LONG-TIME CONVERGENCE OF HARMONIC MAP HEAT FLOWS FROM SURFACES INTO RIEMANNIAN MANIFOLDS

By

Kwangho Choi

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Mathematics

2011

ABSTRACT

LONG-TIME CONVERGENCE OF HARMONIC MAP HEAT FLOWS FROM SURFACES INTO RIEMANNIAN MANIFOLDS

By

Kwangho Choi

We study the long-time convergence of harmonic map heat flows from a closed Riemann surface into a compact Riemannian manifold. P. Topping constructed an example of a flow that does not converge in the infinite-time limit. Motivated by the observation that Topping's flow has accumulation points at which the Hessian of the energy function is degenerate, we prove convergence under the assumptions that (a) the Hessian of the energy at an accumulation point is positive definite, and (b) no bubbling occurs at infinite time. In addition, we present examples of heat flows for geodesics which show that the convexity of the energy function and convergence as $t \to \infty$ may not hold even for 1-dimensional harmonic map heat flows.

ACKNOWLEDGMENT

I sincerely thank my advisor, Professor Thomas Parker for his continued encouragement and thoughtful guidance. This thesis would never have taken shape without his persistent help. My thank goes to Professor Jon Wolfson for his invaluable suggestions and lectures on related topics that helped me improve my knowledge in differential geometry. Thanks are also due to former and current graduate students, Dr. Junho Lee, Dr. Jens Von Bergmann, Oh Hoon Kwon, Harrison Cheuk-Man Hwang, and Manousos Maridakis for numerous helpful discussions. Last but not least, I would like to acknowledge that none of this would have been possible without love and patience of my family in Korea.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 0

Introduction

A map u between two Riemannian manifolds (M, g) and (N, h) is called a harmonic map if it is a critical point of the energy function $E(u) = \frac{1}{2}\int_M |du|^2\, d\mathrm{vol}_g$ in $C^2(\Sigma, N)$. A harmonic map heat flow is a gradient flow of the energy function on a space of maps. If one embeds the target manifold N isometrically into $\mathbb{R}^r$, then the harmonic map heat flow is a weak solution $u\colon M \times [0, \infty) \to N$ to a system of nonlinear parabolic PDEs with initial condition:

$$\begin{aligned} \dot{u} &= \Delta u + A(u)(du, du), \\ u(\cdot, 0) &= u_0(\cdot), \end{aligned} \tag{0.1}$$

where we consider $u\colon M \to N \hookrightarrow \mathbb{R}^r$ to be a map into $\mathbb{R}^r$, and where A denotes the second fundamental form of N in $\mathbb{R}^r$. Intrinsically, the right-hand side of (0.1) is minus the gradient of E for the L^2 Riemannian metric on the space of maps from M to N. Thus one can consider a solution $u = u(t)$ to be a family of maps that starts from u_0 and flows in the energy-minimizing direction.

Standard results show the existence and smoothness of solutions to (0.1) for short time. Eells and Sampson [6] used harmonic map heat flows to prove that if the target manifold N is a Riemannian manifold with non-positive sectional curvature then the solution to (0.1) exists for all time and therefore any smooth map $u_0 \colon M \to N$ is homotopic to a harmonic map.

For dim $M = 2$, the energy E is invariant under the conformal change of metrics on Σ and it is possible that the energy density concentrates at a finite set of points. Sacks and Uhlenbeck [16] discovered a "bubbling phenomena", in which each point of energy concentration yields a harmonic 2-sphere S^2, called a bubble. For sequences of harmonic maps, Parker [13] proved a "bubble tree convergence theorem," in which the maps converge to a limit map together with bubbles so that energy is conserved in the limit and the image of the limit is connected.

For harmonic map heat flows, Struwe [18] showed that a global weak solution $u(t)$ to (0.1) exists and is smooth away from a finite set of "singular points" $(x_1, T_1), \dots, (x_k, T_k)$ in spacetime $\Sigma \times \mathbb{R}$. At each singular point, energy concentrates at x_i and a bubble splits off. Struwe also showed that there is a sequence $t_n \to \infty$ such that the maps $u(t_n)$ converges weakly to a harmonic map u_∞ together with possibly some bubbles (see Section 1.4). Struwe's theorem was refined by Lin and Wang [9] and by Qing and Tian [15] to a bubble tree convergence result. Specifically, they showed that, at both the finite and infinite times, enough bubbles appear to make the energy function continuous and the image of the limit map u_∞ (including bubbles) is connected. Notice that all of these results *require passing to subsequences.*

In general, the limit of the solution $u(t_n)$ as $t_n \to \infty$ is not unique and does depend on the choice of the sequence t_n of time. The uniformity of convergence of the heat flow at infinity is proved for some interesting cases. For example, Topping [19] showed convergence of the harmonic map heat flow $u(t)$ from S^2 to S^2 as $t \to \infty$, assuming that u_∞ and the bubbles are all holomorphic or all anti-holomorphic maps as harmonic maps between two S^2's. In case of real analytic target manifolds, the uniformity of convergence of harmonic map heat flows follows by Simon [17].

In Chapter 2, we present details of the example of P. Topping that motivates our work in this thesis. It is an example of a harmonic map heat flow from a 2-sphere to a compact Riemannian manifold that exists for all time but does not converge as $t \to \infty$. Instead, it exhibits 'winding behavior' — the flow is asymptotic to a flow that moves around a circle of harmonic maps with constant speed.

Topping outlined his construction in Section 5 of [19]. He later gave a more sophisticated example ([20]) for domains with boundary. Section 2.1 shows how the details in [20] yield a complete proof of the construction of [19] (and fix a small error in the original). For this, we consider the manifold $N = \mathbb{R}^2 \times S^2$ with the warped product Riemannian metric used by Topping in [20], and then consider the harmonic map heat flow $u(t)$ from S^2 to N of the form Topping uses in [19], namely $u(q,t) = (p(t), q)$ where

$$\dot{p}(t) = -\nabla f(p(t)). \tag{0.2}$$

This gives the existence of a harmonic map heat flow from S^2 into N having tension uniformly bounded in L^2, but not converging anywhere as $t \to \infty$, although there exist sequences $\{t_n\}$ such that $u(t_n) \to u_\infty$ in C^∞ as $n \to \infty$. Roughly, one can think of $u(t)$ as S^2 fibers "orbiting" the cylinder infinitely many times with orbits asymptotic to the center circle.

In fact, the accumulation points of the above flow are maps u_∞ that lie in the set $\mathcal{M}$ of absolute minima of the energy function. In Section 2.2 we show that $\mathcal{M}$ is a manifold diffeomorphic to $S^1 \times PSL(2,\mathbb{C})$ and we prove that the Hessian Hess(E) of the energy function on the space of maps is degenerate as a bilinear form in the normal bundle along $\mathcal{M}$. The key observation of this thesis is that, in general, convergence is controlled by the degeneracy or non-degeneracy of Hess(E) along the limiting set.

Chapters 3 - 5 develop criteria that insure a harmonic map heat flow will converge as $t \to \infty$. Our approach is to avoid the bad behavior of the above example by controlling Hessian of the energy function $E(u(t))$ along the heat flow. Consider a harmonic map heat flow $u(t)$ from a compact Riemann surface Σ into a compact Riemannian manifold N. From Struwe's theorem, we know that $\sup_\Sigma |du(t)|$ is necessarily unbounded at each finite singular point. We also know that as $t \to \infty$ there is a subsequence $u(t_n)$ that converges weakly in $W^{1,2}$ to a harmonic map u_∞. This limit may not involve bubbles, and there may be a uniform bound on $\sup |du(t)|$ as $t \to \infty$. In fact, if $u(t)$ is very close to a stable harmonic map u_∞ for some large t then one expects that $u(t)$ will flow to u_∞ without bubbles or energy concentration points. For such flows, we can translate time so that $t = 0$ corresponds to a time, beyond the finite singular times, such that we have a uniform sup bound on $|du|$ valid for all $t \geq 0$.

Thus we will suppose that $\sup_{\Sigma\times[0,\infty)} |du(t)| < C$ for some positive constant C. Further, if the Hessian of the energy E is positive definite at u_∞, one expects the exponential convergence of the solution, that is, there should be constants C and λ such that $\mathrm{dist}(u(t), u_\infty) \leq C\, e^{\lambda t}$. Notice that this is an assumption about the Riemannian geometry of the target space N. With these two assumptions, we prove the following result:

Theorem A. *Let $u\colon \Sigma^2 \times [0,\infty) \to N$ be a harmonic map heat flow such that $u(t)$ converges to a harmonic map u_∞ weakly in $W^{1,2}$. Suppose*

(a) the Hessian of the energy E is positive definite at u_∞.

(b) $\sup_\Sigma |du(t)| < C$ for all large $t > T$.

Then $u(t)$ converges to u_∞ exponentially fast in $W^{2,2}$ and hence in C^1.

Consequently, under the hypothesis of Theorem A, $u(t)$ converges uniquely to a harmonic map u_∞, *independent of any choice of subsequence.* The proof of Theorem A is given in Chapter 5.

The organization of this thesis is as follows.

In Chapter 2, we review Topping's construction of a harmonic map heat flow from a two-dimensional domain that fails to converge and that behaves problematically as described above.

In Chapter 3 and thereafter, we assume that the energy density of the harmonic map heat flow is uniformly bounded in $t < \infty$ for technical reasons. We consider the second variation of the energy along the harmonic map heat flow $u(t)$ to justify the definition of a symmetric bilinear tensor B along $u(t)$. Then we prove that if the symmetric bilinear tensor B is positive definite along a harmonic map heat flow $u(t)$ then $u(t)$ converges exponentially to a harmonic map in $W^{2,2}$ norm where a parabolic estimate for harmonic map heat flows plays an important role.

In Chapter 4, we discuss topologies on the space of maps to work with and then prove that the symmetric bilinear tensor B is continuous on the space of maps in the $W^{1,2} \cap C^0$ topology. Roughly, this implies that the symmetric bilinear tensor B is positive definite at u nearby the accumulation point u_∞ under the assumptions (a) and (b) of Theorem A.

Chapter 5 contains the proof of Theorem A. For the proof, we take a sequence $u(t_n)$ converging to u_∞ in $W^{1,2} \cap C^0$ and show by contradiction that, once $u(t_n)$ enters in a $W^{1,2} \cap C^0$ $\frac{\delta}{4}$-neighborhood, it stays in a $W^{1,2} \cap C^0$ δ-neighborhood for all n. Hence our main theorem follows from the exponential convergence in Chapter 3.

Finally, in Chapter 6, we present examples of geodesic heat flows. This case is technically simpler, but still displays interesting non-convergence behavior, and it provided motivation for our work in Chapters 3 - 5. For the 1-dimensional domain, the harmonic map heat flows from S^1 are called geodesic heat flows. Here one can use some facts from Morse theory, working on the (infinite-dimensional) manifold of $W^{1,2}$ loops in N. It is known that Palais-Smale compactness condition is satisfied on this loop space with its $W^{1,2}$ Riemannian metric, but this does not insure convergence of downward $W^{1,2}$ gradient flows (see [1]). The same is true for geodesic heat flows: they may not converge as $t \to \infty$.

We illustrate such behavior by constructing explicit examples which show that the convexity of the energy function and convergence as $t \to \infty$ may not hold. For the first two examples, we consider geodesic heat flows from S^1 into a surface of revolution in $\mathbb{R}^3$ that are equivariant under an S^1 action and then derive a general solution of the geodesic heat flow. As an application, we analyze the solution of the equivariant geodesic heat flow from S^1 into S^2 and show that the energy function is not convex. For the third example, we construct the geodesic heat flow $u(t)\colon S^1 \to T^3$, showing that the convergence fails as $t \to \infty$.

Further studies on the geodesic heat flows, including a C^∞ convergence theorem, are in progress [1].

Chapter 1

Background

We define the basic concepts and notations of harmonic maps and give a short introduction to harmonic map heat flows in the light of the bubble tree convergence. In this thesis we denote by Σ a closed Riemann surface, and by N a compact Riemannian manifold unless specified otherwise.

