

NASH EQUILIBRIA IN THE CONTINUOUS-TIME PRINCIPAL-AGENT PROBLEM
WITH MULTIPLE PRINCIPALS

By

Lening Kang

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

Statistics - Doctor of Philosophy

2013

ABSTRACT

NASH EQUILIBRIA IN THE CONTINUOUS-TIME PRINCIPAL-AGENT PROBLEM WITH MULTIPLE PRINCIPALS

By

Lening Kang

In Chapter 1, we review some basic results of backward stochastic differential equation(BSDE) to prepare for the applications in Chapter 2. BSDE has proven to be a powerful tool in financial mathematics. It was first introduced as a tool to price contingent claims and was later used to model utility functions. The value of a recursive utility function is essentially the solution of a BSDE. We present two versions of comparison lemma for BSDE. The latter one allows quadratic growth in volatility term which is important for its applications in Chapter 2.

In Chapter 2, we study the principal-agent(owner-manager) problem with moral hazard in continuous time with a Brownian filtration, recursive preferences, and multiple principals (one agent for each principal). In simple terms, the problem is defined in two levels, first for the agents and then the principals. The key to the definition of the problem in both levels is Nash equilibrium. The Nash equilibrium among the agents is straightforward and comes through their competing(or cooperative) efforts. The Nash equilibrium among the principals is more complicated, because the connection among them is indirect and comes only through the agents' Nash equilibrium in efforts. The objective is to characterize each principal's equilibrium control over his/her agent, taking into account the control is constrained by the agents' Nash equilibrium in efforts.

In technical terms, different principals' problems are connected, because the effort of each principal's agent affects the common probability measure, and therefore one agent's effort can

impact the cash-flow drifts of all the principals. This could capture, for example, the impact of innovations by agents of one firm on the cash-flow prospects of competing firms. The externality of each agent's effort results in interdependence among the principals' optimal contracting problems. For the class of preferences we consider, solving the equilibrium reduces to computing a system of linked subjective cash-flow value processes, one for each principal. We show that the system has a closed-form solution, when each principal's cash flow is driven by an affine-yield state process. Each principal's optimal pay policy amounts to choosing the component of the subjective cash-flow volatility to transfer to the agent (that is, a volatility sharing rule). The optimal sharing rules are simple functions of each principal's own cash-flow volatility in the case when the impact of aggregate effort on drifts is additive, but are generally functions of all the principals' cash-flow volatilities when the impact of effort on the drift change is diminishing in aggregate effort. We provide a number of closed-form solutions to illustrate.

TABLE OF CONTENTS

Chapter 1	Backwards Stochastic Differential Equation (BSDE)	1
1.1	Introduction: What is BSDE?	2
1.1.1	Existence and Uniqueness of a Solution for BSDEs with Linear Growth	3
1.2	Comparison Lemmas for BSDEs	6
1.2.1	Comparison Lemmas for BSDEs with Linear Growth	6
1.2.2	Comparison Lemmas for BSDEs with Quadratic Growth	10
1.3	Application to Economics	17
1.3.1	Recursive Utility Functions	17
1.3.2	European and American Contingent Claims Valuation	21
Chapter 2	Principal-Agent Problem with Multiple Principals	31
2.1	Introduction	31
2.2	The Setting and Problem	40
2.2.1	The Setup	41
2.2.2	Outline of the Problem and Solution	46
2.2.3	Regularity Conditions and Feasibility	50
2.3	Agent Equilibrium	52
2.3.1	A Necessary and Sufficient Condition for Agent Equilibrium	52
2.3.2	Some Basic Examples of Agent Equilibrium	58
2.3.3	Constant Elasticity of Substitution(CES)	62
2.4	Principal Equilibrium	70
2.5	Closed-Form Solution with Cash Flows Driven by State Processes	86
2.6	Additive Measure Change and Quadratic Penalties	92
2.7	Diminishing Returns to Effort (Concave Φ)	105
2.7.1	Power Measure Change	111
2.7.2	Quadratic Penalty and Cobb-Douglas Measure Change	118
2.7.3	Absolute Effort Penalty and Square-Root Measure Change	122
APPENDICES		128
.1	Appendix 1: Proofs Omitted from the Text	128
.2	Appendix 2: Derivation of Examples	147
BIBLIOGRAPHY		155

Chapter 1

Backwards Stochastic Differential Equation (BSDE)

Backward stochastic differential equation has proven to be a powerful tool in stochastic analysis in the last twenty years. It has been widely applied in the problems of stochastic control and mathematical finance. Its general form was first introduced by Pardoux and Peng in 1990(see Pardoux and Peng (1990)). A lot work has been done since then in both theoretical aspects and applications, such as E.Pardoux and Peng (1992), Peng (1991), Peng (1992) and Peng (1993). In particular, Antonelli (1993) extends BSDE to a forward-backward form(FBSDE). The collection of papers Karoui and Mazliak (1997) summarized some of the earlier results of BSDE with linear growth. More recently, BSDE with quadratic growth was studied in a series of papers(Kobylanski (2000), Briand and Hu (2006), Briand and Hu (2008), and Delbaen, Hu, and Richou (2009) etc.). In this chapter, we will review some fundamentals of BSDE theory and prove a new version of comparison lemma, which will serve as a primary tool for the applications in chapter 2.

All uncertainty is generated by d -dimensional standard Brownian motion B over the finite time horizon $[0, T]$, supported by a probability space (Ω, \mathcal{F}, P) . $\{\mathcal{F}_t : t \in [0, T]\}$ is the augmented filtration(satisfies the usual hypotheses) generated by B . Let $\mathcal{B}([0, T])$ denote the Borel σ -field on $[0, T]$. Let λ be the Lebesgue measure on $[0, T]$ and $P \otimes \lambda$ on $\Omega \times [0, T]$.

The qualification " $P \otimes \lambda$ almost everywhere" is omitted throughout.

For any subset S of Euclidean space, let $L(S)$ denote the set of S -valued \mathcal{F}_T measurable random variables and $\mathcal{L}(S)$ denote the set of S -valued progressively measurable processes¹ w.r.t. $(\Omega \times [0, T], \mathcal{F} \times \mathcal{B}([0, T]), P \otimes \lambda)$.

For this chapter, we will use the following spaces:

Let $L_p(S)$, $\mathcal{L}_p(S)$ and $\mathcal{S}_p(S)$ denote respectively:

$$\begin{aligned} L_p(S) &= \{x \in L(S) : E(|x|^p) < \infty\}, \\ \mathcal{L}_p(S) &= \left\{x \in \mathcal{L}(S) : E \left[\int_0^T |x_t|^p dt \right] < \infty \right\}, \\ \mathcal{S}_p(S) &= \left\{x \in \mathcal{L}(S) : E \left(\sup_{0 \leq t \leq T} |x|^p \right) < +\infty \right\}, \quad 1 \leq p < \infty. \end{aligned}$$

where $|\cdot|$ denotes Euclidean norm.

For any real-valued random or deterministic matrix Z , we will let Z' denote its transpose.

1.1 Introduction: What is BSDE?

A BSDE is an equation of the following type:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z'_s dB_s, \quad 0 \leq t \leq T. \quad (1.1)$$

or equivalently

$$dY_t = -f(t, Y_t, Z_t)dt + Z'_t dB_t, \quad Y_T = \xi.$$

¹In the setting of augmented Brownian filtration, progressively measurable processes are predictable.

where:

- the *terminal value* $\xi : \Omega \rightarrow \mathbb{R}$ is F_T measurable.
- the *aggregator* function $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is progressively measurable with respect to $\mathcal{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$ where \mathcal{P} is the predictable σ -field on $\Omega \times [0, T]$.

The *solution* is a pair of progressively measurable processes (Y, Z) valued in $\mathbb{R} \times \mathbb{R}^d$ that satisfies: $t \rightarrow Y_t$ is continuous and $Z_t \in \mathcal{L}_2(\mathbb{R}^d)$.

For PDEs in deterministic settings, in most situations a backward formulation can be transformed into a forward one through reverse of time argument t . However, we can not simply reverse time argument to transform a BSDE into SDE because of the measurability requirement imposed on the solution. Moreover, unlike in the SDE case where the solution has only one component, a solution of a BSDE consists of two components Y and Z .

1.1.1 Existence and Uniqueness of a Solution for BSDEs with Linear Growth

In the case of $f(s, Y_s, Z_s) = 0$, $s \in [0, T]$, (1.1) reduces to martingale representation theorem(see the examples after Theorem 1 below). For any process $Z \in \mathcal{L}_2(\mathbb{R}^d)$, we know that $M_t = \int_0^t Z'_s dB_s$, $t \in [0, T]$ is a martingale w.r.t. \mathcal{F}_t and $M_T \in L_2(\mathbb{R})$. It is natural to ask whether the converse is true. The answer is Yes and this result is the famous martingale representation theorem.

Theorem 1 (Martingale Representation Theorem for L_2 -Martingales) *Suppose that M_t is a martingale w.r.t. \mathcal{F}_t and $M_T \in L_2(\mathbb{R})$. Then there exists a unique process $Z \in$*

$\mathcal{L}_2(\mathbb{R}^d)$ such that

$$M_t = E[M_T] + \int_0^t Z'_s dB_s, \quad t \in [0, T].$$

Remark 1 *The theorem above can be extended to the case where M_t is a local martingale.*

Before we proceed to the general case of (1.1), let us first look at two instructive examples, which will show how to use Theorem 1 to get a solution to (1.1):

(a) In the case of $f(s, Y_s, Z_s) = 0$, $s \in [0, T]$ and $\xi \in L_2(\mathbb{R})$, there is a unique solution to (1.1). We get the solution by using the theorem above. To see this, let $Y_t = E_t(\xi)$, $t \in [0, T]$, where $E_t(\cdot) \equiv E(\cdot \mid \mathcal{F}_t)$. It is well-known that Y_t is an L_2 -martingale, so we let $M_t = Y_t$ and the theorem gives us the existence of $Z_t \in \mathcal{L}_2(\mathbb{R}^d)$. To see that (Y_t, Z_t) is a solution to (1.1), note that $Y_T = \xi$ and $Y_t = E(\xi) + \int_0^t Z'_s dB_s$, $t \in [0, T]$. In this case, $Y \in \mathcal{L}_2(\mathbb{R})$, because Y is L_2 -martingale and T is finite.

(b) The case of $f(s, Y_s, Z_s) = 0$, $s \in [0, T]$ can be easily extended to $f(s, Y_s, Z_s) = f(s)$, $s \in [0, T]$, i.e. the aggregator f does not depend on the solution (Y, Z) . We also assume $f(s) \in \mathcal{L}_2(\mathbb{R})$. Let $M_t = E_t(\int_0^T f(s)ds + \xi)$, then M_t is an L_2 -martingale. (Note that $f(s) \in \mathcal{L}_2(\mathbb{R})$ implies $\int_0^T f(s)ds \in L_2(\mathbb{R})$.) We get $Z \in \mathcal{L}_2(\mathbb{R}^d)$ by the theorem, i.e. $M_t = E(M_T) + \int_0^t Z'_s dB_s$. The solution Y is given by $Y_t = M_t - \int_0^t f(s)ds = E_t(\int_0^T f(s)ds + \xi)$, $t \in [0, T]$. Note $Y \in \mathcal{L}_2(\mathbb{R})$, because $|Y_t| \leq E_t(\int_0^T |f(s)|ds + |\xi|)$, $t \in [0, T]$ and the right-hand side of the inequality is L_2 martingale.

In the general case (i.e. the aggregator f depends on the solution (Y, Z)), the theorem below taken from Karoui and Mazliak (1997) proves the existence and uniqueness of a solution to (1.1) under certain conditions. Its proof will be based on a fixed point theorem in addition to the martingale representation theorem. We will assume the following on the aggregator and terminal for Theorem 2 below.

Assumption 1 *The aggregator and terminal satisfy: $\xi \in L_2(\mathbb{R})$, $f(\cdot, 0, 0) \in \mathcal{L}_2(\mathbb{R})$ and f is uniformly Lipschitz in Y and Z , i.e. there exists a constant $C > 0$ such that $\forall (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$*

$$|f(t, y_1, z_1) - f(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

Theorem 2 *Assume 1. Then there exists a unique pair $(Y, Z) \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}^d)$ that solves (1.1).*

Sketch of Proof. For any $\beta > 0$ and $\varphi \in \mathcal{L}_2(S)$, let $\|\varphi\|_\beta = E \left[\int_0^T e^{\beta t} |\varphi_t|^2 dt \right]$. We will let $\mathcal{L}_{2,\beta}(S)$ denote the space of $\mathcal{L}_2(S)$ endowed with the norm $\|\cdot\|_\beta$. The proof of this theorem is based on a fixed point theorem for a contraction mapping from $\mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$ into itself.

For any $(y, z) \in \mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$, we will consider the following BSDE:

$$Y_t = \xi + \int_t^T f(s, y_s, z_s) ds - \int_t^T Z'_s dB_s, \quad 0 \leq t \leq T.$$

By the uniform Lipschitz condition, we have $|f(s, y_s, z_s) - f(s, 0, 0)|^2 \leq 2C(|y_s|^2 + |z_s|^2)$. By the assumptions $f(\cdot, 0, 0) \in \mathcal{L}_{2,\beta}(\mathbb{R})$ and $(y, z) \in \mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$, we get $f(s, y_s, z_s) \in \mathcal{L}_{2,\beta}(\mathbb{R})$. By the example (b) above, there's a unique solution $(Y, Z) \in \mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$ to the equation.

We will denote the mapping $(y, z) \rightarrow (Y, Z)$ by $(Y, Z) = \mathcal{T}(y, z)$. For any $(y^1, z^1), (y^2, z^2) \in \mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$, let $(Y^1, Z^1) = \mathcal{T}(y^1, z^1)$ and $(Y^2, Z^2) = \mathcal{T}(y^2, z^2)$. Proposition 2.2 of Karoui and Mazliak (1997) gives the estimate:

$$\|Y_1 - Y_2\|_\beta^2 + \|Z_1 - Z_2\|_\beta^2 \leq \frac{2(2+T)C^2}{\beta} (\|y_1 - y_2\|_\beta^2 + \|z_1 - z_2\|_\beta^2)$$

Choosing $2(2+T)C^2 < \beta$, we see that \mathcal{T} is a contraction mapping from $\mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$ into itself. Thus there exists a fixed point $(Y^*, Z^*) \in \mathcal{L}_{2,\beta}(\mathbb{R}) \times \mathcal{L}_{2,\beta}(\mathbb{R}^d)$, i.e. $\mathcal{T}(Y^*, Z^*) = (Y^*, Z^*)$, which is the unique solution to the BSDE (1.1). ■

1.2 Comparison Lemmas for BSDEs

Since BSDEs are difficult to solve explicitly, a typical comparison lemma for BSDEs is a useful tool in analyzing their solutions. It compares the solutions of BSDEs with different aggregators and terminal values. Upon inspecting (1.1), it is easy to get the intuition that, when the aggregator function and terminal value increase, the solution Y should also increase. A comparison lemma makes this idea precise under certain conditions. In this section, we will review a comparison lemma for BSDEs with linear growth that is a variation of Theorem 2.5 in Karoui and Mazliak (1997). We will also present a comparison lemma for BSDEs with quadratic growth in volatility that extends the result in Briand and Hu (2008). BSDEs with quadratic growth are heavily used in the analysis of recursive utility functions in economics, as will be seen in section 1.3 and chapter 2. When it comes to BSDEs with quadratic growth, uniqueness of the solution is usually proved by using a comparison lemma.

1.2.1 Comparison Lemmas for BSDEs with Linear Growth

The comparison lemma in this subsection is a direct result of the proposition below. It deals with a linear BSDE that has an explicit solution. The solution depends on an adjoint process, which in turn is the solution of a forward SDE.

Proposition 1 *Let (β, γ) be a pair of bounded progressively measurable processes valued in*

$(\mathbb{R}, \mathbb{R}^d)$, $\varphi \in \mathcal{L}_2(\mathbb{R})$ and $\xi \in L_2(\mathbb{R})$. The linear BSDE(LBSDE)

$$dY_t = -[\varphi_t + \beta_t Y_t + Z_t' \gamma_t] dt + Z_t' dB_t, \quad Y_T = \xi. \quad (1.2)$$

has a unique solution $(Y, Z) \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}^d)$. Y is given explicitly by:

$$\Gamma_t Y_t = E_t \left[\xi \Gamma_T + \int_t^T \Gamma_s \varphi_s ds \right] \quad (1.3)$$

where the strictly positive adjoint process satisfies:

$$d\Gamma_t = \Gamma_t (\beta_t dt + \gamma_t' dB_t), \quad \Gamma_0 = 1. \quad (1.4)$$

Moreover, if $\xi \geq 0$ and $\varphi \geq 0$, then $Y_t \geq 0$, $t \in [0, T]$. If in addition, $Y_0 = 0$, then we have $\varphi = 0$ and $Y_t = 0$, $0 \leq t \leq T$.

Proof. By Theorem 2, there exists a unique solution $(Y, Z) \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}^d)$ to (1.2). By Ito's formula, we have $\Gamma_t Y_t + \int_0^t \varphi_s \Gamma_s ds = Y_0 + \int_0^t (\Gamma_s Y_s \gamma_s' + \Gamma_s Z_s') dB_s$, so $\Gamma_t Y_t + \int_0^t \varphi_s \Gamma_s ds$ is a local martingale. By (1.4), the adjoint process is

$$\Gamma_t = \exp \left\{ \int_0^t \left[\gamma_t' dB_t - \frac{|\gamma_t|^2}{2} dt \right] + \int_0^t \beta_t dt \right\}$$

Because β and γ are bounded processes, $\Gamma \in \mathcal{L}_2(\mathbb{R})$ and by Doob's inequality $\sup_{0 \leq t \leq T} |\Gamma_t| \in L_2(\mathbb{R})$.

Using an equivalent form of BSDE (1.2) $Y_t = \xi + \int_t^T [\varphi_s + \beta_s Y_s + Z'_s \gamma_s] ds - \int_t^T Z'_s dB_s$,

we get

$$\sup_{0 \leq t \leq T} |Y_t| \leq |\xi| + \int_0^T |\varphi_s + \beta_s Y_s + Z'_s \gamma_s| ds + \sup_{0 \leq t \leq T} \left| \int_t^T Z'_s dB_s \right|.$$

Because $\varphi, Y \in \mathcal{L}_2(\mathbb{R})$, $Z \in \mathcal{L}_2(\mathbb{R}^d)$ and β, γ are bounded, we have by Holder's inequality

$$\int_0^T |\varphi_s + \beta_s Y_s + Z'_s \gamma_s| ds \in L_2(\mathbb{R}).$$

We also have

$$\sup_{0 \leq t \leq T} \left| \int_t^T Z'_s dB_s \right| \leq \left| \int_0^T Z'_s dB_s \right| + \sup_{0 \leq t \leq T} \left| \int_0^t Z'_s dB_s \right|.$$

By Doob's inequality, we get

$$E \left| \sup_{0 \leq t \leq T} \int_0^t Z'_s dB_s \right|^2 \leq 4E \left| \int_0^T Z'_s dB_s \right|^2 = 4E \int_0^T |Z_s|^2 ds < \infty,$$

so $\sup_{0 \leq t \leq T} \left| \int_t^T Z'_s dB_s \right| \in L_2(\mathbb{R})$. Thus we have $\sup_{0 \leq t \leq T} |Y_t| \in L_2(\mathbb{R})$.

By Holder's inequality, we also have

$$\sup_{0 \leq s \leq T} |Y_s| \cdot \sup_{0 \leq s \leq T} |\Gamma_s| \text{ and } \int_0^T |\varphi_s \Gamma_s| ds \in L_1(\mathbb{R})$$

Thus

$$E \sup_{0 \leq s \leq T} \left\{ \Gamma_t Y_t + \int_0^t \varphi_s \Gamma_s ds \right\} \leq E \sup_{0 \leq s \leq T} |Y_s| \cdot \sup_{0 \leq s \leq T} |\Gamma_s| + E \int_0^T |\varphi_s \Gamma_s| ds < \infty$$

We can conclude that the local martingale $\Gamma_t Y_t + \int_0^t \varphi_s \Gamma_s ds$ is a uniformly integrable martingale, so we have:

$$\Gamma_t Y_t = E_t \left[\xi \Gamma_T + \int_t^T \Gamma_s \varphi_s ds \right]$$

Because the process Γ is strictly positive, it follows directly from the above equation that, if $\xi \geq 0$ and $\varphi \geq 0$, then $Y_t \geq 0$, $t \in [0, T]$. If in addition $Y_0 = 0$, then the expectation of the non-negative random variable $\xi \Gamma_T + \int_t^T \Gamma_s \varphi_s ds$ is 0, so we have $\xi = 0$, $\varphi = 0$ and $Y_t = 0$, $0 \leq t \leq T$. ■ As a direct result, we present the comparison lemma below, for which we have the same assumption on the BSDEs as in Theorem 2. This guarantees the existence of a unique solution for the equations in the theorem.

Theorem 3 (Comparison lemma for BSDEs with linear growth) *Let (f^i, ξ^i) , $i = 1, 2$ be the aggregator and terminal value of the BSDEs*

$$dY_t^i = -f^i(t, Y_t^i, Z_t^i)dt + Z_t^{i'} dB_t, \quad Y_T^i = \xi^i$$

that satisfy Assumption 1.

If $\xi^1 \geq \xi^2$ and $f^1(t, Y_t^1, Z_t^1) \geq f^2(t, Y_t^1, Z_t^1)$ or $f^1(t, Y_t^2, Z_t^2) \geq f^2(t, Y_t^2, Z_t^2)$, then we have $Y_t^1 \geq Y_t^2$, $0 \leq t \leq T$. If in addition $Y_0^1 = Y_0^2$ we have then $Y_t^1 = Y_t^2$, $0 \leq t \leq T$.

Proof. Let $\delta Y = Y_1 - Y_2$ and $\delta Z = Z_1 - Z_2$, thus $(\delta Y, \delta Z)$ will satisfy the LBSDE:

$$d(\delta Y_t) = -\delta f_t dt + \delta Z_t' dB_t, \quad \delta Y_T = \xi_1 - \xi_2. \tag{1.5}$$

where

$$\begin{aligned}
\delta f_t &= f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^2) \\
&= \mathbf{1} \quad f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^1, Z_t^1) \\
&\quad + I_{Y_t^1 \neq Y_t^2} \frac{f^2(t, Y_t^1, Z_t^1) - f^2(t, Y_t^2, Z_t^1)}{\delta Y_t} \delta Y_t \\
&\quad + I_{Z_t^1 \neq Z_t^2} \frac{(f^2(t, Y_t^2, Z_t^1) - f^2(t, Y_t^2, Z_t^2)) \delta Z_t'}{|\delta Z_t|^2} \delta Z_t \\
&= \mathbf{2} \quad I_{Y_t^1 \neq Y_t^2} \frac{f^1(t, Y_t^1, Z_t^1) - f^1(t, Y_t^2, Z_t^1)}{\delta Y_t} \delta Y_t \\
&\quad + I_{Z_t^1 \neq Z_t^2} \frac{(f^1(t, Y_t^2, Z_t^1) - f^1(t, Y_t^2, Z_t^2)) \delta Z_t'}{|\delta Z_t|^2} \delta Z_t \\
&\quad + f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^2, Z_t^2).
\end{aligned}$$

By proposition 1, the LBSDE (1.5) has a unique solution $(\delta Y, \delta Z)$ that satisfies $\delta Y_t \geq 0$, $0 \leq t \leq T$, if $\xi^1 - \xi^2 \geq 0$ and $f^1(t, Y_t^i, Z_t^i) - f^2(t, Y_t^i, Z_t^i) \geq 0$, for $i = 1$ or 2 . (Use equality **1** for $f^1(t, Y_t^1, Z_t^1) - f^2(t, Y_t^1, Z_t^1) \geq 0$ and equality **2** in the other case. Also, use the Lipschitz assumption to bound the coefficients of δY and δZ). If in addition $\delta Y_0 = 0$, then $\delta Y_t = 0$, $0 \leq t \leq T$. ■

1.2.2 Comparison Lemmas for BSDEs with Quadratic Growth

The comparison result in Theorem 3 relies on the assumption that both aggregator functions are uniformly Lipschitz in the corresponding arguments. In typical applications, we often need to deal with the kind of BSDEs whose aggregator is quadratic in the volatility term, such as recursive utility function with quadratic volatility penalty (see next subsection). Some important properties of recursive utility are proved by using comparison lemmas. Reference Kobylanski (2000) was the first to prove a comparison lemma for BSDEs with quadratic

growth in volatility and bounded terminal values. Later Briand and Hu (2006), Briand and Hu (2008) and Delbaen, Hu, and Richou (2009) extended Kobylanski (2000) by allowing unbounded terminal values with exponential moments (the moment generating function is finite on \mathbb{R}). On the other hand, they added the assumption that the aggregator is concave (convex) in the volatility term. In this subsection, we prove a comparison lemma (Theorem 4) for BSDEs with quadratic growth in volatility that extends the result of Briand and Hu (2008). This lemma emphasizes the symmetry between f and \hat{f} (the two aggregators in the two BSDEs that we compare) with regard to assumptions **(a)**, **(b)** and **(c)** below and also allows either f or \hat{f} to be concave or convex in the volatility term. In chapter 2, we will apply this theorem to solve the principal-agent problem with multiple principals. In Theorem 4 below, we will use the following assumption on the aggregator and terminal.

Assumption 2 *There exist two constants $\gamma > 0$, $\beta > 0$ and a process $\alpha(t)$ valued in \mathbb{R}^+ such that,*

- (a)** $\forall t \in [0, T]$ and $y \in \mathbb{R}$, $z \rightarrow f(t, y, z)$ is convex or concave;
- (b)** $\forall (t, z) \in [0, T] \times \mathbb{R}^d$, and $(y, \hat{y}) \in \mathbb{R}^2$, $|f(t, y, z) - f(t, \hat{y}, z)| \leq \beta|y - \hat{y}|$;
- (c)** $\forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$, $|f(t, y, z)| \leq \alpha(t) + \beta|y| + \frac{\gamma}{2}|z|^2$;
- (d)** *The random variables $\int_0^T \alpha(t)dt$ and $|Y_T|$ have exponential moments of all orders.*

Theorem 4 *Suppose (Y_t, Z_t) and (\hat{Y}_t, \hat{Z}_t) solve*

$$\begin{aligned} dY_t &= -f(t, Y_t, Z_t)dt + Z_t' dB_t, & Y_T &= f(T), \\ d\hat{Y}_t &= -\hat{f}(t, \hat{Y}_t, \hat{Z}_t)dt + \hat{Z}_t' dB_t, & \hat{Y}_T &= \hat{f}(T) \end{aligned}$$

where $f, \hat{f} : \Omega \times [0, T) \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $f(T), \hat{f}(T) : \Omega \rightarrow \mathbb{R}$. For all $\omega, t \in \Omega \times [0, T)$, let either (f, Y_T) satisfies Assumption 2 and $\sup_{0 \leq t \leq T} \hat{Y}_t$ has exponential moments of all orders or (\hat{f}, \hat{Y}_T) satisfies Assumption 2 and $\sup_{0 \leq t \leq T} Y_t$ has exponential moments of all orders then

$$\begin{cases} \text{(i)} & Y_T \leq \hat{Y}_T \\ \text{(ii)} & f(t, y, z) \leq \hat{f}(t, y, z), \quad \forall (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \end{cases}$$

implies $Y_t \leq \hat{Y}_t, \forall t \in [0, T]$ ^{2 3}.

Proof of Theorem 4. Suppose that f satisfies Assumption 2 and f is concave in the volatility term. $\forall \theta \in (0, 1)$, let us define $U_t = \theta Y_t - \hat{Y}_t$, $V_t = \theta Z_t - \hat{Z}_t$ and $\delta f_t = f(t, \hat{Y}_t, \hat{Z}_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) \leq 0$.

The aggregator of U_t could be written as: ⁴

$$\begin{aligned} \theta f(t, Y_t, Z_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) &= \theta f(t, Y_t, Z_t) - f(t, Y_t, \hat{Z}_t) + f(t, Y_t, \hat{Z}_t) - f(t, \hat{Y}_t, \hat{Z}_t) \\ &\quad + f(t, \hat{Y}_t, \hat{Z}_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) \end{aligned} \quad (1.6)$$

²If f satisfies Assumption 2 (resp. $\hat{f}(\omega, t, \cdot)$), then it is enough to assume $f(t, \hat{Y}_t, \hat{Z}_t) \leq \hat{f}(t, \hat{Y}_t, \hat{Z}_t)$ (resp. $f(t, Y_t, Z_t) \leq \hat{f}(t, Y_t, Z_t)$). Here we assume uniform dominance over $[0, T] \times \mathbb{R} \times \mathbb{R}^d$ just for presentation convenience.

³Note that if the inequalities (i) and (ii) are reversed, then the inequality in the conclusion is also reversed.

⁴ If \hat{f} satisfies Assumption 2 and \hat{f} is concave in the volatility term, we can define $\delta f_t = f(t, Y_t, Z_t) - \hat{f}(t, Y_t, Z_t) \leq 0$ and write $\theta f(t, Y_t, Z_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) = \theta f(t, Y_t, Z_t) - \theta \hat{f}(t, Y_t, Z_t) + \theta \hat{f}(t, Y_t, Z_t) - \hat{f}(t, Y_t, \hat{Z}_t) + \hat{f}(t, Y_t, \hat{Z}_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t)$. The rest of the steps can be carried out accordingly.

The case of f satisfies Assumption 2 and f is convex was covered by Briand and Hu (2008). If \hat{f} satisfies Assumption 2 and \hat{f} is convex in volatility term, we can define $U_t = Y_t - \theta \hat{Y}_t$, $V_t = \theta Z_t - \theta \hat{Z}_t$ and $\delta f_t = f(t, Y_t, Z_t) - \hat{f}(t, Y_t, Z_t) \leq 0$. We can write $f(t, Y_t, Z_t) - \theta \hat{f}(t, \hat{Y}_t, \hat{Z}_t) = \delta f_t + \hat{f}(t, Y_t, Z_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) + \hat{f}(t, \hat{Y}_t, \hat{Z}_t) - \theta \hat{f}(t, \hat{Y}_t, \hat{Z}_t)$ and carry out the steps in Briand and Hu (2008) accordingly.

We can rewrite

$$\begin{aligned}
f(t, Y_t, \hat{Z}_t) - f(t, \hat{Y}_t, \hat{Z}_t) &= f(t, Y_t, \hat{Z}_t) - f(t, \theta Y_t, \hat{Z}_t) + f(t, \theta Y_t, \hat{Z}_t) - f(t, \hat{Y}_t, \hat{Z}_t) \\
&= f(t, Y_t, \hat{Z}_t) - f(t, \theta Y_t, \hat{Z}_t) + a(t)U_t
\end{aligned}$$

where $a(t) = \left[f(t, \theta Y_t, \hat{Z}_t) - f(t, \hat{Y}_t, \hat{Z}_t) \right] / U_t$, when $U_t \neq 0$ and $a(t) = \beta$, otherwise. By **(b)**, we have $|a(t)| \leq \beta$ and

$$f(t, Y_t, \hat{Z}_t) - f(t, \hat{Y}_t, \hat{Z}_t) \leq (1 - \theta)\beta|Y_t| + a(t)U_t \quad (1.7)$$

Since f is concave in Z , we have

$$\begin{aligned}
f(t, Y_t, \hat{Z}_t) &= f\left(t, Y_t, \theta Z_t + (1 - \theta)\frac{\hat{Z}_t - \theta Z_t}{1 - \theta}\right) \\
&\geq \theta f(t, Y_t, Z_t) + (1 - \theta)f\left(t, Y_t, \frac{\hat{Z}_t - \theta Z_t}{1 - \theta}\right)
\end{aligned}$$

thus by **(c)**

$$\begin{aligned}
\theta f(t, Y_t, Z_t) - f(t, Y_t, \hat{Z}_t) &\leq -(1 - \theta)f\left(t, Y_t, \frac{\hat{Z}_t - \theta Z_t}{1 - \theta}\right) \\
&\leq (1 - \theta)(\alpha(t) + \beta|Y_t|) + \frac{\gamma}{2(1 - \theta)}|\hat{Z}_t - \theta Z_t|^2 \quad (1.8)
\end{aligned}$$

We continue with (1.6). Upon combining (1.7) and (1.8), we have

$$\theta f(t, Y_t, Z_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) \leq a(t)U_t + (1 - \theta)(\alpha(t) + 2\beta|Y_t|) + \frac{\gamma}{2(1 - \theta)}|\hat{Z}_t - \theta Z_t|^2 + \delta f_t \quad (1.9)$$

Let $F_t = \theta f(t, Y_t, Z_t) - \hat{f}(t, \hat{Y}_t, \hat{Z}_t) - a(t)U_t$. We can rewrite (1.9) as

$$F_t \leq (1 - \theta)(\alpha(t) + 2\beta|Y_t|) + \frac{\gamma}{2(1 - \theta)}|\hat{Z}_t - \theta Z_t|^2 + \delta f_t$$

Let $A_t = \int_0^t a(s)ds$. By Ito's formula, we have

$$d(e^{A_t}U_t) = -e^{A_t}F_t dt + e^{A_t}V_t' dB_t \quad (1.10)$$

Let $c \geq 0$ and define $P_t = \exp\{ce^{A_t}U_t\}$. By Ito's formula,

$$dP_t = -G_t dt + Q_t' dB_t,$$

where $Q_t = cP_t e^{A_t} V_t$, and

$$\begin{aligned} G_t &= cP_t e^{A_t} \left(F_t - \frac{ce^{A_t}}{2}|V_t|^2 \right) \\ &\leq cP_t e^{A_t} \{ (1 - \theta)(\alpha(t) + 2\beta|Y_t|) + \delta f_t \} + cP_t e^{A_t} \left(\frac{\gamma}{2(1 - \theta)} - \frac{ce^{A_t}}{2} \right) |V_t|^2 \end{aligned} \quad (1.11)$$

Recall that $|A_t| \leq \beta T$, so if we choose $c(\theta) = \gamma e^{\beta T} / (1 - \theta)$, then

$$\frac{\gamma}{2(1 - \theta)} - \frac{c(\theta)e^{A_t}}{2} \leq 0$$

thus

$$G_t \leq c(\theta)P_t e^{A_t} \{ (1 - \theta)(\alpha(t) + 2\beta|Y_t|) + \delta f_t \} = P_t H_t \quad (1.12)$$

$$\text{where} \quad H_t = e^{A_t} \left\{ \gamma e^{\beta T} (\alpha(t) + 2\beta|Y_t|) + c(\theta) \delta f_t \right\}$$

Let $\tilde{P}_t = D_t P_t$ and $\tilde{Q}_t = D_t Q_t$, where $D_t = \exp\left(\int_0^t H_s ds\right)$. By applying Ito's formula to \tilde{P} and (1.12), we have for any $0 \leq t_1 \leq t_2 \leq T$,

$$\tilde{P}_{t_2} - \tilde{P}_{t_1} \geq \int_{t_1}^{t_2} \tilde{Q}_s dB_s, \quad (1.13)$$

For any fixed $t \in [0, T]$, define the sequence of stopping times τ_n , $n \geq 1$ as:

$$\tau_n = \inf \left\{ u \geq t : \int_t^u |\tilde{Q}_s|^2 ds \geq n \right\} \wedge T$$

By (1.13), we have

$$\tilde{P}_t \leq \tilde{P}_{\tau_n} - \int_t^{\tau_n} \tilde{Q}_s dB_s \quad (1.14)$$

where t Upon taking conditional expectation on (1.14) using $\tilde{P}_t = D_t P_t$, we have

$$P_t \leq E_t \left\{ \exp \left(\int_t^{\tau_n} e^{As} \left[\gamma e^{\beta T} (\alpha(s) + 2\beta |Y_s|) + c(\theta) \delta f_s \right] ds \right) P_{\tau_n} \right\}$$

By integrability condition **(d)**⁵ and monotone convergence theorem, the exponential term on the right-hand side of the above inequality converges.

$$\begin{aligned} P_{\tau_n} &= \exp\{ce^{A\tau_n} U_{\tau_n}\} \\ &\leq \exp\{ce^{\beta T}(\theta Y_{\tau_n} - \hat{Y}_{\tau_n})\} = \exp\{ce^{\beta T}(\theta - 1)Y_{\tau_n} + ce^{\beta T}(Y_{\tau_n} - \hat{Y}_{\tau_n})\} \\ &\leq \exp\{2ce^{\beta T}(\theta - 1)Y_{\tau_n}\}/2 + \exp\{2ce^{\beta T}(Y_{\tau_n} - \hat{Y}_{\tau_n})\}/2 \end{aligned}$$

⁵Corollary 4 in Briand and Hu (2008) shows that **(d)** implies that $\sup_{0 \leq t \leq T} |Y_t|$ has exponential moments of all orders.

$$P_t \leq E_t \left\{ \exp \left(\int_t^T e^{As} \left[\gamma e^{\beta T} (\alpha(s) + 2\beta|Y_s|) + c(\theta)\delta f_s \right] ds \right) P_T \right\}$$

Because $|A_t| \leq \beta T$, we have:

$$\begin{aligned} \exp \left(\frac{\gamma e^{\beta T + A_t}}{1 - \theta} (\theta Y_t - \hat{Y}_t) \right) &\leq E_t \left\{ \exp \left(\int_t^T e^{\beta T} \left[\gamma e^{\beta T} (\alpha(s) + 2\beta|Y_s|) \right. \right. \right. \\ &\quad \left. \left. \left. + \frac{\gamma e^{\beta T}}{1 - \theta} \delta f_s \right] ds + \frac{\gamma e^{2\beta T}}{1 - \theta} (\theta Y_T - \hat{Y}_T) \right) \right\} \end{aligned}$$

By $\theta Y_T - \hat{Y}_T = \theta(Y_T - \hat{Y}_T) + (\theta - 1)\hat{Y}_T \leq \theta(Y_T - \hat{Y}_T) + (1 - \theta)|\hat{Y}_T|$, we have:

$$\begin{aligned} \exp \left(\frac{\gamma e^{\beta T + A_t}}{1 - \theta} (\theta Y_t - \hat{Y}_t) \right) &\leq E_t \left\{ \exp \left[\frac{\gamma e^{2\beta T}}{1 - \theta} \left(\int_t^T \delta f_s ds + \theta(Y_T - \hat{Y}_T) \right) \right. \right. \\ &\quad \left. \left. + \gamma e^{2\beta T} \left(\int_t^T (\alpha(s) + 2\beta|Y_s|) ds + |\hat{Y}_T| \right) \right] \right\} \end{aligned}$$

Because $\beta T + A_t \geq 0$, $\delta f(s) \leq 0$ and $Y_T - \hat{Y}_T \leq 0$, we have:

$$\theta Y_t - \hat{Y}_t \leq \frac{1 - \theta}{\gamma} \log E_t \left\{ \exp \left(\gamma e^{2\beta T} \left(\int_t^T (\alpha(s) + 2\beta|Y_s|) ds + |\hat{Y}_T| \right) \right) \right\}$$

The right hand side is finite due to **(d)** and the simple fact that the class of random variables of all exponential orders is closed under addition. Thus we can let θ go to 1 and get $Y_t \leq \hat{Y}_t$.

■

1.3 Application to Economics

1.3.1 Recursive Utility Functions

In economics, *utility* is a measure of relative satisfaction. Given this measure, one may speak meaningfully of increasing or decreasing utility, and thereby explain economic behavior in terms of attempts to increase one's utility. Utility is often modeled to be affected by consumption of various goods and services, possession of wealth and spending of leisure time, etc. A utility function measures all the objects of choice on a numerical scale and a higher measure on the scale means the consumer likes the object more.

