

CAYLEY GRASSMANNIAN AND DEFORMATIONS IN COMPLEX G_2 MANIFOLDS

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ABSTRACT

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Geometric objects related to the exceptional lie groups G_2 and $\text{Spin}(7)$ have become increasingly popular in the recent years. Especially so after Bryant's (and others') work which showed the existence of riemannian manifolds with holonomy group equal to one of these groups [Bry87]. However, not much attention is given to the complex manifestations of these objects. This thesis consists of two parts which fills some of these gaps.

In the first part of this thesis, we investigate the Cayley Grassmannian (over $\mathbb{C}$) which is the set of four-planes that are closed under a three-fold cross product in $\mathbb{C}^8$. We define a torus action on the Cayley Grassmannian. Using this action, we prove that the minimal compactification is a singular variety. We also show that the singular locus is smooth and has the same cohomology ring as that of $\mathbb{CP}^5$. Furthermore, we identify the singular locus with a quotient of $G_2^{\mathbb{C}}$ by a parabolic subgroup.

In the second part of this thesis, we introduce the notion of (almost) $G_2^{\mathbb{C}}$ -manifolds with compatible symplectic structures. Further, we describe “complexification” procedures for a G_2 manifold $M \subset M_{\mathbb{C}}$. As an application we show that isotropic deformations of an associative submanifold Y of a G_2 manifold inside of its complexification $M_{\mathbb{C}}$ is given by Seiberg-Witten type equations.

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To the memory of my father Halil Yıldırım.

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CHAPTER 1

INTRODUCTION

The existence of Riemannian manifolds with exceptional holonomy groups (G_2 or $\text{Spin}(7)$) is shown by Bryant in 1987 [Bry87]. Later, Bryant and Salamon found complete examples [BS89] and Joyce found compact ones [Joy00]. These advancements naturally motivated many questions among which is how to distinguish two such manifolds. The main methods that may yield to invariants are gauge theoretic approach of counting instantons, counting minimal submanifolds or possibly some combination of the two [DT98, Wal13].

We say that a seven (resp. eight) manifold M has a G_2 (resp. $\text{Spin}(7)$) structure if the manifold is equipped with a special three (resp. four) form ϕ (resp. Φ). These are calibration forms in the sense of [HL82]. So, they define calibrated submanifolds called associative submanifolds in the case of G_2 and Cayley submanifolds in the case of $\text{Spin}(7)$. These calibration forms naturally define subvarieties, called associative Grassmannian and Cayley Grassmannian, of $\text{Gr}(3, 7)$ and $\text{Gr}(4, 8)$, respectively. We can explain these more concretely as follows. The form ϕ (resp. Φ) determines a unique Riemannian metric g on the manifold M . Then, by setting $g(u \times v, w) = \phi(u, v, w)$ (resp. $g(u \times v \times w, z) = \Phi(u, v, w, z)$) one can define a two (resp. three)-fold cross product operation on TM . The associative (resp. Cayley)-submanifolds are three (resp. four)-submanifolds whose tangent spaces are closed under the corresponding cross product operation. Fix a single tangent space $T_p M$, then a three (resp. four)-plane in $T_p M$ is called an associative (resp. Cayley)-plane if it is closed under the corresponding cross product operation. The set of all associative (resp. Cayley)-planes is called the associative (resp. Cayley) Grassmannian.

The associative and Cayley Grassmannians are the homogeneous spaces $G_2/\text{SO}(4)$ and $\text{Spin}(7)/K$ (where $K = (\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)) / \{\pm 1\}$) respectively, and they have been studied in [HL82, SW10, AK16]. Over real numbers, they are compact spaces. Although over complex numbers they are not compact spaces, they admit natural compactifications. Some natural compactifications of the associative Grassmannian have been studied in [AC15].

The first part of this thesis is an investigation of the (compactified) Cayley Grassmannian (over the complex numbers). We study the Cayley Grassmannian using a torus action which has finitely many fixed points. First, we introduce octonions and multiple cross products. Then, using three-fold cross product, we define Cayley planes precisely and describe an equivalent formulation. The latter formulation allows one to express Cayley planes as solutions to some polynomial equations (in $\mathbb{P}(\Lambda^4 \mathbb{C}^8)$ via Plücker embedding). We call the closure of this variety the minimal compactification of the Cayley Grassmannian. We show that the variety is singular, the singular locus is smooth and it is a cohomology $\mathbb{P}^5$. Further, we identify the singular locus as a quotient of $G_2^{\mathbb{C}}$ by a parabolic subgroup P_2 .

The second part of this thesis is a joint work with Akbulut [AY18]. In this part, the main objects of our study are (almost) $G_2^{\mathbb{C}}$ manifolds. An (almost) $G_2^{\mathbb{C}}$ manifold is a (real) 14-manifold whose structure group is $G_2^{\mathbb{C}} \subset GL(14, \mathbb{R})$. In other words, its frame bundle admits a $G_2^{\mathbb{C}}$ subbundle and one such subbundle is fixed. We provide examples of such manifolds through two different constructions. They are described as complexification procedures starting with a G_2 manifold. Given a G_2 manifold (M, φ) , we consider the cotangent bundle T^*M as a complexification of M and construct $G_2^{\mathbb{C}}$ structures $(\varphi_{\mathbb{C}}, B_{\mathbb{C}}, \Omega)$ on it in two different ways. Furthermore, we describe a compatibility condition for $G_2^{\mathbb{C}}$ structures and symplectic structures. In both complexification procedures, we obtain $G_2^{\mathbb{C}}$ manifolds with symplectic structures. However, in one of the procedures the symplectic form is not necessarily closed. On the other hand, we use the canonical symplectic structure ω_{can} on T^*M in the other construction.

It is possible to extend the notion of associative submanifolds to the complex case as well. Further, having a compatible symplectic structure allows one to define a new type of special submanifold which we call isotropic associative submanifolds. These are (real) three dimensional submanifolds which are isotropic with respect to the symplectic form ω and satisfy an associativity condition defined using $\varphi_{\mathbb{C}}$. As an application, we investigate infinitesimal deformations of isotropic associative submanifolds and relate them to Seiberg-Witten type equations.

The organization of the thesis is as follows. In Chapter 2 we collect the background material

common to both parts of the thesis. Chapter 3 is devoted to Cayley Grassmannian and Chapter 4 is devoted to $G_2^{\mathbb{C}}$ manifolds.

CHAPTER 2

PRELIMINERIES

In this preparatory chapter, the background material necessary for both parts of the thesis is given.

2.1 Octonions

In this section, we define an octonion algebra and various cross products. All the constructions of this chapter can be done over $\mathbb{R}$ or over $\mathbb{C}$. In fact, they can be generalized to other fields. However, that is beyond the scope of this thesis.

Definition 2.1.1 ([SV13]). A composition algebra C over a field k is an algebra over k with identity element and a nondegenerate quadratic form N such that

$$N(uv) = N(u)N(v)$$

for $u, v \in C$. The quadratic form N is often referred to as the norm on C , and the associated bilinear form $B(\cdot, \cdot)$ is called the inner product.

A four dimensional composition algebra is called a quaternion algebra, and an eight dimensional composition algebra is called an octonion algebra.

A specific example of an octonion algebra $\mathbb{O}$ (over $\mathbb{R}$) is given as follows. Let $S = \{1, \mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{li}, \mathbf{lj}, \mathbf{lk}\}$ be an orthonormal basis for (the vector space) $\mathbb{O}$. For each (oriented) line (or the circle) in Figure 2.1 from x to y to z , set

$$xy = z = -yx, \quad yz = x = -zy, \quad zx = y = -xz, \quad \text{and} \quad x^2 = y^2 = z^2 = -1.$$

Note that the subspace $\mathbb{H}$ generated by $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is closed under multiplication and therefore, it is a subalgebra. To ease notation later on, we set $e_0 = 1, e_1 = \mathbf{i}, e_2 = \mathbf{j}, e_3 = \mathbf{k}, e_4 = \mathbf{l}, e_5 = \mathbf{li}, e_6 = \mathbf{lj}$ and $e_7 = \mathbf{lk}$. We also set $e^{i_1 \dots i_k} = e^{i_1} \wedge \dots \wedge e^{i_k}$ where $\{e^i\}$ is the dual basis of $\{e_i\}$. The octonions are non-associative but they are alternative, i.e., the subalgebra generated by any two elements is

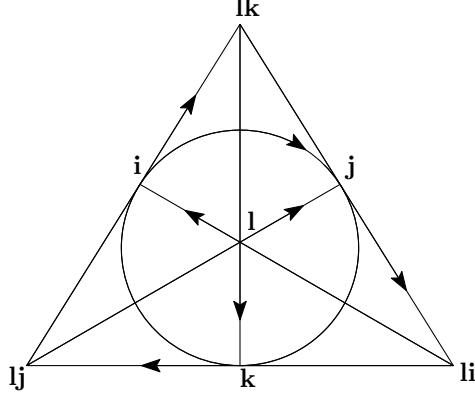


Figure 2.1: Multiplication table for octonions

associative. We denote the projection map from $\mathbb{O}$ to the span of 1 by Re , and projection to the orthogonal complement $1^\perp$ by Im . This allows us to define an involution

$$u \mapsto \bar{u} = \text{Re}(u) - \text{Im}(u).$$

The bilinear form B associated to N can be expressed as $B(u, v) = \text{Re}(\bar{u}v)$. So $N(u) = B(u, u) = \text{Re}(\bar{u}u)$.