1.1 Energy

Let (M, g) be a compact Riemannian manifold without boundary and (N, h) a compact Riemannian manifold. If $u\colon M \to N$ is a smooth map, we consider du to be a u^*TN-valued 1-form of M. The *energy density* $e(u)$ of u is defined to be $e(u) = \frac{1}{2}|du|^2$. In local coordinates (x^α) and (u^i) around x and $u(x)$, we have

$$e(u) \;=\; \frac{1}{2} g^{\alpha\beta} h_{ij}(u) \frac{\partial u^i}{\partial x^\alpha} \frac{\partial u^j}{\partial x^\beta}.$$

The energy $E(u)$ of u is

$$E(u) \;=\; \int_M e(u)\; d\mathrm{vol}_g$$

where $d\mathrm{vol}_g$ is the volume form on (M, g). We note that, as a rule, Greek indices are used for tensors on the domain manifold and Latin ones for tensors on the target manifold and we use the summation convention.

1.2 Euler-Lagrange equation and harmonic maps

In this section, we derive the Euler-Lagrange equation of the energy E.

Let $V \in \Gamma(u^*TN)$ be a vector field along the image of $u\colon M \to N$ and consider a family of maps $u(t)\colon M \to N$ such that $u(0) = u$ and $\frac{\partial u(t)}{\partial t}\big|_{t=0} = V$. We consider the family u as a map $u\colon M \times \mathbb{R} \to N$ and let $\dot{u}$ denote $\frac{\partial u}{\partial t}$. Then we have

$$\begin{aligned}
\frac{d}{dt}E(u(t))\big|_{t=0} \;&=\; \frac{d}{dt}\int_M e(u(t))\; d\mathrm{vol}_g\big|_{t=0} \\
&=\; \frac{1}{2}\int_M \frac{\partial}{\partial t}\langle du(t), du(t)\rangle\; d\mathrm{vol}_g\big|_{t=0} \\
&=\; \int_M \langle \nabla_{\frac{\partial}{\partial t}} du(t), du(t)\rangle\; d\mathrm{vol}_g\big|_{t=0}
\end{aligned}$$

where ∇ is the covariant derivative in $T^*(M \times \mathbb{R}) \otimes u^*TN$. For $X \in TM$, we have

$$\left(\nabla_{\frac{\partial}{\partial t}} du\right) X \;=\; \nabla_X \dot{u} + du([\tfrac{\partial}{\partial t}, X]) \;=\; \nabla_X \dot{u},$$

since $[\frac{\partial}{\partial t}, X] = 0$. Then

$$\begin{aligned} \frac{d}{dt} E(u(t))\big|_{t=0} &= \int_M \langle \nabla \dot{u}(t), du(t) \rangle \ \mathrm{dvol}_g \big|_{t=0} \\ &= \int_M \langle \nabla V, du \rangle \ \mathrm{dvol}_g \\ &= -\int_M \langle V, \tau(u) \rangle \ \mathrm{dvol}_g \end{aligned}$$

where $\tau(u) = \text{trace } \nabla du$ is called the *tension field* of u. Hence the tension field $\tau(u)$ of $u \in C^2(M, N)$ is the negative of the gradient of the energy function with respect to the L^2 Riemannian metric on the space of maps. In local coordinates (x^α) and (u^i), the tension field $\tau(u)$ is given by

$$\tau(u) = \left(\Delta u^i + g^{\alpha\beta} \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial x^\alpha} \frac{\partial u^k}{\partial x^\beta} \right) \frac{\partial}{\partial u^i} =: \tau^i(u) \frac{\partial}{\partial u^i}, \tag{1.1}$$

where $\Delta u^i = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^\alpha} \left(\sqrt{\det g}\, g^{\alpha\beta} \frac{\partial u^i}{\partial x^\beta} \right)$ and Γ^i_{jk} is the Christoffel symbol of N.

Definition 1.1. *A smooth map $u \colon M \to N$ is harmonic if it is a critical point of the energy E.*

Note that u is a harmonic map if and only if $\tau^i(u) = 0$ for all i. For systematic developments of the theory of harmonic maps we refer to [2], [3] and [4].

1.3 Existence of harmonic maps from a surface

As a generalization of the problem of finding harmonic functions, one can consider the problem of the existence of harmonic maps. In many ways, the most interesting case is when the domain is a 2-dimensional Riemannian manifold (Σ, g). Then, given a Riemannian manifold (N, h), the existence problem takes this form:

Given $\varphi\colon \Sigma \to N$, find a harmonic map $u\colon \Sigma \to N$ with u homotopic to φ.

In general, the existence of nontrivial harmonic maps does not hold. For example, it is known that there is no harmonic map of degree one from a smooth torus T^2 to the round sphere S^2 (see Eells and Wood [5]).

A classical result on the existence of harmonic maps from S^2 was given by Sacks and Uhlenbeck [16]. Unlike the 1-dimensional case, the Palais-Smale compactness condition for the energy E in general fails on the space of maps of Σ with any reasonable topology. Hence the direct minimization method does not apply in the 2-dimensional case. To overcome this, Sacks and Uhlenbeck considered a perturbed α-energy E_α: for $\alpha > 1$

$$E_\alpha(u) \;=\; \int_\Sigma (1 + |du|^2)^\alpha \; d\mathrm{vol}_g$$

for u in the separable Banach manifold $W^{1,2\alpha}(\Sigma, N)$ (See (1.3) below). Note that, for $\alpha = 1$, the critical points of E_1 are harmonic maps. For $\alpha > 1$, the α-energy $E_\alpha(u)$ satisfies Palais-Smale compactness condition and hence (smooth) critical maps u_α exist. Thus one can consider a sequence u_k of α_k harmonic maps with $\alpha_k \to 1$ and with $E_{\alpha_k}(u_k)$ uniformly bounded. Sacks and Uhlenbeck showed that there exists a subsequence which converges weakly in $W^{1,2}(S^2, \mathbb{R}^r)$ to a limit u_∞ and a further subsequence u_l converges to u_∞ in C^1 away from a finite set of points $x_1, \ldots, x_K$ in Σ. It follows that (after using a removable singularity theorem) u_∞ is a smooth harmonic map. Furthermore, after renormalizing and passing to a subsequence, they obtained a nontrivial harmonic 2-sphere at each singular point x_i to (partially) capture the energy loss as $u_l \to u_\infty$. As an application, they answered the above problem when the fundamental group $\pi_2(N)$ is trivial: if $\pi_2(N) = 0$, for $\varphi\colon \Sigma \to N$, there exists a smooth harmonic map $u\colon \Sigma \to N$ homotopic to φ.

A nontrivial harmonic map of the 2-sphere $S^2 \to N$ is called a *bubble* when it arises by a renormalization process like the one used by Sacks and Uhlenbeck.

For technical reasons, it is convenient to fix an isometric embedding $N \subset \mathbb{R}^r$ and write the harmonic map equation extrinsically. Then $\tau(u)$ is the tangential component of Δu and satisfies

$$\tau(u) \;=\; \Delta u + A(u)\,(\nabla u, \nabla u)\,, \tag{1.2}$$

where Δu is the Laplacian of $\mathbb{R}^r$ (our sign convention is $\Delta = g^{\alpha\beta}\partial_\alpha\partial_\beta +$lower order terms) and A is the second fundamental form of $N \subset \mathbb{R}^r$ and we abbreviate $A(u)\,(\nabla u, \nabla u) = g^{\alpha\beta} A(u) \left(\frac{\partial u}{\partial x^\alpha}, \frac{\partial u}{\partial x^\beta}\right)$.

To proceed with the extrinsic setting, a natural choice for admissible family of maps is the Sobolev space $W^{1,2}(\Sigma, N)$:

$$W^{1,2}(\Sigma, N) \;=\; \{u \in W^{1,2}(\Sigma, \mathbb{R}^r) \mid u(x) \in N \text{ for a.e. } x \in \Sigma\}, \tag{1.3}$$

where $u \in W^{1,2}(\Sigma, \mathbb{R}^r)$ if and only if $u^i \in W^{1,2}(\Sigma)$ for each $i = 1, \ldots, r$. It is well-known that $W^{1,2}(\Sigma, N)$ is not embedded into $C^0(\Sigma, N)$ though the space $C^\infty(\Sigma, N)$ is dense in $W^{1,2}(\Sigma, N)$. In particular, dim $\Sigma = 2$ is the borderline case of the Sobolev embedding theorem $W^{1,2} \hookrightarrow C^0$. The "bubbling phenomena" of Sacks and Uhlenbeck is related to the borderline failure of this Sobolev embedding and also to the invariance of the energy $E(u)$

under the conformal change of metrics on Σ.

Now we consider a critical point of the energy E among variations of a map in $W^{1,2}(\Sigma, \mathbb{R}^r)$ composed with the nearest point projection onto the compact Riemannian manifold $N \subset \mathbb{R}^r$.

Definition 1.2. *$u \in W^{1,2}(M, N)$ is a weakly harmonic map if*

$$\Delta u + A(u)(\nabla u, \nabla u) = 0 \quad \textit{weakly} \tag{1.4}$$

that is,

$$\int_M -\langle \nabla u, \nabla \varphi \rangle + \langle A(u)(\nabla u, \nabla u), \varphi \rangle \; dvol_g = 0 \tag{1.5}$$

for any $\varphi \in C_c^\infty(M, \mathbb{R}^r)$.

A weakly harmonic map is called *stationary* if it is also a critical point of the energy with respect to domain variations. In general, stationary weakly harmonic maps are not necessarily smooth. We state results on the regularity of stationary weakly harmonic maps without proof below. Proofs and references can be found in Moser [11], and Lin and Wang [10].

Theorem 1.3. *A stationary weakly harmonic map $u\colon M \to N$ is smooth away from a singular set of vanishing $(m-2)$-dimensional Hausdorff measure where dim $M = m$.*

Theorem 1.4. *A continuous weakly harmonic map is smooth.*

For the 2-dimensional case we have by Hélein [7]

Theorem 1.5. *If dim $\Sigma = 2$, every weakly harmonic map of Σ is smooth.*

1.4 Harmonic map heat flow

Another approach on the existence of harmonic maps is the heat flow method. This approach is the main topic of this thesis.

Let M, N be Riemannian manifolds of any dimension. A *harmonic map heat flow* is the downward L^2 gradient flow of the energy $E(u)$:

$$\dot{u}(t) = \tau(u(t)) \tag{1.6}$$

$$u(\cdot, 0) = u_0, \tag{1.7}$$

where $u_0 \in C^\infty(M, N)$ and we abbreviate $u(t) = u(\cdot, t)$ for $t > 0$.

In a seminal paper [6], Eells and Sampson showed that the nonlinear parabolic equation (1.6) with the initial condition (1.7) has a short-time solution which is unique, and that if the target manifold N is a compact Riemannian manifold with non-positive sectional curvature then the solution to (1.6) exists for all time and therefore any smooth map $u_0 \colon M \to N$ is homotopic to a harmonic map.