The utility function that we will use is the continuous-time recursive utility introduced by Duffie and Epstein in Duffie and Epstein (1992b) and Duffie and Epstein (1992a) as an extension of the popular time-additive utility. Skiadas (2008) summarized some important properties of recursive utility and its application to selection of optimal consumption-portfolio. Let us consider an agent who can consume from 0 to T . The set of *consumption plans* is a convex set $\mathcal{C} \subseteq \mathcal{L}_2(\mathbf{C})$, where $\mathbf{C} \subset \mathbb{R}$ (typically $\mathbf{C} = \mathbb{R}^+$). For any $c \in \mathcal{C}$, let $c_t, 0 \leq t < T$ denote the consumption rate at t . There also exists a terminal lump-sum consumption c_T .

Definition 1 *We will let U_t denote the agent's utility at t , $0 \leq t \leq T$. $(U, \Sigma) \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}^d)$ solves the following BSDE:*

$$dU_t = -F(t, c_t, U_t, \Sigma_t)dt + \Sigma_t' dB_t, \quad U_T = F(T, c_T). \quad (1.15)$$

We will assume that the terminal utility $F(T, c_T) : \Omega \times \mathbf{C} \rightarrow \mathbb{R}$ depends only on ω and c_T and the aggregator function ⁶ $F : \Omega \times [0, T] \times \mathbf{C} \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is increasing in c , concave

⁶In the general setup of BSDE (1.1), we allow the aggregator to depend on ω in addition

in (c, U, Σ) and satisfies Assumption 2.

In the next example we present the standard time-additive utility that is widely used in asset pricing theory. It is a special example of recursive utility.

Example 1 Assume that the terminal $F(T, c_T) \in L_1(\mathbb{R})$ and there exists some function $u : \Omega \times [0, T] \times \mathbf{C} \rightarrow \mathbb{R}$ such that $u(t, c_t) \in \mathcal{L}_1(\mathbb{R})$. The aggregator function F satisfies:

$$F(t, c_t, U_t, \Sigma_t) = u(t, c_t) - \beta U_t$$

for some constant $\beta > 0$.

By applying Ito's formula to $U_t e^{-\beta t}$, we have the following closed form expression for U_t .

$$U_t = E_t \left[F(T, c_T) e^{-\beta(T-t)} + \int_t^T e^{-\beta(s-t)} u(s, c_s) ds \right]. \quad (1.16)$$

This is the well-known standard time-additive utility.

In example 1, if in addition we allow β to vary with c , i.e. $\beta : \Omega \times [0, T] \times \mathbf{C} \rightarrow \mathbb{R}^+$, we get the following closed-form expression for U_t that extends (1.16).

$$U_t = E_t \left[F(T, c_T) e^{-\int_t^T \beta(s, c_s) ds} + \int_t^T e^{-\int_t^s \beta(\tau, c_\tau) d\tau} u(s, c_s) ds \right].$$

For any given consumption plan $c \in \mathcal{C}$, we will let $(U(c), \Sigma(c))$ denote the solution of BSDE (1.15) with consumption plan c .

In typical economical applications, we often require that the utility function satisfies

to time t and the solution (Y, Z) . This allows us to include other random processes in the aggregator such as the consumption process c in this section. In our main application in Chapter 2, we will include more processes in aggregator functions.

certain properties, such as:

Monotonicity For $c^1, c^2 \in \mathcal{C}$, if $c^1 \geq c^2$, then $U_t(c^1) \geq U_t(c^2)$, $0 \leq t \leq T$.

Concavity For $c^1, c^2 \in \mathcal{C}$ and $\alpha \in (0, 1)$, $U_t(\alpha c^1 + (1-\alpha)c^2) \geq \alpha U_t(c^1) + (1-\alpha)U_t(c^2)$, $0 \leq t \leq T$.

Dynamic consistency Let $c^1, c^2 \in \mathcal{C}$, $0 \leq s < t \leq T$ and $A \in \mathcal{F}_s$. Assume $c^1 = c^2$ on $A \times [s, t]$ and $U_t(c^1) \geq U_t(c^2)$ (or $U_t(c^1) = U_t(c^2)$) on A , then $U(c^1) \geq U(c^2)$ (or $U(c^1) = U(c^2)$ respectively) on $A \times [s, t]$.

Proposition 2 *The recursive utility function in Definition 1 is monotonically increasing and concave in consumption and satisfies dynamic consistency.*

Proof. The proof of the proposition is based on the comparison lemma(Theorem 4). **Monotonicity** follows directly from Theorem 4, because the aggregator function of BSDE (1.15) is monotonically increasing in c .

For the proof of **Concavity**, let $c^\alpha = \alpha c^1 + (1-\alpha)c^2$, $U^\alpha = \alpha U(c^1) + (1-\alpha)U(c^2)$, and $\Sigma^\alpha = \alpha \Sigma(c^1) + (1-\alpha)\Sigma(c^2)$.

Thus $(U^\alpha, \Sigma^\alpha)$ satisfies(omitting the time argument t for $t \in [0, T)$):

$$\begin{aligned} dU^\alpha &= - \left(\alpha F(c^1, U(c^1), \Sigma(c^1)) + (1-\alpha)F(c^2, U(c^2), \Sigma(c^2)) \right) dt + \Sigma^{\alpha'} dB \\ &= - (F(c^\alpha, U^\alpha, \Sigma^\alpha) - p) dt + \Sigma^{\alpha'} dB, \\ U_T^\alpha &= \alpha F(T, c_T^1) + (1-\alpha)F(T, c_T^2) \\ &= F(T, c_T^\alpha) - p_T. \end{aligned}$$

where

$$\begin{aligned} p &= F(c^\alpha, U^\alpha, \Sigma^\alpha) - \alpha F(c^1, U(c^1), \Sigma(c^1)) - (1 - \alpha) F(c^2, U(c^2), \Sigma(c^2)), \\ p_T &= F(T, c_T^\alpha) - \alpha F(T, c_T^1) - (1 - \alpha) F(T, c_T^2) \end{aligned}$$

Because F is concave in (c, U, Σ) , we have $p_t \geq 0$, $t \in [0, T]$.

We also have that $(U(c^\alpha), \Sigma(c^\alpha))$ satisfies:

$$dU(c^\alpha) = -F(c^\alpha, U(c^\alpha), \Sigma(c^\alpha))dt + \Sigma'(c^\alpha)dB, \quad U_T(c^\alpha) = F(T, c_T^\alpha).$$

Applying Theorem 4 with $(Y, Z) = (U^\alpha, \Sigma^\alpha)$ and $(\hat{Y}, \hat{Z}) = (U(c^\alpha), \Sigma(c^\alpha))$, we get $U_t(c^\alpha) \geq U_t^\alpha$, $0 \leq t \leq T$.

For the proof of **Dynamic Consistency**, we consider the BSDEs for $(U(c^i), \Sigma(c^i))$, $i = 1, 2$ on $A \times [s, t]$ with the aggregators $F(c^1, \cdot) = F(c^2, \cdot)$ and the terminals $U_t(c^1) \geq U_t(c^2)$. Thus it follows from Theorem 4 that $U(c^1) \geq U(c^2)$ on $A \times [s, t]$. By symmetry, the claim still holds with $=$ replacing all the \geq . ■ When maximizing recursive utility with respect to consumption plans, it is enough to maximize U_0 , because the property of dynamic consistency ensures that once an optimal consumption plan is chosen at $t = 0$, the agent does not have incentive to deviate from it at any $t \in [0, T]$. Assume U_0 is maximized by the consumption plan $c \in \mathcal{C}$.⁷ Then for any $t \in [0, T]$, there can not exist a consumption plan \tilde{c} that satisfies $c_s = \tilde{c}_s$, $s \in [0, t)$ and $U_t(\tilde{c}) \geq U_t(c)$. Otherwise, we can define \bar{c} such that $\bar{c}_s = c_s$, $s \in [0, t)$ and $\bar{c}_s = \tilde{c}_s$, $s \in [t, T]$. By dynamic consistency, we have $U_0(\bar{c}) \geq U_0(c)$, which contradicts

⁷In typical applications such as optimal portfolio choice and optimal contracting, c is subject to extra constraints, which is generally expressed as a forward SDE. We will leave out this technicality for the time being and get back to it in Chapter 2.

the optimality of c at time 0. So if $c \in \mathcal{C}$ satisfies $U_0(c) = \max_{\tilde{c} \in \mathcal{C}} U_0(\tilde{c})$, the agent will stick to it as the optimal and not deviate.

Comparing Risk Aversion: In comparison to the standard time-additive utility (Example 1), recursive utility allows us more flexibility to adjust risk-aversion through the dependence of aggregator function on the volatility term. To see this, we will consider two recursive utility functions U^i with aggregators F^i , $i = 1, 2$.

U^1 is more risk-averse than U^2 , if $U_0^1(c) = U_0^2(c)$, for any deterministic plan c and $U_0^1(c) \leq U_0^2(c)$ for any plan $c \in \mathcal{C}$.

For simplicity, we assume that the two aggregators F^i are deterministic functions of the corresponding arguments (t, c^i, U^i, Σ^i) and $F^i(T, c_T^i)$ is a deterministic function of c_T^i , $i = 1, 2$. If $F^1(t, c, y, 0) = F^2(t, c, y, 0)$ for any $t, c, y \in [0, T] \times \mathbf{C} \times \mathbb{R}$, then $U^1(c) = U^2(c)$ for any deterministic plan c . While we can adjust F^i such that $F^1(t, c, y, z) \leq F^2(t, c, y, z)$ for any $t, c, y, z \in [0, T] \times \mathbf{C} \times \mathbb{R} \times \mathbb{R}^d$. By Theorem 4, we get $U_0^1(c) \leq U_0^2(c)$, for any $c \in \mathcal{C}$. Thus The two recursive utilities U_1 and U_2 have the same preference order over deterministic plans while U_1 is more risk-averse. Risk-aversion of the standard time-additive utility is totally governed by the preference order over deterministic plans, since there is no volatility term in the aggregator to adjust risk-aversion.

1.3.2 European and American Contingent Claims Valuation

The following section summarizes some results from Karoui, Peng, and Quenez (1997), Karoui and Mazliak (1997) and Rogers and Talay (1997) about the application of BSDE in pricing European and American contingent claims. In pricing European contingent claims, these results extend the classic risk-neutral pricing results in complete markets. The application of BSDE allows us to use more flexible modeling of wealth equations including

consumption, nonlinear generators and incomplete markets. We also introduce a variation of BSDE, the reflected BSDE(RBSDE) which is closely related to optimal stopping. It is used in pricing of American contingent claims. For simplicity, we will only impose assumption 1 on the BSDEs(RBSDEs) and cover the classic pricing problems with complete markets.

We will adopt the following setting for a complete market. There are $d + 1$ assets. One of them P^0 is a riskless asset. In addition, there are d risky assets P^i , $i = 1, \dots, d$ that do not pay dividends. The $n + 1$ assets follow the equations below.

$$\begin{aligned} dP_t^0 &= P_t^0 r_t dt \\ dP_t^i &= P_t^i \left[b_t^i dt + \sum_{j=1}^d \sigma_t^{i,j} dB_t^j \right], \quad i, j = 1, \dots, d \end{aligned}$$

where r is the short interest rate, $b = (b^1, \dots, b^d)'$ is a $d \times 1$ vector representing the appreciation rates of d stocks and σ is a $d \times d$ volatility matrix.

For simplicity, r , b and σ are assumed to be uniformly bounded and predictable processes. σ has full rank and the inverse σ^{-1} is uniformly bounded as well.

It is also being assumed that there exists a predictable and bounded-valued $d \times 1$ vector process θ that solves the following market price of risk equation.

$$b_t - r_t \mathbf{1} = \sigma_t \theta_t \text{ a.e.}$$

where $\mathbf{1}$ is a vector with every component 1.

Under these assumptions, the market is complete.

Let $\pi = (\pi^1, \dots, \pi^d)'$ denote the amount of wealth in the d risky assets, namely π is a

portfolio. Let V_t denote the value of the portfolio. The pair (V, π) is called self-financing if

$$\int_0^T |\sigma'_t \pi_t|^2 dt < +\infty \text{ a.s.}$$

and the following two equations hold

$$V_t = \sum_{i=0}^d \pi_t^i$$

and

$$\begin{aligned} dV_t &= \sum_{i=0}^d \pi_t^i \frac{dP_t^i}{P_t^i} \\ &= (V_t - \pi'_t \mathbf{1}) r_t dt + \sum_{i=0}^d \pi_t^i (b_t^i dt + \sum_{j=1}^d \sigma_t^{i,j} dB_t^j) \\ &= r_t V_t dt + \pi'_t (b_t - r_t \mathbf{1}) dt + \pi'_t \sigma_t dB_t \\ &= (r_t V_t + \pi'_t \sigma_t \theta_t) dt + \pi'_t \sigma_t dB_t \end{aligned}$$

The case of European option.

First recall that an European contingent claim ξ settled at time T is an \mathcal{F}_T measurable random variable. It is a contract that pays ξ at time T .

Without considering consumption, a hedging strategy against a short position in ξ is defined to be a self-financing trading strategy (V, π) such that $V_T = \xi$. We denote the class of hedging strategies against ξ by $H(\xi)$. The claim ξ is hedgable if $H(\xi)$ is nonempty

The fair price X_0 at time 0 of a claim ξ is defined as

$$X_0 = \inf\{x \geq 0; \exists (V, \pi) \in H(\xi) \text{ such that } V_0 = x\}$$

With the above setup, let ξ be a nonnegative⁸ square-integrable contingent claim. A hedging strategy (X, π) against ξ is a solution of the following linear BSDE

$$dX_t = (r_t X_t + \pi'_t \sigma_t \theta_t) dt + \pi'_t \sigma_t dB_t, \quad X_T = \xi$$

Without restrictions on the solution (X_t, π_t) , the solution to such a BSDE is generally not unique. If we require that the strategy is feasible i.e. $X_t \geq 0$, $t \in [0, T]$, then the solution is unique. As an alternative, we can require as in Theorem 2 that $(X, \sigma'_t \pi_t) \in \mathcal{L}_2(\mathbb{R}) \times \mathcal{L}_2(\mathbb{R}^d)$. In both cases, the solution is given by a standard result on linear BSDE as appears in Proposition 1.

$$X_t = \mathbb{E} \left[H_T^t \xi | \mathcal{F}_t \right]$$

$$\text{where } H_s^t, t \leq s \leq T \text{ satisfies } dH_s^t = -H_s^t [r_s ds + \theta'_s dB_s], H_t^t = 1.$$

The process H above is called "state price density" or "deflator". The above characterization of hedging portfolio agrees with the classic risk-neutral pricing result. By Girsanov's Theorem, there exists Q a risk neutral probability measure so that $W_t + \int_0^t \theta_s ds$ is a standard d -dimensional Brownian Motion that under Q ⁹.

⁸For simplicity, we only consider hedging a short position of a claim that has a nonnegative payoff at T .

⁹Since we assume that θ is uniformly bounded, $\frac{dQ}{dP} = \exp - \left[\int_0^T \theta'_s dB_s + \frac{1}{2} \int_0^T |\theta_s|^2 ds \right]$ is a martingale.

Then the solution of the above BSDE can be written as

$$X_t = E^Q \left[e^{-\int_t^T r_s ds} \xi | \mathcal{F}_t \right]$$

As we see above, the solution of the European option leads to a BSDE with generator that is linear in the wealth and volatility. However, we can apply more general assumptions on the generator of BSDE such as Assumption 2 to allow more flexible modeling.

The case of American option.

To motivate using RBSDE in pricing American contingent claim, let us first consider the problem of hedging European contingent claim with consumption. Let C be an increasing, right-continuous process representing cumulative consumption. A self-financing superhedging strategy is a collection of (V, π, C) where V is the wealth process and π the portfolio process such that

$$\begin{aligned} dV_t &= (r_t V_t + \pi_t^* \sigma_t \theta_t) dt - dC_t + \pi_t' \sigma_t dB_t, \quad V_T = \xi \\ \int_0^T |\sigma_t' \pi_t|^2 dt &< +\infty \text{ a.s.} \end{aligned}$$

A superhedging strategy is called feasible if

$$V_t \geq 0, \quad t \in [0, T] \text{ a.s.}$$

Let $H'(\xi)$ denote the class of superhedging strategy.

The upper price at time 0 of ξ is defined as

$$X'_0 = \inf \{x \geq 0; \exists (V, \pi, C) \in H'(\xi) \text{ such that } V_0 = x\}$$

By incorporating the consumption process C into the generator of the wealth equation, a positive term dC_t/dt ¹⁰ is added to the generator. Given a claim ξ , we see by using the Comparison Lemma 3 that the upper price should be no less than the fair price. In the setting above, the fair price agrees with the upper price i.e. $X'_0 = X_0$. This can be seen by setting $C = 0$ in the above equation.

A feasible superhedging strategy is a special case of RBSDE, if we further require that $\int_0^T V_t dC_t = 0$ and $V \in \mathcal{S}^2$. Although these two last assumptions are not needed for pricing European contingent claim, it turns out that for American contingent claim, due to the early exercise feature, a consumption process is needed for the case that the option holder does not follow an optimal exercise policy. This concludes the motivational part of the presentation.

Definition 2 *A standard data for an RBSDE consists of a terminal value $\xi \in L^2$, a standard generator $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ that satisfy Assumption 1 and a continuous obstacle process $S \in \mathcal{L}^2$*

The solution to a RBSDE with standard data (ξ, f, S) is a tripe of \mathcal{F} -progressively measurable process (Y_t, Z_t, K_t) , $0 \leq t \leq T$ taking values in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^+$ that satisfies

- a. $Z \in \mathcal{L}^2$, $Y \in \mathcal{S}^2$ and $K_T \in L^2$;
- b. $Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + K_T - K_t - \int_t^T Z_s^* dB_s$, $0 \leq t \leq T$;
- c. $Y_t \geq S_t$, $0 \leq t \leq T$;
- d. K_t is continuous and increasing, $K_0 = 0$ and $\int_0^T (Y_t - S_t) dK_t = 0$.

Because of the backward formulation, the above definition might look counter-intuitive at the first glance. Compare an RBSDE for Y_t with a regular BSDE for \bar{Y}_t with the same

¹⁰Assume C is absolutely continuous with respect to the Lebesgue measure

generator and terminal value.

$$-dY_t = f(t, Y_t, Z_t)dt + dK_t - Z_t' dB_t, \quad Y_T = \xi$$

$$-d\bar{Y}_t = f(t, \bar{Y}_t, \bar{Z}_t)dt - \bar{Z}_t' dB_t, \quad \bar{Y}_T = \xi$$

We see that the difference between the two equations is dK_t which represents an upward push of $-dY_t$. Since $Y_t = \xi - \int_t^T dY_s$, increasing $-dY_t$ has the effect of pushing Y_t upward.

Another way of thinking of the effect of K_t is if we let the volatility term $Z_t = 0$ and consider Y_t to represent the price of a zero-coupon bond with a face value ξ at T . Adding K_t to the equation has the effect of decreasing the interest rate and thus increasing the value of the bond.

By condition (d) in the last definition, K_t is continuous and moves upward only when $Y_t = S_t$. This ensures that the minimal push is being used to make condition (c) satisfied.

Similar to the case of classic BSDE, with a set of standard data (ξ, f, S) the RBSDE has a unique solution (Y_t, Z_t, K_t) . The proof of existence part is based on apriori estimates on the solutions of two RBSDEs. The uniqueness part is based on the following comparison theorem for RBSDEs.

Theorem 5 *Let (ξ, f, S) and (ξ', f', S') be two sets of standard data and suppose that*

- a.** $\xi \leq \xi'$
- b.** $f(t, y, z) \leq f'(t, y, z)$ a.e.
- c.** $S_t \leq S'_t$ $0 \leq t \leq T$ a.s.

Let (Y, Z, K) and (Y', Z', K') be the respective solutions of the RBSDEs associated with the

standard data above. Then

$$Y_t \leq Y'_t, \quad 0 \leq t \leq T, \quad a.s.$$

As in the European part, a wealth portfolio before the option is exercised is a pair of processes (X_t, π_t) in $\mathcal{L}^2(\mathbb{R}) \times \mathcal{L}^2(\mathbb{R}^d)$ that satisfies the following SDE

$$dX_t = -b(t, X_t, \pi_t)dt + \pi_t^* \sigma_t dW_t$$

where b is a standard generator. The European option case corresponds to $b(t, x, \pi) = -r_t x - \pi^* \sigma \theta_t$. The volatility matrix σ is assumed to be invertible. Also σ and σ^{-1} are uniformly bounded. We can let σ be identity matrix without loss of generality as we can treat $\sigma^* \pi$ as π .

Let ξ denote the terminal payoff of an American contingent claim in case it is not exercised early. Let S_u denote the intrinsical value of the claim which is the payoff of the claim if it is exercised at time u for any $0 \leq u < T$. Also assume $S \in \mathcal{L}^2$. Let

$$\tilde{S}_u = \xi \mathbf{1}_{u=T} + S_u \mathbf{1}_{u < T}, \quad 0 \leq u \leq T$$

For any $0 \leq t \leq T$, let $\Psi_t = \{\tau; \tau \text{ is a stopping time and } t \leq \tau \leq T\}$. For any $\nu \in \Psi_t$, there exists a unique solution (X_s^ν, π_s^ν) to the following BSDE for the wealth process with terminal at the stopping time ν .

$$-dX_s^\nu = b(s, X_s^\nu, \pi_s^\nu)ds - (\pi_s^\nu)^* dW_s, \quad 0 \leq s \leq \nu, \quad X_\nu^\nu = \tilde{S}_\nu$$

If (X, π) are the solution to a BSDE associated with terminal time T , generator $b(t, x, \pi) \mathbf{1}_{t \leq \nu}$ and terminal value \tilde{S}_ν , then (X^ν, π^ν) is just $(X_s, \pi_s) \mathbf{1}_{s \leq \nu}$. This BSDE represents the wealth

process stopped at ν and replicates the payoff \tilde{S}_ν , if the American contingent claim is exercised at time ν .

Thus the price of the American contingent claim at time t with payoff \tilde{S}_ν at exercise time $\nu \in \Psi_t$ is given by

$$\operatorname{ess\,sup}_{\nu \in \Psi_t} X_t^\nu = \operatorname{ess\,sup}_{\nu \in \Psi_t} \mathbb{E} \left[\tilde{S}_\nu + \int_t^\nu b(s, X_s^\nu, \pi_s^\nu) ds \middle| \mathcal{F}_t \right]$$

The following proposition shows this price is exactly the solution of a RBSDE. It also generalizes the traditional optimal stopping problem and provides a RBSDE that solves the value of the American contingent claim.

Proposition 3 *Let $\{Y_t, Z_t, K_t, 0 \leq t \leq T\}$ be the solution of a RBSDE associated with a set of standard data (ξ, f, S) then for each $t \in [0, T]$*

$$Y_t = \operatorname{ess\,sup}_{\nu \in \Gamma_t} \mathbb{E} \left[\tilde{S}_\nu + \int_t^\nu f(s, Y_s, Z_s) ds \middle| \mathcal{F}_t \right]$$

where $\tilde{S}_u = \xi \mathbf{1}_{u=T} + S_u \mathbf{1}_{u < T}$, $0 \leq u \leq T$

The optimal stopping time in the above equation is achieved by

$$D_t = \inf \{t \leq u \leq T : Y_u = S_u\}$$

Note that in the above proposition, when f is an \mathcal{L}^2 process that is free of (Y, Z) , the solution Y_t corresponds to the value of an optimal stopping problem.

The solution of proposition 3 is compatible with pricing by using risk-neutral probability measure namely, with Q denoting the risk-neutral probability measure, we get that the price of the American contingent claim at time t with payoff \tilde{S}_ν at exercise time $\nu \in \Psi_t$ is given

by

$$\begin{aligned}
& \text{ess sup}_{\nu \in \Psi_t} \mathbb{E} \left[\tilde{S}_\nu + \int_t^\nu b(s, X_s^\nu, \pi_s^\nu) ds | \mathcal{F}_t \right] \\
&= \text{ess sup}_{\nu \in \Psi_t} \mathbb{E}^Q \left[\exp \left(- \int_t^\nu r_s ds \right) \tilde{S}_\nu | \mathcal{F}_t \right]
\end{aligned}$$

Chapter 2

Principal-Agent Problem with Multiple Principals

2.1 Introduction

A nontechnical overview of the problem: We study the principal-agent(owner-manager) problem with moral hazard in continuous time with multiple principals. In simple terms, the problem studies the interaction of a group of principals. Each has one agent working for him/her.

To help understand the problem, we will first recall a simpler problem with just one principal and one agent. In this problem, a utility maximizing principal offers a pay plan to an agent in order to induce effort, which has an impact on both the principal and agent's utilities as well as the principal's cash flow. In the meantime, the principal faces two constraints. The first constraint is an incentive compatibility condition that says the agent will make effort to maximize his/her own utility and not necessarily work in the benefit of the principal. The second is a participation constraint namely, the agent's initial utility must exceed some fixed amount, because of his/her employment opportunities elsewhere. The objective is to characterize the principal's optimal control over the agent, taking into account the two constraints.

In the principal-agent problem with multiple principals, the new element is the assumption that each principal-agent pair's utilities and each principal's cash-flow are affected by the efforts of all agents. When choosing his/her controls, each principal has to take into account not only the two constraints as in the single principal case, but also the impact of his/her control on the other principal-agent pairs and their response. The problem is defined in two levels, first for the agents and then the principals. The key to the definition of the problem in both levels is Nash equilibrium. In simple language, a Nash equilibrium is a relation among a group of utility-maximizing "players", in which each player's strategy is optimal in response to the others' strategies. In other words, in a Nash equilibrium, each player has no incentive to change strategy, when the others do not change theirs. In our problem, the Nash equilibrium among the agents are through their competing(or cooperative) efforts, which have an impact on every agent's utility function. The Nash equilibrium among the principals is more complicated, because the connection among them is indirect and comes only through the agents' Nash equilibrium in efforts. That is each principal's incentive compatibility constraint, from the single principal case, is replaced by a Nash equilibrium condition among the agents. Similar to the single-principal case, each principal also faces the participation constraint of his/her agent. The objective is to characterize each principal's equilibrium control over his/her agent, taking into account the control is constrained by the agents' Nash equilibrium in efforts.

Main ideas of the paper: If agent effort is noncontractible, it is well-know from the principal-agent literature that a principal can induce agent effort by linking pay to the cash flows influenced by that effort. In a single firm with several agents and effort

externalities (one agent’s effort affecting the output of others), the optimal contract for each agent will generally depend on the other agents’ outputs (see Holmstrom (1982) and, in a continuous-time model, Koo, Shim, and Sung (2008)).¹ One reason for this is the impact of an agent’s effort on the output of others (another is simple risk reduction). But such effort externalities occur not only within firms but across firms. For example, innovations (e.g., software, microchips, fracking technology, etc.) by agents at one firm can affect the investment opportunities of other firms. Effort by employees in the service sector could have spillover effects to firms in the same industry. Alternatively, our model could represent a reduced-form model accounting for competitive or complementary industry effects. Within this setting there are moral hazard and potential free-rider problems at two levels. Given the compensation schemes offered by the principals, each agent considers only the impact of his effort on his own compensation, ignoring the benefits to the other agents as well as the principals. The agent may have an incentive to free ride off other agents’ efforts: in one application we show that no more than one agent will exert effort at any time. Each principal designs a compensation scheme to maximize own utility only, while anticipating the impact of the promised pay on the agent equilibrium. Each principal has an incentive to free ride off the other principals, particularly because the assumption of binding agent participation constraints implies that the benefits of effort externalities ultimately accrue to the principal.

We examine the multiple principals (one agent for each principal) problem in a continuous-time setting with a Brownian filtration, recursive preferences, and one agent for each

¹Even absent the effort externality, Holmstrom (1982) shows that such relative performance evaluation can be optimal simply to reduce compensation risk when the task outputs are correlated.

principal. The effort of one agent changes the drifts of the cash flows of other principals/firms as well as their own. We show that optimal contracts in this setting cannot be designed in isolation, because the compensation policy of each principal can affect the distributions of the other principals's cash flows and the incentives of the other agents. The equilibrium is simple to obtain for the class of preferences we consider (a generalization of additive exponential utility), and reduces to computing a system of linked subjective cash-flow value processes, one for each principal. Each principal's optimal pay policy amounts to choosing the component of the subjective cash-flow diffusion, or volatility, to transfer to their agent (that is, the volatility sharing rule), and is solved by maximizing the drift rate of this cash-flow process given the policies of the other principals.

We first examine applications with an additive impact of effort on the Brownian motion drift (which, in turn, implies an additive impact on cash-flow drifts). In the additive case the marginal impact of effort is not affected by other agents' effort. The resulting sharing rules are simple functions of each principal's own subjective cash-flow volatility, but these cash-flow processes must be jointly solved because of the externality of the other agents' effort. When an additive measure change is combined with quadratic effort and risk-aversion penalties, we obtain simple linear volatility sharing rules and closed-form solutions for a class of affine type state-variable dynamics (one example considers Ornstein-Uhlenbeck cash flows, and another example a square-root stochastic cash-flow volatility model). In the simple case of Brownian cash flows and no intermediate pay/consumption, the fixed component of each agent's lump sum terminal consumption is adjusted according to the covariances among the cash-flows. For example, in a setting with two principal-agent pairs and positive covariance between

the cash-flows, then each agent's effort increases the drift of both cash flow processes. This positive externality all accrues to the principals, who can reduce the fixed component of agent pay while still meeting the participation constraint because the fixed fraction of the cash flow paid to each agent becomes more valuable with the larger drift.

We next consider applications in which the impact of effort on the Brownian drift is diminishing in total effort. That is, the measure-change operator is concave in aggregate effort. The sharing rules in this setting are more complicated, with the optimal share of the subjective cash-flow volatility transferred to the agent (via the promised pay) depending on all the principals' subjective cash-flow volatility processes. Each agent's equilibrium optimal pay is therefore influenced by the cash-flow dynamics of all the principals. We obtain an explicit solution with quadratic effort penalties and two risk-neutral principal-agents pairs in which the lump-sum terminal pay of each agent depends on the terminal lump-sum cash-flows of both principals, increasing in their own cash flow and decreasing in the cash flow of the other. We also consider the case of absolute effort penalties (with possibly risk-averse principals and agents), in which the equilibrium results in only one agent working at any moment in time.

In the principal-agent problem with moral hazard, a utility maximizing principal pays a compensation process to an agent in order to induce effort (which, increases expected future cash flows). But the principal faces two constraints. First, because effort is assumed noncontractible, the contract must satisfy an *incentive compatibility* condition that the agent, faced with a particular compensation process, will choose effort to maximize his own utility. Second, because the agent has alternative employment op-

portunities, the agent's initial utility must exceed some fixed amount (the *participation constraint*). In the continuous-time Brownian version, first examined by Holmstrom and Milgrom (1987), the impact of effort choice is often modeled as an equivalent change of measure (that is, the agent's effort changes the probabilities of the states), which changes the drift of the driving Brownian processes. This is a convenient way to model, for example, the impact of effort on the growth rate of a cash flow process.

In the case of multiple principals/agents, each principal chooses the pay process that maximizes his/her own utility subject to incentive compatibility and their agent's participation constraint. At the agent level, we find that optimal effort generally depends on both the volatility of the agent's utility function as well as the effort levels of the other agents. We give necessary and sufficient conditions for a Nash equilibrium among agent effort processes in terms of the joint agent utility-volatility processes. At the principal level, each optimal compensation contract is specified in terms of two controls: his/her own consumption, and the volatility of agent utility. The agent volatility control implies a unique class of pay plans. Each principal chooses the optimal controls to maximize their own utility while fully anticipating the impact of their controls on the equilibrium efforts chosen by the agents (pay cannot be directly contingent on effort). The resulting set of optimal contracts establishes a Nash equilibrium among principals as well as a Nash equilibrium among agent efforts.

The dynamic contracting problem with multiple principals appears formidable, but is simple within the class of translation-invariant (TI) recursive preferences that we consider. The TI class of preferences is essentially as tractable as additive exponential utility, which is a special case, because the agency problem (the fact that effort is

noncontractible) induces recursivity in the principal's utility even in the time-additive case. Furthermore, recursive preferences allow more flexible modeling of risk aversion, as well as distinct modeling of aversion to variability in consumption across states versus across time. We find that the important qualitative aspects of the optimal contracts, as well as equilibrium agent effort, are driven by risk aversion. Preferences for intertemporal substitution enter only indirectly by affecting the drift of the subjective cash-flow value processes. We show in the case of an additive measure change that the most tractable subclass, in which the solution reduces to a set of Riccati ordinary differential equations when uncertainty is characterized by a set of affine-type state variables, is not necessarily time additive. It is characterized by general risk aversion, but elasticity of intertemporal substitution restricted to be infinite (or, alternatively, no intermediate pay/consumption at all).

Literature review: Our paper is part of a broader literature on the impact of interactions among principals/firms or agents on optimal contracting. Aggarwal and Samwick (1999) show if the pay of the manager/agent is increasing in both own-firm and rival-firm profits the incentive to compete is diminished and therefore prices and profits are higher.² Khanna and Schroder (2010) show that if loan non-renewal impacts the future prospects of the rival firm, the optimal debt contract (which induces truthful revelation of profits via the threat of loan non-renewal) is different from a standard debt contract: If default benefits the rival, the optimal contract deters predation with a reduced sensitivity to profits, and if default hurts the rival (say by allowing entry of

²The contract is assumed linear, and is not claimed to be optimal. They also consider a principal-agent problem, but because each agent's effort increases only own-firm profits, but there is no interaction among efforts.

a more efficient firm) then the optimal contract is made more sensitive to profits to increase prices. There is also a literature examining the impact of product market competition on managerial effort in a principal-agent setting (see, for example, Hart (1983), Scharfstein (1988), and Schmidt (1997)), but the level of competition in these papers is assumed to be invariant to the compensation contracts.

A large literature on team contracts allows agent efforts to jointly determine the distribution of output (see Chapter 8 of Bolton and Dewatripoint (2005) for a review). Holmstrom (1982) shows that the role of the principal (implementing pay schemes and extracting the surplus) becomes more important in team settings because of the free rider problem; he also examines the relationship between the information structure (signals and noise) can be used to implement efficient pay schemes. The incentive of any agent to free ride can also be affected by the other agents. For example, Winter (2010) shows that under complementarity of the production function,³ transparency among agents increases the threat against shirking (shirking by one agent can induce retaliatory shirking by his peers), thereby reducing the free rider problem. Edmans, Goldstein, and Zhu (2011) considers a two-period deterministic team problem in which firm output is either zero or one (failure or success) depending on the effort levels (each in $[0, 1]$) of all the agents (they focus on the cases when either output depends on the total efforts, or the minimum of the efforts). Synergy is modeled through effort cost, which is assumed decreasing (for "positive" symmetry), for each agent i , in the weighted sum of other agents' efforts. The principal announces agent wages and then agents simultaneously choose efforts, which constitute a Nash equilibrium. In a

³Their production function is defined as complementarity if the marginal impact of each agent i 's effort increases as the set of other agents who exert effort increases.

continuous time setting, Koo, Shim, and Sung (2008) give a general characterization of agent equilibrium in a team setting with time-additive exponential preferences and Brownian cash flows.

The agent-equilibrium part of our problem is similar the problem with a single principal but multiple agents with multiple tasks (it is most closely related to the agent equilibrium in Koo, Shim, and Sung (2008)). As in the team-contract literature, there is first an equilibrium among agents, but without a single principal to share the aggregate output and coordinate the agents' efforts, resulting in another layer of moral hazard that reduces efficiency. Furthermore the effort externality links the principals' contracting problems, because the incentives created within one firm will indirectly affect the opportunities of other firms. Therefore the optimal contracts set by the principals also constitute a Nash equilibrium. It seems natural in this setting to consider each agent's effort cost to depend on own-effort only (in contrast to the "synergies" modeled in Edmans, Goldstein, and Zhu (2011)), and to consider agent efforts as substitutes rather than complements: The Brownian-motion (and therefore cash-flow) drifts are an increasing and weakly concave function of aggregate efforts.

Our solution approach employs a number of techniques well known from the time-additive-utility case with a single principal. As in Schattler and Sung (1993), we consider a general nonMarkovian Brownian setting, and use a solution technique similar to their first-order approach (see also Williams (2008) and Koo, Shim, and Sung (2008)). These papers contain the key ideas of letting the principal choose agent utility diffusion (or agent effort) in order to solve for lump-sum terminal pay, which is solved by running the agent utility forward from the participation-constraint level

after substituting the optimal diffusion level. This reduces the principal's problem to a dynamic programming problem.

Organization of the paper: The chapter is organized as follows. In Section 2.2, we present the setting and notation, define the translation-invariant (TI) preferences that we use throughout, and outline the contracting problem. In Section 2.3, we develop a necessary and sufficient condition for agent equilibrium and introduce the class of CES(constant elasticity of substitution) measure-change operator. Section 2.4 characterizes principal equilibrium. Section 2.5 presents closed-form solutions to a set of linked BSDEs that represent principals' subjective PV(present value) processes. Section 2.6 and 2.7 present applications with an additive measure-change operator, and a CES operator, respectively. Numerous examples are provided throughout. Finally, the Appendices present proofs omitted from the text and derivations of some examples.

2.2 The Setting and Problem

All uncertainty is generated by d -dimensional standard Brownian motion B over the finite time horizon $[0, T]$, supported by a probability space (Ω, \mathcal{F}, P) . All processes appearing in this paper are assumed to be progressively measurable with respect to the augmented filtration $\{\mathcal{F}_t : t \in [0, T]\}$ generated by B . For any subset S of Euclidean space, let $\mathcal{L}(S)$ denote the set of S -valued processes, and, for any $p \geq 1$,

$$\begin{aligned}\mathcal{L}_p^-(S) &= \left\{ x \in \mathcal{L}(S) : E \left[\int_0^T \|x_t\|^p dt \right] < \infty \right\}, \\ \mathcal{L}_p(S) &= \{ x \in \mathcal{L}_p^-(S) : E [\|x_T\|^p] < \infty \},\end{aligned}$$

where $\|x_t\|$ denotes Euclidean norm.

Define the spaces

$$\begin{aligned}\mathcal{M} &= \{x \in \mathbb{R}: x \text{ is a r.v. such that } E(e^{\kappa x}) < \infty \text{ for all } \kappa > 0\}, \\ \bar{\mathcal{M}} &= \left\{x \in \mathcal{L}(\mathbb{R}): x_T, \int_0^T |x_s| ds \in \mathcal{M}\right\}.\end{aligned}$$

We will let $\mathcal{B}_{[0,T]}$ denote all the Borel sets on $[0, T]$ and λ denote the Lebesgue measure. The qualification " $P \otimes \lambda$ almost surely" is omitted throughout. We will also use the following notation: for $\alpha \in S^N$, we will let $\alpha^{-i} \in S^{N-1}$ denote the collection of all but the i th component (for example, if $\alpha \in \mathbb{R}^N$ then $\alpha^{-1} = (\alpha^2, \dots, \alpha^N)$). Also, for any number $x \in \mathbb{R}$, we denote its positive part by $x^+ = \max(0, x)$.