A key fact one can verify on the basis elements is

$$\overline{uv} = \bar{v}\bar{u}. \quad (2.1)$$

Note that (2.1) implies

$$\overline{\bar{u}u} = \bar{u}u$$

that is $\bar{u}u \in \text{Re}(\mathbb{O})$. Thus,

$$N(uv) = \text{Re}(\overline{uv}uv) = \text{Re}(\bar{v}(\bar{u}u)v) = \text{Re}(\bar{u}u)\text{Re}(\bar{v}v) = N(u)N(v)$$

proving directly that the above multiplication table defines an eight-dimensional composition algebra (also proving $\mathbb{H}$ is a quaternion algebra.) We may complexify $\mathbb{O}$ to get a complex octonion algebra which we denote by $\mathbb{O}^\mathbb{C}$ or if the field is clear from the context simply by $\mathbb{O}$ again. (Here, we complexify B by extending it as a complex bilinear form on both entries and N is extended so that it is the (complex) quadratic form associated to B .)

Next, we would like to define various cross product operations using octonions but, first, we justify their name.

Definition 2.1.2. Let (V, B) be a vector space with a (non-degenerate) symmetric bilinear form. A multilinear map $L : V^r \rightarrow V$ is called an r -fold cross product if

$$N(L(v_1, \dots, v_n)) = N(v_1 \wedge \dots \wedge v_n) \quad (2.2)$$

with the induced norm on $\Lambda^n V$ and

$$B(L(v_1, \dots, v_n), v_i) = 0 \quad \text{for all } i. \quad (2.3)$$

Remark 2.1.3. If L is an alternating multilinear map, then it is enough to check (2.2) on orthogonal vectors in which case (2.2) becomes

$$N(L(v_1, \dots, v_n)) = N(v_1) \dots N(v_n). \quad (2.4)$$

Remark 2.1.4. The usual cross product operation on $\mathbb{R}^3$ is naturally a (two-fold) cross product according to this definition.

In [BG67], Brown and Gray proved that an r -fold cross product exists on an n -dimensional vector space only in the following cases:

1. n is even, $r = 1$
2. n is arbitrary, $r = n - 1$
3. $n = 7, r = 2$
4. $n = 8, r = 3$.

We say that the last two cases are exceptional as they occur only in specific dimensions. Below, we give concrete description of the exceptional two-fold and three-fold cross products using octonions.

Then, we introduce a “four-fold cross product” operation on $\mathbb{O}$. Although it is not a cross product according to Definition 2.1.2, it is conventionally called so [HL82, SW10]. A two-fold cross product (or simply a cross product) can be defined as the restriction of octonionic multiplication to the imaginary part, $\text{Im}(\mathbb{O})$:

Definition 2.1.5. For $u, v \in \text{Im}(\mathbb{O})$, let

$$u \times v = \text{Im}(uv). \quad (2.5)$$

To be able to prove this is a cross product operation, the following lemma is needed.

Lemma 2.1.6. For $u, v, v' \in \mathbb{O}$,

$$N(u)B(v, v') = B(uv, uv') = B(vu, v'u). \quad (2.6)$$

In particular, for unit u , (left or right) multiplication by u is an orthogonal transformation of $\mathbb{O}$.

Proof. Since $B(v, v') = \frac{1}{2} (N(v + v') - N(v) - N(v'))$, we have

$$\begin{aligned} N(u)B(v, v') &= \frac{1}{2} (N(u)N(v + v') - N(u)N(v) - N(u)N(v')) \\ &= \frac{1}{2} (N(uv + uv') - N(uv) - N(uv')) \\ &= B(uv, uv'). \end{aligned}$$

The second equality can be proved similarly. □

Proposition 2.1.7. The map $(u, v) \mapsto u \times v = \text{Im}(uv)$ is a two-fold cross product on $\text{Im}(\mathbb{O})$.

Proof. Since $\bar{u}u \in \text{Re}(\mathbb{O})$, $u \times u = \text{Im}(uu) = -\text{Im}(\bar{u}u) = 0$. So, $u \times v$ is an alternating map. By Remark 2.1.3, we may assume $u, v \in \text{Im}(\mathbb{O})$ are orthogonal, that is, $B(u, v) = 0$. Then, by Lemma 2.1.6 we get

$$\begin{aligned} 0 &= N(u)B(u, v) \\ &= -B(\bar{u}u, uv) \\ &= -B(N(u), uv). \end{aligned}$$

Thus, $uv \in \text{Im}(\mathbb{O})$. This gives us

$$\begin{aligned} N(u \times v) &= N(\text{Im}(uv)) \\ &= N(uv) \\ &= N(u)N(v). \end{aligned}$$

To prove (2.3), we once again use Lemma 2.1.6.

$$\begin{aligned} B(u \times v, u) &= B(\text{Im}(uv), u) \\ &= B(uv - \text{Re}(uv), u) \\ &= B(uv, u) - B(\text{Re}(uv), u) \\ &= N(u)B(v, 1) \\ &= 0 \end{aligned}$$

since u and v are orthogonal to 1. □

Next, following [SW10] we define a three-fold cross product and a four-fold “cross product” as follows:

Definition 2.1.8. For $u, v, w \in \mathbb{O}$, let

$$u \times v \times w = \frac{1}{2} ((u\bar{v})w - (w\bar{v})u). \quad (2.7)$$

Definition 2.1.9. For $u, v, w, x \in \mathbb{O}$, let

$$x \times u \times v \times w = -\frac{1}{4} [(x \times u \times v)\bar{w} - (w \times x \times u)\bar{v} + (v \times w \times x)\bar{u} - (u \times v \times w)\bar{x}]. \quad (2.8)$$

Remark 2.1.10. In fact, the four-fold cross product operation (2.8) does not satisfy (2.3) but it is conventionally called cross product [HL82, SW10]. However, it is alternating and satisfies (2.2). Hence, for orthogonal vectors x, u, v , and w we have

$$N(x \times u \times v \times w) = N(x)N(u)N(v)N(w). \quad (2.9)$$

2.2 Calibration forms and calibrated planes

In this section, we define two calibration forms in the sense of [HL82] and calibrated planes associated to these forms that are relevant to this thesis. The base field is again either $\mathbb{R}$ or $\mathbb{C}$ for this section. Reader may consult [HL82] for the general theory of calibrated geometries.

A k -form $\omega \in \Lambda^k V^*$ is called a calibration form if for every orthonormal set of vectors $\{v_1, \dots, v_k\}$ we have $|\omega(v_1, \dots, v_k)| \leq 1$. Given a k -plane ξ generated by an orthonormal basis $\{v_1, \dots, v_k\}$, ξ is called calibrated if $\omega(v_1, \dots, v_k) = \pm 1$. Using the two-fold (resp. three-fold) cross product, we define a calibration three-form (resp. four-form) ϕ (resp. Φ) called associative (resp. Cayley) calibration on $\text{Im}(\mathbb{O})$ (resp. $\mathbb{O}$) as follows:

Definition 2.2.1. For $u, v, w \in \text{Im}(\mathbb{O})$, let

$$\phi(u, v, w) = B(u, v \times w) \quad (2.10)$$

and for $x, u, v, w \in \mathbb{O}$, let

$$\Phi(x, u, v, w) = B(x, u \times v \times w). \quad (2.11)$$

For a proof of the following proposition see [SW10].

Proposition 2.2.2. The equations (2.10) and (2.11) define calibration forms and they satisfy

$$\phi(u, v, w) = \text{Re}(u \times v \times w) \quad (2.12)$$

and

$$\Phi(x, u, v, w) = \text{Re}(x \times u \times v \times w). \quad (2.13)$$

By (2.4) and (2.12), it is clear that $\phi(u, v, w) = \pm 1$ if and only if $\text{Im}(u \times v \times w) = 0$ for orthonormal $u, v, w \in \text{Im}\mathbb{O}$. Similarly, by (2.9) and (2.13), it is clear that $\Phi(x, u, v, w) = \pm 1$ if and only if $\Xi(x, u, v, w) := \text{Im}(x \times u \times v \times w) = 0$ for orthonormal $x, u, v, w \in \mathbb{O}$.

Definition 2.2.3. A three-plane (resp. four-plane) ξ generated by orthonormal $\{u, v, w\}$ (resp. $\{x, u, v, w\}$) is called an associative (resp. Cayley) plane if $\phi(u, v, w) = \pm 1$ (resp. $\Phi(x, u, v, w) = \pm 1$)

or, equivalently, $\text{Im}(u \times v \times w) = 0$ (resp. $\Xi(x, u, v, w) = 0$). The set of all associative (resp. Cayley) planes is called the associative (resp. Cayley) Grassmannian. We denote the associative Grassmannian by $\text{Gr}(\phi)$ and the Cayley Grassmannian by $\text{Gr}(\Phi)$.

Remark 2.2.4. Over $\mathbb{C}$, not every three-plane (resp. four-plane) is generated by an orthonormal basis (with respect to B). Strictly speaking, this is why the associative (resp. Cayley) Grassmannian is not compact when defined over $\mathbb{C}$.