When the domain is a compact Riemann surface Σ, Struwe [18] showed the following existence theorem:

Theorem 1.6. *Given $u_0 \in C^\infty(\Sigma, N)$, there exists a weak solution $u \in W^{1,2}_{loc}(\Sigma \times [0, \infty), N)$ to the harmonic map heat flow equations*

$$\dot{u} = \Delta u + A(u)(\nabla u, \nabla u) \tag{1.8}$$

$$u(\cdot, 0) = u_0.$$

satisfying

(a) $u\colon \Sigma\times[0,\infty) \to N$ *is smooth on* $\Sigma\times[0,\infty)$ *away from a finite set of points in* $\Sigma\times(0,\infty)$*,*

(b) At each bubble point x_T*, a bubble 'separates', so that the energy function* $E(u(t))$ *is decreasing in* t*,*

(c) Near $t=\infty$ *there exist a sequence of time* $t_n \to \infty$ *and a smooth harmonic map* $u_\infty\colon \Sigma \to N$ *such that* $u_n := u(t_n)$ *converges to* u_∞ *weakly in* $W^{1,2}(\Sigma, N)$*, and strongly in* $W^{2,2}$ *away from a finite set of bubble points* x_∞ *of* Σ *at each of which a bubble 'separates.'*

We note that there are a finite number of bubble points at each singularity time and bubbles separate in the sense that, for each bubble point x_T *with possibly* $T=\infty$*, there are sequences* $x_n \to x_T$*,* $t_n \to T$*,* $0 < R_n \to 0$ *as* $n \to \infty$ *such that, after precomposing exponential map* $\exp_{x_n}$ *at* x_n*, rescaling* $u_n \circ \exp_{x_n}$ *with scale* R_n*, it converges to a nonconstant harmonic map of* $\mathbb{R}^2 \to N$ *with finite energy and hence induces, by the removable singularity theorem [16], a nonconstant harmonic map via a fixed stereographic projection.*

By renormalizing sequences of harmonic maps to obtain "bubbles on bubbles", Parker [13] proved a "bubble tree convergence" theorem in which the limit map and bubbles are connected for a sequence of harmonic maps from a compact Riemann surface Σ.

Qing and Tian [15], also considering maps from surfaces, proved that any Palais-Smale sequence of the energy functional with uniformly L^2 bounded tension fields $\tau(u_n)$ converges, after passing to a subsequence and relabeling, pointwise to the image of the limit map, called the bubble tree map. Building on the "bubble tree convergence" theorem of Parker and Wolfson [14] and Parker [13], Lin and Wang [9] gave another proof of the Qing and Tian's result.

In this thesis we examine the convergence of harmonic map heat flows $u(t)$ from Σ to N as $t \to \infty$. It is natural to study the asymptotic behavior of the harmonic map heat flow as $t \to \infty$. More precisely, one can ask if the weak convergence is actually convergence in C^0 or a stronger norm, and whether the convergence is independent of the choice of subsequences as $t_n \to \infty$. The remainder of this thesis is devoted to answering these questions.

Chapter 2

Degenerate Hessian of the energy

In this chapter, we review an example of P. Topping of a harmonic map heat flow that exists for all time but does not converge as $t \to \infty$. We then observe that Topping's flow is asymptotic to a manifold $\mathcal{M}$ of limit maps u_∞, and that a key feature of the flow is that the Hessian of the energy is degenerate in the normal bundle to $\mathcal{M}$.

2.1 Failure of convergence

Topping's construction is outlined in [19], but for technical reasons it is necessary to replace the flat metric on the torus T^2 used in [19] with the modified metric that Topping uses in [20] for a more sophisticated example. To start, consider the strip $[\text{-}1, 1] \times \mathbb{R}$ with coordinates (w, z) and with the metric

$$dw^2 + 2w^2 dw\, dz + (1 + w^4)\, dz^2.$$

Let C be the cylinder obtained by taking the quotient of $[-1,1] \times \mathbb{R}$ by the group of isometries $\Gamma = \{(w,z) \to (w, z{+}n) \mid n \in \mathbb{Z}\}$, and let N be the warped product $C \times S^2$ with the "warped product" metric

$$h \;=\; dw^2 + 2w^2 dw\, dz + (1+w^4)\, dz^2 \;+\; f(w,z)\left(d\alpha^2 + \sin^2\alpha\, d\theta^2\right)$$

where f is a smooth function of C defined by the equation

$$f(w,z) \;=\; \begin{cases} 1 + e^{-\frac{2\pi}{|w|}}\left(\sqrt{2} + \sin 2\pi\left(\frac{1}{|w|} - z - \frac{1}{8}\right)\right) & \text{if } w \neq 0 \\ 1 & \text{if } w = 0. \end{cases}$$

Using coordinates $x = \dfrac{1}{|w|} - z$ and $y = z$, the metric is

$$h \;=\; (x+y)^{-4} dx^2 + dy^2 + f(x,y)\left(d\alpha^2 + \sin^2\alpha\, d\theta^2\right)$$

where

$$f(x,y) \;=\; 1 \;+\; e^{-2\pi(x+y)}\left(\sqrt{2} + \sin 2\pi(x - \tfrac{1}{8})\right) \quad \text{for } w \neq 0$$

and satisfies $\dfrac{\partial f}{\partial x}(0,\cdot) \equiv 0$. We consider a path of maps $u(t)\colon S^2 \to N$ in (x,y,α,θ) coordinates by

$$u(t)(r,\theta) \;=\; \left(\; 0,\; y(t),\; \arccos\frac{-1+r^2}{1+r^2},\; \theta \;\right)$$

where we parametrize S^2 by stereographic projection, so our domain is $\mathbb{R}^2$ with polar coordinates and with the conformally euclidean metric

$$g \;=\; \frac{4}{(1+r^2)^2}\left(dr^2 + r^2 d\theta^2\right).$$

Theorem 2.1. $y(t) = \frac{1}{2\pi} \ln 2\sqrt{2}\pi^2(t + t_0)$, *for* $t_0 > 0$ *sufficiently large, yields a solution to (1.6) for* $u(t)$*:*

$$\begin{aligned} \dot{y}(t) &= -\frac{\partial f}{\partial y}(0, y) \\ &= \sqrt{2}\pi e^{-2\pi y(t)} . \end{aligned} \tag{2.1}$$

Proof. Each $u(t)$, as a map of $(S^2, g) \to (S^2, g)$, is the identity with respect to the metric g on S^2 and thus harmonic, that is, $\tau^\alpha(u) \equiv 0 \equiv \tau^\theta(u)$. We also have $\Delta x = 0 = \Delta y$, $g_{r\theta} = 0$, and all partial derivatives are zero except for $\frac{\partial \alpha}{\partial r} = \frac{-2}{1+r^2}$, $\frac{\partial \theta}{\partial \theta} = 1$. Using the formula (1.1), we have

$$\begin{aligned} \tau^x(u) &= g^{rr}\Gamma^x_{\alpha\alpha}(u)\left(\frac{\partial \alpha}{\partial r}\right)^2 + g^{\theta\theta}\Gamma^x_{\theta\theta}(u)\left(\frac{\partial \theta}{\partial \theta}\right)^2 \\ &= \frac{(1+r^2)^2}{4}\frac{(-1)}{2}(x+y)^4\frac{\partial f}{\partial x}(0,y)\left(\frac{(-2)}{1+r^2}\right)^2 \\ &\qquad + \frac{(1+r^2)^2}{4r^2}\frac{(-1)}{2}(x+y)^4\frac{\partial f}{\partial x}(0,y)\frac{4r^2}{(1+r^2)^2}1^2 \\ &= -(x+y)^4\frac{\partial f}{\partial x}(0,y) \\ &= 0, \text{ since } \frac{\partial f}{\partial x}(0,\cdot) \equiv 0 \end{aligned}$$

$$\begin{aligned} \tau^y(u) &= g^{rr}\Gamma^y_{\alpha\alpha}(u)\left(\frac{\partial \alpha}{\partial r}\right)^2 + g^{\theta\theta}\Gamma^y_{\theta\theta}(u)\left(\frac{\partial \theta}{\partial \theta}\right)^2 \\ &= \frac{(1+r^2)^2}{4}\frac{(-1)}{2}\frac{\partial f}{\partial y}(0,y)\left(\frac{(-2)}{1+r^2}\right)^2 \\ &\qquad + \frac{(1+r^2)^2}{4r^2}\frac{(-1)}{2}\frac{\partial f}{\partial y}(0,y)\frac{4r^2}{(1+r^2)^2}1^2 \\ &= -\frac{\partial f}{\partial y}(0,y) \\ &= \sqrt{2}\pi e^{-2\pi y}. \end{aligned}$$

Suppose that $y(t)$ solves (2.1). Then $\ddot{y} = -2\pi\dot{y}^2$ and so $\dot{y}(t) = \frac{1}{2\pi(t+t_0)}$ for $t_0 > 0$. Write $y(t) = \frac{1}{2\pi}\left(C_0 + \ln(t+t_0)\right)$. Using (2.1), we set $\dot{y} = \frac{1}{2\pi(t+t_0)} = \frac{\sqrt{2}\pi e^{-C_0}}{t+t_0}$ for $t \geq 0$ and then get $C_0 = \ln\left(2\sqrt{2}\pi^2\right)$. □

The target manifold N is given [20] where Topping considered the harmonic map heat flow from the 2-disc D into his manifold N with a suitable initial boundary condition and proved a finite time singularity is developed at center and is "winding" in the sense of [20].

Given $t_0 > 0$ sufficiently large, the tension field $\tau(u(t))$ satisfies

$$\int_{S^2} |\tau(u(t))|^2 d\mathrm{vol}_{S^2} \;=\; \frac{\mathrm{Vol}(S^2)}{4\pi^2}\frac{1}{(t+t_0)^2} \;<\; C$$

for all $t \geq 0$. Hence we have

Corollary 2.2. *There exists a harmonic map heat flow from S^2 into N having tension uniformly bounded in L^2, but not converging anywhere as $t \to \infty$ although there exist sequences $\{t_n\}$ such that $u(t_n) \to u_\infty$ in C^∞ as $n \to \infty$.*

Proof. For $t_0 > 0$, $t_n = t_0 e^{2\pi n}$ yields such a sequence $u(t_n) \to u_\infty$ as $n \to \infty$. □

2.2 Degenerate Hessian of the energy

Define a family of smooth variations $u(s,t)\colon S^2 \to N$ in (w, z, α, θ) coordinates by

$$u(s,t)(r,\theta) \;=\; \left(s,\; t,\; \arccos\frac{-1+r^2}{1+r^2},\; \theta \right). \tag{2.2}$$

Let $W^{1,2}_1(S^2, N)$ denote the space of $W^{1,2}$ maps from S^2 to N whose restriction to the second component of $N = C \times S^2$ is a degree 1 map $S^2 \to S^2$. Note that the group $PSL(2,\mathbb{C})$ of complex automorphisms of $\mathbf{P}^1 = S^2$ acts conformally on S^2 and acts on $N = C \times S^2$ by acting trivially on the first factor. Composition then gives an action of $PSL(2,\mathbb{C})$ on $W^{1,2}_1(S^2, N)$: for $\gamma \in PSL(2,\mathbb{C})$ and $\phi \in W^{1,2}_1(S^2, N)$ the map $\gamma \cdot \phi$ is the composition $\gamma \circ \phi$.

Lemma 2.3. *The absolute minima for the energy function $E(u)$ on $W^{1,2}_1(S^2, N)$ is the set*

$$\mathcal{M} \;=\; \{\, \gamma \circ u(0,t) \mid \gamma \in PSL(2,\mathbb{C})\},$$

which is diffeomorphic to $S^1 \times PSL(2,\mathbb{C})$.