2.2.1 The Setup

Each of the N principals in our model has a single agent whom they pay to induce costly effort. The N agents' effort together change the probability measure, altering the drift of B and potentially the drifts of the principals' cash flows and cash-flow volatility. It is through the common change of measure that the principal and agent problems are linked.

We will define the following processes that are key to our paper

Consumption The set of *consumption plans* is a convex set $\mathcal{C} \subseteq \mathcal{L}_2(\mathbb{R})$. Let

$c^U = (c^{U1}, \dots, c^{UN}) \in \mathcal{C}^N$ and $c^V = (c^{V1}, \dots, c^{VN}) \in \mathcal{C}^N$ represent the consumption processes of the N agents and N principals respectively. We interpret c_t^{Ui} , $t < T$, as agent i 's consumption rate, and c_T^{Ui} as agent i 's lump-sum terminal consumption.

Similar explanation applies to principal i 's consumption.

Efforts We define the set of *effort plans* as a convex set $\mathcal{E} \subseteq \mathcal{L}_2^-(\mathbf{E})$ for some closed set

$\mathbf{E} \subset \mathbb{R}^d$ (typically $\mathbf{E} = \mathbb{R}^d$ or $\mathbf{E} = \mathbb{R}_+^d$). For any $e = (e^1, \dots, e^N) \in \mathcal{E}^N$, we interpret e_t^i as the time- t effort rate exerted by agent i . We assume that $e_T^i = 0$ (no lump-sum terminal effort).

Interest Rate We assume a bounded deterministic riskless short-rate process r .

Pay Process We define the set of *pay processes* as a convex set $\mathcal{P} \subseteq \bar{\mathcal{M}}$. For any $p =$

$(p^1, \dots, p^N) \in \mathcal{P}^N$, we interpret p_t^i , $t < T$, as intermediate pay and p_T^{Ui} as lump-sum terminal pay by principal i to agent i .

Cash-Flow Process We define the set of *cash-flow processes* as a convex set $\mathcal{X} \subseteq \bar{\mathcal{M}}$. For

any $X = (X^1, \dots, X^N) \in \mathcal{X}^N$, we interpret X_t^i , $t < T$, as intermediate cash-flow rate and X_T^i as lump-sum terminal cash flow of principal i .

Both principal i and agent i are allowed to borrow and lend through a money-market account. As with agent effort, we assume that the agent's money-market account balance is noncontractible. We say that the pay process p^i finances agent consumption c^{Ui} if there is a wealth (money-market balance) process W^{Ui} satisfying the agent's budget equation (2.1); and we say that the cash-flow process X^i finances p^i and principal consumption $c^{Vi} \in \mathcal{C}$ if there is a wealth process W^{Vi} satisfying the principal's budget equation (2.2):

$$W_0^{Ui} = w_0^{Ui}, \quad dW_t^{Ui} = (W_t^{Ui} r_t + p_t^i - c_t^{Ui}) dt, \quad c_T^{Ui} = W_T^{Ui} + p_T^i, \quad (2.1)$$

$$W_0^{Vi} = w_0^{Vi}, \quad dW_t^{Vi} = (W_t^{Vi} r_t + X_t^i - p_t^i - c_t^{Vi}) dt, \quad c_T^{Vi} = W_T^{Vi} + X_T^i - p_T^i. \quad (2.2)$$

Before the terminal date T , principal i invests the cash flow less agent i 's pay and principal i 's consumption, and agent i invests pay less the own consumption. At the terminal

date, principal i 's lump-sum consumption equals the lump-sum terminal cash flow plus the money-market balance minus the terminal lump-sum pay to agent i ; the agent's lump-sum consumption is the sum of the agent's money-market balance and lump-sum pay

Define the discount factor D , as well as the price process Γ of a bond paying a unit coupon rate and unit (lump-sum) par value:

$$D_t = e^{-\int_0^t r_s ds}, \quad \Gamma_t = \frac{1}{D_t} \left(\int_t^T D_s ds + D_T \right) \quad (2.3)$$

Note that

$$d\Gamma_t = (r_t \Gamma_t - 1)dt, \quad \Gamma_T = 1.$$

As in Koo, Shim, and Sung (2008) we model the impact of the collective agent effort $e \in \mathcal{E}^N$ as a change in probability measure to P^e where

$$\frac{dP^e}{dP} = Z_T^e$$

and the exponential supermartingale Z^e is defined by

$$Z_t^e = \exp \left(\int_0^t \Phi(e_s)' dB_s - \frac{1}{2} \int_0^t \|\Phi(e_s)\|^2 ds \right)$$

for some function $\Phi : \mathbb{R}^{d \cdot N} \rightarrow \mathbb{R}^d$ which maps the agents' effort to the change in Brownian drift. We assume throughout that Z^e is a martingale⁴ (equivalently, $EZ_T^e = 1$) for every $e \in \mathcal{E}^N$. By Girsanov's Theorem, $dB_t^e = dB_t - \Phi(e_t) dt$ is a standard d -dimensional Brownian

⁴Imposing the well-known Novikov condition is sufficient.

motion under P^e . The joint impact of effort on the probability measure links the agent problems and the principal problems.

The key idea is that the time- t collection of agent effort rates e_t changes the measure from P to P^e , such $dB_t^e = dB_t - \Phi(e_t)dt$ is Brownian motion under P^e . For example, if the i th principal's cash flow process X^i satisfies $dX_t^i = \mu_t dt + \sigma_t' dB_t$, then its drift under the collective effort process e is augmented, under P^e , by $\sigma_t' \Phi(e_t)$ because $dX_t^i = (\mu_t + \sigma_t' \Phi(e_t))dt + \sigma_t' dB_t^e$.

Definition 3 (Translation-invariant preferences) For any $(c^U, c^V, e) \in \mathcal{C}^N \times \mathcal{C}^N \times \mathcal{E}^N$, the agents' utility functions satisfy the following BSDEs:⁵

$$dU_t^i = - \left\{ h^{Ui} \left(t, x_t^{Ui} \right) + k^{Ui} \left(t, e_t^i, \Sigma_t^{Ui} \right) \right\} dt + \Sigma_t^{Ui'} dB_t^e, \quad U_T^i = c_T^{Ui}, \quad i = 1, \dots, N, \quad (2.4)$$

where

$$x_t^{Ui} = c_t^{Ui} - U_t^i, \quad t \in [0, T),$$

for some deterministic functions

$$h^{Ui} : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \text{ and } k^{Ui} : [0, T] \times \mathbf{E} \times \mathbb{R}^d \rightarrow \mathbb{R}.$$

The principals' utility functions satisfy the following BSDEs:

$$dV_t^i = - \left\{ h^{Vi} \left(t, x_t^{Vi} \right) + k^{Vi} \left(t, \Sigma_t^{Vi} \right) \right\} dt + \Sigma_t^{Vi'} dB_t^e, \quad V_T^i = c_T^{Vi}, \quad i = 1, \dots, N, \quad (2.5)$$

⁵ $(U^i, \Sigma^{Ui}) \in \mathbb{R} \times \mathbb{R}^d$ is the solution of BSDE (2.4). U^i is the utility value and Σ^{Ui} is the utility volatility. Similar explanation applies to (2.5).

where

$$x_t^{Vi} = c_t^{Vi} - V_t^i, \quad t \in [0, T],$$

for some functions

$$h^{Vi} : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \text{ and } k^{Vi} : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}.$$

It is easy to verify that, for any constant $v \in \mathbb{R}$

$$\begin{aligned} U_t^i(c^{Ui} + v, e) &= U_t^i(c^{Ui}, e) + v, & V_t^i(c^{Vi} + v, e) &= V_t^i(c^{Vi}, e) + v. \\ \Sigma_t^{Ui}(c^{Ui} + v, e) &= \Sigma_t^{Ui}(c^{Ui}, e), & \Sigma_t^{Vi}(c^{Vi} + v, e) &= \Sigma_t^{Vi}(c^{Vi}, e). \end{aligned} \quad (2.6)$$

We define the set of *intermediate control* as a convex set $\mathcal{H} \subseteq \mathcal{L}^-(\mathbb{R})$. We will use $x^U = (x^{U1}, \dots, x^{UN}) \in \mathcal{H}$ and $x^V = (x^{V1}, \dots, x^{VN}) \in \mathcal{H}$ as part of the controls of agents and principals respectively throughout the paper.

The following two examples give special cases of TI agent preferences. The case of the principal is analogous, but with no effort penalty. In both examples the effort-penalty function is given by $g : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ (typically assumed convex in e).

Example 2 (Risk-neutral agent) *If*

$$h^{Ui}(t, x) = \beta x, \quad k^{Ui}(t, \epsilon, \Sigma) = -g(t, \epsilon), \quad \beta > 0,$$

then time- t agent utility is

$$U_t = E_t^e \left\{ \int_t^T e^{-\beta(s-t)} \left(\beta c_s^i - g(s, e_s^i) \right) ds + e^{-\beta(T-t)} c_T^i \right\}.$$

Example 3 (Additive exponential) If, for some $\gamma > 0$,

$$h^{Ui}(t, x) = -\frac{1}{\gamma} \exp(-\gamma x), \quad k^{Ui}(t, \epsilon, \Sigma) = -\frac{\gamma}{2} \Sigma' \Sigma - \frac{1}{\gamma} g(t, \epsilon), \quad (2.7)$$

then the ordinally equivalent utility process $u_t = -\exp(-\gamma U_t)$ satisfies (assuming sufficient integrability)

$$\begin{aligned} u_t = & -E_t^e \left\{ \int_t^T \exp \left(- \left[\gamma c_s^i - \int_t^s g(w, e_w^i) dw \right] \right) ds \right. \\ & \left. + \exp \left(- \left[\gamma c_T^i - \int_t^T g(w, e_w^i) dw \right] \right) \right\}. \end{aligned}$$

That is u is a standard time-additive exponential utility with coefficient of absolute risk aversion parameter γ .

2.2.2 Outline of the Problem and Solution

To model moral hazard, it is assumed that the agents' effort processes are not contractible. However, effort can be manipulated by the principal through the pay process. At time-0, each principal i promises a pay process to agent i and selects his/her consumption plan. (The commitment could be enforced by some legal entity.) In response to the pay process, each agent i chooses effort and consumption processes to maximize his/her utility.

Let

$$\mathcal{G}_t = \sigma \left\{ e_s, W_s^U, 0 \leq s \leq t \right\}, \quad 0 \leq t \leq T.$$

We assume the pay process p_t is not adapted to the $\{\mathcal{G}_t\}$ filtration, i.e. there exists t such that $p_t \notin \mathcal{G}_t$. The practical meaning of this assumption is that p can not be expressed as a function of e and W^U .

We can think of the principal and agent choices as occurring in two stages. In the first stage, the principals simultaneously commit to a set of pay processes p and choose their own consumption processes. In the second stage, the agents simultaneously choose efforts and consumption processes.

In general terms, the problem is to describe a Nash equilibrium for the whole system. This means that in both levels of the agents and principals, there is a Nash equilibrium (see Definition 5 for agent equilibrium and Definition 8 for principal equilibrium).

The solution is obtained recursively beginning with the second-stage agent-effort equilibrium. For any joint pay processes p , each agent i chooses effort e^i and control x^{Ui} to maximize utility, $U_0^i(x^{Ui}, e^i)$, given the other agents' strategies (x^{U-i}, e^{-i}) . It turns out that the choice of agents' controls (x^U, e) is equivalent to the more natural choice (c^U, e) . This is why Definition 5 is formulated in terms of (x^U, e) .

In Section 2.3, we define the agents' subjective present value(PV) process (Y, Σ^Y) (see equation (2.9)) and show that the Nash equilibrium joint effort \hat{e} is determined by the agents' joint PV-diffusion processes Σ^Y . Technicalities aside, at time t the Nash equilibrium takes the form $\hat{e}_t = \hat{e}(t, \Sigma_t^Y)$ for some deterministic function $\hat{e}(\cdot) = (\hat{e}^1(\cdot), \dots, \hat{e}^N(\cdot))$.

Theorem 6 shows that a sufficient condition for $\hat{e}(\cdot)$ to be a Nash equilibrium is that, for each agent i and time t , the effort level \hat{e}_t^i maximizes the sum of the risk-effort preference

function plus the effort-induced increase in utility drift:⁶

$$\hat{e}^i(t, \Sigma^Y) \in \arg \max_{e^i \in \mathbf{E}} \Gamma_t k^{U^i} \left(t, e^i, \frac{\Sigma_t^{Y^i}}{\Gamma_t} \right) + \Sigma_t^{Y^i} \Phi(e^i, \hat{e}_t^{-i})$$

for all $\Sigma^Y \in \mathbb{R}^{d \times N}$ and $t \in [0, T]$.

This function is solved in closed form for all our applications. Furthermore, this effort equilibrium is dynamically consistent in the sense that the equilibrium determined by the agents at time-0 will also constitute an equilibrium at any time t in the future.

Having obtained the agent-effort equilibrium (\hat{x}^U, \hat{e}) , we then solve the first-stage principals' problem of choosing optimal pay and principal consumption. Rather than choose p^i directly, we specify the principal i strategies $(x_t^{V^i}, \Sigma_t^{Y^i})_{t < T}$, from which the pay process p^i is computed (as shown in Section 2.4). Choosing x^{V^i} is essentially equivalent to choosing intermediate consumption $\{c_t^{V^i}; t < T\}$.

For any x^{V^i} , we show that the joint choice of Σ^Y implies a unique class of payment plans. Given the form of the agent equilibrium, $\{\hat{e}(t, \Sigma_t^Y); t \in [0, T)\}$, the control Σ^{Y^i} is the natural choice to influence agent effort; it also greatly enhances the tractability of the problem. Taking as given the other principals' strategies $(x^{V^{-i}}, \Sigma^{Y^{-i}})$, principal i chooses (x^{V^i}, Σ^{Y^i}) to maximize utility $V_0^i(x^{V^i}, \hat{e}(t, \Sigma_t^Y))$ subject to agent i 's participation constraint $U_0^i(c^{U^i}, \hat{e}(t, \Sigma_t^Y)) \geq K^i$. Note that principal i 's utility depends on equilibrium efforts by all the agents, and the principal fully anticipates the impact of Σ^Y on the effort equilibrium.

In section 2.4, we define the principals' subjective present value process (Z, Σ^Z) (see equation (2.37)) and shows that the equilibrium strategies $\hat{\Sigma}^Y$ are interdependent and are

⁶We will assume, for simplicity, that the preference functions are deterministic (do not depend directly on ω), and therefore $\hat{e}^i(t, \Sigma)$ will not depend on ω .

solved as a Nash equilibrium as a function of the subjective cash flow diffusion processes Σ^Z .

A sufficient condition for $\hat{\Sigma}_t^Y = \hat{\Sigma}^Y(t, \Sigma^Z)$ to be a Nash equilibrium is

$$\begin{aligned} \hat{\Sigma}_t^{Yi} \in & \arg \max_{\Sigma \in \mathbb{R}^d} \Gamma_t \left\{ k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \Sigma}{\Gamma_t} \right) + k^{Ui} \left(t, \hat{e}^i(t, \Sigma, \hat{\Sigma}_t^{Y-i}) \right) \right\} \\ & + \Sigma_t^{Zi'} \Phi \left(\hat{e} \left(t, \Sigma, \hat{\Sigma}_t^{Y-i} \right) \right), \end{aligned}$$

for each principal i , and all $\Sigma^z \in \mathbb{R}^{d \times N}$ and $t \in [0, T]$.

Each principal i chooses the volatility $\hat{\Sigma}^{Yi}$ to transfer to the agent that maximizes the sum of the principal and agent risk-effort preference functions plus the impact of effort on the principal's utility drift, all holding fixed the controls of the other principals (yet anticipating the impact of the principal's own control on the equilibrium efforts).

The resulting Nash equilibrium among principals is then consistent with a Nash equilibrium among agents' efforts. The principal equilibrium is also time consistent, in the sense that the equilibrium determined by the principals time-0 will also constitute an equilibrium at any (ω, t) in the future after replacing K^i with time- t value of equilibrium agent utility processes U^i .

The final step is to compute the subjective PV processes. Theorem 7 shows that these are given by $Z_t^i = \Gamma_t V_t^i - W_t^{Vi} + Y_t^i$, $i = 1, \dots, N$, at the optimum. Because each cash flow X^i is split between the i th principal/agent pair, it is natural to add the agent PV process to determine the principal PV process. Maximizing principal i 's utility is equivalent to maximizing Z_0^i because the participation constraint binds.⁷ Defining the above drift

⁷Alternatively, we can interpret Z_t^i as a Lagrange multiplier process, incorporating the constraint on the agent's initial utility. It can be shown that the Lagrange multiplier, representing the sensitivity of principal utility to a unit change in agent utility, is always one

functions evaluated at the optimum

$$\hat{\mu}_t^{Zi} = H_t^i + \Gamma_t \left(k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \hat{\Sigma}_t^{Yi}}{\Gamma_t} \right) + k^{Ui} \left(t, \hat{e} \left(t, \hat{\Sigma}_t^Y \right), \frac{\hat{\Sigma}_t^{Yi}}{\Gamma_t} \right) \right) + \Sigma_t^{Zi'} \Phi \left(\hat{e} \left(t, \hat{\Sigma}_t^Y \right) \right),$$

where

$$H_t^i = \Gamma_t \left(h^{Vi}(t, \hat{x}_t^{Vi}) + h_t^{Ui}(t, \hat{x}_t^{Ui}) \right) - \hat{x}_t^{Vi} - \hat{x}_t^{Ui}.$$

Then (Z^i, Σ^{Zi}) , $i = 1, \dots, N$, solve the backward equation system

$$dZ_t^i = -(-r_t Z_t^i + X_t^i + \hat{\mu}_t^{Zi})dt + \Sigma_t^{Zi'} dB_t, \quad Z_T^i = X_T^i, \quad i = 1, \dots, N.$$

Given the solution to this system, we substitute to get the principals' controls and $\hat{\Sigma}^Y(t, \Sigma_t^Z)$. This yields equilibrium effort $\hat{e}(t, \hat{\Sigma}^Y(t, \Sigma_t^Z))$ and finally optimal pay, which is obtained by running the agent subject PV equation forward from its starting value after substituting the equilibrium policy $\hat{\Sigma}^{Yi}$.

2.2.3 Regularity Conditions and Feasibility

This section is purely technical, imposing regularity conditions on the aggregators, and defining the class of feasible consumption and effort plans to ensure existence and uniqueness of the utility functions.

It is assumed that the aggregators satisfy the following condition.

Condition 1 *For all $i = 1, \dots, N$, we have*

with TI preferences.

(a) *There exists a $\beta \in \mathbb{R}_+$ such that*

$$\left| h^{Ui}(t, y) - h^{Ui}(t, \hat{y}) \right| + \left| h^{Vi}(\omega, t, y) - h^{Vi}(\omega, t, \hat{y}) \right| \leq \beta |y - \hat{y}|,$$

for all $(\omega, t, y, \hat{y}) \in \Omega \times [0, T] \times \mathbb{R}^2$;

(b) *$k^{Vi}(\omega, t, \cdot)$ and $k^{Ui}(t, \bar{e}, \cdot)$ are concave functions for all $(\omega, t, \bar{e}) \in \Omega \times [0, T] \times \mathbf{E}$;*

(c) *Both $h^{Ui}(t, \cdot)$ and $h^{Vi}(\omega, t, \cdot)$ are increasing functions for all $(\omega, t) \in \Omega \times [0, T]$.*

Feasibility is defined as follows. Recall x^U and x^V from Definition 3.

Definition 4 *The set of intermediate controls and effort plans $(x^U, x^V, e) \in \mathcal{H}^N \times \mathcal{H}^N \times \mathcal{E}^N$ will be called feasible if there exist $\gamma, \beta \in \mathbb{R}_+$, a process $\alpha \in \mathcal{L}(\mathbb{R}_+)$ such that, for all $i = 1, \dots, N$,*

(a)

$$\begin{aligned} & \left| \tilde{h}^{Ui}(t, y) + k^{Ui}(t, e_t^i, z) \right| + \left| \tilde{h}^{Vi}(\omega, t, y) + k^{Vi}(\omega, t, z) \right| + |z' \Phi(e_t)| \\ & \leq \alpha_t + \beta |y| + \frac{\gamma}{2} |z|^2 \text{ for all } (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d, \end{aligned}$$

where $\tilde{h}^{Ui}(t, y) = h^{Ui}(t, c_t^{Ui} - y)$ and $\tilde{h}^{Vi}(\omega, t, y) = h^{Vi}(\omega, t, c_t^{Vi} - y)$.

(b) $\int_0^T \alpha(t) dt, \left| c_T^{Ui} \right|, \left| c_T^{Vi} \right| \in \mathcal{M}$,

*where c^{Ui} and c^{Vi} are the resulting consumption plans as explained in Section 2.2.2.*⁸

⁸By Corollary 6 of Briand and Hu (2008), there exists a unique solution (U^i, Σ^{Ui}) to the agent utility function in (2.4) and a unique solution (V^i, Σ^{Vi}) to the principal utility function in (2.5).

We denote the set of all feasible intermediate controls and effort plans by $\mathcal{H}_{Uf}^N \times \mathcal{H}_{Vf}^N \times \mathcal{E}_f^N$.

2.3 Agent Equilibrium

The main result in this section is Theorem 6 which gives a necessary and sufficient condition for agent equilibrium. Examples 4, 5 and 6 illustrate the result by obtaining the effort equilibrium in closed form (as a function of agent utility diffusion) for some combinations of quadratic and absolute effort penalties, and linear and concave measure-change operators Φ . In Section 2.3.3, we show that all the examples in Section 2.3.1 with two principal-agent pairs are special cases of the class of constant elasticity of substitution measure change operator Φ . We will impose the feasibility condition as in Definition 4 on any collection of intermediate controls and effort plans referred in this section and thus omit it from the text for simplicity.

2.3.1 A Necessary and Sufficient Condition for Agent Equilibrium

We first define the Nash equilibrium among agents. As explained in Section 2.2.2, the controls are chosen to be (x^U, e) which turns out to be equivalent to (c^U, e) .

Definition 5 (Agent Equilibrium) *Holding fixed a set of pay processes $p \in \mathcal{P}^N$, the set of agent control and effort plans (\hat{x}^U, \hat{e}) where $\hat{e} = \{\hat{e}^i, i = 1, \dots, N\}$ and $\hat{x}^U = \{\hat{x}^{Ui}, i = 1, \dots, N\}$ constitute an agent equilibrium (in the sense of Nash) if, for each $i = 1, \dots, N$,*

$$U_0^i(\hat{x}^{Ui}, \hat{e}) \geq U_0^i(x^{Ui}, (e^i, \hat{e}^{-i})), \text{ for any } e^i \in \mathcal{E}, x^{Ui} \in \mathcal{H}$$

To motivate the characterization of agent equilibrium in Theorem 6 below, define for any feasible policy (x^U, x^V, e) the difference between the dollar utility value and financial

wealth:⁹

$$Y_t^i(x^{Ui}, e) = \Gamma_t U_t^i(x^{Ui}, e) - W_t^{Ui}(x^{Ui}), \quad t \in [0, T]. \quad (2.8)$$

By Ito's lemma, and the identity $\Sigma_t^{Yi} = \Gamma_t \Sigma_t^{Ui}$, we get the dynamics of $Y_t^i = Y_t^i(x^{Ui}, p^i, e)$:

$$\begin{aligned} dY_t^i = & - \left\{ -r_t Y_t^i + p_t^i + \Gamma_t \left(h^{Ui}(t, x_t^{Ui}) + k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Ui} + \Sigma_t^{Yi} \Phi(e_t) \right\} dt \\ & + \Sigma_t^{Yi} dB_t, \quad Y_T^i = p_T^i, \quad i = 1, \dots, N. \end{aligned} \quad (2.9)$$

Theorem 6 (Agent Equilibrium) Assume (Y^i, Σ^{Yi}) solves the set of BSDEs (2.9).

Then a necessary and sufficient condition for (\hat{x}^U, \hat{e}) to constitute a Nash equilibrium among the agents is for each $i = 1, \dots, N$,

$$\begin{aligned} & \Gamma_t \left(h^{Ui}(t, \hat{x}_t^{Ui}) + k^{Ui} \left(t, \hat{e}_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - \hat{x}_t^{Ui} + \Sigma_t^{Yi} \Phi(\hat{e}_t) \\ & \geq \Gamma_t \left(h^{Ui}(t, x_t^{Ui}) + k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Ui} + \Sigma_t^{Yi} \Phi(e_t^i, \hat{e}_t^{-i}) \end{aligned} \quad (2.10)$$

$t \in [0, T]$, for any $e^i \in \mathcal{E}$, $x^{Ui} \in \mathcal{H}$.

Proof.

Sufficiency Holding fixed a set of and principal controls x^V and pay processes p , consider a set of agent control and effort plans (\hat{x}^U, \hat{e}) such that (2.10) is satisfied. Let h_t denote the nonnegative process that represents the following difference

⁹The wealth process $W^{Ui}(x^{Ui})$ satisfies (2.1) after substituting consumption $c_t^{Ui} = x_t^{Ui} + U_t^i(x^{Ui}, e)$.

$$h_t = \left\{ \Gamma_t \left(h^{Ui} \left(t, \hat{x}_t^{Ui} \right) + k^{Ui} \left(t, \hat{e}_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - \hat{x}_t^{Ui} + \Sigma_t^{Yi'} \Phi(\hat{e}_t) \right\} \\ - \left\{ \Gamma_t \left(h^{Ui} \left(t, x_t^{Ui} \right) + k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Ui} + \Sigma_t^{Yi'} \Phi(e_t^i, \hat{e}_t^{-i}) \right\}.$$

With $(\hat{x}^{Ui}, \hat{e}^i)$, the corresponding discounted process $D_t \hat{Y}^i$ solves the following BSDE

$$dD_t \hat{Y}_t^i = -D_t \left\{ h_t + p_t^i + \Gamma_t \left(h^{Ui} \left(t, x_t^{Ui} \right) + k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Ui} + \Sigma_t^{Yi'} \Phi(e_t^i, \hat{e}_t^{-i}) \right\} dt \\ + D_t \hat{\Sigma}_t^{Yi'} dB_t, \quad \hat{Y}_T^i = p_T^i.$$

On the other hand, for any $\{(x^{Ui}, \hat{x}^{U-i}), (e^i, \hat{e}^{-i})\}$ the corresponding discounted process $D_t Y^i$ solves the BSDE

$$dD_t Y_t^i = -D_t \left\{ p_t^i + \Gamma_t \left(h^{Ui} \left(t, x_t^{Ui} \right) + k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Ui} + \Sigma_t^{Yi'} \Phi(e_t^i, \hat{e}_t^{-i}) \right\} dt \\ + D_t \Sigma_t^{Yi'} dB_t, \quad Y_T^i = p_T^i.$$

Theorem 4 implies

$$Y_0^i(\hat{x}^{Ui}, \hat{e}) \geq Y_0^i(x^{Ui}, (e^i, \hat{e}^{-i})). \quad (2.11)$$

By the definition of Y process in (2.8) and the identical initial wealth, we have $U_0^i(\hat{x}^{Ui}, \hat{e}) \geq U_0^i(x^{Ui}, (e^i, \hat{e}^{-i}))$

Necessity Suppose $(\hat{x}^{Ui}, \hat{e}^i)$ is optimal for agent i given $(\hat{x}^{U-i}, \hat{e}^{-i})$ and (2.10) is violated by some (x^{Ui}, e^i) ; that is,

$$\begin{aligned} & \Gamma_t \left(h^{Ui} \left(t, \hat{x}_t^{Ui} \right) + k^{Ui} \left(t, \hat{e}_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - \hat{x}_t^{Ui} + \Sigma_t^{Yi'} \Phi(\hat{e}_t) < \\ & \Gamma_t \left(h^{Ui} \left(t, x_t^{Ui} \right) + k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Ui} + \Sigma_t^{Yi'} \Phi(e_t^i, \hat{e}_t^{-i}) \end{aligned} \quad (2.12)$$

on some subset of $\Omega \times [0, T]$ that belongs to $\mathcal{F} \times \mathcal{B}_{[0, T]}$ with a strictly positive $P \otimes \lambda$ measure.

Let

$$(\bar{x}_t^{Ui}, \bar{e}_t^i) = \begin{cases} (\hat{x}^{Ui}, \hat{e}^i) & \text{if (2.10) is true} \\ (x^{Ui}, e^i) & \text{if (2.12) is true} \end{cases}$$

Then comparison Theorem implies

$$Y_0^i(\hat{x}^{Ui}, \hat{e}) < Y_0^i(\bar{x}^{Ui}, (\bar{e}^i, \hat{e}^{-i}))$$

and $U_0^i(\hat{x}^{Ui}, \hat{e}) < U_0^i(\bar{x}^{Ui}, (\bar{e}^i, \hat{e}^{-i}))$, which contradicts the assumption.

■

From Theorem 6, at any equilibrium corresponding to p , for each $i = 1, \dots, N$ agent i 's time- t optimal control maximizes the (negative) time- t instantaneous drift of Y under the original measure that is

$$\begin{aligned} \hat{e}^i(t, \Sigma^Y) & \in \arg \max_{e^i \in \mathbf{E}} \Gamma_t k^{Ui} \left(t, e^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) + \Sigma_t^{Yi'} \Phi(e^i, \hat{e}_t^{-i}) \\ & \text{for all } \Sigma^Y = (\Sigma^{Y1}, \dots, \Sigma^{YN})' \in \mathbb{R}^{d \times N} \text{ and } t \in [0, T]. \end{aligned}$$

and

$$\hat{x}_t^{Ui} \in \arg \max_{x^i \in \mathbb{R}} \Gamma_t h^{Ui} \left(t, x^i \right) - x^i \quad (2.13)$$

The effort equilibrium at t is determined by only $\left(t, \hat{\Sigma}_t^Y \right)$. The optimal \hat{x}^{Ui} depends only on agent i 's own preferences and not the other agents. Thus in our setting, agent equilibrium is totally characterized by the equilibrium effort processes. At time t , the Nash equilibrium $\hat{e}_t = \hat{e} \left(t, \Sigma_t^Y \right)$ is some deterministic function of t and Σ_t^Y .

We will let $\hat{\mu}^{Yi}(t, \Sigma_t^Y)$ denote

$$\hat{\mu}^{Yi} \left(t, \Sigma_t^Y \right) = \Gamma_t \left(h^{Ui} \left(t, \hat{x}_t^{Ui} \right) + k^{Ui} \left(t, \hat{e}_t^i, \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - \hat{x}_t^{Ui} + \Sigma_t^{Yi'} \Phi(\hat{e}(t, \Sigma_t^Y)), \quad (2.14)$$

so the Y process defined in (2.8) at the equilibrium controls (\hat{x}^U, \hat{e}) follows the following set of BSDEs.

$$dY_t^i = - \left(-r_t Y_t^i + p_t^i + \hat{\mu}^{Yi} \left(t, \Sigma_t^Y \right) \right) dt + \Sigma_t^{Yi'} dB_t, \quad Y_T^i = p_T^i, \quad i = 1, \dots, N \quad (2.15)$$

With the $\hat{\mu}^Y$ of (2.14) and corresponding equilibrium policy functions \hat{e} , \hat{x}^U of (2.13), we can proceed directly to the principals' problem in Section 2.4, but for completeness, the solution of the equilibrium agent-policy sample paths

$\left\{ \hat{x}^U \left(t, \Sigma_t^Y \right), \hat{e} \left(t, \Sigma_t^Y \right); t \in [0, T] \right\}$ given p requires the solution (Y, Σ^Y) to the set of BSDEs (2.15), which then yields the Σ^Y to substitute into the policy functions.

We interpret Y as the subjective present value (PV) of pay process.¹⁰ Note that $\Gamma_t U_t^i(\hat{x}_t^{Ui}, \hat{e}_t)$ represents the dollar cost of financing the optimal excess consumption stream (relative to the zero-utility optimal consumption stream), and therefore $Y_t^i = \Gamma_t U_t^i(\hat{x}_t^{Ui}, \hat{e}_t) - W_t^{Ui}$ represents part of that cost financed by promised pay. The uncertainty driving (Y, Σ^Y) is entirely due to the pay process, because the agents' aggregators are deterministic functions. With \hat{x}^{Ui} of (2.13) and the solution Y^i of (2.15), agent i 's wealth process \hat{W}^{Ui} is obtained from (2.1) after substituting optimal intermediate consumption $\hat{c}_t^{Ui} = \hat{x}_t^{Ui} + (Y_t^i + \hat{W}_t^{Ui})/\Gamma_t$:

$$\hat{W}_0^{Ui} = w_0^{Ui}, \quad d\hat{W}_t^{Ui} = \left(\hat{W}_t^{Ui} \left(r_t - \frac{1}{\Gamma_t} \right) + p_t^i - \hat{x}_t^{Ui} - \frac{Y_t^i}{\Gamma_t} \right) dt. \quad (2.16)$$

Optimal lump-sum terminal consumption is $\hat{c}_T^{Ui} = \hat{W}_T^{Ui} + p_T^i$, and optimal agent utility is $U_t^i(\hat{c}_t^{Ui}, \hat{e}_t^i) = (Y_t^i + \hat{W}_t^{Ui})/\Gamma_t$. Equation (2.16) also shows that (x^U, e) are effective controls, because the process Y^i of (2.8) depends only on (x^{Ui}, p^i, e) .

The solution of (2.15) is not needed to solve the principal's problem, although we show in Section 2.4 that the principals must solve analogous BSDEs representing their subjective cash-flow PV process. We will assume that the equilibrium controls in (2.13) are well defined, as is the case in all our applications. For sufficient conditions for the existence of a solution, see Fudenberg and Tirole (1992).

A Nash equilibrium among the agents is obtained by defining the correspondence $\Gamma :$

$$[0, T] \times \mathbb{R}^{N \cdot d} \rightarrow \mathcal{B}(\mathbf{E}^N) \quad {}^{11}$$

¹⁰More precisely, if we let $Y^i(p)$ denote the solution corresponding to the pay process p , then we can interpret $Y_t^i(p) - Y_t^i(0)$ as the agent's time- t subjective value of pay.

¹¹In our Section 2.7 application there will be cases in which multiple time- t equilibria exist for a given Σ_t^Y (that is, $\Gamma(t, \Sigma_t^Y)$ is indeed set-valued). However, we also show that such a

$$\Gamma(t, \Sigma) = \left\{ \epsilon \in \mathbf{E}^N : \epsilon^i = I^i \left(t, \epsilon^{-i}, \Sigma^i \right), \quad i = 1, \dots, N \right\}. \quad (2.17)$$

where $I^i : [0, T] \times \mathbf{E}^{N-1} \times \mathbb{R}^d \rightarrow \mathbf{E}$ represents agent i 's optimal effort, given the other agents' efforts $\epsilon^{-i} \in \mathcal{E}^{N-1}$, i.e.¹²

$$I^i \left(t, \epsilon^{-i}, \Sigma^i \right) = \arg \max_{\epsilon^i \in \mathbf{E}} \left\{ \Gamma_t k^{Ui} \left(t, \epsilon^i, \frac{\Sigma^i}{\Gamma_t} \right) + \Sigma^{i'} \Phi(\epsilon^i, \epsilon^{-i}) \right\}.$$

2.3.2 Some Basic Examples of Agent Equilibrium

We now give some examples of agent equilibria assuming, for simplicity, additive separability of the risk-aversion and effort penalty terms:

$$k^{Ui}(t, \epsilon, \Sigma) = f^i(t, \Sigma) + g^i(t, \epsilon), \quad t \in [0, T], \quad i = 1, \dots, N, \quad (2.18)$$

for some concave $f^i, g^i : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$. Separability implies that the effort equilibria depend directly only on the effort-penalty functions g^i . The problem of finding equilibrium efforts is reduced to

$$\hat{e} \left(t, \Sigma_t^Y \right) = \arg \max_{e_t^i \in \mathbf{E}} \left\{ \Gamma_t g^i \left(t, e_t^i \right) + \Sigma_t^{Y'} \Phi(e_t) \right\}, \quad t \in [0, T]. \quad (2.19)$$

Examples 4 assume a linear measure-change operator, which implies that equilibrium agent- i effort depends only on agent- i utility diffusion, and not the diffusions of the other Σ_t^Y would never be part of an equilibrium strategy among the principals. That is why we define the function $\hat{e} \left(t, \Sigma_t^Y \right)$ which is some selection from $\Gamma \left(t, \Sigma_t^Y \right)$. In our applications, \hat{e} is always well defined.

¹²In all our applications below I is well-defined: the arg max in (2.3.1) exists and is unique.

agents. Examples 5 and 6 assume a concave measure-change operator, which generally implies that agent- i effort will depend on all the agent diffusion processes. The examples also illustrate the different implications of linear versus quadratic effort penalties. Unlike the case of a quadratic penalty, a linear penalty results in a threshold effect such that agent i 's time- t effort is zero for sufficiently small Σ_t^{Yi} compared to the other agents' time- t diffusion (in the concave case).

Example 4 (Quadratic effort penalties & linear Φ) Suppose (2.18) with a quadratic effort penalty and a linear measure-change operator:

$$g^i(t, e_t^i) = -\frac{1}{2} e_t^{i'} Q_t^{ei} e_t^i, \quad i = 1, \dots, N, \quad \Phi(e_t) = \sum_{i=1}^N \beta_t^i e_t^i,$$

where $\beta^i, Q^{ei} \in \mathcal{L}(\mathbb{R}^{d \times d})$, and Q^{ei} is assumed symmetric positive definite. Each agent's optimal effort in this case does not depend on the other agents' efforts, and there is a unique equilibrium $\hat{e}_t^i = \hat{e}^i(t, \Sigma_t^U)$, $i = 1, \dots, N$. Applying (2.19), we have

If $\mathbf{E} = \mathbb{R}^d$ (no constraints on effort) then $\hat{e}_t^i = (1/\Gamma_t) (Q_t^{ei})^{-1} \beta_t^{i'} \Sigma_t^{Yi}$.

If $\mathbf{E} = \mathbb{R}_+^d$ (nonnegative effort) then $\hat{e}_t^i = (1/\Gamma_t) \max\left(0, (Q_t^{ei})^{-1} \beta_t^{i'} \Sigma_t^{Yi}\right)$ (the maximum operator is applied to each dimension).

If $\mathbf{E} = \left\{v \in \mathbb{R}^d : v_1 = \dots = v_d\right\}$ (effort is restricted to be the same in every dimension) then¹³ $\hat{e}_t^i = \mathbf{1}_d \mathbf{1}_d' \beta_t^{i'} \Sigma_t^{U^i} / \Gamma_t (\mathbf{1}_d' Q_t^{ei} \mathbf{1}_d)$ where $\mathbf{1}_d$ is a length- d vector of 1s.

If, in Example 4, β_t^i is invertible for every i and $t \in [0, T]$ we can redefine effort as $\tilde{e}_t^i = \beta_t^i e_t^i$ (preserving the quadratic penalty form) giving $\Phi(\tilde{e}_t) = \sum_{i=1}^N \tilde{e}_t^i$. We henceforth use this normalization.

¹³The same result is obtained by letting $\mathbf{E} = \mathbb{R}^d$ and replacing obtained Q_t^{ei} with $(\mathbf{1}_d' Q_t^{ei} \mathbf{1}_d) I$ and β_t^i with $\beta_t^i \mathbf{1}_d \mathbf{1}_d'$, where I denotes an identity matrix.