It is helpful to express ϕ , Φ and Ξ in coordinates. The associative calibration form is given by

$$\phi = e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356}$$

the Cayley calibration form is given by

$$\begin{aligned} \Phi = & e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ & - e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367} + e^{4567} \end{aligned}$$

and the imaginary part of the four-fold cross product is given by

$$\begin{aligned} \Xi = & \left(-e^{0247} - e^{0256} + e^{0346} - e^{0357} + e^{1246} - e^{1257} + e^{1347} + e^{1356} \right) e_1 \\ & + \left(+e^{0147} + e^{0156} - e^{0345} - e^{0367} - e^{1245} - e^{1267} + e^{2347} + e^{2356} \right) e_2 \\ & + \left(-e^{0146} + e^{0157} + e^{0245} + e^{0267} - e^{1345} - e^{1367} - e^{2346} + e^{2357} \right) e_3 \\ & + \left(-e^{0127} + e^{0136} - e^{0235} + e^{0567} + e^{1234} - e^{1467} + e^{2457} - e^{3456} \right) e_4 \\ & + \left(-e^{0126} - e^{0137} + e^{0234} - e^{0467} + e^{1235} - e^{1567} + e^{2456} + e^{3457} \right) e_5 \\ & + \left(+e^{0125} - e^{0134} - e^{0237} + e^{0457} + e^{1236} - e^{1456} - e^{2567} + e^{3467} \right) e_6 \\ & + \left(+e^{0124} + e^{0135} + e^{0236} - e^{0456} + e^{1237} - e^{1457} - e^{2467} - e^{3567} \right) e_7. \end{aligned} \quad (2.14)$$

Using these expressions, one can immediately see that

$$\Phi = e^0 \wedge \phi + *\phi \quad (2.15)$$

where $*$ is the Hodge star operator.

2.3 Groups G_2 and $\text{Spin}(7)$

In this section, we define groups G_2 and $\text{Spin}(7)$ (over $\mathbb{R}$ and $\mathbb{C}$). Definitions we give (following Bryant [Bry87]) are somewhat unusual but better suited for our purposes. Then, we describe a (maximal) torus of $\text{Spin}(7)$ and three subgroups of $\text{Spin}(7)$ each of which is isomorphic to $\text{SL}(2, \mathbb{C})$ (over $\mathbb{C}$ and $\text{SU}(2)$ over $\mathbb{R}$). By restriction, we also get a (maximal) torus of G_2 and subgroups that are isomorphic to $\text{SL}(2, \mathbb{C})$ (or $\text{SU}(2)$ depending on the field).

Definition 2.3.1. 1. $\text{Spin}(7, \mathbb{R})$ is the stabilizer of Φ in $\text{SO}(8, \mathbb{R})$,

2. $\text{Spin}(7, \mathbb{C})$ is the identity component of the stabilizer of Φ in $\text{SO}(8, \mathbb{C})$,

3. G_2 is the stabilizer of ϕ in $\text{SO}(7, \mathbb{R})$, and

4. $G_2^{\mathbb{C}}$ is the stabilizer of ϕ in $\text{SO}(7, \mathbb{C})$.

Given $A \in G_2^{\mathbb{C}}$, we can extend it linearly so that it fixes $1 \in \mathbb{O}$. Then, it is easy to see that $G_2^{\mathbb{C}}$ is a subgroup of $\text{Spin}(7, \mathbb{C})$ using (2.15).

By definition, an element of $\text{Spin}(7, \mathbb{C})$ acts on $\mathbb{O}$ preserving orthonormality and the values of Φ . Thus, it takes a Cayley plane to a Cayley plane. In other words, it defines an action on the Cayley Grassmannian $\text{Gr}(\Phi)$.

Consider the following matrix

$$L_\lambda = \begin{pmatrix} P_\lambda & -iM_\lambda \\ iM_\lambda & P_\lambda \end{pmatrix}$$

where $P_\lambda = \frac{\lambda + \lambda^{-1}}{2}$, $M_\lambda = \frac{\lambda - \lambda^{-1}}{2}$, and $\lambda \in \mathbb{C}^*$. Note that its determinant is 1. In fact, it has eigenvalues λ and λ^{-1} with eigenvectors

$$\begin{pmatrix} 1 \\ i \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

respectively.

Considering L_λ as a block matrix, we define the following 8×8 matrices $A_\lambda = L_\lambda \oplus L_\lambda \oplus L_\lambda \oplus L_\lambda$, $B_\mu = L_\mu \oplus L_{\mu^{-1}} \oplus I_4$ and $C_\gamma = I_4 \oplus L_\gamma \oplus L_{\gamma^{-1}}$ and view them as transformations of $\mathbb{O}$ with respect to the standard basis $\{e_i\}$ where I_n is the $n \times n$ identity matrix.

Lemma 2.3.2. The image of $h : (\mathbb{C}^*)^3 \rightarrow \text{SL}(8, \mathbb{C})$ defined by

$$h(\lambda, \mu, \gamma) = A_\lambda B_\mu C_\gamma$$

is a maximal torus T of $\text{Spin}(7, \mathbb{C})$.

Proof. It is easy to prove that $L_\lambda L_\mu = L_{\lambda\mu}$ and if $L_\lambda = I_2$ then $\lambda = 1$. It follows that A_λ, B_μ and C_γ commute with each other. Hence, h is a well defined homomorphism. Furthermore, the kernel of h is given by $\{\pm(1, 1, 1)\}$ and thus, the image $T \cong (\mathbb{C}^*)^3 / \mathbb{Z}_2$ is isomorphic to $(\mathbb{C}^*)^3$. Since the rank of $\text{Spin}(7, \mathbb{C})$ is 3, we only need to show that $T \subset \text{Spin}(7, \mathbb{C})$.

A simple computation shows that $(L_\lambda)^{-1} = L_{\lambda^{-1}} = L_\lambda^T$. In other words, $L_\lambda \in \text{SO}(2, \mathbb{C})$ which implies $T \subset \text{SO}(8, \mathbb{C})$. Finally, we need to show that for $M \in T$, $M^* \Phi = \Phi$. We verify this by a direct computation with the help of a software. $\square$

We identify $\text{SL}(2, \mathbb{C})$ as the subgroup of the multiplicative group of $\mathbb{H}$ with $N = 1$. More explicitly, $u \in \mathbb{H}$ is identified with the matrix A_u given by

$$u = a1 + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mapsto A_u = \begin{pmatrix} a - id & -b + ic \\ b + ic & a + id \end{pmatrix}.$$

Note that $A_u : \mathbb{H} \rightarrow \mathbb{C}^{2 \times 2}$ is a linear isomorphism and it satisfies

$$A_u A_v = A_{uv}.$$

Moreover, $N(u) = \det(A_u)$. Hence, $(\mathbb{C}^{2 \times 2}, \det)$ is a quaternion algebra isomorphic to $(\mathbb{H}, N)$ via $(u \mapsto A_u)$. Thus, $\text{SL}(2, \mathbb{C})$ can be identified with the unit sphere of $\mathbb{H}$, i.e., $\{v \in \mathbb{H} \mid N(v) = 1\}$.

Proposition 2.3.3. There are three $\text{SL}(2, \mathbb{C})$ actions on $\mathbb{O}$ which preserve B and Φ . To describe these actions we express $\mathbb{O}$ as a direct sum $\mathbb{O} = \mathbb{H} \oplus \mathbb{H}$. Let $v = (x, y) \in \mathbb{O}$.

$$1. \quad g \cdot v = (xg^{-1}, y)$$

$$2. \quad g \cdot v = (x, yg^{-1})$$

$$3. \quad g \cdot v = (gx, gy)$$

Proof. Since $g \in \mathrm{SL}(2, \mathbb{C})$ is identified with an element of $\mathbb{H}$ with norm 1, multiplication by g is an orthogonal transformation by Lemma 2.1.6. Thus, B is preserved in all three actions.

To show that Φ is also preserved, we instead look at the corresponding action of the Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \cong \mathrm{Im}(\mathbb{H}) = \langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle$. It is enough to show that $\mathbf{i} \cdot \Phi = \mathbf{j} \cdot \Phi = \mathbf{k} \cdot \Phi = 0$. We verify this by a direct computation for all three actions. $\square$

Remark 2.3.4. Note that all three actions are faithful and thus, provide three different embeddings of $\mathrm{SL}(2, \mathbb{C})$ into $\mathrm{Spin}(7, \mathbb{C})$. Furthermore, we can define an action of the group $(\mathrm{SL}(2, \mathbb{C}))^3$ on $\mathbb{O}$ by

$$(a, b, c) \cdot (x, y) = (cxa^{-1}, cyb^{-1}) \tag{2.16}$$

for $a, b, c \in \mathrm{SL}(2, \mathbb{C})$ and $(x, y) \in \mathbb{O}$. The kernel of this action is $\{\pm(1, 1, 1)\}$.

CHAPTER 3

CAYLEY GRASSMANNIAN

In this chapter we investigate a natural compactification of the Cayley Grassmannian over $\mathbb{C}$. We show that this compactification is singular, its singular set is a cohomology $\mathbb{CP}^5$ and we identify the singular locus as a quotient of $G_2^{\mathbb{C}}$ by one of its parabolic subgroups.

3.1 Charts of $\text{Gr}(4, \mathbb{O})$

We would like to be able to do some computations in order to learn more about the Cayley Grassmannian. For this purpose, we recall some elementary facts about the charts of a Grassmannian in this section.