Proof. Any map u that minimizes $E(u)$ is harmonic, and hence smooth. Writing $u = (u_1, u_2)\colon S^2 \to C \times S^2$, we have

$$\begin{aligned} E(u) \;&=\; \frac{1}{2}\int_{S^2} |du_1|^2_C \;+\; f\,|du_2|^2_{S^2}\ d\mathrm{vol} \\ &\geq\; \frac{1}{2}\int_{S^2}\ f\,|du_2|^2_{S^2}\ d\mathrm{vol} \\ &\geq\; \frac{1}{2}\int_{S^2}\ |du_2|^2_{S^2}\ d\mathrm{vol} \end{aligned}$$

where the first inequality is strict unless u_1 is a map to a point $p \in C$ and the second inequality is strict unless $f(p) = 1$, which occurs only if p lies on the circle $S^1_0 = \{w = 0\}$ in C. Thus any $u \in W^{1,2}_1(S^2, N)$ with minimal energy has the form $u = (p, u_2)$ where $p \in S^1_0$ and u_2 is a degree 1 map $S^2 \to S^2$ with minimal energy. But minimal energy degree 1 maps

$\mathbf{P}^1 \to \mathbf{P}^1$ are holomorphic, and hence are elements of $PSL(2, \mathbb{C})$. $\square$

Proposition 2.4. *Along the minimum set $\mathcal{M}$ of the energy function E on $W_1^{1,2}(S^2, N)$, the Hessian of E is degenerate.*

Proof. We will show that, for each t, the maps $u(s) = u(s,t)$ defined by (2.2) are a smooth 1-parameter family of maps with u_0 in the absolute minimum set $\mathcal{M}$, $\frac{d}{ds}u(s)\Big|_{s=0}$ is not tangent to $\mathcal{M}$, and

$$\frac{d}{ds}E(u(s))\Big|_{s=0} = 0 \qquad \text{and} \qquad \frac{d^2}{ds^2}E(u(s))\Big|_{s=0} = 0. \tag{2.3}$$

The energy density for $u(s)$ is given by $e(u(s)) = \frac{1}{2}e(s)$:

$$e(s) = f(s,0) = \begin{cases} e^{-\frac{2\pi}{s}}\varphi(s) + 1 & \text{for } s > 0, \\ 1 & \text{for } s \leq 0 \end{cases}$$

where $\varphi(s) = \sin 2\pi(\frac{1}{s} - \frac{1}{8}) + \sqrt{2}$ for $s > 0$. Direct calculation shows that the derivatives of $\varphi(s)$ satisfy $\varphi^{(n)} = O(s^{-2n})$ as $s \to 0$, and that

$$\begin{aligned} e'(s) &= e^{-\frac{2\pi}{s}}\left[2\pi s^{-2}\varphi(s) + \varphi'(s)\right] \\ e''(s) &= e^{-\frac{2\pi}{s}}\left[4\pi^2 s^{-4} - 2\pi s^{-3} + 2\pi s^{-2}\varphi'(s) + \varphi''(s)\right]. \end{aligned}$$

Thus as $s \to 0$ we have $e'(s) \to 0$ and $e''(s) \to 0$ (and in fact $e^{(n)}(s) \to 0$ for all n). This gives the variations (2.3). Also note that the description of $\mathcal{M} = S^1 \times PSL(2, \mathbb{C})$ given in the proof of Lemma 2.3 shows that $\frac{d}{ds}u(s)\Big|_{s=0} = (1,0,0,0)$ is not tangent to $\mathcal{M}$. $\square$

Chapter 3

Hessian of the energy and $W^{2,2}$ convergence of harmonic map heat flows

In this chapter, we prove that the harmonic map heat flow $u(t)$ converges exponentially to a harmonic map in $W^{2,2}$ norm if the second variation of $E(u(t))$ is positive definite along $u(t)$ and the energy density $e(u(t))$ is uniformly bounded in t. Key ingredients of the proof are the first and second variations formulas (3.1) and (3.9) for the energy along the harmonic map heat flows, and $W^{2,2}$ estimates (3.4) on harmonic map heat flows. For completeness we record at the end of this chapter the proof of regularity of weakly harmonic maps from Σ under the assumption that $e(u(t))$ is uniformly bounded in t.

3.1 Preliminary lemmas

We start with the following formulas for the harmonic map heat flow.

Lemma 3.1. *For a smooth harmonic map heat flow $u(t)\colon M \times [0,T) \to N$, we have*

$$\frac{d}{dt}E(u(t)) \;=\; -\int_M |\dot{u}(t)|^2 \, dvol_g \tag{3.1}$$

and

$$E(u(t)) + \int_0^t \int_M |\dot{u}(t)|^2 \, dvol\, dt \;=\; E(u_0)\,, \quad \forall t \in [0,T). \tag{3.2}$$

Thus $E(u(t))$ is a nonincreasing function of t.

Proof. Multiplying (1.8) by $\dot{u}$ and integrating over M yields

$$\frac{d}{dt}E(u(t)) + \int_M |\dot{u}(t)|^2 \, \mathrm{dvol}_g \;=\; 0 \quad \forall t \in [0,T)\,. \tag{3.3}$$

Then (3.2) follows by integrating in t. □

For simplicity, we note that $\sup_\Sigma |du|$ are used interchangeably with $|du|_\infty$ and numbered constants below depend only on the geometry of Σ and N and $|du|_\infty$.

Lemma 3.2. *Let $u(t)\colon M \times [0,\infty) \to N$ be a harmonic map heat flow. If $\sup_\Sigma |du(t)| < C$ $\forall t > T$ for some $T \geq 0$, we have*

$$\|u(t) - u(s)\|_{2,2} \;\leq\; c_1 \left(\|\dot{u}(t) - \dot{u}(s)\|_2 + \|u(t) - u(s)\|_{1,2} \right) \quad \forall s,t > T. \tag{3.4}$$

Proof. Regarding $u(t)$ as maps into $\mathbb{R}^r$, let $v := u(t) - u(s)$. Then v is a weak solution of

$$\Delta v = (\dot{u}(t) - \dot{u}(s)) - \Phi(u(t), u(s)) \tag{3.5}$$

where $\Phi(u(t), u(s)) := A(u(t))(du(t), du(t)) - A(u(s))(du(s), du(s))$. By the elliptic estimate for the Laplacian, we have

$$\begin{aligned} \|v\|_{2,2} &\leq c_2 \left(\|\Delta v\|_2 + \|v\|_2\right) \\ &\leq c_2 \left(\|\dot{u}(t) - \dot{u}(s)\|_2 + \|\Phi(u(t), u(s))\|_2 + \|u(t) - u(s)\|_2\right). \end{aligned} \tag{3.6}$$

Since A is a smooth symmetric bilinear tensor,

$$\begin{aligned} \|\Phi(u(t), u(s))\|_2 &\leq \|\left(A(u(t)) - A(u(s))\right)(du(t), du(t))\|_2 \\ &\quad + \|A(u(s))(du(t) - du(s), du(t) + du(s))\|_2 \\ &\leq c_3 \, |\nabla A|_\infty \, |du(t)|^2_\infty \, \|u(t) - u(s)\|_2 \\ &\quad + c_4 \, |A|_\infty \, |du(t) + du(s)|_\infty \, \|u(t) - u(s)\|_{1,2} \\ &\leq c_5 \|u(t) - u(s)\|_{1,2} \quad \forall t, s > T. \end{aligned} \tag{3.7}$$

Thus (3.4) follows from (3.6) and (3.7). □

Lemma 3.3. *If $u(x,t)$ satisfies the harmonic map heat flow equation then*

$$-\frac{1}{2}\partial_t |\dot{u}|^2 = \langle \nabla^* \nabla \dot{u}, \dot{u} \rangle + tr \, \langle R_u(\dot{u}, du)\dot{u}, du \rangle, \tag{3.8}$$

and hence

$$\Delta|\dot u|^2 \;=\; \partial_t|\dot u|^2 + 2\left(|\nabla \dot u|^2 + tr\,\langle R_u(\dot u, du)\dot u, du\rangle\right). \tag{3.9}$$

Proof. By assumption, $u(t)$ satisfies $\dot u = \mathrm{tr}\nabla du$. Pick $(x_0, t_0) \in M \times \mathbb{R}$ and an orthonormal frame $\{e_\alpha\}$ on a neighborhood U of x_0 such that $[e_\alpha, e_t] = 0$, $\nabla_{e_t} e_\alpha = 0$ and $\nabla_{e_t}(\nabla_{e_\alpha} e_\beta) = 0$ at each point in U and we may assume $(\nabla e_\alpha)_{x_0} = 0$ and hence $[e_\alpha, e_\beta]_{x_0} = 0$, $\forall\ \alpha, \beta$. Then at the point (x_0, t_0),

$$\begin{aligned} \nabla_t \dot u \;&=\; \nabla_t\,(\mathrm{tr}\,\nabla du) \\ &=\; \nabla_t\,(\nabla_\alpha \nabla_\alpha u) \\ &=\; \nabla_\alpha\,(\nabla_t \nabla_\alpha u) + R_u(u_* e_t, u_* e_\alpha) u_* e_\alpha \\ &=\; \nabla_\alpha\,(\nabla_\alpha \nabla_t u) + R_u(\dot u, u_* e_\alpha) u_* e_\alpha \\ &=\; -\nabla^* \nabla \dot u + \mathrm{tr}\, R_u(\dot u, du) du, \end{aligned} \tag{3.10}$$

where we abbreviate $\nabla_\alpha = \nabla_{e_\alpha}$ and $\nabla_t = \nabla_{e_t}$. Noting that $\frac{1}{2}\partial_t|\dot u|^2 = \langle \nabla_t \dot u, \dot u\rangle$ gives (3.8) and (3.8) together with

$$\Delta|\dot u|^2 \;=\; -2\langle \nabla^* \nabla \dot u, \dot u\rangle + 2|\nabla \dot u|^2 \tag{3.11}$$

yields (3.9). ☐

Definition 3.4. *For $u \in C^\infty(M, N)$ a symmetric bilinear tensor B on u^*TN is defined as*

$$B_u(V, V) \;=\; \int_M |\nabla V|^2 + tr_g\langle R_u(V, du)V, du\rangle \; dvol_g \quad \forall V \in u^*TN \tag{3.12}$$

Remark 3.5. *If u is a harmonic map then B_u is the Hessian of the energy at u, i.e. the* *usual second variation of the energy at u.*

3.2 $W^{2,2}$ convergence of harmonic map heat flows

The following theorem holds for any closed Riemannian manifold M of any dimension.

Theorem 3.6. *Let $u(t)\colon M \times [0, \infty) \to N$ be a smooth harmonic map heat flow and suppose*

(a) there exist $T \geq 0$ and $\lambda > 0$ such that

$$B_{u(t)}(\dot{u}(t), \dot{u}(t)) \;\geq\; \lambda \, \|\dot{u}(t)\|_2^2 \qquad \forall t > T. \tag{3.13}$$

Then $u(t)$ converges in L^2 exponentially fast in t to a map $u_\infty \in L^2(M, N)$.

(b) If further $\sup_\Sigma |du(t)| < C \; \forall t > T$, the convergence $u(t) \to u_\infty$ is in $W^{2,2}$ and u_∞ is a weakly harmonic map. If $\dim M \leq 3$ then the convergence is in C^0 and if $\dim M = 2$ then u_∞ is a smooth harmonic map.

Proof. (a) Let $E(t)$ denote the energy $E(u(t))$ of the solution $u(t)$. Then

$$E'(t) \;=\; -\int_M \langle \tau(u(t)), \dot{u}(t) \rangle \, d\mathrm{vol}_g \;=\; -\|\dot{u}(t)\|_2^2 \leq 0. \tag{3.14}$$

Integrating both sides of (3.9) (noting that $\partial M = \emptyset$), and then using (3.13), we have

$$E''(t) \;=\; 2B_{u(t)}(\dot{u}(t), \dot{u}(t)) \;\geq\; 2\lambda \|\dot{u}(t)\|_2^2 \;=\; -2\lambda E'(t) \tag{3.15}$$

for all t. Integrating in t,

$$E'(t) \;\geq\; E'(T)\, e^{-2\lambda t} \;=\; -\|\tau(u(T))\|_2^2\, e^{-2\lambda t}, \tag{3.16}$$

and so

$$\|\dot{u}(t)\|_2 \;\leq\; \|\tau(u(T))\|_2\, e^{-\lambda t} \quad \forall t > T. \tag{3.17}$$

Since the map of $t \in [0, \infty) \to u(t) \in L^2(M, \mathbb{R}^r)$ is C^1, we have

$$\|u(t) - u(s)\|_2 \;\leq\; \int_t^s \|\dot{u}(\tau)\|_2\, d\tau \;\leq\; \frac{\|\tau(u(T))\|_2}{\lambda}\, e^{-\lambda t} \tag{3.18}$$

for all $T < t \leq s$ (the first inequality in (3.18) is proved in Palais [12]). Hence $\{u(t)\}$ is Cauchy in L^2 and so $\lim_{t\to\infty} u(t) = u_\infty$ exists and is unique in $L^2(M, \mathbb{R}^r)$. Moreover, $u_\infty(x) \in N$ almost everywhere x, since $u(t)(x) \in N\ \forall (x,t)$.

