DIAGRAMMA AS PROOF AND DIAGRAM IN PLATO

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ABSTRACT

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Like Greek mathematicians long before Euclid, and some mathematicians today, Plato relied on diagrams as essential components of mathematical proof and sometimes as constituting proofs. This fact has not been adequately appreciated among modern interpreters of Plato because the geometric mode of representation of mathematical concepts used by the ancient Greeks has almost completely been superseded by the use of the abstract symbols of algebra in modern mathematical practice. I demonstrate that understanding the geometrical principles and practices with which Plato was operating can help us understand the many mathematical examples in Plato’s texts better—and not only that. I show that a clearer understanding of how and why Plato uses diagrammatic examples can give us insights into his ontological commitments and epistemological theory that are otherwise obscure.

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And of course my thanks to Plato for the inspiration to pursue a doctorate in philosophy and to contribute to the education of the leaders of tomorrow.

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CHAPTER 1:

INTRODUCTION

1.1 The Modern History of Ancient Mathematics

Understanding the geometrical principles and practices with which Plato was operating can clarify the many mathematical examples in Plato's texts. A clearer understanding of how and why Plato uses these examples, especially his use of diagrams (*diagramma*) as proofs, can give us interesting insights into his ontological commitments and epistemological theory—or so I argue. There is a remarkable internal coherence to the geometric examples used by Plato in relation to the research project he proposes for the development of stereometry as a step towards a scientific model of astronomy that can account for the motion of solid spherical bodies. I proffer a structural account of Plato's corpus that articulates his notion of attunement ("harmony of the spheres") as a functional integration of parts into a unified whole.

Those familiar with the philosophical history of Plato's geometrical examples will know that—when scholars could not understand Plato's uses of geometry—he has variously been accused of ignorance, willful deceit, blundering, or joking with his readers. On the contrary, as I hope to show, Plato uses mathematics consistently in *Meno, Theaetetus, Republic*, and *Timaeus*, and can be credited with even more mathematical sophistication than has previously been realized.

Before any of that can be demonstrated, however, I am obliged to take a step back to explain that many of the conclusions I reach are based on the *practice* of ancient Greek geometry, so other relevant precedents—archaeological as well as mathematical—come into play.[1]

Before the middle of the sixth century BCE, architectural drawings were exact and materials were already available: not just drawing with a stick in the sand or on wax-covered wooden slates, but precisely planed marble slabs used in the building trades.[2] And mathematics too, including the use of diagrams as proofs,[3] had reached a high level of sophistication long before Socrates was born. Mathematics was taught in Athenian schools, as the geometry lesson in Plato's *Theaetetus* illustrates, and as the *Republic* curriculum, inter alia, remind us. How then could twentieth-century philologists and philosophers have surmised that Plato had an elementary mathematical understanding?

The answer is reported by historians of mathematics who point to a nadir in the appreciation of geometrical proofs that implicates Plato:[4] As nineteenth-century mathematicians explored the limits of *infinite* processes that defied the visual imagination, suspicion of geometrical intuition took hold. That "visual understanding actually *conflicts* with the truths of analysis" became dogma in the early twentieth century (see Giaquinto 2007, 3–8). Plato and his serious use of diagrams was only one of the casualties of the dogma. Athenian mathematical understanding generally was disparaged for its reliance on geometry; and mathematical discoveries were pushed from the sixth to the fourth century BCE, eclipsing nearly two centuries.

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[1] There has long been a stream of mathematicians who publish on Plato in books and journals that historians of ancient philosophy rarely encounter. Having studied art and ancient Greek mathematics (Knorr 1975, Szabó 1978), on my way to Plato and philosophy, my "solution" to the mystery of Plato's divided line was originally submitted in an undergraduate term paper with a hand-drawn illustration that I still prefer to what I can produce with software.

[2] Artmann (1994, 18). See also Coulton (1976), Haselberger (1985), and Senseney (2011).

[3] Fowler, our most comprehensive source for mathematics as it was practiced in Plato's Academy, notes in passing that "the word *diagramma* seems, in Plato and Aristotle, to refer ambiguously to either a geometrical figure or a proof" (1999, 33). Other philosophers have made similar, though less far-reaching, observations; for example, Patterson (2007, 2) emphasizes the *usefulness* of diagrams in Plato's proofs. That diagrams are essential in geometry is more often pointed out; see Burnyeat (1987), and Netz (2003, 306n22), both cited by Benson (2015, 252n46).

[4] The origins of Greek mathematics is a vast research field, but Artmann (1991), Knorr (1975, 1978), Caveing (1996), Szabó (1978), Heath (1981), Netz (1999), and Unguru (1975) are some of the most useful resources. Heath's notes to his translation of Euclid's *Elements* (1956) remain invaluable.

For example, the discovery of the phenomenon of incommensurability came to be viewed as a crisis of Plato's era; in fact, by the time of Socrates, incommensurability was already providing opportunities for significant mathematical work, and much of it involved construction.[5] Philosophers were slower than historians of mathematics to reestablish what Athenians knew and when. Key to the argument of my dissertation is that the ancient process of constructing figures was essential to proofs; and some diagrams *are* proofs. Moreover, intermediate steps in constructing a figure often reveal connections not apparent in mere algebraic manipulation of symbols in the modern formal sense.

David Fowler (1999, 104) makes another point that, as the following chapters will show, is at the very root of what philosophers want to know about Plato's philosophy of forms:

> [W]hile Plato's principal interest was in dialectic, for which he regarded mathematics only as a preliminary, he does nonetheless show detailed knowledge of important characteristics and problems of technical mathematics, and there is no indication that he could not communicate on equal terms with the mathematicians who seem to have dominated, if not comprised, the group of friends and associates that assembled round him.

The phrase, "he regarded mathematics only as a preliminary," marks my disagreement with Fowler, and I hope to take Plato a step further without disclaiming any other of Fowler's descriptions of the development of geometry and arithmetic in Plato's Academy.

1.2 The Ancient and Modern History of Plato's Philosophy of Mathematics

That mathematics is apprehended with dianoetic reasoning while forms are known *dialectically* is elegantly shown by Hugh Benson (2015), and I do not challenge his insights, here relevant to

[5] See Caveing (1996) for a well-told account of the controversies over ancient mathematics among historians of the subject. Fowler, remarking on "the supposed effects of the discovery of incommensurability," gives the history short shrift: "I find Freudenthal (1966) and Knorr (1975, 306–12) convincing when they argue that, far from being a period of crisis and confusion, the early fourth century was an extraordinary period of creativity, especially in Plato's circle; we have no historical evidence for any of the postulated difficulties of a 'foundation crisis'" (1999, 362). Fowler appropriately agrees with Thesleff (1989, 18n47) in dating the death of Theaetetus in 390 rather than the later date so popular with philosophers, 369.

Republic 7. Both methods move from hypotheses to conclusions, but:

> The contrast consists, rather, in how the two methods treat the hypotheses with which they begin their procedure to a conclusion. Dianoetic treats them as *archai*, as already known, as not needing a *logos*, as already clear to all, when they are not. Dialectic does not. Dialectic treats them as genuine (τῷ ὄντι) hypotheses—as unknown starting points from which one can obtain the conclusion, but which in order to be known require confirmation up to the genuine *archē* of everything. (2015, 247–48)

However, Plato came to see certain aspects of mathematics as constitutive of *formal accounts* or—to put it more provocatively, but more accurately—Plato came to recognize *some* mathematicals as Platonic forms.[6] The equal, for example, was identified as a quintessential form in the *Phaedo* (72e1–77a5). Odd and even are regarded as forms in the *Phaedo* (103d–104b), but as posits of mathematicians in the divided line passage of *Republic* 6. The one (τὸ ἕν) is a candidate Platonic form in the *Parmenides* (129d3, 5, 8–e1) and *Sophist* (251b7–8). It is in the *Statesman* and the *Philebus*, however, that Plato's vocabulary changes into what Aristotle and his commentators—especially Simplicius, Philoponus, and Alexander of Aphrodisias—regard as Platonic.[7]

We know from across the dialogues that Plato regarded forms as existing, real, invisible, indestructible, imperishable, eternal, everlasting, immutable, non-composite, unmixed, simple, and causally responsible for the existence and nature of sensible particulars that *participate* in the forms.[8] All of that would be Platonically true, even if there were no human beings, no intelligent

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[6] In my discussion of Plato's divided line, chapter 4, I suggest further divisions of the line for reasons noted there; this would advance some mathematicals to the level of forms. One was not a number (*arithmos*) for the Greeks, but it was numerical in the sense that it was required to perform arithmetic operations. Sayre (2006, 201) glosses Aristotle (*Metaphysics* A 6, 987b14–18) "as implying that Plato maintained a distinction between mathematical numbers such as 2 and 3, which are eternal and many alike, and the Form Numbers Two and Three, which are eternal and each unique." Gerson (2013, 121n95) cites Sextus Empiricus, *Against the Dogmatists* 10.276 for the similar claim that Plato distinguished two senses of 'one': first principle (πρώτη μόνας) and first number (τὸ ἐν τοῖς ἀριθμοῖς ἕν).

[7] Sayre (2006, Appendix, 241–43) tabulates Aristotle's remarks and those of his commentators to shore up the view he argues throughout the book that there is in Plato's late dialogues a coherent theory of forms recognized by Aristotle and others.

[8] The synonyms merely capture the numerous ways the Greek has been translated. In support of the final clause, we have Aristotle's *Metaphysics*: "the Forms are the causes of all other things" ἐπεὶ δ᾽ αἴτια τὰ εἴδητοῖς ἄλλοις (A 6,

life, or no life at all, in the universe. Because there is intelligent life in the universe, and—to judge from human experience—it is the intellect that grasps the forms, forms are necessary conditions, and in that sense causes, for the *intelligibility* of fluctuating sensible things.

My structural account of Plato's use of mathematics provides a key premise for the late notion of Platonic forms articulated by Kenneth Sayre (2006), but it is not my project to turn specific numbers or magnitudes into forms.[9] Nor is it my project to evaluate Aristotle's remarks in the *Physics* and *Metaphysics* on Plato's forms. I mention the forms here because they have always been the crux by which Platonic ontology and epistemology are assessed. I would only insist that, if *no* mathematicals were forms for Plato himself, important criticisms by Aristotle would miss their mark completely, and that is implausible.[10]

987b18–19); and "the many sensibles which have the same name exist by participating in the corresponding Forms. The only change he made was to use the name 'participation'; for the Pythagoreans say that things exist by imitating numbers, but Plato, changing the name, says that things exist by participating in the Forms" (A 6, 987b8–13). The A 6 passage is important for a second reason. Participation (μέθεξις) poses serious problems for any radical notion of separation. The Pythagoreans, with their copying or imitating of forms (μίμησις), suggest a physical separation that partaking or participating, with its connotations of mixing, lacks.

[9] It is only fair to concede that Sayre was skeptical of my view of the coherence of Plato's mathematical project when we discussed it in person.

[10] We will probably never know which of Aristotle's criticisms were leveled against Plato's own views, and which against other Platonists of the Academy, in and after Plato's lifetime, but Aristotle makes it certain that mathematics was central to philosophy in the last decades of Plato's life and afterwards.

CHAPTER 2:

PLATO'S *MENO*: WHAT DID SOCRATES DRAW?

2.1 Introduction

The passage in Plato's *Meno* in which Socrates gives a slave[1] a geometry lesson is well known. It is also just one of three passages in the dialogue that deal with mathematics, and with geometry in particular (73e3–76a7, 82b9–85d5, 86e1–87b2). These three passages, when taken together, have an internal coherence that is independent of their use for illustrating philosophical issues: they lay out some of the basic principles and objects associated with learning and using plane or two-dimensional geometry.

Each passage utilizes a method and a specific mathematical example from Greek geometry. The notions introduced are similar to those that can be found in the first six books of Euclid's *Elements*, where the elementary concepts of the ancient Greek axiomatic, deductive plane geometry were canonized in the fourth century (in the years following Plato's death in 347).

A significant difference between what we find in the *Elements* and in Plato's examples is that the latter tend to fall somewhere between arithmetic and geometry, in that they involve specific quantitative magnitudes. This is in contrast to the generally non-quantitative nature of

[1] There is a near-ubiquitous tradition of calling Meno's attendant a 'slaveboy'. "Meno's attendant, like any male slave, regardless of age, could expect to be called 'boy' (*pais*); nothing in the dialogue indicates his actual age except perhaps his readiness to participate in Socrates's demonstration and his success at the task assigned him." Meno had brought several attendants with him from Thessaly (82a) in anticipation of following Cyrus into Persia on campaign. He would hardly be traveling with very old or very young slaves (Nails, 2002, 204 and 298).

the geometry in most of Euclid.[2] In modern mathematics, Euclid's treatment is called the synthetic approach. Such an approach relies on axioms rather than on measurements.[3] I argue in this chapter that Plato's treatment is closer to what is typically called the metric approach, in which the quantitative sizes of lines, areas, and angles can be used in proofs (*diagramma*) and theorems. The examples I examine in the *Meno*—and from *Theaetetus*, *Republic*, and *Timaeus* in later chapters—all involve particular irrational quantities.[4] The most significant of these are the square roots of integers and the cube root of two.

In what follows, I show that the *Meno*'s mathematical passages serve as an introduction to metric plane geometry for a student of Plato's Academy or reader of the dialogues. I also show that each of the basic geometrical principles we find in the *Meno* has a philosophical counterpart: a basic philosophical notion or tool that is required for a more sophisticated consideration of other philosophical matters. Since the *Meno*'s philosophical and geometrical content introduce the elementary tools and notions needed for philosophy and geometry respectively, there is an intriguing analogy between the dialogue's mathematical and philosophical content. Hence an investigation of the correlation between the mathematical examples and philosophical content

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[2] David Fowler, whose magisterial *The Mathematics of Plato's Academy* has done much to demonstrate that important advances in mathematics, including the discovery of incommensurability, occurred during and even before Plato's time, characterizes the type of geometry I am invoking here: "There is a style of mathematics that I shall call 'arithmetised'. In geometry lines become endowed with a length, the area of a rectangle is a product of the lengths of its base and height" (1999, 8).

[3] As pointed out by Courant and Robbins (1941, 165), synthetic geometry is usually contrasted with analytical geometry, which is based on numerical coordinates and uses the techniques of algebra. Klein (1965, 62) regards Euclid as a late example of the synthetic method. Klein, however, conjectures that Meno learned the geometrical *technē*—one that "prides itself on never using 'unknown' terms"—from Gorgias. The *technē* itself "begins with something agreed upon as true (*alēthes ti homologoumenon*) and reaches, through agreed consequences, an indubitable result," citing Heath (1926, 1, 137–38 and 3, 442).

[4] While Plato's approach differs from Euclid's by considering the size or magnitude of geometric objects, it is consistent with Euclidean geometry in all other respects.

can provide insight into the dialogue itself.[5]

2.2 The First Example: The Notion of Definition 73e3–76a7

The three geometrical passages in the *Meno* cover several key aspects of the geometry mentioned in Book I of Euclid's *Elements*. The first passage points to the preliminary materials that are prior even to the first of the propositions. These include definitions, postulates, and axioms—which, in modern terminology, fall under the general category of axioms.[6] As I explain below, the importance of definition in Greek mathematics shows itself most clearly in the axiomatic system of the *Elements*.

The context in which the *Meno*'s first mathematical example appears is Socrates's search for a definition of 'virtue'. When Socrates asks Meno "What is virtue?" Meno tells Socrates that the task is easy and then proceeds to list several different kinds of virtue (71e). Seeing that Meno has not understood his question, Socrates asks him to name a single characteristic (73d1) that

[5] Systematic attempts to make explicit and to elaborate the close connection between mathematical and philosophical method in Plato have a venerable history that took its modern start in English with Cook-Wilson (1903, 1904) and reached book-length efforts in German with Stenzel (1917, 1924). In *Greek Mathematical Thought and the Origin of Algebra,* Klein asserts the connection vigorously (1968, 61–62): "We have so far avoided coming to terms with the ontological point of view which from the very first determined the form taken by the Greek doctrine of the *arithmoi.* Yet any attempt to understand Greek mathematics as a self-sufficient science must fail. It is impossible to disregard the ontological difficulties which fundamentally determine its problems, its presentation, and its development, especially in its beginnings. It is just as impossible, on the other hand, to understand Greek ontology without reference to its specifically 'mathematical' orientation. For the characterization of 'mathematical' truths as *mathēmata*, i.e., as things 'to be learned,' serves as a model for all teachable and learnable knowledge—and it is knowledge as understood which determines the sphere of Greek ontological inquiry."

[6] The importance of this first passage is not lost on historians of mathematics. Fowler calls it "our *first* direct, explicit, extended piece of evidence about Greek mathematics" (1999, 7). Scott (2006, 31–45) offers an extended discussion of the passage. Sharples (1985, 127–34) is reliable for the history of philological controversies in the Greek text here and elsewhere in the *Meno*.

would govern all cases that fall under the notion of virtue. He uses the example of *schēma* (figure) to help make clear the distinction between the form of virtue and its various instances. Roundness, he says (74b6–7), is not figure (*schēma*) but *a* figure (*schēma ti*). This is because there are other figures besides roundness. The point he makes here is that mere enumeration of kinds of figures does not constitute the definition that Socrates is seeking. Similarly, since there are many virtues, the enumeration of virtues (74a) will not be an adequate answer to the question "What is virtue?" that Socrates asks.

Since the term 'figure' applies just as well to the round as to the straight, an adequate answer to the question "What is figure?" will have to cover both (74d–75a).[7] Socrates asks Meno if he is familiar with the notion of "the end … as a limit or boundary, for all those are, I say, the same thing. Prodicus might disagree" (75e1–3).[8] He then shifts to more technical terms: 'surface' and 'solid', which properly belong to the discipline of geometry. Here he explains that a figure is where a solid ends: "a figure is the limit of a solid" (76a7).[9]

The mathematical example, Socrates tells Meno, is "practice for your answer about virtue" (75a8–9), while also showing that answering a "What is x?" question is a more difficult task than Meno has presupposed. Socrates often encounters interlocutors who at first believe that

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[7] Socrates then suggests that color might be the only thing that accompanies a figure. This does not satisfy a scientific or technical definition of 'figure' because color is not in fact a characteristic of figure as shape, so Socrates calls it a naïve definition. (Aristotle remarks parenthetically at *De Sensu* 439a30–31 that the Pythagoreans called a surface a *chrōma.* See also *Physics* 210b1–8.)

[8] Prodicus of Ceos was the closest thing the ancient world had to a linguist, and was famous for making distinctions among words. Later in the *Meno* (96d5–7), Socrates implies that he learns the use of technical terminology from Prodicus (Nails 2002, 254–56).

[9] I use the translation of Grube (1997), occasionally modified as noted in context. Here I translate *schēma* 'figure' in place of Grube's (less mathematical) 'shape' (1997); but I allow 'shape' as a loose synonym below. Other translators use 'boundary' instead of 'limit'—but that difference is not important here. Grube based his translation of the OCT edited by John Burnet.

he is asking them to do something very easy.[10] But it transpires that understanding what the "What is x?" question *means* is a crucial philosophical starting point—one that must be grasped before any progress in answering the question can be made. The *Meno*'s discussion of figure provides a model for the beginner, or for one who does not yet understand what the philosopher is seeking.

2.3 Analysis of the First Example

In this first passage, Socrates challenges Meno to define *schēma*. Socrates holds that definitions are fundamental for knowledge. Until you know what a thing is, you cannot answer other questions about it.[11] The two geometrical problems we find later in the *Meno*—the doubling of the square and the inscription of a triangle inside a circle—both amount to constructing figures. But you cannot solve the problem of doubling a square figure until you have answered a more fundamental "What is figure?" question.

In Greek mathematics, as in philosophy, definitions are fundamental, preliminary material.[12] In the axiomatic system found in Euclid's *Elements*, the definitions—placed at the beginning of *Elements* I—set out the properties of the mathematical objects of plane geometry. It should be noted that while Euclid does not cite the definitions in his constructions, theorems and proofs, knowing the nature of the objects he addresses is nevertheless essential to understanding

[10] For example, Laches, just like Meno, says that "it's not hard to say" (*ou chalepon eipein*) what courage is (*Laches* 191e4, which is similar to *Meno* 72e1). And, like Meno, Laches's first attempt at defining 'courage' shows that he has not understood Socrates's question.

[11] Irwin (1977, 136–38) dubs this 'the dialectical requirement', and the appellation has stuck.

[12] The other preliminary materials are the postulates, which state what is constructible, and the common notions, which are general truths applicable to mathematics.

what is going on in his text. Hence the place of definitions in Greek geometry is analogous to their place in philosophy: they are necessary foundations for building systematic knowledge.

For remote insight into what use Plato could have made of this, we may turn to the *Seventh Letter*.[13] The author describes five aspects of knowing in a mathematical context:

> [T]ake a particular example, and think of all other objects as analogous to it. There is something called a circle, and its name is this very word we have just used. Second, there is its definition, composed of nouns and verbs. "The figure whose extremities are everywhere equally distant from its center" is the definition of precisely that to which the names 'round', 'circumference', and 'circle' apply.[14] Third is what we draw or rub out, what is turned or destroyed; but the circle itself to which they all refer remains unaffected, because it is different from them. In the fourth place are knowledge (*epistēmē*), reason (*nous*), and right opinion (which are in our minds, not in words or bodily shapes, and therefore must be taken together as something distinct both from the circle itself and from the three things previously mentioned); of these, reason is nearest the fifth in kinship and likeness, while the others are further away.—Morrow, trans. (342a7–d3)

The first two—name and account or definition—are crucial for developing an axiomatic mathematical system. This shows that the author was well aware of the requirements for such a system. This makes good sense, given that Plato's *philosophical* interest in mathematics seems focused on the structure and methods of the system (e.g. the priority of definition, proof as giving an account, geometric similarity). This of course is not to deny that Plato has *mathematical* interests in details of the objects of mathematics (e.g. the length of a side of a particular figure). Philosophically, Plato is interested in the geometric form of the Pythagorean theorem, that is, the configuration of the shapes in the diagram and the ratios between the lines; but mathematically he has an interest in specific magnitudes of a subset of instances of this

[13] Since the letter is generally agreed to have been written by someone knowledgeable about affairs of the Academy and Sicily at the time of Plato's later life, it does not matter for my purposes whether Plato himself was the author. Although Lloyd (1990) seeks to shed light on the question of the letter's authenticity, his article traces mathematical developments around Archytas as well. For the application to *Meno*, see Lloyd (1992).
[14] See *Timaeus* 33b and *Parmenides* 137.

diagram, for example, the ones constructed with ($1,1,\sqrt{2}$) and (3,4,5) triangles.

Socrates will ultimately identify *schēma* with the technical term *epipedon* (plane, 76a1); and the notion of the plane in Euclid's geometry, that is, a flat two-dimensional surface, has become commonplace. Nevertheless, it is in fact an abstract notion that is not instantiated anywhere in the physical realm, and Socrates's initial suggestion that "shape is that which alone of existing things always follows color" (75b9–11), had called to mind sensible things. His revised definition is mathematical, and is more clearly about something intelligible: the plane, rather than mere shape.[15]

The plane in Greek geometry is of primary importance. Euclid devotes five separate definitions to describe the properties of surface and figure. All of Greek geometry—even three-dimensional (solid) geometry—is carried out with two-dimensional figures. For example, for Archimedes, cones and spheres are "figures of rotation"—that is, three-dimensional figures produced by the rotation around a fixed axis of a given two-dimensional figure (e.g. a right-angled triangle or a semicircle; see Heath, 1897, clxix). The five regular solids (what we call "Platonic solids") in Euclid are composed of polygonal faces; and the interior geometry on the basis of which the more complex regular solids (e.g. dodecahedron, icosahedron) are constructed is entirely worked out with two-dimensional figures (see Heath 1956, 3, 438–511). And the most famous problem in Greek stereometry—the problem of the duplication of the cube (dubbed the Delian problem by neoplatonists)—is solved with two-dimensional constructions (though it goes beyond the Euclidean limitations of compass and straightedge constructions).[16] Both

[15] For a detailed comparison of this section of the *Meno* with portions of Euclid's *Elements*, and of both texts to *Republic* 6, see Klein (1965, 64–65).

[16] Fowler insists on an additional method of construction, *neusis*-construction, in use in Plato's time though not described before Simplicius's commentary on Aristotle's *Physics*: "a line is drawn 'verging' or 'inclining' (*neuein*) towards—that is, passing through—a given point and intercepting two given curves or straight lines in a given

Menaechmus's and Archytas's solutions to doubling the cube amount to finding intersections of curves representing sections of solid geometric objects.[17] Hence, in seeking the definition of *schēma*, Socrates seeks what is foundational to Greek geometry.

Socrates tells Meno that in order to know a thing it is essential to define the object that one seeks to understand. Thus it is no accident that he uses *schēma* in his discussion of definition. He will need to construct *schēmata* (squares and triangles) to solve the linear problem that is coming up, which requires familiarity with the *schēma* as a geometrical concept. He starts by defining the object of plane geometry (*schēma* or figure, and plane surface). The starting point is to define the field; this makes it possible then to perform geometric proofs and constructions, and to state theorems.

While Plato does not give specific instructions for constructing figures in the two-dimensional plane, we will see that the second *Meno* example (the doubling of the square) involves producing a diagram that displays some of the most basic properties of the field of two-dimensional geometry.[18]

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segment" (1999, 283–89). With this and other examples, Fowler opposes the widespread notion that geometry was in its infancy in Plato's time, a notion that—as we will see in chapter 4—plays havoc with understanding the construction of Plato's divided line.

[17] This results in conic sections in the first case; and a circle, triangle, and rectangle (which produce the torus, cone, and cylinder, respectively) in the second (Knorr 1975, 49–65).

[18] A note about the figures that feature so prominently throughout my work is in order: supplying figures to accompany Plato's Greek text is suggested both by the characters' use of terms taken from geometry and arithmetic, and by their use of common words—here, over, beside, this, bottom—most naturally understood as referring to the characters' discussion of actual drawn figures. The history of drawing Platonic figures goes back at least to the tradition of textual scholia; if there were diagrams in the original papyri, they are lost to us and, as Brumbaugh (1954, 10) notes, "no set of diagrams derived from Plato's original manuscripts was known to Hellenistic schools." In fact, some early figures do such disservice to the Greek text that a scholiast's misunderstanding is patent. Brumbaugh's 108 Platonic figures—some with several subordinate figures—is impressive, enlarging the work of Greene (1938), who amassed a broad collection of diagrams, both mathematical and interpretive, from scholiasts, neoplatonists, Renaissance scholars, and others who (sometimes in the margins of manuscripts) sought to provide illustrations corresponding to passages in Plato's texts. In more recent times, geometers and others have added many diagrams for the same illustrative purpose. Some of my diagrams derive from the existing tradition, and

2.4 The Second Example: Doubling the Square 82b9–85d5[19]

The problem of doubling the area of a given square is the two-dimensional analogue of a problem of doubling the area of a cube. Plato's choice of this particular example is pregnant with associations to key aspects of geometry and theorems also traditionally linked to him. The most prominent are its description of a mathematical proof and its dependence on the Pythagorean theorem.

The numerical solution of the problem is finding the geometric mean between 1 and 2, which gives a geometric proportion of $1 : \sqrt{2} :: \sqrt{2} : 2$. The rigorous deductive proof of the fact that the diagonal equals the magnitude of a line that doubles the square in which it is contained is given by the Pythagorean theorem. This theorem (Euclid I.47), is the climax of the book.[20] The continuous geometric proportion of $1 : \sqrt{2} :: \sqrt{2} : 2$ and its generic form of $a : \sqrt{ab} :: \sqrt{ab} : b$ is the first instance of the notion of proportional magnitude that is fundamental in the mathematical examples examined.

The intuitive proof that emerges from Socrates's lesson uses an arithmetical method (counting triangular units of area) formed by drawing diagonals in the square. The specific triangular figures serve as units of area that can be used to show that the four interior triangles of the larger square are twice the number of triangles formed by the diagonal in the original two-foot square.

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others are my own.

[19] No figure proposed for a text of Plato has attracted so many commentators as my second example. Among the most important analyses and interpretations, most of them illustrated, are Brumbaugh (1954, 21–32) who considers previous contributions by Greene (1938), Jowett (1892), and Heath (1981); Klein (1965, 99–107); Brown (1967); and Fowler (1990; and 1999, 3–29, 366–69).

[20] While I.47 is the penultimate theorem in the book, the last proposition (I.48) simply gives its converse.

To make this visually obvious, Socrates draws a square (*ABCD*) and the lines joining the middle of its sides that pass through the center of the square (*EF* and *GH*; see Figure 2.4.1).[21] It is left unsaid that Socrates draws this figure in a medium such as a sandy patch of ground or on a waxed slate as he speaks.

Socrates first establishes that the slave already knows that multiplying the length and width of a rectangle gives its area, such that a two-foot by one-foot rectangle has an area of two square feet (82c7–8), a two-foot square has an area of four square feet (82d4), and a square double that area is eight square feet (82d8). The slave's initial response to Socrates's question—

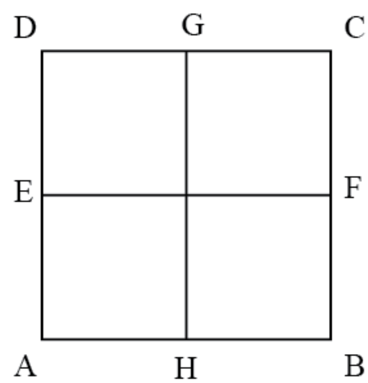

Figure 2.4.1: The Original Square

[21] Even so early in the passage, however, there is controversy in the literature over the term 'middle' for Plato's διὰ μέσου (82c3): Following a suggestion made in passing by Mugler (1948), Ebert (1973, 181n18) argues that commentators have "gone wrong" in their interpretation of the text, understanding the lines "to be transversals, not diagonals." Ebert summarizes (2007, 190–91n11): "Since Socrates has stated that the figure drawn by him has four equal sides, any figure with (only) four equal sides (i.e. a rhombus) must have transversals of equal length. Thus this specification would not add anything mathematically useful to what has been said so far. Only a rectangular figure, however, has two equal diagonals. Hence only if the lines Socrates refers to at 82c2–3 are diagonals has he given a definition of a square, and a quite elegant one at that: it makes use only of the concept of equal length of lines. Moreover, the slave-boy [*sic*] will need this definition of a square to be able to recognize the figure Socrates draws at the end of the geometry lesson as a square (cf. 84e–85b). This figure is a square because it has four equal sides, i.e. the diagonals of the original square, and two diagonals of equal length, i.e. twice the side of the original square." Boter (1988) and Weiss (2001, 84–85) have accepted Ebert's interpretation of the text.

how long is the side of a square that is twice the area of a four-foot square?—is that it will be a line twice the length of the side of that square. Socrates remarks to Meno that he does not *teach* the slave how to find the correct answer but will question him so that he can recollect or realize

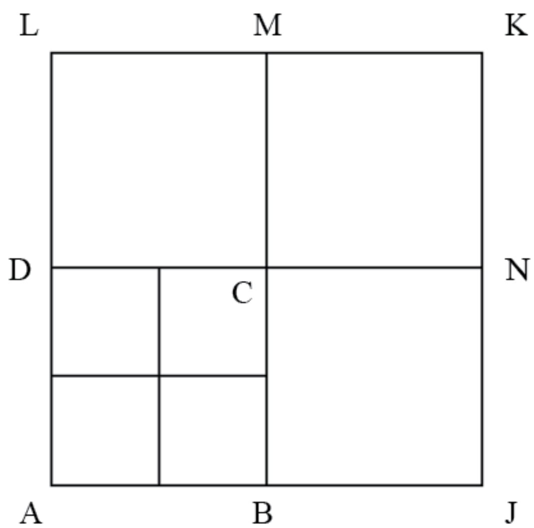

Figure 2.4.2: Doubling the Side

things in a certain order and find the answer for himself (82e4–6). He then proceeds to point out to the slave that a line twice the length of the side of the original square does not give a square twice its area, but gives a square four times the area of the original square, that is, a square equal to sixteen square feet. Through the use of an extension of his original diagram, Socrates illustrates his point (see Figure 2.4.2).

Socrates uses his next set of questions to establish that the eight-foot square is double the four-foot square and half the sixteen-foot square, and that its side is longer than two feet and shorter than four feet (83d4–5)—the length of the sides of the four-foot square and the sixteen-foot square, respectively. In light of these answers, Socrates once again asks the slave what the length of the side will be. The slave responds: “Three feet.” Socrates examines this response by noting that a three-foot side is equal to the sum of one half the length of the side of the four-foot square added to the side of that square itself. Again the slave realizes his answer is incorrect, because a side of three feet produces a nine-foot square (83e8), which is still larger than the desired eight-foot square.

Clearly this line of questioning is not leading to the correct length, so Socrates tells the slave that if he cannot work it out using this approach, then he should "show" Socrates the line (84a1), implying that the slave should point to a segment of the correct length on the physical diagram.[22] Socrates tells Meno that the slave has reached a state of perplexity concerning the solution of the problem: he has realized that making informed guesses concerning the desired length is not working; the closest he is able to come is to say that the answer is less than three but greater than two.

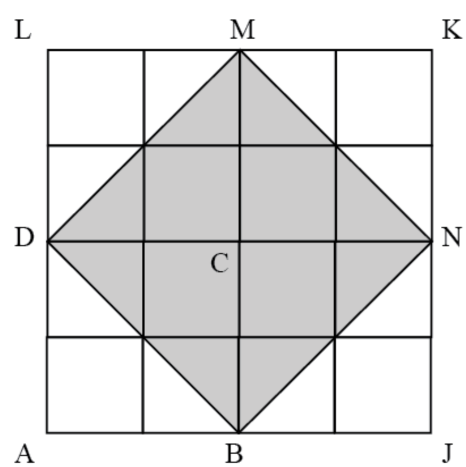

Figure 2.4.3: Square with Diagonals

[22] For philosophers concerned with the history of mathematics, Socrates's instruction to Meno's slave—"if you do not want to work it out, show me from what line"—is an exciting discovery. Malcolm Brown won the *Review of Metaphysics* dissertation award in 1967 for his insightful paper on Socrates's switch from geometry to arithmetic with the easily missed remark. David Fowler thanks Wilbur Knorr for memorably pointing it out "in January 1991 in a train somewhere between Verona and Venice" (1999, 366n12); Fowler writes of *Meno* 82c as "the only arithmetized passage I know, anywhere up to Archimedes and beyond."

Socrates proceeds to lay out another diagram similar to Figure 2.4.2 which is a sixteen-foot square that is produced by joining two four-foot squares, then a third, and finally filling "the space in the corner" to complete a sixteen-foot square containing four four-foot squares. Then Socrates draws the diagonals of the four inside squares that cut each of the four-foot squares in half, which results in a square at a forty-five degree angle to the sides of the sixteen-foot square (Figure 2.4.3). He asks the slave how many of the triangular halves (e.g. *DBC*) are in one of the figures equal to the original four-foot square (*ABCD*). Using the slave's answers to his next series of questions, Socrates elicits the response that there are two triangles in the four-foot squares and that four of these triangles, taken together, equal double the area of the original square. From this, the slave determines that the square on the bisecting line is equal to an area of eight square feet, thus providing the solution to the problem posed by Socrates. To conclude the demonstration, Socrates informs the slave that mathematicians call this line the 'diagonal' (85b4–6). The passage ends with Socrates telling Meno that the answer was found by the slave's using knowledge he already possessed, brought out only by the questions Socrates asked to guide him in their recollection.

2.5 Analysis of the Second Example

Socrates uses the example of doubling the square to illustrate the notion of *anamnesis.* He insists that he is not *teaching* the slave anything, which seems to mean that he is not informing him of anything new, but rather using *only* the knowledge or intuitions he already has to show him something he did not recognize that he knew. This mirrors exactly the axiomatic deductive structure of geometry. Each proposition relies exclusively on prior, i.e. previously established, elements (definition, postulates, propositions, etc.). The connection between what Socrates is doing and geometry is made explicit at 85e1–3: "he will perform the same way about all geometry, and all other knowledge."

Socrates's experiment with the slave, undertaken in response to Meno's paradox (80d5–8),[23] frames the paradox in geometrical terms. Geometry serves not merely as a convenient example; rather, Socrates's choice of a geometrical illustration shows that obtaining knowledge in general involves the same process as obtaining knowledge in the science of geometry.

