LEARNING TO DEFINE IN MATHEMATICS: THE EFFECTS OF A SEQUENCE OF TASKS ON PRESERVICE TEACHERS' ABILITIES TO CONSTRUCT HIGH QUALITY DEFINITIONS

By

Darlene Esther Kohrman

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ABSTRACT

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Definitions and definitional reasoning are central to the learning of mathematics and to the teaching of mathematical content. Definitions are more than sentences that merely describe a memorized concept. According to Berger (2005), examining how individuals make personal meaning of a mathematical object or an idea is basis for how students learn mathematics. Definitions are a fundamental part of the logical structure or nature of mathematics. Although some studies have examined how preservice and in-service teachers view definitions, little research has examined the deductive structure of mathematical definitions through the use of five principles. This study uses the concept image to examine the relationship between preservice teachers' conceptions of mathematical definitions and their concept images.

In this qualitative study, seven preservice teachers at various stages in their mathematical content preparation to teach grades K – 8, were engaged in a series of tasks to systematically engage with definitions. The five logical principles served to guide the preservice teachers as they negotiated and refined the meanings of and wrote high quality definitions for the four quadrilaterals. As the preservice teachers interacted with the tasks, their discussions were recorded and coded to determine the extent to which they used these principles.

The findings indicated the strength of the concept image influence the preservice teachers abilities to write high quality definitions for the quadrilaterals. The findings also indicated that the preservice teachers hold intuitive values about that often match the five principles. Through the examination of examples of high quality definitions for quadrilaterals, the preservice teachers could rewrite definitions that demonstrated the use of the principles. However, throughout the evolution of the sequence of tasks, the interplay between the five principles and their total cognitive structure were put into conflict. The cognitive structure included prior learning experiences and their own personal reconstruction of the definitions for quadrilaterals. Changes in the use of the principles demonstrated that the nature and role of mathematics is teachable and that the five principles can support such learning.

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CHAPTER 1: INTRODUCTION AND OVERVIEW

Definitions are central to mathematics, and they play a significant role in the learning of mathematics (Ouvrier-Buffet, 2004). Definitions express and communicate mathematical ideas and provide opportunities for mathematical reasoning. Students encounter mathematical definitions very early in their learning. This fact places the initial responsibility for that learning on elementary teachers but this continues for all teachers K - 20. The key role played by these teachers requires special attention as they introduce basic mathematical ideas, including definitions, to students thus requiring a profound understanding of mathematics (Ma, 1999). Mathematical definitions do more than just describe concepts; they also provide the necessary logical connections needed for deductive reasoning. One of the primary roles attributed to the role of definitions is their use in proving theorems (Parameswaran, 2010). This application of definitions requires students to “unpack the definitions of the concepts involved, including their logical structure” (Harel, Selden, & Selden, 2006, p. 4). Definitions by their nature relate concepts through the use of previously defined concepts and necessary and sufficient conditions. Therefore, for students to use definitions, the concepts embodied in the definitions must be understood.

Definitions play a significant role in the case of geometry. Definitions contribute to the learning of geometry in significant and related ways: in the description of geometric figures, in the support of conjectures and arguments that lead to formal proof of geometric theorems, and in the development of the hierarchical relationships in geometric shapes (Chesler, 2012; Edwards & Ward, 2004; Rasmussen & Zandieh, 2000; Usiskin & Griffin, 2008; Zaslavsky & Shir, 2005). Research often demonstrates that the role of definition in the learning of geometry is problematic and often approached from a vocabulary position in elementary grades. Students encounter definitions in their years of study without having opportunities to create definitions. NCTM calls for attention to the learning of vocabulary; they write: "Although a facility with the language of geometry is important, it should not be the focus of the geometry program but rather should grow naturally from exploration and experience" (NCTM, 1989, p. 48). This can occur as students have opportunities to classify geometric shapes and to use their conjectures to support rational conclusions leading to relational and hierarchical opportunities. These activities would be perfect opportunities for creating formal mathematical definitions of these geometric shapes. Such activities could simultaneously contribute to both the hierarchical understanding of the shapes and how definitions are written so demonstrate and support this hierarchy.

In order for students to participate in the variety of roles that definitions support in mathematics, they must engage in processes that emulate the behavior of mathematicians when creating definitions. Such behaviors include the use of images, appropriate examples, and analyzing conflicts of equivalent definitional forms that in turn stimulate cognitive reflection. According to Vinner and Hershkowitz, a parallel exists between the construction of a concept and the construction of its definition, that is, definitions help to form concept images (Vinner & Hershkowitz, 1980). The concept image for Vinner and Hershkowitz in this quote refers to a formal mathematical definition. This parallel is problematic as most clearly stated by Vinner: "Definition creates a serious problem in mathematics learning. It represents perhaps, more than anything else, the conflict between the structure of mathematics as conceived by professional mathematicians and the cognitive process of concept acquisition" (Vinner, 1991, p. 65). This quote sets up the cognitive reflection that became the focus of my sequence of tasks used with the preservice teachers as they refined and wrote definitions. One must then consider the types of opportunities students have that enable them to engage in this conflict.