(b) Now suppose that $|du(t)|_\infty < C \quad \forall t > T$. By interpolation, let $C = C(\epsilon) > 0$ be a positive constant such that

$$\|u(t) - u(s)\|_{1,2} \;\leq\; \epsilon\|u(t) - u(s)\|_{2,2} + C(\epsilon)\|u(t) - u(s)\|_2. \tag{3.19}$$

Taking $\epsilon = \frac{1}{2c_1}$ and using (3.4), we have

$$\begin{aligned} \|u(t) - u(s)\|_{2,2} \;&\leq\; c_6\,(\|\dot{u}(t) - \dot{u}(s)\|_2 + \|u(t) - u(s)\|_2) \\ &\leq\; c_7\, e^{-\lambda t} \quad \forall t \leq s, \end{aligned} \tag{3.20}$$

where $c_7 = c_6\|\tau(u(T))\|_2 \left(2 + \frac{1}{\lambda}\right)$. Thus $u(t)$ is also Cauchy in $W^{2,2}$ and so converges to u_∞ in $W^{2,2}$ as $t \to \infty$. Moreover, (3.17) implies

$$\|\Delta u_\infty + A(du_\infty, du_\infty)\|_2 = \lim_{t\to\infty} \|\Delta u(t) + A(du(t), du(t))\|_2 = \lim_{t\to\infty} \|\dot{u}(t)\|_2 = 0.$$

If $\dim M \leq 3$ then C^0 convergence follows from the Sobolev embedding theorem. If $\dim M = 2$ then the well-known theorem of Hélein [7], or Lemma 3.8 below, implies that u_∞ is a smooth harmonic map.

□

Remark 3.7. *Under the hypotheses of Theorem 3.6 with $M = \Sigma^2$, u_∞ is the unique harmonic map in the weak $W^{1,2}$ closure of the flow $\{u(t) \mid t > T\}$. In particular, the weakly convergent subsequences $\{u_n\}$ of Struwe's Theorem 1.6 all converge to u_∞.*

Instead of using Hélein's theorem, we can use the following bootstrap argument in the last sentence of the proof of Theorem 3.6.

Lemma 3.8. *If $\dim \Sigma = 2$, any weakly harmonic $W^{1,2}$ map $u\colon \Sigma \to N$ with $\sup_\Sigma |du| < C$ is smooth.*

Proof. Using $\sup_\Sigma |du| < C$ and Hölder inequality, we have $u \in W^{2,p}(\Sigma, N)$ $\forall p \in (2, \infty)$, since

$$\begin{aligned} \|u\|_{2,p} &\leq c_8(\|\Delta u\|_p + \|u\|_p) \\ &\leq c_9(\|A(u)(du, du)\|_p + \|u\|_p) \\ &\leq c_{10}(1 + \|u\|_2), \end{aligned} \tag{3.21}$$

where the constant c_{10} depends on $p > 2$ and $\mathrm{Vol}(\Sigma)$. By the Sobolev embedding theorem

$W^{2,p} \hookrightarrow C^1$, $u \in C^1(\Sigma, N)$.

Using the Bochner formula for functions,

$$
\begin{aligned}
|\Delta \nabla_\alpha u| &= |\nabla_\alpha \Delta u + R_{\alpha\beta} \nabla_\beta u| \\
&\leq \big|\nabla_\alpha \big(A(u)(du, du)\big)\big| + \big|R_{\alpha\beta} \nabla_\beta u\big| \\
&\leq |\nabla A|_\infty |du| + 2|A|_\infty |\nabla^2 u|\,|du| + |R|_\infty |du| \\
&\leq c_{11} \left(|\nabla^2 u|^2 + |du|^2 + |du|^2 \right),
\end{aligned}
$$

where the last inequality holds by Young's inequality. By (3.21), we have

$$
\|u\|_{3,p} \leq c_{12} \|u\|_{2,2p}^2 < \infty \tag{3.22}
$$

for all $p < \infty$. Hence $u \in C^2(M, N)$. By induction, we have

$$
\Delta \nabla_{\alpha_1} \cdots \nabla_{\alpha_{k+1}} u = \nabla_{\alpha_1} \cdots \nabla_{\alpha_{k+1}} \Delta u + \text{lower order terms.}
$$

Hence $\|u\|_{k+1,p} \leq c_{13} \sum_{l=0}^{k} \|u\|_{l,p}$ $\forall p$ and thus $u \in C^k(M, N)$ for each k. $\square$

Chapter 4

Continuity of Hessian

In this chapter we discuss topologies on the space of maps to work with and then prove that the symmetric bilinear tensor B is continuous on the space of maps in the $W^{1,2} \cap C^0$ topology. Roughly, this implies that the symmetric bilinear tensor B is positive definite at u nearby the accumulation point u_∞ under the assumptions (a) and (b) of Theorem A.

4.1 Sobolev spaces of maps

For a closed Riemann surface Σ let $\mathcal{M}$ denote the completion of $C^\infty(\Sigma, N)$ with respect to the $W^{1,p}$ norm $(p > 2)$ where N is isometrically embedded in $\mathbb{R}^r$ as above. For $\dim \Sigma = 2$, the Sobolev embedding theorem implies that $\mathcal{M}$ embeds continuously in $C^{1-\frac{2}{p}}(\Sigma, N)$ and hence $\mathcal{M}$ is a C^2 separable Banach manifold. Then $T_u\mathcal{M}$ is the set of $W^{1,p}$ vector fields along image:

$$T_u\mathcal{M} = \left\{V \in \Gamma\left(u^*TN\right) \mid \|V\|_{1,p}^p < \infty\right\}, \tag{4.1}$$

where $\|V\|_{1,p}^p = \int_M |\nabla V|^p + |V|^p \, d\mathrm{vol}_g$. Hence $\mathcal{M}$ carries a weak L^2 Riemannian metric given by

$$\langle V, W\rangle_2 \;=\; \int_\Sigma \langle V, W\rangle \, d\mathrm{vol}_g$$

and a weak $W^{1,2}$ Riemannian metric given by

$$\langle V, W\rangle_{1,2} \;=\; \int_\Sigma \langle \nabla V, \nabla W\rangle + \langle V, W\rangle \, d\mathrm{vol}_g \tag{4.2}$$

for $V, W \in T_u\mathcal{M}$. Note that $T_u\mathcal{M}$ is not complete with respect to either L^2 or $W^{1,2}$ topology.

Lemma 4.1. *Suppose that the symmetric bilinear form B is positive definite at $u \in \mathcal{M}$ with respect to $\langle \cdot, \cdot \rangle_2$: there exists $\lambda > 0$ such that*

$$B_u(V, V) \;\geq\; \lambda \|V\|_2^2 \qquad \forall V \in u^*(TN). \tag{4.3}$$

Then B is positive definite at u with respect to $\langle \cdot, \cdot \rangle_{1,2}$: there exists $\mu > 0$ depending on Σ, N, λ and $|du|_\infty$ such that

$$B_u(V, V) \;\geq\; \mu \|V\|_{1,2}^2 \qquad \forall V \in u^*(TN). \tag{4.4}$$

The constant $\mu > 0$ is uniform on any set of maps with a uniform bound $|du|_\infty < C$.

Proof. Using the Hölder inequality and the interpolation inequality,

$$\begin{aligned}
\int_\Sigma |\nabla V|^2 + |V|^2 \, d\mathrm{vol}_g \;&\leq\; B_u(V, V) + 2|R|_\infty \int_\Sigma |V|^2 |du|^2 \, d\mathrm{vol}_g + \int_\Sigma |V|^2 \, d\mathrm{vol}_g \\
&\leq\; B_u(V, V) + 2|R|_\infty \|V\|_4^2 \|du\|_4^2 + \int_\Sigma |V|^2 \, d\mathrm{vol}_g \\
&\leq\; B_u(V, V) + c_{14}\, \epsilon \|\nabla V\|_2^2 + (1 + C(\epsilon))\, \|V\|_2^2,
\end{aligned}$$

where $c_{14} = c_{14}\left(|R|_\infty,\ |du|_\infty, \mathrm{Vol}(\Sigma)\right)$. Taking $\epsilon = \dfrac{1}{2c_{14}}$ and absorbing the second term to the left hand side,

$$\begin{aligned}\|V\|_{1,2}^2 &\leq 2B_u(V,V) + c_{15}\|V\|_2^2 \\ &\leq \left(2 + \frac{c_{15}}{\lambda}\right) B_u(V,V).\end{aligned}$$

□

4.2 Continuity of Hessian

Fix an isometric embedding $N \hookrightarrow \mathbb{R}^r$ with a second fundamental form A. For small $\epsilon > 0$ and let N_ϵ be an ϵ-neighborhood of N in $\mathbb{R}^r$ such that the nearest projection map $\mathrm{proj}\colon N_\epsilon \to N$ is well-defined. Let $u, v \in C^\infty(\Sigma, N)$. For $V \in \Gamma(u^*TN)$, there is a corresponding $\hat{V} \in v^*TN$ such that $\hat{V} = \pi_v V$ via the composition

$$u^*TN \hookrightarrow u^*T\mathbb{R}^r \backsimeq \Sigma \times \mathbb{R}^r \backsimeq v^*T\mathbb{R}^r \stackrel{\pi_v}{\to} v^*TN$$

where $\pi_v\colon v^*\mathbb{R}^r \to v^*TN$ is the orthogonal projection to the image of v. Let D denote the Levi-Civita connection on $\mathbb{R}^r$. For $X \in T\Sigma$, $A(u)(u_*X, V)$ is the normal component of $D_X V$ such that

$$\begin{aligned}\pi_u &= d\mathrm{proj}_u, \\ D\pi_u(X,V) &= A(u)(u_*X, V), \\ \nabla_X V &= D_X V - A(u)(u_*X, V).\end{aligned}$$

Note the sign convention.

Now extend A smoothly to a tensor in a neighborhood N_ϵ of N in $\mathbb{R}^r$. For Theorem 4.3 we estimate the difference between the second fundamental forms at two points of N.

Lemma 4.2. *There exists a constant $c_{16} > 0$ depending on the geometry of N such that*

$$\begin{aligned} \big| |A(u)(V,W)| - |A(v)(\hat{V},\hat{W})| \big| \\ \leq\; c_{16}\big(|u-v|\,|V|\,|W| + |V-\hat{V}|\,|W| + |\hat{V}|\,|W-\hat{W}|\big), \end{aligned} \tag{4.5}$$

where $c_{16} > 0$ depends on the geometry of N.

Proof. Because N is compact, both $|A|$ and $|\nabla A|$ are bounded. We can then write

$$\begin{aligned} & A(u)(V,W) - A(v)(\hat{V},\hat{W}) \\ &= (A(u)-A(v))(V,W) + A(v)(V-\hat{V},W) + A(v)(\hat{V},W-\hat{W}), \end{aligned}$$

and note that the Mean Value Theorem implies that $|A(u)-A(v)| \leq C\,\mathrm{dist}(u,v)$ for some $C > 0$. Lemma 4.2 follows because the Riemannian distance $\mathrm{dist}(u,v)$ in N is uniformly equivalent to the euclidean distance $|u-v|$. □

Define the $W^{1,2} \cap C^0$ topology by the norm

$$|||u-v||| \;:=\; \|u-v\|_{1,2} + \sup_{x\in\Sigma} |u(x)-v(x)| \tag{4.6}$$

for maps $u, v\colon \Sigma \to N \hookrightarrow \mathbb{R}^r$.