In geometrical terms, the paradox could be restated as follows. Either you do not have the definitions and axioms, and so cannot learn the rest of the *Elements* (because the propositions and theorems all depend on the definitions and axioms); or, by already having all the definitions and axioms, you already know the rest of the *Elements* (and so cannot *learn* it).

While the role of the example in the dialogue is to illustrate the notion of *anamnesis* or recollection, the example suggests that the knowledge that is recollected is implicit: the slave's ability to grasp counting and spatial recognition is already an implicit understanding of the foundational concepts of arithmetic and geometry. The solution to Meno's paradox is to recognize the distinction between explicit and implicit knowledge. What we know implicitly we know but have not yet applied. If you already have all the definitions and axioms (as implicit innate knowledge concerning spatial recognition) required for making progress in geometry, you are in a position to use these as starting points for learning, which in geometry amounts to developing and proving theorems by building on definitions and axioms.

[23] Meno, using what Socrates recognizes as a debater's tactic, asks about their mutual quest for the answer to the question of what virtue (*aretē*) is: "How will you look for it, Socrates, when you do not know at all what it is? How will you aim to search for something you do not know at all? If you should meet with it, how will you know that this is the thing that you did not know?"

The philosophical notion of recollection is paired with a prototypical proof of arguably the most famous theorem in mathematics: the Pythagorean theorem, in which the square on the diagonal of a right-angled triangle is equal to the sum of the squares on the two legs of the triangle adjacent to the right angle (in the right-angled triangle a, b, c: $a^2 + b^2 = c^2$).[24] Fowler notes that this is "the first direct, explicit, extended piece of evidence about Greek mathematics" and that it provides "a lucid and straightforward example of some of the problems, styles, and techniques of early Greek mathematics" (1999, 7–8).

There are hundreds of known proofs for the Pythagorean theorem. The geometric proof given by Euclid (I.47) involves a step-by-step transformation of the areas of the squares of the two sides into two rectilinear figures that are superimposed onto the square on the hypotenuse. This proof involves a large number of intermediate stages. While it uses a logically sound

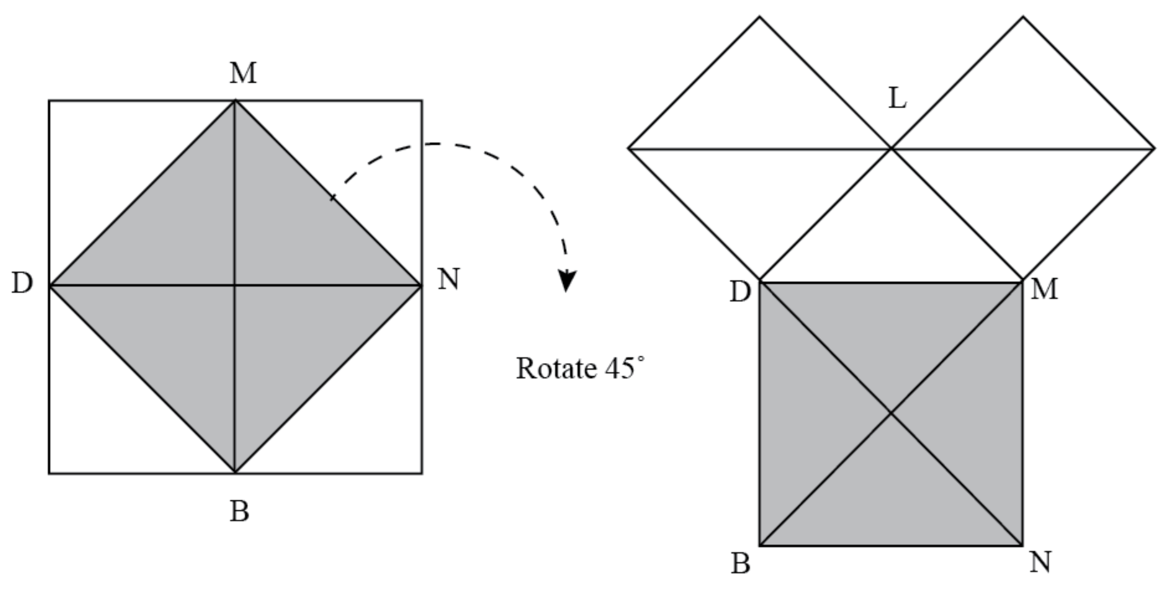

Figure 2.5.1: A Minimal Rearrangement

[24] Euclid I.47. There is a more general form of this theorem for similar figures other than squares at VI.31, but the name 'Pythagorean theorem' is now firmly associated with squares alone.

construction process, it is far from intuitive or accessible to anyone not already familiar with the theorems of Euclidean geometry. This second example is significant in the history of Greek geometry: it gives an arithmetical proof of the geometrical problem of doubling the square, gives an arithmetical and highly intuitive proof for the Pythagorean theorem, and plays a role in the development of the notion of irrational number in that √2 is the length of the relevant side.

As Figure 2.5.1 shows, Socrates's diagram (left) mirrors a geometric proof of the theorem (right) using the simplest case from the geometric point of view—itself equivalent to the problem of doubling the square. An isosceles right-angled triangle or half square is a special case for the Pythagorean theorem, because the dissected areas that will be equated are all triangles of the same shape and size. A proof is easily recognized in Figure 2.5.1: a minimal rearrangement of some of the triangles makes the proof visually apparent by the mere counting of the triangles.

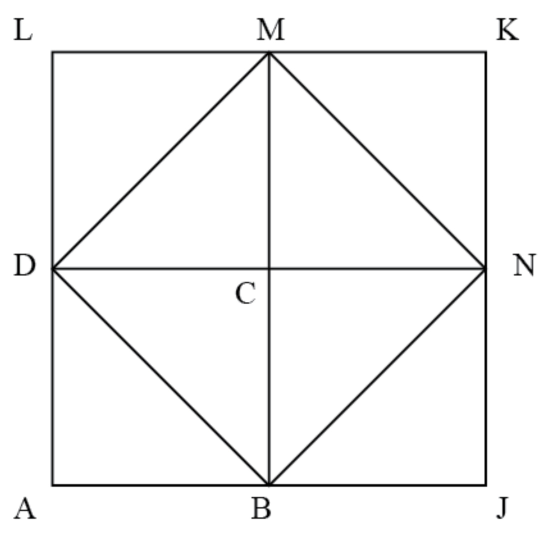

Figure 2.5.2: A Comparison to Al-Khwārizmī

While the rearrangement aids in the visual recognition of the proof, the doubling of the square diagrams contain the elements of the proof as it stands in the *Meno*. This was demonstrated by the Persian mathematician Al-Khwārizmī (1831, 75).[25] He gives a basic arithmetical proof of the Pythagorean theorem in his book on algebra. As I will show, his proof bears a striking similarity to Socrates's construction of the doubling the square diagram for the slave in our second *Meno* passage. Al-Khwārizmī's proof uses a rhetorical equation (an equation written in words rather than in symbols) to describe the theorem, stating that "in every rectangular triangle the two short sides each multiplied by itself and the products added together equal the product of the long side multiplied by itself."

His proof begins with the drawing of a square with equal sides and equal angles (*AJKL*). This corresponds to the larger square diagram Socrates constructs for doubling the square (Figure 2.5.2).

The line JK is divided into two halves at point N. Line ND is drawn parallel to LK and AJ. The line KL is divided in the same way as JK at point M, and another parallel line MB is drawn—a step equivalent to the construction of the transversals for Figure 2.5.2 when doubling the square. This divides square AJKL into four squares with equal sides, angles and areas (the squares CK, CJ, CL, and CA).

A line is then drawn from point N to point M, which divides the square CK into two equal parts, giving two triangles KMN and CMN. Lines are drawn from M to D, from D to B and from B to N, dividing their respective squares in the same manner so that each of the four smaller squares contains two equal triangles. This is the same move Socrates makes when he draws the diagonals of the inside squares, which results in a square at a forty-five degree angle to the sides of the sixteen-foot square (Figure 2.5.2).

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[25] An interesting historical note about Al-Khwārizmī (†850 CE) is that he introduced Hindu numerals to the Islamic world, and the word 'algorism' derives from his name, which was associated with those who adopted the of use of Arabic numerals.

Al-Khwārizmī then states that these four triangles make the square BDMN, which must be equal to half of the large square AJKL. When the lines KM and KN are multiplied by themselves (i.e. squared) each equals the area of two of the triangles, and their sum is equal to four triangles. From this observation he concludes that the Pythagorean theorem is proved because the area of *KM* squared plus the area of *KN* squared equals the area of the square *BDMN*.

A final note about *Meno* 82b9–85d5: the shift Socrates makes from geometry to an arithmetic-based proof is also an example of the notion of transposing (reducing) one problem to another that is more accessible, or to another that one already knows how to solve. The prime example of this in Greek mathematics is the discovery by Hippocrates of Chios, a rough contemporary of Socrates, that the solution to doubling the cube could be "reduced" to finding two mean proportionals between a line segment and a second line segment twice the length of the first.[26] A significant aspect of this reducti that Hippocrates transformed the stereometric cube duplication problem into one that could be handled in plane geometry. The doubling of the square example is similar to the problem of doubling the cube in that both involve finding means between two numbers. The former involves finding a single geometric mean between two plane numbers, while the latter involves finding two means between two solid numbers.

This proportional relationship between two plane numbers with a single geometric mean, and that of two solid numbers with two geometric means, was considered significant enough to be deemed a theorem.[27] These proportional relationships are generalized as the *one mean*

[26] In modern notation, with given segments of lengths a and $2a$, the duplication problem is equivalent to finding segments of lengths r and s so that $a/r = r/s = s/2a$ with the side of the doubled cube being $r = \sqrt[3]{2}$.

[27] This was labelled a Platonic theorem in Greek arithmetic because of its mention at *Timaeus* 32a–b (Nicomachus II.6 = 1926, 272n2).

between two extremes proportion (1M2E) and the *two means between two extremes* proportion (2M2E) in discussions of Plato's possible contribution to the problem of doubling the cube. As we will see, the diagram for the third *Meno* example is a precursor for connecting the 1M2E and 2M2E proportions geometrically and quantitatively in a single figure.

2.6 The Third Example: The Method of Hypothesis 86e1–87b2[28]

Plato's third example in the *Meno* introduces two important methods: hypothesis, and the application of areas. The method of hypothesis, as used in the example, operates on the fact that certain conditions must be met if a given figure is to be constructible.[29] More concretely, to determine whether a figure of given area can be constructed as a triangle in a given circle, one's first step is to determine whether certain conditions can be met. The close connection between this method and Plato's use of diagrams as proofs will increase in importance across his corpus.

In the present passage, Socrates proposes to Meno that they adopt a method of reasoning frequently used by geometers: they should investigate the question of virtue's teachability by

[28] This example is well trodden scholarly territory, though no consensus has been reached. Some of the more influential contributions to the literature are those of Cook-Wilson (1903), Robinson (1953, 93–95), Bluck (1961, 75–107, *pace* Cook-Wilson), Klein (1965, 179–90), Myers (1988), Lloyd (1992), Menn (2002), Scott (2006, 129–44), Wolfsdorf (2008b, *pace* Menn), and Ebrey (2013). Stahl (1971, 184–95, *pace* Robinson) uses the passage to set out the beginnings of propositional logic. Brumbaugh (1954, 32–38), attending as before to accompanying figures for the dialogue, considers previous work by Benecke (1867), Gow (1884), and Heath (1981), attempting to adjudicate disagreements among them.

[29] The crucial philosophical significance of this point is developed in Szabó (1978) and also, with specific attention to Plato, and to the hypothesis passage in *Meno* in particular, in Knorr's (1983) "Construction as Existence Proof in Ancient Geometry." Knorr credits Zeuthen (1896) on the same topic for its historical reliability. Gow (1884, 176) provides a historical note: "The hypothesis in this case serves as a partial *diorismos* in setting out a condition for the problem's solvability. The Eudemian summary credits the invention of the *diorismos* to Leon, but this passage gives a clear example of a *diorismos* which appears to have originated with Plato. It is probable therefore that the whole systematization of analysis is due to Plato. Both Proclus and Diogenes Laertius state that Plato invented the method of proof by analysis."

means of a hypothesis. Socrates then offers an example drawn from plane geometry to illustrate by analogy the nature of this method. In this example, the question asked concerns the construction of a diagram in which a specific area is inscribed as a triangle within a given circle.

The hypothesis offered by the geometer involves a particular condition that, if met, determines that the inscription is possible, and if not met, determines that it is impossible. The condition is that the given area (*X*) must be constructible as a rectangle (*ABCD*) on the diameter of the circle (*CE*) such that it falls short of the length of the diameter (*CE*) by an area contained in a rectangle of similar shape (*BFDE*) to the one containing the given area (see Figure 2.6.1). 'Similar' in this case means that the *ratio* of corresponding sides of a rectilinear figure must be the same, resulting in the same shape.

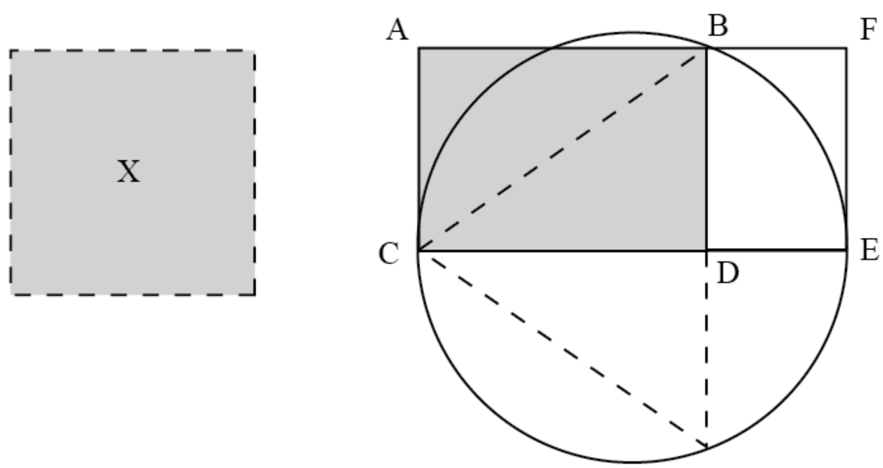

Figure 2.6.1: Inscribing a Triangle in a Circle

It needs to be added that the side *BD* of the rectangle must meet the circle at *B*. The placement of these rectangles in this manner on the diameter of the circle is an instance of the method known as the application of areas, and we will see its use in the *Republic* as well. I describe this as *X*'s being 'elliptic', borrowing for convenience Wolfsdorf's term for this property (2008a, 174). If two rectangles of similar shape cannot be constructed, then *X* is not elliptic, and so the inscription is impossible.

The mathematical construction, application of areas, is credited to the Pythagoreans and was one of the most powerful methods utilized in ancient Greek mathematics. While fundamentally geometrical in nature, it also serves as the basis for the proto-algebraic methods of solving quadratic equations found in Euclid II. Its importance to Apollonius's theory of conic sections is attested by the fact that the parabola, hyperbola, and ellipse are each named after one of the three types of applications of area (parabolic, hyperbolic, and elliptical), which can be used for plotting the points that produce conic sections. Interestingly, despite the fact that the elliptic property is involved in applying the area to the diameter line, point *B* on the circle (Figure 2.6.1) is given by the hyperbola plotted by using the corners of a series of rectangles that have the same area as *X*; hence the shape of the desired rectangle is determined.

2.7 Analysis of the Third Example

The third example is of historical interest in that it presents a problem that may have been unsolvable in Socrates's lifetime because it requires a hyperbola to successfully apply the given area to the diameter and inscribe a triangle of the required size in the circle. The earliest development of a theory of curves from conic sections is usually credited to Menaechmus, one of several mathematicians associated with Plato's Academy, thus after Socrates's death in 399.[30] It

[30] Proclus (fifth century CE) provides a list of these Academic mathematicians in his summary at the head of Euclid's *Elements*, a summary most likely taken from the late fourth century BCE account of Eudemus, also lost, but probably derived by Proclus from Pappus, who wrote in the third century CE (Nails 2002, 277).

is possible that Socrates knew that his hypothesis was literally hypothetical and not a practical, constructible solution; and this fact may have suited his use of it to illustrate his philosophical point concerning how to investigate the question of virtue's teachability. But my point is certainly not to argue for the historicity of Socrates's words in any Platonic dialogue. I suggest rather that Plato saw how this example could serve another purpose: namely, the most plausible diagram for the desired inscription is a precursor of the diagram instrumental in future solutions to doubling the cube.[31]

If this is not readily obvious, consider that the solution illustrated by Figure 2.7.1 is determined by the points of intersection of the hyperbola and the circle; this is most like the problem of finding two mean proportionals (2M2E), which can be solved in a similar manner to Menaechmus's finding the point of intersection between a hyperbola and an ellipse in his effort to determine the cube root of 2 for the proposed doubling the cube. Heath (1981, 1, 299–302) offers a comprehensive account of the mathematics behind the problem, the hypothesis, and a plausible solution. My Figure 2.7.1 follows his diagram and lettering.

The application of areas is a key condition in the inscription hypothesis and can be used to plot the proper hyperbola required in the solution. It is an important link between the solutions of these two problems due to the nature of the constructions it entails. The theoretical foundation for the application of areas lies in the method of transformation of the area of a rectangle into that of a square figure (Euclid I.46) which is closely related to the method of constructing the geometric mean between two magnitudes. The "applying" or construction of rectilinear figures on a given line serves as the geometric equivalent for solving linear and (most importantly)

[31] Knorr (1986, 73) suggests: "Plato's emphasis on the possibility of the inscription might be taken to signify that geometers had then discovered the diorism, but not the actual solution of this problem."

quadratic equations.[32] It will be shown in chapter 4 that constructing a triangle determined by the construction of geometric means is instrumental in developing potential solutions for doubling the cube (and thus solving a cubic equation).

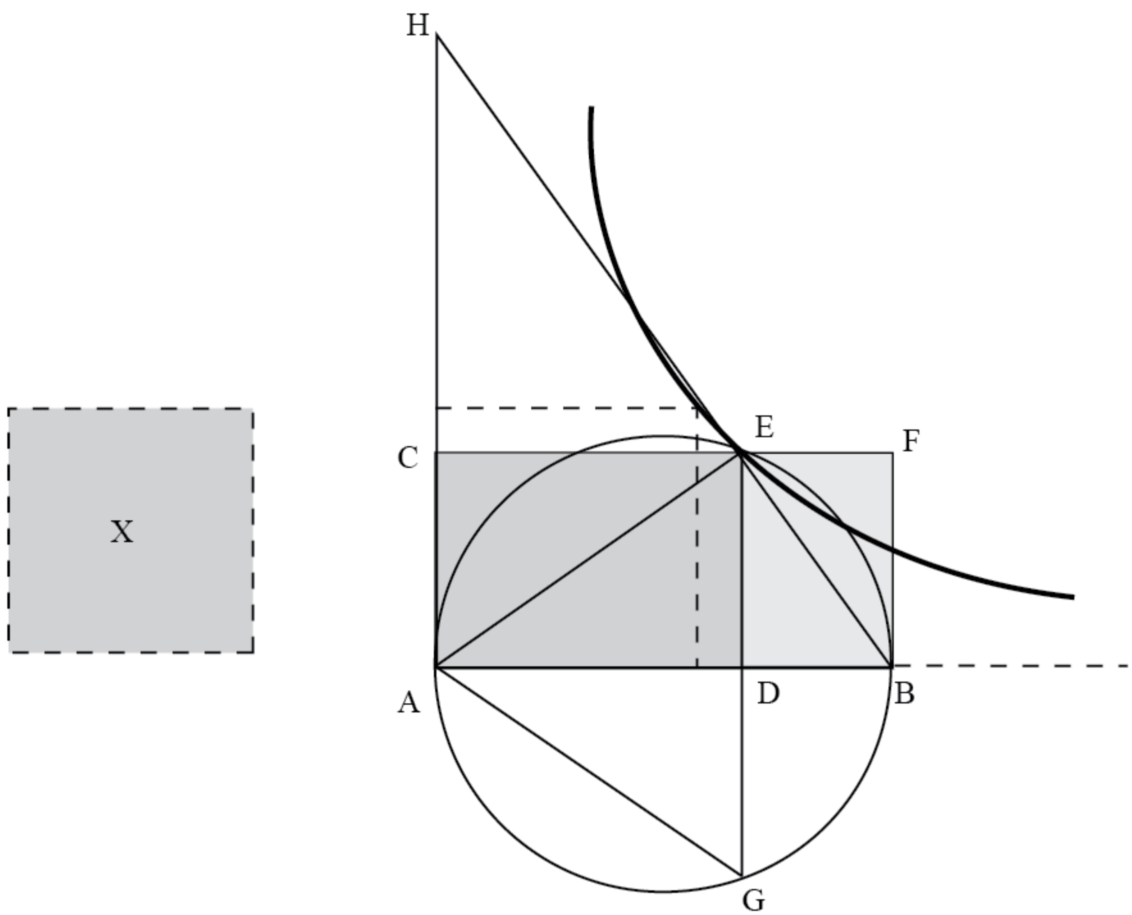

Figure 2.7.1: Solving a Cubic Equation

It can be asked whether the inclusion of this hypothesis might be anachronistic due to its association with conic sections. While the general solution requires the use of a conic curve, the

[32] Taisbak (2003, 299–300): **"**To apply an area to a (straight) line" always means "to construct a parallelogram along that line." The parallelogram may have the line segment as one of its sides, in which case it is known as the 'parabolic application' of area (*Elements* I.44); it may also exceed the segment, in which case it is the 'hyperbolical application' (VI.29); or finally, it may fall short of it, in which case it is the 'elliptical application' (VI.28)."

conditions for the hypothesis of inscribability can be determined from the diagrams of cases of specific areas that can be inscribed as triangles. An examination of the diagrams for the two standard cases—an equilateral or a right-angled isosceles triangle inscribed in a circle—show that *X* has the elliptical property. The diagrams of these two cases follow directly from the construction of the circumcircles of three- and four-sided regular polygons and are easily produced with a compass and straightedge. These standard cases allow for the development of the hypothesis that *whenever X* has the elliptical property, it will be inscribable as a triangle in a given circle (Figure 2.6.1). While the practical application of *X* in non-standard cases, i.e. for non-right-angled isosceles triangles, will require the use of conic sections, the hypothesis is not theoretically dependent on the use of conics.[33]

Plato's choice of this example tells us something about the practical versus theoretical value of using certain hypotheses in both philosophy and mathematics. The method of hypothesis can be put to philosophical use in the establishment of conditions to determine whether something is possible. Socrates is proposing that an attribute of virtue is that it be teachable. Since if virtue is knowledge then it is teachable, the apparent absence of teachers of virtue means that it is not immediately apparent that virtue is teachable. Thus Socrates and Meno will investigate the question of virtue's teachability by inquiring whether virtue is knowledge. Analogously, since, if the area *X* is elliptical it is inscribable, and it is not immediately apparent whether *X* is inscribable, the geometer will investigate the question of area *X*'s inscribability by inquiring whether *X* is elliptical.

This method does not necessarily give an answer to how to inscribe the area or the

[33] The solution requires rectilinear figures, so scalene triangles will not give the necessary figure although a scalene triangle's area could be transformed into the shape of an isosceles triangle.

answer to how to determine whether X is elliptical. It only provides a condition to help determine whether the problem is unsolvable (unanswerable) if a certain set of conditions is *not* met. This may be considered an instance of mathematical *aporia* that produces an impasse, while at the same time yielding insight into the path forward for solving the problem.

The fact that the theory of conic sections was still in its infancy at this time does not preclude the possibility of establishing the conditions for the inscription of the triangle in the circle. This can be seen from this mathematical example, where there are two specific cases for which the construction is possible and certain conditions become apparent from their construction. One of these cases involves an equilateral triangle of a specific size that is easily inscribed in a circle. This is the case that gives the maximum area that can be inscribed in a circle as a triangle which is used in Figures 2.6.1 and 2.7.1.[34]

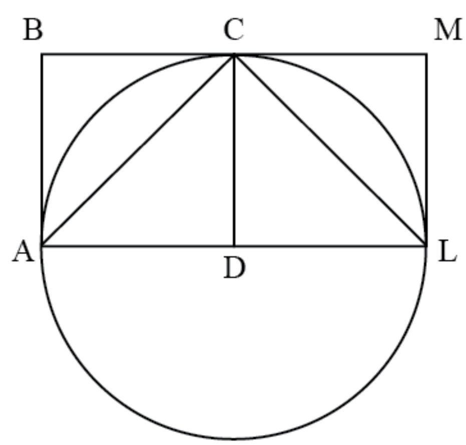

Figure 2.7.2: Isosceles Right-angled Triangle

The other involves an isosceles right-angled triangle inscribed in the circle, using a figure that once again mirrors the doubling of the square diagram. This applied shape will be a square

[34] For a solution in real numbers, the given area cannot be larger than the equilateral triangle inscribed in the circle, i.e. less than $3\sqrt{3} \cdot a^2/4$, with a being the radius of the circle.

that falls short by an area that is not just similar, but equal, to the area with which we started. In Figure 2.7.2, *ABCD* is the area to be inscribed in the circle and applied to the diameter *ADL*. It falls short by the area *DCML*, which is similar (and in this case equal) to *ABCD*.[35] Hence the area *ABCD* can be inscribed as a triangle in the given circle.[36]

2.8 Conclusion

In sum, I have shown that Plato's mathematical examples in the *Meno* use a number of foundational geometrical concepts whose significance becomes more apparent when taken together. They broadly define the plane or two-dimensional geometric space while introducing two highly significant notions in Greek geometry, linking these to the construction of figures that have irrational square roots as a metrical property ($\sqrt{2}$ and $\sqrt{3}$).[37]

Paradigmatic cases of the second and third *Meno* examples, shown in diagram, will lend support to my thesis about the construction of the *Republic*'s divided line in chapter 4. The proof for doubling the square relies both on the Pythagorean theorem and on finding $\sqrt{2}$, which is directly related to finding the geometric mean between one and two.

The geometric mean theorem is another notion frequently used by Greek geometers for both solving problems and proving complex theorems.[38] The geometric mean also plays a role in

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[35] Application of areas can utilize the equation $yz = x^2$ and is another appearance of the geometric mean.

[36] This was suggested first by Benecke (1867); Heath (1981, 1, 302) cites Bluck (1981, 447–48) for the point.

[37] It also establishes the foundation for the field of constructible numbers (those involving addition, subtraction, multiplication, division, and finding square roots) that can be drawn with a compass and straightedge (which in turn make manifest the mathematical objects of the straight line and circle). The ancient Greeks relied on diagrams to establish this field of constructability (whereas we can see the relations from algebraic formulae derived within the Cartesian coordinate system).

[38] These are found in Euclid as a corollary to VI.8 and as a method for squaring a rectangle in II.14.

the application of areas in the third example, as well as in the search for a solution to doubling the cube, one of the three great problems in Greek mathematics,[39] and the one neoplatonists associated with Plato.

Plato introduces us in the *Meno* to notions linking lines and areas that are crucial for developing his mathematical analogs to knowing and being elsewhere in the corpus. Moreover, learning geometrical truths is, for Plato, an instance of grasping truths more generally. As later chapters will show, these simple examples from *Meno* are essential building blocks for Plato's construction of the divided line in *Republic* 6 as well as puzzles such as the "nuptial number" of *Republic* 8.

[39] The other two are squaring the circle and trisecting an angle.

CHAPTER 3:

WHAT DID THEODORUS DRAW?

3.1 The Mathematical Passage in Context

In Plato's *Theaetetus* (147d4–148b3), the youth Theaetetus describes his tutor's use of a diagram to demonstrate a point about incommensurability. The brief description occurs in the midst of Theaetetus's efforts to answer Socrates's broader question: *What is knowledge?* Having just returned from Theodorus's mathematics class with his friend—the young Socrates,[1] namesake of the philosopher asking the questions—Theaetetus wonders whether a mathematical proof qua diagram might be a good model for answering Socrates's question satisfactorily.[2]

> THEAETETUS: Theodorus was using diagrams to illustrate a point for us about powers,[3] in relation to a figure of three square feet and one of five square feet, namely that they are not commensurable in length with a figure of one square foot; and he proceeded in this way case by case until he reached a figure of seventeen square feet, where somehow or other he came to a halt. Well, this sort of thing occurred to us—given that the powers were apparently unlimited in number, we should try to combine them into one, so that we'd have something to call all these powers.
> SOCRATES: And did you find something like that?
> THEAETETUS: I think we did, but see what you think.
> SOCRATES: Go on.
> THEAETETUS: We divided the whole of number into two. Any number that can be produced by multiplying two equal numbers we compared to a square figure, and

[1] Young Socrates (a.k.a Socrates the Younger, and Socrates Junior) appears also in the *Sophist* and *Statesman*. He is mentioned by Aristotle in the *Metaphysics* (1036b25) and in the pseudo-Platonic Letter 11. Young Socrates was later an associate in Plato's Academy (Nails 2002, 269).

[2] The texts of the *Theaetetus* and the *Statesman* are from the OCT edition of E. A. Duke et al., and the translations are those of Christopher Rowe. Passages quoted include Rowe's footnotes.

[3] Which Theaetetus will shortly attempt to define as 'oblong numbers'. To understand this, we need to recognize at least the following: (a) that the mathematics of the time does without 'irrational' numbers; (b) that it therefore has to deal with what we call the roots of non-square numbers in a special way; (c) that this special way is geometrical in form; but (d) that the whole exercise is not just about numbers, but geometry too.

> called it ‘square’ or ‘equal-sided’.
> SOCRATES: Good; well done.
> THEAETETUS: So then any number between these, namely three, and five, and any other that can’t be produced by multiplying two equal numbers, but only by multiplying a greater by a less or a less by a greater, and is always contained by a side that’s greater and a side that’s less—this we compared to an oblong figure, and called it an ‘oblong’ number.
> SOCRATES: Very fine! And what was your next step?
> THEAETETUS: Lines that as sides of a square produce the ‘equal-sided’ plane numbers we marked off as ‘lengths’, and those that produce the ‘oblong’ ones as ‘powers’—on the grounds that while these are not commensurable in length with the other sort of lines, they *are* commensurable in the plane figures they have the power to produce. And we made another, similar distinction in relation to solids.—Rowe, trans. (147d4–148b3)

The main interlocutor in Plato’s *Theaetetus* is the boyhood version of the renowned Greek mathematician, Theaetetus of Sunium, who is credited with the early development of the theory of irrationals canonized in Euclid’s *Elements* X, and with important discoveries concerning the five regular polyhedra of *Elements* XIII. His geometry tutor was Theodorus of Cyrene, another noted figure in the history of Greek mathematics who was a rough contemporary of Socrates. Theaetetus’s important work on irrationals was performed in the context of a more general effort by Greek mathematicians, many of them associates in Plato’s Academy, to develop a mathematical notion of number that was broader than the ancient Greek notion of *arithmos* (‘whole number’, ‘positive integer’). The brief mathematical passage in the *Theaetetus* recounts what has been viewed as the initial motivation behind the development of the theory of irrationals.

The passage has also raised a number of issues that remain in dispute among commentators—philosophers and mathematicians alike. They include: (i) Plato’s use of the term *dunamis* and its cognates in the passage (d4, d9, e1, e6, a1); (ii) the method of proof—if proof it was—used by Theodorus to establish the incommensurability of certain square roots; and (iii) the *reason* Theodorus stopped with √17. In the course of my discussion, I will examine these overlapping issues in pursuit of an appropriate diagram that Theodorus could in fact have drawn in Socrates's lifetime.

A prior question, however, is whether Theodorus should be assumed to have been showing the boys the actual proofs of incommensurability, or rather drawing something less

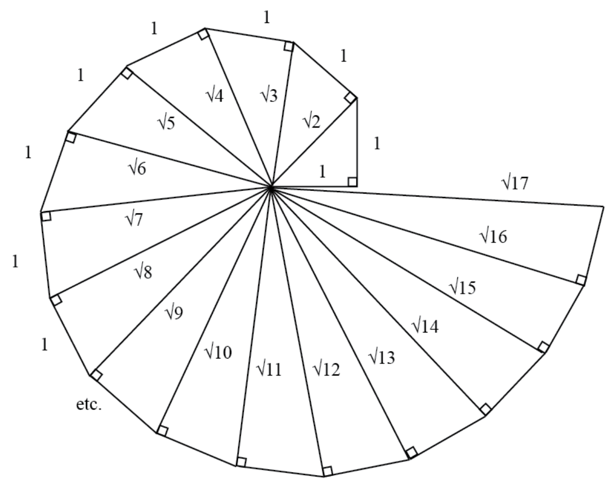

Figure 3.1.1: Preview

rigorous to display results of his own previous investigations.[4] I will offer reasons for preferring the latter, while arguing that a particular drawing (Figure 3.1.1) could well be *the* drawing that Plato imagined Theodorus producing for the boys. This illustration, widely quoted and reproduced in mathematics textbooks and on the Internet, graphically displays the relationship among the square roots of the first seventeen whole numbers, and supports the specific interpretation of the three disputes I identified above as living issues among philosophers of

[4] Everyone has something to say on the issue. Among the principals, Knorr (1975) denies that Theodorus offered proofs; and Burnyeat (1978) insists that Theodorus did offer proofs. Fowler (1999, 380) alludes to the apt point that the dialogue is about apodicticity (what is knowledge?). Plato is not likely to have thought that apodictic proof can be achieved below the level of the forms themselves.

ancient mathematics—or so I argue in this chapter.[5]

The need for an understanding of irrationals arose in the context of quantifying geometric magnitudes associated with two and three-dimensional objects. As quantification in ancient Greece was historically restricted to the notion of whole numbers, there was no established terminology to apply to quantities that were *not* expressible using whole numbers. The term *dunamis*—which has the everyday meaning of 'power' 'capacity', or 'ability'—was adapted to refer to an aspect of a range of mathematical notions that included two-dimensional space, plane geometry, and the magnitudes for measuring areas. Discussion of how the term is used in this passage offers an insight into the diagram Theodorus could have drawn.

The importance of translating the Greek term *dunamis* correctly in mathematical contexts—and especially at *Theaetetus* 147d4–148b3—cannot be overestimated. Høyrup calls *dunamis* "among the most debated single terms in ancient Greek mathematics" (1990, 202). More recent commentators and translators translate 'square' rather than the 'square root' that was often used in the earlier part of the twentieth century.[6] But the controversies run much deeper than that. Complaining that "immense heat has been generated over the terminology of the passage through failure to make any distinction between meaning and application" (1978, 496), Burnyeat comprehensively cites the scores of authors, articles, and books articulating positions on *dunamis* in the *Theaetetus.* Excising numerous citations, footnotes, and less

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[5] The diagram has passed into the public domain without attribution; 'spiral of Theodorus' and 'Theodoran spiral' have become common terms among authors writing on ancient and contemporary mathematics, and Plato's dialogue is always credited, but the term has not been taken up in the Platonic literature. The diagram's interesting history is summarized in 3.4.