In the next two examples the marginal impact of any agent's effort is diminishing in aggregate effort, which we model with a measure-change operator $\Phi(\cdot) = (\Phi_1(\cdot), \dots, \Phi_d(\cdot))'$ that is concave in aggregate effort along each dimension:.

$$\Phi_k(e_t) = \frac{\left(\sum_{i=1}^N e_k^i(t)\right)^{1-\delta}}{1-\delta}, \quad \delta \in [0, 1), \quad k = 1, \dots, d. \quad (2.20)$$

Unlike the additive case, equilibrium effort of agent i depends on the utility-diffusion processes of all agents.

Example 5 (Quadratic effort penalties & concave Φ) Suppose (2.18); $\mathbf{E} = \mathbb{R}_+^d$ (non-negative effort); a quadratic effort penalty

$$g^i(t, e_t^i) = -\frac{1}{2} \sum_{k=1}^d Q_k^{ei} e_k^i(t)^2, \quad i = 1, \dots, N,$$

with $Q_k^{ei} > 0$ for all i, k ; and the power measure-change operator (2.20).

Equilibrium agent- i 's effort in the k th dimension is uniquely given by

$$\hat{e}_k^i(\Sigma^Y) = \frac{(\Sigma_k^{Yi})^+}{\Gamma_t Q_k^{ei}} \left(\sum_{j=1}^N \frac{(\Sigma_k^{Yj})^+}{\Gamma_t Q_k^{ej}} \right)^{\frac{-\delta}{1+\delta}}, \quad k = 1, \dots, d. \quad (2.21)$$

Agent i 's effort is increasing in his/her own utility diffusion value but diminishing in the diffusion values of the others.

The final example combines a linear effort penalty with a concave measure-change operator, resulting in equilibria with only a single agent working in each dimension when

penalty-scaled diffusions are different, and multiple equilibria with more than one agent working in each dimension when penalty-scaled diffusions match.

Example 6 (Linear effort penalties & concave Φ) Suppose preferences satisfy (2.18);

$\mathbf{E} = \mathbb{R}_+^d$ (nonnegative effort); a linear effort penalty

$$g^i(t, e_t^i) = - \sum_{k=1}^d q_k^i e_k^i(t), \quad i = 1, \dots, N,$$

with $q_k^{ei} > 0$ for all i, k ; and the power measure-change operator (2.20). With two principal-agents pairs, $i \in \{a, b\}$, we obtain for each dimension k and time t (henceforth omitting time arguments) the following possible Nash equilibria $e_k \in \Gamma_k(t, \Sigma^Y)$ in efforts:¹⁴

$$\begin{aligned} e_k^a &= 0, & e_k^b &= \left\{ \left(\Sigma_k^{Yb} \right)^+ / \Gamma_t q_k^b \right\}^{1/\delta} & \text{if } \Sigma_k^{Ya} / q_k^a &\leq \Sigma_k^{Yb} / q_k^b \\ e_k^b &= 0, & e_k^a &= \left\{ \left(\Sigma_k^{Ya} \right)^+ / \Gamma_t q_k^a \right\}^{1/\delta} & \text{if } \Sigma_k^{Yb} / q_k^b &\leq \Sigma_k^{Ya} / q_k^a \\ e_k^a, e_k^b &> 0, & e_k^a + e_k^b &= \left\{ \left(\Sigma_k^{Ya} \right)^+ / \Gamma_t q_k^a \right\}^{1/\delta} & \text{if } \Sigma_k^{Ya} / q_k^a &= \Sigma_k^{Yb} / q_k^b \end{aligned} \quad (2.22)$$

The agent with the smaller scaled diffusion value will not work, whereas the agent with the larger value will work if that value is positive. Multiple time- t equilibria exist only if $\Sigma_k^{Ya} / q_k^a = \Sigma_k^{Yb} / q_k^b$ for some dimension k .¹⁵

¹⁴Given the other agents' efforts, optimal agent- i effort in dimension k is

$$I_k^i(t, e_k^{-i}, \Sigma_k^{Yi}) = \left(\left\{ \left(\Sigma_k^{Yi} \right)^+ / \Gamma_t q_k^i \right\}^2 - \sum_{j \neq i} e_k^j \right)^+.$$

¹⁵For $N > 2$ the results are analogous: Total effort in dimension k is

$$\sum_{k=1}^d e_k^i(t) = \left(\max_{i=1, \dots, N} \left(\frac{\Sigma_k^{Yi}}{\Gamma_t q_k^i} \right)^+ \right)^{1/\delta},$$

Derivation. See Section .2 in the Appendix. ■

The marginal cost of each agent i 's effort is constant, and the marginal benefit is the product of the agent's diffusion and the derivative of the common measure-change operator. Each working agent therefore equates the ratio of diffusion and penalty term to the same quantity: the inverse of the common derivative. Each agent whose ratio falls short of the maximum will find the fixed marginal cost of effort too high at any effort level, and will therefore shirk, free riding off the other agents' effort.

2.3.3 Constant Elasticity of Substitution(CES)

In this section, we define a new class of measure change operator: CES production function. The CES class deals with the case of two principal-agent pairs. It covers the linear and concave Φ in Section 2.3.1 as special cases and many other production functions in Economics. Lemma 1 below presents closed-form solutions for agent equilibrium with CES measure-change operator and quadratic effort penalty. We achieved additional relaxation of the restrictions on parameters by working with quasiconcavity and FOC in the proof of Lemma 1.

We assume two principal-agents pairs, $i \in \{a, b\}$ throughout this section.

Definition 6 (Constant Elasticity of Substitution) *For the measure change operator $\Phi(\cdot) = (\Phi_1(\cdot), \dots, \Phi_d(\cdot))$, we define*

$$\Phi_k(e) = \kappa \left\{ \alpha (e_k^a)^\gamma + (1 - \alpha) (e_k^b)^\gamma \right\}^{\frac{v}{\gamma}}, \quad \kappa > 0, \alpha \in (0, 1), 0 \neq \gamma \leq 2,$$

$$0 < v < 2, 1 \leq k \leq d.$$

and positive effort is exerted only by those agents whose utility to penalty ratio equals the maximum ratio.

Elasticity of substitution is $1/(1 - \gamma)$, and v is the elasticity of scale.

Some special cases follow:

Cobb Douglas production function:

$$\Phi_k(e) = \kappa \left\{ (e_k^a)^\alpha (e_k^b)^{1-\alpha} \right\}^v. \text{ This is achieved by letting } \gamma \longrightarrow 0.$$

Leontief production function (or perfect complements):

$$\Phi_k(e) = \kappa \min(e_k^a, e_k^b). \text{ This is achieved by letting } v = 1 \text{ and } \gamma \longrightarrow -\infty.$$

Infinite elasticity (linear production if $v = 1$; diminishing returns to scale if $v < 1$):

$$\Phi_k(e) = \left\{ e_k^a + e_k^b \right\}^v. \text{ This is achieved by letting } \gamma = 1, \alpha = \frac{1}{2}, \text{ and } \kappa = 2^v.$$

Lemma 1 Assume preferences satisfy (2.18); $\mathbf{E} = \mathbb{R}_+^d$ (nonnegative effort); a quadratic effort penalty

$$g^i(t, e_t^i) = -\frac{1}{2} \sum_{k=1}^d Q_k^{ei} e_k^i(t)^2,$$

with $Q_k^{ei} > 0$ for all i, k ; $\Phi(e)$ satisfies (2.23).

Suppose $\Sigma^{Ya}, \Sigma^{Yb} > 0$ and define

$$S^a = \frac{\alpha v \Sigma^{Ya}}{\Gamma_t Q^{ea}}, \quad S^b = \frac{(1 - \alpha) v \Sigma^{Yb}}{\Gamma_t Q^{eb}}.$$

a. $0 \neq \gamma < 2$. The unique Nash equilibrium among the agents' efforts is

$$\begin{aligned} e^a &= \kappa^{1/(2-v)} (S^a)^{1/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}}, \\ e^b &= \kappa^{1/(2-v)} (S^b)^{1/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}}. \end{aligned}$$

b. $\gamma = 2$. The Nash equilibria are

$$\begin{aligned} e^a &= \alpha^{\frac{v}{2(2-v)}} \left(\frac{v \Sigma^{Ya} \kappa}{\Gamma_t Q^a} \right)^{\frac{1}{2-v}}, \quad e^b = 0, \quad \text{if } \frac{\alpha \Sigma^{Ya}}{Q^a} > \frac{(1-\alpha) \Sigma^{Yb}}{Q^b}, \\ e^b &= (1-\alpha)^{\frac{v}{2(2-v)}} \left(\frac{v \Sigma^{Yb} \kappa}{\Gamma_t Q^b} \right)^{\frac{1}{2-v}}, \quad e^a = 0, \quad \text{if } \frac{\alpha \Sigma^{Ya}}{Q^a} < \frac{(1-\alpha) \Sigma^{Yb}}{Q^b}, \\ \alpha (e^a)^2 + (1-\alpha) (e^b)^2 &= \left(\frac{\alpha v \Sigma^{Ya} \kappa}{\Gamma_t Q^a} \right)^{\frac{2}{2-v}}, \quad \text{if } \frac{\alpha \Sigma^{Ya}}{Q^a} = \frac{(1-\alpha) \Sigma^{Yb}}{Q^b}. \end{aligned} \quad (2.23)$$

c.(Cobb Douglas) $\gamma \rightarrow 0$. The Nash equilibria are

$$\begin{aligned} e^a &= \kappa^{1/(2-v)} (S^a)^{\frac{2-(1-\alpha)v}{2(2-v)}} (S^b)^{\frac{(1-\alpha)v}{2(2-v)}}, \\ e^b &= \kappa^{1/(2-v)} (S^a)^{\frac{\alpha v}{2(2-v)}} (S^b)^{\frac{2-\alpha v}{2(2-v)}}, \end{aligned}$$

and the additional equilibrium $e^a = e^b = 0$.

d.(Leontief) $\gamma \rightarrow -\infty$. The Nash equilibria are

$$e^a = e^b = K, \text{ for any } K \in \left[0, \frac{\kappa}{\Gamma_t} \min \left(\frac{\Sigma^{Ya}}{Q^a}, \frac{\Sigma^{Yb}}{Q^b} \right) \right].$$

Proof.

By Theorem 6, each agent solves, for each dimension k ,

$$\max_{e_k^i \in \mathbf{E}} -\frac{\Gamma_t}{2} Q_k^{ei} (e_k^i)^2 + \Sigma_k^Y \kappa \left\{ \alpha (e_k^a)^\gamma + (1 - \alpha) (e_k^b)^\gamma \right\}^{\frac{v}{\gamma}}, \quad (2.24)$$

holding fixed the other's effort. Henceforth omit the k subscript. Consider agent a 's problem

(2.24). The first derivative of the RHS of (2.24) w.r.t effort is

$$\begin{aligned} \frac{\partial}{\partial e^a} &= -\Gamma_t Q^a e^a + \alpha v \Sigma^{Ya} \kappa \left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}^{\frac{v}{\gamma}-1} (e^a)^{\gamma-1} \\ &= -\Gamma_t Q^a e^a + \alpha v \Sigma^{Ya} (e^a)^{\gamma-1} \frac{\Phi(e)}{\left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}}. \end{aligned}$$

First suppose $\Sigma^{Ya} \leq 0$ then $\frac{\partial}{\partial e^a} \leq 0$. The maximum is attained by $e^a = 0$, if $\gamma \geq 0$. The maximum is unattainable, if $\gamma < 0$. In this case, the Nash equilibrium does not exist and instead the ϵ -Nash equilibrium exists. However, in all our applications, we have $\Sigma^{Yi} > 0$, $i \in \{a, b\}$.

Now suppose $\Sigma^{Ya} > 0$, $\Sigma^{Yb} > 0$.

The stationary point is positive, because $v > 0$. The second derivative is

$$\frac{\partial^2}{(\partial e^a)^2} = -\Gamma_t Q^a + \alpha v \Sigma^{Ya} \frac{\partial}{\partial e^a} \left[\left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}^{\frac{v}{\gamma}-1} (e^a)^{\gamma-1} \right]$$

where

$$\begin{aligned} & \frac{\partial}{\partial e^a} \left[\left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}^{\frac{v}{\gamma}-1} (e^a)^{\gamma-1} \right] \\ &= \frac{\Phi(e) (e^a)^{\gamma-2}}{\left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}^2} \left[(\gamma - 1) \left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\} + \alpha (v - \gamma) (e^a)^\gamma \right] \end{aligned}$$

A sufficient condition for concavity is that the square bracketed term is nonpositive. That is,

$$\alpha (v - 1) (e^a)^\gamma + (\gamma - 1) (1 - \alpha) (e^b)^\gamma \leq 0,$$

which is satisfied if $v, \gamma \leq 1$ and because of our parameter restrictions. So the FOCs are sufficient for optimality.

More generally, from

$$\frac{\partial}{\partial e^a} = e^a \left\{ -\Gamma_t Q^a + \alpha v \Sigma^{Y^a} (e^a)^{\gamma-2} \frac{\Phi(e)}{\{\alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma\}} \right\}.$$

we get *quasiconcavity* (A function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is called quasiconcave, if the upper-level set $\{x \in \mathbb{R}^n : f(x) \geq r\}$ is convex, for any $r \in \mathbb{R}$.) if the term in the brackets is decreasing in e^a , because then the objective function is either always decreasing or first increasing and then decreasing as e^a increases. Assuming $\Sigma^{Y^a} > 0$ the term in parenthesis is decreasing if and only if

$$h(e^a) = (e^a)^{\gamma-2} \frac{\Phi(e)}{\{\alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma\}} = \kappa (e^a)^{\gamma-2} \left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}^{\frac{v}{\gamma}-1}$$

decreasing in e^a . From

$$h'(e^a) = \kappa (e^a)^{\gamma-3} \left\{ \alpha (e^a)^\gamma + (1 - \alpha) (e^b)^\gamma \right\}^{\frac{v}{\gamma}-2} \left[(v - 2) \alpha (e^a)^\gamma + (\gamma - 2) (1 - \alpha) (e^b)^\gamma \right]$$

we get

$$h'(e^a) \leq 0 \quad \Longleftrightarrow \quad (v - 2) \alpha (e^a)^\gamma + (\gamma - 2) (1 - \alpha) (e^b)^\gamma \leq 0 \quad (2.25)$$

This holds if

$$\gamma, v \leq 2 \quad (\text{strictly for } e^a > 0 \text{ if } v < 2, \gamma \leq 2)$$

We now deal with each case separately.

a. $\gamma < 2$. The FOCs for the two agents can be written (if $\gamma < 2$)

$$\frac{(e^a)^{2-\gamma}}{S^a} = \frac{(e^b)^{2-\gamma}}{S^b} = \frac{\Phi(e)}{\alpha(e^a)^\gamma + (1-\alpha)(e^b)^\gamma} = \kappa \left\{ \alpha(e^a)^\gamma + (1-\alpha)(e^b)^\gamma \right\}^{\frac{v}{\gamma}-1},$$

Substituting out e^b yields

$$\frac{(e^a)^{2-\gamma}}{S^a} = \kappa (e^a)^{v-\gamma} \left\{ \alpha + (1-\alpha) \left(\frac{S^b}{S^a} \right)^{\gamma/(2-\gamma)} \right\}^{\frac{v}{\gamma}-1}$$

so the FOC implies

$$\hat{e}^a = \kappa^{1/(2-v)} (S^a)^{1/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}}.$$

Use the equality $\frac{(e^b)^{2-\gamma}}{S^b} = \frac{(e^a)^{2-\gamma}}{S^a}$ to get the solution

$$\hat{e}^b = \kappa^{1/(2-v)} (S^b)^{1/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}}$$

The next step is to verify that \hat{e}^a is indeed a unique maximum point given \hat{e}^b and vice versa.

This is achieved by noting that strict monotonicity of h (see (2.25)) implies $\frac{\partial}{\partial e^a} > 0$ for

any $e^a \in (0, \hat{e}^a)$ and $\frac{\partial}{\partial e^a} < 0$ for any $e^a \in (\hat{e}^a, \infty)$.

b. $\gamma = 2$ The first derivative of the RHS of agent a 's problem (2.24) w.r.t effort is

$$\frac{\partial}{\partial e^a} = e^a \left(-\Gamma_t Q^a + \alpha v \Sigma^{Y^a} \kappa \left\{ \alpha (e^a)^2 + (1 - \alpha) (e^b)^2 \right\}^{\frac{v}{2}-1} \right)$$

(it is easy to confirm quasiconcavity because $\left\{ \alpha (e_k^a)^2 + (1 - \alpha) (e_k^b)^2 \right\}^{\frac{v}{2}-1}$ is monotonically decreasing in e^a). The FOC (also sufficient) for a and b are, respectively,

$$\left(\frac{\alpha v \Sigma^{Y^a} \kappa}{\Gamma_t Q^a} \right)^{\frac{2}{2-v}} = \alpha (e^a)^2 + (1 - \alpha) (e^b)^2, \quad (2.26)$$

$$\left(\frac{(1 - \alpha) v \Sigma^{Y^b} \kappa}{\Gamma_t Q^b} \right)^{\frac{2}{2-v}} = \alpha (e^a)^2 + (1 - \alpha) (e^b)^2. \quad (2.27)$$

If $\alpha (e^a)^2 > \left(\frac{(1 - \alpha) v \Sigma^{Y^b} \kappa}{\Gamma_t Q^b} \right)^{\frac{2}{2-v}}$, then from (2.27) the optimal $e^b = 0$, (2.26) implies the first possible equilibrium below.

Similarly, the second possible equilibrium holds if $(1 - \alpha) (e^b)^2 > \left(\frac{\alpha v \Sigma^{Y^a} \kappa}{\Gamma_t Q^a} \right)^{\frac{2}{2-v}}$.

The last possible equilibrium follows if $\alpha (e^a)^2, (1 - \alpha) (e^b)^2 \leq \left(\frac{(1 - \alpha) v \Sigma^{Y^b} \kappa}{\Gamma_t Q^b} \right)^{\frac{2}{2-v}} = \left(\frac{\alpha v \Sigma^{Y^a} \kappa}{\Gamma_t Q^a} \right)^{\frac{2}{2-v}}$.

Thus all the possible equilibria are (2.23).

c.(Cobb Douglas) $\gamma \rightarrow 0$.

$$\begin{aligned}
& \lim_{\gamma \rightarrow 0} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}} \\
&= \exp \left[\lim_{\gamma \rightarrow 0} \frac{(v-\gamma)}{(2-v)} \left\{ \frac{\ln \left(\alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right)}{\gamma} \right\} \right] \\
&= \exp \left[\frac{v}{2(2-v)} \left\{ \alpha \ln S^a + (1-\alpha) \ln S^b \right\} \right] \\
&= (S^a)^{\frac{v\alpha}{2(2-v)}} (S^b)^{\frac{v(1-\alpha)}{2(2-v)}}
\end{aligned}$$

d.(Leontief) $\gamma \rightarrow -\infty$. If $\Phi(e) = \kappa \min(e^a, e^b)$, then

$$e^a = \min \left(\frac{\kappa \Sigma^{Ya}}{\Gamma Q^a}, e^b \right) \text{ and } e^b = \min \left(\frac{\kappa \Sigma^{Yb}}{\Gamma Q^b}, e^a \right).$$

Therefore any equal nonnegative effort level less than or equal to $\min \left(\frac{\kappa \Sigma^{Ya}}{\Gamma Q^a}, \frac{\kappa \Sigma^{Yb}}{\Gamma Q^b} \right)$ is an equilibrium.

■

The CES measure change operator with quadratic effort penalty includes power measure change operator with linear effort penalty as a special case. The notation change defined below transforms CES with quadratic penalty into Example 6.

First define $\tilde{e}_k^i = \sqrt{2e_k^i}$, $i \in \{a, b\}$.

For any $\delta \in [0, 1)$, let

$$\gamma = 2, \quad v = 1 - \delta, \quad \alpha = 1/2 \text{ and } \kappa = 1/(1 - \delta).$$

With the new notations, the measure change operator (2.23) agrees with (2.20).

The agent equilibrium solution in (2.23) agrees with (2.22).

2.4 Principal Equilibrium

Having solved the agent equilibrium efforts as function of the agent subjective pay PV diffusion, we now solve the equilibrium in the principal level. The principal equilibrium is defined based on agent equilibrium and subject to participation constraints. The participation constraint

$$U_0^i \geq K^i, \quad i = 1, \dots, N. \quad (2.28)$$

is equivalent to $Y_0^i \geq \Gamma_0 K^i - w_0^{Ui}, i = 1, \dots, N$.

To simplify notations, for any set of Σ^Y , we will let $\Pi(\Sigma^Y)$ denote the set of agent equilibrium efforts under Σ^Y , i.e.

$$\Pi(\Sigma^Y) = \left\{ e \in \mathcal{E}^N : \text{for any } t \in [0, T), e_t \in \Gamma(t, \Sigma_t^Y) \right\} \quad (2.29)$$

,

where $\Gamma(\cdot)$ is defined in (2.17).

Recall our definition, $x_t^{Vi} = c_t^{Vi} - V_t^i, t < T$.

We denote by

$$x^V = (x^{V1}, \dots, x^{VN}) \quad \text{and} \quad p = (p^1, \dots, p^N)$$

the collection of principals' strategies. As we explained in Section 2.2.2, choosing x^V is essentially equivalent to choosing c^V .

Because of the availability of money-market trading, the class of pay processes p that generates the same agents' and principals' utilities is not unique. This is well known, but shown in Lemma 2 below for completeness.

The following lemma shows that there is no unique optimal pay process because with unrestricted trading in the money-market security, both principal and agent are indifferent between shifting some intermediate pay to the money-market account, and modifying terminal pay. The next lemma applies to any principal-agent pair i , $i = 1, \dots, N$, so for simplicity we will omit the superscript i .

Lemma 2 *Suppose X finances (p, c^V) , and p finances c^U . Let \tilde{p}_t , $t < T$, be some intermediate pay process. If*

$$\tilde{p}_T = p_T + \int_0^T e^{\int_t^T r_s ds} (p_t - \tilde{p}_t) dt, \quad (2.30)$$

then X finances (\tilde{p}, c^V) , and \tilde{p} finances c^U . That is, the same agent and consumption streams are attained by investing the difference in intermediate pay in the money-market account and adding the terminal money-market balance to lump-sum terminal pay.

Proof. Fix the consumption processes c^U and c^V , and let W^k and \tilde{W}^k , $k \in \{U, V\}$, denote the money-market balances corresponding to pay processes p and \tilde{p} , respectively. Then $\Delta_t^i = W_t^i - \tilde{W}_t^i$, $i \in \{U, V\}$, satisfy

$$\begin{aligned} \Delta_0^U &= 0, & d\Delta_t^U &= \left(\Delta_t^U r_t + p_t - \tilde{p}_t \right) dt, & 0 &= \Delta_T^U + p_T - \tilde{p}_T, \\ \Delta_0^V &= 0, & d\Delta_t^V &= \left(\Delta_t^V r_t - p_t + \tilde{p}_t \right) dt, & 0 &= \Delta_T^V - p_T + \tilde{p}_T, \end{aligned}$$

which have the solutions

$$\Delta_t^U = \int_0^t e^{\int_s^t r_u du} (p_s - \tilde{p}_s) ds, \quad \Delta_t^V = -\Delta_t^U, \quad t \in [0, T].$$

If (2.30) holds then (X, p) and (X, \tilde{p}) finance the same principal and agent consumption processes.

■

We will say two feasible pay processes p and \tilde{p} are **equivalent** if (2.30) is satisfied.

Based on the above lemma, the set of pay \mathcal{P} is a union of mutually exclusive equivalent classes. We will treat the pay processes that belong to the same equivalent class as the same throughout the paper. Any pay process can be replaced by its peers in the same equivalent class.

Denote by $V_0^i(x^{Vi}, p, e)$ the initial principal i utility, given by the solution of the BSDE (2.5). We now give our initial formulation of principal equilibrium. The final formulation will appear later, see Definition 8.

An initial formulation of principal equilibrium: The set of principal strategies $\hat{p} =$

$\{\hat{p}^i, i = 1, \dots, N\} \in \mathcal{P}^N$ and $\hat{x}^V = \{\hat{x}^{Vi}, i = 1, \dots, N\} \in \mathcal{H}^N$ constitute a principal equilibrium (in the sense of Nash), if for all $(x^V, p) \in \mathcal{H}^N \times \mathcal{P}^N$ and each $i = 1, \dots, N$,

$$V_0^i(\hat{x}^{Vi}, \hat{p}, \hat{e}) \geq V_0^i(x^{Vi}, (p^i, \hat{p}^{-i}), e),$$

given that \hat{p} and (p^i, \hat{p}^{-i}) induce agent equilibrium efforts \hat{e} and e , respectively. Also the participation constraint (2.28) is satisfied. As we saw in Section 2.3.1, agent equilibrium

and participation constraint can be expressed by the following:

$$\begin{aligned}\hat{e} &\in \Pi(\Sigma^Y(\hat{p})), \\ e &\in \Pi(\Sigma^Y(p^i, \hat{p}^{-i})) \quad \text{and}\end{aligned}$$

$$Y_0^i(\tilde{p}) \geq \Gamma_0 K^i - w_0^{Ui}, \quad \tilde{p} \in \left\{ \hat{p}, (p^i, \hat{p}^{-i}) \right\}$$

where $(Y^i = Y^i(\tilde{p}), \Sigma^{Yi} = \Sigma^{Yi}(\tilde{p})), i = 1, \dots, N$ satisfy

$$dY_t^i = - \left(-r_t Y_t^i + \tilde{p}_t^i + \hat{\mu}^{Yi} \left(t, \Sigma_t^Y \right) \right) dt + \Sigma_t^{Yi'} dB_t, \quad Y_T^i = \tilde{p}_T^i, \quad i = 1, \dots, N,$$

and $\hat{\mu}^{Yi}(\cdot)$ is defined in (2.14).

In the formulation above, (Y^i, Σ^{Yi}) represents the agent i 's subjective PV process with pay plan \tilde{p} and evaluated at the agent equilibrium solution. As we discussed in Section 2.3.1, the uncertainty driving (Y, Σ^Y) is entirely due to the set of pay processes \tilde{p} , in addition to the driving Brownian motion B .

We follow the solution approach as in the single principal/agent case examined in Schroder (2013). The paper shows that letting principals choose (x^V, p) is essentially equivalent to letting principals choose $(\hat{x}^V, \hat{\Sigma}^Y)$. By using the strategy (x^{Vi}, Σ^{Yi}) , the principal's problem is amenable to a simple dynamic programming solution. Furthermore, with this choice of principal strategy, principal i need not consider whether his/her strategies will impact the

participation constraints of other principal-agent pairs¹⁶; all participation constraints are shown to be binding under the TI preferences. The added complexity of the multiple principal problem is the interdependence of the problems and the determination of the equilibrium strategies among principals and agents.

The first step in applying the dynamic programming approach is to confirm that the participation constraints will bind at principal equilibrium.

Lemma 3 *The participation constraints all bind in any principal equilibrium.*

Proof. Suppose $(\hat{x}^V, \hat{p}) \in \mathcal{H}^N \times \mathcal{P}^N$ is a principal equilibrium, and there exists i , such that under the resulting agent equilibrium (\hat{x}^U, \hat{e}) , $U_0^i(\hat{x}^{U^i}, \hat{e}) > K^i$. Let (\hat{c}^U, \hat{c}^V) be the implied sets of agents' and principals' consumptions and $\epsilon = U_0^i(\hat{x}^{U^i}, \hat{e}) - K^i$. By the wealth equations (2.1) and (2.2), the same principal cash flow process would finance a principal consumption rate $\hat{c}^{V^i} + \epsilon$ and a pay plan rate of $\hat{p}^i - \epsilon$. This pay plan would finance an agent consumption rate $\hat{c}^{U^i} - \epsilon$ and , while Σ^Y , and therefore $\hat{e}(t, \Sigma_t^Y)$ remain unchanged¹⁷. By quasilinearity (2.6), the resulting agent i 's utility is $U_0^i(\hat{c}^{U^i}, \hat{e}) = K^i$, a binding participation constraint. The resulting principal i 's utility is $V_0^i(\hat{c}^{V^i}, \hat{e}) + \epsilon > V_0^i(\hat{c}^{V^i}, \hat{e})$. Therefore (\hat{x}^V, \hat{p}) can not be a principal equilibrium. ■

Upon substituting the binding participation constraints $U_0^i = K^i$, which implies $Y_0^i =$

¹⁶If a strategy choice by principal i were to cause a violation of agent j 's participation constraint, then principal i would have to account for the effect of agent j 's rejection of the contract.

¹⁷This can be seen from (2.15). Because Y^i is in the TI form, a constant change in pay process results a deterministic change in Y^i and Σ^{Y^i} remains unchanged.

$\Gamma_0 K^i - w_0^{Ui}$, into (2.15), we have the following set of forward equations

$$dY_t^i = - \left(-r_t Y_t^i + p_t^i + \hat{\mu}^{Yi}(t, \Sigma_t^Y) \right) dt + \Sigma_t^{Yi'} dB_t, \quad Y_0^i = \Gamma_0 K^i - w_0^{Ui}, \quad i = 1, \dots, N \quad (2.31)$$

where $\hat{\mu}^{Yi}(t, \Sigma_t^Y)$ is given by (2.14). Observe that the agent equilibrium solution (\hat{x}^U, \hat{e}) is part of (2.14), so it is already built into (2.31).

With a binding participation constraint, for principal i , choosing p^i is essentially equivalent to choosing Σ^{Yi} . Once the set of Σ^Y has been chosen, principal i can choose any feasible intermediate pay process $(p_s^i, s < T)$ is feasible, if $\int_0^T p_s^i ds \in \mathcal{M}$.) with terminal pay implied by (2.31) i.e. for any Σ^Y and feasible intermediate pay process $p_t^i, t < T$, the terminal pay is

$$p_T^i = Y_0^i e^{\int_0^T r_s ds} - \int_0^T e^{\int_t^T r_s ds} \left[p_t^i + \hat{\mu}^{Yi}(t, \Sigma_t^Y) \right] dt + \int_0^T e^{\int_t^T r_s ds} \Sigma_t^{Yi'} dB_t \quad (2.32)$$

Each set of Σ^Y corresponds to a class of equivalent pay processes for each principal i . This is because for any other feasible intermediate pay processes $\tilde{p}_t, t < T$, let the terminal pays \tilde{p}_T^i be implied by the forward equation (2.31) with the same initial value $Y_0^i = \Gamma_0 K^i - w_0^{Ui}$ and $\Sigma_t^Y, t < T$ chosen by the N principals as in (2.32), i.e.

$$\tilde{p}_T^i = Y_0^i e^{\int_0^T r_s ds} - \int_0^T e^{\int_t^T r_s ds} \left[\tilde{p}_t^i + \hat{\mu}^{Yi}(t, \Sigma_t^Y) \right] dt + \int_0^T e^{\int_t^T r_s ds} \Sigma_t^{Yi'} dB_t \quad (2.33)$$

Upon subtracting (2.32) from (2.33), we see that (2.30) is satisfied. Thus p and \tilde{p} are equivalent.

Based on the above argument, we will define Σ^Y as principals' strategies instead of p .

From now on, we denote by

$$x^V = (x^{V1}, \dots, x^{VN}) \quad \text{and} \quad \Sigma^Y = (\Sigma^{Y1}, \dots, \Sigma^{YN})$$

the collection of principals' strategies.

The following definition specifies the set of implementable principal strategies such that the resulting equilibrium effort and control plans are feasible.

Definition 7 *For a set of principals' strategies $(x^V, \Sigma^Y) \in \mathcal{H}^N \times \mathcal{L}_2(\mathbb{R}^d)^N$, let (\hat{x}^U, \hat{e}) be the resulting agent equilibrium control and effort plans in (2.13).*

If $(\hat{x}^U, x^V, \hat{e}) \in \mathcal{H}_{Uf}^N \times \mathcal{H}_{Vf}^N \times \mathcal{E}_f^N$, we will call (x^V, Σ^Y) implementable and denote the set of the principals' implementable strategies by \mathcal{I}^N .

We will impose the above implementability condition on any collection of principals' strategies and thus omit it from the text. Each principal i chooses the optimal (x^{Vi}, Σ^{Yi}) to maximize utility holding fixed the strategies of the other principals, (x^{V-i}, Σ^{Y-i}) , while anticipating the impact of their strategy on the equilibrium efforts of all the agents.

Based on the explanation above, for any set of principals' strategies (x^V, Σ^Y) , we will let $\Psi^i(\Sigma^Y)$ denote the class of equivalent pay plans of principal i induced by Σ^Y , i.e.

$$\Psi^i(\Sigma^Y) = \left\{ p^i \in \mathcal{P} : p_T^i \text{ satisfies (2.32) for any feasible } p_t^i, t < T \right\} \quad (2.34)$$

We will also let $\Lambda(\Sigma^Y)$ denote the pay plans of all principals induced by Σ^Y , i.e.

$$\Lambda(\Sigma^Y) = \left\{ p \in \mathcal{P}^N : p^i \in \Psi^i(\Sigma^Y), i = 1, \dots, N \right\} \quad (2.35)$$

Since we have used (x^V, Σ^Y) as principals' strategies, we give the following appropriate Definition for principal equilibrium.

Definition 8 (Principal Equilibrium) *A set of strategies $(\hat{x}^V, \hat{\Sigma}^Y) \in \mathcal{H}^N \times \mathcal{L}_2(\mathbb{R}^d)^N$ constitutes a principal equilibrium, if for all $(x^{Vi}, \Sigma^{Yi}) \in \mathcal{H} \times \mathcal{L}_2(\mathbb{R}^d)$ and each $i = 1, \dots, N$,*

$$V_0^i(\hat{x}^{Vi}, \hat{\Sigma}^Y) \geq V_0^i(x^{Vi}, \Sigma^{Yi}, \hat{\Sigma}^{Y-i}),$$

subject to

$$V_0^i(\hat{x}^{Vi}, \hat{\Sigma}^Y) = V_0^i(\hat{x}^{Vi}, \hat{p}, \hat{e}), \quad \text{where } \hat{e} \in \Pi(\hat{\Sigma}^Y) \text{ and } \hat{p} \in \Lambda(\hat{\Sigma}^Y),$$

and

$$V_0^i(x^{Vi}, \Sigma^{Yi}, \hat{\Sigma}^{Y-i}) = V_0^i(x^{Vi}, p, e), \quad \text{where } e \in \Pi(\Sigma^{Yi}, \hat{\Sigma}^{Y-i}) \text{ and } p \in \Lambda(\Sigma^{Yi}, \hat{\Sigma}^{Y-i}).$$

The main result of the section is Theorem 7 and its corollary below, which show that the determination of the equilibrium pay contracts reduces to the computation of a set of subjective present-value (PV) processes and associated diffusion processes for the principals, (Z^i, Σ^{Zi}) , $i = 1, \dots, N$ (see (2.36) for the definition). The principals' equilibrium controls $\hat{\Sigma}^Y$ take the form $\hat{\Sigma}_t^Y = \hat{\Sigma}^Y(\omega, t, \Sigma^Z)$. The equilibrium time- t diffusion that principal

i chooses for his/her agent is a function of the time- t subjective PV diffusion of all the principals. In the case of a linear measure-change operator Φ , the equilibrium simplifies because $\hat{\Sigma}_t^{Yi}$ depends only i th PV diffusion value Σ_t^{Zi} . But even in the linear case, the principals' optimal contracts are linked because the subjective PV processes Σ^Z must be solved jointly as the drift of each depends on the diffusion processes of the others. This follows because even though agent i 's effort under the equilibrium compensation contract will depend only on Σ_t^{Yi} , this effort affects the cash-flow drifts and therefore the subjective PV processes of all the principals.

To motivate the solution of principal equilibrium, for any implementable principal policy (x^V, Σ^Y) define for principal i , $Z_t^i(x^{Vi}, \Sigma^Y)$ as his/her subjective PV process, which is dollar utility value plus the subjective pay liability minus the principal's financial wealth:

$$Z_t^i(x^{Vi}, \Sigma^Y) = \Gamma_t V_t^i(x^{Vi}, \Sigma^Y) - W_t^{Vi}(x^{Vi}) + Y_t^i(\Sigma^Y). \quad (2.36)$$

Applying Ito's lemma, with (2.5), (2.31), and (2.2), we get the dynamics of $Z_t^i = Z_t^i(x^{Vi}, \Sigma^Y)$:

$$dZ_t^i = -\left(-r_t Z_t^i + X_t^i + \mu_t^{Zi}(x^{Vi}, \Sigma^Y)\right) dt + \Sigma_t^{Zi'} dB_t, \quad Z_T^i = X_T^i, \quad i = 1, \dots, N. \quad (2.37)$$

where

$$\begin{aligned} \mu_t^{Zi}(x^{Vi}, \Sigma^Y) &= \Gamma_t \left(h^{Vi}(t, x_t^{Vi}) + k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \Sigma_t^{Yi}}{\Gamma_t} \right) \right) - x_t^{Vi} \\ &+ \Gamma_t \left(h^{Ui}(t, \hat{x}_t^{Ui}) + k^{Ui} \left(t, \hat{e}(t, \Sigma_t^Y), \frac{\Sigma_t^{Yi}}{\Gamma_t} \right) \right) - \hat{x}_t^{Ui} + \Sigma_t^{Zi'} \Phi(\hat{e}(t, \Sigma_t^Y)), \end{aligned} \quad (2.38)$$

$$\Sigma_t^{Zi} = \Gamma_t \Sigma_t^{Vi} + \Sigma_t^{Yi}, \quad (2.39)$$

and \hat{x}_t^{Ui} and $\hat{e}(t, \Sigma_t^Y)$ are the agent equilibrium given by Theorem 6 for any set of Σ^Y .

Theorem 7 below shows that each principal's optimality is essentially equivalent to maximizing the drift term of his/her cash-flow PV processes.

Theorem 7 (Principal optimality) *Suppose (Zi, Σ^{Zi}) , $i = 1, \dots, N$ solves the BSDE system (2.37) with a set of principals' strategies $(x^V, \Sigma^Y) \in \mathcal{H}^N \times \mathcal{L}_2(\mathbb{R}^d)^N$. Then $(x^{Vi}, \Sigma^{Yi}) \in \mathcal{H} \times \mathcal{L}_2(\mathbb{R}^d)$ is optimal for principal i holding fixed other principals' strategies, i.e. $V_0^i(x^{Vi}, \Sigma^Y) \geq V_0^i(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi}, \Sigma^{Y-i})$, if and only if for any other strategy $(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi}) \in \mathcal{H} \times \mathcal{L}_2(\mathbb{R}^d)$,*

$$\mu_t^{Zi}(x^{Vi}, \Sigma^Y) \geq \mu_t^{Zi}(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi}, \Sigma^{Y-i}), \quad t \in [0, T]. \quad (2.40)$$

where $\mu_t^{Zi}(\cdot)$ is defined in (2.38).

Proof.