Theorem 4.3. *Suppose there exists $\lambda > 0$ such that*

$$B_v(V,V) \ \geq \ \lambda\|V\|_2^2 \quad \forall V \in \Gamma(v^*TN).$$

For $C > 0$ given, there exist $\mu = \mu(\Sigma, N, \lambda, C) > 0$ and a $W^{1,2} \cap C^0$ neighborhood $\mathcal{U}$ of v such that

$$B_u(V,V) \ \geq \ \mu\|V\|_2^2 \quad \forall V \in \Gamma(u^*TN)$$

for all $u \in \mathcal{U}_C := \mathcal{U} \cap \{\sup_\Sigma |du| < C\}$.

Proof. Fix $v\colon \Sigma \to N$ and let $\delta_1 < \frac{1}{2}\mathrm{inj}(N)$ be a small positive number to be chosen later and set $\delta := \sup_{x\in\Sigma} \mathrm{dist}(u(x), v(x)) < \delta_1$. Write

$$\begin{aligned} & B_u(V,V) - B_v(\hat{V},\hat{V}) \\ &= \int_\Sigma |\nabla V|^2 - |\nabla \hat{V}|^2 \, d\mathrm{vol}_g + \int_\Sigma \mathrm{tr}_g\langle R_u(V,du)V, du\rangle - \mathrm{tr}_g\langle R_v(\hat{V},dv)\hat{V}, dv\rangle \, d\mathrm{vol}_g \\ &= I \ + \ II. \end{aligned} \tag{4.7}$$

(I) Let $\{e_\alpha\}$ be a local orthonormal frame. For $V \in \Gamma(u^*TN)$, we have

$$\begin{aligned} \nabla_\alpha \hat{V} &= D_\alpha \hat{V} - A(v)(v_* e_\alpha, \hat{V}) \\ &= D_\alpha(\pi_v V) - A(v)(v_* e_\alpha, \hat{V}) \\ &= D\pi_v(v_* e_\alpha, V) + \pi_v(D_\alpha V) - A(v)(v_* e_\alpha, \hat{V}) \\ &= A(v)(v_* e_\alpha, V - \hat{V}) + \pi_v(D_\alpha V), \end{aligned}$$

where

$$\begin{aligned}\pi_v(D_\alpha V) &= (\pi_v - \pi_u)(D_\alpha V) + \pi_u(D_\alpha V)\\ &= (\pi_v - \pi_u)(D_\alpha V) + \nabla_\alpha V.\end{aligned}$$

Applying the Mean Value Theorem with $\pi = d\text{proj}$, we have the bounds

$$\begin{aligned}|\pi_v - \pi_u| &\leq c_{17}\,\delta,\\ |V - \hat{V}| = |(\pi_v - \pi_u)V| &\leq c_{18}\,\delta\,|V|.\end{aligned} \tag{4.8}$$

Noting

$$|D_\alpha V| \leq |\nabla_\alpha V| + |A|_\infty\,|du|_\infty |V| \leq c_{19}(|\nabla_\alpha V| + |V|),$$

we have

$$\begin{aligned}\big|\,|\nabla_\alpha V| - |\nabla_\alpha \hat{V}|\,\big| &\leq |A(v)(v_* e_\alpha, V - \hat{V})| + |(\pi_v - \pi_u)(D_\alpha V)|\\ &\leq c_{20}\,\delta(|\nabla_\alpha V| + |V|),\end{aligned} \tag{4.9}$$

where $c_{20} = c_{20}(|A|_\infty, |du|, |dv|)$. And also,

$$\begin{aligned}|\nabla_\alpha V| + |\nabla_\alpha \hat{V}| &\leq 2\,|\nabla_\alpha V| + c_{21}\,\delta(|\nabla_\alpha V| + |V|)\\ &\leq c_{22}\,(|\nabla_\alpha V| + |V|),\end{aligned} \tag{4.10}$$

where $c_{22} = c_{22}(|A|_\infty, |du|, |dv|)$. Then (4.9) and (4.10) yield

$$\begin{aligned} I &\leq \int_\Sigma (|\nabla V| + |\nabla \hat{V}|)\,\big|\,|\nabla V| - |\nabla \hat{V}|\,\big|\; d\mathrm{vol}_g \\ &\leq c_{23}\,\delta \int_\Sigma (|\nabla V| + |V|)^2 \; d\mathrm{vol}_g \\ &\leq c_{24}\,\delta\,\|V\|^2_{1,2}. \end{aligned} \tag{4.11}$$

(II) For $N \hookrightarrow R^r$ the Gauss-Codazzi equation becomes

$$\langle R(X,Y)Z,W\rangle \;=\; A(X,W)\cdot A(Y,Z) - A(X,Z)\cdot A(Y,W)$$

for $X, Y, Z, W \in TN$. Hence we have

$$\big|\langle R(u)(V, u_* e_\alpha)V, u_* e_\alpha\rangle - \langle R(v)(\hat{V}, v_* e_\alpha)\hat{V}, v_* e_\alpha\rangle\big| \;\leq\; A_1 \;+\; A_2,$$

where

$$\begin{aligned} A_1 &:= \big|\,|A(u)(V, u_* e_\alpha)|^2 - |A(v)(\hat{V}, v_* e_\alpha)|^2\,\big|, \\ A_2 &:= \big|A(u)(V,V)\cdot A(u)(u_* e_\alpha, u_* e_\alpha) - A(v)(\hat{V},\hat{V})\cdot A(v)(v_* e_\alpha, v_* e_\alpha)\big|. \end{aligned}$$

(A_1) Noting (4.8), $|\hat{V}| \;\leq\; |V|$ and Lemma 4.2, we have

$$\begin{aligned} A_1 &\leq \big(|A(u)(V, u_* e_\alpha)| + |A(v)(\hat{V}, v_* e_\alpha)|\big)\,\big|\,|A(u)(V, u_* e_\alpha)| - |A(v)(\hat{V}, v_* e_\alpha)|\,\big| \\ &\leq c_{25}\,|V|\,(\,\delta\,|V| + |du - dv|\,|V|) \\ &= c_{25}(\,\delta\,|V|^2 + |du - dv|\,|V|^2), \end{aligned} \tag{4.12}$$

where $c_{25} = c_{25}(|\nabla A|_\infty, |A|_\infty, |du|, |dv|)$.

(A_2) Using Lemma 4.2,

$$\begin{aligned} A_2 &\leq \big|A(u)(V,V) - A(v)(\hat{V},\hat{V})\big|\,\big|A(u)(u_*e_\alpha, u_*e_\alpha)\big| \\ &\quad + \big|A(v)(\hat{V},\hat{V})\big|\,\big|A(u)(u_*e_\alpha, u_*e_\alpha) - A(v)(v_*e_\alpha, v_*e_\alpha)\big| \\ &\leq c_{26}\,\delta\,|V|^2 + c_{27}|V|^2(\delta + |du - dv|) \\ &\leq c_{28}\,(\delta|V|^2 + |du - dv|\,|V|^2), \end{aligned} \tag{4.13}$$

where $c_{28} = c_{28}(|\nabla A|_\infty, |A|_\infty, |du|, |dv|)$. But the Hölder inequality and the Sobolev embedding $W^{1,2} \hookrightarrow L^4$ give

$$\begin{aligned} \int_\Sigma |du - dv|\,|V|^2\,d\mathrm{vol}_g &\leq \|du - dv\|_2\,\|V\|_4^2 \\ &\leq c_{29}\,\|u - v\|_{1,2}\,\|V\|_{1,2}^2\,. \end{aligned} \tag{4.14}$$

Combining (4.12), (4.13) and (4.14), we have

$$II \leq c_{30}\left(\delta + \|u - v\|_{1,2}\right)\|V\|_{1,2}^2\,. \tag{4.15}$$

Hence, from (4.11) and (4.15),

$$B_u(V,V) \geq B_v(\hat{V},\hat{V}) - c_{31}\left(\delta + \|u - v\|_{1,2}\right)\|V\|_{1,2}^2, \tag{4.16}$$

where $c_{31} = c_{31}(\Sigma, N, |dv|_\infty, |du|_\infty)$.

Now we prove Theorem 4.3. Using Lemma 4.1 and (4.8), one can choose a positive number $\mu_1 > 0$, depending on $|du|_\infty$, such that

$$B_v(\hat{V}, \hat{V}) \;\geq\; \mu_1 \|V\|_{1,2}^2 \quad \forall V \in \Gamma(u^* TN). \tag{4.17}$$

Let $\delta_1 > 0$ be a positive number with $\delta_1 < \dfrac{\mu_1}{2\, c_{31}}$. If $\delta + \|u - v\|_{1,2} < \delta_1$, we have, from (4.16) and (4.17),

$$\begin{aligned} B_u(V, V) \;&\geq\; \mu_1 \, \|V\|_{1,2} - c_{31} \, \delta_1 \, \|V\|_{1,2}^2 \\ &\geq\; \frac{\mu_1}{2} \, \|V\|_{1,2}^2. \end{aligned}$$

□

Chapter 5

Proof of the main theorem

In this chapter we complete the proof of our main result Theorem A. To that end, we consider a harmonic map heat flow $u(t) : \Sigma \to N$ whose energy density is uniformly bounded:

$$\sup_{\Sigma} |du(t)| \ \leq \ C \quad \text{for all } t > T_0. \tag{5.1}$$

Lemma 5.1 below shows that this bound implies that the flow is adherent to a smooth harmonic map u_∞ at $t \to \infty$. We then add the hypothesis that this limit u_∞ is stable, that is, the Hessian of the energy function is positive definite (in the sense of Definition 3.4) at u_∞. With these hypotheses, Proposition 5.2 implies that at some large time, the flow enters and remains in a neighborhood of u_∞ in which the symmetric bilinear tensor B is positive definite. Theorem A then follows from the exponential convergence proved in Chapter 3.

To start, consider a harmonic map heat flow $u(t) : \Sigma \to N$ satisfying (5.1). This assumption implies that the maps $u(t)$ are uniformly bounded in $W^{1,p}$ for any p, so by the compactness of the Sobolev embedding $W^{1,p} \to C^0$ for $p > 2$, there is a sequence $t_n \to \infty$

such that the maps $u(t_n)$, which we denote as u_n, converge in C^0. In fact, using the heat flow, we can also assume that the u_n converges in $W^{1,2}$ as follows.

Lemma 5.1. *For any harmonic map heat flow $u(t) : \Sigma \to N$ satisfying (5.1) , there is a sequence $t_n \to \infty$ such that the maps $u_n = u(t_n)$ converge in $W^{1,2} \cap C^0$ to a smooth harmonic map u_∞.*

Proof. As above let $t_n \to \infty$ be a sequence such that u_n converges in C^0 and so does in L^2. By Lemma 3.1 we have

$$\int_t^{t+1} \int_\Sigma |\dot{u}(t)|^2 \, d\mathrm{volg} \to 0 \quad \text{as } t \to \infty. \tag{5.2}$$

Hence, after passing to a subsequence and relabeling, there exists a sequence $t_n \to \infty$ such that $\|\dot{u}_n\|_2 \to 0$ as $n \to \infty$. Under the assumption (5.1), the elliptic estimate for Δ shows

$$\|u_n\|_{2,2} \ \leq \ c_{32} \left(\|\dot{u}_n\|_2 + \|u_n\|_{1,2} \right).$$

Interpolating the second term and absorbing to the left side, we have

$$\|u_n\|_{2,2} \ \leq \ c_{33} \left(\|\dot{u}_n\|_2 + \|u_n\|_2 \right).$$

Hence u_n converges to u_∞ in $W^{2,2}$, and so in $W^{1,2}$. Moreover, u_∞ is a harmonic map, since $\|\dot{u}_n\|_2 \to 0$ as $n \to \infty$. □

The $W^{1,2} \cap C^0$ norm $|||u||| = \|u\|_{1,2} + \sup_{x \in \Sigma} |u(x)|$ used in (4.6) defines neighborhoods of u_∞

$$\mathcal{U}_\delta \ := \ \{u \mid |||u - u_\infty||| < \delta\}.$$

Proposition 5.2. *Let $u(t)\colon \Sigma \times [0,\infty) \to N$ be a harmonic map heat flow and let $\delta_1 > 0$ be the constant in the proof of Theorem 4.3. Suppose that*