[6] Translating *dunamis* 'square root' are Thomas (1939, 381–82), van der Waerden (1954, 141–42), McDowell (1973, 116); translating 'square' are Burnyeat 1970; Knorr 1974; Levett 1978; Szabó 1978; Fowler, 1999; and Rowe 2015. This is, of course, only a sample; and it would need further division to account for the difference between *translating* for a largely student audience and *commenting* for one's specialist colleagues.

pressing details, what I quote below will bring us up-to-date on two thousand years of scholarship to 1978, leaving me free to take up the discussion earlier and later, i.e. with Plato's text itself in the context of ancient Greek mathematics, and with scholars writing about Theodorus's drawing *after* Burnyeat's 1978 state-of-the-question article:[7]

> … while δύναμις is applied to incommensurable lines in the definition at 148b1, it is also … used earlier to specify what Theodorus's demonstrations were about, so it has been a matter of controversy whether at that earlier stage [147d4–5] the term stands for the *sides* of a series of squares or for the *squares* themselves …. The latter view … aims to avoid what would, it is felt, be an intolerable shift of meaning (from 'square' to 'incommensurable side'), but has to concede a narrowing of meaning (from 'side' to 'incommensurable side') and adduces no parallel for δύναμις in the sense of 'side, whether commensurable or incommensurable'. It incurs, in addition, a grammatical objection. … [for which] no one has been able to supply a satisfactory parallel or explanation. In any case, the difficulty these scholars aim to avoid is illusory: δύναμις is *applied* to incommensurable lines without meaning 'line' or 'incommensurable line'. Given what we have yet to confirm, that the word means 'square', it is precisely in virtue of this meaning that it can be adapted to serve as a name for incommensurable lines: in its naming function it alludes to the fact that the lines in question are commensurable in square but not in length, just as the lines which are commensurable in length as well as in square are termed 'lengths' or 'length lines'.
>
> Ancient scholars made the reverse mistake. Evidently they could not conceive how a word meaning 'square' might be applied to something that was not actually a square. … This ancient testimony, mistaken as it is about the defined use of δύναμις at the end of the passage, is for that very reason strong grounds for accepting that δύναμις at 147d4 means 'square.' … Later Greek scholars plainly had no inkling that at an earlier period the word might have meant something different, and in paraphrasing 147d4 they automatically take δύναμις as 'square' …. The correctness of their assumption can be confirmed from Greek mathematical usage itself … and the matter is clinched, so far as the

[7] The article itself addresses other issues at stake in understanding the mathematics of *Theaetetus*. Against Szabó (1963, 1966, and 1969), who argues that the mathematics reported by Theaetetus is routine Pythagorean doctrine, nothing more than schoolboys would learn in Socrates's time, and thus that neither Theodorus nor Theaetetus and young Socrates are represented by Plato as developing mathematical theory, Burnyeat (1978, 194–95) defends the more widely held view that original mathematical developments attributed to Theaetetus himself are acknowledged by Plato's choice to make him a respondent to Socrates. As I pointed out in chapter 1, philosophers tend to push mathematical advances forward in history, and mathematicians to find them in evidence earlier. In the quotation below, Burnyeat was referring to the Burnet OCT so, where the two differ, I have amended his line numbers to that the Duke et al. OCT.

> earlier period is concerned, by Plato, *Timaeus* 31c, where δύναμις signifies a 'square' number in contrast to ὄγκος, a 'cube' number. We must settle, then, for δύναμις at 147d4 meaning 'square.' (Burnyeat 1978, 496–98)

Consider Plato's text. To determine what would constitute an inappropriate shift, and whether a shift is real or apparent, we can start by examining the context of the first appearance of the term *dunamis* in the passage. Literally translated, *dunamis ... tēs tripodos* (d4–5) is "power of three feet," but both 'three feet' and 'power' can have more than one meaning. The most straightforward meaning of *tripodos* is "measuring three feet"; but LSJ cites *Theaetetus* 147d as its paradigm for the mathematical sense: "the side of a square three feet in area" because a foot was a unit of linear measurement in ancient Greek, just as it is in English. Theaetetus says that "three feet" is not commensurable with "one foot" (*podiaios*), which only makes mathematical sense if he is referring to square figures with *areas* of three feet and one foot, respectively. If "three feet" and "one foot" meant linear rather than square units, then "three feet" and "one foot" would be commensurable. Thus part of the passage's ambiguity stems from the use of terms such as *tripodos* and *podiaios*, which can be used to designate both linear and area measurements.[8]

But we do not yet have an explanation for why *dunamis* could refer in this passage to either the square root of a number or the resulting product of multiplying two equal factors, i.e. $\sqrt{3}$ or $3 = \sqrt{3} \times \sqrt{3} = (\sqrt{3})^2$ in mathematical notation. This brings us to the second part of the puzzle: unravelling the meaning of 'power' or *dunamis* in the expression "power of three feet."

[8] The issue is not unique to the *Theaetetus*. For example, in the *Meno*, close attention to context is needed to determine whether *podos* is used to designate feet as linear or square units. At *Meno* 82c5–8, "how many feet would the whole be?" refers to square units while "if it were two feet this way, and only one foot that way" (*duoin podoin* and *enos podos*) clearly refers to linear feet as these are measurements of the sides of a rectangle. Further, at *Meno* 83d4–5 Socrates says: "The line on which the eight-foot square is based must then be longer than this one of two feet and shorter than that one of four?" Here 'eight-foot' (*octopodos*) refers to an area or space, while 'two foot' and 'four foot' refer to lines (*dipodos* and *tetrapodos*), which they have just determined cannot be the length of the side of the eight-foot area figure.

Plato's use of 'foot' to refer to either a measurement of length or a measurement of area has not been considered controversial. What is of interest here is that the reader must depend on context to determine which is meant. I contend that a possible resource for disambiguating *dunamis* lies in a parallel relation between length/area and square root/squared number. There is a complementary nature to the notions of square roots and squared numbers in their geometric form. While in modern mathematics these notions are abstract and without spatial connotations, in Greek mathematics they were conceptualized geometrically as, respectively, the side and areas of actual square figures. And unlike modern mathematics, where the concept of number has a broad meaning, in Plato's time, as we have seen, *arithmos* designates positive integers (or, less technically, whole numbers and, by extension, ratios of whole numbers—which gives rise to the term 'rational numbers').

One of the uses of *dunamis* in this passage is to distinguish sides of square figures that are irrational magnitudes, and so not expressible in whole numbers, from sides that *are* expressible in whole numbers. Since a quantity such as $\sqrt{2}$ is irrational, it is not expressible in ratios of whole numbers. Consequently, it would not fit in the category of number. The realization that many geometric magnitudes could not be expressed by rational numbers (*arithmoi*), which was understood as non-commensurability, was problematic. The formulation of a solution required conceiving of certain lengths, such as a magnitude equaling $\sqrt{2}$, as *geometric* entities. The need to address magnitudes of such lengths naturally emerged in geometry: the diagonal of the one unit square was easily constructed, but it was impossible to assign it an exact magnitude by measurement.[9]

[9] As illustrated in the *Meno*, this diagonal, which is $\sqrt{2}$ in length, is a line on which to construct a square with an area of two square units; the construction of this second square is dependent on first constructing a unit square (or a right-angled triangle with one-unit legs, which is a step in constructing a unit square). The notions of a square figure and

The fact that the construction of a linear magnitude equal to √2 is dependent on the construction of a unit square suggests that there is an organic connection between such a length and a square figure and, by extension, an area measured in square units. Thus, in the case of our search for the correct meaning of *dunamis* in these passages, it can be said that the identity of a √2 length is inextricably tied to that of a two-foot² area. The geometer must work using a two-dimensional construction to produce this one-dimensional object, as shown in Figure 3.1.2 where a line (AB) of √2 unit length constructed as the diagonal of a unit square. This is in contrast to a √4 length, which is not inextricably tied to a four-foot² area since a √4 length can be exactly expressed by the rational number 2, a magnitude that can be constructed simply by extending the length of a line from one unit to two using a compass and straightedge as shown in Figure 3.1.3 where a line (AC) of 2-unit length is constructed with compass. The geometer need not reach into two dimensions to construct a figure to determine such a length.

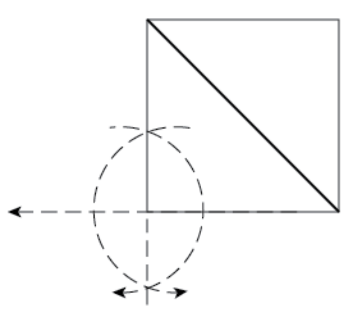

Figure 3.1.2: Diagonal Construction

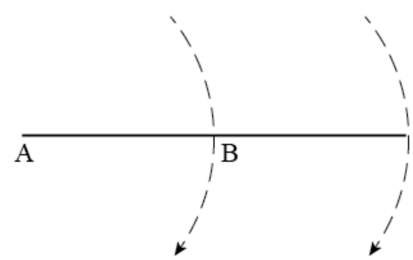

Figure 3.1.3: Compass Construction

Thus the measurement assigned to areas is not only in terms of square units but can also be thought of as another order of quantification, 'plane numbers' in the broader modern sense of

the measurement of area in square units are independent because any shaped area can be measured in square units.

number. To work with quantities inexpressible by rational numbers required an initial move to a more complex notion of number that could include the operation of finding square roots as well as ways to conceptualize mathematical objects that fall into the category of quadratic equations in modern algebraic thinking. In the absence of the latter, mathematicians developed a geometric conception of quadratic equations and finding square roots. The role of algebraic equations would be filled by proportional magnitudes in the form of ratios and proportions of the sides of rectilinear figures. But let us return to the language of Plato's text.

Typically, *dunamis* occurs in the dative case (*dunamei*) when it means 'by its ability' (or 'by its square'). But in the *Theaetetus* passage, we find it in the genitive case because it is governed by *peri* (concerning [the powers] …), which here takes the genitive case. Still, *dunamis* in the *Theaetetus* passage must have the dative meaning because, as we have seen, the *dunamis* (nominative case) of three feet, i.e. three squared (or three to the second power), is commensurable with the *dunamis* (nominative case) of one foot, that is, 1^2. The entities being declared incommensurable are lengths, not areas. But they are a special kind of length whose very identity, as I have argued, is tied to the two-dimensional figure that must be constructed for the length to be produced.

The point I am making is that the geometrical notion of *dunamis* (power) seems to cover both the finding of a square root and the squaring of a magnitude. While square roots and squares are usually stated as magnitudes, these magnitudes are referred to using these terms as the result of specific mathematical operations. These two operations are related in just the way that subtraction and addition, or division and multiplication, are related: the one operation is the inverse of the other. Given the tight connection between inverse operations, it might have seemed natural in early times to use a single term to refer to both. Thus *dunamis* refers to *both* of the directions of the relationship between a square and its sides, the inverse relation between squares and square roots.

When *dunamis* is translated 'powers', therefore, the notions of a squared number and a square root of a number (e.g. $(\surd 3)^2$ and $\surd 3$), share a commonality in that they are manifested by the inverse operations of squaring (multiply a number by itself) and extracting a square root, respectively. This relationship is reflected in the modern notional expression of square and square root as n^2 and $n^{1/2.}$

The relationship between the side and diagonal of a square is an acknowledged constant in mathematics as the ratio of $1n : \surd 2n$ that is invariant for a square of any size. The notion of side and diagonal numbers, as we have already seen, was known to Plato and figured in ancient Greek mathematics in the investigation of irrational numbers. Thus it is easily supposed that a similar relationship was theorized between a square and its square root (n^2 and $\surd n$).

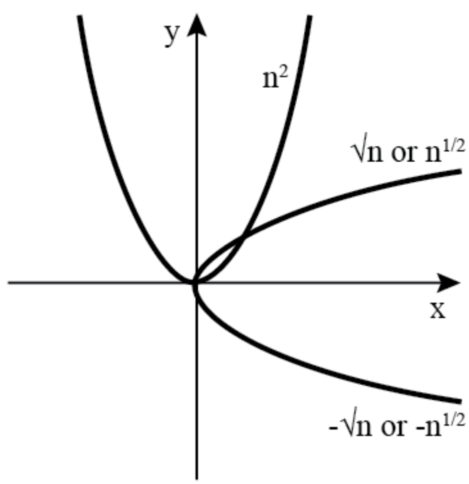

Figure 3.1.4: Graph of n^2 and $\surd n$

The connection between the two operations becomes even more apparent when their values are graphed in co-ordinate geometry (Figure 3.1.4). The foci of their points trace similar conic curves; n^2 generates a parabola along the y axis while $\sqrt{n}$ generates one of the same shape on the x axis. While ancient Greek mathematicians conceptualized parabolas as the edge of a plane sliced out of a three dimensional cone, the curve itself was plotted by using the application of areas method to plot the values of a series rectilinear figures equal to n^2 or $\sqrt{n}$ in relationship to ordinates relative to the shape of the cone.

This insight into the nature of the mathematical concept of *dunamis* in *Theaetetus* 147d4–148b3 will prove to be useful in considering what diagram Plato may have envisioned Theodorus drawing in his demonstration for the boys.

3.2 What a Plausible Diagram Requires

There are a number of theories about the identity of the diagram or diagrams that Theodorus drew for Theaetetus and young Socrates. Any plausible reconstruction must meet certain requirements set by the text and by then-current geometrical practice, while avoiding introducing more mathematical complexity than is actually justified by the text. Here, I first enumerate those requirements before turning in section 3.3 to the evaluation of how well various proposed diagrams fulfill them.

Requirement 1. Theodorus must draw a figure worthy of his reputation as an expert in geometry (145a6–7). I am not making grand, anachronistic claims about Plato's intentions regarding historical accuracy or literary fidelity,[10] but noting that some of Plato's characters

[10] Many commentators write as if they are able to channel Plato, at least sometimes, and that his purposes and

(Meno, to take one notorious example) are cast as miscreants in the text itself. This is not so of Theodorus, who is immediately credited with expertise "in astronomy, arithmetic, music, and everything else that goes to make an educated person" (145a8–10). His friendship with Protagoras (161b9–10) marks him as an intellectual of his era, shown also by his return the following day to participate in the discussions of the *Sophist* and its sequel, the *Statesman*, where Socrates remarks that Theodorus is "the best arithmetician and geometer" (257a6–8).

Requirement 2. The drawing should be something Theodorus could reasonably have drawn while the boys were present for their lesson. That is, the drawing must be one that can be produced in a relatively short timeframe. If a proposed diagram or series of diagrams would take Theodorus several hours to draw for the boys, it does not fulfill requirement 2.

Requirement 3. The proposed diagram should be one we have good reason to think Plato would have been able to conceive and construct—a drawing he could have had in mind as he dictated the text to his scribe. An obvious restriction is that it not deploy mathematical concepts and operations that had not yet been discovered or developed in Socrates's (or even Plato's) time. Less obvious, but crucial, is the matter of Greek mathematical history I discussed in chapter 1: Euclid, Archimedes, and Apollonius canonized mathematics long after Plato wrote his dialogues, but it is evident that notions canonized or proved by later Greek mathematicians had been utilized long before.

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intentions—in even the restricted domain of a single passage such as the one I am here considering—are perspicuous. Burnyeat (1978, 490) is one of those who sees so clearly: "all agree that we are dealing with an original contribution to science by Theaetetus, building on results previously attained by Theodorus"; and Burnyeat calls in evidence "a few of the distinguished names" who share his "reading of Plato's purpose": van der Waerden (1954, 142), Vogt (1909–10, 155), Sachs (1914, chapter 1), Heath (1921 = 1971, 1, 202–204), von Fritz (1934a, 1351–52 and 1934b, 1811–12), and Heller (1956–58, 1–58). As I have already noted, however, Szabó's articles and books of the 1960s channel Pythagoreans instead.

Requirement 4. The diagram must somehow account for Theaetetus's report that Theodorus ran into difficulties and stopped when he reached the √17 (147d7–8). There has been considerable speculation about this assertion, which has in part driven the theories concerning what Theodorus drew for Theaetetus and young Socrates.

Requirement 5. A final requirement is that the diagram should suggest in some way to the boys what Theaetetus reports: (a) that all of the square roots of whole numbers form a class or are somehow one in kind (147d9–e1), and (b) that this class can in fact be divided into two subclasses (147e5).

Each of the above requirements narrows the possible candidates from among those in the literature, but the literature has not valued or even itemized my particular set of requirements. By introducing Figure 3.1.1, I have already put my cards on the table, but it remains for me to demonstrate that mine is the winning hand, that the figure I provide really does meet all the requirements.

3.3 What Diagram Meets the Requirements?

Given his reputation as a geometer, Theodorus would have used geometric rather than arithmetic methods to make his discoveries and conduct his proofs. I will treat requirements 1 and 2 together, since we are looking for a diagram that is both something a geometer is likely to have drawn (requirement 1) and likely to have drawn *in front of* other geometers (requirement 2).

It should not be presupposed that Theodorus's drawing constitutes a rigorous diagrammatic *proof:* a diagram with the elements of each step constructed according to enumerable geometric principles. The text only states that he used a drawing (*egraphe*) to show that these square roots were “not commensurable in length with a figure of one square foot” (147d6–7). A proof would require that each step be constructed according to geometric principles using only a compass and straightedge. This would require a great number of auxiliary lines to justify the move from one step to the next. By contrast, a drawing could illustrate the final result of the constructed proof without including all the steps of the proof and their auxiliary construction lines. That the diagram(s) Theodorus drew need not have reached the level of *proofs* is reasonable given the fact that there is, on the one hand, the geometer’s discovery of a proof rigorous enough to be mathematically convincing and to satisfy him of its validity and, on the other hand, a drawing that displays his results, with the aim of informing other geometers of his discovery.[11]

Netz (1999, 54) observes that, in Greek mathematics, proof via construction of a figure is a precise practice requiring the drawing of auxiliary circles and lines. He rightly points out that it is impossible to construct a figure such as the isosceles triangle in proposition I.1 the *Elements* without auxiliary circles. The geometer may assume that auxiliary circles have been constructed in increasingly more complex constructions, but in that case he must “acknowledge the shadow of a possible construction without actually performing it.”[12]

Consider for example the difference between the construction a geometer would produce to *prove* that a pentagon is constructible with compass and straightedge (Figure 3.3.1) versus the mere construction of a pentagon with compass and straightedge (Figure 3.3.2). The first shows the rather messy figure that emerges from a proof that carries within itself what must be known to draw the second, cleaner figure. Yet one need not include all the auxiliary construction lines

[11] As we will see in the discussion of requirement five, although Theodorus is not here giving a rigorous proof, this will not prevent the boys from making discoveries of their own.

[12] The point that Netz makes is essential to the construction of the divided line in chapter 4.

every time one wants to draw a pentagon because the cleaner method comes to light from the process of proving that a pentagon can be constructed with the geometric method, which is in

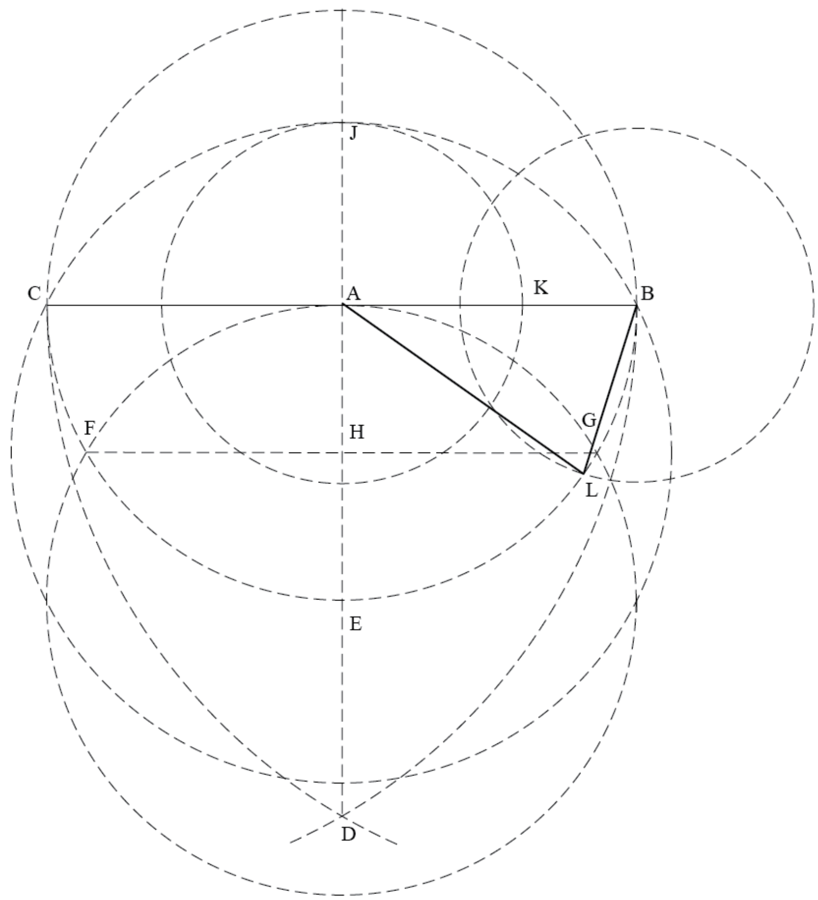

Figure 3.3.1: Compass and Straightedge Pentagon Proof

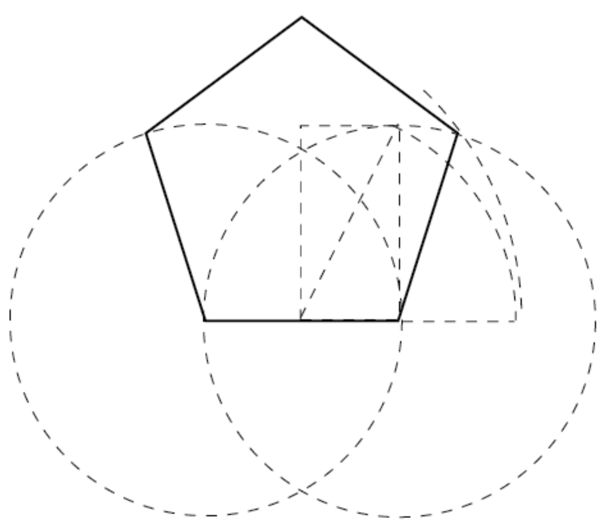

Figure 3.3.2: Pentagon Drawing

effect a simplification of the *synthesis* part of an ancient Greek mathematical proof.[13]

A much discussed candidate diagram is the Euclidean algorithm in its geometric guise of anthyphairesis (from the Greek verb *anthuphairein,* 'to take away in return') found at Euclid X.2,[14] which is applied to magnitudes (rather than numbers as in VII.2). Contemporaries are more likely to be familiar with the term 'continued fractions' or 'reciprocal subtraction'.[15] Fowler (1979, 817, modified) describes the process without diagrams:

> Given two homogeneous magnitudes A and B (think of A and B as line segments) with B smaller than A, suppose B goes into A some number n_0 times leaving a remainder A_1 *less* than B; now repeat the procedure with B and A_1 giving rise to a second number n_1, and magnitude B_1. If, at some stage, the current smaller magnitude goes precisely into the larger magnitude, "measures it," then this current smaller magnitude measures the magnitude before, which then measures the magnitude before it, and so on, and so it measures both original magnitudes, which are therefore commensurable, and the process terminates. Conversely, if the magnitudes are commensurable, so have a common measure C, the process will terminate. This follows because we can see that any common measure of A and B must also measure the remainder A_1; hence the first step replaces A and B by a smaller pair B and A_1 which also have common measure C. Repeating this process sufficiently many times will decrease the magnitudes until, it can be shown, they become less than C, at which a contradiction is manifest.

Anthyphairesis is a possible method for achieving proofs, but it does not follow that Plato would then have had Theodorus drawing diagrams using the anthyphairetic process when he depicts him as sharing his results with Theaetetus and young Socrates. As I have already emphasized in my discussion of requirements 1 and 2, any insistence that the diagrams must have constituted

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[13] I use the term 'geometric method' (rather than 'Euclidean method') to avoid an anachronism: this construction is not found in the *Elements.*

[14] Illustrated steps in the process are given by Artmann (1994, 5–6) who prefers the alternative term, *antanairesis.* Zeuthern (1896) first proposed the method of anthyphairesis for Theodorus. See Knorr (1975, 118–19) for other proposed anthyphairetic diagrams, but all attempt *proofs*, and none succeed in all the requirements I have set out, so I do not rehearse them here. More recent diagrammatic efforts, e.g., Ksenia 2017, invoke computer assistance in determining more exact values for the irrationals that goes some way toward showing numerically why some are more difficult to construct by the anthyphairetic method than others.

[15] Fowler cautions that "the Euclidean algorithm is now generally construed as a division process, whereas anthyphairesis is based on repeated subtraction" (1999, 30).

actual *proofs* of the irrationality of the series of square roots (√3, √5, √6, √7, √8, √10, √11, √12, √13, √14, √15, √17) is not required by Plato's text.[16]

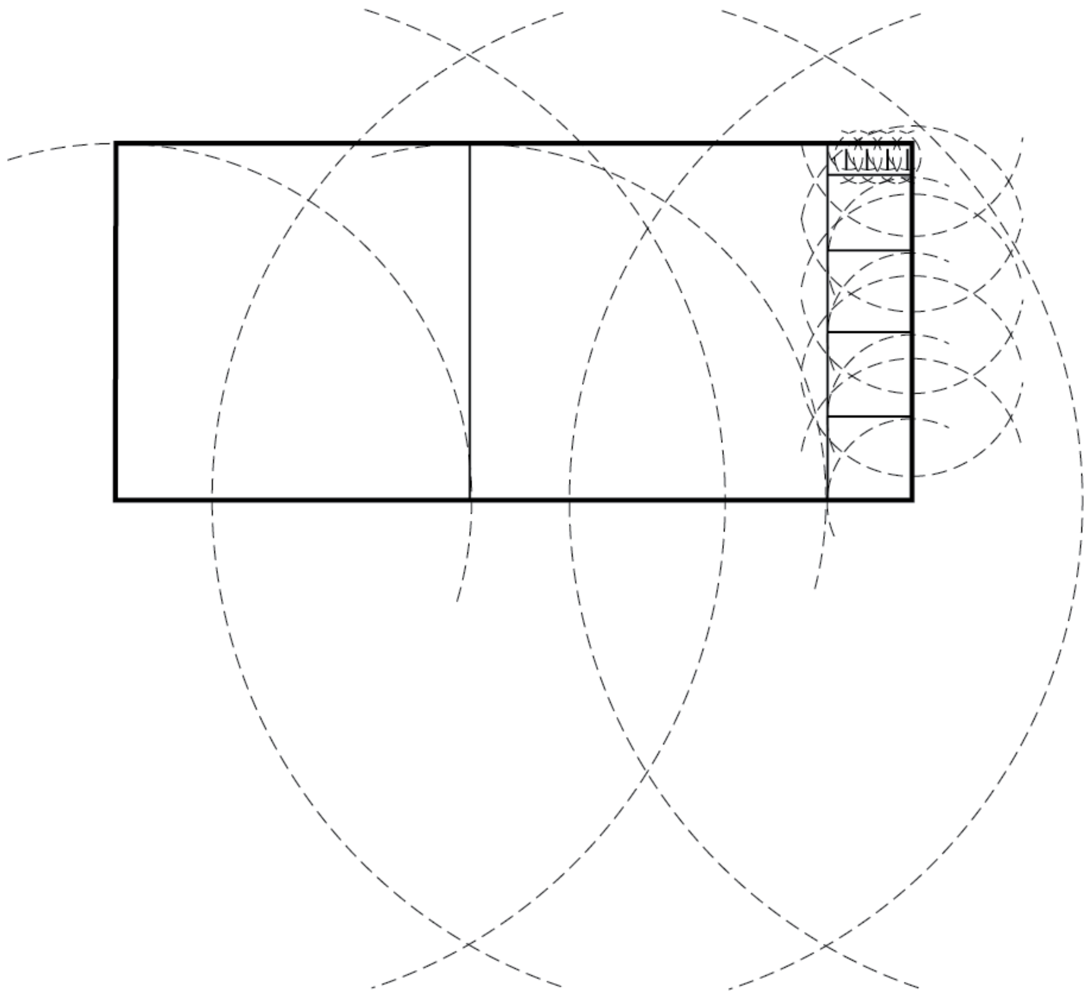

Figure 3.3.3: Anthyphairetic Proof of √5

In my view, it is highly unlikely that Theodorus would have used anthyphairesis to display his results to geometry students—not only because a proof is not necessary for such a display, but (1) the method, already well known, would not have exalted Theodorus as a preeminent geometer; and (2) the drawing of the diagrams themselves would have taken an inordinate

[16] *Contra* Burnyeat (1978, 496): "he proved the irrationality of the square roots of each of the integers."

amount of time. Thus the view that Theodorus was engaged in the process of anthyphairesis with the boys fails to fulfill requirements 1–2. The labor involved in producing a *single* diagram of geometric anthyphairetic proof is illustrated for the √5 in Figure 3.3.3.

Any geometric anthyphairetic proof would require a considerable number of steps as each of the squares and rectangles dividing the interior of the initial rectangle would need to be constructed individually. This would entail the dropping of an arc to mark off the side distance on the base of the rectangle and constructing a line perpendicular to the base to establish the side of the square, using the method that Euclid later canonizes.[17] This construction involves drawing two similar arcs that intersect on either side of the line on which the perpendicular line is to be

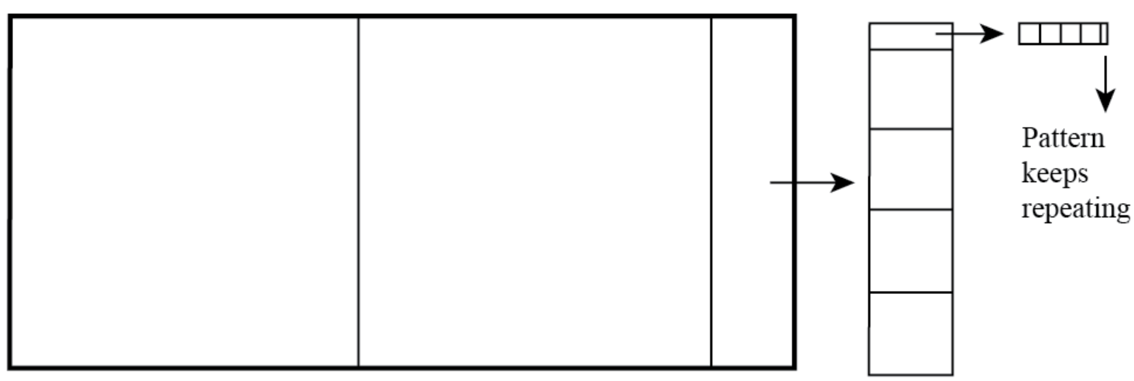

Figure 3.3.4: Anthyphairetic Proof Generalized

constructed (shown in Figure 3.3.3 as dashed lines). This process would have to be repeated until no more squares could be added to the base, leaving a rectilinear area as a remainder. The process would again be applied to this smaller rectangle using the side length of the initial triangle as the base, and the remainder distance of the initial base as the new side.

Now consider that in addition to √2, which is already understood to be irrational, the

[17] Euclid I.11, "To draw a straight line at right angles to a given straight line from a given point on it."

method of anthyphairesis would require twelve separate diagrams to prove the irrationality of the remaining twelve irrational square roots between √1 and √17. The technique involved in the proof is the same in each case; yet the resultant figures are distinct, with each requiring the construction of at least eight individual lines using eighteen construction arcs.[18] This is after the construction of rectangles with two sides equaling the appropriate irrational square roots. If Theodorus drew such diagrams for the boys, laying out each individual anthyphairesis, the boys would have stood around watching him while he constructed the appropriate squares that are used to subtract continually the interior of the rectangles into smaller and smaller squares until it became apparent that the process would not terminate—in *each case*.

While it is true that anthyphairesis can be used to find the greatest common denominator between two numbers (if they are whole numbers), it is also included in *Elements* X as a method of proof that a quantity or magnitude is irrational. Despite the attention given to it by commentators, no one really wants to commit to saying that this is what Theodorus must have drawn in the context of the dialogue.[19] Nevertheless, following through with the requirements given at 3.3.2, let us ask how the process fares. Requirement 3, that the diagram could have been conceived and constructed by Plato is easily met.

Requirement 4, that there be some special difficulty that would prevent Theodorus from continuing after √17 is more difficult to assess. Difficulties at √10 and √13 are reported by Artmann,[20] and Fowler says, "I have just set out a sequence of proofs based on diagrams that

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[18] This is the number for the proof of the √3; the √5 would need 10 and 30, respectively.

[19] Knorr (1975, 118–26) lays out objections to the whole class of anthyphairetic interpretations. His own method (1975, chapter 6), which I do not here pursue, is based on Pythagorean number triples. Like anthyphairesis, however, Knorr's method requires separate drawings for each case—a long process—though it has the virtue of failing at 17.

[20] Artmann expressed difficulty in producing the full series of drawings of an anthyphairetic proof for the twelve irrational roots, saying, "I was quite surprised by the easiness of the case of √7 and the difficulties for √10. This is

snarls up at the case of √19." But he then notes, "In truth, most other anthyphairetic techniques already run into difficulties with √13, or they change to a new method for this case, or they avoid it by invoking an unacceptable use of algebraic manipulation" (1999, 379). Trouble with the √19 is supposed to satisfy the text in that Theodorus stopped at √17 because he could foresee becoming entangled at √19. Unguru (1977, 217) agrees, "At the 17-foot square he came to a standstill because of a difficulty just ahead at 19."

A philological point leads many commentators to suppose that something special needs to occur at exactly √17: Hackforth (1978, 128) cites commentators who "take the Greek ἐνέσχετο to mean *stopped*" (in the phrase ἐν δὲ ταύτῃ πως ἐνέσχετο = somehow or other he came to a halt), but points out that LSJ quotes *only* the *Theaetetus* passage for "came to a standstill" and cites precedent for "entangled by" which would give the sense that Theodorus became entangled or had difficulties at √17. Burnyeat sums up the choices: (a) "at that point for some reason [*sc.* for no particular reason that Theaetetus knows of] he stopped"; (b) "at that point for some reason [*sc.* for some particular reason] he stopped"; "at that point he somehow got tied up."[21] The jury must apparently remain out on how well the method of anthyphairesis diagrams fulfill requirement 4.

According to requirement 5, the diagram must suggest the two implications that the boys take from the lesson: that all of the square roots of whole numbers form a class or are somehow one in kind (147d9–e1); and that this class can in fact be divided into two subclasses (147e5).

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something one cannot learn from theory" (1994, 12; for √13, see 15). However, Artmann did not use auxiliary construction lines that would have been required, suggesting that he used a ruler rather than the Euclidean method of compass and straightedge.

[21] Burnyeat (1978, 379) using the translation of McDowell 1973 for (c). What Fowler called a "regrettable altercation" (1999, 379n21) between heavyweights Knorr and Burnyeat (made immortal in their doubly authored 1979) concerned just this phrase.

Virtually all scholars agree that there would have been no problem for the boys would have been able to generalize to a definition of 'incommensurability' and to divide the class into 2 subclasses. In short, proofs by anthyphairesis meet only about half of our desiderata.

3.4 The Spiral of Theodorus

Since I am concerned here with *what Theodorus drew,* and not especially with whether or how he may have proved the irrationality of the square root of a whole number, I will take the discussion further and in a different direction by investigating the possibility that a diagram (or diagrams), rather than a proof, is what we are looking for.

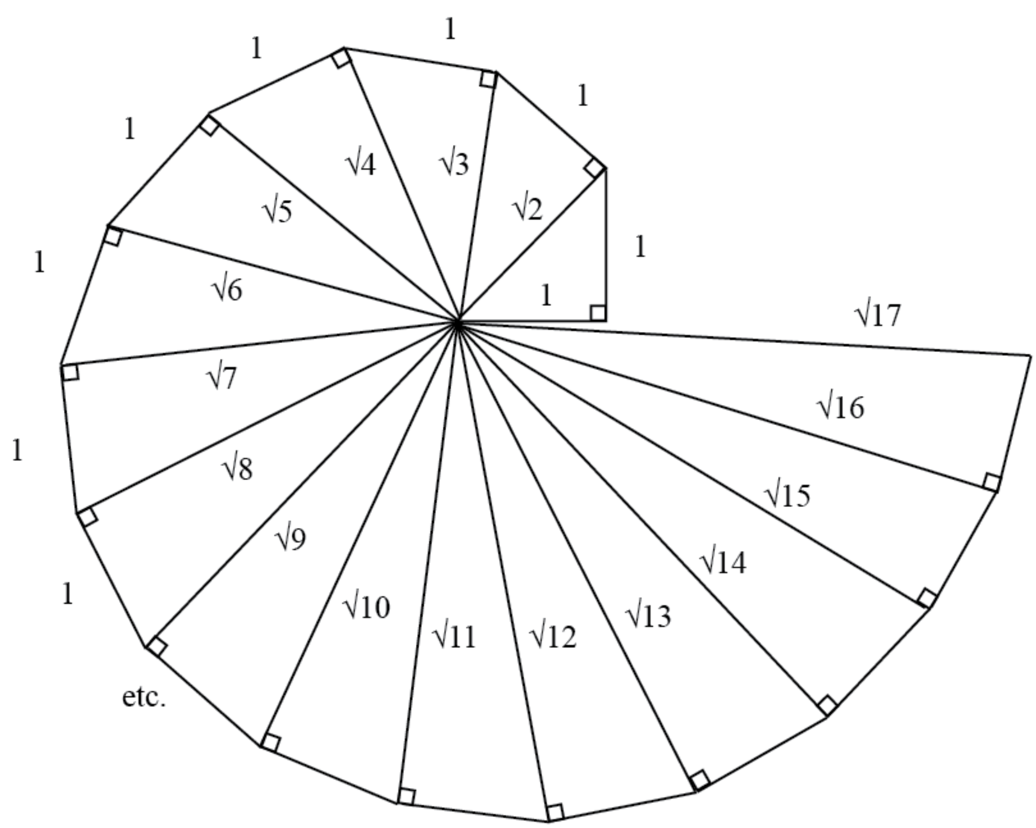

Figure 3.4.1: The Spiral of Theodorus

If the drawing was not a large number of rectangles (as in anthyphairesis), what diagram could the group have been looking at? The serial construction of square roots generating a Theodoran spiral diagram provides both an easy and simple construction of each root by the same method, as well as a means to illustrate clearly that there are two classes of magnitudes in the series of all the square roots from one to seventeen, including both those in whole numbers and the irrational quantities in between. It is a construction based on facts established by the Pythagorean theorem—yielding the diagram I first introduced in 3.1.