Set $p_{ijkl} = e^{ijkl}$ so they are coordinate functions on $\Lambda^4 \mathbb{O}$ and recall that $\text{Gr}(4, \mathbb{O})$ can be thought of as a subvariety of $\mathbb{P}(\Lambda^4 \mathbb{C}^8)$ cut out by the Plücker relations (see [KL72]):

$$p_{i_1 i_2 i_3 j_1} p_{j_2 j_3 j_4 j_5} = p_{i_1 i_2 i_3 j_2} p_{j_1 j_3 j_4 j_5} - p_{i_1 i_2 i_3 j_3} p_{j_1 j_2 j_4 j_5} + p_{i_1 i_2 i_3 j_4} p_{j_1 j_2 j_3 j_5} - p_{i_1 i_2 i_3 j_5} p_{j_1 j_2 j_3 j_4} \quad (3.1)$$

Since $\text{Gr}(\Phi)$ lives inside $\text{Gr}(4, \mathbb{O})$, $\text{Gr}(\Phi)$ is also a subvariety of $\mathbb{P}(\Lambda^4 \mathbb{O})$. More precisely, $\text{Gr}(\Phi)$ lie in the intersection of $\text{Gr}(4, \mathbb{O})$ and the zero locus of Ξ . By (2.14), the zero locus of Ξ is given by these seven linear equations:

$$f_1 := -p_{0247} - p_{0256} + p_{0346} - p_{0357} + p_{1246} - p_{1257} + p_{1347} + p_{1356} = 0 \quad (3.2)$$

$$f_2 := +p_{0147} + p_{0156} - p_{0345} - p_{0367} - p_{1245} - p_{1267} + p_{2347} + p_{2356} = 0 \quad (3.3)$$

$$f_3 := -p_{0146} + p_{0157} + p_{0245} + p_{0267} - p_{1345} - p_{1367} - p_{2346} + p_{2357} = 0 \quad (3.4)$$

$$f_4 := -p_{0127} + p_{0136} - p_{0235} + p_{0567} + p_{1234} - p_{1467} + p_{2457} - p_{3456} = 0 \quad (3.5)$$

$$f_5 := -p_{0126} - p_{0137} + p_{0234} - p_{0467} + p_{1235} - p_{1567} + p_{2456} + p_{3457} = 0 \quad (3.6)$$

$$f_6 := +p_{0125} - p_{0134} - p_{0237} + p_{0457} + p_{1236} - p_{1456} - p_{2567} + p_{3467} = 0 \quad (3.7)$$

$$f_7 := +p_{0124} + p_{0135} + p_{0236} - p_{0456} + p_{1237} - p_{1457} - p_{2467} - p_{3567} = 0. \quad (3.8)$$

We call the Zariski closure of this variety the minimal compactification of the Cayley Grassmannian and denote it by $X_{\min}$. It is at least 12-dimensional as it contains the Cayley Grassmannian which is 12-dimensional [HL82]. Later, we shall see that $X_{\min}$ is indeed 12-dimensional (see Theorem 3.2.3).

Once we choose a chart $U_{stun} = \{x \in \mathbb{P}(\Lambda^4 \mathbb{O}) \mid p_{stun}(x) \neq 0\}$, we use the following notation for local coordinates (suppressing the indices s, t, u, n).

$$q_{ijkl} = \frac{p_{ijkl}}{p_{stun}}.$$

For example, over U_{0123} , using Plücker relations (3.1), we have

$$\frac{p_{4567}}{p_{0123}} = \frac{p_{0456} p_{1237}}{p_{0123} p_{0123}} - \frac{p_{1456} p_{0237}}{p_{0123} p_{0123}} + \frac{p_{2456} p_{0137}}{p_{0123} p_{0123}} - \frac{p_{3456} p_{0127}}{p_{0123} p_{0123}} \quad (3.9)$$

or, more concisely,

$$q_{4567} = q_{0456} q_{1237} - q_{1456} q_{0237} + q_{2456} q_{0137} - q_{3456} q_{0127}. \quad (3.10)$$

After fixing a chart U_{stun} , one can show that any coordinate function can be expressed only in terms of the variables q_{ijkl} with $|\{i, j, k, l\} \cap \{s, t, u, n\}| = 3$ by using (3.1) (repeatedly if necessary). There are exactly 16 such variables corresponding to the fact that $\dim(\text{Gr}(4, \mathbb{O})) = 16$ and they give us the charts of $\text{Gr}(4, \mathbb{O})$, see [KL72, AC15].

3.2 Torus fixed points and the minimal compactification

Recall that $\text{Spin}(7, \mathbb{C})$ acts on $\text{Gr}(\Phi)$ and we have described a specific torus of $\text{Spin}(7, \mathbb{C})$ earlier. By restricting the action to the torus T , we would like to obtain more information about $X_{\min} = \text{Gr}(\Phi)$. In this section, we prove some facts about this torus action.

From our discussion in Section 2.3 for L_λ , it is easy to find eigenvalues and eigenvectors for $h(\lambda, \mu, \gamma)$. They are given in the following table.

eigenvalue	eigenvector
$\lambda\mu$	$1 + i\mathbf{i}$
$\lambda^{-1}\mu^{-1}$	$1 - i\mathbf{i}$
$\lambda\mu^{-1}$	$\mathbf{j} + i\mathbf{k}$
$\lambda^{-1}\mu$	$\mathbf{j} - i\mathbf{k}$
$\lambda\gamma$	$\mathbf{l} + i\mathbf{li}$
$\lambda^{-1}\gamma^{-1}$	$\mathbf{l} - i\mathbf{li}$
$\lambda\gamma^{-1}$	$\mathbf{l}\mathbf{j} + i\mathbf{l}\mathbf{k}$
$\lambda^{-1}\gamma$	$\mathbf{l}\mathbf{j} - i\mathbf{l}\mathbf{k}$

(3.11)

We set $\tilde{e}_0 = 1 + i\mathbf{i}$, $\tilde{e}_1 = 1 - i\mathbf{i}$, $\tilde{e}_2 = \mathbf{j} + i\mathbf{k}$, $\tilde{e}_3 = \mathbf{j} - i\mathbf{k}$, $\tilde{e}_4 = \mathbf{l} + i\mathbf{li}$, $\tilde{e}_5 = \mathbf{l} - i\mathbf{li}$, $\tilde{e}_6 = \mathbf{l}\mathbf{j} + i\mathbf{l}\mathbf{k}$, and $\tilde{e}_7 = \mathbf{l}\mathbf{j} - i\mathbf{l}\mathbf{k}$. We also set $\tilde{e}_{pqrs} = \tilde{e}_p \wedge \tilde{e}_q \wedge \tilde{e}_r \wedge \tilde{e}_s$ and consider the action of $h(\lambda, \mu, \gamma)$ on $\Lambda^4\mathbb{O}$ (or its projectivization). If we denote by $\tilde{p}_{pqrs}$ the transformed Plücker coordinates, the equations (3.2)-(3.8) can be rewritten as

$$\tilde{f}_1 := \tilde{p}_{0257} - \tilde{p}_{1346} = 0 \quad (3.12)$$

$$\tilde{f}_2 := \tilde{p}_{0146} - \tilde{p}_{0157} - \tilde{p}_{0245} - \tilde{p}_{0267} + \tilde{p}_{1345} + \tilde{p}_{1367} + \tilde{p}_{2346} - \tilde{p}_{2357} = 0 \quad (3.13)$$

$$\tilde{f}_3 := \tilde{p}_{0146} + \tilde{p}_{0157} - \tilde{p}_{0245} - \tilde{p}_{0267} - \tilde{p}_{1345} - \tilde{p}_{1367} + \tilde{p}_{2346} + \tilde{p}_{2357} = 0 \quad (3.14)$$

$$\tilde{f}_4 := \tilde{p}_{0127} - \tilde{p}_{0136} + \tilde{p}_{0235} - \tilde{p}_{0567} - \tilde{p}_{1234} + \tilde{p}_{1467} - \tilde{p}_{2457} + \tilde{p}_{3456} = 0 \quad (3.15)$$

$$\tilde{f}_5 := \tilde{p}_{0127} + \tilde{p}_{0136} - \tilde{p}_{0235} + \tilde{p}_{0567} - \tilde{p}_{1234} + \tilde{p}_{1467} - \tilde{p}_{2457} - \tilde{p}_{3456} = 0 \quad (3.16)$$

$$\tilde{f}_6 := \tilde{p}_{0125} - \tilde{p}_{0134} - \tilde{p}_{0237} + \tilde{p}_{0457} + \tilde{p}_{1236} - \tilde{p}_{1456} - \tilde{p}_{2567} + \tilde{p}_{3467} = 0 \quad (3.17)$$

$$\tilde{f}_7 := \tilde{p}_{0125} + \tilde{p}_{0134} + \tilde{p}_{0237} - \tilde{p}_{0457} + \tilde{p}_{1236} - \tilde{p}_{1456} - \tilde{p}_{2567} - \tilde{p}_{3467} = 0 \quad (3.18)$$

Theorem 3.2.1. The following eigenvectors of $h(\lambda, \mu, \gamma)$ lie in $X_{\min}$.