Sufficiency Consider a set of principals' strategies (x^V, Σ^Y) and let the process Z^i be defined as in (2.36) with its dynamics specified by (2.37) and (2.38). If principal i 's strategy (x^{Vi}, Σ^{Yi}) is switched to $(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi})$, analogously define

$$\tilde{Z}_t^i(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi}, \Sigma^{Y-i}) = \Gamma_t V_t^i(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi}, \Sigma^{Y-i}) - W_t^{Vi}(\tilde{x}^{Vi}) + Y_t^i(\tilde{\Sigma}^{Yi}, \Sigma^{Y-i}).$$

and it has the following dynamics

$$d\tilde{Z}_t^i = - \left(-r_t \tilde{Z}_t^i + X_t^i + \mu_t^{Zi}(\tilde{x}^{Vi}, \tilde{\Sigma}^{Yi}, \Sigma^{Y-i}) \right) dt + \tilde{\Sigma}_t^{Zi'} dB_t, \quad \tilde{Z}_T^i = X_T^i.$$

Using (2.40), define the nonnegative process

$$h_t = \mu_t^{Z^i} \left(x^{Vi}, \Sigma^Y \right) - \mu_t^{Z^i} \left(\tilde{x}^{Vi}, \tilde{\Sigma}^{Y^i}, \Sigma^{Y-i} \right), \quad t < T.$$

The discounted processes $D_t Z_t^i$ and $D_t \tilde{Z}_t^i$ follow the dynamics below.

$$\begin{aligned} dD_t Z_t^i &= -D_t(h_t + X_t^i + \tilde{\mu}_t^{Z^i})dt + D_t \Sigma_t^{Z^i} dB_t, \quad Z_T^i = X_T^i, \\ dD_t \tilde{Z}_t^i &= -D_t(X_t^i + \tilde{\mu}_t^{Z^i})dt + D_t \tilde{\Sigma}_t^{Z^i} dB_t, \quad \tilde{Z}_T^i = X_T^i. \end{aligned} \quad (2.41)$$

The comparison theorem (Theorem 5 of Briand and Hu (2008)) implies

$$Z_0^i \geq \tilde{Z}_0^i.$$

By the definition of Z and \tilde{Z} , the identical initial financial wealth and a binding participation constraint for both, we have $V_0^i(x^{Vi}, \Sigma^Y) \geq V_0^i(\tilde{x}^{Vi}, \tilde{\Sigma}^{Y^i}, \Sigma^{Y-i})$.

Necessity Suppose that (x^{Vi}, Σ^{Y^i}) is the optimal strategy for principal i and there exists some other strategy $(\tilde{x}^{Vi}, \tilde{\Sigma}^{Y^i})$ such that the process $h < 0$ on some subset of $\Omega \times [0, T]$ that belongs to $\mathcal{F} \times \mathcal{B}_{[0, T]}$ with a strictly positive $P \otimes \lambda$ measure. Here h is defined the same as in the sufficiency part. We define

$$(\bar{x}^{Vi}, \bar{\Sigma}^{Y^i}) = \begin{cases} (x^{Vi}, \Sigma^{Y^i}) & \text{if } h \geq 0, \\ (\tilde{x}^{Vi}, \tilde{\Sigma}^{Y^i}) & \text{otherwise.} \end{cases} \quad (2.42)$$

Analogously, define

$$\bar{Z}_t^i(\bar{x}^{Vi}, \bar{\Sigma}^{Y^i}, \Sigma^{Y-i}) = \Gamma_t V_t^i(\bar{x}^{Vi}, \bar{\Sigma}^{Y^i}, \Sigma^{Y-i}) - W_t^{Vi}(\bar{x}^{Vi}) + Y_t^i(\bar{\Sigma}^{Y^i}, \Sigma^{Y-i}).$$

then the discounted process $D_t \bar{Z}_t$ solves

$$dD_t \bar{Z}_t^i = -D_t \left(X_t^i + \mu_t^{Zi} \left(\bar{x}^{Vi}, \bar{\Sigma}^{Yi}, \Sigma^{Y-i} \right) \right) dt + D_t \bar{\Sigma}_t^{Zi'} dB_t, \quad \bar{Z}_T^i = X_T^i.$$

By (2.42), $\mu_t^{Zi} \left(\bar{x}^{Vi}, \bar{\Sigma}^{Yi}, \Sigma^{Y-i} \right) > \mu_t^{Zi} \left(x^{Vi}, \Sigma^Y \right)$ on some subset of $\Omega \times [0, T]$ with a strictly positive measure. Upon comparing the above BSDE with the BSDE for $D_t Z_t^i$ in (2.41), the comparison Theorem implies $Z_0^i < \bar{Z}_0^i$ and thus $V_0^i(x^{Vi}, \Sigma^Y) < V_0^i(\bar{x}^{Vi}, \bar{\Sigma}^{Yi}, \Sigma^{Y-i})$, which contradicts the optimality of (x^{Vi}, Σ^{Yi}) .

■

The corollary below characterizes principal equilibrium based on Theorem 7.

Corollary 1 (Principal equilibrium) *Suppose $(\hat{x}^V, \hat{\Sigma}^Y) \in \mathcal{H}^N \times \mathcal{L}_2(\mathbb{R}^d)^N$ satisfies*

$$\begin{aligned} \hat{x}_t^{Vi} &\in \arg \max_{x \in \mathbb{R}} \Gamma_t h^{Vi}(t, x) - x, \\ \hat{\Sigma}_t^{Yi} &\in \arg \max_{\Sigma \in \mathbb{R}^d} \Gamma_t \left\{ k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \Sigma}{\Gamma_t} \right) + k^{Ui} \left(t, \hat{e}^i(t, \Sigma, \hat{\Sigma}_t^{Y-i}), \frac{\Sigma}{\Gamma_t} \right) \right\} \\ &\quad + \Sigma_t^{Zi'} \Phi \left(\hat{e} \left(t, \Sigma, \hat{\Sigma}_t^{Y-i} \right) \right) \quad i = 1, \dots, N, \quad \text{all } (\omega, t) \in \Omega \times [0, T], \end{aligned} \tag{2.43}$$

where (Z, Σ^Z) solves the BSDE system

$$dZ_t^i = -(-r_t Z_t^i + X_t^i + \hat{\mu}^{Zi}(t, \Sigma_t^Z))dt + \Sigma_t^{Zi'} dB_t, \quad Z_T^i = X_T^i, \quad i = 1, \dots, N. \tag{2.44}$$

where we have defined

$$\begin{aligned}\hat{\mu}_t^{Zi} = & H_t^i + \Gamma_t \left(k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \hat{\Sigma}_t^{Yi}}{\Gamma_t} \right) + k^{Ui} \left(t, \hat{e} \left(t, \hat{\Sigma}_t^Y \right), \frac{\hat{\Sigma}_t^{Yi}}{\Gamma_t} \right) \right) \\ & + \Sigma_t^{Zi} \Phi \left(\hat{e} \left(t, \hat{\Sigma}_t^Y \right) \right),\end{aligned}\tag{2.45}$$

$$H_t^i = \Gamma_t \left(h^{Vi}(t, \hat{x}_t^{Vi}) + h_t^{Ui}(t, \hat{x}_t^{Ui}) \right) - \hat{x}_t^{Vi} - \hat{x}_t^{Ui},$$

and $\hat{x}_t^{Ui}, \hat{e} \left(t, \hat{\Sigma}_t^Y \right)$ are the agent equilibrium given by Theorem 6 for the set $\hat{\Sigma}^Y$.

Then $(\hat{x}^V, \hat{\Sigma}^Y)$ is a set of equilibrium strategies among the principals.

The equilibrium x_t^{Vi} only depends on principal i 's own preferences, while $\hat{\Sigma}_t^{Yi}$ depends on the diffusion of cash-flow PV of all the principals, Σ^Z . The equilibrium among the principals is constructed by first solving (2.43) for \hat{x}_t^{Vi} and $\hat{\Sigma}_t^{Yi}$; the equilibrium thus takes the form

$$\hat{x}^{Vi}(\omega, t) \quad \text{and} \quad \hat{\Sigma}_t^{Yi} = \hat{\Sigma}^{Yi} \left(\omega, t, \Sigma_t^Z \right), \quad t \in [0, T], \quad i = 1, \dots, N,\tag{2.46}$$

for some functions $\hat{x}^{Vi} : \Omega \times [0, T] \rightarrow R$ and $\hat{\Sigma}^{Yi} : \Omega \times [0, T] \times \mathbb{R}^{d \times N} \rightarrow \mathbb{R}$. These are substituted into (2.45) to obtain the BSDE system (2.44). In Section 2.5, we obtain closed-form solutions for (2.44), when the cash-flow dynamics for X^i are driven by affine-yield state variables. The functions $\hat{x}^V(\cdot)$ and $\hat{\Sigma}^Y(\cdot)$ evaluated at the BSDE solution (Z, Σ^Z) yields a principal equilibrium.

Note that the final "payoff" of Z_T^i is principal i 's lump-sum terminal cash flow, and the intermediate cash flow enters the drift of Z^i . If the agent and principal preference functions do not depend explicitly on ω as in all our applications, then the only source of

uncertainty driving the Z^i 's is the cash-flow uncertainty. This justifies the interpretation of Z^i as principal i 's subjective PV of the cash-flow process X^i .

The subjective PV process defined by (2.44) is within a multidimensional version of the TI class of the utility functions and inherits their quasilinearity property: each unit increase in the cash-flow process X^i (including the terminal lump-sum component) results in a deterministic increase in Z^i .

Recalling the identity $\Sigma_t^{Zi} = \Gamma_t \Sigma_t^{Vi} + \Sigma_t^{Yi}$ in (2.39), the equilibrium diffusion strategy can be interpreted as a diffusion "sharing rule" of the subjective PV process Z^i , with $\hat{\Sigma}^{Yi}(\omega, t, \Sigma_t^Z)$ of the time- t risk allocated to agent i and $\Sigma_t^{Zi} - \hat{\Sigma}^{Yi}(\omega, t, \Sigma_t^Z)$ allocated to principal i . Unlike the single principal-agent case, principal i 's optimal sharing rule can depend on all the subjective diffusion processes, $\Sigma^Z = (\Sigma^{Z1}, \dots, \Sigma^{ZN})$.

As seen from the corollary, it is simple to compute the equilibrium controls \hat{x}^V because the problems are unlinked across the principals. We will henceforth assume that the optimal x_t^{Vi} is well defined for each i . Also, the equilibrium controls \hat{x}^V have no direct impact on either equilibrium agent effort or on the equilibrium rule for sharing the subjective cash-flow diffusion processes Σ^Z among principals and agents. As a simple example, if for some $\gamma > 0$, $h^{Vi}(t, x) = -\frac{1}{\gamma} \exp(-\gamma x)$, then $\hat{x}_t^{Vi} = \frac{1}{\Gamma} \ln(\Gamma_t)$.

The more interesting problem is the computation of the equilibrium strategies $\hat{\Sigma}^Y$, which must be jointly solved, and each component $\hat{\Sigma}^{Yi}$ may depend on joint diffusion processes Σ^Z . Furthermore it is the set of $\hat{\Sigma}^Y$ that determine equilibrium agent effort, $\{\hat{e}(t, \Sigma_t^Y); t \in [0, T)\}$. Our applications will focus on this control, for which we find explicit solutions.

Having solved for (Z, Σ^Z) and the equilibrium principal strategies $(\hat{x}^V, \hat{\Sigma}^Y)$, substitute

$\hat{\Sigma}^Y$ into (2.31) to get

$$\hat{Y}_t^i = Y_0^i e^{\int_0^t r_s ds} - \int_0^t e^{\int_s^t r_u du} \left[p_s^i + \hat{\mu}^{Yi}(s, \hat{\Sigma}_s^Y) \right] ds + \int_0^t e^{\int_s^t r_u du} \Sigma_s^{Yi'} dB_s.$$

for any feasible intermediate consumption process. Then the terminal pay is

$$\hat{p}_T^i = \hat{Y}_T^i, \quad (2.47)$$

equilibrium principal i 's utility is $\hat{V}_t^i = (Z_t^i - \hat{Y}_t^i + \hat{W}_t^{Vi}) / \Gamma_t$, and equilibrium principal i 's wealth satisfies (2.2) after substituting $\hat{c}_t^{Vi} = \hat{x}_t^{Vi} + \hat{V}_t^i$:

$$\hat{W}_0^{Vi} = w_0^{Vi}, \quad d\hat{W}_t^V = \left(\hat{W}_t^{Vi} \left(r_t - \frac{1}{\Gamma_t} \right) + X_t^i - p_t^i - \hat{x}_t^{Vi} - \frac{Z_t^i - \hat{Y}_t^i}{\Gamma_t} \right) dt,$$

and equilibrium lump-sum terminal consumption is $\hat{c}_T^{Vi} = \hat{W}_T^{Vi} + X_T^i - \hat{p}_T^i$.

Remark 2 (Terminal consumption only) *The following modifications are required if there is no intermediate consumption (by either principal or agent). Let $c^{Ui} = c^{Vi} = 0$ in the wealth equations (2.1) and (2.2), omit the excess consumption arguments in the aggregators, and replace the bond price Γ_t by $D_T/D_t = e^{-\int_t^T r_s ds}$ in the following equations.*

The corresponding equations (2.9) for the Y process are changed to

$$dY_t^i = - \left\{ -r_t Y_t^i + p_t^i + \frac{D_T}{D_t} k^{Ui} \left(t, e_t^i, \frac{\Sigma_t^{Yi}}{D_T/D_t} \right) + \Sigma_t^{Yi'} \Phi(e_t) \right\} dt + \Sigma_t^{Yi'} dB_t, \quad Y_T^i = p_T^i, \\ i = 1, \dots, N.$$

The agent equilibrium condition (2.13) is the same except that the maximization over

intermediate consumption is dropped.

$$\hat{e}^i(\omega, t, \Sigma^Y) \in \arg \max_{e^i \in \mathbf{E}} \frac{D_T}{D_t} k^{Ui} \left(t, e^i, \frac{\Sigma_t^{Yi}}{D_T/D_t} \right) + \Sigma_t^{Yi'} \Phi(e^i, \hat{e}_t^{-i})$$

The corresponding equations (2.37) for the Z process are changed to

$$dZ_t^i = -(-r_t Z_t^i + X_t^i + \mu_t^{Zi})dt + \Sigma_t^{Zi'} dB_t, \quad Z_T^i = X_T^i, \quad i = 1, \dots, N.$$

where

$$\mu_t^{Zi} = \frac{D_T}{D_t} \left(k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \Sigma_t^{Yi}}{D_T/D_t} \right) + k^{Ui} \left(t, \hat{e} \left(t, \Sigma_t^Y \right), \frac{\Sigma_t^{Yi}}{D_T/D_t} \right) \right) + \Sigma_t^{Zi'} \Phi \left(\hat{e} \left(t, \Sigma_t^Y \right) \right)$$

The principal equilibrium condition (2.43) is the same except that the maximization over intermediate consumption is dropped.

$$\begin{aligned} \hat{\Sigma}_t^{Yi} \in & \arg \max_{\Sigma \in \mathbb{R}^d} \frac{D_T}{D_t} \left\{ k^{Vi} \left(t, \frac{\Sigma_t^{Zi} - \Sigma}{D_T/D_t} \right) + k^{Ui} \left(t, \hat{e}^i(t, \Sigma, \hat{\Sigma}_t^{Y-i}), \frac{\Sigma}{D_T/D_t} \right) \right\} \\ & + \Sigma_t^{Zi'} \Phi \left(\hat{e} \left(t, \Sigma, \hat{\Sigma}_t^{Y-i} \right) \right) \end{aligned}$$

2.5 Closed-Form Solution with Cash Flows Driven by State Processes

In the previous sections, we showed that the key step to solving principal equilibrium is solving a subjective PV of cash flow process denoted by Z and given by (2.44). The equilibrium strategies Σ^Y are then functions of the corresponding subjective diffusion process Σ^Z . In general, this diffusion process is itself a stochastic process, with dynamics dependent on the preferences of the principal and agent. We show in this section that if the principals' aggregators in (2.5) are deterministic functions, and uncertainty is driven by state processes in (2.48) with affine dynamics, then the subjective cash-flow diffusion Σ^Z is deterministic, solvable in closed form, and invariant to preferences (i.e., invariant to the form of the aggregators). Of course the subjective cash-flow process will depend on preferences, but it will always be affine in the state process for this class of state dynamics. The cash-flow-diffusion solution below can be used in the subsequent sections to obtain explicit solutions for the equilibrium policies under various specifications for the aggregators. We follow the main result of this section with two examples, in which current Brownian shock have short-run and long-run impacts on future cash-flows.

We introduce N state processes $\zeta^i \in \mathcal{L}(\mathbb{R}^n)$ with dynamics

$$d\zeta_t^i = \left(\mu^i + \beta^i \zeta_t^i \right) dt + \Sigma^{\zeta^i} dB_t, i = 1, \dots, N \quad (2.48)$$

where¹⁸ $\mu^i \in \mathbb{R}^n$, $\beta^i \in \mathbb{R}^{n \times n}$, and $\Sigma^{\zeta^i} \in \mathbb{R}^{d \times n}$. Assume that the cash-flow process X

¹⁸All the results, except the closed-form expression for Θ in Proposition 4, apply with time-varying deterministic parameters replacing the constant parameters throughout.

satisfies

$$X_t^i = M_t^{i'} \zeta_t^i, \quad i = 1, \dots, N \quad (2.49)$$

where $M^i \in \mathcal{L}(\mathbb{R}^n)$ is deterministic. Our examples will all assume constant M^i , but by allowing time-dependency we can also model the case of no intermediate cash flow by letting $M_t^i = 0$ for $t < T$. The following proposition gives the closed-form solution (Z^i, Σ^{Zi}) for the subjective cash-flow PV processes. Z^i is affine in the state process ζ and Σ^{Zi} is affine in the state process diffusion Σ^ζ .

Proposition 4 (Closed-form solution with affine state process) *Suppose the principals aggregators h^{Vi} , k^{Vi} , $i = 1, \dots, N$ are deterministic functions, the state process ζ satisfies (2.48), and the cash flow X^i satisfies (2.49). Furthermore, let the deterministic vector process $\Theta^i \in \mathcal{L}(\mathbb{R}^n)$, $i = 1, \dots, N$ solve the linear ODE system:*

$$\dot{\Theta}_t^i + (\beta^{i'} - r_t I) \Theta_t^i = -M_t^i, \quad t < T, \quad \Theta_T^i = M_T^i, \quad i = 1, \dots, N. \quad (2.50)$$

The system has the closed-form solution

$$\Theta_t^i = \exp((T-t)\beta^{i'}) e^{-\int_t^T r_s ds} M_T^i + \int_t^T \exp((s-t)\beta^{i'}) e^{-\int_t^s r_u du} M_s^i ds.$$

Finally, let

$$\theta_t^i = \int_t^T \frac{D_s}{D_t} \left\{ \Theta_s^{i'} \mu^i + \hat{\mu}^{Zi} \left(t, \Sigma^{\zeta 1} \Theta_s^1, \dots, \Sigma^{\zeta N} \Theta_s^N \right) \right\} ds \quad (2.51)$$

where $\hat{\mu}^{Zi}(\cdot)$ is defined in (2.45).

Then the solution (Z, Σ^Z) to the BSDE system (2.44) satisfy

$$Z_t^i = \theta_t^i + \Theta_t^{i'} \zeta_t^i, \quad \Sigma_t^{Zi} = \Sigma^{\zeta i} \Theta_t^i, \quad t \in [0, T], \quad i = 1, \dots, N. \quad (2.52)$$

The equilibrium strategies Σ^Y are therefore deterministic and only the lump-sum components of terminal pay p_T are stochastic.

Because Σ^{Zi} is independent of preferences, it matches the diffusion obtained using risk-neutral discounting of cash flows (that is, solving (2.44) with $\hat{\mu}^{Zi} = 0$). Risk aversion enters only the θ^i of Z^i .

The first example considers Ornstein-Uhlenbeck cash flows. As mean reversion increases, the impact of current Brownian shocks (and therefore current effort) diminishes more quickly, and has only a transient impact on future cash flows. Higher mean reversion therefore implies a smaller $|\Sigma^{Zi}|$.

Example 7 (Short-run effort impact) Suppose r is constant (for simplicity), and the cash-flow dynamics are

$$dX_t^i = \left(\eta^i - \kappa^i X_t^i \right) dt + \sigma^{i'} dB_t, \quad i = 1, \dots, N \quad (2.53)$$

for some $\sigma^i \in \mathbb{R}^d$, $\eta^i \in \mathbb{R}$, and $\kappa^i \in \mathbb{R}_+$. This is just a special case of Proposition 4 by letting $M^i = (1, \dots, 1)'$, $\mu^i = \eta^i/n(1, \dots, 1)'$, $\beta^i = -\kappa^i I$ and $\sigma^i = \Sigma^{\zeta i} M^i$.

Then

$$Z_t^i = \Phi_0^i(t) + \Phi_1^i(t) X_t^i, \quad \Sigma_t^{Zi} = \Phi_1^i(t) \sigma^i, \quad i = 1, \dots, N$$

where

$$\begin{aligned}\Phi_1^i(t) &= \frac{1}{r + \kappa^i} - e^{-(r + \kappa^i)(T-t)} \left(\frac{1}{r + \kappa^i} - 1 \right), \\ \Phi_0^i(t) &= \int_t^T e^{-r(s-t)} \left\{ \eta^i \Phi_1^i(s) + \hat{\mu}^{Z^i} \left(s, \Sigma_s^{Z^1}, \dots, \Sigma_s^{Z^N} \right) \right\} ds.\end{aligned}\tag{2.54}$$

Note that $\Phi_1(t) = \Gamma_t(r + \kappa^i)$, where $\Gamma(r + \kappa^i)$ denotes the bond price with interest rate $r + \kappa^i$ replacing r . If $\kappa^i = 0$ then $\Phi_1^i(t) = \Gamma_t$, which follows because a unit time- t shock in the k th dimension of B_t increases the present value of future cash flows by $\Gamma_t \sigma^i[k]$, and therefore increases Z_t^i by this amount (by the quasilinearly property of Z_t^i). As κ^i increases, the impact of effort diminishes through additional discounting, and as $\kappa^i \rightarrow \infty$, the contribution to present value vanishes.

The second example considers long-run Brownian shock impact, modeled along the lines of Bansal and Yaron (2004). Current Brownian shocks (and therefore current effort) is allowed to affect not only the current cash-flow shock, but also the drift of the cash-flow drift.

Example 8 (long-run effort impact) Suppose a constant r (for simplicity) and the cash flow dynamics

$$dX_t^i = \alpha_t^i dt + \sigma^{i'} dB_t, \quad d\alpha_t^i = \left(\eta^i - \kappa^i \alpha_t^i \right) dt + \Sigma^{i'} dB_t,\tag{2.55}$$

for some $\sigma^i, \Sigma^i \in \mathbb{R}^d$, $\mu^i \in \mathbb{R}$, and $\kappa^i \in \mathbb{R}_+$.

This is a special case of Proposition 4 by letting

$$\zeta_t^i = (\alpha_t^i, X_t^i)', \quad \mu^i = (\eta^i, 0)', \quad \beta_t^i = \begin{pmatrix} -\kappa^i & 0 \\ 1 & 0 \end{pmatrix}, \quad \Sigma_t^{\zeta^i} = (\Sigma^i, \sigma^i), \quad M_t^i = (0, 1).$$

Then

$$Z_t^i = \Phi_0^i(t) + \Phi_1^i(t) \alpha_t^i + \Gamma_t X_t^i, \quad \Sigma_t^{Z^i} = \Phi_1^i(t) \Sigma^i + \Gamma_t \sigma^i,$$

where ¹⁹

$$\begin{aligned} \Phi_1^i(t) &= \int_t^T e^{-(r+\kappa^i)(s-t)} \Gamma_s ds, \\ \Phi_0^i(t) &= \int_t^T e^{-r(s-t)} \left\{ \eta^i \Phi_1^i(s) + \hat{\mu}^{Z^i} \left(s, \Sigma_s^{Z^1}, \dots, \Sigma_s^{Z^N} \right) \right\} ds. \end{aligned} \tag{2.56}$$

A time- t unit shock in the time- t Brownian motion in dimension k has two impacts: 1) it increases the future cash-flow path by $\sigma^i[k]$, which increases its present value by $\Gamma_t \sigma^i[k]$; and 2) it increases the time- s cash flow drift, for each $s \geq t$, by $e^{-\kappa^i(s-t)} \Sigma^i[k]$, resulting in a time- s present-value increment of $e^{-(r+\kappa^i)(s-t)} \Sigma^i[k] \Gamma_s$. This second effect reflects a higher-order persistence effect.

Assuming $r < 1$ (less than 100%), then both Γ_t and $\Phi_1^i(t)$ are increasing $T - t$, time remaining to the terminal date. Letting $T \rightarrow \infty$, then Γ_t is the time- t present value of a

¹⁹The assumption of constant r implies

$$\Gamma_t = \frac{1}{r} - e^{-r(T-t)} \left(\frac{1}{r} - 1 \right), \quad \Phi_1^i(t) = \frac{1}{r(r+\kappa^i)} + \frac{e^{-r(T-t)}}{\kappa^i} \left[e^{-\kappa^i(T-t)} \left\{ \frac{1}{r+\kappa^i} - 1 \right\} - \left(\frac{1}{r} - 1 \right) \right].$$

unit perpetuity and $\Phi_1^i(t)$ is the time- t present value of a growing perpetuity

$$\left\{ \left(1 - e^{-\kappa^i(s-t)}\right) / \kappa^i; s \geq t \right\}: \Gamma_t \rightarrow \frac{1}{r}, \Phi_1^i(t) \rightarrow \frac{1}{r(r+\kappa^i)}, \text{ and therefore}$$

$$\Sigma_t^{Zi} = \frac{1}{r} \left(\frac{1}{r + \kappa^i} \Sigma^i + \sigma^i \right), \quad \text{as } T \rightarrow \infty.$$

Examples 7 and 8 show that increased persistence of current Brownian shocks on future cash-flow increments results in higher subjective cash-flow diffusion (i.e., higher sensitivity of the cash-flow PV process to Brownian shocks). The diffusion is sensitive to mean reversion and the interest rate. Lower interest rates imply a larger impact of current shocks on the subjective cash-flow PV, and therefore a larger diffusion, particularly with the long-run dynamics. The sensitivity of the diffusion to the interest rate also increases with a lower interest rate.

Because Σ^Z is determined by the impact of current Brownian shocks on the present value of future cash flows, the terminal-date effects are driven by the principal's lifespan, not the agent's. The same diffusion process would be obtained in a model with a short-lived agent employed by a long-lived principal.

2.6 Additive Measure Change and Quadratic Penalties

We assume throughout this section unrestricted effort²⁰, $\mathbf{E} = \mathbb{R}^d$, and the following quadratic penalties and linear measure change specification:

$$\begin{aligned} k^{Ui} \left(t, e_t^i, \Sigma_t^{Ui} \right) &= -\frac{1}{2} \Sigma_t^{Ui'} Q_t^{Ui} \Sigma_t^{Ui} - \frac{1}{2} e_t^{i'} Q_t^{ei} e_t^i, \\ k^{Vi} \left(\Sigma_t^{Vi} \right) &= -\frac{1}{2} \Sigma_t^{Vi'} Q_t^{Vi} \Sigma_t^{Vi}, \quad i = 1, \dots, N, \\ \Phi(e_t) &= \sum_{i=1}^N e_t^i, \end{aligned} \tag{2.57}$$

where the deterministic $Q^{ei}, Q^{Ui}, Q^{Vi} \in \mathcal{L}(\mathbb{R}^{d \times d})$ are assumed symmetric positive definite. From Example 4, the set of equilibrium effort plans as functions of agent utility diffusion is uniquely given by

$$\hat{e}_t^i = (1/\Gamma_t)(Q_t^{ei})^{-1} \Sigma_t^{Yi}, \quad t \in [0, T], \quad i = 1, \dots, N. \tag{2.58}$$

That is, each agent's effort in equilibrium is linear in the the PV process diffusion. The following proposition gives the equilibrium principal controls $\hat{\Sigma}^Y$ and the resulting equilibrium effort and subjective cash-flow PV processes.

Proposition 5 *Under the linear measure change and quadratic penalties (2.57), equilibrium agent utility diffusion and agent effort are*

$$\hat{\Sigma}_t^{Yi} = W_t^i \Sigma_t^{Zi}, \quad \hat{e}_t^i = (1/\Gamma_t) \left(Q_t^{ei} \right)^{-1} W_t^i \Sigma_t^{Zi}, \quad i = 1, \dots, N,$$

²⁰Except in Example 12 where effort is allowed only in one dimension.

where

$$W_t^i = \left((Q_t^{ei})^{-1} + Q_t^{Vi} + Q_t^{Ui} \right)^{-1} \left((Q_t^{ei})^{-1} + Q_t^{Vi} \right),$$

and $(Z^i, \Sigma_t^{Zi}; i = 1, \dots, N)$ solve the system of linked BSDEs

$$\begin{aligned} dZ_t^i = & - \left\{ -r_t Z_t^i + X_t^i + H_t^i - \frac{1}{2\Gamma_t} \Sigma_t^{Zi'} Q_t^{Zi} \Sigma_t^{Zi} \right. \\ & \left. + \frac{1}{\Gamma_t} \Sigma_t^{Zi'} \sum_{j \neq i} (Q_t^{ej})^{-1} W_t^j \Sigma_t^{Zj} \right\} dt + \Sigma_t^{Zi'} dB_t, \quad Z_T^i = X_T^i, \quad i = 1, \dots, N, \end{aligned} \quad (2.59)$$

where

$$Q_t^{Zi} = W_t^{i'} Q_t^{Ui} - (Q_t^{ei})^{-1}, \quad (2.60)$$

$$H_t^i = \Gamma_t \left(h^{Vi}(t, \hat{x}_t^{Vi}) + h_t^{Ui}(t, \hat{x}_t^{Ui}) \right) - \hat{x}_t^{Vi} - \hat{x}_t^{Ui},$$

and \hat{x}^{Vi} is part of principal i 's equilibrium strategy defined in (2.43) and \hat{x}^{Ui} is part of agent i 's equilibrium strategy defined in (2.13).

For any intermediate pay p_t^i , $t < T$ chosen by the principal,

the terminal pay

$$\begin{aligned} p_T^i = & (\Gamma_0 K^i - w_0^{Ui}) e^{\int_0^T r_s ds} - \int_0^T e^{\int_t^T r_s ds} \left[p_t^i + \left(\Gamma_t h^{Ui}(t, \hat{x}_t^{Ui}) - \hat{x}_t^{Ui} \right) \right. \\ & \left. - \frac{1}{2\Gamma_t} \Sigma_t^{Zi'} Q_t^{pi} \Sigma_t^{Zi} + \frac{1}{\Gamma_t} \Sigma_t^{Zi'} W_t^{i'} \sum_{j \neq i} (Q_t^{ej})^{-1} W_t^j \Sigma_t^{Zj} \right] dt + \int_0^T e^{\int_t^T r_s ds} \Sigma_t^{Zi'} W_t^{i'} dB_t \end{aligned}$$

where

$$Q_t^{pi} = W_t^{i'} \left(Q_t^{Ui} - (Q_t^{ei})^{-1} \right) W_t^i.$$

Proof of Proposition 5.

With the preferences and Φ in (2.57), after substituting the equilibrium agent i effort $\hat{e}_t^i = (1/\Gamma_t) (Q_t^{ei})^{-1} \Sigma_t^{Yi}$, we have

$$\hat{\mu}_t^{Zi} = H_t^i + k^{Zi}(t, \Sigma_t^{Yi}, \hat{\Sigma}_t^{Y-i}, \Sigma_t^{Zi})$$

where

$$\begin{aligned} k^{Zi}(t, \Sigma_t^{Yi}, \hat{\Sigma}_t^{Y-i}, \Sigma_t^{Zi}) &= -\frac{1}{2\Gamma_t} \left(\Sigma_t^{Zi} - \Sigma_t^{Yi} \right)' Q_t^{Vi} \left(\Sigma_t^{Zi} - \Sigma_t^{Yi} \right) \\ &\quad - \frac{1}{2\Gamma_t} \Sigma_t^{Yi'} \left\{ Q_t^{Ui} + \left(Q_t^{ei} \right)^{-1} \right\} \Sigma_t^{Yi} + \frac{1}{\Gamma_t} \Sigma_t^{Zi'} \left(Q_t^{ei} \right)^{-1} \Sigma_t^{Yi} \\ &\quad + \frac{1}{\Gamma_t} \Sigma_t^{Zi'} \sum_{j \neq i} \left(Q_t^{ej} \right)^{-1} \hat{\Sigma}_t^{Yj}. \end{aligned} \quad (2.61)$$

Maximizing over Σ_t^{Yi} , the FOC for principal i (which is necessary and sufficient) is

$$Q_t^{Vi} \left(\Sigma_t^{Zi} - \hat{\Sigma}_t^{Yi} \right) - \left\{ Q_t^{Ui} + \left(Q_t^{ei} \right)^{-1} \right\} \hat{\Sigma}_t^{Yi} + \left(Q_t^{ei} \right)^{-1} \Sigma_t^{Zi} = 0, \quad (2.62)$$

which has the solution $\hat{\Sigma}_t^{Yi} = W_t^i \Sigma_t^{Zi}$.

Multiplying (2.62) by $\frac{1}{2} \hat{\Sigma}_t^{Yi'}$, we get

$$\frac{1}{2} \hat{\Sigma}_t^{Yi'} Q_t^{Vi} \left(\Sigma_t^{Zi} - \hat{\Sigma}_t^{Yi} \right) - \frac{1}{2} \hat{\Sigma}_t^{Yi'} \left\{ Q_t^{Ui} + \left(Q_t^{ei} \right)^{-1} \right\} \hat{\Sigma}_t^{Yi} + \frac{1}{2} \hat{\Sigma}_t^{Yi'} \left(Q_t^{ei} \right)^{-1} \Sigma_t^{Zi} = 0.$$

Upon substituting this into (2.61) we have

$$\begin{aligned}
k^{Zi}(t, \hat{\Sigma}^{Yi}, \hat{\Sigma}_t^{Y-i}, \Sigma_t^{Zi}) &= -\frac{1}{2\Gamma_t} \Sigma_t^{Zi'} Q_t^{Vi} (\Sigma_t^{Zi} - \hat{\Sigma}_t^{Yi}) + \frac{1}{2\Gamma_t} \Sigma_t^{Zi'} (Q_t^{ei})^{-1} \hat{\Sigma}_t^{Yi} \\
&\quad + \frac{1}{\Gamma_t} \Sigma_t^{Zi'} \sum_{j \neq i} (Q_t^{ej})^{-1} \hat{\Sigma}_t^{Yj} \\
&= -\frac{1}{2\Gamma_t} \Sigma_t^{Zi'} \left(W_t^{i'} Q_t^{Ui} - (Q_t^{ei})^{-1} \right) \Sigma_t^{Zi} \\
&\quad + \frac{1}{\Gamma_t} \Sigma_t^{Zi'} \sum_{j \neq i} (Q_t^{ej})^{-1} W_t^{j'} \Sigma_t^{Zj},
\end{aligned}$$

where we used the identity $Q_t^{Vi} (I - W_t^i) - (Q_t^{ei})^{-1} W_t^i = W_t^{i'} Q_t^{Ui} - (Q_t^{ei})^{-1}$ for the last equality.

For any intermediate pay p_t^i , $t < T$ chosen by the principal, recall the expression of (2.32) and the definition of $\hat{\mu}^{Yi}$ in (2.14). We get the terminal pay (2.61).

■

The optimal diffusion sharing rule is simple: Principal i lays off the (matrix-valued) proportion W_t^i of the appreciated subjective-cash-flow diffusion Σ_t^{Zi}/Γ_t to the agent, and bears the rest (from $\hat{\Sigma}_t^{Vi} = (I - W_t^i) \Sigma_t^{Zi}/\Gamma_t$). The weight process W_t^i depends only on the risk aversion and effort penalties, and is invariant to the cash-flow dynamics. In the extreme case of agent- i risk neutrality ($Q^{Ui} = 0$) then $W^i = \mathbf{I}$ and all the risk of Z is transferred to the agent. At the other extreme, as the agent becomes infinitely risk averse, then $W^i \rightarrow 0$, the principal bears all the risk of Z , and the agent exerts no effort.

Each agent i 's equilibrium time- t effort depends only on his/her own subjective cash-flow diffusion Σ_t^{Yi} , but the principal problems are nonetheless linked because of the common impact of each agent's effort on the measure (and therefore the distribution of cash flows). This linkage appears in the summation term in the drift of Z , which adds the distorted

covariances of the subjective cash-flow processes. Optimal agent terminal pay above has the following components: a) a fixed component to satisfy the participation constraint; b) an adjustment for utility from intermediate consumption and pay (through the h^{Ui} term); c) compensation for agent risk aversion; d) an adjustment depending equilibrium effort of the other agents; e) a martingale part typically driven by innovations in the cash-flow process.

The following example obtains an expression for equilibrium lump-sum terminal pay in terms of the lump-sum component of terminal cash flow in the case of constant diagonal preference parameters.

Example 9 *Assume constant diagonal preference parameters: $Q_t^{ei} = q^{ei}I$, $Q_t^{Ui} = q^{Ui}I$, and $Q_t^{Vi} = q^{Vi}I$, for some $q^{ei}, q^{Ui}, q^{Vi} \in \mathbb{R}_{++}$, $i = 1, \dots, N$. Then $W_t^i = w^iI$, $Q_t^{Zi} = q^{Yi}I$, and $Q_t^{pi} = q^{pi}I$ where*

$$w^i = \frac{1 + q^{ei}q^{Vi}}{1 + q^{ei}q^{Vi} + q^{ei}q^{Ui}}, \quad q^{Zi} = q^{Ui}w^i - \frac{1}{q^{ei}}, \quad q^{pi} = (w^i)^2 \left(q^{Ui} - \frac{1}{q^{ei}} \right). \quad (2.63)$$

Principal i 's equilibrium control and agent i 's equilibrium effort are

$$\hat{\Sigma}_t^{Yi} = w^i \Sigma_t^{Zi}, \quad \hat{e}_t^i = \frac{w^i}{\Gamma_t q^{ei}} \Sigma_t^{Zi},$$

and equilibrium terminal pay for agent i is

$$\begin{aligned} p_T^i &= w^i \left[X_T^i + \int_0^T e^{\int_t^T r_s ds} X_t^i dt \right] + \left[(1 - w^i) e^{\int_0^T r_s ds} (\Gamma_0 K^i - w_0^{Ui}) \right. \\ &\quad \left. - w^i e^{\int_0^T r_s ds} (\Gamma_0 V_0^i - w_0^{Vi}) \right] + \int_0^T e^{\int_t^T r_s ds} \xi_t^i dt + \int_0^T \frac{e^{\int_t^T r_s ds} w^i (1 - w^i)}{2\Gamma_t q^{ei}} \Sigma_t^{Zi} \Sigma_t^{Zi} dt \\ &\quad - \int_0^T e^{\int_t^T r_s ds} p_t^i dt \end{aligned} \quad (2.64)$$

where

$$\xi_t^i = w^i \left(\Gamma_t h^{Vi}(t, \hat{x}_t^{Vi}) - \hat{x}_t^{Vi} \right) - (1 - w^i) \left(\Gamma_t h^{Ui}(t, \hat{x}_t^{Ui}) - \hat{x}_t^{Ui} \right).$$

The first component of terminal pay is a proportion w^i of the terminal cash flow and the cumulative intermediate cash flow. The second and third terms adjust for the participation constraint and intermediate consumption. The fourth term compensates the agent for the cumulative risk of the cash-flow process. The last term subtracts the cumulative intermediate pay. The effort of other agents affects the solution (Z^i, Σ^{Zi}) , which then impacts terminal pay through $Z_0^i = \Gamma_0 V_0^i - w_0^{Vi} + \Gamma_0 K^i - w_0^{Ui}$ and the quadratic variation term in (2.64).