Before defending Figure 3.4.1 in relation to the five requirements I provide in 3.2, I digress briefly to mention the obscure history of the now-popular diagram. Much as diagrams used in ancient mathematics, and perhaps philosophy (a few still visible on papyri), did not survive millennia of scribes and copyists, diagrams after the invention of the printing press in the West have suffered under the material conditions of publication.

In 1877, in German, Hermann Schmidt published the first known ancestor of the current version of the spiral of Theodorus, Figure 3.4.2. Although it is not quite right in its details, one can see that Schmidt had the seminal idea for how Theodorus might have proceeded. In 1883, Lewis Campbell—a scholar remembered now chiefly for his division of Plato's works into chronological categories—offered Figure 3.4.3, which clarifies Schmidt's version for the English-speaking world without making it rigorous.[22]

Anyone who has attempted to chase down the original source in the twentieth century of the *rigorous* version of Theodorus's spiral will have ground to a halt with, at best, a partial

[22] Google Books offers a complete free edition of Schmidt, where the diagram appears (1877, 441): <https://play.google.com/books/reader?id=8QPch6LEWEIC&hl=en&pg=GBS.PA442 > and of Campbell, where the diagram appears (1883, 21–22n3): <https://play.google.com/books/reader?id=usYUAAAAQAAJ&hl=en&pg=GBS.PA22>. The figure is blurry in the original.

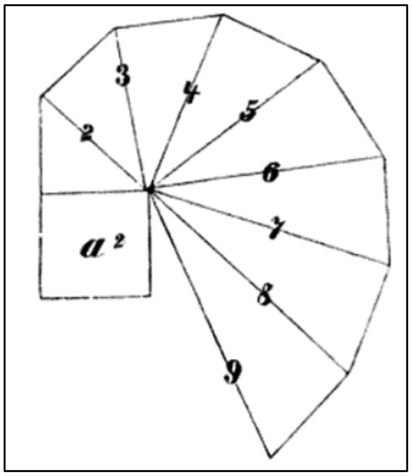

Figure 3.4.2:
Schmidt 1877

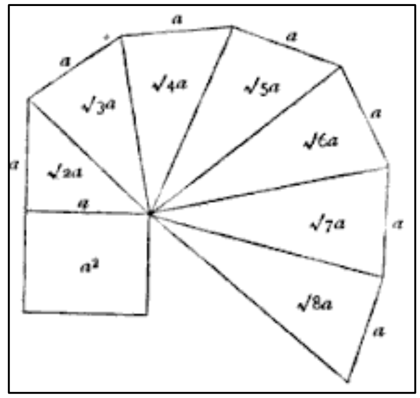

Figure 3.4.3:
Campbell 1883

reference to J. H. Anderhub 1941. Complete information is surprisingly hard to find and can take years of searching, but dissertation research was never meant to be easy, and giving proper credit seems especially important in the present case. It transpires that Jakob Heinrich Anderhub (1894–1946) was a jurist and commercial publisher whose avocations included the fine arts[23] and Plato. Anderhub's accurate version of what may have been Theodorus's drawing (1941, 205) is elusive because the chapter in which it appears, "Genetrix Irrationalium: Platonis *Theaetetus* 147d," is tucked into the back of a collection of short stories, privately published as a gift for friends of the publishing house at the beginning of World War II.[24] Since the whole book is in

[23] The Yale Center for British Art sold off most of his collection of prints in 1963; but some were offered at an "AHA!" sale in Germany by Lüder H. Niemeyer (sellers) in July 2018: <https://www.luederhniemeyer.com/aha.luederhniemeyer/aha1807e.php>

[24] An insert explains that Kalle & Co. distributed such a gift annually at Christmas, in this case "cheerful stories" and "a scientific problem for those who enjoy 'cracking nuts'." The copy I purchased online from a German bookseller had a second enclosure: a carbon copy of a letter written by Anderhub in reply to one Professor Dr. F. Sugar who had received a copy of the book, shared it, and praised it. After addressing other matters (a colleague, *dunamis*), Anderhub agrees with Sugar that "was Sie am Schluß Ihres Briefes ausführten, daß wir nämlich allen Grund haben, auch außerhalb des Fachgelehrtenkreises für den unersetzlichen Wert der Antike einzutreten," (we have every reason to stand up for the irreplaceable value of antiquity even outside the circle of scholars), but the letter is signed "Heil Hitler."

German gothic script, the chapter has so far defied search engines and browsers, though it is well worth reading: it is studded with calculations and interpretive figures based on *Theaetetus*, and *inter alia* chronicles previous efforts by fifty-five scholars (1941, 188–93), starting with Ficino, to address the problem occasioned by the phrase ἐν δὲ ταύτῃ πως ἐνέσχετο (= somehow or other he came to a halt)—the very phrase that Hackforth was soon (1957) to realize put a spanner in the works of numerous interpretations. It was Anderhub's recognition that *there really is a*

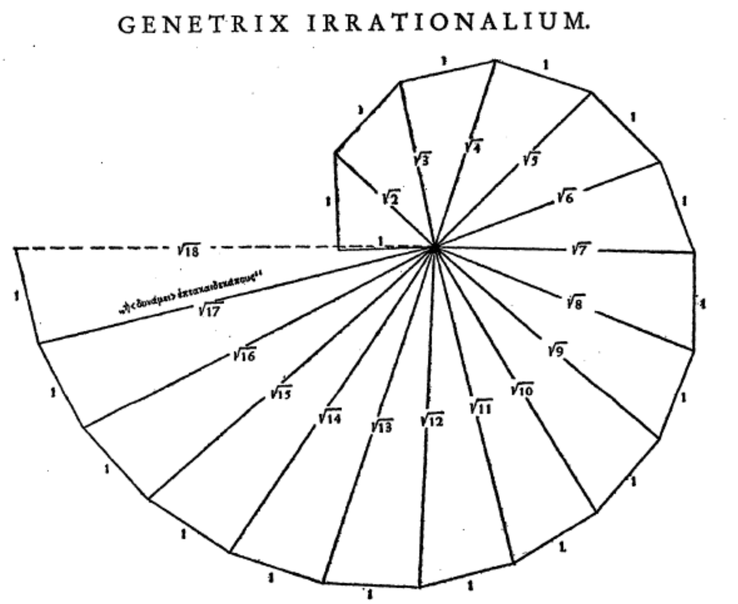

Figure 3.4.4: Anderhub 1941

special problem at √17 that accounts for the way his own Figure 3.4.4 is shown, drawing attention with his dotted line to the overlap that would have resulted if Theodorus had continued drawing.[25]

Returning now to the present purpose of defending Figure 3.4.1, note in its favor that its construction would not take all day. The time taken to draw the spiral accurately amounts to

[25] Anderhub quotes Schmidt's diagram and mentions Campbell's as well. The barely legible Greek phrase is still barely legible at 1600% magnification but seems to say something like "the possibility melts away."

constructing, at most, two anthyphairetic proofs, and the simplicity of the process recommends it. Starting with a right-angled triangle with legs of one unit, each succeeding triangle is constructed by extending a line of unit length from the outer end of the hypotenuse of the prior triangle.

The square-root generating spiral satisfies the first two requirements: it displays the results of Theodorus's important discovery concerning powers, reflecting actual geometric

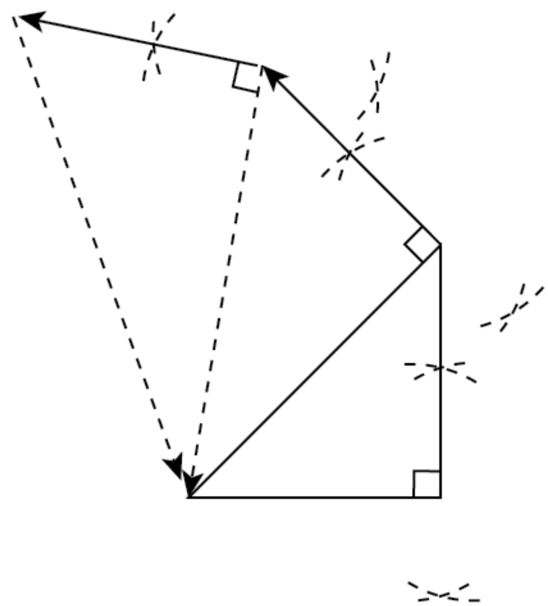

Figure 3.4.5: Detail

practice, but without going through laborious and time consuming demonstrations to show each non-perfect square is irrational by using a geometric anthyphairetic proof.

The third requirement, that the proposed diagram must be something we have good reason to think would lie within Plato's range of mathematical knowledge, can be shown by examining a pun (*Statesman* 266b1–3) that removes any doubt that Plato was aware that a geometric relationship involving square roots could produce a spiral figure.

It will be recalled that, at the end of the *Theaetetus,* Socrates must rush away to answer Meletus's indictment at the king archon's court, but he arranges to meet Theodorus and the others "in the morning" to continue their discussion. When they reconvene, they have been joined by a visitor to Athens from Elea who does most of the questioning. The *Sophist* and then the *Statesman*, but not the "lost" dialogue *Philosopher*[26] are the result. I include the *Statesman* pun in its context of the classification of animals:

> VISITOR: Now those creatures that are tame and live in herds have pretty well all now been cut into their pieces, except for two classes. For it is not worth our while to count the class of dogs as among creatures living in herds.
> YOUNG SOCRATES: No indeed. But what are we to use to divide[27] the two classes?
> VISITOR: Something that is absolutely appropriate for Theaetetus and you to use in your distributions, since it's geometry the two of you engage in.[28]
> YOUNG SOCRATES: What is it?
> VISITOR: The diagonal, one could say, and then again the diagonal of the diagonal.
> YOUNG SOCRATES: What do you mean?
> VISITOR: The nature which the family or class of us humans possesses surely isn't endowed for the purpose of going from place to place any differently from the diagonal that has the power of two feet?[29]
> YOUNG SOCRATES: No.
> VISITOR: And what's more, the nature of the remaining class has in its turn the power of the diagonal of our power, if indeed it is endowed with two times two feet.
> YOUNG SOCRATES: Of course it is—and I actually almost understand what you want to show.
> VISITOR: And there's more—do we see, Socrates, that there's something else resulting in our divisions that would itself have done well as a comic turn?

[26] Davidson (1985, 15–17) argues that *Philebus* is, or is a draft for, *Philosopher.*

[27] Reading *diairōmen* at a5.

[28] See *Theaetetus* 147c–e.

[29] In Greek mathematical parlance, 'having the power of two feet' is the way of expressing the length of the diagonal of a one-foot square (i.e., in modern terms, $\sqrt{2}$); the expression reflects the fact that a square formed on this line will have an area of two square feet. The diagonal of this square will then 'have the power' of four feet—the 'power of the diagonal of our power' in the Visitor's next remark. All this is for the sake of the pun on 'power' and 'feet': we humans are enabled to move by having two feet, while the members of 'the remaining class' from which we are being distinguished—pigs—have four. (On the mathematical use of 'power' see *Theaetetus* 147d–148b.)

YOUNG SOCRATES: What's that?
VISITOR: That our human class has shared the field and run together with the noblest and also most easy-going class of existing things?[30]
YOUNG SOCRATES: I see it turning out very oddly indeed.
VISITOR: Well, isn't it reasonable to expect the slowest—or sow-est—to come in last?
YOUNG SOCRATES: Yes, I can agree with that.
VISITOR: And don't we notice that the king looks even more ridiculous, when he continues to run, along with the herd, and has traversed convergent paths, with the man who for his part is best trained of all for the easy-going life?[31]
YOUNG SOCRATES: Absolutely right.—Rowe, trans. (266a1–d3)

The Eleatic visitor's pun is a humorous way of appealing to his interlocutor's keen interest in geometry, depending on word play between the different meanings of both *dunami*s and feet—exactly as we saw in the passage quoted from the *Theaetetus*.[32]

In the pun, the two-legged/two-footed humans are characterized by the diagonal of the unit square (two-footedness), while the four-footedness/four-leggedness of some animals is characterized by a diagonal constructed on a square whose side is the diagonal of the unit square as shown in Figure 3.4.3. This diagonal produces the numerical value of our human *dunamis*

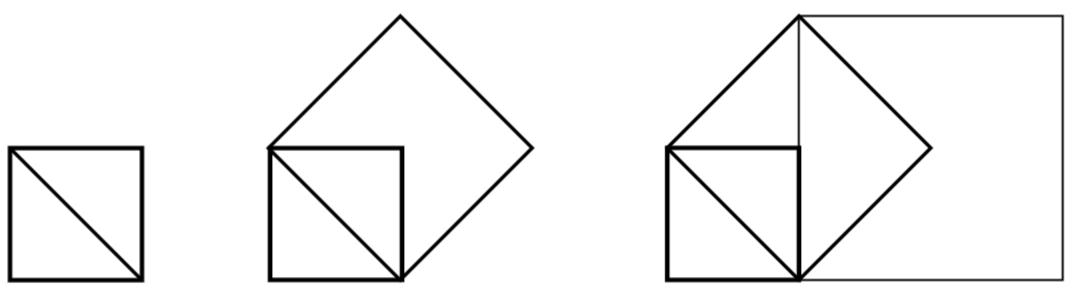

Figure 3.4.6: The Visitor's Pun

[30] I.e., pigs, as the Visitor makes clear in his next question, by punning on the Greek word for 'pig'.
[31] The swineherd.
[32] The pun has been well understood and described since ancient times, but Szabó (1978: 70) explains aspects of it in greater detail than Rowe's footnote, providing Figure 3.4.3 that I quote below. The phrase "ability (to walk on) two feet" would likely be expressed in the everyday Greek by *dunamei dipous*. A "geometer would immediately think of the diagonal of the unit square on hearing this expression, for this linearly incommensurable straight line was also denoted by *dunamei dipous*, i.e. 'two feet when measured by the area of the square constructed on it'. The pun is in fact made possible by the ambiguity of the expressions *dunamei* ('ability' and 'square') and *dipous* ('two legged' and 'measuring two feet')."

(ability) to walk on two feet. The diagonal of the second square is a linearly commensurable quantity that can be described as having a length of two feet. Yet the Visitor's pun calls for this second diagonal to be measured "by its square" (*dunamei* or 'by its ability'); and when measured by its square, it has twice two feet, or an area of four. Giving the magnitude of a line by the square that can be constructed on it is much like using the square root sign where it would usually be considered redundant, e.g. writing 4 as $(\sqrt{4})^2$ when it is not required in the context of a mathematical equation.

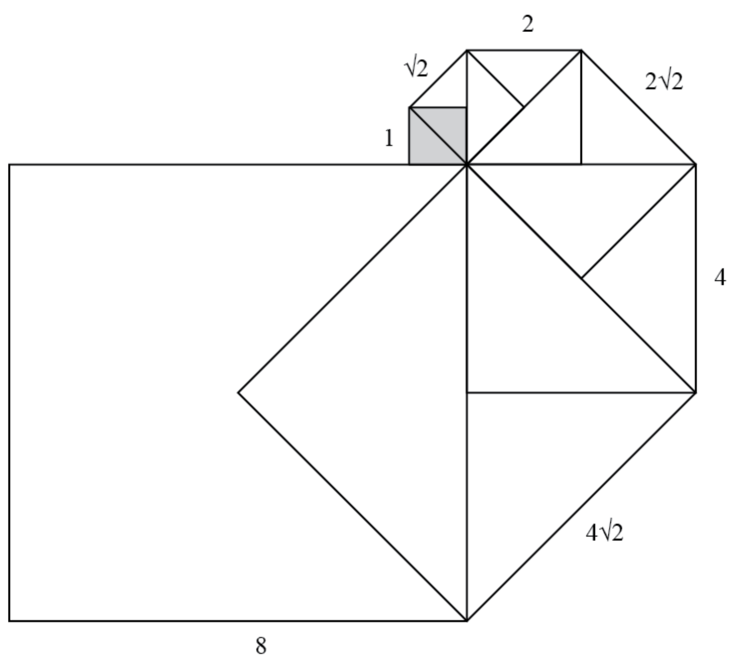

Figure 3.4.7: Spiral-like Mathematical Progression

The diagonal/square procedure in the *Statesman* mirrors the hypotenuse/triangle procedure of the *Theaetetus* that generates the spiral. By extending the *Statesman* diagram through the continuation of the same procedure (as in Figure 3.4.7), a repetition of the diagonal-of-a-diagonal figure, it starts to look spiral-like.

The spiral diagram (Figure 3.4.1) satisfies our fourth requirement by providing a straightforward reason for Theodorus to stop after 17 that emerges naturally from its construction. The hypotenuse of base √17 triangle (which equals the √18) would overlap the base 1 triangle making it difficult to clearly distinguish the sides of the base 1 and base √17 triangles from each other. Each subsequent triangle would make it more difficult to see clearly the relationship between the triangles. It is true that stopping at 17 works for the method of anthyphairesis as well, if the thought is that the problem of coming up with √19 has been spotted at √17. Both Fowler (1999, 379)[33] and Artmann (1994, 19)[34] give the details of why √19 does not work. But there remains the issue of individual diagrams for each case versus a single diagram showing the relationship of the cases as a sequence of the square roots of integers.

Requirement 5—that the diagram suggest both (a) that the square roots of whole numbers form a class and (b) that this class contain of two distinct subclasses—is also fulfilled by the square root generating the spiral diagram. The boys watching Theodorus perform repeated iterations of the same operation while constructing the spiral diagram might well surmise that the square roots of whole numbers form a single class. The sequence of square roots of 1 to 17 can be said to have a unity as a mathematical progression formed by repeating the same operation, but repeating the triangle construction gives this class a concrete representation that might be apparent if their numerical values were compared. The diagram itself shows only how the roots can be seen as a single class. The square roots of whole numbers fall into two distinct kinds of magnitudes: those commensurable in square and line (√4, √9, and √16) and those

[33] Fowler adds, "I have just set out a sequence of proofs based on diagrams that snarls up at the case of √19, and so may be a possible candidate for anthyphairetic explanation of this passage."

[34] Artmann says the geometric method applied using anthyphairesis works for the case of √17 but cannot solve the case of √19 (1994, 17); he refers to Burnyeat (1978, 502–5, 512n18) for a detailed discussion. About Knorr's method Artmann says, "Knorr's method fails at 17, the geometric proof fails at 19. Of the philologists, von Fritz (1978, 99) has no objections to 17 as the first failure, whereas Burnyeat (1978, 13) vehemently opposes for reasons of both the translation and the context" (1994, 20).

commensurable in square only (e.g. √3 and √5). Theaetetus notices that that there are two kinds of shapes that follow the previous distinction: square/equilateral and oblong.

The spiral diagram itself, however, suggests that there are two classes of lines among the square roots of whole numbers. All one needs to do to see this is to add circles with radii having whole-number values—something a geometer might do. A modern geometer can easily be imagined as going up to the board and adding lines/circles to show additional relationships,

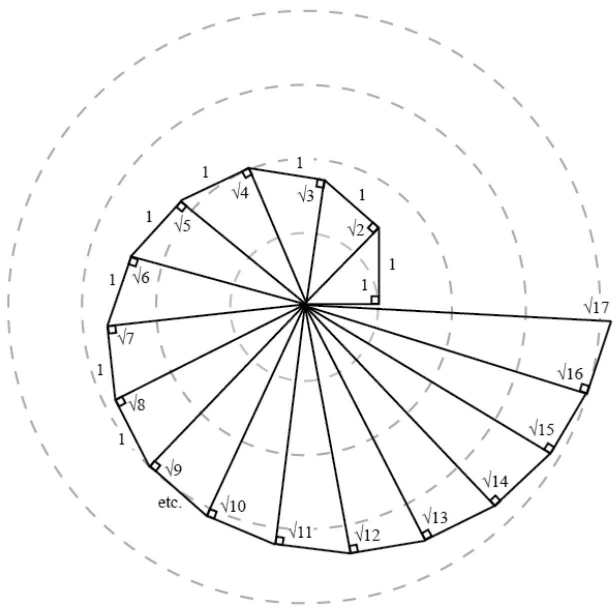

Figure 3.4.8: Additional Relationships

marking a discovery by playing with Theodorus's diagram. Adding circles with radii in whole numbers that increase by one unit clearly marks off the square roots that are commensurable *in line* from the intermediate, incommensurable ones. The right angle of the triangle of each commensurable in-line root touches one of the circles; ones that do not are incommensurable.

3.5 Connections

An important aspect of Theaetetus's discovery is the method he uses; it involves making an irrational side of a square into a rational side (that has the value of a whole number) of a rectangle. One needs to work in two dimensions to construct irrational one-dimensional lengths and understand that these lengths are not rational magnitudes (not expressible with numbers). This fact connects the examples in the *Meno* and *Theaetetus* to the *Republic* and gives support to my view that the divided line diagram is a two-dimensional diagram, and not simply a line. The move from arithmetic (expressing quantity only) to two dimensions points to the next step of understanding magnitudes in three dimensions. This should be seen to be a theme for Plato: just as we needed to move from one dimension to two dimensions in mathematical thinking, so we need to develop our mathematical thinking to work with three dimensions.

Plato uses mathematical examples that show how to cross the boundaries between geometry and arithmetic. In a sense, he is arithmetizing geometry, though not in the objectionable sense where arithmetic is considered sufficient for working with the full spectrum of real numbers.[35] It might be more accurate to say that the problems are numericized by Plato. Notions such as doubling and halving are essentially arithmetical, not geometrical. Plato is using geometry to get the right kind of units, ones that can be expressed as rational numbers so that he can understand and operate arithmetically.

In the *Theaetetus*, the square root of three is not an *arithmos*, but geometrically it is the side of a square with an area equal to three (as $\sqrt{3} \times \sqrt{3}$). It can be transformed into an oblong

[35] Fowler says that arithmetized mathematics "is characterised by the use of some idea of number that is sufficiently general to describe some model of what is now frequently called 'the positive number line' *and its arithmetic*" (1999, 8).

rectangle with an area of one by three, both of which are rational *arithmoi*. This is a way of turning a shape with irrational sides into a shape with rational sides—so that something that could only be understood geometrically ($\sqrt{3}$) can now be understood arithmetically (3 and 1).[36]

Van der Waerden's characterization (1954, 169) of the proofs in Euclid X offers a good description of this way of mathematical conception:

> All these proofs are based on one fundamental idea which runs as a guiding thread through the entire book: *to prove properties of any type of line, one constructs a square on this line and one investigates the properties of this square.* For instance, to prove that a binomial cannot be a medial, it is shown that the square on a binomial cannot be a medial area.
>
> Properly speaking, this basic idea already turns up in the first part of *Elements* X and in the dialogue *Theaetetus*, for Theaetetus derived the incommensurability of certain line segments from the ratio of their squares.

The ancient commentators were no less eager to describe a plausible diagram than the scores of scholars who made efforts over the past two centuries. The anonymous commentator on Plato's *Theaetetus* (P.Berol. inv. 9782), for example, suggests even wider connections for the *Theaetetus* passage: to musical theory (to explain Theodorus's stop at $\sqrt{17}$), to solids and cubes, and to his commentary on Plato's *Timaeus*.[37]

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[36] This operation is seen in the *Meno* as well: the diagonal (a geometrical notion) is used to get the right unit (by halving squares to produce a triangle); then arithmetic is used with the new units.

[37] See sections XXXIV–XXXV and XLII–XLIV in Boys-Stones 2015.

CHAPTER 4:

WHAT DID GLAUCON DRAW?

A DIAGRAMMATIC PROOF FOR PLATO'S DIVIDED LINE

4.1 The Division and the Mystery

Elaborating the analogy between the sun and the good, Plato's Socrates tells Glaucon to divide a line αβ into two unequal segments at γ. The result is that αγ represents what is intelligible and γβ what is visible.[1] Then Glaucon is to divide each of the two segments by the same ratio as he used in the original division (*Republic* 6.509d6–8).[2] Whatever proportion he used to make the cuts γ, δ, and ε in the divided line, generating its four segments, the geometrical implication is that the two middle segments must be equal in length. As both Nicholas D. Smith and Richard Foley have emphasized, when Socrates reiterates the characteristics of the line at 534a3–5, *transposing*

[1] In crucial respects, beginning here where the intelligible segment is longer than the visible segment, I regard Smith's arguments and description (1996, 42–43) as dispositive. He concludes, "Plato's divided line is a vertical line, divided unequally with the largest segment on top. These two segments represent the intelligible realm (at the top) and the visible realm (at the bottom)." Smith usefully canvasses scores of previous constructions and discussions of the divided line, from which I have learned much. He shows clearly that twenty-three then-recently published images of the line—including the well-known representations by Bloom (1968), Brumbaugh (1954), and Grube (1974)—do violence to the text of Plato's *Republic*. Popular illustrations of the line in more recent translations continue the injustice: Griffith and Ferrari (2000), Reeve (2004), Allen (2006), and Badiou (2012). Some commentators' diagrams so ill fit their own words that, so as not to be uncharitable, one is tempted to blame deficiencies in the history of printing and illustrating.

[2] For the sake of exposition, I am using the term 'ratio' in the modern sense. The Greek notion of *proportion* (*a* is to *b* as *c* is to *d*) differed from their notion of ratio; and there were at least three competing definitions of 'ratio': from music theory, astronomy, and mathematics. See Fowler (1979) and A. Thorup (1992). Euclid makes no effort to reconcile the different senses of proportion given in books V and VII; and Plato's *Philebus* 25a introduces two senses as well, as pointed out by Artmann (1991, 6).

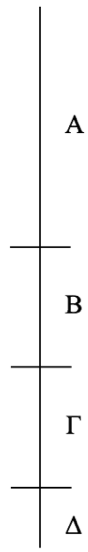

Figure 4.1.1: Segments

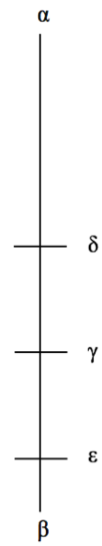

Figure 4.1.2: Cuts

δγ and γε, there should be no doubt that Plato knew the two middle segments were equal.[3] Plato, like Euclid, used 'segment/s' (τμῆμα, pl. τμήματα) broadly to denote not just parts of lines but parts of circles and polygons as well (e.g. *Symposium* 191d6). The divided line passage is replete with instances of its cognates but, as in English, the reader—or observer—must determine its referents in context. This point will become crucial when I introduce what Glaucon drew. The equality of middle segments constitutes half the mystery of the divided line that my interpretation attempts to solve. But let us follow the text awhile.

"[D]ifferences in relative clarity and obscurity" (d9)[4] determine what is assigned to each part of the line, its contents, now labeled with uppercase letters. The lower part of what is visible, Δ, represents what is most obscure: images, shadows and reflections; the next part, Γ, represents spatiotemporal objects, sensible particulars, such as animals, plants, and artifacts. The lower part of what is intelligible, B, represents hypotheses such as those used in geometry and arithmetic;

[3] Smith (1996, 42) and Foley (2008, 7).

[4] *Republic* translations are those of Christopher Rowe, and references to the Greek text are to the OCT edition of S. R. Slings—neither of whom includes a diagram of the divided line.

and the upper intelligible part, A, represents “what reason itself grasps through the power of dialectic” (511b3), namely, Platonic forms. Corresponding to each sort of content is a cognitive state (511d8–e1): A for the contents of highest segment, αδ, νόησις or ἐπιστήμη; B for δγ, διάνοια; Γ for γε, πίστις; and Δ for εβ, εἰκασία, the lowest of the four segments.

After Glaucon shows that he has understood Socrates’s description of the parts of the line, their contents, and their relations of dependence on one another—that each is the image of the part above it—Socrates says, “arrange them in proportion to their clarity, taking degree of clarity as corresponding to the degree that the things they are assigned to[5] share in truth” (511e2–4).

If clarity and truth increase in proportion to the contents of each part of the divided line, as Socrates claims they do, then one would expect the degree of increase to be visible in the diagram. Therein lies the other half of the mystery: There is an apparent contradiction between the initial instructions for dividing the line (509d6–8) and the later description of the line’s divisions (511e2–4).[6] In short, then, if the two middle segments of the divided line are, and must be, equal, in conformity with Socrates’s explicit instructions, then the illustration fails to exhibit the greater degree of clarity and truth as one moves up the line. On the other hand, (a) if the two middle segments contain the very same things, even if used differently;[7] or (b) if the stuff of an object *just is* given by its formal characteristics, if its formal structure and its matter are two

[5] Rowe notes that “the language is that of the argument with the sight- and sound-lovers in 475d–480a.”

[6] In a masterful and comprehensive updating of the literature on the matter, Foley (2008) concentrates on the incompatibility between 509d6–8 and 511e2–4 and how that incompatibility is intensified by 534a, carefully combing through forty-eight philosophical and philological attempts to address the problem.

[7] Pritchard (1995, 92) argues that the objects of the two middle segments are the same, except that in the higher, “they are now being used as images of something else.” Pritchard is not alone in making the claim; I mention him because he escaped the very wide nets of both Smith and Foley.

aspects of the same one thing,[8] then Β = Γ without remainder, and the claim of a greater degree of clarity and truth is, at the very least, seriously misleading. Either way, the mystery is evident in all previous *representations* of Plato's divided line.

Foley helpfully discusses four types of "imperfect solutions" to the contradiction, which he dubs the "overdetermination problem." (i) Early *revisionist* philologists challenged the text of *Republic*, suggesting emendations to edit away the contradiction. (ii) Those attempting *demarcation* denied that the two middle segments of the line were ever intended by Plato to be compared. (iii) A third group argued that Plato made a *gaffe* in his design of the line, even if he did recognize the equality of Β and Γ. My interpretation will rule out those first three approaches. (iv) Finally, *dissolution* scholars variously emphasized the equality of Β and Γ, arguing that Plato wanted especially to draw attention to it.[9] Some of the fourth group denied the existence of mathematical intermediaries, a topic with a vast literature of its own; for Foley's purposes, it is more salient to point out that this fourth imperfect solution disarmed the ontological version of the overdetermination problem—equality of the middle segments. The epistemic version remained because, although διάνοια must be clearer than πίστις, the line gives them equal clarity. Foley credits Smith with being the only philosopher previously to recognize the *remaining* epistemic problem with the "dissolution" account. I acknowledge their insight that existing representations of the divided line give rise to both aspects of the overdetermination

[8] To take a simple example, one's imagined cue ball is less cognitively reliable than one's perceptions of the cue ball as it is held, felt, and seen. But the cue ball as weighed and measured to the very limits of our most precise instrumentation, is but an imperfect image of the sphere itself, about which one could say in truth that its surface is $4 \times \pi \times r^2$. Just *how* true is a further issue. In his discussion of Plato's divided line, Rescher (2010, 154) notes that the highest kind of knowledge requires a scientific framework that is thus best captured by Spinoza's notion of adequacy. If π is an infinite series, one might question whether the formula is a *complete* truth.

[9] For discussion of revisionists, see Foley (2008, 8–9); demarcation advocates (2008, 9–12); those claiming a blunder (2008, 12–15); and dissolutionists (2008, 15–17, citing Smith, 1996, 40n34). For clarity, here and below, I refer to labels of the line I introduced above, rather than to labels other authors assigned.

problem, and that a diagram worth its salt must address both satisfactorily, but I stop short of accepting Smith's or Foley's conclusions, though I am sympathetic to how they reached them.[10]

Growing up with algebra, and knowing in advance what one expects of Plato's divided line, one can easily produce the proportions that have the *appearance* required by Socrates's initial instructions to Glaucon at 509d6–8. In drawing Figure 4.1.2, the desired ratios

$$\alpha\gamma : \gamma\beta :: \alpha\delta : \delta\gamma :: \gamma\varepsilon : \varepsilon\beta$$

are achieved effortlessly by thinking in units: 6 : 3 :: 4 : 2 :: 2 : 1.[11] Getting the ratios right, however, regardless of the length of the units, yields a contradiction with the desiderata of 511e2–4 that has bedeviled discussions of the divided line:

$$(\delta\gamma = \gamma\varepsilon) \ \& \ (\delta\gamma > \gamma\varepsilon)$$

For us, it is straightforward, quick—not to mention pedagogically attractive—to illustrate the line as 4–3–2–1: four units for forms, three for hypotheticals, two for objects, and one unit for shadows. But it would be wrong.[12]

The divided line passages present a challenge in the interpretation of Plato's philosophy in that, on the one hand, the divided line itself can be seen as a mere analogy to illustrate the

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[10] Smith says, "I am tempted to think that Plato might have purposefully woven this subtle flaw into the intricate fabric of his own image, because he wished to avoid the sin of perfection. According to his own philosophy, images can never be perfect, and Plato's divided line is, after all, only an image" (1996, 43). Foley says, "Plato presents the divided line in a contradictory fashion because in so doing he forces the reader to follow the paradigmatic course in the procurement of true wisdom: from mathematics to philosophy (2008, 23).

[11] Exactly the same procedure would be available if one wanted different ratios, say, 30 : 6 :: 25 : 5 :: 5 : 1 to expand the length of the segment of intelligibles.

[12] Foley (2008, 21n43) draws attention to the connection between physically drawing Plato's line and recognizing that a problem exists: "I discovered the overdetermination problem drawing the line in front of a class. I simply could not get the middle two subsegments to look right, since I believed that one should be longer than the other, given what Plato says about the corresponding types of mental states." Denyer (2007, 292–93) contributes amusing visualizations of the line on behalf of Proclus and Plutarch, respectively, both of which generate the appearance of a 4–2–2–1 line, keeping the middle segments equal, and noting that Plutarch's version differs in making εἰκασία (fancy) the longest segment.

relationship between ontological and epistemological concerns, yet on the other—given the crucial importance of mathematics for one's understanding of these relations in this part of the *Republic*[13]—it should strike one as odd that the effectiveness of the analogy should break down as soon as one attempts to produce a diagram of its relations. For it calls immediately into question Socrates's or Plato's own grasp of elementary mathematics because what calls for a straightforward correspondence between increasing clarity and increasing quantity is complicated by ratios that fail to produce the required image.

I will demonstrate that this pressing difficulty is naturally overcome by shifting from a simple linear representation to the more complex two-dimensional figure that would already have been visible in Glaucon's drawing as a result of the *constructions* required for making cuts δ and ε. That is to say, I allow the mathematical implications to be the driving force behind the illustration of the divided line: the desired *proportions* must be visible and the two middle segments of the line must be and appear to be equal. The result of this approach, if I am right, is that the apparent contradiction that Foley rightly identifies as having "exerted a powerful influence on the development of interpretations of the divided line" (2008, 8) is resolved both ontologically and epistemically.

Plato's theory of images obviates the expectation that any image could be perfect, but two advantages of reproducing Glaucon's actual drawing, as I hope to show, are that philosophers no longer need be distracted by a non-problem, and that a whole range of previous emendations and interpretations of the Plato's text are ruled out. The remaining difficulties are challenging enough, even after the visual aid to the passage has been restored.

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[13] Mathematics is crucial throughout the corpus. See Burnyeat (2000).

4.2 Stepping Stones

The solution I offer below, if I am right, overturns or problematizes from a different angle most of the solutions surveyed by Smith and Foley. However, as I have said, the solution I propose is based on the *practice* of ancient Greek geometry that I described in chapter 1 and should now be kept especially in mind. Recall that *diagramma* operates as both proof and diagram.