eigenvalue	eigenvector
1	$\tilde{e}_{0123}, \tilde{e}_{0145}, \tilde{e}_{0167}, \tilde{e}_{2345}, \tilde{e}_{2367}, \tilde{e}_{4567}$
γ^{-2}	$\tilde{e}_{0156}, \tilde{e}_{2356}$
$\gamma^{-2}\lambda^{-2}$	$\tilde{e}_{1356}$
$\gamma^{-2}\lambda^2$	$\tilde{e}_{0256}$
$\gamma^{-2}\mu^{-2}$	$\tilde{e}_{1256}$
$\gamma^{-2}\mu^2$	$\tilde{e}_{0356}$
$\gamma^{-1}\lambda^{-2}\mu^{-1}$	$\tilde{e}_{1235}, \tilde{e}_{1567}$
$\gamma^{-1}\lambda^{-2}\mu$	$\tilde{e}_{0135}, \tilde{e}_{3567}$
$\gamma^{-1}\lambda^2\mu^{-1}$	$\tilde{e}_{0126}, \tilde{e}_{2456}$
$\gamma^{-1}\lambda^2\mu$	$\tilde{e}_{0236}, \tilde{e}_{0456}$
$\gamma\lambda^{-2}\mu^{-1}$	$\tilde{e}_{1237}, \tilde{e}_{1457}$
$\gamma\lambda^{-2}\mu$	$\tilde{e}_{0137}, \tilde{e}_{3457}$
$\gamma\lambda^2\mu^{-1}$	$\tilde{e}_{0124}, \tilde{e}_{2467}$
$\gamma\lambda^2\mu$	$\tilde{e}_{0234}, \tilde{e}_{0467}$
γ^2	$\tilde{e}_{0147}, \tilde{e}_{2347}$
$\gamma^2\lambda^{-2}$	$\tilde{e}_{1347}$
$\gamma^2\lambda^2$	$\tilde{e}_{0247}$
$\gamma^2\mu^{-2}$	$\tilde{e}_{1247}$
$\gamma^2\mu^2$	$\tilde{e}_{0347}$
λ^{-4}	$\tilde{e}_{1357}$

eigenvalue	eigenvector
$\lambda^{-2}\mu^{-2}$	$\tilde{e}_{1257}$
$\lambda^{-2}\mu^2$	$\tilde{e}_{0357}$
$\lambda^2\mu^{-2}$	$\tilde{e}_{1246}$
$\lambda^2\mu^2$	$\tilde{e}_{0346}$
λ^4	$\tilde{e}_{0246}$
μ^{-2}	$\tilde{e}_{1245}, \tilde{e}_{1267}$
μ^2	$\tilde{e}_{0345}, \tilde{e}_{0367}$

Proof. Once we evaluate all the eigenvectors on the transformed defining equations (3.12)-(3.18), we see that exactly the above list of vectors satisfy them. $\square$

Theorem 3.2.2. The fixed point set $X_{\min}^T$ of the maximal torus action is only the above set of points.

Proof. We only need to verify that in eigenspaces of dimension greater than one, there are no other fixed points. We prove this for the eigenspace associated to the eigenvalue 1. Let

$$\tilde{x} = \tilde{c}_{0123}\tilde{e}_{0123} + \tilde{c}_{0145}\tilde{e}_{0145} + \tilde{c}_{0167}\tilde{e}_{0167} + \tilde{c}_{2345}\tilde{e}_{2345} + \tilde{c}_{2367}\tilde{e}_{2367} + \tilde{c}_{4567}\tilde{e}_{4567} \in X_{\min}$$

be a torus fixed point that is different from $\tilde{e}_{0123}, \tilde{e}_{0145}, \tilde{e}_{0167}, \tilde{e}_{2345}, \tilde{e}_{2367}$, and $\tilde{e}_{4567}$. Then, at least two of the coordinates $\tilde{c}_I$, and $\tilde{c}_J$ are nonzero for index sets $I \neq J$. Let $I = \{i_1 i_2 i_3 j_1\}$, and $J = \{j_2 j_3 j_4 j_5\}$. If $\tilde{x} \in X_{\min} \subset \text{Gr}(4, \mathbb{O})$ then it has to satisfy the Plücker relation (3.1)

$$\tilde{c}_{i_1 i_2 i_3 j_1} \tilde{c}_{j_2 j_3 j_4 j_5} = \tilde{c}_{i_1 i_2 i_3 j_2} \tilde{c}_{j_1 j_3 j_4 j_5} - \tilde{c}_{i_1 i_2 i_3 j_3} \tilde{c}_{j_1 j_2 j_4 j_5} + \tilde{c}_{i_1 i_2 i_3 j_4} \tilde{c}_{j_1 j_2 j_3 j_5} - \tilde{c}_{i_1 i_2 i_3 j_5} \tilde{c}_{j_1 j_2 j_3 j_4}.$$

Now, the left hand side of this relation is nonzero by choice. However, the right hand side has to be zero as the index sets of eigenvectors associated to eigenvalue 1 differ by at least two elements. This means that $\tilde{x}$ does not belong to $\text{Gr}(4, \mathbb{O})$ and hence, $\tilde{x} \notin X_{\min}$.

The same argument also works for all the other eigenspaces of dimension greater than one. $\square$

Theorem 3.2.3. $X_{\min}$ is a singular variety of dimension 12. Moreover, among the torus fixed points listed in Theorem 3.2.1, all but the following six of them are smooth points.

$$\tilde{e}_{0246}, \tilde{e}_{0347}, \tilde{e}_{0356}, \tilde{e}_{1247}, \tilde{e}_{1256}, \tilde{e}_{1357}$$

Proof. We need to analyze neighborhoods of fixed points by using affine charts. We start with the fixed point $m = \tilde{e}_{0123}$, which lies on the open chart $\tilde{U}_{0123}$ as its origin. Recall that $X_{\min}$ is cut-out on $\tilde{U}_{0123}$ by the vanishing of the seven linear equations (3.12)-(3.18). At first, it may seem that it is necessary to express these equations in local variables $\tilde{q}_{0124}, \tilde{q}_{0125}, \tilde{q}_{0126}, \tilde{q}_{0127}, \tilde{q}_{0134}, \tilde{q}_{0135}, \tilde{q}_{0136}, \tilde{q}_{0137}, \tilde{q}_{0234}, \tilde{q}_{0235}, \tilde{q}_{0236}, \tilde{q}_{0237}, \tilde{q}_{1234}, \tilde{q}_{1235}, \tilde{q}_{1236}$, and $\tilde{q}_{1237}$. However, the Plücker relations (in local coordinates) replace linear terms with higher order terms (see, for example, (3.10)) and we compute Jacobian at the origin. So, there will be no contribution to Jacobian matrix from other variables. The Jacobian matrix at m is given by

$$\tilde{J}_{0123} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

which is of rank four. So, it is at most 12-dimensional. However, it contains the 12-dimensional Cayley Grassmannian. So, it must be 12-dimensional, that is, codimension four in $\text{Gr}(4, \mathbb{O})$. Hence, $\tilde{e}_{0123}$ is a smooth point of $X_{\min}$.

We repeat this computation for the other points and see that their Jacobian matrices are all rank four, except for the six points we have listed above. Thus, they are all smooth points of $X_{\min}$. $\square$

3.3 Singular locus

Next, we turn our attention to the singular locus $\Sigma := \text{Sing}(X_{\min})$ of $X_{\min}$. The torus action on $X_{\min}$ restricts to Σ . By Theorem 3.2.3, we know that there are six points in Σ^T . We quote the

following lemma from [BBCM02].

Lemma 3.3.1. If $Y \subset \mathbb{P}(V)$ is a projective T -variety, then Y^T contains at least $\dim Y + 1$ points.

Therefore, we have the following corollary.

Corollary 3.3.2. The singular locus Σ is at most five-dimensional.

As we did with $X_{\min}$, we can check whether Σ is singular or not, by using Jacobian criterion on these six torus fixed points Σ^T .

Theorem 3.3.3. The singular locus Σ is smooth and five-dimensional.

Proof. We start with the point $\tilde{e}_{0246}$. This point lies at the origin of the chart

$$\tilde{U}_{0246} = \left\{ x \in \mathbb{P}(\Lambda^4 \mathbb{O}) \mid \tilde{p}_{0246}(x) \neq 0 \right\}.$$

On this chart the local variables are $\tilde{q}_{0124}, \tilde{q}_{0126}, \tilde{q}_{0146}, \tilde{q}_{0234}, \tilde{q}_{0236}, \tilde{q}_{0245}, \tilde{q}_{0247}, \tilde{q}_{0256}, \tilde{q}_{0267}, \tilde{q}_{0346}, \tilde{q}_{0456}, \tilde{q}_{0467}, \tilde{q}_{1246}, \tilde{q}_{2346}, \tilde{q}_{2456}$, and $\tilde{q}_{2467}$ where $\tilde{q}_{ijkl} = \tilde{p}_{ijkl} / \tilde{p}_{0246}$. Since the codimension of $X_{\min}$ in $\text{Gr}(4, \mathbb{O})$ is four, Σ is locally the vanishing locus of the equations (3.12)-(3.18) localized to $\tilde{U}_{0246}$ and all 4×4 minors of the Jacobian of these localized equations. These equations by themselves do not generate a radical ideal, so we take the radical ideal generated by those equations with the help of a software called Singular. It turns out the ideal is generated by

$$\begin{aligned} &\tilde{q}_{1246}, \quad \tilde{q}_{0346}, \quad \tilde{q}_{0267} - \tilde{q}_{2346}, \quad \tilde{q}_{0256}, \quad \tilde{q}_{0247}, \quad \tilde{q}_{0245} - \tilde{q}_{2346}, \quad \tilde{q}_{0236} - \tilde{q}_{0456}, \\ &\tilde{q}_{0234} - \tilde{q}_{0467}, \quad \tilde{q}_{0146} - \tilde{q}_{2346}, \quad \tilde{q}_{0126} - \tilde{q}_{2456}, \quad \text{and} \quad \tilde{q}_{0124} - \tilde{q}_{2467}. \end{aligned}$$

So, Σ is just cut out by some hyperplanes in $\tilde{U}_{0246} \cap \text{Gr}(4, \mathbb{O})$. Therefore, it is clearly smooth at the origin and of dimension five. We repeat this computation for the other fixed points and see they are all smooth points of Σ . Hence, Σ is smooth and five-dimensional. $\square$

Theorem 3.3.4. The singular locus Σ has the same cohomology ring (over $\mathbb{Q}$) as $\mathbb{CP}^5$.