Derivation. See Section .2 in the Appendix. ■

In the single principal/agent case ($N = 1$), the BSDE (2.59) is of the same form as the BSDE (21) in Schroder and Skiadas (2005) (which applies to the optimal portfolio problem). They provide sufficient conditions on the BSDE parameters and Markovian state-variable processes such that the Z will be an affine function of the state variables, the coefficients of which satisfy a set of Riccati ordinary differential equations (ODEs). In Section 2.5, we showed that an analogous result holds in the multiple principal/agent case, in which the solution reduces to a tractable linked system of ODEs. The examples below follow the Examples 7 and 8 in Section 2.5. We will assume the the same diagonal preferences as in Example 9 throughout.

The next example follows Example 7 and focuses on the role of mean reversion and its impact on equilibrium controls and optimal terminal pay. We obtain a simple sharing rule for the terminal lump-sum cash flow, proportional to the volatility sharing rule. Neither sharing rule depends on the distribution of cash flows. Therefore an increase in cash-

flow mean reversion, which causes the impact of effort to become more transient and reduces equilibrium effort rates, has no impact on the sharing rules. In fact, additional compensation may be necessary in order to satisfy the participation constraint.

Example 10 (Ornstein-Uhlenbeck cash flow) *This example follows Example 7 in Section 2.5 where the cash-flow dynamics are*

$$dX_t^i = \left(\eta^i - \kappa^i X_t^i \right) dt + \sigma^{i'} dB_t, \quad i = 1, \dots, N$$

for some $\sigma^i \in \mathbb{R}^d$, $\eta^i \in \mathbb{R}$, and $\kappa^i \in \mathbb{R}_+$.

Assume the same constant diagonal preference parameters as in Example 9 and that r is constant (for simplicity).

In Example 7, we showed that

$$Z_t^i = \Phi_0^i(t) + \Phi_1^i(t) X_t^i, \quad \Sigma_t^{Zi} = \Phi_1^i(t) \sigma^i, \quad i = 1, \dots, N$$

The equilibrium principal i 's control and agent effort are

$$\hat{\Sigma}_t^{Yi} = w^i \Phi_1^i(t) \sigma^i, \quad \hat{e}_t^i = \frac{w^i}{\Gamma_t q^{ei}} \Phi_1^i(t) \sigma^i$$

where

$$\begin{aligned} \Phi_1^i(t) &= \frac{1}{r + \kappa^i} - e^{-(r + \kappa^i)(T-t)} \left(\frac{1}{r + \kappa^i} - 1 \right) \\ \Phi_0^i(t) &= \int_t^T e^{-r(s-t)} \left[\eta^i \Phi_1^i(s) + H_s^i - \frac{1}{2\Gamma_s} \Phi_1^i(s)^2 q^{Zi} \sigma^{i'} \sigma^i + \frac{1}{\Gamma_s} \Phi_1^i(s) \Phi_1^j(s) \sigma^{i'} \sum_{j \neq i} \frac{w^j}{q^{ej}} \sigma^j \right] ds \end{aligned} \tag{2.65}$$

In the absence of mean reversion, with an interest rate $r \in (0, 1)$ the $\Phi_1^i(t)$ increases as

time to the terminal date increases so are the absolute agent diffusions and efforts.

$$\Phi_1^i(t) \uparrow \frac{1}{r}, \quad \hat{\Sigma}_t^{Y^i} \rightarrow \frac{w^i}{r} \sigma^i, \quad \hat{e}_t^i \rightarrow \frac{w^i}{\Gamma_t q^{ei} r} \sigma^i \quad \text{as } T \rightarrow \infty.$$

With a positive mean reversion ($\kappa^i > 0$) that is large enough such that $r + \kappa^i > 1$, $\Phi_1^i(t)$ decreases as time to the terminal date increases, and therefore absolute agent diffusion and absolute effort also decrease.

$$\text{If } r + \kappa^i > 1, \text{ then } \Phi_1^i(t) \downarrow \frac{1}{r + \kappa^i}, \quad \hat{\Sigma}_t^{Y^i} \rightarrow \frac{w^i}{r + \kappa^i} \sigma^i, \quad \hat{e}_t^i \rightarrow \frac{w^i}{\Gamma_t q^{ei} (r + \kappa^i)} \sigma^i \text{ as } T \rightarrow \infty.$$

This is because the drift impact of effort is more likely to be reversed by mean reversion (in fact both converge to zero as $\kappa^i \rightarrow \infty$).

The terminal pay is

$$\begin{aligned} p_T^i &= w^i \left[L^i - \sum_{j \neq i} \frac{w^j}{q^{ej}} \text{Cov}(X^i, X^j) \right] + e^{rT} (\Gamma_0 K^i - w_0^{Ui}) \\ &\quad + \frac{(w^i)^2}{2} \left(q^{Ui} - \frac{1}{q^{ei}} \right) \text{Var}(X^i) - \int_0^T e^{r(T-t)} \left(p_t^i + \Gamma_t h^{Ui}(t, \hat{x}_t^{Ui}) - \hat{x}_t^{Ui} \right) dt \end{aligned} \quad (2.66)$$

where

$$\begin{aligned}
L^i &= \int_0^T e^{r(T-t)} \Phi_1^i(t) \sigma^{i'} dB_t \\
&= X_T^i + \int_0^T e^{r(T-t)} X_t^i dt - e^{rT} \Phi_1^i(0) X_0^i - \eta^i \int_0^T e^{r(T-t)} \Phi_1^i(t) dt \\
&= \frac{1}{r} \int_0^T e^{r(T-t)} dX_t^i - \left(\frac{1}{r} - 1 \right) X_T^i + \left[\left(\frac{1}{r} - \frac{1}{r + \kappa^i} \right) e^{rT} + \left(\frac{1}{r + \kappa^i} - 1 \right) e^{-\kappa^i T} \right] X_0^i \\
&\quad - \eta^i \int_0^T e^{r(T-t)} \Phi_1^i(t) dt,
\end{aligned} \tag{2.67}$$

$$\begin{aligned}
Cov(X^i, X^j) &= \int_0^T \frac{e^{r(T-t)} \Phi_1^i(t) \Phi_1^j(t)}{\Gamma_t} dX_t^i dX_t^j, \\
Var(X^i) &= \int_0^T \frac{e^{r(T-t)} \Phi_1^i(t)^2}{\Gamma_t} dX_t^i dX_t^i.
\end{aligned}$$

The mean reversion term κ^i affects the deterministic function $\Phi_1^i(t)$.

The impact of other agents' effort is through the deterministic the terms $\Phi_1^j(t)$, $j \neq i$.

In the case of terminal consumption only ($c^{U^i} = c^{V^i} = 0$) and zero cash flow drift ($\eta^i = \kappa^i = 0$), the solution is ²¹

$$\begin{aligned}
\Phi_1^i(t) &= \Gamma_t = \frac{1}{r} - e^{-r(T-t)} \left(\frac{1}{r} - 1 \right) \\
\Phi_0^i(t) &= \left(-\frac{q^{Z^i} \sigma^{i'} \sigma^i}{2} + \sigma^{i'} \sum_{j \neq i} \frac{w^j}{q^{e^j}} \sigma^j \right) \int_t^T e^{r(T-2s+t)} \Phi_1^i(s)^2 ds
\end{aligned}$$

and the terminal pay is

²¹As discussed in Remark 2, the Γ_s in (2.65) needs to be replaced by $e^{-r(T-s)}$. Although $\Phi_1^i(s)$ equals the old Γ_s , it should not be replaced.

$$\begin{aligned}
p_T^i &= w^i \left[L^i - \sum_{j \neq i} \frac{w^j}{q^{ej}} Cov(X^i, X^j) \right] + e^{rT} (\Gamma_0 K^i - w_0^{Ui}) \\
&\quad + \frac{(w^i)^2}{2} \left(q^{Ui} - \frac{1}{q^{ei}} \right) Var(X^i) - \int_0^T e^{r(T-t)} p_t^i dt
\end{aligned} \tag{2.68}$$

where

$$L^i = \int_0^T e^{r(T-t)} \Phi_1^i(t) dX_t^i = \frac{1}{r} \int_0^T e^{r(T-t)} dX_t^i - \left(\frac{1}{r} - 1 \right) (X_T^i - X_0^i) \tag{2.69}$$

$$\begin{aligned}
Cov(X^i, X^j) &= \int_0^T \left(e^{r(T-t)} \Phi_1^i(t) \right)^2 dX_t^i dX_t^j = \sigma^{i'} \sigma^j S(2) \\
Var(X^i) &= \int_0^T \left(e^{r(T-t)} \Phi_1^i(t) \right)^2 dX_t^i dX_t^i = \sigma^{i'} \sigma^i S(2)
\end{aligned}$$

with ²²

$$S(k) = \int_0^T \left(e^{r(T-t)} \Phi_1^i(t) \right)^k dt.$$

If the cash-flow covariance is positive, the j th agent's effort increases the drift of X^i thereby increasing the value of agent i 's share of the terminal value of the cash flow, $w^i X_T^i$. This allows principal i to reduce the fixed component of pay, while still satisfying the partic-

²²Here $S(2) = \left(e^{2rT} - 1 - 4(1-r)(e^{rT} - 1) \right) / \left(2r^3 \right) + \left(\frac{1}{r} - 1 \right)^2 T$

ipation constraint.

Example 11 (long-run effort impact) *This example follows Example 8 in Section 2.5*

where the cash flow dynamics are

$$dX_t^i = \alpha_t^i dt + \sigma^{i'} dB_t, \quad d\alpha_t^i = \left(\eta^i - \kappa^i \alpha_t^i \right) dt + \Sigma^{i'} dB_t,$$

for some $\sigma^i, \Sigma^i \in \mathbb{R}^d$, $\mu^i \in \mathbb{R}$, and $\kappa^i \in \mathbb{R}_+$.

Assume the same constant diagonal preference parameters as in Example 9. and that r is constant (for simplicity),

In Example 8, we showed that

$$Z_t^i = \Phi_0^i(t) + \Phi_1^i(t) \alpha_t^i + \Gamma_t X_t^i, \quad \Sigma_t^{Z^i} = \Phi_1^i(t) \Sigma^i + \Gamma_t \sigma^i,$$

The equilibrium principal i 's control and agent effort are

$$\hat{\Sigma}_t^{Y^i} = w^i(\Phi_1^i(t) \Sigma^i + \Gamma_t \sigma^i), \quad \hat{e}_t^i = \frac{w^i}{\Gamma_t q^{ei}} (\Phi_1^i(t) \Sigma^i + \Gamma_t \sigma^i)$$

where

$$\begin{aligned} \Phi_1^i(t) &= \int_t^T e^{-(r+\kappa^i)(s-t)} \Gamma_s ds \\ \Phi_0^i(t) &= \int_t^T e^{-r(s-t)} \left[\eta^i \Phi_1^i(s) + H_s^i - \frac{q^{Z^i}}{2\Gamma_s} \left(\Phi_1^i(s) \Sigma^i + \Gamma_s \sigma^i \right)' \left(\Phi_1^i(s) \Sigma^i + \Gamma_s \sigma^i \right) \right. \\ &\quad \left. + \frac{1}{\Gamma_s} \sum_{j \neq i} \frac{w^j}{q^{ej}} \left(\Phi_1^i(s) \Sigma^i + \Gamma_s \sigma^i \right)' \left(\Phi_1^j(s) \Sigma^j + \Gamma_s \sigma^j \right) \right] ds \end{aligned}$$

The terminal pay is

$$\begin{aligned}
p_T^i &= w^i \left[L^i - e^{rT} \Phi_0^i(0) \right] + e^{rT} (\Gamma_0 K^i - w_0^{Ui}) \\
&\quad + \frac{w^i(1-w^i)}{2qe^i} \int_0^T \frac{e^{r(T-t)}}{\Gamma_t} \left(\Phi_1^i(t) \Sigma^i + \Gamma_t \sigma^i \right)' \left(\Phi_1^i(t) \Sigma^i + \Gamma_t \sigma^i \right) dt \\
&\quad + \int_0^T e^{r(T-t)} (\xi_t^i - p_t^i) dt
\end{aligned} \tag{2.70}$$

where L^i is given by (2.69).

As the time to the terminal date T increases, both $\Phi_1^i(t)$ and Γ_t increases.

$$\Phi_1^i(t) \uparrow \frac{1}{r(r + \kappa^i)}, \quad \Gamma_t \uparrow \frac{1}{r} \quad \text{as } T \rightarrow \infty.$$

The equilibrium principal and efforts approach the following limits.

$$\hat{\Sigma}_t^{Yi} \rightarrow \frac{w^i}{r} \left\{ \frac{1}{r + \kappa^i} \Sigma^i + \sigma^i \right\}, \quad \hat{e}_t^i \rightarrow \frac{w^i}{qe^i} \left\{ \frac{1}{r + \kappa^i} \Sigma^i + \sigma^i \right\} \quad \text{as } T \rightarrow \infty.$$

Example 12 (Square-root state-variable dynamics) Assume the same constant diagonal preference parameters as in Example 9. We introduce a state process $\zeta \in \mathcal{L}(\mathbb{R}^n)$ with dynamics²³

$$d\zeta_t = \left(\mu^\zeta + \beta^\zeta \zeta_t \right) dt + \Sigma' \text{diag} \left(\sqrt{v + V \zeta_t} \right) dB_t. \tag{2.71}$$

where²⁴ $\mu^\zeta \in \mathbb{R}^n$, $\beta^\zeta \in \mathbb{R}^{n \times n}$, $v \in \mathbb{R}^d$ and $V, \Sigma \in \mathbb{R}^{d \times n}$. Let $m^i \in \mathbb{R}$ and $M^i \in \mathbb{R}^n$ and

²³For any length- N vector $\delta = (\delta_1, \dots, \delta_N)'$, we define $\sqrt{\delta} = (\sqrt{\delta_1}, \dots, \sqrt{\delta_N})'$ and let $\text{diag}(\delta)$ denote the $N \times N$ matrix with i th diagonal element δ_i .

²⁴All the results apply with time-varying deterministic parameters replacing the constant parameters throughout.

assume that the cash-flow process X^i satisfies:

$$X_t^i = m_t^i + M_t^{i'} \zeta_t.$$

We obtain a solution for the subjective cash-flow value process Z^i in equation (2.59) which is affine in the state variable:

$$Z_t^i = \theta_t^i + \Theta_t^{i'} \zeta_t, \quad \Sigma_t^{Zi} = \text{diag} \left(\sqrt{v + V \zeta_t} \right) \Sigma \Theta_t^i \quad i = 1, \dots, N,$$

for deterministic $\theta_t^i \in \mathcal{L}(\mathbb{R})$ and $\Theta_t^i \in \mathcal{L}(\mathbb{R}^n)$ satisfying the Riccati ODE system

$$\begin{aligned} 0 &= \dot{\theta}_t^i + \mu^{Zi} \Theta_t^i - r_t \theta_t^i + m_t^i + H_t^i + \sum_{j \neq i} \frac{w^j}{q^{ej} \Gamma_t} \Theta_t^{i'} \Sigma' \text{diag}(v) \Sigma \Theta_t^j \\ &\quad - \frac{q^{Zi}}{2\Gamma_t} \Theta_t^{i'} \Sigma' \text{diag}(v) \Sigma \Theta_t^i, \\ 0 &= \dot{\Theta}_t^i + \beta^{Zi} \Theta_t^i - r_t \Theta_t^i + M_t^i + \sum_{j \neq i} \frac{w^j}{q^{ej} \Gamma_t} \left(\Theta_t^{i'} \Sigma' \text{diag} \left(\Theta_t^{j'} \Sigma' \right) V \right)' \\ &\quad - \frac{q^{Zi}}{2\Gamma_t} \left(\Theta_t^{i'} \Sigma' \text{diag} \left(\Theta_t^{i'} \Sigma' \right) V \right)' \end{aligned} \tag{2.72}$$

with the terminal conditions $\theta_T^i = m_T^i$ and $\Theta_T^i = M_T^i$, $i = 1, \dots, N$.

From Example 9, the equilibrium principal controls and agent efforts are

$$\hat{\Sigma}_t^{Yi} = w^i \text{diag} \left(\sqrt{v + V \zeta_t} \right) \Sigma \Theta_t^i, \quad \hat{e}_t^i = \frac{w^i}{\Gamma_t q^{ei}} \text{diag} \left(\sqrt{v + V \zeta_t} \right) \Sigma \Theta_t^i, \quad i = 1, \dots, N,$$

and terminal pay/consumption is

$$\begin{aligned}
p_T^i &= w^i \left[X_T^i + \int_0^T e^{\int_t^T r_s ds} X_t^i dt - e^{\int_0^T r_s ds} (\theta_0^i + \Theta_0^{i'} \zeta_0) \right] + e^{rT} (\Gamma_0 K^i - w_0^{Ui}) \\
&\quad + \frac{w^i(1-w^i)}{2q^{ei}} \int_0^T \frac{e^{\int_t^T r_s ds}}{\Gamma_t} \left[\Theta_t^{i'} \Sigma' \text{diag}(v) \Sigma \Theta_t^i + V' \text{diag}(\Theta_t^{i'} \Sigma') \Sigma' \Theta_t^i \zeta_t \right] dt \\
&\quad + \int_0^T e^{r(T-t)} (\zeta_t^i - p_t^i) dt.
\end{aligned}$$

The impact of the other principal-agents pairs enters both the fixed component of pay (via θ_0^i and Θ_0^i) and the path dependent state-variable term (the ζ_t part of the third term). Note that $\Theta = (\Theta^1, \dots, \Theta^N)$ are jointly solved from (2.72), and each Θ^i generally depend on the preference parameters of all agents and principals. Once the system Θ is solved, it is a simple matter to solve the first-order linear ODE for each θ^i .

Derivation. See Section .2 in the Appendix. ■

2.7 Diminishing Returns to Effort (Concave Φ)

Section 2.6 showed that with a linear Φ , the sharing rule allocating the time- t subjective PV volatility, $\Sigma_t^{Y^i}$, between the i th principal-agent pair depends on $\Sigma_t^{Y^i}$, but not on the other principals' subjective PV volatility terms. In this section, we build on Lemma 1 and Examples 5 and 6. We solve the principal equilibria under a CES measure-change operator Φ . A CES measure change operator implies that the marginal impact of an agent's effort is diminishing in the aggregate effort of all agents. As a result, each agent's optimal effort generally depends on the volatility-controls of all the principals, and therefore each principal's optimal (own-agent) volatility control generally depends on the subjective PV

volatility processes of all the principals.

In this Section, we will assume two pairs of principals/agents ($N = 2$), labeled $\{a, b\}$ and the CES measure-change operator as defined in Definition 6 throughout.

$$\Phi_k(e) = \kappa \left\{ \alpha (e_k^a)^\gamma + (1 - \alpha) (e_k^b)^\gamma \right\}^{\frac{v}{\gamma}}, \quad \kappa > 0, \quad \alpha \in (0, 1), \quad 0 \neq \gamma \leq 2, \quad 0 < v < 2. \quad (2.73)$$

We will also assume the diagonal quadratic penalties below, except in Section 2.7.3.

$$\begin{aligned} k^{Ui} \left(t, e_t^i, \Sigma_t^{Ui} \right) &= -\frac{1}{2} \sum_{k=1}^d Q_k^{ei} (e_k^i)^2 - \frac{1}{2} \sum_{k=1}^d Q_k^{Ui} (\Sigma_k^{Ui})^2, \\ k^{Vi} \left(t, \Sigma_t^{Vi} \right) &= -\frac{1}{2} \sum_{k=1}^d Q_k^{Vi} (\Sigma_k^{Vi})^2, \end{aligned} \quad (2.74)$$

with $Q_k^{ei}, Q_k^{Ui}, Q_k^{Vi} > 0$ for all i, k .

Before proceeding to each special case, we first present a general solution of principal equilibrium for risk-neutral principal-agent pairs in the following Lemma. A relaxation of the restrictions on parameters is achieved by working with quasiconcavity and FOC in the Proof of the Lemma.

Lemma 4 *Assume the aggregator functions satisfy (2.74) and $\Phi(\cdot)$ satisfies (2.73). Suppose that in addition to our usual parameter restrictions on Φ ($\kappa > 0, \alpha \in (0, 1), 0 \neq \gamma \leq 2, 0 < v < 2$) we assume $\gamma \leq v$ or*

$$2 > \gamma > v \text{ and } \gamma(2 - \gamma)(2 - v) + v - \gamma > 0. \quad (2.75)$$

Note that (2.75) is implied by $0 < v < \gamma \leq 1$.

Henceforth, omit the dimensional argument. Let

$$S^a = \frac{\alpha v \Sigma^{Ya}}{\Gamma_t Q^{ea}}, \quad S^b = \frac{(1 - \alpha) v \Sigma^{Yb}}{\Gamma_t Q^{eb}}.$$

If the agents and principals are risk-neutral i.e. $Q^{Ui} = Q^{Vi} = 0$, $i \in \{a, b\}$, then principal equilibrium satisfies

$$\begin{aligned} & \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} \frac{2-\gamma}{2-v} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\} S^a \\ &= \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\} \frac{\alpha \Sigma_t^{Za} v (2-\gamma)}{\Gamma_t Q^{ea} (2-v)} \end{aligned} \quad (2.76)$$

$$\begin{aligned} & \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \frac{2-\gamma}{2-v} \right\} S^b \\ &= \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\} \frac{(1-\alpha) \Sigma_t^{Zb} v (2-\gamma)}{\Gamma_t Q^{eb} (2-v)} \end{aligned}$$

Proof.

We showed in Lemma 1 that if $\gamma < 2$, agent equilibrium is

$$\begin{aligned} e^a &= \kappa^{1/(2-v)} (S^a)^{1/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}} \\ e^b &= \kappa^{1/(2-v)} (S^b)^{1/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{(v-\gamma)/\{\gamma(2-v)\}} \end{aligned}$$

and therefore

$$\Phi(e) = \kappa^{2/(2-v)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{\frac{v(2-\gamma)}{\gamma(2-v)}}$$

The principal a 's problem is

$$\begin{aligned} \max_{S^a} \quad & -\frac{1}{2} \Gamma_t Q^{ea} \kappa^{2/(2-v)} (S^a)^{2/(2-\gamma)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{2(v-\gamma)/\{\gamma(2-v)\}} \\ & + \Sigma_t^{Za} \kappa^{2(2-v)} \left\{ \alpha (S^a)^{\gamma/(2-\gamma)} + (1-\alpha) (S^b)^{\gamma/(2-\gamma)} \right\}^{\frac{v(2-\gamma)}{\gamma(2-v)}} \end{aligned}$$

Let

$$z = (S^a)^{\gamma/(2-\gamma)}, \quad K = (1-\alpha) (S^b)^{\gamma/(2-\gamma)}$$

and then the above problem is equivalent to

$$\max_z -\frac{1}{2} \Gamma_t Q^{ea} \kappa^{2/(2-v)} z^{2/\gamma} \{\alpha z + K\}^{\frac{2(v-\gamma)}{\gamma(2-v)}} + \Sigma_t^{Za} \kappa^{2/(2-v)} \{\alpha z + K\}^{\frac{v(2-\gamma)}{\gamma(2-v)}}$$

The derivative of the RHS is (using the abbreviation $\{\} = \{\alpha z + K\}$)

$$\begin{aligned} \frac{\partial}{\partial z} = \quad & \left\{ -\frac{1}{2} \Gamma_t Q^{ea} \kappa^{2/(2-v)} \right\} \left\{ \frac{2}{\gamma} z^{2/\gamma-1} \{\}^{\frac{2(v-\gamma)}{\gamma(2-v)}} + \alpha z^{2/\gamma} \frac{2(v-\gamma)}{\gamma(2-v)} \{\}^{\frac{2(v-\gamma)}{\gamma(2-v)}-1} \right\} \\ & + \alpha \Sigma_t^{Za} \kappa^{2/(2-v)} \frac{v(2-\gamma)}{\gamma(2-v)} \{\}^{\frac{2(v-\gamma)}{\gamma(2-v)}} \end{aligned}$$

This is equivalent to

$$\frac{\partial}{\partial z} = \left(\{-\Gamma_t Q^{ea}\} \frac{1}{\gamma} z^{2/\gamma-1} \left\{ 1 + \alpha z \frac{(v-\gamma)}{(2-v)} \{\alpha z + K\}^{-1} \right\} + \alpha \Sigma_t^{Za} \frac{v(2-\gamma)}{\gamma(2-v)} \right) \{\}^{\frac{2(v-\gamma)}{\gamma(2-v)}} \kappa^{2/(2-v)}$$

To prove quasiconcavity, it is sufficient the term in the large parentheses above is monotonically decreasing in z . This is equivalent to showing $h(z)$ is increasing in z where

$$h(z) = \left\{ 1 + \alpha z \frac{(v-\gamma)}{(2-v)} \{\alpha z + K\}^{-1} \right\} \frac{1}{\gamma} z^{2/\gamma-1} = \left\{ 1 + \frac{(v-\gamma)}{(2-v)} \frac{\alpha z}{\alpha z + K} \right\} \frac{1}{\gamma} z^{2/\gamma-1}$$

The first derivative

$$h'(z) = \frac{2-\gamma}{\gamma^2} z^{\frac{2}{\gamma}-2} + \frac{v-\gamma}{2-v} \frac{\alpha z^{\frac{2}{\gamma}-1} [(2-\gamma)\alpha z + 2K]}{(\alpha z + K)^2 \gamma^2} \quad (2.77)$$

It is obvious that $h'(z) \geq 0$ if $0 \neq \gamma \leq v < 2$.

Now suppose $2 > \gamma > v > 0$.

Let $\varepsilon = \frac{\alpha z}{\alpha z + K}$. From (2.77), we have

$$\begin{aligned} h'(z) &= \frac{z^{\frac{2}{\gamma}-2}}{\gamma^2} \left\{ 2 - \gamma + \frac{v-\gamma}{2-v} \left[(2-\gamma)\varepsilon^2 + 2\varepsilon(1-\varepsilon) \right] \right\} \\ &= \frac{z^{\frac{2}{\gamma}-2}}{\gamma^2} \frac{(2-\gamma)(2-v) + 2(v-\gamma)\varepsilon + \gamma(\gamma-v)\varepsilon^2}{2-\gamma} \end{aligned}$$

And $h'(z) \geq 0$ is equivalent to

$$g(\varepsilon) = (2-\gamma)(2-v) + 2(v-\gamma)\varepsilon + \gamma(\gamma-v)\varepsilon^2 \geq 0$$

for any $0 \leq \varepsilon \leq 1$.

$g'(\varepsilon) = 0$ yields $\varepsilon = \frac{1}{\gamma}$.

If $1 \geq \gamma > v > 0$, then the minimal of $g(\varepsilon)$ is attained by $\varepsilon = 1$ and

$$g(1) = (2 - \gamma)^2 > 0 \text{ when } 1 \geq \gamma > v > 0.$$

If $\gamma > 1$ and $\gamma > v > 0$, then the minimal of $g(\varepsilon)$ is attained by $\varepsilon = \frac{1}{\gamma}$ and

$$g\left(\frac{1}{\gamma}\right) = \frac{\gamma(2 - \gamma)(2 - v) + v - \gamma}{\gamma}$$

$g(\varepsilon) > 0$ is equivalent to

$$\gamma(2 - \gamma)(2 - v) + v - \gamma > 0.$$

So quasiconcavity holds for any $0 \neq \gamma \leq v < 2$ or $0 < v < \gamma \leq 1$ or

$1 < \gamma$ and $v < \gamma$ such that

$$\gamma(2 - \gamma)(2 - v) + v - \gamma > 0.$$

Note that the above inequality is implied by $0 < v < \gamma \leq 1$.

Under these conditions, the optimality for principal a , holding fixed principal b , is equiv-

alent to a positive solution z (there can be only one, because of the monotonicity of $h(z)$)
to the principal's FOC, which is equivalent to a positive solution z to

$$\left\{ \alpha z \frac{(2-\gamma)}{(2-v)} + (1-\alpha) \left(S^b \right)^{\gamma/(2-\gamma)} \right\} z^{2/\gamma-1} = \left\{ \alpha z + (1-\alpha) \left(S^b \right)^{\gamma/(2-\gamma)} \right\} \frac{\alpha \Sigma_t^{Za} v (2-\gamma)}{\Gamma_t Q^{ea} (2-v)}$$

which is a quadratic polynomial in z when $\gamma = 1$.

If $\gamma = 1$, $\alpha = 1/2$ and $\kappa = 2^v$, then

the measure change function is

$$\Phi_k(e) = \left\{ e_k^a + e_k^b \right\}^v$$

and the FOC is

$$\left\{ \frac{S^a}{2(2-v)} + \frac{S^b}{2} \right\} S^a = \frac{1}{2} \left\{ S^a + S^b \right\} \frac{\alpha \Sigma_t^{Za} v}{\Gamma_t Q^{ea} 2-v}$$

■

2.7.1 Power Measure Change

For any $\delta \in [0, 1)$, let

$$\gamma = 1, \quad v = 1 - \delta, \quad \alpha = 1/2 \quad \text{and} \quad \kappa = 2^{1-\delta}/(1-\delta).$$

The measure change operator is (2.20).

In Example 5, we show the agent equilibrium solution is (2.21) in the case of two principal-agent pairs $i \in \{a, b\}$.

Proposition 6 (Risk neutrality) *Suppose Φ satisfies the above specification, and preferences satisfy the risk-neutral specification (2.74) with $Q^{Ui} = Q^{Vi} = 0$. There are two pairs of principal and agent $i \in \{a, b\}$. Assume $\Sigma_t^{Za}, \Sigma_t^{Zb} > 0$ all t and define the processes*

$$\Delta_k = \frac{\Sigma_k^{Za}}{Q_k^{ea}} - \frac{\Sigma_k^{Zb}}{Q_k^{eb}}, \quad S_k = \frac{\Sigma_k^{Za}}{Q_k^{ea}} + \frac{\Sigma_k^{Zb}}{Q_k^{eb}}.$$

The unique principal equilibrium is at each dimension k ²⁵

$$\frac{\hat{\Sigma}_k^{Ya}}{Q_k^{ea}} = \frac{S_k + \sqrt{S_k^2 + (2 + \delta) \delta \Delta_k^2}}{2(2 + \delta)} + \frac{\Delta_k}{2}, \quad \frac{\hat{\Sigma}_k^{Yb}}{Q_k^{eb}} = \frac{S_k + \sqrt{S_k^2 + (2 + \delta) \delta \Delta_k^2}}{2(2 + \delta)} - \frac{\Delta_k}{2} \quad (2.78)$$

and the corresponding equilibrium agent-effort processes are

$$\hat{e}_k^i(t, \hat{\Sigma}_t^Y) = \frac{\hat{\Sigma}_k^{Yi}}{\Gamma_t Q_k^{ei}} \left(\frac{\hat{\Sigma}_k^{Ya}}{\Gamma_t Q_k^{ea}} + \frac{\hat{\Sigma}_k^{Yb}}{\Gamma_t Q_k^{eb}} \right)^{\frac{-\delta}{1+\delta}}, \quad i \in \{a, b\} \quad k = 1, \dots, d. \quad (2.79)$$

Total equilibrium agent effort is

$$\hat{e}_k^a + \hat{e}_k^b = \left(\frac{S_k + \sqrt{S_k^2 + (2 + \delta) \delta \Delta_k^2}}{(2 + \delta) \Gamma_t} \right)^{\frac{1}{1+\delta}}.$$

Proof. Applying Lemma 4 with $\gamma = 1$, $v = 1 - \delta$, $\alpha = 1/2$ (note that the restriction on the parameters is obviously satisfied), the FOC (2.76) implies

²⁵As shown in the proof, when $\Sigma_k^{Za} \leq 0$ the optimum is $\hat{\Sigma}_k^{Ya} = 0$ and $\hat{\Sigma}_k^{Yb} = \Sigma_k^{Zb}$, and vice versa.

$$\begin{aligned}
(S_k^a)^2 + (1 + \delta)S_k^a S_k^b &= (S_k^a + S_k^b) \frac{\Sigma_k^{Za}(1 - \delta)}{2\Gamma_t Q_k^{ea}} \\
(S_k^b)^2 + (1 + \delta)S_k^a S_k^b &= (S_k^a + S_k^b) \frac{\Sigma_k^{Zb}(1 - \delta)}{2\Gamma_t Q_k^{eb}}
\end{aligned} \tag{2.80}$$

Subtracting the second equation above from the first yields

$$S_k^a - S_k^b = \frac{1 - \delta}{2\Gamma_t} \left(\frac{\Sigma_k^{Za}}{Q_k^{ea}} - \frac{\Sigma_k^{Zb}}{Q_k^{eb}} \right)$$

Using the fact that $S_k^a = A \frac{1 - \delta}{2\Gamma_t}$ and $S_k^b = B \frac{1 - \delta}{2\Gamma_t}$, where $A = \frac{\Sigma_k^{Ya}}{Q_k^{ea}}$ and $B = \frac{\Sigma_k^{Yb}}{Q_k^{eb}}$, we get

$$A - B = \Delta_k. \tag{2.81}$$

Plugging (2.81) into the first equation in (2.80), we get an equation for A below

$$(2 + \delta)A^2 - [(2 + \delta)\Delta_k + S_k]A + \frac{\Delta_k(S_k + \Delta_k)}{2} = 0.$$

This yields the solution (2.78).

For verification, in Section .1 in the Appendix, we provide another proof by directly applying Corollary 1. ■

Lemma 5 (Comparative statics) *Under the assumptions of Proposition 6, the equilibrium optimal controls $(\hat{\Sigma}_t^{Ya}, \hat{\Sigma}_t^{Yb}, \hat{e}^a, \hat{e}^b)$ as functions of $(\Sigma_t^{Za}, \Sigma_t^{Zb}, \delta)$, satisfy, for all $t \in [0, T]$ and $i, j \in \{a, b\}$, $j \neq i$, the following:*

a)

$$\frac{1}{1+\delta} \leq \frac{d\hat{\Sigma}_k^{Yi}}{d\Sigma_k^{Zi}} \leq 1, \quad -\frac{\delta}{1+\delta} \frac{Q_k^{ei}}{Q_k^{ej}} \leq \frac{d\hat{\Sigma}_k^{Yi}}{d\Sigma_k^{Zj}} \leq 0,$$

b)

$$\frac{1}{Q_k^{ea}} \frac{d\hat{\Sigma}_k^{Ya}}{d\delta} = \frac{1}{Q_k^{eb}} \frac{d\hat{\Sigma}_k^{Yb}}{d\delta} \leq 0, \quad (2.82)$$

c)

$$\frac{d\hat{e}_k^i}{d\delta} \leq 0,$$

d)

$$\frac{d\hat{e}_k^i}{d\Sigma_k^{Zi}} \geq 0, \quad \frac{d\hat{e}_k^i}{d\Sigma_k^{Zj}} \leq 0.$$

The inequalities in b)-d) are all strict if $\Sigma_k^{Za}, \Sigma_k^{Zb} > 0$.

Proof. See Section .1 in the Appendix. ■

In the k th dimension holding fixed Σ_k^{Zb} , an increase in Σ_k^{Za} implies an increase in both principal a 's control, $\hat{\Sigma}_k^{Ya}$, and agent a 's effort, \hat{e}_k^a ; but a decrease in principal b 's control, $\hat{\Sigma}_k^{Yb}$, and agent b 's effort (though total time- t effort exerted increases). (Analogous results hold for an increase in Σ_k^{Yb} .) The special case of $\delta = 0$ corresponds to an additive measure change and the optimally bearing all the cash-flow risk under the principal's optimal policy: $\hat{\Sigma}_k^{Yi} = \Sigma_k^{Zi}$; the corresponding optimal agent effort is $\hat{e}_k^i = \hat{\Sigma}_k^{Zi}/(\Gamma_t Q_k^{ei})$, which is the first-best effort level for principal i . We also obtain $\hat{\Sigma}_k^{Ya} = \Sigma_k^{Za}$ if $\Sigma_k^{Yb} = 0$, and the corresponding optimal agent effort is $\hat{e}_k^a = \left(\hat{\Sigma}_k^{Ya}/(\Gamma_t Q_k^{ea})\right)^{1/(1+\delta)}$, which is again first-best for principal a (given zero effort by agent b). Finally, an increase in the concavity parameter δ implies a decrease in both principals' controls and both agents' efforts.

The negative dependence of principal a 's sharing rule on principal b 's cash-flow volatility is different from seemingly related results in some team contract settings. Here the interde-

pendence is a free rider problem. With diminishing marginal productivity to effort, higher sensitivity of firm b 's cash flows in any dimension will cause b to offer its agent a larger cash-flow-volatility share in that dimension, inducing more effort by that agent. Higher effort by agent b causes agent a to work less. Principal a responds to the diminished marginal value of its own agent's effort by reducing the cash-flow-volatility share it offers.

The following example illustrates the interdependence of the contracts in a two-dimension setting with constant-volatility cash flows.

Example 13 (Terminal consumption only) *Suppose risk-neutral preferences, the Brownian dimension $d = 2$ and terminal consumption only. The cash flows satisfy*

$$dX_t^i = \sigma^{i'} dB_t, \quad \sigma^i > 0, \quad i \in \{a, b\}, \quad t \in [0, T]$$

, $Q_k^{ea} = Q_k^{eb} = Q_k$, $k = 1, 2$ and a constant interest rate r . By Example 7 (letting $\eta^i = \kappa^i = 0$), the subjective cash-flow volatilities match the discounted actual cash-flow volatilities, $\Sigma^{Zi} = \Gamma_t \sigma^i$, $i \in \{a, b\}$, where $\Gamma_t = 1/r - (1/r - 1)e^{-r(T-t)}$ and the optimal principal volatility controls are

$$\begin{aligned} \hat{\Sigma}_k^{Ya} &= \Gamma_t \left\{ \frac{\sigma_k^a + \sigma_k^b + \sqrt{(\sigma_k^a + \sigma_k^b)^2 + (2 + \delta) \delta (\sigma_k^a - \sigma_k^b)^2}}{2(2 + \delta)} + \frac{\sigma_k^a - \sigma_k^b}{2} \right\}, \\ \hat{\Sigma}_k^{Yb} &= \Gamma_t \left\{ \hat{\Sigma}_k^{Ya} - (\sigma_k^a - \sigma_k^b) \right\}, \quad k = 1, 2. \end{aligned}$$

There exist constants $v_i^i \in (0, 1)$ and $v_j^i \in (-1, 0)$, $i \neq j$, such that the control $\hat{\Sigma}^{Yi}$ is a linear

combination of the cash-flow diffusions:²⁶

$$\hat{\Sigma}^{Yi} = \Gamma_t(v_a^i \sigma^a + v_b^i \sigma^b), \quad i \in \{a, b\}.$$

That is, each principal's volatility sharing rule depends on the volatility of both principals' cash flows. With any feasible intermediate pay p_t^i , $t < T$, the lump-sum terminal pay for agent i is affine in the terminal cash flows of both firms²⁷ :

$$p_T^i = Y_0^i e^{rT} - \int_0^T e^{r(T-t)} p_t^i dt - \Pi^i + v_a^i L^a + v_b^i L^b \quad i \in \{a, b\},$$

where the constant Π^i is defined as²⁸

$$\Pi^i = S \left(\frac{2}{1+\delta} \right) \sum_{k=1}^2 \frac{Q_k^{-\left(\frac{1-\delta}{1+\delta}\right)}}{1-\delta} \left(\frac{\left(\frac{1+\delta}{2}\right) \left(v_a^i \sigma_k^a + v_b^i \sigma_k^b\right)^2 + (v_a^i \sigma_k^a + v_b^i \sigma_k^b)(v_a^j \sigma_k^a + v_b^j \sigma_k^b)}{\left[(v_a^a + v_a^b) \sigma_k^a + (v_b^a + v_b^b) \sigma_k^b\right]^{\frac{2\delta}{1+\delta}}} \right)$$

with

²⁶With $d > 2$ we cannot generally replicate $\hat{\Sigma}^{Yi}$ with a linear combination of cash-flow diffusion vectors because of the nonlinear relationship given in (2.78). With $d = 2$, let the 2×2 matrix $\sigma = [\sigma^a, \sigma^b]$, then $v^i = (v_a^i, v_b^i)'$ is the solution of the linear equation $\sigma v^i = \Sigma^{Yi}$, $i \in \{a, b\}$.