An excellent example of diagrammatic proof is nested in the divided line. A Greek schoolboy in Socrates's time could have proved the two middle segments of the divided line must be equal, but almost certainly not with Klein's proof "in the Greek manner" (1965, 119) based Euclid's *Elements* V:

Let there be given a line subdivided into four sections.
Let these sections be designated by the letters A, B, Γ, Δ respectively.
Let the division be made according to the prescription: (A + B) : (Γ + Δ) :: A : B :: Γ : Δ.
From (A + B) : (Γ + Δ) :: Γ : Δ follows *alternando* (Euclid V.16)
    [1] (A + B) : Γ :: (Γ + Δ) : Δ.
From A:B :: Γ : D follows *componendo* (Euclid V.18)
    [2] (A + B) : B :: (Γ + Δ) : Δ.
Therefore (Euclid V.11)
    [3] (A + B) : Γ :: (A + B) : B.
and consequently (Euclid V.9)
    [4] Γ = B.

There is nothing wrong with the proof, and Heath (1956, 2, 112–13) takes *Elements* V as very likely the work of Eudoxus (±390–340) but, in section 4.4 below, I offer a Euclidean proof based on *Elements* I that could have been used much earlier, and which reaches the same conclusion.

So far as I have been able to find, we owe the earliest modern attempt at a rigorous description of the proportions of Plato's divided line to the scientific polymath, William Whewell. On November 10, 1856, this former Cambridge second Wrangler addressed the

Cambridge Philosophical Society on the subject of Plato's "*diagram* by which he illustrates the different degrees of knowledge."[14] After citing and translating the Greek, Whewell summarizes, "The four segments might be as 4 : 2 :: 2 : 1; or as 9 : 6 :: 6 : 4; or generally, as $a : ar :: ar : ar^2$" (1860, 444n6, with notation slightly modified). Rescher takes up what he calls the "Whewell Relations" to urge a shift from seeing Plato's line as an analogy to a robust appreciation of its mathematical proportionalities, and what they imply for human reasoning and knowledge (2010, 152 and 156–63).[15]

A later stalwart, Brumbaugh, notably unwilling to lay ignorance or a blunder at Plato's feet, and equally unwilling to assign blame to his copyists, rightly points out what Plato acknowledges himself: images are imperfect. He pays respect to a Plato who has struggled with the illustration of philosophically significant ontological and epistemological material. In his earliest work on the line, Brumbaugh identified the cause of the overdetermination problem (though he did not call it that) as the "interference of metaphor," in which two lines of thought require two different concrete images and neither is appropriate for the other: "if, for example, four things in a set are unequal in respect to property P, yet tightly connected by property Q, a

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[14] See Robson and Cannon (1964, 169). Rescher misremembered Whewell's accomplishment, making him senior Wrangler (2010, 152n46)—an error well-spotted by an anonymous referee for *Journal of the History of Philosophy*). Whewell was later to include the address, entitled "Of the Intellectual Powers According to Plato," as an appendix (1860, 440–8), itself more about Plato than about any other single topic. For Whewell, it was "the study of the exact sciences in a comprehensive spirit" that had the power to make a person dialectical, and enable the ascent to first principles (1860, 438). He proposed that the highest level of the divided line must include the axioms of mathematics, though its definitions and theorems are properly relegated to the level of hypotheticals because "the Axioms of Arithmetic and Geometry belong to the Higher Faculty, which ascends to First Principles" (1860, 441n3). Rescher's paper on Plato's line, which discusses Proclus, Whewell, and Henry Sidgwick—eschewing twentieth-century Platonists—proposes that the mathematical level "may encompass symbolically mediated thought in general" (2010, 135).

[15] Continuing with the implications of Rescher's approach to the interesting parallels he draws among the sun, line, and cave is far from my present task. Another post-Foley paper should be mentioned in the same vein (informative and seminal, but outside my current task): Benson (2010), superseded now by Benson (2015, chapter 9).

linear representation of these four things as segments cannot adequately present P and Q as simultaneous principles of order." In other words, there is a conflict between the metaphor of proportion and that of disparity and, thus, someone making the diagram "must then choose which of the two properties, P or Q, he intends to make central and which peripheral to his illustration."[16] He concludes, "In short, to schematize the passage adequately, the reader should have two distinct diagrams, and be mistrustful of both" (1952, 533–34). Brumbaugh was right about two issues crucial to what Glaucon drew: his diagram could not have been linear, and it must have enabled the observer to distinguish what is central from what is peripheral.

4.3 The Divided Line Proportion

Understanding how Plato would have conceived an actual line divided *in the proper proportions*, with the mathematical knowledge and tools available to him, requires a return to the constructive methods of geometric proof that were in the process of being codified during his lifetime and that were compiled soon after in the form we know today as Euclid's *Elements*. Translating Plato's descriptions into a diagram that makes good on both the desiderata of what has seemed for centuries to be a contradiction reveals a complex relationship that would otherwise be invisible—has been invisible. The topographical relations given by a diagram add another dimension for the representation of concepts that may seem ambiguous in everyday language.[17]

The construction of the divided line proportion (DLP) depends on notions developed in

[16] The quotations above are from Brumbaugh (1954, 91–92). Brumbaugh wrestled with the divided line problem throughout his life and published other contributions over decades, but his discussions of the early 1950s seem to me most perceptive.
[17] For a discussion on the use of topographical concepts in Plato see Sattler (2012).

Euclid's *Elements* II, which continues with the theme of transformation and application of areas with the now disputed concept of a geometric "algebra" that allowed for solutions to problems that contained the general form of quadratic equations.[18] Five of the fourteen propositions begin with the accustomed phrase, "If a straight line be cut at random" (2.2–4, 7–8). Despite the lack of a general theory of proportion, the use of rectangular parallelograms and parallelogrammic areas makes possible additional diagrammatic proofs of equal areas and equal segments.

The DLP, using as a unit the smallest whole number, for simplicity, is the familiar 4 : 2 :: 2 : 1 (or the proportion $p^2 : pq :: pq : q^2$).[19] It is closely related to the geometric mean or mean proportional; and the construction of a continuous geometrical proportion is central to demonstrating how Plato's divided line would have been drawn by those familiar with the geometric methods of the time.[20] Since the construction of geometric means is the mathematical concept that justifies my diagram of Plato's divided line, it requires some elaboration here.

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[18] The notion that the ancient Greeks had a form of algebra that used geometric elements (lines, areas, angles) as "symbols" for formal reasoning rather than letters and operational symbols (=, –, +, etc.) is widely held, especially among mathematicians working on foundational aspects of mathematics. Although this view has been vigorously contested by Unguru (1975), the dispute may well resolve into a matter of semantics that depends on one's conception of algebra. Greek geometric "algebra" was not as formally abstract as modern algebra in that it also utilized visual reasoning and concepts we now associate with spatial notions and aspects of physics, but supporters of the notion of a "geometric algebra" emphasize the functional role it played in the investigation of mathematical structures and thus its similarity to algebra in the more general, contemporary use of the term.

[19] Any geometric sequence that is a continued proportion in which the consequent of each ratio is the antecedent of the next could be used. The DLP is deceptively simple and very uninformative, especially if stated in a modern algebraic form where letters designate each segment on the line: $p^2 / pq = pq / q^2$. It can be reduced to a continuous proportion that more clearly shows one aspect of the relationship but loses the four-part aspect: $p^2 : pq : q^2$. The geometric aspect can be shown by $p^2q^2 = (pq)^2$ or by $pq = \sqrt{p^2q^2}$.

[20] In a sense, the DLP can be considered somewhat similar to the classic problems of duplicating the cube and squaring the circle, though the scope of the DLP is much broader. Its construction and proof involve the theory of proportions and irrational numbers that were foundational issues addressed adequately only when the first ten books of Euclid's *Elements* appeared; even they give only part of the ancient Greek geometrical thinking (structures behind harmonics and astronomy are not fully included). It is beyond the scope of this chapter to plot the connections between the DLP and propositions in *Elements*, or other passages in Plato that support my view of his grasp of the fundamental issues of mathematics.

The actual division of the line turns out to be straightforward, yet has a considerable amount of theory behind it because the proportion must apply to any line arbitrarily divided that could produce irrational quantities. Supplying a proof or calculating particular quantities to satisfy the ratios is much more complicated than the actual division. Dividing Plato's line could even seem anti-climatic to some, for it is as simple as applying ratios from one line to another using similar triangles involving parallel lines, and dropping perpendicular lines from a point to mark a length on another line. This construction, using the geometric mean, has the advantage over others that a right-angled triangle inherently sets up a series of parallels showing the replication of the ratios, following from the construction lines through the application of what is now called the triangle proportionality theorem.[21]

4.4 What Glaucon Drew

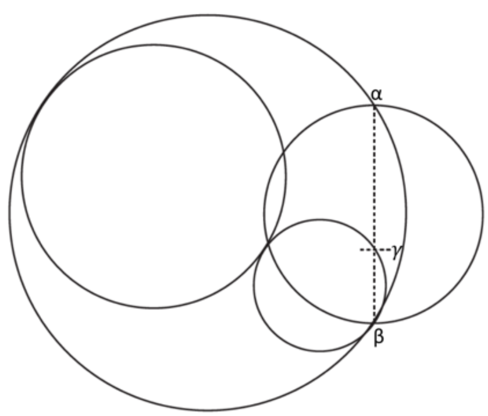

Figure 4.4.1: Reduced Representation

Like Greek mathematicians long before Euclid or even Socrates, and like some mathematicians

[21] This, like *Meno* 82b–85b, is an occasion when Plato gives an accessible demonstration of higher mathematics from more intuitive notions.

today,[22] Plato used diagrams (διάγραμμα) as proofs.[23] The fact bears repeating because its importance has not been adequately appreciated. Symbolic algebra has largely superseded geometry in modern mathematical thinking, but familiarity with some of the geometrical principles with which Plato was operating are essential for understanding Plato's text and, moreover, for *seeing* what Glaucon drew. I concede that it looks complicated at first glance, but that is because we spend little time in school with compasses. Facility is a matter of practice, and one quickly becomes adept.

I proffer the uncontroversial facts that (i) symbolic algebra was not available in the fifth and fourth centuries BCE, when (ii) diagrams were constructed with a compass and straightedge. But these were not today's compass and straightedge: the compass was collapsible, so the radius was lost after each inscription; and the straightedge was not a ruler for it had no unit marks.

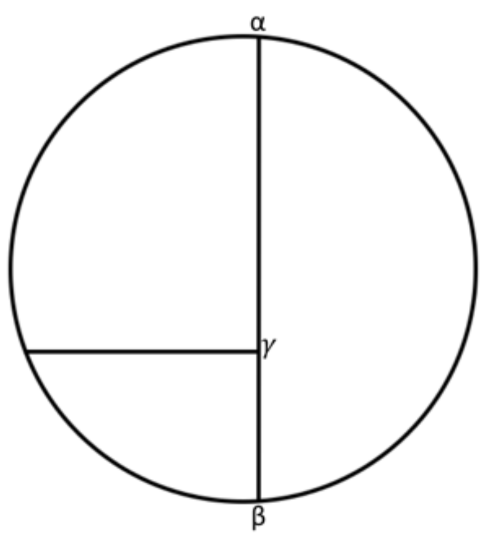

Figure 4.4.2: How the Line Began

Moreover, as is widely repeated in the mathematics literature, Plato used only a compass.[24] If so,

[22] The chief proponent of diagrammatic proof is Brown (2004 and 2008). Denyer, while acknowledging the use of diagrams as proofs, cautions against becoming promiscuous about it (2007, 294–303).

[23] See Fowler (1999, 33), Artmann (1994), Knorr (1983a), Mueller (1992, especially 184–85), and Zeuthen (1896).

[24] See Weisstein (2002, 1185–86), which adds, "It turns out that all constructions possible with a compass and straightedge can be done with a compass alone, as long as a line is considered constructed when its two endpoints

let us not be unduly surprised that drawing the divided line began with the circles required to determine its end points and establish its proportions (Figure 4.4.2).

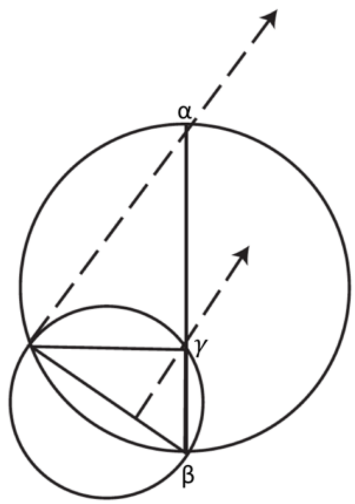

Figure 4.4.3: Another Step

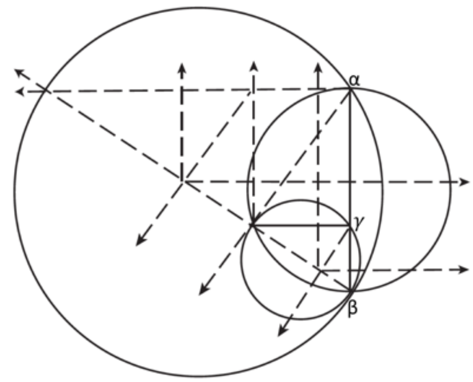

Figure 4.4.4: Still Constructing

It is difficult not to utilize the modern conceptions of abstract mathematical notions because this has been the language of discourse in which most of us have been introduced to them. Moreover, the geometric approach requires a visual graphic interface to enable us to think in this way, and it requires a step by step construction to actualize. The accompanying figures (4.4.2–4.4.5) attempt to replicate that process, though I have omitted most of the repetitive steps of the construction because of their sheer number.

Moreover, because omitting the role of the straightedge in a construction with so many circles would be almost impossible to apprehend, I include the triangles that are inscribed there. The core argument in favor of the DLP's being represented by a diagram that includes triangles is that the *construction* of a line divided in the proper proportions requires the auxiliary

are located." Thanks to David C. Royster for advice about early sources for this information.

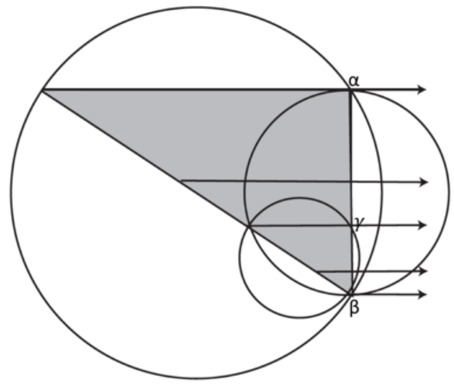

Figure 4.4.5: The Crucial Triangle

construction of triangles to carry out the division.[25] There are several different, closely related, ways to perform this division; the one I offer reiterates the use of parallel lines for finding the mean proportional between two lengths.

The first figure in the above sequence, Figure 4.4.1, shows the construction of the geometric mean, which is the first step in the process: from the first cut, γ, a perpendicular line is drawn to connect to the circumference of a circle whose diameter is the original line being divided. As *Elements* VI.13 shows, this perpendicular line is the mean proportional to the two segments of the original line.[26] The remaining figures, 4.4.2–4.4.5, illustrate the iterative process that would have been used to determine the successive geometric means by constructing rays (illustrated here with dashed lines) that project the original ratio onto a larger triangle, and back onto the line itself; that is, the auxiliary construction lines project the initial ratio onto the diagonal side of the triangle and back onto the line as the proportion $p^2 : pq :: pq : q^2$. In short, determining where Glaucon should make cuts δ and ε *requires* the series of constructions that

[25] Geometers will recall that *Elements* I.1 uses circles to construct a triangle on a straight line. Advantages of a triangle for the ascent from the cave in *Republic* 7 (see Jackson, 1882); analogous passages in *Symposium* and *Phaedrus*; and mathematical passages in *Meno*, *Theaetetus,* and *Timaeus*, are not relevant to the present discussion.
[26] See *Elements* II.14 for an alternative proof for the correctness of the construction.

include the triangle, shaded in Figure 4.4.5.

Figure 4.4.4, the third in the sequence, enables me to provide the geometric proof for the equality of the two middle segments of Plato's divided line that I promised in section 4.2. To see it in Plato's double sense of *diagramma*—diagram qua proof—consider the third figure, but focus on the original line, the triangle, and the auxiliary parallel ray constructed immediately to the left of the divided line itself. The equality of the two middle segments would have been obvious to any Greek schoolboy looking at the diagram and understanding the meaning of 'parallelogram'—but it almost certainly requires a proof in words in our own time. Applying *Elements* I.33, parallelograms between the same parallel lines, and with the same base, are equal in area. The original divided line and the ray are straight and parallel, so δζ, γη, and εθ are equal, forming the two parallelograms δζηγ and γηθε. According to *Elements* I.34,[27] since ζδ (ηγ) is equal to ηγ (θε), and δη (γθ) is common, the two sides ζδ (ηγ), δη (γθ) are equal to the two sides γη (θε), ηδ (θγ) respectively; and the angle ζδη (ηγθ) is equal to the angle δηγ (γθε); therefore the base ζη (ηθ) is also equal to γδ (εγ), and the triangle ζδη (ηγθ) is equal to the triangle γηδ (εθγ). Therefore, the diameter δη (γθ) bisects the parallelogram ζηγδ (ηθεγ), forming a third parallelogram δηθγ bisected by ηγ. Now we reach the clincher: since γ and ηθ must be equal, and ηθ must be equal to γε, then the two middle segments of the divided line, δγ and γε, must be equal in length.[28]

My final illustration of the divided line, Figure 4.4.6, with most of the constructions removed but implicit, displays the DLP as a continuous geometric proportion, able to depict the equality and the increase between the two middle sections simultaneously. The divided line—

[27] Only the variables differ from a verbatim quotation of *Elements* I.34.

[28] Derivatively, the corresponding segments on the diagonal ray that forms the triangle are equal in length.

constructed as rays of an angle within a triangle, one ray corresponding to cognition and the other to the objects of cognition—results in a divided line where increasing *areas*, not γε and δγ themselves, illustrate the increase in truth and clarity from Γ to Β. I resort to the term ‘areas’ instead of ‘segments’ for τμήματα—despite my footnote 3—to highlight what is made visible by the epistemological diagonal ray, and to distinguish it from the ontological vertical ray that has usually stood by itself as *the* divided line. My interpretation leaves one crucial construction line in view, an elongated reflection, as it were, of the vertical.

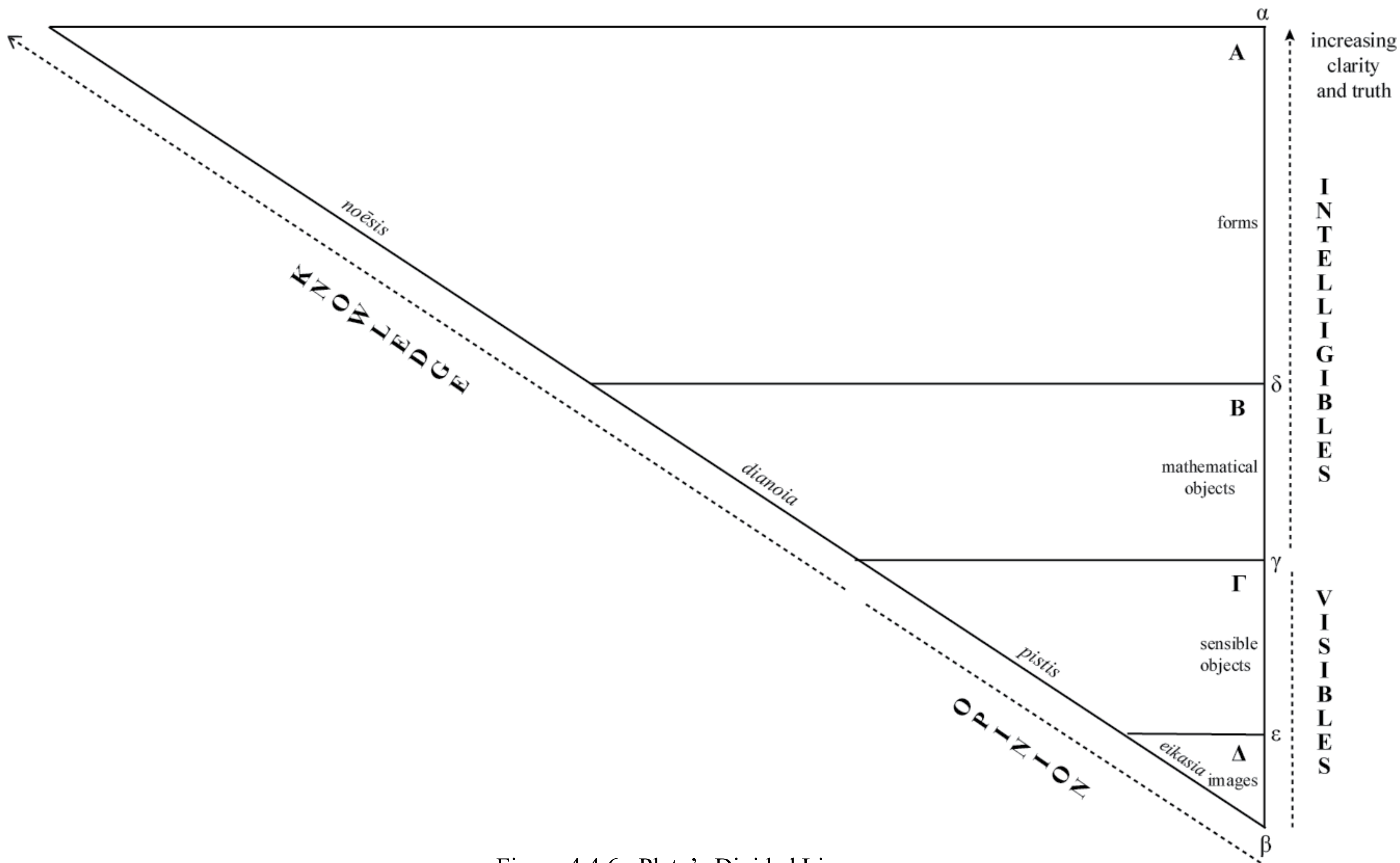

Figure 4.4.6: Plato’s Divided Line

4.5 Implications and Puzzles

Some of the initial implications of what Glaucon drew have already been mentioned in passing: that no textual emendations are required, that Plato's knowledge of mathematics was more advanced than twentieth-century philosophers realized, and that the process of constructing the divided line is crucial to understanding it. Somewhat more can be said, however tentatively.

I agree with Foley that Plato is indicating "that the serious reader should analyze what these further difficulties might be"; but Plato is *not* "signaling that the core issue surrounding the divided line is the issue of contradiction" (2008, 18–19) and challenging the reader to discover this contradiction. If, as I have argued, the τμήματα we see at 511d are segments of the above triangle, that is, areas rather than line segments, then the overdetermination problem disappears; there is no contradiction to be discovered. This leaves us with a quite different interpretation of 534a, where Socrates recommends to Glaucon that they not work out all of the ratios at this time, "to prevent its costing us many times the number of words we've used in discussing the preceding topics" (534a5–8). For Foley, Plato is here signaling that he is aware of the overdetermination problem (2008, 22). But if there is no overdetermination problem, then Plato's transposition of δγ and γε at 534a3–5 may be innocuous, a gesture to the equality of δγ and γε.

Even Plato's innocuous gestures, however, can raise devilish questions of interpretation. Why even mention a required lengthy discussion if anyone with a basic understanding of geometrical diagramming would recognize that δγ and γε are equal in length? I suggest that Plato is addressing not mathematicians but philosophers with his comment. One possibility is that descriptions in words of simple mathematical operations visible to the eye would be a distraction in the context of the conversation—much as I omitted repetitive steps in my illustration of the construction above. Another possibility is that it is not to the illustrated ratios that Plato refers, but to the *further* ratios that would be required to generate the hierarchies of relationships among the entities each segment is said to represent. To put it another way, what if further divisions of each segment, by the same ratio, were necessary for complete clarity—or such clarity as humans can reach—about the relations among entities? Consider for a moment those two middle segments and how they might be populated. If, for example, the being of a tree is different from the being of a shield, one might call for a division of γε, sensible objects, to account for the difference between natural objects and artefacts. Since some natural objects are alive, some are not only the objects of sensation but subjects who sense, natural objects might be further divided. More controversial would be further divisions of δγ, mathematical objects—unless Plato would have thought it appropriate to lump together willy-nilly the number 4, the isosceles triangle, the even and the odd, formulae with pi, the hypotenuse, definitions, axioms, and propositions. Working out *all* the ratios is what would involve a lengthy discussion, detracting from the more general subject under discussion.

With Brumbaugh, I admitted earlier that Glaucon's line could not be linear and that it would illustrate visually the difference between what is central and what peripheral. Therein lies what I regard as an important implication of the passage as we have it, and of Glaucon's diagram: insofar as the vertical line is the explicit goal of the construction, ontology is prior to epistemology. Further, and according to the diagram, there are mathematical intermediates that are clearer and truer than sensible objects.[29] Making the case that being is in fact prior to

[29] What those intermediates *are* remains open to dispute: recall that Whewell counted the axioms of mathematics among the forms (1860, 441n3), in a sense dividing one of the line's "objects." The more common—though also disputed—approach is to hold the objects fixed while allowing more fluidity to cognition, that is, to permit both

knowing, however, or that Plato consistently held that ontology is prior to epistemology, or even that propositions about the sphere itself are superior to—clearer and truer than—propositions about the cue ball in my hand, would make my paper "many times the number of words" it is already, for supporting evidence would need to be drawn not only from the text but from other dialogues, mostly from the late group.[30] For now, my aim is more limited.

Complex and fundamental metaphysics is not *settled* by diagrams because, as is true of all diagrams since they are all images, they can never get us all the way to complete and genuine understanding. One might then wonder whether I have not circled right back to conclusions about the inadequacy of the divided line diagram that others have offered, namely, that it is deliberately flawed (Smith) or contradictory (Foley). I have not: the correctly drawn image illustrates what the dialogue's characters say without distortion. Yet, Plato is right to criticize the inadequacy of all images, in this case because the four levels do not illustrate *all* that we want to understand about being and knowing.

One is nevertheless entitled to frustration that Plato was not more direct about the relationships depicted in the diagram, that the text does not provide more clues, more hints, about how its correct interpretation might move us further along toward the goal of understanding.[31] At the very least, the diagram I have offered is a visual *aid* to the resolution of both the ontological

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knowledge of sensibles and beliefs about forms. See Smith (1996, 34–35 with n24).

[30] While this is not the place for a full discussion of the relationship between mathematical and sensible objects, I think Plato recognized that they are intertwined in a way that makes it difficult to understand one without the other. As we might now say, it is a relationship sharing in complexity the difficulties that are encountered when numerical concepts are applied to geometric magnitudes or the reverse.

[31] Responses with which the literature is littered will be familiar: Plato saves his most complex views for initiates; Plato is only providing a backdrop for politics and moral psychology, so it is unreasonable to expect him to express his more fundamental positions in detail; Plato, at the time of writing the *Republic*, still had not settled all the facets of his mature view of mathematics; Plato would have needed further divisions of the divisions of the line to accomplish a comprehensive picture, and would thereby have detracted from the line's parallel to the cave. I have sided implicitly with the last.

and epistemic aspects of the overdetermination problem, that is, the elimination of the seeming contradiction between 509d6–8 (with 534a3–5) and 511e2–4. It also provides a better understanding of how the mathematical practices of Plato's time may inform our interpretations of his texts. It is another matter whether Plato actually envisioned exactly this diagrammatic representation, but I have provided a justification that reflects accurately the concepts of Plato's text in a diagram Glaucon could have drawn. It could plausibly have been conceived by Plato himself.[32]

[32] Earlier versions of parts of this chapter were presented to the Society for Ancient Greek Philosophy in 2008 and the Ancient Philosophy Circle at Michigan State University in 2013. I thank both audiences for their comments. I am especially grateful to Debra Nails and Emily Katz for the invaluable conversations and shared insights that greatly broadened my interpretation of the construction, and also to the two readers at the *Journal of the History of Philosophy* for their careful comments.

CHAPTER 5:

THE GEOMETRIC DIAGRAM DERIVED FROM THE NUPTIAL NUMBER

5.1 Procreation of the Guardians

People who remember little else about Plato's *Republic* will recall that a grand city is proposed by Socrates in which males and females are trained and educated together (*Republic* 5.451c3–462e3),[1] and those among them who evince philosophical aptitude are given further education, its curriculum described in *Republic* 7.521d4–7.540c9, to equip them to rule the city as guardians (5.473c11–e4)—foregoing the trappings of wealth, fame, and raw power—for the common good. Guardians also relinquish nuclear family life (5.457c10–465b), living in barracks-style quarters, their sex lives determined during their reproductive years by mating festivals (5.459e6) that the city's officers in charge of such matters claim to be lottery-driven when they are in fact stage-managed to pair the best with the best. Socrates's Athenian interlocutors—avid breeders of livestock, dogs, and birds (5.459a1–c1)—are quick to agree that the city's best women and men would produce the best offspring,[2] thereby ensuring the future of the best city.

Although this summary may appear remote from the Platonic mathematics that is my focus, the so-called nuptial number passage, the subject of this chapter, takes up the regulation of the procreation of guardians. Socrates had earlier stated his intention to discuss four forms of

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[1] Hostility to the proposal, and to the notion of equality of the sexes, has characterized the overwhelming bulk of *Republic* scholarship from the Renaissance to the present, and for a wide variety of reasons—some paternalistic, some misogynistic, many based on claims about what is "natural." Strauss (1964, 127), and many in his wake, Saxonhouse (1976) most prominently, assert that Socrates's proposal is proof that Plato himself viewed the city of the *Republic* as impossible.

[2] In *Theaetetus*, it is the midwife's task to pair the best with the best (149d5–8).

corruption in the city (*Republic* 5.449d), but had been forced by Polemarchus, Adeimantus, Glaucon, and even Thrasymachus to drop the subject that he now takes up again in *Republic* 8. If the city can be corrupted, then something must have gone wrong with either the festive breeding program or the educational curriculum—the two devices that were proposed to help ensure the city's future. The mating festivals were supposed to have been regulated by a formula that is not given in the original discussion; it appears later in the *Republic* and is attributed to the muses.

The muses' speech at *Republic* 8.546a1–547c4, which concerns cycles associated with human procreation, embeds a set of numbers that are supposed to engender better or worse births among the guardians of the city. But it is opaque with respect to the practical application of these numbers. The commentary on the purely mathematical aspects of the passage has a very long history—beginning with Aristotle—and a well-established, controversial literature that has produced plausible interpretations of the numbers involved;[3] most translators offer a key to the numbers in their notes.[4] Although the numbers have historically attracted mystical associations, the passage is nevertheless of pure scientific interest, as it is an intermediate step on the way from doubling the square to doubling the cube. In mathematics, doubling the cube leads clearly

[3] Brumbaugh (1954, 107–50) declares that no translation of *Republic* 8.546b4–d3 can be neutral, that all make presuppositions; he provides examples of several competing translations and interpretations, on all of which he comments; he then gives an intricately detailed commentary on the philosophy, philology, and mathematics of Plato's text, studded with 8 geometrical figures (including a scholium), additional diagrams, and snippets of Babylonian astronomy and Pythagorean astrology. Finally, Brumbaugh provides a précis of all scholarship from the sixteenth century to the nineteenth, though these involve no geometrical figures. Readers who find themselves obsessed with fifteen lines of Greek text in *Republic* 8 are not alone; they might turn to Brumbaugh for a plethora of destinations toward which to direct their interests. However, much additional literature has been added since 1954, much of it harking back to neoplatonic and neopythagorean sources—and my own interpretations derive from these later contributions.

[4] Corresponding to the quotation below, Rowe (2012, 422–23nn544–45). See also the widely used Grube translation (1992: 216–17n10) revised by Reeve.

to conic sections.

My aim here is to establish that the *geometric* representation of the numbers described at 546a1–d7 has a distinctive similarity to the diagrams I used in chapter 4 to illustrate the construction of Plato's divided line. While the specific lengths and ratios of the segments of each diagram differ (we have the ratio of 9 : 16 for the two segments on the base of the triangle of book 8, and the ratio 1 : 2 for the same two segments of the divided line), the diagrams share certain key structural properties. Most strikingly: (1) both are right-angled triangles; (2) both presuppose Euclid VI.13 in the construction of their internal divisions, as these divisions produce three lengths, one of which is the geometric mean of the other two; and (3) both use several iterations of (2)—finding a single mean—to determine a proportion with two means between the extremes.[5] Like other mathematical passages in Plato's dialogues, the nuptial number passage is pregnant with associations to a variety of mathematical concepts and problems prominent in Greek mathematics in Plato's day.

5.2 The Nuptial Number Passage

The relevant portion of the long speech of the muses, including notes supplied with the translation, reads:

> The cycle for divine birth is captured by a perfect number, whereas for mortal birth the number is that in which the controlled potencies of growth first realize the three dimensions, and four limits, of the things that bring about likeness and unlikeness, and growth and decay, in such a way as to render everything mutually translatable and rational; of which a base of four over three, combined with five and raised to the fourth power, offers two harmonious figures, the first with two sides equal, taken a hundred

[5] Hippocrates of Chios shows that doubling the cube can be reduced to finding just such a proportion (Fowler 1999, 283).

> times over, the second having one side equal but the other longer, and consisting of a hundred times the square on the rationalized diameter of a square with side five, less one, or the square on the irrational diameter less two, and a hundred times the cube on side three .[6] All of this constitutes a geometrical number[7] governing the sort of thing in question, namely, whether the new generation will be better or worse, and if ever the guards' ignorance of this causes them to have brides cohabit with bridegrooms out of due season, the resulting offspring will not be well endowed by nature, nor fortunate.—Rowe, trans. (8.546b4–d3).

The passage has historically been a challenge to understand, in part due to the brevity of the description but also because much of the terminology appears to be derived from the Pythagorean arithmetical tradition preserved, or further interpreted, only by later writers.

Proclus (412–485 CE), who headed Plato's Academy for a time, was the most important of these later writers; and most of the literature on the nuptial number passage begins with Proclus's commentary on Plato's *Republic—In Platonis Rem Publicam Commentarii*—the thirteenth essay of which covers the nuptial number. Proclus's commentary is in fact the *only* extant commentary on Plato's *Republic* from ancient times, and its translation into English is an

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[6] That is, it seems, 604* = 12,960,000 = 3,6002 = (4,800† x 2700). For what it is worth, 12,960,000 is also 100 x 3602, i.e., the square of (roughly) the number of days in a human life multiplied by the number of years in that life (see 615a, where Socrates will use 100 years as some kind of measure of the human lifespan).

> * i.e. (4 x 3 x 5 [the sides of a 'Pythagorean' triangle]) 'raised to the fourth power' ('three times increased' in the Greek);
>
> † where 4,800 = (7 x 7 – 1) x 100, or (√502 – 2) x 100, with 7 representing the 'rationalized' diagonal of a square side 5.

[7] This, i.e., apparently, 12,960,000 (with all its potentialities: see preceding note) is the notorious Platonic 'nuptial number', which has been and remains the subject of intense speculation. Whatever may, or may not, lie behind it in terms of serious mathematics (or, alternatively or additionally, mathematical mysticism), some general points are clear: (1) working out the number has to be difficult, since if it were easy there would be no reason for anyone to get it wrong, and the city could sail on undisturbed forever; (2) the description as a whole is based on the mathematical construction of solids, as being what underlies the complex and changing world of the senses; (3) if the description ends up with *two* figures 'in tune' (literally, just 'atunements', *harmoniai*), that too is no surprise, if the number in question has to do with the regulation of sexual coupling; (4) the introduction of multiples of a hundred should perhaps be equally unsurprising, given that what is at issue is the regulation of coupling on a large scale; and (5) the number, whatever it is, clearly attempts to eliminate, or at least reduce, the element of irrationality in its constructions. 'Irrationality' here is, strictly speaking, incommensurability, or what cannot be expressed in terms of whole numbers. But in the context it will also, presumably, stand for irrationality in a wider sense, including and especially irrationality in human behavior—so that if the mathematics of the passage is obscure (and after all, the Muses are 'playing … teasing us as if we were children'), its figurative sense is less so.

ongoing project.[8]

An even earlier and anonymous source has come to light in the twenty-first century: the marginalia of Oxyrhynchus Papyrus XV 1808, an edition and commentary on which McNamee and Jacovides published in 2003. The papyrus is the oldest extant version of the nuptial-number fragment of the *Republic*, preserving a second-century CE scholar's eleven marginal annotations. If the annotator is well-informed and astute—as appears to be the case—the papyrus is an ancillary resource for understanding the mathematical meaning of Plato's terms. There is a fly in the ointment, however: McNamee and Jacovides use Proclus's text to explicate the comments of the anonymous annotator who preceded Proclus by three centuries, inadvertently undermining the independence of the former's value as a source—and verging on circular reasoning.[9] Caution is essential.[10]

I pause to introduce some of the terminology in Plato's text—with Rowe's translations—in the hope that doing so will help to identify the numerical elements and operations required for following Plato's suggested geometrical constructions and to clarify the mathematics being addressed in the obscure poetic language of the muses. Unavoidably, such identification and clarification involves the fragmented and inconsistent commentary tradition; my intention below is to lay out my own position on the passage rather than to trace what two thousand years of

[8] Baltzly, Finemore, and Miles (2018) is the first volume of the first English translation of Proclus. The second volume (wherein the nuptial number passage appears) is projected for 2020.