Proof. By Theorem 3.3.3, we see that Σ is a smooth projective variety and hence, it is Kähler. Thus, $2i^{th}$ Betti number is at least one for $i = 0, \dots, 5$. So, the sum of its Betti numbers is at least six. On the other hand, there is a torus action on Σ (induced from T -action on $X_{\min}$) with exactly six fixed points. Thus, by Białyński-Birula decomposition [BB73], the sum of Betti numbers is exactly six. $\square$

Next, we restrict the action of $\text{Spin}(7, \mathbb{C})$ on Σ to the subgroup $G_2^{\mathbb{C}}$. A maximal torus for $G_2^{\mathbb{C}}$ is given by $T \cap G_2^{\mathbb{C}}$. Let $\mathfrak{g}$ denote the Lie algebra of $G_2^{\mathbb{C}}$ and $\mathfrak{h}$ denote the Cartan subalgebra corresponding to our choice of maximal torus. Choose a set of positive roots S^+ so that $\tilde{e}_{0246}$ has the highest weight and let $\Delta = \{\alpha_1, \alpha_2\}$ be the set of simple roots corresponding to this choice where α_2 is the longer root. Let P_i be the parabolic subgroup of $G_2^{\mathbb{C}}$ whose Lie algebra is $\mathfrak{g}_{-\alpha_i} \oplus \mathfrak{h} \oplus (\oplus_{\alpha \in S^+ \setminus \alpha_i} \mathfrak{g}_{\alpha})$ where $\mathfrak{g}_{\alpha} = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$. A straight-forward, albeit lengthy, calculation shows that the stabilizer subgroup of $\tilde{e}_{0246}$ is P_2 . Hence, we get the following theorem.

Theorem 3.3.5. $G_2^{\mathbb{C}}$ acts on the singular locus Σ and the stabilizer group of $\tilde{e}_{0246}$ is P_2 . So, by dimensional reasons, $\Sigma = G_2^{\mathbb{C}}/P_2$.

CHAPTER 4

DEFORMATIONS IN COMPLEX G_2 MANIFOLDS

This chapter is based on a joint work with Akbulut [AY18]. The main object of this chapter are (almost) $G_2^{\mathbb{C}}$ manifolds. These are manifolds with a fixed $G_2^{\mathbb{C}}$ subbundle of their frame bundle. The constructions in this chapter are first described in a linear algebraic setting. Then, we extend them to a global setting. More specifically, compatibility of symplectic and $G_2^{\mathbb{C}}$ structures are introduced which allows us to define the notion of isotropic associative submanifolds. Examples of $G_2^{\mathbb{C}}$ manifolds are provided by “complexifying” any (almost) G_2 manifold. In fact, this complexification also admits a compatible symplectic structure. Therefore, we can consider isotropic associative submanifolds of the complexification. Finally, we discuss the (infinitesimal) deformations of such submanifolds.

4.1 Linear algebra

4.1.1 Non-degenerate three-forms in seven-space

Let (V, Ω) be an oriented seven-dimensional vector space over $\mathbb{C}$.

Definition 4.1.1. An alternating three form $\varphi \in \Lambda^3 V^*$ is called non-degenerate if for every pair of linearly independent vectors (u, v) there exists $w \in V$ such that

$$\varphi(u, v, w) \neq 0. \tag{4.1}$$

Example 4.1.2. The associative calibration ϕ is a non-degenerate three-form on $\text{Im}(\mathbb{O})$.

We define a symmetric bilinear form B using the equation

$$(\iota(u)\varphi) \wedge (\iota(v)\varphi) \wedge \varphi = 6B(u, v)\Omega. \tag{4.2}$$

For the rest of the discussion, we will only consider non-degenerate φ which defines a non-degenerate B .

Note that if we scale Ω by λ , we scale B by λ^{-1} . Furthermore, B induces a norm on $\Lambda^n V^*$. So by scaling Ω by a positive constant, we may require that the norm of Ω is of magnitude 1. We will implicitly assume this for the rest of the thesis. Also, to simplify our notations and discussions later on, we abbreviate the following quadruple (V, φ, Ω, B) satisfying (4.2) as $G_2^\mathbb{C}$ -(vector) space.

Remark 4.1.3. One can also define real G_2 -spaces in a similar manner. In fact, over $\mathbb{R}$, φ determines both a metric and a volume form uniquely. In that case the metric need not be positive definite. The 3-form φ is called positive if the metric is positive definite.

4.1.2 The complexification of a G_2 -space

In this section, we exhibit the linear version of some constructions starting with a real 7-dimensional vector space with a positive 3-form φ . Although it is possible to do a similar construction with any non-degenerate 3-form, in this section and for the rest of the thesis, we will focus on positive φ (see Remark 4.1.3). Recall that φ determines a (real) G_2 -space (V, φ, Ω, g) . Let $V_\mathbb{C} = V \oplus iV$. Furthermore, we can extend all of the structures complex linearly and the equation (4.2) continues to hold. This implies that the complexified three-form is still non-degenerate. Therefore, we get a (complex) G_2 -space $(V_\mathbb{C}, \varphi_\mathbb{C}, \Omega_\mathbb{C}, g_\mathbb{C})$ where we extend each form complex linearly in every entry.

We can also extend g as a hermitian form h . Explicitly, we define

$$h(x + iy, z + iw) = g(x, z) + g(y, w) + i(g(y, z) - g(x, w)).$$

Then, the real part of h is a positive definite metric and the imaginary part is a symplectic form ω on $V_\mathbb{C}$.

If V is a half dimensional subspace of W with an almost complex structure J such that $V \oplus JV = W$, we could use J in place of i in the above construction. This flexibility will be important later on.

4.1.3 Compatible structures on a $G_2^{\mathbb{C}}$ -space

Kähler geometry is often said to be at the intersection of Riemannian geometry, symplectic geometry and complex geometry because it comes with these three structures that are compatible with each other. Moreover, any (compatible) two of those structures determines the third one. At the group level, we can state this as follows

$$\mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{O}(2n) \cap \mathrm{Sp}(2n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{U}(n), \quad (4.3)$$

see [MS17]. Our construction (see subsection 4.1.2) of a positive-definite metric g , a symplectic form ω and a (complex) non-degenerate three-form $\varphi_{\mathbb{C}}$ from a given (real) non-degenerate three-form φ allows us to talk about compatibility between these structures related to G_2 geometry. In this section, we describe this relation for a complex 7-dimensional vector space (V, J) .

Definition 4.1.4. We say that the triple $(g, \omega, \varphi_{\mathbb{C}})$ is compatible if there is a real 7 dimensional subspace Λ of V and a positive φ on Λ (determining a metric g' on Λ) such that

1. $V = \Lambda \oplus J\Lambda =: \Lambda_{\mathbb{C}}$
2. $\varphi_{\mathbb{C}}$ is the complex linear extension of φ
3. $g + i\omega$ is the hermitian extension of g' .

In this case, we say they are induced from (Λ, φ, J) .

Proposition 4.1.5.

$$G_2^{\mathbb{C}} \cap \mathrm{U}(7) = G_2$$

Proof. It is clear from the definition that $G_2 \subset G_2^{\mathbb{C}}$. Since $G_2 \subset \mathrm{O}(7, \mathbb{R}) \subset \mathrm{U}(7)$, $G_2 \subset G_2^{\mathbb{C}} \cap \mathrm{U}(7)$.

For the converse, first note that $\mathrm{U}(7) \cap \mathrm{O}(7, \mathbb{C}) = \mathrm{O}(7, \mathbb{R})$ since a matrix whose inverse is both its conjugate transpose and transpose, must be a real matrix. Therefore, $G_2^{\mathbb{C}} \cap \mathrm{U}(7) \subset \mathrm{O}(7, \mathbb{R})$, since

$G_2^{\mathbb{C}} \subset O(7, \mathbb{C})$. So, the intersection consist of real 7×7 matrices preserving $\varphi_{\mathbb{C}}$. In particular, they preserve φ and we get

$$G_2^{\mathbb{C}} \cap U(7) = G_2.$$

□

Now, using (4.3) and Proposition 4.1.5, it is easy to see that we have

$$G_2^{\mathbb{C}} \cap O(14) = G_2^{\mathbb{C}} \cap Sp(14) = G_2.$$

We will need the following technical lemma later.