(a) B is positive definite at u_∞

(b) $\sup_\Sigma |du(t)| < C \quad \forall t.$

For each $\delta < \delta_1$ there exist a $T > 0$ such that if $t > T$, $u(t)$ lies in the neighborhood $\mathcal{U}_{\delta,C}$ of u_∞ defined by

$$\mathcal{U}_{\delta,C} \ := \ \mathcal{U}_\delta \cap \{u \,|\sup_\Sigma |du| < C\}.$$

Proof. Choose a sequence $t_n \to \infty$ such that u_n converges to a harmonic map u_∞ in $W^{1,2} \cap C^0$ as in Lemma 5.1. By Theorem 4.3 with $v = u_\infty$ we have

$$B_u(V,V) \ \geq \ \mu \|V\|_2^2 \quad \forall V \in \Gamma(u^*TN) \tag{5.3}$$

for $u \in \mathcal{U}_{\delta,C}$. After passing to a subsequence, we can assume that

$$\|u(t_n) - u_\infty\| \ < \ \frac{\delta_1}{4} \quad \forall n.$$

Note that Theorem 1.6 implies that u is smooth on $\Sigma \times (0,\infty)$ except for finitely many points of $\Sigma \times (0,\infty)$ and hence we can assume $u \in C^\infty(\Sigma \times [t_1,\infty), N)$. For each n, let

$$T_n \ := \ \sup\{t \mid \|u(s) - u_\infty\| < \delta_1 \ \forall s \in [t_n, t]\}.$$

If $T_n = \infty$ for some n, we are done. Suppose $T_n < \infty$ for each n. We then have $T_n > t_n$ and $\|u(t_n) - u_\infty\| = \delta_1$ for each n, since $\|u(s) - u_\infty\|$ is a continuous function in s on

$[t_1, \infty)$. Note that $B_{u(s)}$ satisfies the condition (5.3) for all $s \in [t_n, T_n)$. Then by the Sobolev inequality and (3.20) we have

$$\|u(s) - u(t_n)\| \;\leq\; c_{34}\, \|u(s) - u(t_n)\|_{2,2} \;\leq\; c_{35}\, e^{-\mu t_n} \quad \forall s \in [t_n, T_n) \qquad (5.4)$$

for each n.

Choose n large enough that the right side of (5.4) is less than $\frac{1}{4}\delta_1$. Then

$$\|u(s) - u_\infty\| \;\leq\; \|u(s) - u(t_n)\| + \|u(t_n) - u_\infty\| \;<\; \frac{\delta_1}{4} + \frac{\delta_1}{4} \;=\; \frac{\delta_1}{2}$$

for all $s \in [t_n, T_n)$. Hence, we have $\|u(T_n) - u_\infty\| \leq \frac{1}{2}\delta_1$, a contradiction. □

Theorem A. *Let $u\colon \Sigma^2 \times [0, \infty) \to N$ be a smooth harmonic map heat flow such that $u(t)$ converges to a harmonic map u_∞ weakly in $W^{1,2}$. Suppose that*

(a) the Hessian of the energy E is positive definite at u_∞ and

(b) $\sup_\Sigma |du(t)| < C$ for all large $t > T$.

Then $u(t)$ converges to u_∞ exponentially fast in $W^{2,2}$ and hence in C^1.

Proof. By Proposition 5.2 and Theorem 4.3, one can find a $T > 0$ so that the symmetric bilinear tensor B is positive definite along $u(t)$ for all $t > T$. Hence Theorem A follows from Theorem 3.6. □

Chapter 6

Examples of geodesic heat flows

A classical result in Riemannian geometry asserts that every map $u_0\colon S^1 \to M$ into a closed Riemannian manifold (M, h) is homotopic to a closed geodesic. Intuitively, this can be proven by deforming u_0 by following the flow of the downward gradient vector field of the energy function

$$E(u) \;=\; \frac{1}{2}\int_{S^1} |du|^2 \, d\theta \tag{6.1}$$

on the free loop space of maps $u\colon S^1 \to M$. It is standard to do this using the $W^{1,2}$ gradient flow on the space of maps, but alternatively one can use the heat flow. Specifically, given an $W^{1,2}$ map $u_0\colon S^1 \to M$, we can consider the "geodesic heat flow"

$$\dot{u} \;=\; \nabla_T T \qquad\qquad u(\theta, 0) = u_0(\theta) \tag{6.2}$$

One expects that this heat flow produces maps $u(t)\colon S^1 \to M$ that converge to a closed geodesic at $t \to \infty$. Standard results (cf. [8], [1]) show that the flow exists for all time, that $u(t)$ is smooth for each $t > 0$ and that there is a sequence $t_n \to \infty$ so that $u(t_n)$ converges in $W^{1,2}$ to a closed geodesic. But, surprisingly, it is not known whether the flow converges (without extracting a sequence).

Actually, convergence is claimed in [8]. Unfortunately, the proof relies on the assertion that the energy $E(t) = E(u(t))$ along the flow is a convex function of t, and the proof of this assertion in [8] has an error. In this chapter we give examples of geodesic heat flows that show that

- the energy function $E(t)$ need not be convex, and
- the geodesic flow may not converge.

Convergence results for the geodesic flow will be given in [1].

6.1 Geodesic heat flows on surfaces of revolution

In this section, we derive the equation for the harmonic map heat flow $u(t)$ from S^1 into a surface of revolution $M \subset \mathbb{R}^3$, for $u(t)$ being equivariant for the standard S^1 action that fixes the z-axis.

Consider a surface of revolution M in $\mathbb{R}^3$ obtained by rotating a curve $r = f(z) \geq 0$ about the z-axis and imagine a circle S^1 sitting in M as in Figure 6.1 below. Introducing the angle coordinate θ to S^1 and the cylindrical coordinates (r, θ, z) to $\mathbb{R}^3$, one writes

$$u(\theta, t) \ = \ (\, f(z(t))\,,\ \theta\,,\ z(t)\,) \in M \subset \mathbb{R}^3 \tag{6.3}$$

Since the surface of revolution M is the graph of the map $\varphi\colon [0, 2\pi) \times \mathbb{R} \to M$ given by

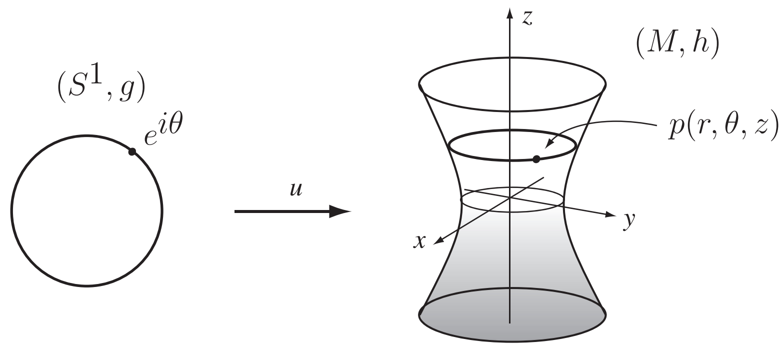


Figure 6.1: $u\colon (S^1, g) \to (M, h)$

$\varphi(\theta, z) = (f(z), \theta, z)$, we can use φ^{-1} as (global) coordinates for M as

$$\varphi^{-1} \circ u(\theta, t) \;=\; (\,\theta\,,\, z(t)\,) \;:=\; (\,u^1(\theta, t)\,,\, u^2(\theta, t)\,). \tag{6.4}$$

In coordinates θ for S^1 and (u^1, u^2) for M, the metric on the domain is given by

$$g \;=\; d\theta^2, \tag{6.5}$$

and the metric on M is given by

$$\begin{aligned} h \;&=\; \left(dr^2 \,+\, r^2 d\theta^2 \,+\, dz^2\right)|_M \\ &=\; \left(\frac{df}{dz}\right)^2 dz^2 \,+\, f^2 d\theta^2 \,+\, dz^2 \\ &=\; f^2\, d\theta^2 \,+\, \left((f')^2 + 1\right) dz^2 \end{aligned} \tag{6.6}$$

for $f' = \frac{df}{dz}$. Then the geodesic heat flow equation $\dot{u} \;=\; \Delta u$ is given in coordinates by

$$\dot{u} \;=\; g^{\theta\theta} \sum_{i=1}^{2} \left(\frac{\partial^2 u^i}{\partial \theta^2} \;+\; \sum_{j,k=1}^{2} \Gamma^i_{jk}(u) \, \frac{\partial u^j}{\partial \theta} \, \frac{\partial u^k}{\partial \theta} \right) \frac{\partial}{\partial u^i}. \tag{6.7}$$

Using the definition of u^1 and u^2, the right side of (6.7) reduces to

$$\begin{aligned} \dot{u} \;&=\; \Gamma^1_{11} \frac{\partial}{\partial u^1} \;+\; \Gamma^2_{11} \frac{\partial}{\partial u^2} \\ &=\; 0 \, \frac{\partial}{\partial u^1} \;-\; \frac{1}{2} h^{22} \, h_{11,2} \, \frac{\partial}{\partial u^2} \\ &=\; - \, \frac{f \, f'}{(f')^2 + 1} \, \frac{\partial}{\partial u^2} \end{aligned} \tag{6.8}$$

where we used the formula for the Christoffel symbols for M.

On the other hand, we have $\dot{u} = \dot{z} \, \frac{\partial}{\partial u^2}$. Hence the harmonic map heat flow equation is a nonlinear first order ODE: for $z = z(t)$,

$$\dot{z} \;=\; - \, \frac{f(z) \, f'(z)}{\big(f'(z)\big)^2 + 1}. \tag{6.9}$$

6.2 The energy function $E(t)$ needn't be convex.

We will find the solution of a S^1 equivariant harmonic map heat flow $u(t)$ from S^1 into S^2 whose energy function $E(u(t))$ is not convex. For future use, we present two ways of finding the solution via cylindrical and spherical coordinates.

(Method 1) In cylindrical coordinates (r, θ, z), S^2 as a surface of revolution about the z-axis is obtained by rotating the graph $r = \sqrt{1 - z^2}$. Applying (6.9) with $f(z) = \sqrt{1 - z^2}$, the S^1 equivariant harmonic map heat flow equation is

$$\dot{z} \;=\; z(1-z^2). \tag{6.10}$$

Separating variables and taking integrals, we have

$$\int \frac{dz}{z(1-z^2)} \;=\; \int dt \;=\; t+C \tag{6.11}$$

for some constant C. Using the partial fraction expansion, the left integral of (6.11) is

$$\int \frac{dz}{z(1-z^2)} \;=\; \ln \frac{|z|}{\sqrt{1-z^2}}.$$

Noting $z = \cos\alpha$ in spherical coordinates (θ, α) on S^2, we have

$$e^{t+C} \;=\; \frac{\pm\cos\alpha}{\sqrt{1-\cos^2\alpha}} \;=\; \cot\alpha.$$

Thus we have, for a constant C,

$$\begin{aligned} \alpha &= \operatorname{arccot} e^{t+C}, \\ z &= \cos \operatorname{arccot} e^{t+C}. \end{aligned} \tag{6.12}$$

(Method 2) Let (M, h) be the unit sphere S^2 in $\mathbb{R}^3$. In spherical coordinates (ρ, θ, α) in $\mathbb{R}^3$, the harmonic map heat flow equation for maps $u\colon S^1 \times [0,\infty) \to S^2$ is

$$\begin{aligned} \dot{u} &= \tau(u) \\ &= g^{\theta\theta} \sum_{i=1}^{2} \left(\frac{\partial^2 u^i}{\partial \theta^2} + \sum_{j,k=1}^{2} \Gamma^i_{jk}(u) \frac{\partial u^j}{\partial \theta} \frac{\partial u^k}{\partial \theta} \right) \frac{\partial}{\partial u^i} \end{aligned} \tag{6.13}$$

where $u(\theta,t) = (\ \theta\ ,\ \alpha(t)\ ) := (\ u^1(\theta,t)\ ,\ u^2(\theta,t)\ )$ in spherical coordinates (θ,α) of S^2 (see Figure 6.2).