[9] Occasionally, they write as if the source of the anonymous annotator is Proclus: e.g. "We have no view about what Plato really means here, but we think the annotator agrees with Proclus" (2003, 33). Hence, I take a grain of salt with their assertion (2003, 45), in agreement with Vogt (1909–10), that the Kroll edition (1901) marked the end of possible doubt that Proclus's interpretation of the nuptial number passage is the correct one. They go on to say that Brumbaugh 1954, 125–27, 134–35) and Ehrhardt (1986, 409) decline to agree.

[10] Nevertheless, I rely on the account of McNamee and Jacovides, who say that they "lean on Proclus" (2003, 40), for the terminological section below. In fact, they cite a host of ancient sources—mathematicians and philosophers—for the information they provide; but chasing down the contexts for those many sources would take us far afield.

commentary have produced.[11] The sheer number of LSJ paradigms in fifteen lines of text indicates how difficult the passage is.

546b6 αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι—"controlled potencies of growth": The phrase apparently refers to the hypotenuse and sides of a right-angled triangle. Pythagorean terminology is preserved by the early third-century CE peripatetic, Alexander of Aphrodisias: "Since … the hypotenuse is equal in power to both the other sides together, for this reason it is called 'the one ruling' (ἡ δυναμένη), and the others are 'the ones controlled' (δυναστευόμεναι), and it is five" (McNamee and Jacovides 2003, 38). Hence we should understand that the numbers associated with the hypotenuse and the sides of a right-angled triangle, respectively, are to be increased or multiplied. The passive form, δυναστευόμεναι, is an LSJ paradigm.

b6–7 τρεῖς ἀποστάσεις, τέτταρας δὲ ὅρους—"three dimensions, and four limits": This phrase should be understood as a line divided by four points into three segments. The segments represent the connecting ratios, and these segments are divided (or bounded) by the four points, which represent numbers in a mathematical progression such as 27—36—48—64.

b7 ὁμοιούντων τε καὶ ἀνομοιούντων—"likeness and unlikeness": Nicomachus, Theon of Smyrna, and Proclus all give the same arithmetical meaning for these words (McNamee and Jacovides 2003, 40). A 'like number' is either a square or cube (e.g. 3^2, 2^3), while an 'unlike number' has two or three different factors (e.g. 2 x 3, 3 x 4 x 5).

c1–2 πάντα προσήγορα[12] καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν—"everything mutually translatable

[11] The contemporary sources I cite are replete with references to the commentary tradition and can easily be consulted by the curious.

[12] Another LSJ paradigm is given for προσήγορα: "of things, agreeing."

and rational": Both b7 and c1–2 are names for numbers that can be algebraically expressed by $a^n b^n$, where a can be lesser or greater than b, or equal to it. In general, an unlike number can either increase or decrease, depending on the values of a and b, but in this passage it appears to refer specifically to a progression of the form a^3, a^2b, ab^2, b^3, where the power of a decreases in the progression while the power of b increases (McNamee and Jacovides 2003, 41). I take "mutually translatable and rational" to mean that the numbers are in a mathematical ratio, or a mathematical progression such as the one just mentioned, using natural numbers.[13]

c2–3 ὧν ἐπίτριτος πυθμὴν πεμπάδι συζυγεὶς—"a base of four over three, combined with five": This is widely taken as a reference to the 3,4,5 right-angled triangle, whose legs are in a 4 : 3 ratio and whose hypotenuse is five units in length. The combination of these numbers in a triangle may refer to the Pythagorean's use of the 3,4,5 triangle as a symbol of marriage. Since Plato could have used a different set of numbers to satisfy the mathematical relations he describes in the passage (e.g. 1,2,4,8), he may well have chosen this particular diagram with the Pythagorean meaning in mind (Brumbaugh 1954, 119). Two of the terms in the phrase are LSJ paradigms: ἐπίτριτος = containing an integer and one-third (1 + 1/3), i. e. in the ratio of 4 : 3, ἐπίτριτος πυθμήν; and πυθμήν = in arithmetic, base of a series, i.e. lowest number possessing a given property, ἐπίτριτος πυθμήν the first couple of numbers giving the ratio 4 : 3.

c3 δύο ἁρμονίας παρέχεται τρὶς αὐξηθείς—"offers two harmonious figures, the first with two sides equal" (in his endnote, Rowe gives the literal sense, "three times increased"): The precise nature of the 'harmonies' is far from clear. This may refer to the three

[13] See Heath (1981, 1, 306) for such alternative forms of the equation as $a, b, b^2/a, b^3/a^2$.

multiplications of the sides of the triangle to reach the desired dimensions to get the two harmonies (e.g. 3 x 3 = 9, 3 x 9 = 27, 100 x 27 = 2700). They could also be the two geometric proportions 27 : 36 : 48 and 36 : 48 : 64. In a mathematical context, a harmony required at least three terms, which may rule out their being, as Proclus theorized, a single number. Hence, while I use Proclus's diagrams to show the geometric relations of the relevant numbers (2700, 3600, 4800, 6400), I cannot agree with his view that the harmonies just are single numbers—specifically, certain sums of the relevant numbers. The following four specifications refer to the two "harmonies" given as rectilinear figures:

c3–4 τὴν μὲν ἴσην ἰσάκις ἑκατὸν τοσαυτάκις[14]—"the first with two sides equal, taken a hundred times over": This refers to a square (ἰσάκις—LSJ paradigm) with sides of length

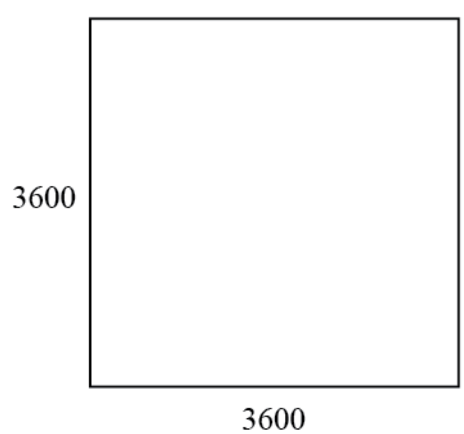

Figure 5.2.1: A Square 3600 x 3600

6 x 6 = 36, which is then multiplied by 100. That is, 6 x 6 x 100 = 3600. This square emanates from the fact that the geometric mean of 27 and 48 is 36, which is found with

[14] τοσαυτάκις is another LSJ paradigm; it is the adverbial form of τόσᾰκῐς = so many times.

the formula $b = \sqrt{ac}$ with $ac = 27 \times 48 = 1298$ and $b = \sqrt{1298} = 36$.

c4–5 τὴν δὲ ἰσομήκη μὲν τῇ, προμήκη δέ—"the second having one side equal but the other longer": This describes a rectangle. A rectangle is of equal length one way, that is, the opposite sides are equal to one another; but it is oblong in shape, as the two *sets* of opposite sides are not equal to one another. It refers to the rectangle that will have 27 x 100 and 48 x 100 for its sides as set out in c5–6 and c6–7 and shown in Figure 5.2.4.

c5–6 ἑκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων ῥητῶν πεμπάδος, δεομένων ἑνὸς ἑκάστων, ἀρρήτων δὲ δυοῖν—"consisting of a hundred times the square on the rationalized diameter of a square with side five, less one, or the square on the irrational diameter less two": First, "side five" refers to the long sides (the ones with the numerical value of 4800) of the rectangle described in c4–5. The description of this side that follows is perhaps more complicated than strictly required. Plato seems to be using this as an opportunity to bring up again the notions of side and diagonal numbers.[15] 'Diameter' just means the diagonal of a square. The 'rational diameter' is a diagonal whose length is a rational number; and the 'irrational diameter' is a diagonal whose length is an irrational number.

Plato proceeds to explain in two different ways how to arrive at the numerical value of the long side of the square given at c3–4. The first is by starting with the diagonal of a square that is a rational number—a diagonal of length 7. This length, 7, is then squared; 1 is then subtracted from the resulting square number (49); and finally the difference (48) is multiplied by a hundred. That is, $(7^2 - 1) \times 100 = 4800$.

[15] See chapter 3 on *Theaetetus* for its relation to the *Statesman* and *Meno*. The notion of rational and irrational diameters (referring to the diagonals of the squares) was connected to that of 'side and diagonal numbers' that were used for approximating the numerical value of $\sqrt{2}$ and could also be utilized in a proof of the irrationality of it. Fowler (1999, 290) notes that Theon of Smyrna and Proclus are led by *Republic* 8.546b–d to describe side and diagonal numbers.

The second way starts with the diagonal of a 5 x 5 square, which has the irrational value of the square root of fifty. This value is also squared and then diminished by two and multiplied again by one hundred or $((\sqrt{50})^2 - 2) \times 100 = 4800$.

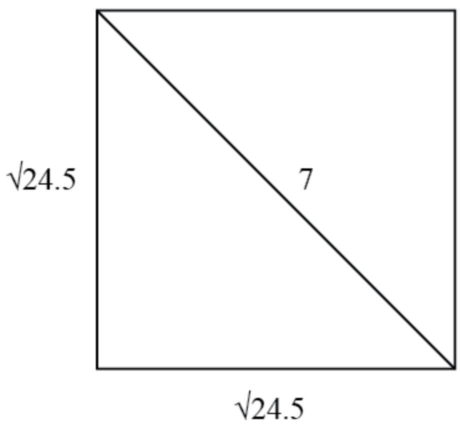

Figure 5.2.2: A Square √24.5 x √24.5

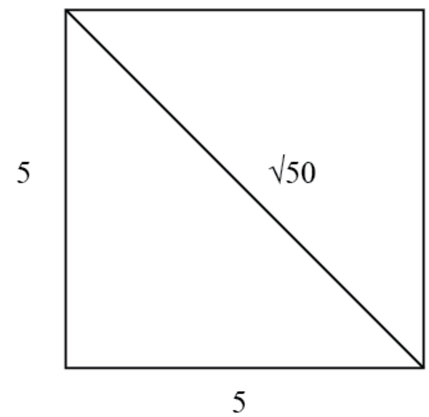

Figure 5.2.3: A Square 5 x 5

c6–7 ἑκατὸν δὲ κύβων τριάδος—"a hundred times the cube on side three": one hundred times three cubed: 3 x 3 x 3 x 100 = 2700. This length, together with the length described in c5–6, yields the 2700 x 4800 rectangle (introduced as longer in c4–5).

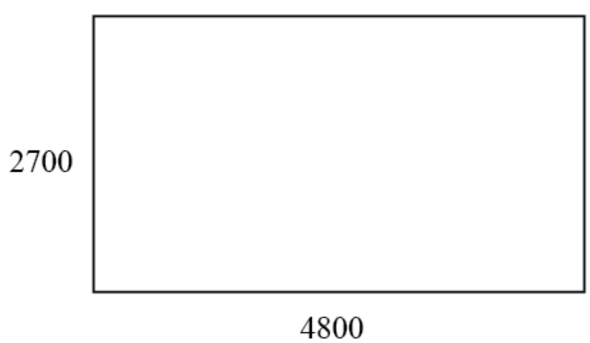

Figure 5.2.4: A Rectangle 2700 x 4800

If the area is calculated for each of the two harmonies (Figures 5.2.1 and 5.2.4), each will be 12,960,000. Whether this is the nuptial number itself is not relevant to the diagram that emerges from these numbers. I share the view expressed by Brumbaugh that, if this is the geometrical number, its significance lies in the fact that it summarizes the proportionate relations of the geometrical image and specifies how the number should be derived by giving the formula or pattern of the interacting factors.[16] What is relevant for my thesis is the mathematics of the passage, not its application to procreation in the ideal city.

The words describing mathematical objects that Plato uses are drawn from a tradition in which the terminology was often more metaphoric or symbolic than literal. It also evidently predates any attempts at standardizing, or at least a common acceptance of terms used by mathematicians. Thus any attempt to interpret the passage without reference to the ancient commentators would be undertaken at a disadvantage.

5.3 The Commentator Tradition

Commentary on Plato's mathematical passages begins with Aristotle. In the *Politics* (1316a), he refers to the nuptial number passage in the discussion of changing cycles of the state: "He [Plato] only says that the cause is that nothing is abiding, but all things change in a certain cycle; and

[16] As Brumbaugh (1954, 132) puts it, "[I]n a 'geometrical number,' we should expect the relation to be given by geometrical construction, which would not necessarily be represented arithmetically as a product or sum. When scholars have asked, therefore, "whether the numbers 3, 4, and 5 are to be added and cubed, or multiplied and cubed, or cubed and added," they have overlooked the possibility that these numbers refer to geometric elements, and that the construction need not limit itself to simple addition or multiplication. Second, in a passage as elliptical as this one, a careful author will use coordinating particles very carefully to indicate elision and organization, so that a careful interpreter should start with these, rather than beginning, as some have, by "deciphering" 10,000 or 4,800 or 7,500 as a number referred to in one part of the passage."

that the origin of the change consists in those numbers 'of which 4 and 3, married with 5, furnish two harmonies' (he means when the number of this figure becomes solid)."

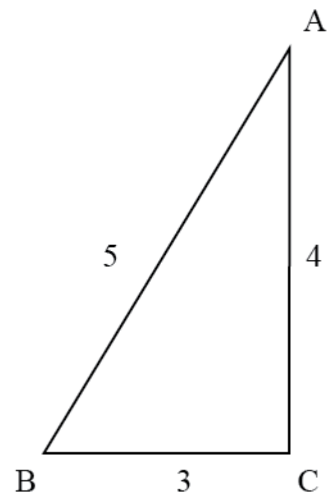

Figure 5.3.1: A Simple Pythagorean Triangle

While some commentators think that this suggests that the final figure itself becomes a three-dimensional diagram or object, Aristotle clearly states that he takes it to be the numerical value associated with the figure that becomes solid, and so not the figure itself. I am following Aristotle's lead and examining the passage under the assumption that it is cubed numbers associated with plane figures that are to be sought. Drawing on the various attempts to work out the mathematical significance of the passage, the arithmetical and geometrical aspects of the nuptial number will be seen to be linked to Plato's interest in developing methods to solve cubic equations such as those that underlie the problem of doubling the cube. The geometric mean triangle plays a significant role in his program.

The fullest ancient account of Plato's calculation is that of Proclus, who also compiles earlier explanations of the significance of Plato's geometrical number.[17] Proclus's approach to understanding the numbers of the muses is initially geometrical, as he starts with the 3,4,5 right-angled triangle associated with the Pythagoreans. Successive augmentations of the sides and the hypotenuse of this triangle generate increasingly larger triangles with the same proportions but with greater numerical values. Proclus uses a series of diagrams that show the sides of right triangles as genuine augmentations in size.

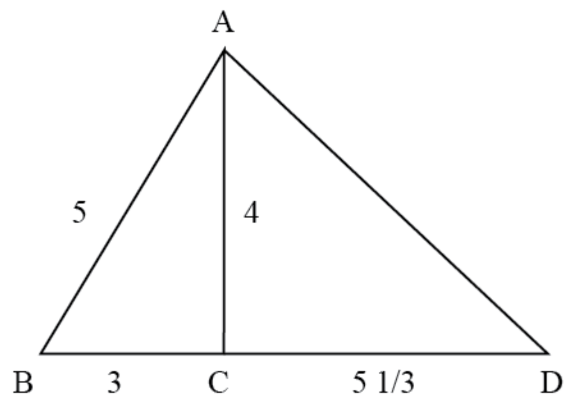

Figure 5.3.2: Proclus Continues the Triangle

The procedure he sets out is as follows. Starting with a triangle ABC with its sides being AC = 4, BC = 3, AB = 5 (Figure 5.3.2), the line BC is extended to D and a line AD is drawn to the upright line AC. In the right triangle ABD the line AC is the altitude, and it follows that AC is then the proportional mean between BC and CD. The ratio AC : CB is 4 : 3, so it follows that CD is also in ratio 4 : 3 with AC because they are similar triangles. This gives a numerical value of 5⅓ for CD. Ploclus then multiplies these numbers by 3 to make the fraction disappear, giving BC = 9, AC = 12, AB = 15, and CD =16 (Figure 5.3.3).[18]

[17] Hultsch (in Kroll 1901, 384–413) clarifies some of the particularly difficult aspects of Proclus's discussion of the passage in his commentary of Plato's *Republic*.

[18] McNamee and Jacovides (2003, 39) remark, "The multiplication by 3 may seem arbitrary, but it is in accordance

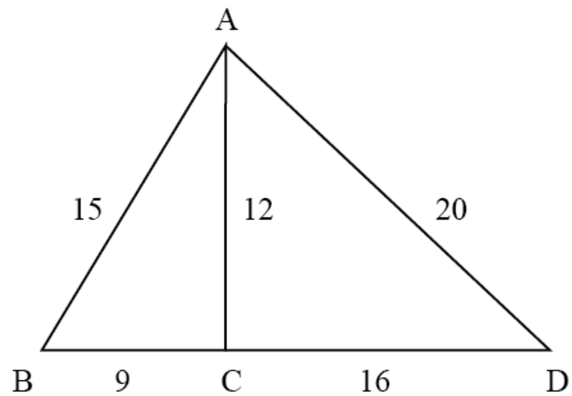

Figure 5.3.3: Expansion

He then expands the triangle by extending a line DF perpendicular to CD from D until it meets the extension of AB at F. AE is drawn parallel to BD. Since ADF is a right triangle and

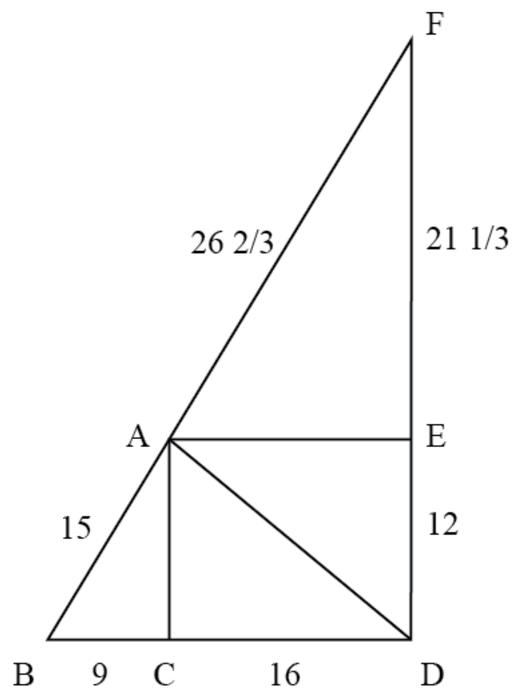

Figure 5.3.4: The Mean Proportional

with a remark Plato makes in book 7 of the *Republic*. Clever calculators—those, that is, who study calculation "for the sake of knowledge rather than selling merchandise"—will not permit fractions. Socrates tells us, 'If, in conversation, someone tries to cut up the one, they laugh at him and do not allow it. Rather, if you fragment it, they multiply, taking care lest the one ever seem not to be one but many parts' (525d–e)."

AE is perpendicular to DF, line AE is the mean proportional between DE and EF (Figure 5.3.4). As AE = 16 and DE = 12 are in the ratio 4 : 3, the line EF is also to AE in the same ratio due to AED and FEA being similar triangles with EF = 21⅓ and AE = 16. Plato's rule for clever calculators is then applied again, multiplying all these values by 3 to eliminate the fraction ⅓, giving EF = 64, LA = 48, LZ = 36, and BG = 27.

The final triangle that Proclus is seeking is Figure 5.3.5. In his interpretation, the two harmonies are found through arithmetic by manipulating the 4 : 3 ratio to produce the sequence 27, 36, 48, 64. These match his geometric diagram with the lengths CB, CD, BD, and EF, which are 27, 36, 48, and 64, respectively. These four numbers are needed to produce his interpretation of what the two harmonies should be. Proclus derives these two harmonies by adding together the two segments of each of the legs of the triangle (27 + 48 and 36 + 64), combining CB and CD together to form BD = 75 and ED with EF for DF = 100. Increasing them both 100-fold

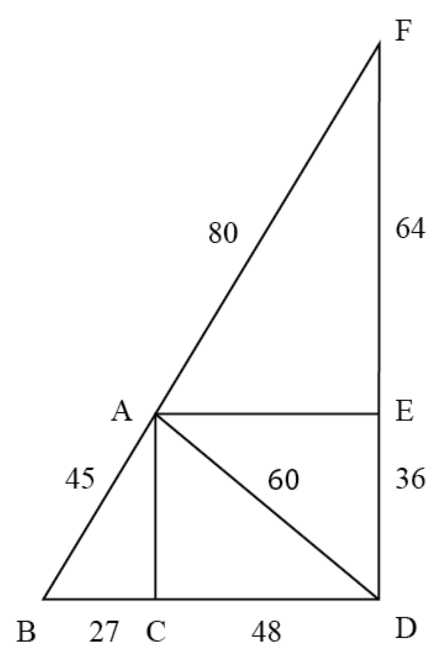

Figure 5.3.5: The Two Harmonies

gives the harmonies as 7,500 and 10,000.

The geometric progression a^3, a^2b, ab^2, b^3 (or a, b, b^2/a, b^3/a^2) directly follows from the method of finding geometric means in Euclid. It also plays a role in subsequent investigations of how to double the cube. The diagram graphically displays the two sets of geometric means (27 : 36 : 48 and 36 : 48 : 64) that are contained within the progression, and mirrors the construction using Euclid II.14 (see Figures 5.3.6 and 5.3.7). AC is constructed on line CB + CD using a right-angled triangle inscribed in a semicircle. AE is constructed in a similar manner. The shift from the horizontal base to the vertical leg in this construction sets up a significant feature of Proclus's triangle, which is the zig-zag structure of the relationship of the lines CB, CA, AE, and EF (see Figure 5.3.8).

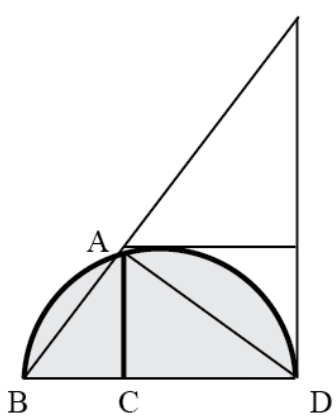

Figure 5.3.6:
Geometric Mean AC

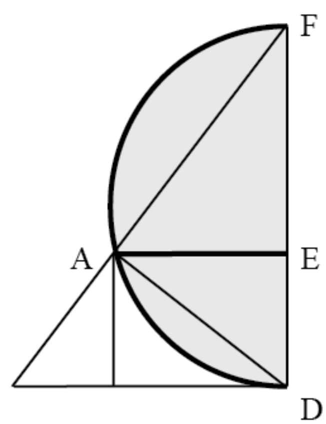

Figure 5.3.7:
Geometric Mean AE

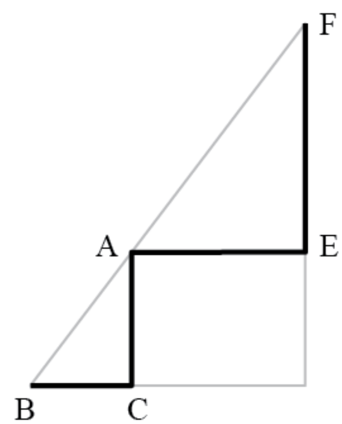

Figure 5.3.8:
Zig-zag Structure

5.4 Significance

Three aspects of the nuptial number passage are of mathematical significance: (1) the numerical relations given to establish the two harmonies; (2) the structure of the triangle that forms the geometric figure; and (3) the classification of diameters as either rational or irrational. The notion of irrational diameters is a link to the doubling of the square in the *Meno* addressed in chapter 2. The *Meno* proof (see 2.4 and 2.5 in chapter 2) offered a prototype for a mixed arithmetical and geometrical solution to the quadratic equation $x^2 = 2$ or $x = \sqrt{2}$ based on the diagonal-to-side ratio of a half-square isosceles triangle.[19] The notion of irrational diameters addressed in this passage serves as a bridge to mixed arithmetical-geometrical solutions based on the diagonal-to-side ratios of triangles derived from quadrangles and proportions using these ratios; they apply to cubic equations as well.

Both (1) and (2) can be taken as arithmetical and geometrical counterparts of the relation of two mean proportionals between two given numbers that form the extremes of a geometric progression (a^3, a^2b, ab^2, b^3)—2M2E.[20] The exploration of the relationship between these numbers can be seen as a preliminary step toward discovering a geometric construction of the cube root of two. In the mid to late fifth century BCE, Hippocrates of Chios offered a hypothesis that the solution to the problem of doubling the cube could be reduced to the problem of finding 2M2E. But as the nuptial number example shows, not *all* sets of numbers in 2M2E proportion yield a cube root between the two extremes, so the construction of a triangle similar to the

[19] The papyrus edited by McNamee and Jacovides (2003) contains fragmentary sentences that make reference to the *Meno* in connection to rational and irrational diameters. It references the subtracting of 1 from a squared number, showing a portion of a figure from the *Meno* (doubling the square on the diagonal). Adding or subtracting 1 from a square number is part of the method of side-to-diagonal numbers used to approximate a value for √2.

[20] Here I use the abbreviations introduced in chapter 2.5.

nuptial number triangle may not be sufficient for solving the problem. Hippocrates specifies that a solution requires the two extremes to be in a 1 : 2 ratio. The nuptial number triangle has a continued geometric proportion of the general form $a : b^{1/3} : b^{2/3} : b$ (alternatively, a^3, a^2b, ab^2, b^3), which gives the 2M2E proportion; but 27 : 64 does not reduce to a 1 : 2 ratio. And we can see that the second magnitude (36) is not the cube root of 64, the magnitude of *b*, or the last term. I suggest that a comparison of the nuptial number and divided line triangles shows that the additional necessary condition for solving the problem is in fact addressed in the nuptial number triangle: a general form for the nuptial number triangle can set the extremes in a proportion that can yield any cube root using 1 : *b* as the extreme magnitudes.

There is more than just similarity between the nuptial number and divided line triangles. Because the numerical values and ratios between the numbers differ, we cannot say they are strictly the same. However, on a more abstract structural level, we can say, from a Platonic point of view, that they are different instantiations of the same mathematical form. As mentioned earlier, the divided line triangle has an initial division that can be reduced to a ratio of the form $1 : n^2$ (e.g. 1 : 2, if it is desired to have the final proportion expressible in whole numbers where $n^2 = (\sqrt{2})^2 = 2$)—while the nuptial number has a base ratio of 9 : 16. While both require that the numerical values for their segments be in continuous geometric proportions, there is a major difference on the quantitative level between the two triangles. My divided line triangle uses the lowest whole number combination that produces the proportion of $a : b :: b : c$ for the triangle's base and gives the $1 : b^{1/3} : b^{2/3} : b$ proportion for the zig-zag structure. The general form of the base ratio of the triangle for finding a cube root is $1 : b$, where b is the number whose cube root is being sought. Using this lowest whole number combination for the $1 : b^{1/3} : b^{2/3} : b$ proportion, the triangle will give the solution for finding the cube root of $2\sqrt{2}$, which is $\sqrt{2}$. While the whole number value of the base adds up to 9, reducing the ratio to its lowest form gives $1 : \sqrt{2} : 2 : 2\sqrt{2}$, where the $\sqrt{2}$ is the second term of $1 : b^{1/3} : b^{2/3} : b$.

A major limitation of the divided line triangle is that it only gives a specific case of finding a cube root and, although other specific cases can be found (such as for the cube root of eight, it cannot be used in general to find cube roots. However, it does provide for the general form of a triangle giving the 2M2E proportion, which plays a foundational role in the Greek quest for a solution to the problem of doubling the cube and cubic equations in general. The nuptial number triangle may well be an early attempt to express 2M2E geometrically in a way that that could be used to find any cube root if the use of *neusis* or conic sections is accepted as a legitimate method for the construction of the desired line.

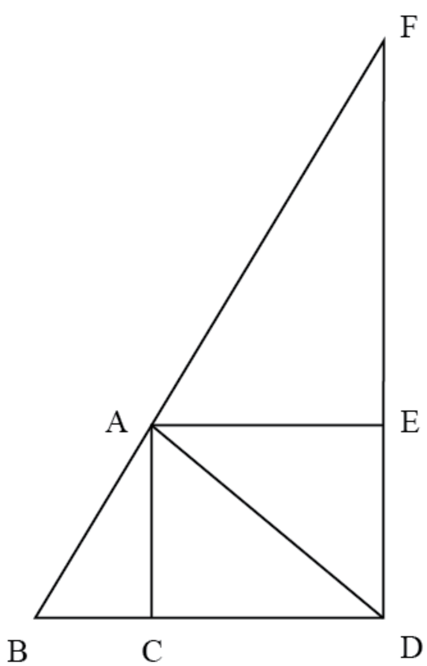

Figure 5.4.1: Building the Nuptial Triangle

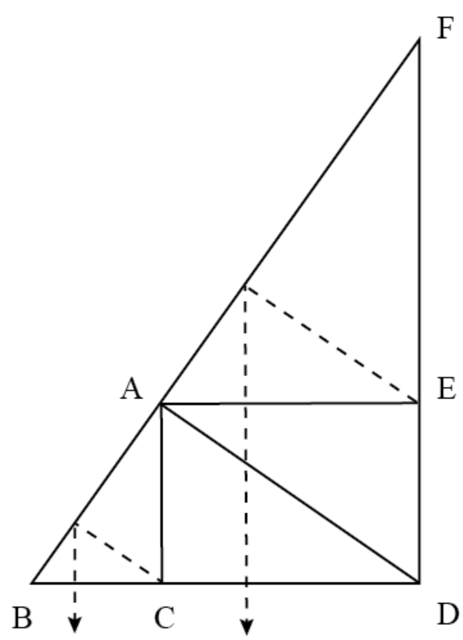

Figure 5.4.2: Projecting Back and Forth

The geometrical mean is the core relationship of both diagrams, both in the constructions and in the numerical relations the diagrams depict.[21] The nuptial number triangle is based on two manifestations of the geometric mean construction. The divided line triangle has both manifestations and multiple others as the proportion is projected back and forth to finally divide the base according to Plato's instructions (see Figures 5.4.1 and 5.4.2). There is a great deal more to say about the relations among the various parts of this triangular structure in connection with ancient solutions to the problem of doubling the cube, showing that the structural relations inherent in the divided line and nuptial number triangles are more significant than their context in the *Republic* might imply—but that project would take us beyond Plato's texts.

Our understanding of the significance of the divided line is enhanced by understanding the construction required to divide the line in the given proportion using geometric methods. As I argued in chapter 4, the division itself necessitates the construction of a two-dimensional figure in order to utilize properties of triangles, parallel lines, and proportion theory. It can readily be seen that the construction gives rise to a series of geometrical means that are the cornerstone of the given proportion.

But what about philosophical significance? Or even the significance for ruling a city, if we return to Plato's text—beyond simple and instrumental, tactical, or strategic calculations? As Burnyeat puts it provocatively,[22] "How, we may ask, will knowing how to construct an icosahedron (Figure 5.4.3) help them when it comes to regulating the ideal market or understanding the Platonic Theory of Forms?"

[21] *Elements* II.14, VI.8, and VI.13.

[22] Burnyeat (2000, 2), including my Figure 5.4.3 that is his Figure 1. Burnyeat's argument that mathematics is a good in itself that is good for the soul is a remarkable step forward for the topic of Platonic mathematics as it was conceived before his widely appreciated article.

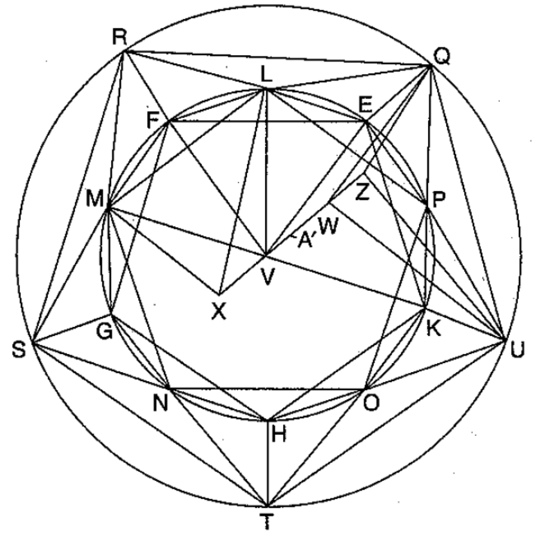

Figure 5.4.3: The Icosahedron

5.5 Higher Education

The curriculum for the guardians that Plato describes between the divided line at the end of *Republic* 6 and the nuptial number at the beginning of *Republic* 8 illustrates the use of higher education to escape from the cave, to escape ignorance. This is not "putting knowledge into souls that lack it, like putting sight into blind eyes" (7.518b9–c2), but the turning of both mind and body toward the light (7.518c4–d1). The curriculum itself, as I will show, is an intriguing clue to the coherence and importance of Plato's geometrical project as an all-encompassing project in epistemology and ontology.

All children in the ideal city of Plato's *Republic* learn arithmetic and geometry—prerequisites to dialectic—through play (compulsion being anathema to free individuals, 7.536d4–e3);[23] but those men and women later chosen to guard the city enjoy a ten-year period

[23] Greek education in the fifth century was not so child-friendly; and Burnyeat (2000, 26) notes that, in the *Laws,* 819a–c, the idea of playfully presenting mathematics was regarded as anti-Greek, an import from Egypt—which may be an implicit rebuke of the methods of Theodorus (in the *Theaetetus*), whose city of origin, Cyrene (in present-

from the age of twenty, studying higher mathematics to gain "a synoptic view of the relatedness of these subjects, both to each other and to the nature of things as they truly are" (7.537c1–3). The subjects are, in order, arithmetic, plane geometry, stereometry, astronomy, harmonics, and dialectic.[24] For my purposes, the focus should be on stereometry because of its role in demonstrating that, for Plato, some *diagramma* are not mere illustrations to be judged by the extent to which they *clarify* a set of claims, but themselves constitute proofs. This is my chief departure from Brumbaugh's otherwise sympathetic appreciation of Plato's use of diagrams. Whereas he says, "the purpose of a mathematical image in Plato's writing is to schematize and clarify dialectical relations in the context, and if it fails to do so, it is dialectically nonfunctional" (1954, 112), I maintain that diagrams can sometimes *advance* the dialectic themselves. That focus will take us into my final chapter; here I will do little more than introduce stereometry.

Plato emphasizes that there is a discipline between geometry and astronomy, where 'geometry' refers specifically to plane geometry. Socrates tells Glaucon that "solids in rotation come after plane surfaces" but corrects himself, emphasizing instead: "solids in and of themselves. The correct order is to go on from two dimensions to three, which presumably takes us to where we find cubes, and anything that has depth" (7.528a9–b2). That is, the discipline after plane geometry is stereometry, solid or three-dimensional geometry. And it is clear from Glaucon's response that the study of stereometry is relatively undeveloped.

Indeed, stereometry has a number of additional difficulties not faced by plane geometry. Diagrams have the limitation of being confined to two dimensions. The typical constructions

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day Libya), worshiped the Egyptian god Ammon. Plato was rather ahead of his time in realizing the pleasures of learning mathematics.

