \cup K_H$ cannot be a Hopf link so that $H_1 \cup E_1$ is isotopic to H_1 or E_1 in S^3. Let $\{K_a, K_b\}$ and f_1 be defined just as in the previous paragraph. The induction hypothesis applies to the source K_b of f_1 so that K_b is a graph knot and also, there exists a graph kit Φ_b of K_b satisfying the properties stated in the theorem for f_1. If $K = K_b$, then this graph kit Φ_b works for f as well. Otherwise, $K \succ K_b$ so that K is a graph knot and $\Phi := \Phi_b \cup \{K_b\}$ is a graph kit of K. Set the solid tori $R_{K_b} := \tilde{H}_1$ and $R_{\Gamma(K)} := \tilde{H}_1$. The collection of these two solid tori together with the solid tori associated to the graph knots in Φ_b proves the theorem.

Assume now $K = K_1$. Our previous argument shows that P_1 is a graph knot and when P_1 is not trivial, there exists a graph kit Φ_1 of P_1 satisfying the properties stated in the theorem. If P_1 is trivial, then take Φ_1 to be the empty set. Since $K \succ P_1$, the knot K is a graph knot and $\Phi_1 \cup \{P_1\}$ is a graph kit of K. The collection of the solid tori corresponding to the graph knots in Φ_1 together with $R_{P_1} := E_1$ and $R_{\Gamma(K)} := E_1$ proves the theorem.

Assume now $K \subseteq \tilde{H}_1$. If $P_1 \cup K_H$ is a Hopf link, we define f_1 by $f_1(p) := f(p)$ for $p \in \tilde{H}_1$ so that $S^3 - \tilde{H}_1$ becomes a k-model neighborhood of the source P_1. We then apply the induction hypothesis as usual. If $P_1 \succ K_H$ or $K_H \succ P_1$, we can define $\{K_a, K_b\}$ and f_1

just as before and our previous argument shows that K_b is a graph knot. Also, we get a graph kit Φ_b of K_b satisfying the properties stated in the theorem for f_1. Let $\{U_a, U_b\} := \{E_1, \tilde{H}_1\}$ be such that U_a and U_b are tubular neighborhoods of K_a and K_b respectively. If $K_H = K_b$, we simply define $\Phi_H := \Phi_b$. If $K_H = K_a$, we define a graph kit $\Phi_H := \Phi_b \cup \{K_b\}$ of K_H and the solid tori $R_{K_b} := U_b$ and $R_{\Gamma(K_H)} := U_b$. The graph kit Φ_H of K_H with its associated solid tori satisfies then the properties stated in the theorem. We now embed $\tilde{H}_1$ into S^3 by $g : \tilde{H}_1 \to S^3$ such that $g(\tilde{H}_1)$ becomes a standard, unknotted solid torus in S^3. Such an embedding g onto a standard, unknotted solid torus has been studied before in the $P_1 \neq P_2$ situation. We can similarly prove the theorem in this situation by combining the graph kits Φ_H and Φ_g of $g(K)$. This completes the proof of the theorem. □

Proof of Theorem 1. Lemma 8 proves one side of the theorem and Theorem 9 proves the other side since any k-function f can be made ordered by modifying it within a tubular neighborhood of its critical link without changing the set of critical points of f. □

Chapter 5

Conclusion

The classifications of graph knots in [**11**], [**12**] are better than ours in several respects. Their classifications are stronger as they classify all the graph links in S^3 or in a homology 3-sphere. Such a classification attempt demands a global picture of the whole graph link rather than a small picture of a component of a graph link. Our own narrow perspective to classify just the graph knots but not the graph links has limited us to work in a smaller, local setting with deficient information where we have gathered extra technical machinery (Lemma 7 and parts (i), (ii) and (iii) of Theorem 9) to overcome these deficiencies. As such deficiencies do not exist in the global settings in [**11**] and [**12**], their studies and proofs seem more natural than ours.

We conclude our work with the following final remarks. If K and J are two non-isotopic graph knots, then one way to qualitatively distinguish them is to look at the minimal numbers k and j such that $K \in \mathbb{S}_k$ and $J \in \mathbb{S}_j$. The bigger the difference $|k - j|$ gets, then one can interpret that K and J become more distinct from each other.

Suppose now that a knot K is not a graph knot so that we don't have a k-function to study it directly. How can we measure its deviation from being a graph knot? One trivial approach is to consider all the knot diagrams of K. The over or under crossings of a given diagram can be interchanged until the produced diagram becomes an unknot where an unknot is a graph knot. Therefore, there exists a minimal positive integer n (similar to the unknotting number of K) such that one produces a graph knot after making n crossing changes on a knot diagram of K. The bigger n gets, then one can think that K deviates more from being a graph knot.

The perspective of [**11**] gives a conceptually better answer to our above question. When K is not a graph knot, the JSJ-decomposition of the complement of K has at least one atoroidal (non-Seifert fibered) piece. The more atoroidal pieces there are, the more K deviates from being a graph knot.

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