Since $\rho \equiv 1$ on S^2, the metric h on S^2 is given by

$$\begin{aligned} h &= \left(d\rho^2 + \rho^2 \sin^2\alpha\, d\theta^2 + \rho^2\, d\alpha^2 \right) |_M \\ &= \sin^2\alpha\, d\theta^2 + d\alpha^2. \end{aligned} \tag{6.14}$$

Then the same computation as in (6.8) shows

$$\dot{u} = \dot{\alpha} \frac{\partial}{\partial u^2} = -\frac{1}{2} h^{22} h_{11,2} \frac{\partial}{\partial u^2} = -\frac{1}{2} \sin 2\alpha \frac{\partial}{\partial u^2}. \tag{6.15}$$

Thus the harmonic map heat flow equation is

$$\dot{\alpha} = -\frac{1}{2} \sin 2\alpha. \tag{6.16}$$

Using the relation $z = \cos\alpha$ between two coordinates on S^2, it is straightforward that (6.16) is equivalent to (6.10).

We compute the energy function $E(t)$ with the solution $\alpha(t) = \operatorname{arccot} e^{t+C}$ where the constant C will be determined later. From (6.5) and (6.14), the energy density $e(u(t))$ of

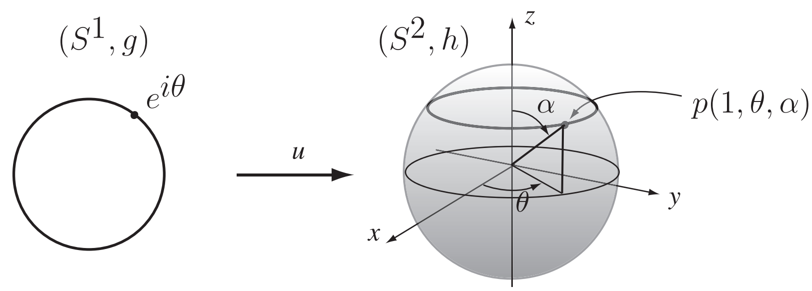


Figure 6.2: $u\colon (S^1, g) \to (S^2, h)$

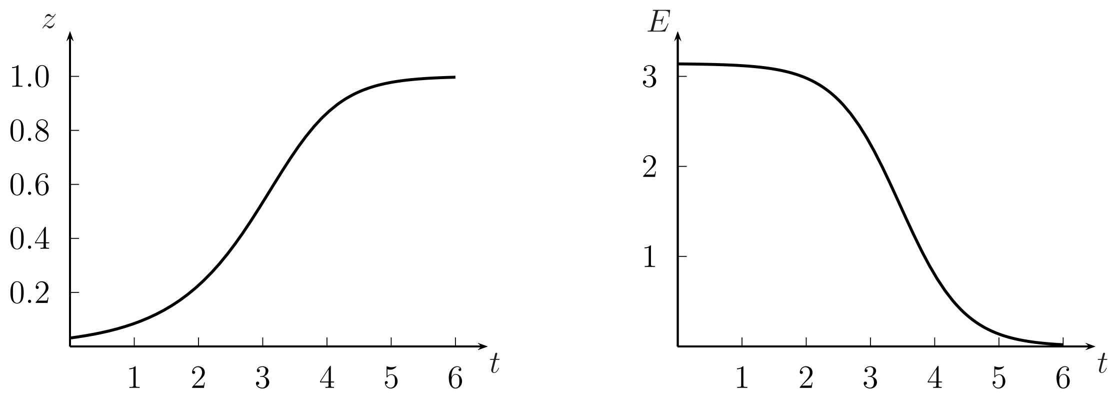
$u(\theta, t) = (\, u^1(\theta, t)\, ,\, u^2(\theta, t)\,) = (\theta, \alpha(t))$ is

$$e(u(t)) \;=\; \frac{1}{2}\,|du(t)|^2 \;=\; \frac{1}{2}\,h_{11} = \;\frac{1}{2}\,\sin^2\alpha,$$

since $\dfrac{\partial u^1}{\partial\theta} = 1$ and $\dfrac{\partial u^2}{\partial\theta} = 0$. Consequently, the energy $E(t)$ is

$$E(t) \;=\; \frac{1}{2}\int_{S^1} |du(t)|^2\, d\theta \;=\; \pi\,\sin^2\alpha(t) \;=\; \frac{\pi\, e^{-2(t+C)}}{1 + e^{-2(t+C)}}.$$

This formula implies that if the initial map $u(0)$ is a circle close to the north pole, say $\alpha(0) = .01$, then $u(t)$ approaches the point map to the north pole exponential fast, and the energy function is convex. However, if the initial map is a circle close to the equator in the northern hemisphere, say $\alpha(0) = 0.49\pi = \operatorname{arccot} e^C$, then $z(t)$ increases slowly at first, then rapidly, and then approaches $z = 1$ exponentially as in Figure 6.3(a). Correspondingly, the energy $E(t)$ decreases slowly at first, then rapidly, and then approaches 0 exponentially as in Figure 6.3(b). In particular, *the energy function $E(t)$ is not convex.*



(a) $z(t)$ for $\alpha(0) = 0.49\pi$ (b) $E(t)$ for the same flow.

Figure 6.3: $E(t)$ is not convex.

6.3 A non-converging geodesic heat flow

In this final section, we construct an explicit geodesic heat flow $u(t)\colon S^1 \to (M, h)$ that does not converge as $t \to \infty$. In this example, (M, h) is the 3-dimensional torus T^3 with a warped product metric — essentially the metric used for the harmonic map heat flow described in Chapter 2. Again, this is based on the example of Topping in [20].

Let (w, z, θ) be the standard coordinates for the flat 3-torus T^3 and fix a smooth cutoff function η with support on $[-2, 2]$, with $0 \leq \eta \leq 1$ and with $\eta \equiv 1$ on $[-1, 1]$. Define a metric h on T^3 as

$$h \;=\; dw^2 + 2\eta(w)w^2 dw\, dz + \left(1 + \eta(w)w^4\right) dz^2 + \psi(w, z)d\theta^2$$

where $\psi(w, z)$ is the smooth function given by

$$\psi(w, z) \;=\; \begin{cases} 1 & \text{if } w = 0, \pm\pi \\ 1 + \eta(w)e^{-\frac{1}{|w|}} \left(1 + \sin\left(\frac{1}{w} - z\right)\right) & \text{otherwise.} \end{cases}$$

If $x = \frac{1}{w} - z$ and $y = z$ with $0 < w < 1$, the metric is written as

$$h \;=\; (x+y)^{-4}dx^2 + dy^2 + \psi(x,y)d\theta^2$$

where $\psi(x,y) = 1 + e^{-x-y}(1+\sin x)$.

We consider the geodesic heat flow for maps $u\colon S^1 \to T^3$ of the form

$$u(t)(\theta) \;=\; (\;0,\; y(t),\; \theta) \tag{6.17}$$

in (x,y,θ) coordinates. Then the image of $u(t)$ lies in the product of the curve $\{(0,y)\} \subset T^2$ and S^1. Noting $\frac{\partial y}{\partial \theta} = 0$ and $\frac{\partial \theta}{\partial \theta} = 1$, we have

$$\tau^x(u(t)) \;=\; g^{\theta\theta}\,\Gamma^x_{\theta\theta}\left(\frac{\partial\theta}{\partial\theta}\right)^2 \;=\; -\frac{1}{2}\,h^{xx}\,h_{\theta\theta,x} = 0, \tag{6.18}$$

since $h_{\theta\theta,x} = \frac{\partial\psi}{\partial x}(0,y) = 0$. Similarly,

$$\begin{aligned}\tau^y(u(t)) \;&=\; g^{\theta\theta}\,\Gamma^y_{\theta\theta}\left(\frac{\partial\theta}{\partial\theta}\right)^2 \;=\; -\frac{1}{2}\,h^{xx}\,h_{\theta\theta,y} \;=\; -\frac{1}{2}\,\frac{\partial\psi}{\partial y}(0,y) = e^{-y}\\ \tau^\theta(u(t)) \;&=\; g^{\theta\theta}\,\Gamma^\theta_{\theta\theta}\left(\frac{\partial\theta}{\partial\theta}\right)^2 \;=\; h_{\theta\theta,\theta} \;=\; 0.\end{aligned}$$

Since $\dot{u} = (0,\dot{y},0)$, the geodesic heat flow equation is

$$\dot{y} \;=\; e^{-y}. \tag{6.19}$$

We take $y(0) = \pi$. Then the geodesic heat flow $u(t)$ starts from $(w,z) = (\frac{1}{\pi},\pi)$ and lies in the set where $\eta \equiv 1$. Solving (6.19) for y, we have $y(t) = \ln(t + e^\pi)$ and so the geodesic heat flow is

$$u(t)(\theta) \;=\; (w, z, \theta) \;=\; \left(\frac{1}{\ln(t + e^{\pi})},\; \ln(t + e^{\pi}),\; \theta \right).$$

This flow does not converge anywhere as $t \to \infty$ although there exist sequences $\{t_n\}$ such that $u_n \to u_\infty$ in C^∞ as $n \to \infty$.

BIBLIOGRAPHY

[1] K. Choi and Thomas H. Parker, *Convergence of the heat flow for closed geodesics*, Preprint, (2010).

[2] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc., Vol. **10**, (1978), 1–68.

[3] J. Eells and L. Lemaire, *Selected topics in harmonic maps*, C.B.M.S. Regional Conf. Series **50**, Amer. Math. Soc., Vol. (1983).

[4] J. Eells and L. Lemaire, *Another report on harmonic maps*, Bull. London Math. Soc., Vol. **20**, (1988) 385–524.

[5] J. Eells and J. C. Wood, *Restrictions on harmonic maps of surfaces*, Topology **15** (1976), 263–266.

[6] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds*, Amer. J. Math., Vol. **86**, (1964), 109–160.

[7] Frédéric Hélein, *Régularité des applications faiblement harmoniques entre une surface et une variété Riemannienne*, C. R. Acad. Sci. Paris, Vol. **312**, (1991), 591–596.

[8] Jürgen Jost, *Riemannian geometry and geometric analysis*, 5th ed., Springer-Verlag, Berlin, 2008.

[9] F. Lin and C. Wang, *Energy identity of harmonic map flows from surfaces at finite singular time*, Calculus of Variations and Partial Differential Equations, Vol. **6**, (1998), 369–380.

[10] F. Lin and C. Wang, *The analysis of harmonic maps and their heat flows*, World Scientific Publishing, 2008.

[11] R. Moser, *Partial regularity for harmonic maps and related problems*, World Scientific Publishing, 2005.

[12] R. S. Palais, *Morse theory on Hilbert manifolds*, Topology **2** (1963), 299–340.

[13] T. Parker, *Bubble tree convergence for harmonic maps*, J. Diff. Geom. **44** (1996), 595–633.

[14] T. H. Parker and J. G. Wolfson, *Pseudo-holomorphic maps and bubble trees*, J. Geom. Anal. **3** (1993), 63–98.

[15] Jie Qing and Gang Tian, *Bubbling of the heat flows for harmonic maps from surfaces*, Comm. Pure Appl. Math. **50** (1997), 295–310.

[16] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. Math. **113** (1981), 1–24.

[17] L. Simon, *Asymptotics for a class of non-linear evolution equations, with applications to geometric problems*, Ann. of Math. **118** (1983), 525–571.

[18] M. Struwe, *On the evolution of harmonic mappings of Riemannian surfaces*, Comment. Math. Helvetici **60** (1985), 558–581.

[19] P.M. Topping, *Rigidity in the harmonic map heat flow*, J. Differential Geom. **45** (1997), 593–610.

[20] P.M. Topping, *Winding behaviour of finite-time singularities of the harmonic map heat flow*, Math. Zeit. **247** (2004), 279–302.