[24] Fowler (1999, 290) points to *Laws* 7.817e–820e (embedding the passage cited in the immediately preceding note) where a dumbed-down but similar curriculum is a source of shame to Greeks.

using straightedge and compass do not translate directly for use in three dimensions. This limitation is overcome by making cuts in 3-dimensional space that yield 2-dimensional figures—enabling geometers then to work with compass and straightedge—a procedure in some respects similar to modern co-ordinate geometry. However, working with cubic equations and finding cube roots is a necessary component of stereometry, and these cannot be accomplished with compass and straightedge alone, presenting a challenge not fully addressed until the sixteenth century.

There can be little doubt that the mathematical significance of the nuptial number passage is related to the construction of 2M2E in virtue of the numerical values of its proportion. While it is said that the problem can be solved using conic sections,[25] this is not entirely true. Conic sections cannot be directly drawn without the aid of some mechanical device. The ancient Greeks had to rely on step-by-step constructions of points to plot one of these curves. In the case of the hyperbola, this was accomplished via the construction of a series of rectilinear figures with equal areas but different dimensions based on two ordinate lines. These constructions also utilize the method of constructing geometrical means, which are intrinsically related to the construction of the divided line and nuptial number triangles.

The triangles for the divided line and nuptial number bear a similarity to at least a part of the geometric figures later utilized in efforts to double the cube. They are also closely related to the *Meno*'s third mathematical example (see chapter 2), that is, to the application of areas for determining whether it is possible to inscribe a given triangle in a circle with a given diameter.

[25] See, for example, Heath (1981, 1, 300).

CHAPTER 6:

MATHEMATICAL FOUNDATION AND THE *TIMAEUS*

6.1 Introduction

The most prominent mathematical passages in Plato's dialogues appear in *Meno*, *Theaetetus*, *Republic,* and *Timaeus.* While there are notable differences in how Plato utilizes his mathematical examples, all share a common characteristic: they involve geometric figures whose shapes are largely determined by specific ratios or proportions that usually include irrational magnitudes. As this commonality involves quantitative relations that straddle the Greek mathematical disciplines of arithmetic and geometry, both plane and solid, I refer to the relations collectively as proportional magnitudes.

The account in the *Timaeus* of the construction of the world by a craftsman or demiurge (δημιουργός, 28a6) involves aspects of arithmetic, geometry, harmonics, and astronomy—familiar from the curriculum of *Republic* 7—whose interconnections are far from clearly explained when approached from the perspective of proportionality and proportional magnitudes. The *Timaeus* is set dramatically immediately after the *Republic*, effecting a broad sweep from individual to *polis* to *cosmos.* The dialogue's account is qualified famously soon after it opens when the Pythagorean, Timaeus of Locri, tells Socrates:[1]

> Don't be surprised then, Socrates, if it turns out that we won't be able to produce accounts on a great many subjects—on gods or the coming to be of the universe—that are completely and perfectly consistent and accurate. Instead, if we can come up with accounts no less likely than any, we ought to be content, keeping in mind that both I, the speaker, and you, the judges, are only human. So we should

[1] I use the Zeyl (1997) translation, based on the Burnet OCT.

> accept the likely tale [τὸν εἰκότα μῦθον] on these matters. It behooves us not to look for anything beyond this.—Zeyl, trans. (29c4–d3)

The Pythagorean goes on to narrate the entire *Timaeus*.[2] I suggest that the notion of proportional magnitudes provides the structural foundation that links the various mathematical elements that are utilized in all four dialogues, and that this accords with Plato's claims about the significance of proportions in general in his dialogues.[3]

I argue that another point of commonality among the examples is that they draw attention to material for the exploration of various aspects of plane geometry useful for expanding the reasonably well-developed geometric science of two dimensions into a science that works in three dimensions. The critical need for such a move is expressed in the *Republic* (7.528a–e) when Plato's Socrates discusses the characteristics of each of the mathematical sciences: research on a science of stereometry requires a director if any discoveries are to be made on the subject, but finding a director for such a project would be difficult, and "even if he could be found, those who currently do research in this field would be too arrogant to follow him" (528b8–c1). In the absence of an organized research project with a competent director, Plato may

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[2] As a result, "a likely story" has been used as shorthand in many accounts of the dialogue's creation myth—where the term *muthos* is the same as in the myth of the metals in the *Republic*—usually denoting irony, but sometimes downright dismissal. At *Timaeus* 56a1, however, εἰκότα λόγον is used, suggesting the stakes have been raised during the discussion. Johansen (2004, 62–64) provides a plausible explanation for Plato's Timaeus's use of *muthos* and *logos*. As on so many aspects of the *Timaeus*, Broadie (2012, 31–38) offers a thoughtful commentary on the very possibility of a human natural science.

[3] It may not be obvious why proportion should play such a key role in Plato's philosophy as a whole and, while the details of that role are beyond my scope, a few indications may be useful. The Platonic forms that scholars have found most difficult to associate with mathematics (*pace* Burnyeat 2000) are the virtues, but the moderation that characterizes all the citizens of the ideal city in the *Republic* is a good place to begin: theirs is a proper proportion of reasoning, spirit, and appetites. Justice might likewise be viewed as proportional reciprocity: fairness in distribution of benefits in proportion to each person's talents, abilities, and actions. Courage (*Laches* 198b) results from balancing fear and confidence proportionately (which Aristotle will develop into his doctrine of the mean in *Nicomachean Ethics* 2.6–8). Beauty is defined by proper proportions, so spheres are beautiful because every point on the surface is equidistant from a single point. I allude to a few examples in 6.3.

have included in his dialogues certain mathematical examples carefully chosen to help stimulate research in the right direction by marking the significance of mathematical notions for founding such a project. Plato's role as a driving force behind the development of mathematics and science in his era may be disputed, but the specific elements of mathematics he chooses to include in his dialogues suggest an awareness of the path to greater generalization that mathematics did eventually follow, freed from the restraints of reference to the physical world for the objects of its study.

In the preceding chapters, I have examined the mathematics behind each of these examples by looking at how diagrams can be constructed according to the rules of Greek geometry. The constructions that produce these diagrams provide essential mathematical information for researchers wishing to understand the common foundations of plane and solid geometry. In what follows, I show how three primary mathematical notions from the *Timaeus* involving the bonding proportions of the body of the cosmos (31b4–32c4), the world soul and harmonics (35c2–36b5, 43c7–d6), and the Platonic solids (53c4–55c6), are connected to the mathematics of the other three dialogues and thereby give us insight into the depth of Plato's understanding of mathematics.[4] All three point to the structural function that the notion of proportional magnitude plays in constructing the parts that make up the whole of the cosmos as well as the interrelations of mathematical concepts contained in them.

[4] Neoplatonists—some of whom were also neopythagoreans—were fascinated by *Timaeus* and contributed early commentaries that are worth our attention today; of particular note are those by Plutarch and Proclus. After centuries of relative neglect, contemporary mainstream philosophers have begun to recognize the importance of Plato's use of geometry in the *Timaeus*: Johansen (2004), a few chapters in Mohr and Sattler (2010), Broadie (2012, especially her "The Geometrical Account,"), and Zeyl and Sattler (2017, especially in their section on physics) have all made progress and are introduced in their turns below, though there is still work for me to do.

6.2 From Planes to Solids

In his evaluation of the mathematical disciplines in *Republic* 7, Plato mentions the underdeveloped state of the science of stereometry (solid geometry) that should come after the geometry of plane surfaces and prior to the study of the *motion* of solids. The mathematical science of astronomy as discussed in *Republic* is not a science directly concerned with celestial spheres visible to the senses, but a science of the geometrical relations thought to underlie their apparent motion in the sky.[5]

Greek mathematics in the time of Plato did not reach the level of generalization provided by algebra and analytical geometry that has proved be so useful in modern mathematics. Multiplication of two and three factors was approached as the length, width, and depth of planes and solids. Squared and cubed numbers were conceptualized in the same way in relation to two- and three-dimensional objects; by contrast, we now have a general form of exponentiation that covers a wide variety of cases independent of spatial reference. In a similar manner, proportions and ratios fulfilled some of the roles in ancient Greek mathematics as quadratic and cubic equations. Work with quadratic equations involving irrational magnitudes could make use of geometrical constructions, but since the compass and straightedge limit constructions to equations of the second degree, such constructions could not be used directly (that is, without also involving curves) to solve cubic equations.

The reliance on constructions by compass and straightedge alone made the move from a mathematics of two dimensions to that of three dimensions more of a challenge than the previous

[5] Fowler (1999, 117–25) traces the move from *Republic* 7 to the *Timaeus* in his discussion of the study of astronomy in Plato's Academy. Noting Plato's view that astronomy and music theory are 'kindred sciences', Fowler goes on to do the same for the study of music theory in the Academy (125–48).

move from an arithmetic of rational numbers to the realm of plane geometry with irrational magnitudes (as the latter could be generally subsumed under the limitations of such constructions). The problems of doubling the square and doubling the cube exemplified the challenges of these moves, as they involve finding square and cube roots respectively, with the latter not directly solvable by compass and straightedge constructions. Whatever the historical timeline, the development of stereometry appears to have been more of a challenge than working with irrational quantities.

6.3 Mathematical Examples in the Dialogues

Burnyeat (2000), mentioned several times already, campaigns on the view that Plato's mathematical preoccupations have two chief causes—one ontological and the other epistemological: (i) mathematics provides the unique structure of the whole universe, right down to human action and invisibly small particles; also, (ii) the study of mathematics leads the intellect to an appreciation for structure, order, truth, beauty, and the forms generally. That seems right to me, but I am operating here at the more focused level of *diagramma,* how diagrams, and especially diagrams qua proofs, operate in the dialogues to effect the epistemological goal.

There are two main reasons Plato brings diagrams into the dialogues. One is to illustrate philosophical notions, and the other is to reflect what he takes to be the nature of the real world. The examples in the *Meno* and *Theaetetus* illustrate the structure of virtue qua knowledge. The examples in the *Republic* and *Timaeus* move on to Pythagorean physics and metaphysics, addressing ordering principles and claims about the world. As we have seen already, the divided line passage in the *Republic* describes the structure of the world and mind: levels of being or reality, and levels of cognition. The divided line comports well with Burnyeat's notion of the practical use of mathematics in preparing citizens for lives based on a clear understanding of the well-ordered individual and the well-ordered city. In *Timaeus*, a possible structure of the cosmos is described utilizing such notions as the construction of the cosmos and the world soul, and the harmony of the celestial spheres.[6]

The guiding mathematical idea behind the structure of the cosmos in Plato is the notion of proportion (ἀναλογία), and proportions are treated diagrammatically. Plato's emphasis on proportions is of course not unique to the *Timaeus,* though he is clear when he says that proportions are the *best* bond for accomplishing the unity of the cosmos and the elements of which it is constituted (*Timaeus* 32c2–4). We have already seen that proportion is used *passim* in the *Republic,* but most of Plato's sustained invocations of the concept of proportion are concentrated in the group of late dialogues, of which *Timaeus* is one.[7]

An exception to the appearance of discussions of proportion in late dialogues is the

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[6] Aristotle did not consider mathematics a cause or foundational principle (as he makes especially clear in *Metaphysics* M–N). For him, mathematical objects and numbers are *properties* of certain things, not themselves substances. Hence, like Plato, he makes use of mathematics to clarify or illustrate philosophical concepts and distinctions; but unlike Plato, he does not also use it for metaphysical purposes.

[7] At *Sophist* 228a10–d11, disproportion (ἀμετρία) describes both ugliness and wrongdoing. The passage is particularly interesting as an introduction to what Aristotle will later develop in his ethical works as an explanation for wrongdoing: missing the mark. The *Statesman* opens with remarks about proportion (257b2–4) and sometimes appears to use 'measure' as a synonym for it (see 283c3–d1). In the *Philebus*, Socrates says, "if we cannot capture the good in one form, we will have to take hold of it in a conjunction of three: beauty, proportion [συμμετρία], and truth. Let us affirm that these should by right be treated as a unity and be held responsible for what is in the mixture, for its goodness is what makes the mixture itself a good one" (65a1–5). In the *Laws*, dialectical conversation is almost completely abandoned, leaving the Athenian free to make such assertions as "What is equal is equal and what is proportional is proportional, and this does not depend on anyone's opinion that it is so, nor does it cease to be true if someone is displeased at the fact" (2.668a); "citizens must be esteemed and given office, so far as possible, on exactly equal terms of 'proportional inequality', so as to avoid ill-feeling" (5.744c); and "Any just action we do has the quality of being 'good' roughly in proportion to the degree to which it has the quality of justice" (9.859e).

Gorgias, to take an example from an intense dispute about justice and injustice, where Socrates says,

> Yes, Callicles, wise men claim that partnership and friendship, orderliness, self-control, and justice hold together heaven and earth, and gods and men, and that is why they call this universe a *world order*, my friend, and not an undisciplined world-disorder. I believe that you don't pay attention to these facts, even though you're a wise man in these matters. You've failed to notice that proportionate equality has great power among both gods and men, and you suppose that you ought to practice getting the greater share. That's because you neglect geometry.
> —Zeyl, trans. (507e7–508a7)

Returning now to the *Timaeus,* the relations with which the demiurge sought to endow the cosmos were undoubtedly as mathematically precise as possible: "the things we see were in a condition of disorderliness when the god introduced as much proportionality into them and in as many ways—making each thing proportional both to itself and to other things-as was possible for making them be commensurable and proportionate" (69b2–5). As I show below, the notion of mathematical proportion in Plato covers a set of related concepts that were later generalized by algebraic equations (perhaps, unfortunately, at the expense of some of the uniqueness presumed for pre-modern mathematics).

Plato's inclusion of the five regular polyhedra or Platonic solids in the *Timaeus* is confirmation of his interest in solid geometry and the development of the science of stereometry as a stepping stone on the way to the mathematics of the science of astronomy, which Plato characterizes as the study of solids in motion. The ultimate solid in Platonic mathematics is the sphere, with its most excellent and complete structure that gave form to the cosmos and the world soul (33b4–7, 62c8–63a4).

Spheres are of course central to the study of astronomy, as the position and motion of the planets and stars were plotted, as they still are today, in relation to the interior of a sphere projected onto the sky. Each Platonic solid can be inscribed so that its vertices touch the interior of a sphere. In the concluding book of Euclid's *Elements*, all five are analyzed in relation to a sphere determining the ratio of side to diameter of each. The study of such ratios and the interior architecture provided by the construction of the solids within a sphere can be seen as a precursor to the development of celestial coordinate systems used to study the positions of the heavenly bodies in relation to the earth.

6.4 Proportional Magnitudes

A common feature of the examples I have examined from the *Meno*, *Theaetetus, and Republic* (as well as the ones I will introduce in the *Timaeus*) is that all involve in some way the notion of proportion. I introduce the term 'proportional magnitude' as a convenient way to refer a class of mathematical relations used in ancient Greek mathematics that does not map directly onto a specific type of relations in modern mathematics. It includes ratios (designated by the ':' symbol), proportions (designated by the '::' symbol), means (three-term ratios[8] of the type $a : b : c$), and certain progressions or sequences whose members can be determined by a formula. While these relations can generally be expressed by algebraic equations, the class of mathematical objects I refer to as 'proportional magnitudes' often have a geometric component that eludes reduction to algebraic expression. An example of this would be the relationships contained in the square root-generating spiral diagram in *Theaetetus*, where the shapes of the triangles and the magnitudes of their sides ($1 : 1 : \sqrt{2}$, $1 : \sqrt{2} : \sqrt{3}$, $1 : \sqrt{3} : \sqrt{4}$, etc.) form a sequence similar to Pythagorean triplets but using irrational magnitudes, for which there is no modern equivalent.

[8] Strictly speaking, a ratio is a relation between two terms.

The notion of proportional magnitudes allows me to avoid the conceptualization of a "geometric algebra" that has been embraced by many scholars working on ancient Greek mathematics since Zeuthen (1896) pioneered this approach. While translating the geometric concepts and methods into algebraic equations is useful for understanding aspects of Greek mathematical thinking for the modern reader, the language of ratio and proportion in conjunction with diagrams based on the methods of geometric construction can offer more accurate explanations. According to van der Waerden (1988, 119), a good portion of the theory of polygons and polyhedra, and the whole theory of conic sections, depends on the methods of *geometric algebra* with Theaetetus, Archimedes and Apollonius "perfect virtuosos on this instrument." If 'proportional magnitudes' is substituted for 'geometric algebra', then the significance I place on proportions in Plato's mathematical examples gains further support.

My notion of proportional magnitude is not as strictly defined as, say, the operations that link the terms of an algebraic equation, as it can include the Greek notion of figurate number, which arranged unit markers ("pebbles") to classify numbers according to geometric shapes—sometimes called arithmogeometry (numbers set out in spatial patterns according to a formula). A variation is the so-called lambda arrangement of musical ratios in the *Timaeus*—mimicking the shape of the Greek uppercase Λ. According to Fowler (1982, 157), the notion of ratio is not well defined anywhere in the surviving corpus of Greek mathematics, with Euclid giving only a vague description of the word in *Elements* V.3 (Definition). Euclid's definition of proportion (V.5) as a relation among four magnitudes can be seen as circular in that the relation was generally viewed as the *equality* of two ratios that are themselves ill defined. Proportions in Plato can, at least in a broad sense, cover any analogous relation of the form "as A is to B, so C is to D" that could be interpreted in mathematical proportions as A : B ∷ C : D.[9]

Another distinguishing feature of a proportional magnitude is that it involves two or more terms or quantities in a specific relationship, rather than just a single term or quantity such as a number (whether the ancient Greek or the modern notion of number). Whereas in modern mathematics $\sqrt{2}$ is considered an independent entity (a real number), it would appear in the context of proportional magnitudes as part of the geometric mean ratio (or progression) of $1 : \sqrt{2} : 2$ or in the side-to-diagonal relation of a square as the constant $1n : \sqrt{2}n$ ratio.

The three Pythagorean means—the geometric, arithmetic, and harmonic—also fall within the class of proportional magnitudes. They play a significant role in Greek musical theory, giving rise to the special four-term musical proportion that Nicomachus (1.5) called the "most perfect" proportion.[10] But the geometric mean progression is the most prominent example of a proportional magnitude, and it was used extensively in Greek mathematics. It makes an appearance in the three- and four-term progressions that Plato considered important in how things that make up the world are bound together in the account given in the *Timaeus.* While there is some ambiguity in Plato's account among the musical proportion, the four-term geometric mean progression, and how the four elements (earth, air, fire, and water) are bound together in their polyhedral form, the passages already cited emphasize that proportion is a foundational element in the creation of the world and our understanding of it.

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[9] The pattern can be used in decidedly non-mathematical circumstances as well. In the *Gorgias,* Socrates develops an elaborate analogical argument in conversation with Polus, comparing things as the geometers do, by setting out that, as cosmetics is to gymnastics, pastry baking is to medicine, and as sophistry is to legislation, oratory is to justice.

[10] Nicomachus of Gerasa (60–120 CE) is a towering figure in mathematics, noted for his *Introduction to Arithmetic*; he is known to the West by the better-known philosopher, Iamblichus of Syria (±245–325 CE), more of whose work is extant. The proportion is also known as the di-diapason or quadruple ratio.

The propositions and theorems of proportion found in various books of Euclid's *Elements* would have served a similar purpose for Greek geometers as that served in modern mathematics by algebraic equations, formulae, and algorithms. What Plato's mathematical examples show is that the notion of proportion already played a foundational in Greek mathematical thinking prior to the canonization of a general theory in *Elements* V that is credited to Eudoxus.

6.5 Mathematics in the *Timaeus*

The passage on the proportions that bind the elements of the body of the cosmos together (31b4–32c4) involves the two proportions (1M2E and 2M2E) discussed in earlier chapters in relation to doubling the square, doubling the cube, and the geometric mean triangle.

The ratios and proportions that generate the harmonic aspect of the world soul (35c2–36b5, 43c7–d6) provide the prime example of the correlation between numbers and the physical world (sound frequencies and string lengths) that inspired the Pythagorean belief that all is number or all is like number—depending on the source.[11] It contains the formula generating a sequence of scalar octaves based on interlocking sequences of mathematical progressions that is

[11] Aristotle (*Metaphysics* 985b23–986a8) is most often cited for the explanation of the Pythagoreans' belief, reduced to the familiar slogans in my text: "The Pythagoreans, as they are called, devoted themselves to mathematics; they were the first to advance this study, and having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being—more than in fire and earth and water (such and such a modification of numbers being justice, another being soul and reason, another being opportunity—and similarly almost all other things being numerically expressible); since, again, they saw that the attributes and the ratios of the musical scales were expressible in numbers; since, then, all other things seemed in their whole nature to be modelled after numbers, and numbers seemed to be the first things in the whole of nature, they supposed the elements of numbers to be the elements of all things, and the whole heaven to be a musical scale and a number. And all the properties of numbers and scales which they could show to agree with the attributes and parts and the whole arrangement of the heavens, they collected and fitted into their scheme; and if there was a gap anywhere, they readily made additions so as to make their whole theory coherent."

not widely recognized as what it is: a fascinating and complex example of arithmogeometry. As the lambda diagram discussed below will show, proportional magnitudes were not seen as independent entities, but as part of an interrelated complex of mathematical objects arising from numbers, and linked by spatial relations.

Finally, the construction of four of the Platonic solids (53c4–55c6) can be linked to Theaetetus's work on irrationals and the regular polyhedra that appear in Euclid's *Elements* X and XIII, respectively.

6.6 The Bonding Proportion 31b4–32c4

The proportions that bind the elements (earth, water, air, and fire) of the body of the cosmos together are described at 31b4–32c4.[12] The passage begins with the claim that, to be properly combined, two things require a third. That is, they require some kind of bond to unite them. The best bond is one that unifies the parts into a true whole or unity, and the best way to accomplish this is through the use of proportion. The basic proportion for linking three numbers requires that the middle term is such that, what the first term is to the middle, the middle is to the last ($a : b :: b : c$); and, conversely, the middle is to the first as the last term is to the middle. In other words, the middle term becomes both the first and last term, and the first and last terms become the middle terms ($b : c :: a : b$). If all the terms maintain the same relationship to each other, they are unified as a continuous geometric proportion, which I have designated the 1M2E proportion. The relation between the terms is such that the middle term is the geometric mean or mean

[12] Fowler (1999, 387) uses this passage to argue that Plato did not use 'elements' (τὰ στοιχεῖα) in the sense later deployed by Euclid and other writers in the Greek mathematical style known as *protasis,* on which see Netz (1999, 252–61). Klein (1968, 69–79) argues at length that Plato regarded *mathematics* as the bond.

proportional $(ac = b^2)$. This proportion is intended to show the link between two square numbers and can be written as $a^2 : ab :: ab : b^2$.

The lowest integer progression that fulfills Plato's specification ($a^2 : ab :: ab : b^2$) for the two dimensional plane is 1, 2, 4.[13] Given as the proportion 1 : 2 :: 2 : 4, the first term is to the middle term as the middle term is to the last term, and vice versa. This is a continuous proportion where the middle term is the geometric mean of the extreme terms. The extremes are plane or square numbers ($1 = 1^2$ and $4 = 2^2$). The ratios remain the same even when rearranged as in 2 : 1 :: 4 : 2 so that the middle term becomes the first and last terms, and the first and last terms turn out to be middle terms, although, in this case, the middle terms are not the means of the two extremes.

As the body of the world is three-dimensional, its components must be represented by four terms (32b–c). The formula requires two solid or cubed numbers for the extreme terms and two means as middle terms ($a^3 : a^2b :: a^2b : b^2 a :: b^2 a : b^3$). I have categorized this as a 2M2E proportion but, as we have seen, it is formed from two sets of the geometric means of the 1M2E type ($a^3 : a^2b :: a^2b : b^2 a$ and $a^2b : b^2 a :: b^2 a : b^3$).

The lowest examples in whole numbers is 1 : 2 :: 2 : 4 :: 4 : 8, in which the two sets of the 1M2E proportions can be clearly seen as 1 : 2 :: 2 : 4 and 2 : 4 :: 4 : 8. The two geometric means between 1 and 8 are easily determined as the sequence progresses by doubling the terms. However, in most cases, finding the two means turns out to be problematic. The simple case above uses numbers that are perfect squares and perfect cubes, so that the answer is immediately obvious. For all other cases, a method for solving cubic equations would be required; the Greeks

[13] It is also the lowest integer progression that fits the divided line formula (which is a proportional magnitude of the same type, i.e., $a^2 : ab :: ab : b^2$).

had no such method.

Plato applies this same proportion in a qualitative manner in the theory of the four elements.[14] He sets out the four elements of fire, water, air, and earth in a sequence similar to the 2M2E proportion, with water and air analogous to the two means in a quantitative proportion. This gives the proportion "fire : air ∷ air : water ∷ water : earth" which can be also reduced to a continuous proportion of fire : air : water : earth. He also gives the formulations "fire : air ∷ air : water" and "air : water ∷ water : earth" which mirrors the two sets of 1M2E shown above. The demiurge constructs the sensible universe by fusing together these four constituents of the body of the world, making it a "symphony of proportion" (32c2).

This kind of qualitative application of a four term proportion is prominent in a later passage recapping the divided line in the *Republi*c (534a3–5), where the relation *noesis* : *dianoia* : *pistis* : *eikasia* is given. In addition, in their respective ratios, *epistēmē* and *dianoia* correspond to being (in the being : becoming ratio), while *pistis* and *eikasia* correspond to becoming. The proportions become interlinked as being : becoming ∷ knowledge : opinion and knowledge : opinion ∷ *noesis* : *pistis* ∷ *dianoia* : *eikasia*. The divided line formula adds another layer of complexity to this relationship as the specifications for the relations between the initial four terms $a : b : c : d$ yield the $a^2 : ab :: ab : b^2$ type of proportion.

6.7 The World Soul and Harmonics 35c2–36b5, 43c7–d6[15]

The musical proportion in this example of Plato's can be categorized as a 2M2E proportion,

[14] This is well known, uncontroversial, and explained in virtually all notes and commentaries on Plato's text.
[15] Baltzly 2009, Introduction.

though it utilizes all three of the Pythagorean means (geometric, arithmetic, and harmonic) rather than (as in the bonding proportion) two geometric means. This variation provides strong evidence for why a foundational role was ascribed to the 2M2E type of proportion in the previous passages. The interlocking sequences of ratios and numerical progressions generated by the musical proportion more than justified the appellation of "most perfect" proportion used by Nicomachus and later by Iamblichus.

The first step of setting out numbers in the musical proportion used by Plato is based on squares of numbers in geometric progressions based on the geometric mean. The generation of the sequence is given as dividing one portion away from the whole, the first; then a second one twice as large; followed by a third that is one-and-a-half times as large as the second, and three times as large as the first. The fourth portion is twice as large as the second ($4 = 2^2$); the fifth, three times as large as the third ($9 = 3^2$); the sixth, eight times that of the first ($8 = 2^3$); and finally the seventh, twenty-seven times that of the first ($27 = 3^3$). These can be arranged in a line in the order given—1, 2, 3, 4, 9, 8, 27—or as a sequence of the powers of two and three in a triangular configuration that has become known as the lambda diagram,[16] as shown in figure 6.7.1. Both legs of the triangle are progressions of geometric means as well as progressions of the powers of two and three.

After this, the double (n^2) and triple (n^3) intervals were filled by two middle terms, one determined by finding the harmonic mean with the formula $2ab/(a + b)$, and the other by the arithmetical mean that is equidistant from each extreme given by the formula $(a+b)/2$. This would give the basic form of the musical proportion that could be expressed with E_1 and E_2

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[16] The lambda diagram is credited to Crantor (±335–275 BCE) in the *Timaeus* commentaries, according to Plutarch (*Moralia* 13.70 1027d). There is a large contemporary mathematical literature on the lambda triangle. For a recent assessment, with references to previous studies devoted to Plato's *Timaeus,* see Shannon and Horadam (2002).

representing the extremes, HM the harmonic mean, and AM the arithmetical mean:

$$(1)\ E_1 : \mathrm{HM}(\text{of } E_1 \text{and } E_2) :: \mathrm{AM}(\text{of } E_1 \text{and } E_2) : E_2$$

When a proportional magnitude with 1 : x : 2 as its base form gives an arithmetical progression (so that the arithmetical mean x = 3/2), the strings of corresponding lengths give the ground-tone, descending fifth, and descending octave. When 1 : x : 2 gives a harmonic progression where x = 4/3, the strings give the ground-tone, descending fourth, and descending octave, giving the basic musical portion as 1 : 4/3 : 3/2 : 2.

This process continues, giving various combinations of the intervals of 3 : 2, 4 : 3, and 9 : 8 within the previous intervals, and finishes by filling in all the 4 : 3 intervals with 9 : 8. There remains a small portion as each tetrachord is formed with the ratio of 256 : 243. From this mixture, individual octaves with extremes in the ratio of 2 : 1 are eventually completely filled (35b4–36b5). The ensuing set of ratios for the intervals in the world soul is given as:

(2) 9 : 8, 9 : 8, 256 : 243, 9 : 8, 9: 8, 9 : 8, 256 : 243

This gives the musical intervals for an octave of the Dorian scale, consisting of two tetrachords linked to a 9 : 8 interval.

Ancient musical theorists preferred to work with whole numbers, so they first multiplied

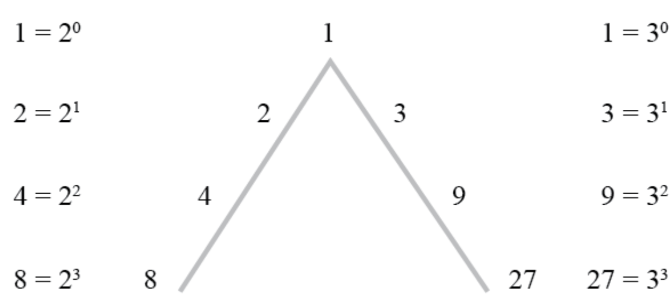

Figure 6.7.1: Lambda Diagram

each 1 : 4/3 : 3/2 : 2 by six to clear the fractions; 6 and 12 are the lowest pair of integers that can work as double extremes, producing the numerical values of the musical proportion as 6 : 8 ∷ 9 : 12. This would not suffice for the filling out an octave with integers, so the lowest value of the progression was set by later commentators at 384. This allowed the four-octave-plus-a-sixth range to be generated by expanding from the initial ratios of the original seven numbers given by Plato to be realized[17] and gives the sequence (2) above as:

(3) 384 : 432 : 486 : 512 : 576 : 648 : 729 : 768

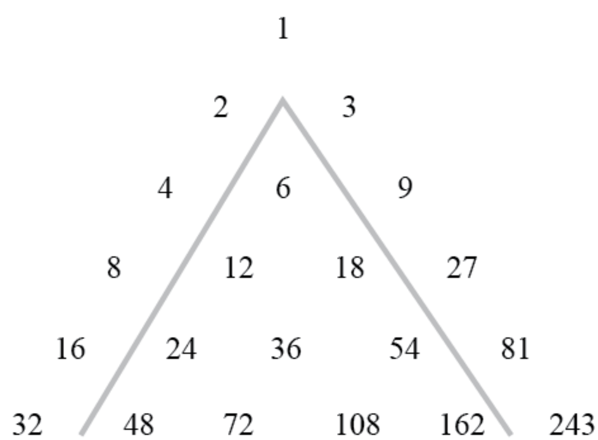

Figure 6.7.2: Expanded Lambda Diagram

The triangular configuration of the lambda diagram is similar in the Pythagorean *tetraktus* that sets out the first four numbers as "pebbles" but it could also show the arrangement for the triangular number ten. When figure 6.7.1 is filled in and expanded by the formula Plato

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[17] According Plutarch (*Moralia* 13.70 1020c) Eudorus followed Crantor in using 384, the lowest number capable of clearing the fractions, making it the first number of the sequence. See also Creese (2010, 265–66) for 384. Setting 384 as the lowest number would also allow the sequence to be extended using only integers for twelve octaves in a cycle of fifths.

provides, a pyramid of numbers similar to Pascal's triangle is formed (Figure 6.7.2), a similarity that extends beyond their shape when given generalized forms.

This same configuration of numbers became part of Greek arithmetical theory related to

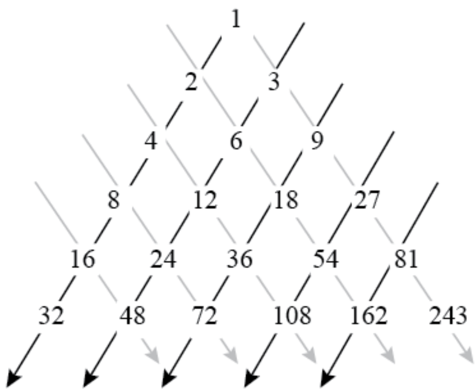

Figure 6.7.3: Geometric Means

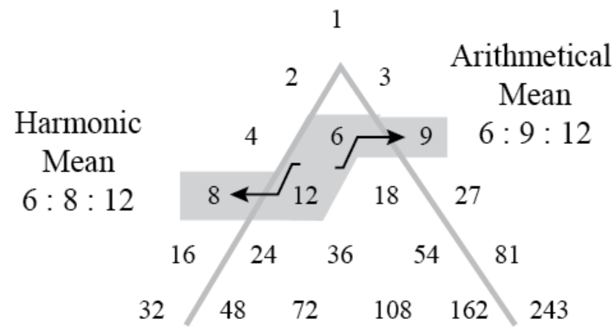

Figure 6.7.4: Link between Means

figurate numbers set out in a table of perpendicular rows and columns (Nicomachus 2.2–3).

The triangular configuration reveals visually that the progressions derived from the arithmetic, geometric, and harmonic means form an interlocking set of patterns (displayed in the figures below). The terms of the arithmetic and harmonic means are generated from two numbers in the left-leaning diagonal column (i.e. geometric means of double numbers). The arithmetic mean is to the right of the top number, and the harmonic mean to the left of the bottom number.

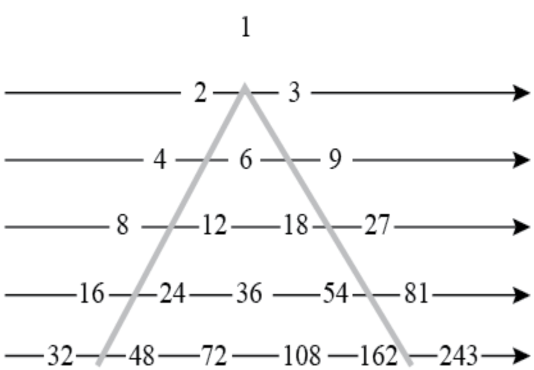

Figure 6.7.5: Harmonic Means

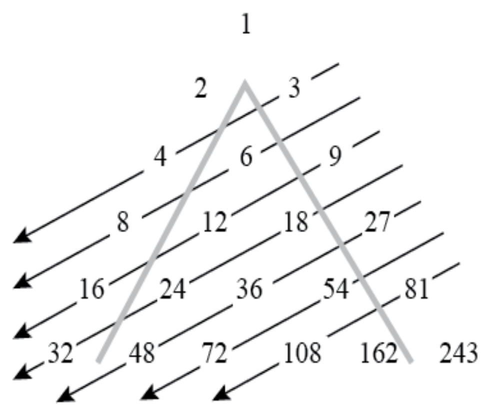

Figure 6.7.6: Arithmetical Means

A sequence of harmonic means forms the horizontal rows that are also ratios for the musical interval of fifths. Similarly, a sequence of arithmetic means is found in the left-leaning oblique columns that are ratios for the musical interval of fourths. The lambda diagram shows the close relationship among the three Pythagorean means and their connection to the naturally occurring intervals of sound frequencies emanating from vibrating strings and other physical objects, providing a prime example of the Pythagorean belief that, as the slogan has it, all is number. The *Timaeus* examples show that Plato understood that the patterns of number could be extended to patterns of irrational magnitudes and to three-dimensional bodies.

The most obvious link is to the two mean and extreme proportions I have been discussing. When reduced to a general form, this interlocking set of patterns reveals an underlying structure related to the bonding proportion. Just as Plato describes the move from the plane to solids with the expansion of the 1M2E ($a^2 : ab :: ab : b^2$) to the 2M2E ($a^3 : a^2b :: a^2b : b^2 a :: b^2 a : b^3$) proportion, so these two can be seen as part of a larger pattern. The Greeks at this time had not conceptualized powers beyond the second and third, but they clearly recognized the sequences of numbers continually multiplied by themselves such as $a^2 : a^2 : a^3 : a^4 : a^5$.