Lemma 4.1.6. Given a symplectic form ω on $\mathbb{R}^{14}$, a Lagrangian subspace Λ and a positive 3-form φ on Λ , let $\mathcal{J}(\omega, \Lambda, \varphi)$ be the space of almost complex structures J such that the triple $(g', \omega', \varphi_{\mathbb{C}})$ induced from (Λ, φ, J) satisfies

1. $\omega = \omega'$
2. $g'|_{\Lambda} = g$

Then, $\mathcal{J}(\omega, \Lambda, \varphi)$ is contractible.

The proof of this lemma will follow from the next lemma.

Lemma 4.1.7. Given a symplectic form ω on $\mathbb{R}^{2n}$, a Lagrangian subspace Λ and a metric g on Λ , let $\mathcal{J}(\omega, \Lambda, g)$ be the space of almost complex structures compatible with ω and $g(x, y) = \omega(x, Jy)$ for $x, y \in \Lambda$. Then, $\mathcal{J}(\omega, \Lambda, g)$ is contractible.

Remark 4.1.8. Lemma 4.1.7 says that the set of almost complex structures compatible with a given symplectic form and a fixed metric on some Lagrangian subspace is contractible.

Proof. First, we choose an orthonormal basis $\{e_i\}$ for Λ and extend it to ω -standard basis $\{e_i, f_i\}$. So, $\omega_0 = \sum_{i=1}^n e^i \wedge f^i$. We think of $J \in \mathcal{J}(\omega, \Lambda, g)$ as an $2n \times 2n$ matrix with respect to this basis. Note that $J \in \mathcal{J}(\omega, \Lambda, g)$ if and only if

1. $J^2 = -I_{2n}$,
2. $J^\top J_{2n} J = J_{2n}$ where $J_{2n} = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$
3. $-J_{2n} J = \begin{pmatrix} I_n & B \\ B^\top & C \end{pmatrix}$ is symmetric positive definite.

Let $P = -J_{2n} J$. Note that

$$\begin{aligned} P^\top J_{2n} P &= -J^\top J_{2n} J_{2n} J_{2n} J \\ &= J^\top J_{2n} J \\ &= J_{2n}. \end{aligned}$$

This implies $C = I + BB^\top$. Define the path $P_t = \begin{pmatrix} I_n & tB \\ tB^\top & I + t^2 BB^\top \end{pmatrix}$. Clearly, $P_t^\top = P_t$. Next, we check if P_t is a symplectic matrix.

$$\begin{aligned} &\begin{pmatrix} I_n & tB^\top \\ tB & I_n + t^2 BB^\top \end{pmatrix} \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} I_n & tB \\ tB^\top & I_n + t^2 BB^\top \end{pmatrix} \\ &= \begin{pmatrix} I_n & tB^\top \\ tB & I_n + t^2 BB^\top \end{pmatrix} \begin{pmatrix} -tB^\top & -I_n - t^2 BB^\top \\ I_n & tB \end{pmatrix} \\ &= \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \end{aligned}$$

Therefore, P_t is invertible for all t . Since it is always symmetric and at $t = 0$ (or $t = 1$) it is positive definite, P_t is positive definite for all t . Hence, $J_{2n} P_t$ is a path in $\mathcal{J}(\omega, \Lambda, g)$ from J_{2n} to J . Clearly, the path depends continuously on J . $\square$

Proof of Lemma 4.1.6. The first two properties imply that $\mathcal{J}(\omega, \Lambda, \varphi) = \mathcal{J}(\omega, \Lambda, g)$ where g is the metric induced from φ on Λ . Thus, Lemma 4.1.7 shows that it is contractible. The third property is trivially satisfied by definition of complex linear extension. $\square$

4.2 $G_2^{\mathbb{C}}$ manifolds and complexification of a G_2 manifold

In this chapter we define (almost) $G_2^{\mathbb{C}}$ manifolds and provide examples of them. The examples are obtained by complexifying a G_2 manifold (M, ϕ) . In fact, two different complexification procedures are described. The advantage of the first procedure is that an almost complex structure, a metric and a symplectic form on the complexification can be written explicitly. However, the symplectic form is not necessarily closed. In the second procedure one obtains a closed symplectic form at the cost of losing some control over the corresponding almost complex structure and metric.

Definition 4.2.1. A (real) 14-dimensional manifold M is called an (almost) $G_2^{\mathbb{C}}$ -manifold if its frame bundle admits a reduction to a principal $G_2^{\mathbb{C}}$ -bundle.

Proposition 4.2.2. A $G_2^{\mathbb{C}}$ -manifold M naturally has the following structures

- an almost complex structure $J \in \Gamma(M; \text{End}(TM))$
- a $\mathbb{C}$ -linear three-form $\varphi \in \Omega^3(M; \mathbb{C})$
- a $\mathbb{C}$ -linear seven-form $\Omega \in \Omega^7(M; \mathbb{C})$
- a symmetric bilinear form $B \in \Gamma(M; S^2(TM) \otimes \mathbb{C})$
- two signature (n, n) pseudo-Riemannian metrics $g_1 = \text{Re}B$ and $g_2 = \text{Im}B$.

Proof. Since $G_2^{\mathbb{C}}$ preserves each one of these structures, one may pull them back onto M by using a $G_2^{\mathbb{C}}$ -frame. $\square$

Next, we reformulate the above definition. Since $G_2^{\mathbb{C}}$ is the stabilizer of ϕ in $\text{SO}(\text{Im}\mathbb{O})$ (by Definition 2.3.1), one may also use the following definition of (almost) $G_2^{\mathbb{C}}$ -manifolds.

Definition 4.2.3. A (real) 14 dimensional manifold $(M, J, \varphi, \Omega, B)$ with an almost complex structure J , a $\mathbb{C}$ -multilinear three form φ , a $\mathbb{C}$ -multilinear seven-form Ω and a symmetric $\mathbb{C}$ -bilinear form B is called an (almost) $G_2^{\mathbb{C}}$ -manifold if for every $m \in M$, there is an $\mathbb{R}$ -linear isomorphism $(T_m M, J, \varphi, \Omega) \cong (\text{Im}(\mathbb{O}), i, \varphi_0, \Omega_0)$.

Next, we describe examples of $G_2^{\mathbb{C}}$ manifolds with compatible (almost) symplectic structures. We start with a usual G_2 manifold and construct two different $G_2^{\mathbb{C}}$ manifold structures on its cotangent bundle.

Our first construction is as follows. Let (M, φ) be a (real) 7-dimensional G_2 manifold. Recall that M is naturally equipped with a Riemannian metric g and a volume form Ω satisfying

$$\iota(u)\varphi \wedge \iota(v)\varphi \wedge \varphi = 6g(u, v)\Omega. \quad (4.4)$$

We can think of the Levi Civita connection on the cotangent bundle as a horizontal distribution and hence, it induces the isomorphism

$$T_\alpha T^*M \cong T_p M \oplus T_p^* M \quad (4.5)$$

where $\alpha \in T_p^* M$ and $p \in M$. To define an almost complex structure on TT^*M , we view the metric as a vector bundle isomorphism $g : TM \rightarrow T^*M$ and we set

$$J(X + \beta) = -g^{-1}(\beta) + g(X) \quad (4.6)$$

for $(X, \beta) \in T_p M \oplus T_p^* M = T_\alpha T^* M$. Clearly, $J^2 = -I_{TT^*M}$.

Next, we “extend φ complex linearly” to TT^*M , i.e. we define $\varphi_{\mathbb{C}}$ to be the unique $\mathbb{C}$ -valued 3-form satisfying

1. $\varphi_{\mathbb{C}}(X, Y, Z) = \varphi(X, Y, Z)$ and
2. $\varphi_{\mathbb{C}}(J(X), Y, Z) = i\varphi(X, Y, Z)$

for horizontal vectors X, Y, Z ; where we identify $T_p M$ with horizontal part of $T_\alpha T^* M$ using (4.5). Similarly, we extend g and Ω complex linearly and we denote the complexifications by B and $\Omega_{\mathbb{C}}$, respectively. Then, from (4.4), we immediately get

$$\iota(\xi)\varphi_{\mathbb{C}} \wedge \iota(\varepsilon)\varphi_{\mathbb{C}} \wedge \varphi_{\mathbb{C}} = 6B(\xi, \varepsilon)\Omega_{\mathbb{C}} \quad (4.7)$$

for $\xi, \varepsilon \in TT^*M$. Note that B is non-degenerate and $\Omega_{\mathbb{C}}$ is a non-vanishing complex volume form. Therefore, by (4.7), $\varphi_{\mathbb{C}}$ is non-degenerate. We extend g as a hermitian form h as well. So, $\text{Re}h$ is a

positive definite metric and $\omega = \text{Im}h$ is an almost symplectic form on T^*M . More explicitly,

$$\omega(X + \alpha, Y + \beta) = \alpha(Y) - \beta(X).$$

From the construction it is clear that $\varphi_{\mathbb{C}}$ is compatible with ω .

In the above example, the symplectic form we obtained is not necessarily closed. Our next example is a similar construction but the symplectic form we obtain at the end is the canonical symplectic form on T^*M . We obtain this result at the cost of losing some control of the almost complex structure.