²⁷If $\delta = 0$ then $v_a^i = 1$, $v_b^i = 0$, $i \neq j$ and solution matches (2.68).

²⁸The Γ_t in equation (2.79) needs to be replaced by $D_T/D_t = e^{-r(T-t)}$ throughout the calculation. Apply (2.32) with

$$\hat{\mu}^{Yi}(t, \hat{\Sigma}_t^Y) = -\frac{e^{-r(T-t)}}{2} \sum_{k=1}^2 Q_k e_k^i(t, \hat{\Sigma}_k^Y)^2 + \sum_{k=1}^2 \hat{\Sigma}_k^{Yi} \Phi_k \left(e_k(t, \hat{\Sigma}_k^Y) \right)$$

$$S(k) = \int_0^T \left(e^{r(T-t)} \Gamma_t \right)^k dt$$

and

$$L^i = \int_0^T e^{r(T-t)} \Gamma_t dX_t^i = \frac{1}{r} \int_0^T e^{r(T-t)} dX_t^i - \left(\frac{1}{r} - 1 \right) (X_T^i - X_0^i).$$

Agent a's terminal pay is increasing in L^a and decreasing in L^b (and analogously for agent b). An increase in either cash-flow volatility (in either dimension) results a reduction in the fixed component of both agent's pay because the increase in aggregate effort increases the cash-flow drifts.

Relaxing the $N = 2$ assumption, in the special case of identical cash-flow processes $X_t = \sigma' B_t$ and identical Q^{ei} s (letting Q^e denote the common effort penalty and Σ^Z the common subjective PV diffusion process), the optimal controls and effort are

$$\hat{\Sigma}_t^{Yi} = \left(1 + \left(1 - \frac{1}{N} \right) \delta \right)^{-1} \Gamma_t \sigma, \quad \hat{e}_t^i = N^{\frac{-\delta}{1+\delta}} \left(\frac{\sigma}{Q^e \left(1 + \left(1 - \frac{1}{N} \right) \delta \right)} \right)^{\frac{1}{1+\delta}}, \quad i = 1, \dots, N.$$

When $\delta = 0$ (additive Φ), the effect of N vanishes. As $N \rightarrow \infty$ the optimal control $\hat{\Sigma}_t^{Yi}$ converges to $(1 + \delta)^{-1} \Gamma_t \sigma$ (and Σ_t^{Ui} converges to $(1 + \delta)^{-1} \sigma$ and Σ_t^{Vi} converges to $[1 - (1 + \delta)^{-1}] \sigma$) and individual agent effort converges to zero (though aggregate effort is of order $N^{1/(1+\delta)}$).

The next proposition shows that in the case of a strictly concave measure-change operator,

as N goes to infinity the optimal principal policies are motivated solely by risk-sharing, and individual agent effort goes to zero. The sharing rules converge, as N gets large, to the sharing rule of Proposition 5 with linear Φ and infinite effort cost.

Proposition 7 *Assume, for simplicity, terminal consumption only, deterministic principal aggregator functions h^{Vi} , k^{Vi} for each i , and Ornstein-Uhlenbeck cash flows: $X_t^i = (\eta^i - \kappa^i X_t^i) + v^{i'} B_t$, for each i and $t \in [0, T]$, where $v^i \in \mathbb{R}_+^d$ and the power measure-change specification (2.20) with $\delta > 0$. Also assume*

$$Q^{Vi} \geq \frac{1}{\kappa}, \quad Q^{ei}, Q^{Ui} \leq \kappa, \quad i = 1, 2, \dots, \quad \lim_{N \rightarrow \infty} \sum_{i=1}^N v_k^i = \infty, \quad k = 1, \dots, d, \quad (2.83)$$

for some constant $\kappa > 0$. Then the limiting equilibrium principal and agent controls as $N \rightarrow \infty$ are given by

$$\lim_{N \rightarrow \infty} \hat{\Sigma}_k^{Yi} = \frac{Q^{Vi} \Phi_1^i(t)}{Q^{Vi} + Q^{Ui}} v_k^i \text{ and } \lim_{N \rightarrow \infty} \hat{e}_k^i \rightarrow 0, \quad k = 1, \dots, d.$$

$$\text{where } \Phi_1^i(t) = \frac{1}{r + \kappa^i} - e^{-(r + \kappa^i)(T-t)} \left(\frac{1}{r + \kappa^i} - 1 \right).$$

Derivation. See Section .1 in the Appendix. ■

2.7.2 Quadratic Penalty and Cobb-Douglas Measure Change

For any $\delta^a, \delta^b \in [0, 1)$, $\delta^a + \delta^b > 0$, let $\kappa = 1$, $\alpha v = 1 - \delta^a$ and $(1 - \alpha)v = 1 - \delta^b$.

As $\gamma \rightarrow 0$, the measure change operator converges to

$$\Phi_k(e) = (e_k^a)^{1-\delta^a} (e_k^b)^{1-\delta^b}, \quad k = 1, \dots, d$$

In Section 2.3.3, we obtain for each dimension k and time t the following possible Nash equilibria

$$\begin{aligned} e_k^a &= \left(\frac{\Sigma_k^{Y^{a+}}(1 - \delta^a)}{\Gamma_t Q_k^{ea}} \right)^{\frac{1+\delta^b}{2(\delta^a+\delta^b)}} \left(\frac{\Sigma_k^{Y^{b+}}(1 - \delta^b)}{\Gamma_t Q_k^{eb}} \right)^{\frac{1-\delta^b}{2(\delta^a+\delta^b)}} \\ e_k^b &= \left(\frac{\Sigma_k^{Y^{b+}}(1 - \delta^b)}{\Gamma_t Q_k^{eb}} \right)^{\frac{1+\delta^a}{2(\delta^a+\delta^b)}} \left(\frac{\Sigma_k^{Y^{a+}}(1 - \delta^a)}{\Gamma_t Q_k^{ea}} \right)^{\frac{1-\delta^a}{2(\delta^a+\delta^b)}} \end{aligned}$$

or

$$e_k^a = 0, \quad e_k^b = 0.$$

Proposition 8 (Risk neutrality) *Suppose Φ satisfies the above specification, and preferences satisfy the risk-neutral specification (2.74) with $Q^{Ui} = Q^{Vi} = 0$. The principal equilibrium solution is*

$$\Sigma^{Ya} = \frac{2\Sigma^{Za+}}{1 + \delta^b}, \quad \Sigma^{Yb} = \frac{2\Sigma^{Zb+}}{1 + \delta^a}$$

Proof. From the parameter specification $\alpha v = 1 - \delta^a$ and $(1 - \alpha)v = 1 - \delta^b$, we get

$$v = 2 - (\delta^a + \delta^b) \text{ and } \alpha = \frac{1 - \delta^a}{2 - (\delta^a + \delta^b)}.$$

Applying Lemma 4 with the above values and the identity $S^a = \frac{\alpha v \Sigma^{Ya}}{\Gamma_t Q^{ea}}$ and letting $\gamma \rightarrow 0$, (2.76) implies the following equation for Σ^{Ya} .

$$\left[\frac{1 - \delta^a}{2 - (\delta^a + \delta^b)} \frac{2}{\delta^a + \delta^b} + \frac{1 - \delta^b}{2 - (\delta^a + \delta^b)} \right] \Sigma^{Ya} = \Sigma^{Za} + \frac{2}{\delta^a + \delta^b}$$

Using the identity $2(1 - \delta^a) + (1 - \delta^b)(\delta^a + \delta^b) = [2 - (\delta^a + \delta^b)](1 + \delta^b)$, we get the solution $\Sigma^{Ya} = \frac{2\Sigma^{Za}}{1 + \delta^b}$. Similarly, we can get the solution for Σ^{Yb} .

For verification, in Section .1 in the Appendix, we provide another proof by directly applying Corollary 1. ■

In the case of identical principal-agent pairs and constant return to scale i.e.

$$Q^{Ua} = Q^{Ub} = Q^U, \quad Q^{Va} = Q^{Vb} = Q^V, \quad Q^{ea} = Q^{eb} = Q^e, \quad \delta^a = \delta^b = \frac{1}{2}$$

The solution is

$$\Sigma^Y = \frac{16Q^e Q^V + 4}{16Q^e(Q^V + Q^U) + 3} \Sigma^{Z+}$$

Example 14 (Terminal consumption only) *Assume the cash-flow dynamics are*

$$dX_t^i = \sigma^{i'} dB_t, \quad i \in \{a, b\}$$

for some $\sigma^i \in (\mathbb{R}^+)^d$, $Q^{Vi} = 0$, $Q^{Ui} = 0$, $i \in \{a, b\}$ and terminal consumption only.

The equilibrium principal i 's control and agent effort are

$$\Sigma^{Yi} = \frac{2\Gamma_t \sigma^i}{1 + \delta^j}, \quad \hat{e}_k^i = \left(\frac{2\sigma_k^i(1 - \delta^i)}{(1 + \delta^j)Q_k^{ei}} \right)^{\frac{1+\delta^j}{2(\delta^i+\delta^j)}} \left(\frac{2\sigma_k^j(1 - \delta^j)}{(1 + \delta^i)Q_k^{ej}} \right)^{\frac{1-\delta^j}{2(\delta^i+\delta^j)}} \left(e^{r(T-t)} \Gamma_t \right)^{\frac{1}{\delta^i+\delta^j}}$$

where

$$\Gamma_t = \frac{1}{r} - e^{-r(T-t)} \left(\frac{1}{r} - 1 \right).$$

The terminal pay is

$$\begin{aligned} p_T^i &= (\Gamma_0 K^i - w_0^{Ui}) e^{rT} - \int_0^T e^{r(T-t)} p_t^i dt \\ &\quad - S \left(\frac{2}{\delta^i + \delta^j} \right) \sum_{k=1}^d (C^i)^{\frac{1-\delta^i}{\delta^i+\delta^j}} (C^j)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \frac{1+\delta^i}{1+\delta^j} (\sigma_k^i)^{\frac{1+\delta^j}{\delta^i+\delta^j}} (\sigma_k^j)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \\ &\quad + \frac{2}{1+\delta^j} L^i. \end{aligned} \tag{2.84}$$

where

$$S(k) = \int_0^T \left(e^{r(T-t)} \Gamma_t \right)^k dt$$

and

$$L^i = \int_0^T e^{r(T-t)} \Gamma_t dX_t^i = \frac{1}{r} \int_0^T e^{r(T-t)} dX_t^i - \left(\frac{1}{r} - 1 \right) (X_T^i - X_0^i).$$

Derivation. See Section .2 in the Appendix. ■

2.7.3 Absolute Effort Penalty and Square-Root Measure Change

In this section we build on Example 6 and solve for the principal equilibrium with an absolute effort penalty and, to obtain a closed form solution, a square-root measure change (i.e., the power measure-change operator (2.20) with $\delta = 1/2$). Unlike the quadratic-effort-penalty solution in the previous section, any equilibrium with absolute effort penalty has at most one agent working in any given dimension at any moment. Furthermore, there are regions with more than one possible principal equilibrium.

For simplicity, we assume throughout this section one-dimensional Brownian motion²⁹ ($d = 1$), nonnegative effort ($\mathbf{E} = \mathbb{R}_+$), and two principals/agents ($N = 2$), which we label a and b . We also assume the preferences and measure-change operator

$$k^{Ui} \left(t, e_t^i, \Sigma_t^{Ui} \right) = -\frac{1}{2} Q_t^{Ui} (\Sigma_t^{Ui})^2 - q_t^i e_t^i, \quad k^{Vi} \left(t, \Sigma_t^{Vi} \right) = -\frac{1}{2} Q_t^{Vi} (\Sigma_t^{Vi})^2, \quad i \in \{a, b\},$$

$$\Phi(e) = 2\sqrt{e_t^a + e_t^b},$$

where $q^i, Q^{Ui} \in \mathcal{L}(\mathbb{R}_{++})$ and $Q^{Vi} \in \mathcal{L}(\mathbb{R}_+)$. When $N = 1$ this specification is equivalent to the quadratic/linear specification³⁰ (2.57) with $\mathbf{E} = \mathbb{R}_+$.

²⁹The extension to $d > 1$ with diagonal preference parameters is simple: as in the agent equilibrium in Example 6, the principal equilibrium below applies to each dimension.

³⁰This is seen by redefining effort as $\tilde{e}_t = 2\sqrt{e_t}$.

Define, for $i \in \{a, b\}$,

$$\underline{w}_t^i = \frac{Q_t^{Vi}}{Q_t^{Vi} + Q_t^{Ui}}, \quad \bar{w}_t^i = \frac{\frac{2}{q_t^i} + Q_t^{Vi}}{\frac{2}{q_t^i} + Q_t^{Vi} + Q_t^{Ui}}, \quad \tilde{w}_t^i = \frac{1}{4} q_t^i Q_t^{Vi} (\bar{w}_t^i - \underline{w}_t^i) + \frac{1}{2} \bar{w}_t^i.$$

We assume that $Q^{Va}, Q^{Vb} \geq 0$ are sufficiently small that

$$\left(\frac{1}{2} Q_t^{Vi} + \frac{1}{q_t^i} \right) \bar{w}_t^i > \left(\frac{1}{2} Q_t^{Vi} + \frac{2}{q_t^i} \right) \underline{w}_t^i, \quad t \in [0, T], \quad i \in \{a, b\}. \quad (2.85)$$

Condition (2.85) implies $\tilde{w}_t^i \in (\underline{w}_t^i, \bar{w}_t^i)$.

The agent equilibrium was given by (2.22) in Example 6. The following proposition characterizes the principal equilibrium controls $(\hat{\Sigma}^{Ya}, \hat{\Sigma}^{Yb})$ and agent equilibrium efforts (\hat{e}^a, \hat{e}^b) .

Proposition 9 *If $\Sigma_t^{Za}, \Sigma_t^{Zb} \leq 0$ then $\hat{\Sigma}_t^{Yi} = \underline{w}_t^i \Sigma_t^{Zi}$ for each $i \in \{a, b\}$. If $\Sigma_t^{Zi} > 0$ and*

$$\Sigma_t^{Zi} > \Sigma_t^{Zj} \left(\frac{q_t^i}{q_t^j} \right) \max \left(\frac{\tilde{w}_t^j}{\bar{w}_t^i}, \frac{\underline{w}_t^j}{\tilde{w}_t^i} \right), \quad i, j \in \{a, b\}, \quad i \neq j,$$

then there is an equilibrium with $\hat{\Sigma}_t^{Yi} = \bar{w}_t^i \Sigma_t^{Zi}$ and $\hat{\Sigma}_t^{Yj} = \underline{w}_t^j \Sigma_t^{Zj}$. The corresponding agent-effort equilibria are

$$\begin{aligned} \hat{\Sigma}_t^{Yi} &= \underline{w}_t^i \Sigma_t^{Zi} &\implies \hat{e}_t^i &= 0, \\ \hat{\Sigma}_t^{Yi} &= \bar{w}_t^i \Sigma_t^{Zi} &\implies \hat{e}_t^i &= \left(\frac{\hat{\Sigma}_t^{Yi}}{\Gamma_t q_t^i} \right)^2, \quad i \in \{a, b\}, \end{aligned}$$

and the BSDE system for (Z^i, Σ^{Zi}) satisfies (2.44) with

$$\begin{aligned} \hat{\mu}^{Zi}(t, \Sigma_t^Y) &= \left[-\frac{1}{2} Q_t^{Vi} (1 - \bar{w}_t^i)^2 - \frac{1}{2} \left(Q_t^{Ui} + \frac{2}{q_t^i} \right) (\bar{w}_t^i)^2 + 2 \frac{\bar{w}_t^i}{q_t^i} \right] \frac{(\Sigma_t^{Zi})^2}{\Gamma_t} 1_{\{\hat{e}_t^i > 0\}} \\ &+ \left[-\frac{1}{2} \left\{ Q_t^{Vi} (1 - \underline{w}_t^i)^2 + Q_t^{Ui} (\underline{w}_t^i)^2 \right\} \frac{(\Sigma_t^{Zi})^2}{\Gamma_t} + 2 \frac{\bar{w}_t^j}{q_t^j} \frac{\Sigma_t^{Yi} (\Sigma_t^{Yj})^+}{\Gamma_t} \right] 1_{\{\hat{e}_t^i = 0\}}. \end{aligned} \quad (2.86)$$

Derivation. See Section .1 in the Appendix. ■

An equilibrium with principal- i sharing the proportion \underline{w}_t^i of Σ_t^{Zi} corresponds to zero equilibrium agent- i exert, and the higher proportion \bar{w}_t^i corresponds to strictly positive equilibrium agent- i effort. The lower proportion is motivated purely by risk sharing, and the higher proportion by both risk sharing and agent-effort incentive. If Σ_t^{Zi} is not positive, then the only principal equilibrium is with $\hat{\Sigma}_t^{Yi} = \underline{w}_t^i \Sigma_t^{Zi}$ and zero agent- i effort. If $\Sigma_t^{Zi} > 0$ then there is at least one principal equilibrium depending on the magnitude of $\Sigma_t^{Zi}/\Sigma_t^{Zj}$. Defining $\underline{\lambda}_t = \left(\frac{q_t^b}{q_t^a} \right) \max \left(\frac{\tilde{w}_t^a}{\bar{w}_t^b}, \frac{w_t^a}{\tilde{w}_t^b} \right)$ and $\bar{\lambda}_t = \left(\frac{q_t^b}{q_t^a} \right) \min \left(\frac{\tilde{w}_t^a}{\underline{w}_t^b}, \frac{\bar{w}_t^a}{\bar{w}_t^b} \right)$, there is a time- t equilibrium with only agent b working if $\Sigma_t^{Zb}/\Sigma_t^{Za} > \underline{\lambda}_t$ and an equilibrium with only agent a working if $\Sigma_t^{Zb}/\Sigma_t^{Za} < \bar{\lambda}_t$. Because $\bar{\lambda}_t > \underline{\lambda}_t$ we have both equilibria possible if $\Sigma_t^{Zb}/\Sigma_t^{Za} \in (\underline{\lambda}_t, \bar{\lambda}_t)$.

For example, if $Q^{Va} = Q^{Vb} = 0$ (risk-neutral principals) and the preference parameters are identical ($Q_t^{Ua} = Q_t^{Ub}$ and $q_t^a = q_t^b$), then $\underline{\lambda}_t = 1/2$ and $\bar{\lambda}_t = 2$. Risk-neutrality of principal i implies that agent- i diffusion is zero in a zero-effort- i equilibrium because there is no risk-sharing motive on the part of the principal to share cash-flow risk with the agent.

For comparison, in a team setting with a single principal, identical agents and absolute effort penalties, it can be shown that the optimal principal policy is to choose identical agent

diffusions, resulting in a continuum of agent equilibria.

Example 15 (Equilibrium with only agent a working) *We assume constant preference parameters and interest rate r , terminal consumption only and constant cash-flow volatility:*

$$dX_t^i = \sigma^i dB_t, \quad \sigma^i > 0, \quad i \in \{a, b\}. \quad (2.87)$$

Note that by Example 7, $\Sigma^{Zi} = \Gamma_t \sigma^i$, where $\Gamma_t = \frac{1}{r} - e^{-r(T-t)}(\frac{1}{r} - 1)$.

Assume condition (2.85) as well as

$$\frac{\sigma^a}{\sigma^b} > \left(\frac{q^a}{q^b} \right) \max \left(\frac{\tilde{w}^b}{\bar{w}^a}, \frac{w^b}{\tilde{w}^a} \right).$$

Then Proposition 9 implies that equilibrium principal and agent controls are

$$\begin{aligned} \hat{\Sigma}^{Ya} &= \bar{w}^a \Gamma_t \sigma^a, \quad \hat{\Sigma}^{Yb} = \underline{w}^b \Gamma_t \sigma^b, \\ \hat{e}_t^a &= \left\{ \frac{\bar{w}^a \sigma^a}{q^a} \Gamma_t e^{r(T-t)} \right\}^2, \quad \hat{e}_t^b = 0. \end{aligned}$$

Equilibrium terminal pay are

$$P_T^a = \bar{w}^a L^a + \frac{1}{2} \left(Q^{Ua} - \frac{2}{q^a} \right) (\bar{w}^a \sigma^a)^2 S(2) + e^{rT} (\Gamma_0 K^a - w_0^{Ua}) \quad (2.88)$$

$$\begin{aligned} & - \int_0^T e^{r(T-t)} p_t^a dt, \\ P_T^b &= \underline{w}^b \left(L^b - 2 \frac{\bar{w}^a}{q^a} \sigma^b \sigma^a S(2) \right) + \frac{1}{2} Q^{Ub} \left(\underline{w}^b \sigma^b \right)^2 S(2) + e^{rT} (\Gamma_0 K^b - w_0^{Ub}) \quad (2.89) \\ & - \int_0^T e^{r(T-t)} p_t^b dt. \end{aligned}$$

where

$$S(k) = \int_0^T \left(e^{r(T-t)} \Gamma_t \right)^k dt$$

and

$$L^i = \int_0^T e^{r(T-t)} \Gamma_t dX_t^i = \frac{1}{r} \int_0^T e^{r(T-t)} dX_t^i - \left(\frac{1}{r} - 1 \right) (X_T^i - X_0^i)$$

Derivation. See Section .2 in the Appendix. ■

If, for simplicity, agent preferences are identical, and principal preferences are identical, then $\sigma^a = \sigma^b$ implies that principal a has a higher utility in an equilibrium with agent b working rather than an equilibrium with only agent a working. This follows because the same effort is exerted in either equilibrium, but principal b pays for the disutility of agent effort if agent b works. However, if σ^b is sufficiently close to zero, then principal a can be better off with his/her own agent working because under the agent- b -working equilibrium insufficient effort is exerted.

The example is easily extended to multiple dimension ($d > 1$), in which case the same results holds dimension by dimension, and equilibria are characterized by only one agent exerting effort in each dimension at any point in time.

APPENDICES

.1 Appendix 1: Proofs Omitted from the Text

Proof of Proposition 4.

Matching the diffusion term of (2.44) and (2.52), we get

$$\Sigma_t^{Zi} = \Sigma_t^{\zeta^i} \Theta_t^i$$

Matching the drifts, we get

$$\dot{\theta}_t^i + \Theta_t^{i'} \mu + (\dot{\Theta}_t^i + \beta^{i'} \Theta_t^i)' \zeta_t^i = r_t \theta_t^i - \mu^{Zi}(t, \Sigma_t^{\zeta^i} \Theta_t^i) + (r_t \Theta_t^i - M_t^i)' \zeta_t^i$$

Matching coefficients of ζ_t^i yields the ODE (2.50), and matching the other terms yields the ODE

$$\dot{\theta}_t^i - r_t \theta_t^i = -\mu^{Zi} \left(t, \Sigma^{\zeta^1} \Theta_t^1, \dots, \Sigma^{\zeta^N} \Theta_t^N \right) - \Theta_t^{i'} \mu^i, \quad \theta_T^i = 0,$$

which has the solution (2.51). It is easy to then confirm that (2.52) solve the BSDE system (2.44).

Below is a derivation of the closed form solution for Θ^i .

To solve

$$\dot{\Theta}_t = -(\beta' - r_t I) \Theta_t - M_t, \quad t < T, \quad \Theta_T = M_T$$

where Θ_t , M_t are $n \times 1$ vectors, β is an $n \times n$ matrix, r_t is a function, I is an identity matrix.

To avoid confusion, for the exponential functions we will use $\exp(\cdot)$ with matrix arguments

and e^\cdot with scalar arguments.

First note

$$de^{-\int_0^t r_s ds} \Theta_t = \left[-\beta' e^{-\int_0^t r_s ds} \Theta_t - e^{-\int_0^t r_s ds} M_t \right] dt$$

Substituting $\xi_t = e^{-\int_0^t r_s ds} \Theta_t$ and $N_t = e^{-\int_0^t r_s ds} M_t$, we get the equation

$$d\xi_t = [-\beta' \xi_t - N_t] dt, \quad \xi_T = e^{-\int_0^T r_s ds} M_T$$

Let $y_t = \exp(t\beta') \xi_t$.

$$\begin{aligned} \frac{dy_t}{dt} &= \beta' \exp(t\beta') \xi_t + \exp(t\beta') \frac{d\xi_t}{dt} \\ &= \beta' \exp(t\beta') \xi_t + \exp(t\beta') [-\beta' \xi_t - N_t] dt \\ &= -\exp(t\beta') N_t dt \end{aligned}$$

where we have used $\beta' \exp(t\beta') = \exp(t\beta') \beta'$.

Integrating the above equation from t to T and use $y_T = \exp(T\beta') e^{-\int_0^T r_s ds} M_T$ to get

$$y_t = \exp(T\beta') e^{-\int_0^T r_s ds} M_T + \int_t^T \exp(s\beta') N_s ds$$

Substitute in $y_t = \exp(t\beta') e^{-\int_0^t r_s ds} \Theta_t$ and $N_s = e^{-\int_0^s r_u du} M_s$ to get

$$\exp(t\beta')e^{-\int_0^t r_s ds}\Theta_t = \exp(T\beta')e^{-\int_0^T r_s ds}M_T + \int_t^T \exp(s\beta')e^{-\int_0^s r_u du}M_s ds$$

Left multiply both sides by $\exp(-t\beta')e^{\int_0^t r_s ds}$ and use the fact that $t_1\beta'$ and $t_2\beta'$ are commutable where t_1 and t_2 are any arbitrary real numbers to get

$$\Theta_t = \exp((T-t)\beta')e^{-\int_t^T r_s ds}M_T + \int_t^T \exp((s-t)\beta')e^{-\int_t^s r_u du}M_s ds$$

■

Proof of Proposition 6.

Let

$$A = \frac{(\Sigma_k^{Ya})^+}{\Gamma_t Q_k^{ea}}, \quad B = \frac{(\Sigma_k^{Yb})^+}{\Gamma_t Q_k^{eb}}$$

For notational simplicity, omit the dimension subscripts k . From equation (2.43) and (2.21), at principal equilibrium given B

$$A \in \arg \max_{A \in \mathbb{R}^+} f(A)$$

where

$$f(A) = -\frac{\Gamma_t Q^{ea}}{2} A^2 (A+B)^{\frac{-2\delta}{1+\delta}} + \frac{\Sigma^{Za}}{1-\delta} (A+B)^{\frac{1-\delta}{1+\delta}}$$

If $\Sigma^{Za} \leq 0$, then it is easy to check the optimal $A = 0$, $B = \frac{\Sigma^{Zb}}{\Gamma_t Q^{eb}}$ and vice versa. We will assume $\Sigma^{Za} > 0$, $\Sigma^{Zb} > 0$ and calculate the following derivatives.

$$f'(A) = -\Gamma_t Q^{ea} A(A+B)^{\frac{-2\delta}{1+\delta}} + \frac{\Gamma_t Q^{ea} \delta A^2}{1+\delta} (A+B)^{\frac{-3\delta-1}{1+\delta}} + \frac{\Sigma^{Za}}{1+\delta} (A+B)^{\frac{-2\delta}{1+\delta}}$$

$$f''(A) = \frac{(A+B)^{\frac{-4\delta-2}{1+\delta}}}{(1+\delta)^2} \left[\Gamma_t Q^{ea} g(A) - 2\delta \Sigma^{Za} (A+B) \right]$$

where

$$g(A) = 4\delta(1+\delta)A(A+B) - (1+\delta)^2(A+B)^2 - \delta(3\delta+1)A^2$$

It is easy to verify that $g(A) \leq 0$ and thus $f''(A) \leq 0$.

Let $\sigma^a = \frac{\Sigma^{Za}}{Q^{ea}}$ and $\sigma^b = \frac{\Sigma^{Zb}}{Q^{eb}}$. Finding the equilibrium is equivalent to solving the following FOC equations

$$-\Gamma_t(1+\delta)A(A+B) + \Gamma_t \delta A^2 + \sigma^a(A+B) = 0$$

$$-\Gamma_t(1+\delta)B(A+B) + \Gamma_t \delta B^2 + \sigma^b(A+B) = 0$$

Upon subtracting the second equation from the first, we get $A-B = \frac{\sigma^a - \sigma^b}{\Gamma_t}$. Substituting this equality back to get

$$(2+\delta)(\Gamma_t A)^2 - [(2+\delta)\Delta + S]\Gamma_t A + \frac{(S+\Delta)\Delta}{2} = 0$$

where $S = \sigma^a + \sigma^b$ and $\Delta = \sigma^a - \sigma^b$.

This yields the solution (2.78).

■

Proof of Lemma 5.

Each proof is for $i = a$ and $j = b$ (the other case is proved by reversing the roles of the two labels).

Proof of a) From the expression (2.78) we have

$$\frac{d\hat{\Sigma}_t^{Ya}}{d\Sigma_t^{Za}} = \frac{1}{2(2+\delta)} \{1 + g(\Delta_t, S_t)\} + \frac{1}{2}$$

where

$$g(\Delta_t, S_t) = \frac{S_t + (2+\delta)\delta\Delta_t}{\sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2}}. \quad (90)$$

Holding fixed S_t we have $\partial g(\Delta_t, S_t) / \partial \Delta_t \geq 0$ and therefore $g(-S_t, S_t) \leq g(\Delta_t, S_t) \leq g(S_t, S_t)$. Substituting $g(-S_t, S_t) = \frac{(2+\delta)(1-\delta)}{1+\delta} - 1$ and $g(S_t, S_t) = 1 + \delta$ yields the first inequality. Note that $d\hat{\Sigma}_t^{Ya}/d\Sigma_t^{Za}$ is minimized at $\Delta_t = -S_t$ (i.e., at $\Sigma_t^{Ya} = 0$) and maximized at $\Delta_t = S_t$ (i.e., $\Sigma_t^{Yb} = 0$), holding fixed S .

Again from (2.78) we have

$$d\left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}}\right)/d\left(\frac{\Sigma_t^{Zb}}{Q_t^{eb}}\right) = \frac{1}{2(2+\delta)} \{1 + g(-\Delta_t, S_t)\} - \frac{1}{2} \quad (91)$$

Because g is increasing in Δ_t it follows that the left side is decreasing in Δ_t , and therefore maximized at $\Delta_t = -S_t$ (i.e., at $\Sigma_t^{Ya} = 0$) and minimized at $\Delta_t = S_t$ (i.e., at $\Sigma_t^{Yb} = 0$).

Substituting the above expressions for $g(-S_t, S_t)$ and $g(S_t, S_t)$ give the bounds.

Proof of b) From the solution (2.78) we have

$$\begin{aligned} \frac{2}{Q_t^{ea}} \frac{d\hat{\Sigma}_t^{Ua}}{d\delta} &= \frac{d}{d\delta} \frac{S_t + \sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2}}{2+\delta} \\ &= \frac{\frac{1}{2}(2+\delta)(2+2\delta)\Delta_t^2}{(2+\delta)^2 \sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2}} - \frac{S_t + \sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2}}{(2+\delta)^2} \end{aligned}$$

Derivative is negative if and only if

$$(2+\delta)\Delta_t^2 \leq S_t \sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2} + S_t^2$$

which follows easily. (The derivative equals zero when $S_t = \Delta_t$.)

Proof of c) Differentiating (2.79) and using the equality in (2.82)

$$\begin{aligned} (\Gamma_t)^{1/(1+\delta)} \frac{\partial \hat{e}^a}{\partial \delta} &= \frac{\partial \left(\hat{\Sigma}_t^{Ya}/Q_t^{ea} \right)}{\partial \delta} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right)^{\frac{-\delta}{1+\delta}-1} \left\{ \frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} - 2 \frac{\delta}{1+\delta} \frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} \right\} \\ &= \frac{\partial \left(\hat{\Sigma}_t^{Ya}/Q_t^{ea} \right)}{\partial \delta} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right)^{\frac{-\delta}{1+\delta}-1} \left\{ \frac{S_t + \sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2}}{(2+\delta)(1+\delta)} - \frac{\delta}{1+\delta} \Delta_t \right\} \end{aligned}$$

The braced term is positive iff

$$S_t + \sqrt{S_t^2 + (2+\delta)\delta\Delta_t^2} > (2+\delta)\delta\Delta_t.$$

This is obviously true for $\Delta_t \leq 0$. Now consider $\Delta_t > 0$. Because the left side is increasing in S and $S_t \geq \Delta_t$ the proof is completed by showing the inequality holds at $S_t = \Delta_t$, which follows because $\delta < 1$.

Proof of d) We first have

$$\frac{d}{d(\Sigma_t^{Za}/Q_t^{ea})} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right) = \frac{1}{2+\delta} \{1 + g(\Delta_t, S_t)\}$$

where g is defined in (90). Differentiating (2.79) yields

$$\begin{aligned} (\Gamma_t)^{1/(1+\delta)} \frac{d\hat{e}^a}{d(\Sigma_t^{Za}/Q_t^{ea})} &= \left[\frac{1}{2(2+\delta)} \{1 + g(\Delta_t, S_t)\} + \frac{1}{2} \right] \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right)^{\frac{-\delta}{1+\delta}} \\ &\quad - \frac{\delta}{1+\delta} \frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right)^{\frac{-\delta}{1+\delta}-1} \frac{1}{2+\delta} \{1 + g(\Delta_t, S_t)\} \end{aligned}$$

and therefore

$$\begin{aligned} &(\Gamma_t)^{1/(1+\delta)} \left\{ \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right)^{\frac{\delta}{1+\delta}+1} \right\} \frac{d\hat{e}^a}{d(\Sigma_t^{Za}/Q_t^{ea})} \\ &= \frac{1}{2} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right) + \left\{ \frac{1 + g(\Delta_t, S_t)}{2+\delta} \right\} \left\{ \frac{1}{2} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right) - \frac{\delta}{1+\delta} \frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} \right\}. \end{aligned}$$

Positivity of $d\hat{e}^a/d\Sigma_t^{Ya}$ follows because $\frac{\delta}{1+\delta} < 1/2$.

Now use

$$\frac{d}{d(\Sigma_t^{Zb}/Q_t^{eb})} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right) = \frac{1}{2+\delta} \{1 + g(-\Delta_t, S_t)\}$$

and (91) to get

$$\begin{aligned} &(\Gamma_t)^{1/(1+\delta)} \left\{ \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right)^{\frac{\delta}{1+\delta}+1} \right\} \frac{d\hat{e}^a}{d(\Sigma_t^{Zb}/Q_t^{eb})} \\ &= -\frac{1}{2} \left(\frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} + \frac{\hat{\Sigma}_t^{Yb}}{Q_t^{eb}} \right) \left\{ 1 - \frac{1 + g(-\Delta_t, S_t)}{2+\delta} \right\} - \left\{ \frac{1 + g(-\Delta_t, S_t)}{2+\delta} \right\} \frac{\delta}{1+\delta} \frac{\hat{\Sigma}_t^{Ya}}{Q_t^{ea}} \end{aligned}$$

The inequality $g(-\Delta_t, S_t) \leq 1 + \delta$ implies that \hat{e}^a is decreasing in Σ_t^{Yb} .

■

Proof of Proposition 7.

The maximization problem can be handled dimension by dimension. In each dimension k , omitting the dimension argument k let $y^i = \frac{\Sigma^{Yi}}{\Gamma_t Q^{ei}}$, $i = 1, \dots, N$, $y^{-i} = \sum_{j \neq i} y^j$ and $\sigma^i = \frac{\Sigma^{Zi}}{Q^{ei}}$. The following Lemma implies the following bounds on principal i 's optimal policy: $\hat{y}^i \in \left(\frac{Q^{Vi}}{Q^{Vi} + Q^{Ui}} \frac{1}{(1+\delta)\Gamma_t} \sigma^i, \frac{\sigma^i}{\Gamma_t} \right)$.

By Corollary 1, the principal i 's problem is

$$\max_{y^i} f(y^i)$$

where

$$\begin{aligned} f(y^i) = & \Gamma_t \left\{ -\frac{Q^{Vi}(Q^{ei})^2}{2} \left(\frac{\sigma^i}{\Gamma_t} - y^i \right)^2 - \frac{Q^{Ui}}{2} (y^i Q^{ei})^2 - \frac{Q^{ei}}{2} (y^i)^2 (y^i + y^{-i})^{\frac{-2\delta}{1+\delta}} \right\} \\ & + \sigma^i Q^{ei} \frac{(y^i + y^{-i})^{\frac{1-\delta}{1+\delta}}}{1-\delta} \end{aligned} \quad (92)$$

Lemma 6 Assume $\sigma^i, Q^{Vi}, \delta > 0$. Then the function f in (92) satisfies

$$f'(y^i) > 0 \text{ for } y^i \in \left[0, \frac{\sigma^i}{\Gamma_t(1+\delta)} \frac{Q^{Vi}}{Q^{Vi} + Q^{Ui}} \right] \quad \text{and} \quad f'(y^i) < 0 \text{ for } y^i \geq \frac{\sigma^i}{\Gamma_t}.$$

Proof.