The relation of the generalized lambda triangle (Figure 6.7.7) to that of a generalized form of Pascal's triangle used by Newton for the coefficient of binomial expansion is striking. The abstract lambda triangle does not have coefficients, but the pattern of the variables is exactly the same—despite the fact that the Pascal triangle is made up of a different, but similar sequence of numbers. They differ also in that the abstract lambda triangle is made up of sequences or progressions of terms, while those in the generalized Pascal triangle are mathematical series where the terms are joined by addition. The binomial triangle shows that equations of different degrees also form a pattern. The belief that an investigation into the 2M2E proportion could be based on the properties of the 1M2E proportion was an insight that probably did not anticipate that equations of all degrees were part of the same pattern.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & a & & b & & & & \\
 & & & a^2 & & ab & & b^2 & & & \\
 & & a^3 & & a^2b & & ab^2 & & b^3 & & \\
 & a^4 & & a^3b & & a^2b^2 & & ab^3 & & b^4 & \\
a^5 & & a^4b & & a^3b^2 & & a^2b^3 & & ab^4 & & b^5
\end{array}
$$

Figure 6.7.7: Abstract Lambda Triangle

$$
\begin{array}{c}
1 \\
a + b \\
a^2 + 2ab + b^2 \\
a^3 + 3a^2b + 3ab^2 + b^3 \\
a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{array}
$$

6.7.8: Binomial Expansion Triangle

6.8 Platonic Solids 53c4–55c6

The five regular polyhedra—tetrahedron, octahedron, cube, icosahedron, and dodecahedron—have become known as the Platonic solids.[18] Their inclusion in Greek mathematics is credited to the Pythagoreans, with Theaetetus often named as a major contributor to their study.

Plato's Timaeus tells his listeners that the elements "fire, earth, water and air are bodies. Now everything that has bodily form has depth. Depth, moreover, is of necessity comprehended within surface, and any surface bounded by straight lines is composed of triangles" (53c4–7). He says further that all the triangles needed for the construction of the faces of the polyhedra of the four elements can be derived from two fundamental triangles, each having one right angle and two acute angles. The first is the right-angled isosceles triangle with 45º angles (equal parts of a right angle) at two of its vertices; there is only one kind of right-angled isosceles triangle that is a half square with a constant ratio for sides of $1 : 1 : \sqrt{2}$.

The second is a scalene right-angled triangle that has unequal parts of a right angle at its other two vertices, giving the angles 30º, 60º, and 90º for the whole triangle. There are infinitely many scalene triangles (54a1–2) and the determination that *this* one is the most beautiful and excellent, surpassing all others (54a5–7), was taken to be a positive achievement. The beauty of this particular scalene triangle stems from the fact that it is half of an equilateral triangle. Just as the first triangle was half an equilateral quadrilateral (with perfect symmetry as a plane figure) the second has a similar relation to the equilateral triangle which has only one form and perfect

[18] They have found their way into popular culture. Oliver Sacks, in *The Man Who Mistook his Wife for a Hat,* reports: "Dr. P.'s temporal lobes were obviously intact: he had a wonderful musical cortex. What, I wondered, was going on in his parietal and occipital lobes, especially in those areas where visual processing occurred? I carry the Platonic solids in my neurological kit, and decided to start with these."

symmetry as well. The longer side of this right-angled scalene triangle, when squared, is always three times its shorter side squared, giving a constant ratio of its sides as 1 : √3 : 2 (which can be read as √1 : √3 : √4).

The primary solid was said to be the tetrahedron, which has the smallest structure with four faces made up of equilateral triangles with four vertices. Each equilateral triangle is in turn composed of six of the fundamental scalene right-angled triangles. The tetrahedron divides the entire circumference of the sphere in which it is inscribed into equal, similar parts. (The process is described at 54d5–55a6.) This is the only distinct reference made in the *Timaeus* to the inscription of the regular solids inside a sphere. The only way the entire circumference of the sphere can be divided into equal and similar parts by an inscribed tetrahedron is by spherical triangles connecting the vertices. The important implications of the claim that the sphere can be divided into equal and similar parts has been overlooked by commentators. Spherical geometry was known to the ancient Greeks, with Autolycus of Pitane writing *On the Rotating Sphere* in the late fourth century BCE in the decades after Plato's death in 347; but it has not been an

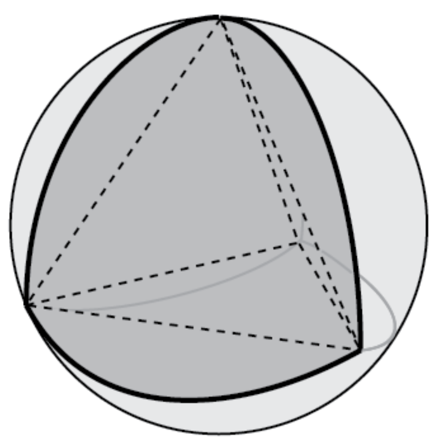

Figure 6.8.1: Inscribed Tetrahedron

aspect of mathematics with which Plato is credited as having familiarity.

Timaeus continues his description with what was considered the second solid, the octahedron, constructed from the same scalene right-angled triangles arranged in eight equilateral triangles and having six vertices. The icosahedron is the third, made up of a combination of one hundred and twenty of the scalene right-angled triangles with twelve vertices each enclosed by five plane equilateral triangles, giving the icosahedron twenty equilateral triangular faces. Sattler and Zeyl (2017, 22–23) raise a question at this point:[19]

> Timaeus does not say why each face is composed of six such triangles, when in fact two, joined at the longer of the two sides that contain the right angle, will more simply constitute an equilateral triangle. The faces of the cube are squares composed of four elemental isosceles right-angled triangles and again, it is not clear why four should be preferred to two. Given that every right-angled triangle is infinitely divisible into two triangles of it own type (by dropping a perpendicular from the right-angle vertex to the hypotenuse, the resulting two smaller right triangles are both similar to the original triangle) the equilateral or square faces of the solids and thus the stereometric solids themselves have no minimal size.

Their conjectural reply is that "the choice of six component triangles for the equilateral and four for the square is intended to prevent the solid particles from becoming vanishingly small." My own view—likewise a conjecture—is based on the notion that composing the faces from so many triangles does not seem to stem directly from, i.e. to be required by, any constructions or proofs. Rather, the triangles are meant to *converge*, marking the center of the square and equilateral triangle, giving the center of the face (which is where an inscribed sphere would touch the faces of the solid). Certain solids can be inscribed in others, but a sphere provides a common "solid" for these first three regular polygons.

The fourth solid body is the cube, the faces of which are squares made up of isosceles

[19] They note other conjectures by Cornford (1937, 231–39) and Artmann and Schäfer (1993, *pace* Cornford).

right-angled triangles arranged in sets of four with their right angles coming together at the center to produce a single equilateral quadrangle or square; its faces are six of these quadrangles with eight vertices. The construction of the fifth regular polygon, the dodecahedron—which has twelve pentagons for its faces and twenty vertices—is not discussed by Timaeus though he mentions that the demiurge used it for the whole universe, "embroidering figures on it" (55c6). The dodecahedron approximates the spherical shape of the universe.[20] The celestial sphere holding the fixed stars mirrors the spherical shape of the Earth, leading Taylor (1928, 377) to suggest that the twelve faces of the dodecahedron may have been used to divide the heavens into sectors in order to map the constellations and other groups of stars. While this is conjecture, the vertices do divide up the surface of a sphere in a manner similar to that attributed to the tetrahedron.

The dodecahedron is the largest in volume of the five regular polyhedra and closely approximates the volume of the circumscribing sphere. The inscription of the dodecahedron in a sphere is the penultimate proposition of *Elements* XIII followed by a proposition about the ratios of the edges of the solids in the sphere.[21] As mentioned earlier, the study of such ratios and the inscription of solids within a sphere can be seen as steps toward the development of celestial coordinate systems using what we might call an "interior architecture" to determine the magnitudes of lines linking points on a sphere.[22]

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[20] As Cornford (1937, 219) notices, Socrates says in the *Phaedo* that the spherical earth, if viewed from above, would look like "one of those balls made of twelve pieces of leather" (110b).

[21] Euclid has propositions for the construction of all the regular polyhedra: tetrahedron, octahedron, cube, icosahedron, and dodecahedron (XIII.13–17).

[22] These would later be systematized by Hipparchus of Nicaea (±190–±120 BCE) and Ptolemy (100–160 CE), using trigonometric tables of chords. Kepler famously utilized the relations of the Platonic solids inscribed in spheres in his *Harmonices Mundi* (*The Harmony of the World*, 1619) that would lead to his discovery of the third law of planetary motion.

The solid polyhedra give explicit structure to the elements of fire, earth, water, and air. Broadie remarks on the elegance of Timaeus's physical theory: a "small handful of geometrical assumptions" provides "a great deal of empirical variety under just four headings" (2012, 192). This is in part because the four materials can transform into one another, and in part because they come in different sizes. I would emphasize that it is the mathematical features of the two basic triangles that generate the specific properties of each element.

Nevertheless, since the correlations among the polyhedra and elements underpins a

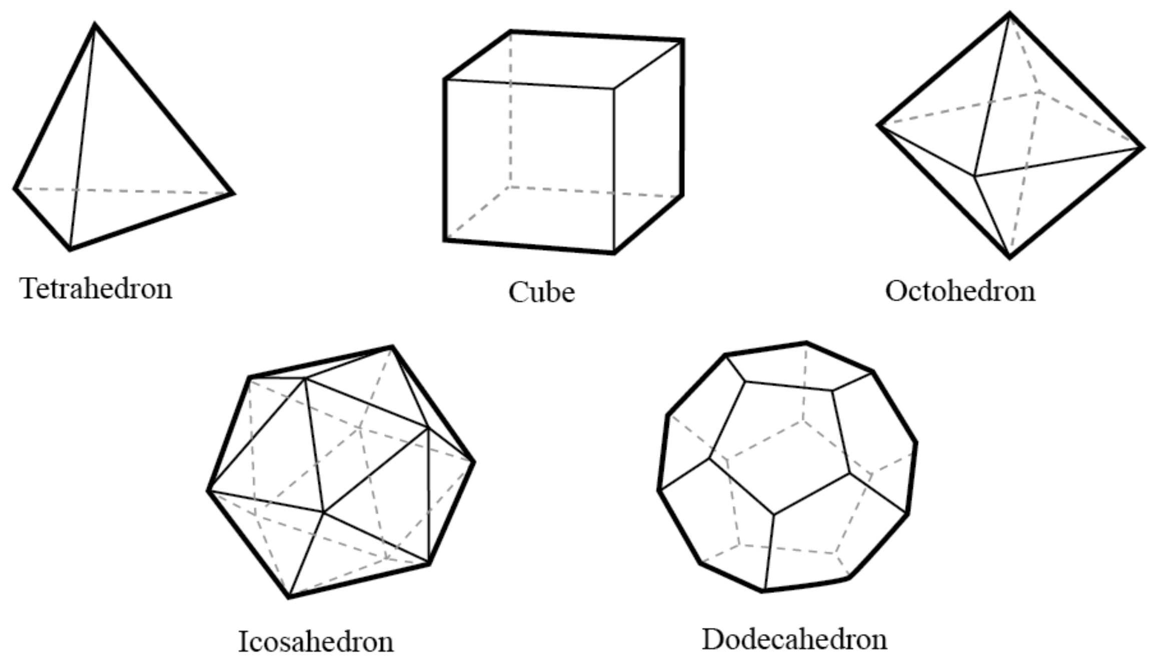

Figure 6.8.2: Platonic Solids

speculative physical theory, I am obliged to move back to the more purely mathematical examples.[23] Besides, Timaeus reiterates at 56a1 that the composition of the elements is only a

[23] Timaeus's likely account, it has often been noted, anticipates the geometric configurations of atoms in molecules in modern chemical theory at least as well as the ancient atomists' models.

provisional hypothesis.

6.9 The Most Beautiful Triangles 54a1–7

It is the composition of the faces of the polyhedra that gives the initial appearance of the use of proportions in the construction of the Platonic solids. The right-angled isosceles triangle has a constant ratio for its sides of 1 : 1 : √2, and the scalene right-angled triangle has a constant ratio for its sides of 1 : √3 : 2.

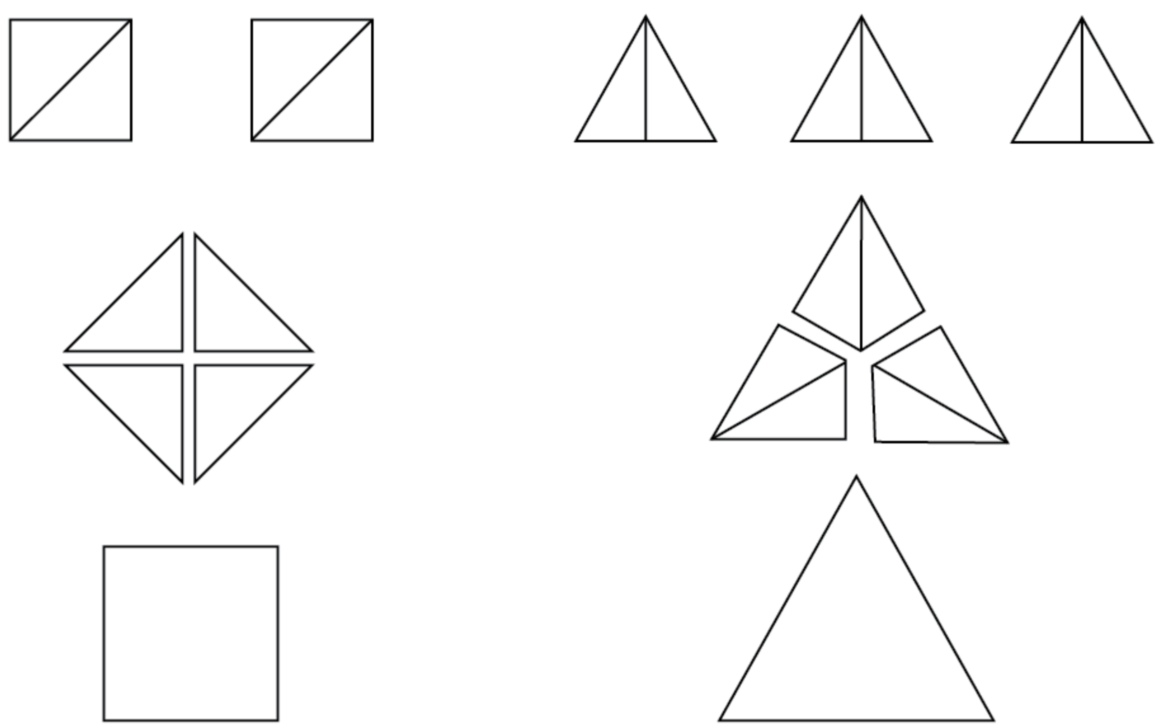

Figure 6.9.1: Decomposition

Artmann and Schäfer (1993) offer a theory to account for the unusual way in which the triangles (half-equilateral triangles) are not required for the construction of the polyhedra by

Euclidean methods; and the authors find no satisfactory explanation in the commentator tradition on the *Timaeus* for this method of composition of the faces. Their proposed explanation is that the constructions are based on those for the duplication of the square and the triplication of the equilateral triangle.

It is the manner in which the triangles can be decomposed and rearranged that is the basis

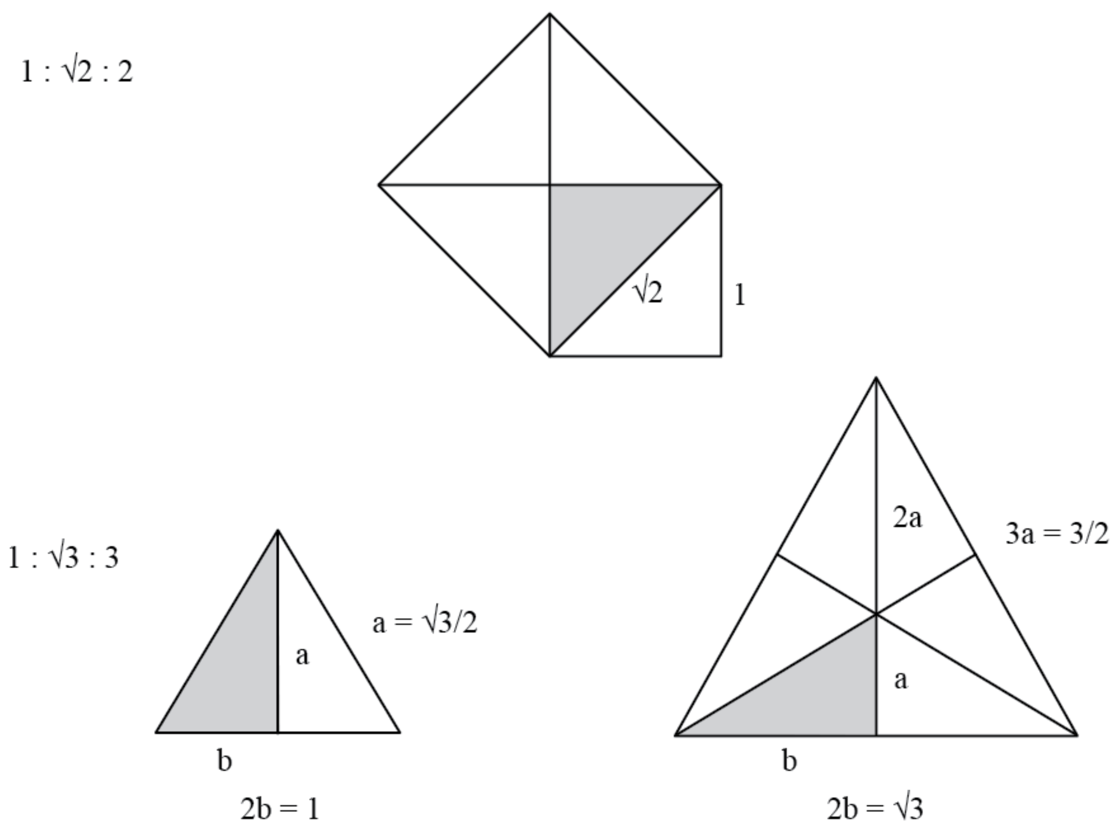

Figure 6.9.2: Doubling and Tripling

for their conjecture. As they point out, Plato's construction of the square by combining four isosceles right-triangles resembles the figure used to prove the doubling of the square in the *Meno* (Figure 2.4.3). There is an analogous extension of this procedure to the tripling of the area of an equilateral triangle.[24] The triplication of the equilateral triangle does not have the pedigree of doubling the square, but it is interesting for the broader purpose of this dissertation. Artmann and Schäfer write:

> Similarly, the composition of an equilateral face out of six halves of an equilateral triangle (*Timaeus* 54e) demonstrates the tripling of that triangle. To repeat: Plato's strange prescription for the composition of an equilateral face, the building block of three of the elements (fire, air, and water), makes very good sense when viewed as a demonstration of the tripling of an equilateral triangle; as such, it is analogous to the demonstration of the doubling of the square. (1993, 259)

Harte (2002, 244) points out that when the square and equilateral triangle faces of the solids are constructed according to the instructions of Artmann and Schäfer, they contain the same kind of continued geometric proportion that Plato used in the account of the construction of the body of the cosmos. If the face of the cube is comprised of four elementary half-squares in the way they propose, then:

$$(1)\ \ a : b\ \ b : 2a \ \text{ or } \ 1 : \sqrt{2} :: \sqrt{2} : 2$$

And if, in similar manner, the face of the tetragon is comprised of six elementary half-equilateral triangles then:

$$(2)\ \ a : b :: b : 3a \ \text{ or } 1 : \sqrt{3} :: \sqrt{3} : 2$$

Artmann and Schäfer seek to demonstrate that Plato saw a continuity of the series of square roots of integers found in the *Theaetetus* with the examples in *Timaeus*: the two right triangles connected to four of the polyhedra incorporating the "first" incommensurable segments (1993, 262). They show through a series of diagrams (Figures 6.9.1–6.9.3, redrawn) that the "first" three polygons that are faces of the regular polyhedra all have sequences of line segments (sides of subdividing triangles) that contain irrational magnitudes: $a, a, a\sqrt{2}$ for the square, $a,$

[24] This is similar to the move from the 1M2E proportion for doubling the square to the 2M1E hypothesis for doubling the cube.

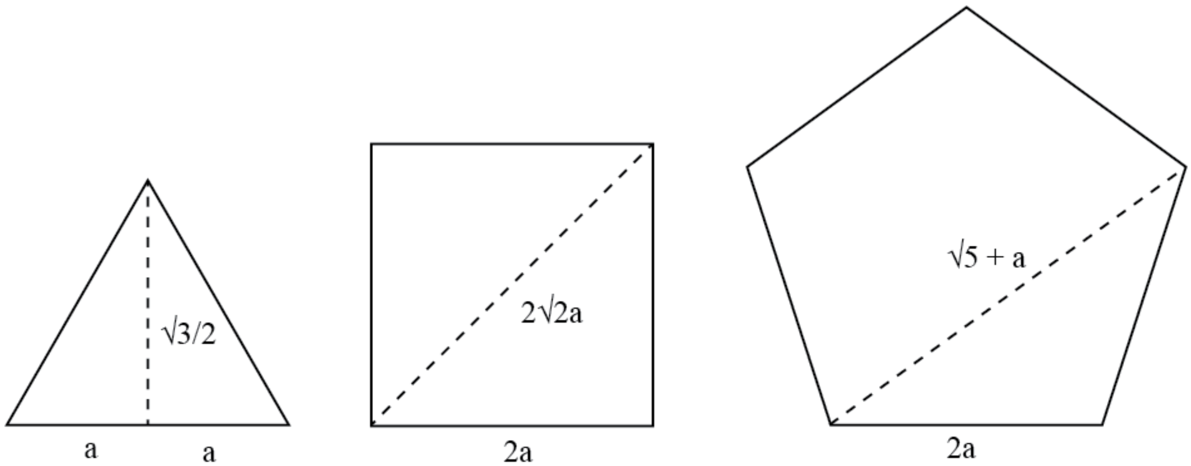

Figure 6.9.3: First Regular Polygons

a√*3, 2a* for a equilateral triangle, and *a, 2a, a* + √*5* for the pentagon. They conclude that Plato's interest in these triangles was related to his interest in the notion of incommensurability explored in the *Theaetetus*, where the √3 and √5 are specifically mentioned. These three triangles, with their incommensurable sides, appear naturally in the first regular polygons as shown.

6.10 Theaetetus and the Mathematics of the *Timaeus*

To fully appreciate the mathematical examples in the *Timaeus,* we must consider their connections to the most prominent figure in fourth-century Greek mathematics, Theaetetus. His work on the theory of irrationals and the five Platonic solids are two steps towards a comprehensive science of stereometry. Burnyeat (1978, 508) cites van der Waerden's conclusion with approval: "The author of Book XIII knew the results of Book X, but … moreover, the theory of Book X was developed with a view to its applications in Book XIII. This makes inevitable the conclusion that the two books are due to the same author. We already know his name: Theaetetus” (1954: 173–74).

Elements X classifies surds systematically into thirteen kinds, one of which is the medial, six of which are the binomial, and six of which are the apotome. I say a little more below about the connection between the medial, binomial, and apotome line types and the three Pythagorean means.

Among Theaetetus’s achievements in solid geometry—if van der Waerden is correct in his attribution—was finding the relation between the dimensions of the respective solids and the circumscribing spheres and, specifically, developing constructions for the octahedron and icosahedron, and a method for inscribing them in a sphere. He is also credited with discovering the relationship between the edges of these solids and the diameter of their circumscribing spheres as well as proving the theorem that only five polyhedra can be constructed with equal, equilateral, and equiangular figures.[25] His theory of irrationals would provide a method for expressing the quantitative values of the magnitudes of the sides the polygons that make up the faces of the regular polyhedra. For example, proposition XIII.11 says that the side of the pentagon inscribed in a circle with a rational diameter is an irrational straight line classified as “minor” or the fourth apotome. Using this proposition, it can be shown that the side of a pentagonal face of the dodecahedron constructed in XIII.17 is an irrational straight line belonging to the apotome class. The side of a pentagon is $(10–2\sqrt{5})/2$ (when the radius is one), a formula for an irrational number far more complicated than irrational square roots.

If our ancient sources report accurately—mindful of Knorr’s warning (1983b, 44) that it is feckless to speculate beyond the evidence available—Theaetetus’s work also has a direct connection to the Pythagorean musical proportion. He apparently made use of the three

[25] These were canonized in *Elements* XIII.

Pythagorean means to develop the categories of his theory of irrationals, the geometric mean for the medial line, the arithmetic mean for the binomial line, and the harmonic mean for the apotome. Medial straight lines are the products of expressible straight lines commensurable in square only. Square roots were considered expressible even when they are irrational magnitudes. Binomials are the sums of pairs of expressible straight lines that are commensurable in square only. An apotome is formed by cutting a straight line in two, where one part is commensurable in square only to the whole, and the other is the apotome.[26]

Finding a geometric mean when one extreme is an irrational number gives a medial line:

$$(1)\ 1 : \sqrt[4]{2} : \sqrt{2}$$

Finding an arithmetic mean when one extreme is an irrational number gives an binomial (subtractive irrationals):

$$(2)\ 1 : (1+\sqrt{2})/2 : 2$$

Finding a harmonic mean when one extreme is an irrational number gives an apotome (subtractive irrationals):

$$(3)\ 1 : 2(2\sqrt{2}) : 2$$

Knorr (1983b, 56) sees this as belonging to the early stage of the development of the theory, where certain irrational lines were formed as geometric, arithmetical and harmonic means. He considers the proportionality linking the means (arithmetical : geometric ∷ geometric : harmonic) to be an indispensable instrument for Theaetetus's theory, but which is noticeably absent in

[26] It is a bit more complicated. Knorr adds, "In the Euclidean theory, the *apotome* irrational is defined as $a - b$, and its irrationality is proved via consideration of the ratio $(a - b)^2 : a^2$, parallel to the manner given above for the arithmetic mean. It thus happens that Euclid treats the apotome independently of the binomial and relegates to a postscript the property that any binomial $(a + b)$ and its associated apotome $(a - b)$ have a rational product (namely, $a^2 - b^2$). By contrast, the analogue of this property would be the chief instrument for reducing the harmonic to the arithmetic case within the means-based theory of Theaetetus" (1983b, 44).

Euclid's *Elements* X. Pappus of Alexandria (±290–±350 CE), gives the most detailed account of the use of the three means as a precursor to X:

> We accept these propositions, since Theaetetus enunciated them, but we add thereto, in the first place, that the geometric mean [in question] is [and only is] the mean (or medial) line between two lines rational and commensurable in square whereas the arithmetical mean is one or other of the [irrational] lines that are formed by addition [binomial], and the harmonic mean one or other of the [irrational] lines that are formed by subtraction [apotome] and, in the second place, that the three kinds of proportion produce all the irrational lines."
>
> —Thomson, trans. (138)

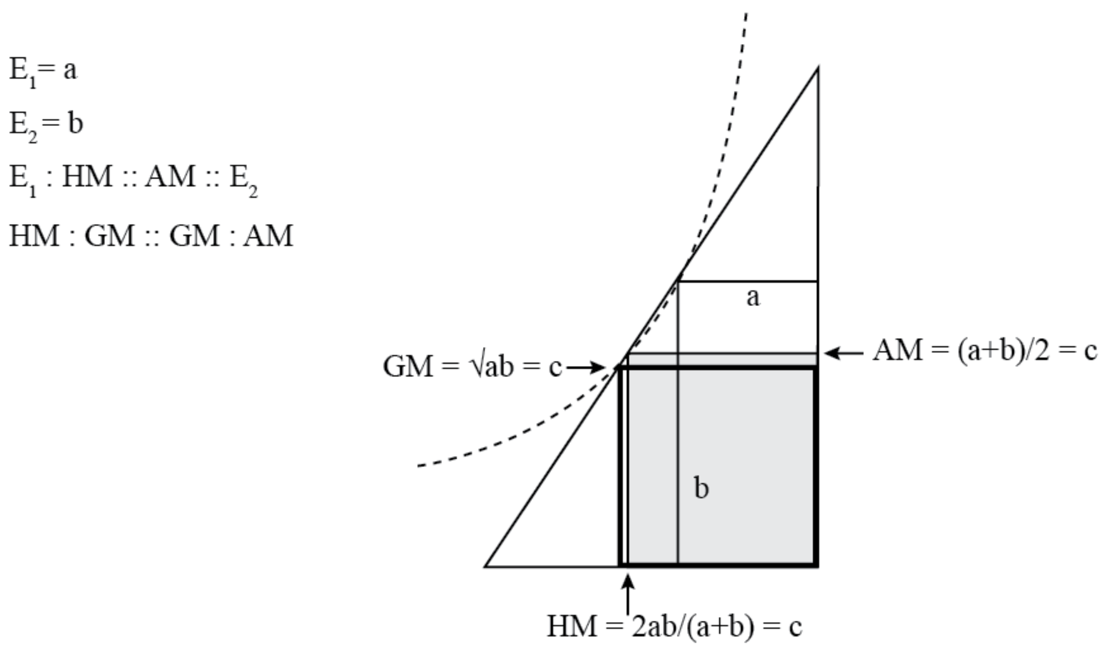

Figure 6.10.1: The Three Means Triangle

The relation of the three means can be shown with a triangle that is similar to those examined in earlier chapters. The relations of the means as quadrilaterals is similar to the diagram for the third hypothesis in the *Meno*, which is based on the application of areas.

CHAPTER 7:

CONCLUSION

I have argued that the mathematical examples in Plato's dialogues are connected, and that they form the groundwork for the research project he proposes in the *Republic*—the project of expanding geometric knowledge beyond the two-dimensional to the three-dimensional. The examples used by Plato point to the need for a methodical, systematic analysis of the structural elements of two-dimensional space via plane geometry that might offer insight into the elements that govern the structure of three-dimensional space through solid geometry. In this way, the mathematics of the dialogues is a model for all knowledge acquisition—that is, acquisition of the kind of knowledge that does *not* consist of unconnected pieces of information but, rather, is integral to a structure. If knowledge is or requires the justification of true belief, there must be a method for finding such a justification that yields a position of certainty.

The diagrams examined in the previous chapters can be linked thematically through Euclid's method of constructing a geometric mean at *Elements* II.14. Closely related is proposition VI.8, which is now known as the geometric mean theorem or right triangle altitude theorem. The examples in the *Meno* draw upon fundamental aspects of two-dimensional geometry, yielding two diagrams that involve the Pythagorean theorem and the application of areas—a theorem and a method, respectively, that can be used in proofs of the geometric mean theorem. Most applications of this theorem and this method involve the geometer in problems that include irrational and incommensurable magnitudes that are inherent in the figures of plane geometry.

As I argued in chapter 3, the mathematical example in the *Theaetetus* suggests a diagram containing a sequence of consecutive square roots of integers involving incommensurable magnitudes. The resulting diagram is a spiral composed of right-angled triangles. The relationship between the sides of these triangles can be demonstrated using the geometric mean construction. As for the *Republic*'s mathematical examples, I showed in chapters 4 and 5 that these provide two diagrams that incorporate proportions of a right-angled triangle that connects the geometric mean construction to the two means two extremes (2M2E) proportion—the proportion considered essential to the exploration of cubic equations and the extension of the Greek understanding of magnitudes from two to three dimensions. I have called these triangles 'geometric mean triangles' because the many divisions marked on them are the product of repeated applications of the geometric mean construction.

The mathematics of the *Timaeus* is connected to the geometric mean construction via a close link to the *Theaetetus*: the triangular faces of four of the regular polyhedra described in the *Timaeus* follow directly from the sequence found in the *Theaetetus*. The theory of irrationals begun in the *Theaetetus* contributes to an understanding of the irrational magnitudes that are discovered by the study of the five regular polyhedra in *Elements* XIII. Finally, the numerical triangular array of the lambda diagram in the *Timaeus* connects the sequences of magnitudes generated by the various means and proportions found in the diagrams of the examples into a unified structure of interlocking mathematical relations. Modern mathematics would see this structure as a system of algebraic equations arising from the binomial theorem; but the qualitative values of these equations' variables are ultimately determined through the geometric mean triangle—the very triangle that is a key element in the construction and proof of all the diagrams I have examined.

The prominence of mathematics in Plato's dialogues, and the connections between the mathematical passages, show that, for Plato, mathematical inquiry is an epistemological process *par excellence*. In contrast to Aristotle, Plato's mathematical passages do more than merely illustrate or clarify a point being made. They have been devised to serve as the groundwork for the expansion of geometrical knowledge beyond the confines of two-dimensional to three-dimensional space. The paradigm of a *logos* or account in Greek mathematics was proof by deduction from the axioms, etc., of geometry (exemplified by Euclid's *Elements*). Such proof is not merely verbal; it involves the production of geometrical drawings according to a precise set of instructions. As Netz and other historians of mathematics have shown, the same word (*diagramma*) is used for both the geometrical drawing (what we would call a 'diagram') and the proof or proposition itself. For the Greeks, this is "because the diagram is the proof, it is the essence of the proof for the Greek, the metonym of the proof" (1998, 37–38). Taken together, the proofs, theorems, and propositions of the *Elements* form a complex system that exemplifies the interrelational model of knowledge that, as Fine has argued (1999, 1–35), is at play in Plato's epistemological theories (even if he did not write in the standard *protasis* style).

Of course it is not just geometry, but also all the sciences of mathematics taken together, that form a complex system. As Burnyeat points out, students must master each individual science and then form a synoptic view of all the mathematical disciplines; they must also understand how this complex interrelated system is connected to reality itself. He writes: "the subjects they learned in no particular order as children they must now bring together to form a unified vision of their kinship both with one another and with the nature of that which is" (2000, 67). In the terms of the *Republic*, the ability to achieve such a unity is a test of who is thinking dialectically and who is not.

I have demonstrated that the mathematical examples in the dialogues reflect a unified vision of mathematics. This finding stands regardless of whether the dialogues represent Plato's "last word" on the connection between mathematics and metaphysics or cosmology. In the *Timaeus*, perhaps the most mathematical of his dialogues, Plato characterizes his account of the universe as a likely tale. This suggests that the Pythagorean-influenced description of a cosmic order based on mathematical proportion and harmony in the *Timaeus* may not be Plato's definitive view of the nature of the universe. He may rather be offering a plausible account of how it might be. Keeping in mind Gadamer's view that "the truth which the narrative claims for itself is explicitly limited to what is 'probable' both in respect to what is presented as a story (*muthos*) and in respect to what is presented in rational argument (*logos*)" (1980, 159), the speculative nature of Plato's account does not in any way rule out the veracity of the mathematical proportions and the notions of musical harmony it utilizes.

A final word on the state and significance of the mathematics in the dialogues. I would have to disagree with both Taylor's view (1928) that the mathematics in the dialogues reflected fifth-century Pythagorean doctrine, and with Cornford's claim (1937) that Plato's mature views on mathematics are found in the dialogue. Rather, we see a transitional stage that anticipates further work leading to a greater understanding of stereometry. Such understanding is itself needed for developing the dynamic mathematical theory that would allow astronomy to be studied at a higher level. We have seen, *contra* Taylor, that the mathematics of Plato's dialogues does move beyond its Pythagorean origin: the importance placed on the irrational magnitudes in the ratios of the sides of triangular components of the faces of the regular polyhedra shows that we have left behind the number theory ("pebble arithmetic") associated with the Pythagoreans. However, *contra* Cornford, we have also seen that the method in the *Timaeus* for constructing four of the polyhedra is not as sophisticated as the method Theaetetus is said to have used with the Platonic solids (as recorded in *Elements* XIII). As Plato is acquainted with Theaetetus's work, it is unlikely that he takes the mathematics of the *Timaeus* to be the final word on the regular polyhedra. While I have demonstrated the unity of Plato's mathematical examples, this by no means implies that he takes mathematical inquiry to have been completed. As we know, the foundations laid by the Greeks would not fully bear fruit until the sixteenth and seventeenth centuries with Galileo, Kepler, and Newton.

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