Again, we start with a (real) 7-dimensional G_2 manifold (M, φ) and we think of g as an isomorphism between TM and T^*M . Using this isomorphism, we think of φ as an element of $\Gamma(\Lambda^3 TM)$. Therefore, (T_p^*M, φ) is a G_2 -space. The vertical subspace of $T_{\alpha}T^*M$ is canonically defined and isomorphic to $T_{\pi(\alpha)}^*M$. The vertical subbundle defines a Lagrangian 7-plane distribution on $(T^*M, \omega_{\text{can}})$. The space of compatible almost complex structures on $(T_{\alpha}^*TM, \Lambda = T_{\pi(\alpha)}^*M, \varphi, \omega_{\text{can}})$ is contractible by Lemma 4.1.6. Therefore, one can find a global almost complex structure J such that the complexification of (Λ, φ) with respect to J gives us a compatible triple $(\omega_{\text{can}}, \varphi_{\mathbb{C}}, g)$. Compatibility here means compatibility at every point in the sense of subsection 4.1.3.

4.3 Isotropic associative submanifolds and their deformations

Recall the following two definitions that are well-known in the literature. A submanifold X of a symplectic manifold (N, ω) is called isotropic if $\omega|_X = 0$. Also, a three-submanifold Y of a G_2 manifold (M, φ) is called associative if the restriction $\varphi|_Y$ is the riemannian volume form on Y . In this section, we define what we call isotropic associative submanifolds of a $G_2^{\mathbb{C}}$ manifold with a compatible symplectic structure. The definition is a little subtle. The natural notion of associative submanifold of a $G_2^{\mathbb{C}}$ manifold is, strictly speaking, a complex three-submanifold but complex submanifolds are not isotropic with respect to a compatible ω . Instead we consider the “real part” of an associative submanifold. The notion of real part can be made precise using the symplectic form.

Definition 4.3.1. Let L be a (real) 3-dimensional subspace of $\text{Im}\mathbb{O} = \mathbb{C}^7$. We call L isotropic associative if

1. $\omega|_L \equiv 0$,
2. $B|_{L_{\mathbb{C}}}$ is non-degenerate.
3. $\phi(u, v, w) = \pm 1$ for $u, v, w \in L_{\mathbb{C}}$ orthonormal (with respect to B)

where $L_{\mathbb{C}} = L \oplus iL$. We denote the space of all isotropic associative planes by $I_3^{\varphi} \subset \text{Gr}^{\mathbb{R}}(3, 14)$.

Moreover, let Y be a (real) 3-dimensional submanifold of a $G_2^{\mathbb{C}}$ -manifold M . We call Y isotropic associative submanifold if $T_p Y$ is an isotropic associative plane in $T_p M$ for every p .

Note that an associative submanifold Y of a G_2 manifold M , naturally sits as an isotropic associative submanifold in the zero section of T^*M . We consider the infinitesimal deformations of Y in which Y stays isotropic associative. We obtain Seiberg-Witten type equations from these deformations.

We denote the normal bundle of Y in M (resp. T^*M) by $\nu_{\mathbb{R}}Y$ (resp. $\nu_{\mathbb{C}}Y$) and set $\mathbb{V} = \nu_{\mathbb{R}}Y \oplus J\nu_{\mathbb{R}}Y$. Then we have the following decomposition

$$\nu_{\mathbb{C}}Y = JTY \oplus \mathbb{V}. \quad (4.8)$$

Let $\sigma_t : Y \rightarrow T^*M$ be a one parameter family of embeddings. Without loss of generality, we may assume that $\dot{\sigma}_0$ is a section of $\Gamma(\nu_{\mathbb{C}}Y)$. Let $f \in \Gamma(JTY)$, $v \in \Gamma(\mathbb{V})$ with $\eta := f + v = \dot{\sigma}_0$. Also, let $\tilde{G} := \text{Gr}(3, TT^*M) \rightarrow T^*M$ denote the Grassmann 3-plane bundle over T^*M . We can lift the embedding $Y \hookrightarrow T^*M$ to $Y \hookrightarrow \tilde{G}$ using the Gauss map. Then, the infinitesimal deformation of Y by η induces an infinitesimal deformation of the lift as in [AS08].

For a tangent space $L = T_x Y = \langle e_1, e_2, e_3 \rangle$, infinitesimal deformation is given by

$$\dot{L} = \sum_{i=1}^3 e^i \otimes \mathcal{L}_{\eta}(e_i) \in T_L \tilde{G}.$$

So, the conditions for Y to stay isotropic associative are given by

$$1. \sum e_i \times \mathcal{L}_v(e_i) = 0$$

$$2. \sigma_t^* \omega = 0$$

[AS08, AY18].

Using the Levi-Civita connection ∇ of (T^*M, g) , we define a Dirac type operator

$$\begin{aligned} \not{D}_{A_0} : \Omega^0(v_{\mathbb{C}}Y) &\rightarrow \Omega^0(v_{\mathbb{C}}Y) \\ \not{D}_{A_0}(v) &= \sum e_i \times \nabla_{e_i}(v). \end{aligned} \tag{4.9}$$

Note that in the role of Clifford multiplication we are using the cross product operation. So, the first condition can be expressed as

$$\begin{aligned} 0 &= \sum e_i \times \mathcal{L}_v(e_i) \\ &= \sum e_i \times (\nabla_v e_i - \nabla_{e_i} v) \\ &= \sum e_i \times \nabla_v e_i - \sum e_i \times \nabla_{e_i} v \end{aligned}$$

We set the perturbation parameter $a(v) = -\sum e_i \times \nabla_v e_i$. So, the last equation becomes

$$\not{D}_A(v) = \not{D}_{A_0}(v) + a(v) = 0 \tag{4.10}$$

where $A = A_0 + a$.

For the isotropy condition, we choose a standard coordinate chart (q^i, p^i) for the symplectic form so that $\omega = \sum dq^i \wedge dp^i$ where (q^i) are coordinates on the base space and (p^i) are fiber directions. Write $\sigma_t^i = \sigma_t^i(x^1, x^2, x^3) = q^i(\sigma_t(x^1, x^2, x^3))$ for $1 \leq i \leq 7$ and $\sigma_t^j = \sigma_t^j(x^1, x^2, x^3) = p^j(\sigma_t(x^1, x^2, x^3))$ for $8 \leq j \leq 14$ where (x^1, x^2, x^3) are local coordinates on Y . Note that (possibly after reparametrization) we may assume that $(\sigma_t^1, \sigma_t^2, \sigma_t^3) = (x^1, x^2, x^3)$. Furthermore, since the image of σ_0 lies in the 0-section of T^*M , we may also assume $\sigma_0^j = 0$ for $8 \leq j \leq 14$.

During the deformation Y stays isotropic if $\sigma_t^* \omega = 0$. Since

$$\begin{aligned}
\sigma_{t*} \frac{\partial}{\partial x^i} &= \sum_{j=1}^7 \frac{\partial \sigma_t^j}{\partial x^i} \frac{\partial}{\partial q^j} + \frac{\partial \sigma_t^{j+7}}{\partial x^i} \frac{\partial}{\partial p^j} \\
&= \sum_{j=1}^3 \delta_i^j \frac{\partial}{\partial q^j} + \sum_{j=4}^7 \frac{\partial \sigma_t^j}{\partial x^i} \frac{\partial}{\partial q^j} + \sum_{j=1}^7 \frac{\partial \sigma_t^{j+7}}{\partial x^i} \frac{\partial}{\partial p^j} \\
&= \frac{\partial}{\partial q^i} + \sum_{j=4}^7 \frac{\partial \sigma_t^j}{\partial x^i} \frac{\partial}{\partial q^j} + \sum_{j=1}^7 \frac{\partial \sigma_t^{j+7}}{\partial x^i} \frac{\partial}{\partial p^j},
\end{aligned}$$

we have

$$\begin{aligned}
0 &= \omega(\sigma_{t*}(\frac{\partial}{\partial x^i}), \sigma_{t*}(\frac{\partial}{\partial x^j})) \\
&= \frac{\partial \sigma_t^{i+7}}{\partial x^j} - \frac{\partial \sigma_t^{j+7}}{\partial x^i} + \sum_{k=4}^7 \frac{\partial \sigma_t^k}{\partial x^i} \frac{\partial \sigma_t^{k+7}}{\partial x^j} - \frac{\partial \sigma_t^k}{\partial x^j} \frac{\partial \sigma_t^{k+7}}{\partial x^i}.
\end{aligned} \tag{4.11}$$

Note that the last equation is of the form $da = -q(\psi_1 \otimes \psi_2)$ where a is a 1-form on Y given by

$$a = \sigma_t^8 dx^1 + \sigma_t^9 dx^2 + \sigma_t^{10} dx^3,$$

ψ_1 and ψ_2 are spinors living as sections of $\Omega^1(\nu_{\mathbb{R}} Y)$ and $\Omega^1(J\nu_{\mathbb{R}} Y)$ given by

$$\begin{aligned}
\psi_1 &= \sum_{i=1}^3 \frac{\partial}{\partial x^i} (\sigma_t^4, \dots, \sigma_t^7) dx^i, \\
\psi_2 &= \sum_{j=1}^3 \frac{\partial}{\partial x^j} (\sigma_t^{11}, \dots, \sigma_t^{14}) dx^j,
\end{aligned}$$

and q is a bilinear map given by

$$q(\psi_1 \otimes \psi_2) = \psi_1 \times \psi_2$$

here the cross product is taken in the 1-form parts with metric identification.

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