The first derivative is

$$\begin{aligned}
f'(y^i) &= (Q^{ei})^2 \left(Q^{Vi} \sigma^i - \Gamma_t (Q^{Vi} + Q^{Ui}) y^i \right) \\
&\quad + \frac{(y^i + y^{-i})^{\frac{-2\delta}{1+\delta}}}{1+\delta} Q^{ei} \left[\sigma^i - \Gamma_t y^i - \Gamma_t y^{-i} \delta \frac{y^i}{y^i + y^{-i}} \right]
\end{aligned} \tag{93}$$

If $y^i \in \left[0, \frac{\sigma^i}{\Gamma_t(1+\delta)} \frac{Q^{Vi}}{Q^{Vi} + Q^{Ui}} \right]$ then both braced term are positive, the second because

$$\sigma^i - \Gamma_t y^i - \Gamma_t y^{-i} \delta \frac{y^i}{y^i + y^{-i}} \geq \sigma^i - \Gamma_t (1 + \delta) y^i > 0.$$

If $y^i \geq \sigma^i$, then both braced terms are negative. ■

The second derivative of $f(y^i)$ is

$$\begin{aligned}
f''(y^i) &= -\Gamma_t (Q^{ei})^2 (Q^{Vi} + Q^{Ui}) - \frac{(y^i + y^{-i})^{\frac{-2\delta}{1+\delta}-2} Q^{ei}}{1+\delta} \left[\frac{2\delta \sigma^i (y^i + y^{-i})}{1+\delta} \right. \\
&\quad \left. + \Gamma_t \left(\frac{1-\delta}{1+\delta} (y^i)^2 + 2(1-\delta) y^i y^{-i} + (1+\delta) (y^{-i})^2 \right) \right] \leq 0
\end{aligned}$$

The concavity of f together with differentiability imply that i 's optimal policy \hat{y}^i is uniquely given by the FOC $f'(\hat{y}^i) = 0$. The next Lemma bounds principal i 's optimal policy as a function of the other principals' policies,

Lemma 7 *The optimal y^i satisfies*

$$\hat{y}^i = \frac{Q^{Vi} \sigma^i}{(Q^{Vi} + Q^{Ui}) \Gamma_t} + \varepsilon(y^{-i}) \tag{94}$$

where $\varepsilon(y^{-i})$ is of order $(y^{-i})^{\frac{-2\delta}{1+\delta}}$ given by

$$\begin{aligned} & -\frac{1}{Q^{ei}(Q^{Vi} + Q^{Ui})} \frac{\delta}{(1+\delta)\Gamma_t} \sigma^i(y^{-i})^{\frac{-2\delta}{1+\delta}} \leq \varepsilon(y^{-i}) \\ & \leq \frac{1}{Q^{ei}(Q^{Vi} + Q^{Ui})} \frac{1}{(1+\delta)\Gamma_t} \sigma^i(y^{-i})^{\frac{-2\delta}{1+\delta}} \end{aligned}$$

Proof. Substituting (94) into (93) and equating to zero yields

$$Q^{ei}(Q^{Vi} + Q^{Ui}) \Gamma_t \varepsilon(y^{-i}) = \frac{1}{1+\delta} \left\{ \sigma^i - \Gamma_t \hat{y}^i - \Gamma_t \delta y^{-i} \frac{\hat{y}^i}{\hat{y}^i + y^{-i}} \right\} (\hat{y}^i + y^{-i})^{\frac{-2\delta}{1+\delta}}. \quad (95)$$

The inequality $\sigma^i - \Gamma_t y - \Gamma_t \delta y^{-i} \frac{y}{y+y^{-i}} < \sigma^i$ implies

$$\varepsilon(y^{-i}) \leq \frac{1}{Q^{ei}(Q^{Vi} + Q^{Ui})} \frac{1}{(1+\delta)\Gamma_t} \sigma^i(y^{-i})^{\frac{-2\delta}{1+\delta}}.$$

Substitute

$$\sigma^i - \Gamma_t \hat{y}^i - \Gamma_t \delta y^{-i} \frac{\hat{y}^i}{\hat{y}^i + y^{-i}} = \sigma^i - \left(\frac{\hat{y}^i + (1+\delta)y^{-i}}{\hat{y}^i + y^{-i}} \right) \Gamma_t \hat{y}^i \geq \sigma^i - (1+\delta) \Gamma_t \hat{y}^i > -\delta \sigma^i$$

into (95) (the last inequality because $\Gamma_t \hat{y}^i < \sigma^i$) to get

$$Q^{ei}(Q^{Vi} + Q^{Ui}) \Gamma_t \varepsilon(y^{-i}) \geq -\frac{\delta}{1+\delta} \sigma^i (\hat{y}^i + y^{-i})^{\frac{-2\delta}{1+\delta}} \geq -\frac{\delta}{1+\delta} \sigma^i (y^{-i})^{\frac{-2\delta}{1+\delta}}.$$

■

Because $\varepsilon(y^{-i}) \rightarrow 0$ as $y^{-i} \rightarrow \infty$, and $\sum_{j \neq i} \hat{y}^j \rightarrow \infty$ as $N \rightarrow \infty$ (by the lower bound in Lemma 6 and the assumptions of the proposition), we get $\varepsilon(\sum_{j \neq i} \hat{y}^j) \rightarrow 0$ and

therefore $\hat{y}^i \rightarrow \frac{Q^{Vi} \sigma^i}{(Q^{Vi} + Q^{Ui}) \Gamma_t}$ as $N \rightarrow \infty$, which is equivalent to $\Sigma^{Yi} \rightarrow \frac{Q^{Vi} \Sigma^{Zi}}{Q^{Vi} + Q^{Ui}}$. Finally, the assumptions of the proposition and Example 7 imply $\Sigma^{Zi}(t) = \Phi_1^i(t) v^i$, $i = 1, \dots, N$, where $\Phi_1^i(t) = \frac{1}{r + \kappa^i} - e^{-(r + \kappa^i)(T-t)} \left(\frac{1}{r + \kappa^i} - 1 \right)$.

■

Proof of Proposition 9.

We omit the arguments (ω, t) throughout the proof. From the expression of equilibrium efforts above, we obtain

$$\sqrt{\hat{e}^a(\Sigma^U) + \hat{e}^b(\Sigma^U)} = \max \left(0, \frac{\Sigma^{Ya}}{q^a}, \frac{\Sigma^{Yb}}{q^b} \right) / \Gamma_t.$$

Fixing some $\Sigma^{Yb} \in \mathbb{R}$, and suppressing the dependence on Σ^{Yb} in the notation, by Corollary 1, principal a 's problem is $\max_{\Sigma^{Ya} \in \mathbb{R}} J^a(\Sigma^{Ya})$ where

$$\begin{aligned} J^a(\Sigma^{Ya}) = & -\frac{1}{2} \left\{ Q^{Va} (\Sigma^{Za} - \Sigma^{Ya})^2 + Q^{Ua} (\Sigma^{Ya})^2 \right\} + 2\Sigma^{Za} \max \left(0, \frac{\Sigma^{Ya}}{q^a}, \frac{\Sigma^{Yb}}{q^b} \right) \\ & - q^a \left\{ \left(\frac{(\Sigma^{Ya})^+}{q^a} \right)^2 - (\Gamma_t)^2 e^b \right\} 1_{\{\Sigma^{Ya}/q^a = \Sigma^{Yb}/q^b\}} \\ & - q^a \left(\frac{(\Sigma^{Ya})^+}{q^a} \right)^2 1_{\{\Sigma^{Ya}/q^a > \Sigma^{Yb}/q^b\}} \end{aligned} \quad (96)$$

(note that $e^b \in \left[0, \left(\frac{(\Sigma^{Ya})^+}{q^a \Gamma_t} \right)^2 \right]$ on $\{\Sigma^{Ya}/q^a = \Sigma^{Yb}/q^b\}$). Switching a and b gives principal b 's problem.

We first consider, in the following lemma, the case of nonpositive Σ^{Ya} .

Lemma 8 For any fixed Σ^{Zb} ,

$$\Sigma^{Za} \leq 0 \implies \hat{\Sigma}^{Ya} = \underline{w}^a \Sigma^{Za}.$$

Proof. We first conclude that $\hat{\Sigma}^{Ya} \leq 0$, because it is easy to be seen from (96) that $J^a(x) > J^a(-x)$ if $x < 0$. Principal a 's problem therefore simplifies to

$$\max_{\Sigma^{Ya} \leq 0} -\frac{1}{2}Q^{Va} \left(\Sigma^{Za} - \Sigma^{Ya} \right)^2 + 2\Sigma^{Za} \left(\frac{\Sigma^{Yb}}{q^b} \right)^+ - \frac{1}{2}Q^{Ua} \left(\Sigma^{Ya} \right)^2.$$

From the FOC

$$\frac{\partial J^a}{\partial \Sigma^{Ya}} = Q^{Va} \Sigma^{Za} - \left\{ Q^{Ua} + Q^{Va} \right\} \Sigma^{Ya} = 0$$

(Note that $J^a(x)$ is concave on $x \leq 0$ and the right-hand derivative at 0 is negative) we get the result. ■

From Lemma 8 we also get that $\Sigma^{Zb} \leq 0$ and $\Sigma^{Za} > 0$ imply $\hat{\Sigma}^{Ya} = \bar{w}^a \Sigma^{Za}$. This follows because $\hat{\Sigma}^{Yb} = \underline{w}^b \Sigma^{Zb} \leq 0$ and (from (96)) $J^a(x) < J^a(0)$ for any $x < 0$; therefore principal a 's problem simplifies to

$$\max_{\Sigma^{Ya} \geq 0} -\frac{1}{2}Q^{Va} \left(\Sigma^{Za} - \Sigma^{Ya} \right)^2 + 2\Sigma^{Za} \left(\frac{\Sigma^{Ya}}{q^a} \right) - \frac{1}{2} \left(Q^{Ua} + \frac{2}{q^a} \right) \left(\Sigma^{Ya} \right)^2.$$

(Note that $J^a(x)$ is concave on $x \geq 0$ and the right-hand derivative at zero is positive). The result is obtained from the FOC

$$\frac{\partial J^a}{\partial \Sigma^{Ya}} = \left(Q^{Va} + \frac{2}{q^a} \right) \Sigma^{Za} - \left(Q^{Va} + Q^{Ua} + \frac{2}{q^a} \right) \Sigma^{Ya} = 0.$$

The next step, in the following Lemma, is to solve principal a 's problem given any $\Sigma^{Yb} \geq 0$ chosen by principal b .

Lemma 9 *Let $\hat{\Sigma}^{Ya}$ denote principal a 's optimal control given $\Sigma^{Yb} \geq 0$. For any $\Sigma^{Za} > 0$ then $\Sigma^{Yb} > \tilde{w}^a \left(\frac{q^b}{q^a} \right) \Sigma^{Za}$ implies $\hat{\Sigma}^{Ya} = \underline{w}^a \Sigma^{Za}$ (which corresponds to zero agent- a effort) and $0 \leq \Sigma^{Yb} < \tilde{w}^a \left(\frac{q^b}{q^a} \right) \Sigma^{Za}$ implies that $\hat{\Sigma}^{Ya} = \bar{w}^a \Sigma^{Za}$ (which corresponds to positive agent- a effort).*

Proof. Fixing some $\Sigma^{Za} > 0$ throughout and define principal a 's objective function corresponding to zero and positive agent- a effort, respectively:

$$\begin{aligned} J^n(\Sigma; \Sigma^{Yb}) &= -\frac{1}{2} Q^{Va} (\Sigma^{Za} - \Sigma)^2 - \frac{1}{2} Q^{Ua} \Sigma^2 + 2\Sigma^{Za} \frac{\Sigma^{Yb}}{q^b}, \quad \Sigma^{Yb} \geq 0, \\ J^p(\Sigma) &= -\frac{1}{2} Q^{Va} (\Sigma^{Za} - \Sigma)^2 - \frac{1}{2} \left(Q^{Ua} + \frac{2}{q^a} \right) \Sigma^2 + 2\Sigma^{Za} \frac{\Sigma}{q^a}. \end{aligned}$$

Principal a solves (note that $\Sigma^{Ya}/q^a = \Sigma^{Yb}/q^b$ cannot be optimal unless principal a knows with certainty that agent b will exert all the joint effort at that point)

$$\sup_{\Sigma} J^n(\Sigma) 1_{\left\{ \Sigma/q^a < \Sigma^{Yb}/q^b \right\}} + J^p(\Sigma) 1_{\left\{ \Sigma/q^a > \Sigma^{Yb}/q^b \right\}}. \quad (97)$$

The maximum of $J^n(\Sigma; \Sigma^{Yb})$ occurs at $\hat{\Sigma}^n = \underline{w}^a \Sigma^{Za}$ and

$$J^n(\hat{\Sigma}^n; \Sigma^{Yb}) = -\frac{1}{2} Q^{Va} (1 - \underline{w}^a) (\Sigma^{Za})^2 + 2\Sigma^{Za} \frac{\Sigma^{Yb}}{q^b}.$$

where we have used the equality

$$Q^{Va}(1 - \underline{w}^a)^2 + Q^{Ua}(\underline{w}^a)^2 = Q^{Va}(1 - \underline{w}^a)$$

The maximum of $J^p(\Sigma)$ occurs at $\hat{\Sigma}^p = \bar{w}^a \Sigma^{Za}$ and

$$J^p(\hat{\Sigma}^p) = \left\{ -\frac{1}{2}Q^{Va}(1 - \bar{w}^a) + \frac{\bar{w}^a}{q^a} \right\} (\Sigma^{Za})^2.$$

where we have used the equality

$$Q^{Va}(1 - \bar{w}^a)^2 + (Q^{Ua} + \frac{2}{q^a})(\bar{w}^a)^2 - \frac{2\bar{w}^a}{q^a} = Q^{Va}(1 - \bar{w}^a)$$

Define

$$\begin{aligned} f(\Sigma) &= J^n(\hat{\Sigma}^n; \Sigma) - J^p(\hat{\Sigma}^p) \\ &= \left\{ \frac{1}{2}Q^{Va}(\underline{w}^a - \bar{w}^a) - \frac{\bar{w}^a}{q^a} \right\} (\Sigma^{Za})^2 + 2\Sigma^{Za} \frac{\Sigma}{q^b} \end{aligned}$$

which is continuous and strictly increasing in Σ . It is easily confirmed that

$f\left(\frac{q^b}{q^a}\tilde{w}^a\Sigma^{Za}\right) = 0$ and therefore principal a 's optimal control is $\hat{\Sigma}^{Ya} = \hat{\Sigma}^p$ if $\Sigma^{Yb} < \frac{q^b}{q^a}\tilde{w}^a\Sigma^{Za}$ and $\hat{\Sigma}^{Ya} = \hat{\Sigma}^n$ if $\Sigma^{Yb} > \frac{q^b}{q^a}\tilde{w}^a\Sigma^{Za}$. ■

We now apply Lemma 9 to obtain the Nash equilibria among principals when $\Sigma^{Za}, \Sigma^{Zb} > 0$.

Case (i) (on agent b works): Lemma 9 implies that $\hat{\Sigma}^{Ya} = \underline{w}^a\Sigma^{Za}$, $\hat{\Sigma}^{Yb} = \bar{w}^b\Sigma^{Zb}$ holds

if $\Sigma^{Yb} > \tilde{w}^a \left(\frac{q^b}{q^a} \right) \Sigma^{Za}$ and $\Sigma^{Ya} < \tilde{w}^b \left(\frac{q^a}{q^b} \right) \Sigma^{Zb}$; that is

$$\frac{\Sigma^{Zb}}{\Sigma^{Za}} > \left(\frac{q^b}{q^a} \right) \max \left(\frac{\tilde{w}^a}{\tilde{w}^b}, \frac{\underline{w}^a}{\underline{w}^b} \right).$$

Case (ii) (on agent a works): Lemma 9 implies that $\hat{\Sigma}^{Ya} = \bar{w}^a \Sigma^{Za}$, $\hat{\Sigma}^{Yb} = \underline{w}^b \Sigma^{Zb}$ holds if $\Sigma^{Yb} < \tilde{w}^a \left(\frac{q^b}{q^a} \right) \Sigma^{Za}$ and $\Sigma^{Ya} > \tilde{w}^b \left(\frac{q^a}{q^b} \right) \Sigma^{Zb}$; that is

$$\frac{\Sigma^{Zb}}{\Sigma^{Za}} < \left(\frac{q^b}{q^a} \right) \min \left(\frac{\tilde{w}^a}{\underline{w}^b}, \frac{\bar{w}^a}{\tilde{w}^b} \right).$$

The same Lemma easily rules out equilibria with both agents working.

■

Proof of Proposition 8.

Let

$$A = \frac{\Sigma^{Ya}}{\Gamma_t}, \quad B = \frac{\Sigma^{Yb}}{\Gamma_t}$$

and

$$f^a = \frac{1 - \delta^a}{Q^{ea}}, \quad f^b = \frac{1 - \delta^b}{Q^{eb}}$$

By (2.43), at each dimension k principal a is solving holding fixed B (omitting dimension arguments)

$$\max_{A \in \mathbb{R}^+} f(A)$$

where

$$\begin{aligned} f(A) = & -\frac{\Gamma_t Q^{Va}}{2} \left(\frac{\Sigma^{Za}}{\Gamma_t} - A \right)^2 - \frac{\Gamma_t Q^{Ua}}{2} A^2 - \frac{\Gamma_t Q^{ea}}{2} (A f^a)^{\frac{1+\delta^b}{\delta^a+\delta^b}} (B f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} \\ & + \Sigma^{Za} (A f^a)^{\frac{1-\delta^a}{\delta^a+\delta^b}} (B f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} \end{aligned}$$

If $\Sigma^{Za} \leq 0$, the optimal $A = 0$. We will assume $\Sigma_k^{Za} > 0$ and calculate the following derivatives.

$$\begin{aligned} f'(A) = & \Gamma_t Q^{Va} \left(\frac{\Sigma^{Za}}{\Gamma_t} - A \right) - \Gamma_t Q^{Ua} A - \frac{\Gamma_t Q^{ea}}{2} (B f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} (f^a)^{\frac{1+\delta^b}{\delta^a+\delta^b}} \frac{1+\delta^b}{\delta^a+\delta^b} A^{\frac{1-\delta^a}{\delta^a+\delta^b}} \\ & + \Sigma^{Za} (B f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} (f^a)^{\frac{1-\delta^a}{\delta^a+\delta^b}} \frac{1-\delta^a}{\delta^a+\delta^b} A^{\frac{1-2\delta^a-\delta^b}{\delta^a+\delta^b}} \end{aligned}$$

$$\begin{aligned} f''(A) = & -\Gamma_t Q^{Va} - \Gamma_t Q^{Ua} - \frac{\Gamma_t Q^{ea}}{2} (B f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} (f^a)^{\frac{1+\delta^b}{\delta^a+\delta^b}} \frac{(1+\delta^b)(1-\delta^a)}{(\delta^a+\delta^b)^2} A^{\frac{1-2\delta^a-\delta^b}{\delta^a+\delta^b}} \\ & + \Sigma^{Za} (B f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} (f^a)^{\frac{1-\delta^a}{\delta^a+\delta^b}} \frac{(1-\delta^a)(1-2\delta^a-\delta^b)}{(\delta^a+\delta^b)^2} A^{\frac{1-3\delta^a-2\delta^b}{\delta^a+\delta^b}} \end{aligned}$$

$f''(A) < 0$ if

$$- \frac{(1 + \delta^b)(1 - \delta^a)\Sigma^{Ya}}{2} + \Sigma^{Za}(1 - 2\delta^a - \delta^b) < 0 \quad (98)$$

Similarly, for principal b the solution has to satisfy

$$- \frac{(1 + \delta^a)(1 - \delta^b)\Sigma^{Yb}}{2} + \Sigma^{Zb}(1 - 2\delta^b - \delta^a) < 0 \quad (99)$$

Assuming $\Sigma^{Zi} > 0$, $i \in \{a, b\}$, (98) and (99) are satisfied if

$$\begin{aligned} 2\delta^a + \delta^b &> 1 \\ 2\delta^b + \delta^a &> 1 \end{aligned} \quad (100)$$

(100) holds if for example $\delta^a + \delta^b > 1$.

We will solve the following FOC equations.

$$\begin{aligned} \Gamma_t Q^{Va} \left(\frac{\Sigma^{Za}}{\Gamma_t} - A \right) - \Gamma_t Q^{Ua} A - \frac{\Gamma_t Q^{ea}}{2} (B f^b) \frac{1 - \delta^b}{\delta^a + \delta^b} (f^a) \frac{1 + \delta^b}{\delta^a + \delta^b} \frac{1 + \delta^b}{\delta^a + \delta^b} A \frac{1 - \delta^a}{\delta^a + \delta^b} \\ + \Sigma^{Za} (B f^b) \frac{1 - \delta^b}{\delta^a + \delta^b} (f^a) \frac{1 - \delta^a}{\delta^a + \delta^b} \frac{1 - \delta^a}{\delta^a + \delta^b} A \frac{1 - 2\delta^a - \delta^b}{\delta^a + \delta^b} = 0 \\ \Gamma_t Q^{Vb} \left(\frac{\Sigma^{Zb}}{\Gamma_t} - B \right) - \Gamma_t Q^{Ub} B - \frac{\Gamma_t Q^{eb}}{2} (A f^a) \frac{1 - \delta^a}{\delta^a + \delta^b} (f^b) \frac{1 + \delta^a}{\delta^a + \delta^b} \frac{1 + \delta^a}{\delta^a + \delta^b} B \frac{1 - \delta^b}{\delta^a + \delta^b} \\ + \Sigma^{Zb} (A f^a) \frac{1 - \delta^a}{\delta^a + \delta^b} (f^b) \frac{1 - \delta^b}{\delta^a + \delta^b} \frac{1 - \delta^b}{\delta^a + \delta^b} B \frac{1 - 2\delta^b - \delta^a}{\delta^a + \delta^b} = 0 \end{aligned} \quad (101)$$

Case 1 Both pairs of principal and agent are risk-neutral.

By letting $Q^{Vi} = 0$, $Q^{Ui} = 0$, $i \in \{a, b\}$, (101) yields solution

$$\Sigma^{Ya} = \frac{2\Sigma^{Za+}}{1 + \delta^b}, \quad \Sigma^{Yb} = \frac{2\Sigma^{Zb+}}{1 + \delta^a}$$

Case 2 Only pair a of principal and agent are risk-neutral.

Let $Q^{Va} = 0$, $Q^{Ua} = 0$ and assume $Q^{Vb} > 0$, $Q^{Ub} > 0$. (101) yields solution

$$\Sigma^{Ya} = \frac{2\Sigma^{Za+}}{1 + \delta^b}$$

If $\Sigma^{Za} \leq 0$, then

$$\Sigma^{Yb} = \frac{Q^{Vb}\Sigma^{Zb+}}{Q^{Vb} + Q^{Ub}}$$

If $\Sigma^{Za} > 0$, B is the solution to the nonlinear equation

$$\begin{aligned} \Gamma_t Q^{Vb} \left(\frac{\Sigma^{Zb}}{\Gamma_t} - B \right) - \Gamma_t Q^{Ub} B - \frac{\Gamma_t Q^{eb}}{2} (A f^a)^{\frac{1-\delta^a}{\delta^a+\delta^b}} (f^b)^{\frac{1+\delta^a}{\delta^a+\delta^b}} \frac{1+\delta^a}{\delta^a+\delta^b} B^{\frac{1-\delta^b}{\delta^a+\delta^b}} \\ + \Sigma^{Zb} (A f^a)^{\frac{1-\delta^a}{\delta^a+\delta^b}} (f^b)^{\frac{1-\delta^b}{\delta^a+\delta^b}} \frac{1-\delta^b}{\delta^a+\delta^b} B^{\frac{1-2\delta^b-\delta^a}{\delta^a+\delta^b}} = 0 \end{aligned}$$

where $Af^a = \frac{2\Sigma^{Za+(1-\delta^a)}}{\Gamma_t Q^{ea}(1+\delta^b)}$

Case 3 Both pairs of principal and agent are risk-averse

The nonlinear equation system (101) has to be solved to obtain principal equilibrium.

In the case of identical principal-agent pairs and constant return to scale i.e.

$$Q^{Ua} = Q^{Ub} = Q^U, \quad Q^{Va} = Q^{Vb} = Q^V, \quad Q^{ea} = Q^{eb} = Q^e, \quad \delta^a = \delta^b = \frac{1}{2}$$

(101) simplifies to

$$Q^V \Sigma^Z - \Gamma_t(Q^V + Q^U)A - \frac{3\Gamma_t}{4}Q^e(f^a)^2A + \frac{f^a}{2}\Sigma^Z = 0$$

which yields solution

$$\Sigma^Y = \frac{16Q^e Q^V + 4}{16Q^e(Q^V + Q^U) + 3}\Sigma^Z$$

■

.2 Appendix 2: Derivation of Examples

Derivation of Example 6.

By Theorem 6, at each dimension k agent i holds fixed $e_k^j \geq 0$ and seeks e_k^i to maximize

$$\max_{e_k^i \geq 0} f(e_k^i)$$

where

$$f(e_k^i) = -\Gamma_t q_k^i e_k^i + \Sigma_k^{Yi} \frac{(e_k^i + e_k^j)^{1-\delta}}{1-\delta}, \quad i, j \in \{a, b\}, \quad i \neq j.$$

The first derivative is

$$f'(e_k^i) = -\Gamma_t q_k^i + \Sigma_k^{Yi} (e_k^i + e_k^j)^{-\delta}$$

If $\Sigma_k^{Yi} \leq 0$, the maximum is achieved by $e_k^i = 0$. So we assume $\Sigma_k^{Yi}, \Sigma_k^{Yj} > 0$.

The second derivative is

$$f''(e_k^i) = -\delta \Sigma_k^{Yi} (e_k^i + e_k^j)^{-\delta-1}$$

If $e_k^j = 0$, the FOC implies the maximum is achieved by $e_k^i = \left(\frac{\Sigma_k^{Yi}}{\Gamma_t q_k^i} \right)^{1/\delta}$ and vice versa.

If $e_k^j > 0$, the FOC implies

$$(e_k^i + e_k^j)^\delta = \frac{\Sigma_k^{Yi}}{\Gamma_t q_k^i}$$

If $e_k^j > \left(\frac{\Sigma_k^{Yi}}{\Gamma_t q_k^i} \right)^{1/\delta}$, the above FOC can not be satisfied. It is easy to see that $f'(e_k^i) < 0$, so the maximum is achieved by $e_k^i = 0$. Thus the agent equilibrium is

$$e_k^i = 0, \quad e_k^j = \left(\frac{\Sigma_k^{Yj}}{\Gamma_t q_k^j} \right)^{1/\delta}, \quad \text{if } \frac{\Sigma_k^{Yj}}{q_k^j} > \frac{\Sigma_k^{Yi}}{q_k^i}$$

If $e_k^i, e_k^j > 0$, symmetry implies $\frac{\Sigma_k^{Yi}}{q_k^i} = \frac{\Sigma_k^{Yj}}{q_k^j}$. So the agent equilibrium is

$$e_k^i + e_k^j = \left(\frac{\Sigma_k^{Yi}}{\Gamma_t q_k^i} \right)^{1/\delta}, \quad \text{if } \frac{\Sigma_k^{Yi}}{q_k^i} = \frac{\Sigma_k^{Yj}}{q_k^j}$$

■

Derivation of Example 9.

Applying Ito's formula on $D_t Z_t^i$, using (2.44) and integrating from 0 to T , we get

$$\begin{aligned} \int_0^T e^{\int_t^T r_s ds} \Sigma_t^{Zi'} dB_t &= X_T^i - e^{\int_0^T r_s ds} (\Gamma_0 V_0^i - w_0^{Vi} + Y_0^i) + \int_0^T e^{\int_t^T r_s ds} (X_t^i + H_t^i) dt \\ - \int_0^T \frac{e^{\int_t^T r_s ds}}{2\Gamma_t} q^{Zi} \Sigma_t^{Zi'} \Sigma_t^{Zi} dt &+ \int_0^T \frac{e^{\int_t^T r_s ds}}{\Gamma_t} \Sigma_t^{Zi'} \sum_{j \neq i} \frac{w^j}{q^{ej}} \Sigma_t^{Zj} dt \end{aligned} \quad (102)$$

Substituting (102) into (2.61) to get

$$\begin{aligned}
p_T^i &= w^i \left[X_T^i + \int_0^T e^{\int_t^T r_s ds} X_t^i dt \right] + w^i \left[\int_0^T e^{\int_t^T r_s ds} \left(\Gamma_t h^{Vi}(t, \hat{x}_t^{Vi}) - \hat{x}_t^{Vi} \right) dt \right. \\
&\quad \left. - e^{\int_0^T r_s ds} (\Gamma_0 V_0^i - w_0^{Vi}) \right] + (1 - w^i) \left[e^{\int_0^T r_s ds} Y_0^i - \int_0^T e^{\int_t^T r_s ds} \left(\Gamma_t h^{Ui}(t, \hat{x}_t^{Ui}) - \hat{x}_t^{Ui} \right) dt \right] \\
&\quad + \int_0^T \frac{e^{\int_t^T r_s ds} w^i (1 - w^i)}{2\Gamma_t q^{ei}} \Sigma_t^{Zi'} \Sigma_t^{Zi} dt - \int_0^T e^{\int_t^T r_s ds} p_t^i dt
\end{aligned}$$

■

Derivation of Example 12.

Apply Ito's formula to

$$Z_t^i = \theta_t^i + \Theta_t^{i'} \zeta_t,$$

and match diffusion to get

$$\Sigma_t^{Zi} = \text{diag} \left(\sqrt{v + V \zeta_t} \right) \Sigma \Theta_t^i$$

Match drift to get (2.72).

Substitute the above into (2.64) to get (2.70) ■

Derivation of Example 15.

Apply (2.32)

$$p_T^i = Y_0^i e^{\int_0^T r_s ds} - \int_0^T e^{\int_t^T r_s ds} \left[p_t^i + \hat{\mu}^{Yi}(t, \Sigma_t^Y) \right] dt + \int_0^T e^{\int_t^T r_s ds} \Sigma_t^{Yi'} dB_t, \quad i = a, b.$$

with

$$\Sigma^{Ya} = \bar{w}^a \Gamma_t \sigma^a, \quad \Sigma^{Yb} = \underline{w}^b \Gamma_t \sigma^b$$

and

$$\begin{aligned} \hat{\mu}^{Ya}(t, \Sigma_t^Y) &= e^{-r(T-t)} \left\{ -\frac{1}{2} Q^{Ua} \left(\Sigma^{Ya} e^{r(T-t)} \right)^2 - q^a \left(\frac{\Sigma^{Ya} e^{r(T-t)}}{q^a} \right)^2 \right\} \\ &\quad + \Sigma^{Ya} \frac{2 \Sigma^{Ya} e^{r(T-t)}}{q^a} \\ \hat{\mu}^{Yb}(t, \Sigma_t^Y) &= e^{-r(T-t)} \left\{ -\frac{1}{2} Q^{Ub} \left(\Sigma^{Yb} e^{r(T-t)} \right)^2 \right\} + \Sigma^{Yb} \frac{2 \Sigma^{Ya} e^{r(T-t)}}{q^a} \end{aligned}$$

to get (2.88).

■

Derivation of Example 14.

The change of drift function is

$$\Phi_k(\hat{e}_k) = \left(\frac{2\sigma_k^a(1-\delta^a)}{(1+\delta^b)Q_k^{ea}} \right)^{\frac{1-\delta^a}{\delta^a+\delta^b}} \left(\frac{2\sigma_k^b(1-\delta^b)}{(1+\delta^a)Q_k^{eb}} \right)^{\frac{1-\delta^b}{\delta^a+\delta^b}} \left(e^{r(T-t)}\Gamma_t \right)^{\frac{2-(\delta^a+\delta^b)}{\delta^a+\delta^b}}$$

In Example 7, we showed that

$$Z_t^i = \Phi_0^i(t) + \Phi_1^i(t) X_t^i, \quad \Sigma_t^{Zi} = \Phi_1^i(t) \sigma^i, i = 1, \dots, N$$

where

$$\Phi_1^i(t) = \Gamma_t = \frac{1}{r} - e^{-r(T-t)} \left(\frac{1}{r} - 1 \right)$$

The Φ_0^i is given by

$$\begin{aligned}
\Phi_0^i(t) &= \int_t^T e^{-r(s-t)} \left\{ -\frac{e^{-r(T-s)}}{2} \sum_{k=1}^d Q_k^{ei} \left(\frac{2\sigma_k^i(1-\delta^i)}{(1+\delta^j)Q_k^{ei}} \right)^{\frac{1+\delta^j}{\delta^i+\delta^j}} \left(\frac{2\sigma_k^j(1-\delta^j)}{(1+\delta^i)Q_k^{ej}} \right)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \right. \\
&\quad \left(e^{r(T-s)} \Gamma_s \right)^{\frac{2}{\delta^i+\delta^j}} + \Gamma_s \sum_{k=1}^d \sigma_k^i \left(\frac{2\sigma_k^i(1-\delta^i)}{(1+\delta^j)Q_k^{ei}} \right)^{\frac{1-\delta^i}{\delta^i+\delta^j}} \left(\frac{2\sigma_k^j(1-\delta^j)}{(1+\delta^i)Q_k^{ej}} \right)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \\
&\quad \left. \left(e^{r(T-s)} \Gamma_s \right)^{\frac{2-(\delta^i+\delta^j)}{\delta^i+\delta^j}} \right\} ds. \\
&= \left\{ -\frac{1}{2} \sum_{k=1}^d Q_k^{ei} \left(\frac{2\sigma_k^i(1-\delta^i)}{(1+\delta^j)Q_k^{ei}} \right)^{\frac{1+\delta^j}{\delta^i+\delta^j}} \left(\frac{2\sigma_k^j(1-\delta^j)}{(1+\delta^i)Q_k^{ej}} \right)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \right. \\
&\quad \left. + \sum_{k=1}^d \sigma_k^i \left(\frac{2\sigma_k^i(1-\delta^i)}{(1+\delta^j)Q_k^{ei}} \right)^{\frac{1-\delta^i}{\delta^i+\delta^j}} \left(\frac{2\sigma_k^j(1-\delta^j)}{(1+\delta^i)Q_k^{ej}} \right)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \right\} F(t) \\
&= F(t) \sum_{k=1}^d (C^i)^{\frac{1-\delta^i}{\delta^i+\delta^j}} (C^j)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \frac{\delta^i+\delta^j}{1+\delta^j} (\sigma_k^i)^{\frac{1+\delta^j}{\delta^i+\delta^j}} (\sigma_k^j)^{\frac{1-\delta^j}{\delta^i+\delta^j}}
\end{aligned}$$

where

$$\begin{aligned}
F(t) &= \int_t^T e^{-r(s-t)} \left(e^{r(T-s)} \right)^{\frac{2}{\delta^i+\delta^j}-1} (\Gamma_s)^{\frac{2}{\delta^i+\delta^j}} ds \\
C^i &= \frac{2(1-\delta^i)}{(1+\delta^j)Q_k^{ei}}, \quad C^j = \frac{2(1-\delta^j)}{(1+\delta^i)Q_k^{ej}}
\end{aligned}$$

and the terminal pay is

$$p_T^i = Y_0^i e^{rT} - \int_0^T e^{r(T-t)} \left[p_t^i + \hat{\mu}^{Yi}(t, \hat{\Sigma}_t^Y) \right] dt + \int_0^T e^{\int_t^T r_s ds} \hat{\Sigma}_t^{Yi'} dB_t$$

where

$$\begin{aligned} \hat{\mu}^{Y^i}(t, \hat{\Sigma}_t^Y) = & \sum_{k=1}^d \left\{ -\frac{e^{-r(T-t)} Q_k^{ei}}{2} \left(C^i \sigma_k^i \right)^{\frac{1+\delta^j}{\delta^i+\delta^j}} \left(C^j \sigma_k^j \right)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \left(e^{r(T-t)} \Gamma_t \right)^{\frac{2}{\delta^i+\delta^j}} \right. \\ & \left. + \frac{2\Gamma_t \sigma_k^i}{1+\delta^j} \left(C^i \sigma_k^i \right)^{\frac{1-\delta^i}{\delta^i+\delta^j}} \left(C^j \sigma_k^j \right)^{\frac{1-\delta^j}{\delta^i+\delta^j}} \left(e^{r(T-t)} \Gamma_t \right)^{\frac{2-(\delta^i+\delta^j)}{\delta^i+\delta^j}} \right\} \end{aligned}$$

Thus we get the expression (2.84). ■

BIBLIOGRAPHY

BIBLIOGRAPHY

- AGGARWAL, R., AND A. SAMWICK (1999): “Executive Compensation, Strategic Competition, and Relative Performance Evaluation: Theory and Evidence,” *Journal of Finance*, pp. 1999–2043.
- ANTONELLI, F. (1993): “Backward-forward stochastic differential equations,” *The Annals of Applied Probability*, 3(3), 777–793.
- BANSAL, R., AND A. YARON (2004): “Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles,” *Journal of Finance*, 59, 1481–1509.
- BOLTON, P., AND M. DEWATRIPOINT (2005): *Contract Theory*. MIT Press, Cambridge.
- BRIAND, P., AND Y. HU (2006): “BSDE with quadratic growth and unbounded terminal value,” *Probability Theory and Related Fields*, pp. 604–618.
- (2008): “Quadratic BSDEs with convex generators and unbounded terminal conditions,” *Probability Theory and Related Fields*, 141, 543–567.
- DELBAEN, F., Y. HU, AND A. RICHOU (2009): “On the uniqueness of solutions to quadratic bsdes with convex generators and unbounded terminal conditions,” *Ann. Inst. Henri Poincaré Probab. Stat.*
- DUFFIE, D., AND L. G. EPSTEIN (1992a): “Asset Pricing with Stochastic Differential Utility,” *Review of Financial Studies*, 5, 411–436.
- (1992b): “Stochastic Differential Utility,” *Econometrica*, 60, 353–394.
- EDMANS, A., I. GOLDSTEIN, AND J. Y. ZHU (2011): “Contracting with Synergies,” *NBER Working Paper 17606*.
- E. PARDOUX, AND S. PENG (1992): “Backward stochastic differential equations and quasilinear parabolic partial differential equations,” *Lecture Notes in CIS*, Springer, 176, 200–217.
- FUDENBERG, D., AND J. TIROLE (1992): *Game Theory*. MIT Press, Cambridge.
- HART, O. (1983): “The market mechanism as an incentive scheme,” *Bell Journal of Economics*, 14, 366–82.

- HOLMSTROM, B. (1982): “Moral Hazard in Teams,” *Bell Journal of Economics*, 13, 324–340.
- HOLMSTROM, B., AND P. MILGROM (1987): “Aggregation and Linearity in the Provision of Intertemporal Incentives,” *Econometrica*, 55, 303–328.
- KAROUI, N. E., AND L. MAZLIAK (eds.) (1997): *Backward Stochastic Differential Equation*. Addison Wesley Longman Inc.
- KAROUI, N. E., S. PENG, AND M.-C. QUENEZ (1997): “Backward stochastic differential equations in finance,” *Mathematical Finance*, 7, 1–71.
- KHANNA, N., AND M. SCHRODER (2010): “Optimal debt contracts and product market competition with replacement,” *Journal of Economic Theory*, 145, 156188.
- KOBYLANSKI, M. (2000): “Backward Stochastic Differential Equations and Partial Differential Equations with Quadratic Growth,” *The Annals of Probability*, 28, 558–602.
- KOO, H. K., G. SHIM, AND J. SUNG (2008): “Optimal Multi-Agent Performance Measures For Team Contracts,” *Mathematical Finance*, 18:4, 649–667.
- PARDOUX, E., AND S. PENG (1990): “Adapted Solution of a Backward Stochastic Differential Equation,” *Systems and Control Letters*, 14, 55–61.
- PENG, S. (1991): “A generalized dynamic programming principle and hamilton-jacobibellman equation,” *Stochastics*, 37, 61–74.
- (1992): *A nonlinear feynman-kac formula and applications*, *World Scientific, Singapore*. Proceedings of Symposium of System Sciences and Control Theory.
- (1993): “Backward stochastic differential equation and it’s application in optimal control,” *Appl. Math. Optim.*, pp. 125–144.
- ROGERS, L. C. G., AND D. TALAY (eds.) (1997): *Numerical Methods in Finance*. Cambridge University Press.
- SCHATTLER, H., AND J. SUNG (1993): “The First-Order Approach to the Continuous-Time Principal-Agent Problem with Exponential Utility,” *Journal of Economic Theory*, 61, 331–371.

- SCHMIDT, K. (1997): “Managerial incentives and product market competition,” *Review of Economic Studies*, 64, 191–213.
- SCHRARFSTEIN, D. (1988): “Product market competition and managerial slack,” *Rand Journal of Economics*, 19, 147–155.
- SCHRODER, M. (2013): “The Dynamic Continuous-Time Principal-Agent Problem with Moral Hazard and Translation-Invariant Recursive Preferences,” .
- SCHRODER, M., AND C. SKIADAS (2005): “Lifetime Consumption-Portfolio Choice under Trading Constraints and Nontradeable Income,” *Stochastic Processes and their Applications*, 115, 1–30.
- SKIADAS, C. (2008): “Dynamic Portfolio Choice and Risk Aversion,” in *Handbooks in OR & MS*, Vol. 15, ed. by J. R. Birge, and V. Linetsky, chap. 19, pp. 789–843. Elsevier.
- WILLIAMS, N. (2008): “On Dynamic Principal-Agent Problems in Continuous-Time,” *working paper*.
- WINTER, E. (2010): “Transparency and incentives among peers,” *RAND Journal of Economics*, 41, 504–523.