Taking the process of definition-construction into the context of the classroom is the work of Lakatos and others. Lakatos describes a definitional procedure as a procedure of concept formation (Lakatos, 1976). This suggests a connection between the words in a definition and the images that the definition describes. The reality of the classroom as argued by Stephan et al. is that an essential core practice of mathematicians underemphasized in the education of mathematics students is the activity of defining (Stephan, McManus, Dickey, & Arb, 2012). Students are given definitions as finished products to memorize but not the opportunities to create definitions. They support this argument by noting that the prior school experiences of students include with beginning a geometry lesson with the definitions supplied them as an imposed body of knowledge not to be discussed or changed (Stephan et al., 2012). In addition, this approach may enforce the false notion that only one definition exists for a given concept (Stephan et al., 2012). However, definitions are man-made inventions and the existence of several correct definitions may exist for one geometric concept (Craine & Rubenstein, 1993; Linchevsky, Vinner, & Karsenty, 1992; NCTM, 2009). The fifth principle that states definitions are arbitrary or more than one correct definitions may exist for a concept. This study provides the opportunity for the preservice teachers to discuss this fact.

Furthermore, researchers have noted that an implied incongruence exists between being given a definition and then expecting that students will learn to define automatically. According to Mariotti and Fischbein, "[d]efining is a basic component of geometrical knowledge, and learning to define is a basic problem of mathematical education" (Mariotti & Fischbein, 1997, p. 219). This suggests that teaching students to define is a priority in the curriculum. Contributing to this situation is the concern that once a definition is determined in a curriculum, its existence then influences how a teacher approaches the teaching of that concept to students (Zazkis & Leikin, 2008). These researchers seem to suggest that teachers as well as students could be limited to the definitions given by a curriculum if that teacher does not know how to engage students in creating definitions nor understand the fact that several correct definitions can exist for a concept. This implies that the resulting concept image of a mathematical concept for a student is shaped by the way the teacher uses given definitions only as presented in the curriculum without any opportunities to discuss how students understand the meanings of the definitions.

The result of limited access to definitions suggests that students may develop incomplete or inaccurate understandings of mathematical concepts. This concern is reiterated in the work of Ouvrier-Buffet who sees a constructive parallel between the definition and the concept. Limiting a concept strictly to its definition should not be the focus of learning concepts when teachers engage students in learning definitions (Ouvrier-Buffet, 2003, p. 1). This research suggests the need for professional development in definition-construction for teachers as well as teachers also could benefit from engaging in creating definitions.

Another challenge with definition-construction is consensus of those engaged in the process on the wording and the meaning. Mathematicians work towards consensus.

Ball's work concerning the mathematical knowledge for teaching (MKT) suggests that teachers have both competences in definition usage as well as an awareness of the arbitrary nature of definitions in order to make appropriate adjustments to instruction when they encounter equivalent definitions in textbook materials (Ball, 2003). Her statement suggests that teachers have a sophisticated level of definition understanding. This research study is designed to determine if preservice teachers have an awareness of the arbitrariness of definitions. Linchevsky et al. (1992) have previously examined this question. Their research indicated that many preservice teachers do not understand the arbitrary nature of definitions nor do they see the economical value of minimal definitions that as a result will use only necessary and sufficient conditions (Linchevsky et al., 1992). My study examines this issue. For preservice teachers to understand the nature of definitions and their use in the learning of mathematics, it seems necessary that they first understand the characteristics of a mathematical or high quality definition as embraced by the mathematical community. Of equal importance is their knowledge and understanding of the roles that definitions play in the learning of mathematical content. These roles include particular actions or behaviors used by mathematicians such as the use of examples and nonexamples that lead to the creation of mathematical definitions. These actions and behaviors imply that preservice teachers then have opportunities to engage in the process of defining during their mathematical educational experiences that lead them to degrees in teaching.

Problem Statement and Purpose of the Study

The purpose of this study is to contribute to research on how preservice teachers understand the role and nature of definitions by investigating if the use of a sequence of tasks demonstrates a difference in the abilities of elementary/middle school preservice teachers to define quadrilaterals that exhibit the principles of mathematical definitions. Mathematical definitions of this type will be known as high quality. This purpose keeps in mind the work and findings of current and past researchers about how preservice teachers understand the role and nature of mathematical definitions.

An implication of this study is that preservice teachers must have an opportunity to engage in the process of creating definitions during their training. However, definitions are often handed provided to students from textbooks or teachers in the classroom without student participation in the process of their creation or evaluation. This disconnect between what should happen with definitions and what does happen has been noted by previous research. For example, Stigler and Hiebert (1999) reported that mathematical lessons in the United States place much emphasis on the *definition-of* terms, discussed in Chapter 3 and 5, but less on the underlying rationale of the definitions. They add that the presentation of definitions is important in the learning of mathematics as a communication avenue, but what is done with the definition in the classroom is more important. They suggest that only when definitions are used to explore mathematical properties and relationships are students engaged in the learning of mathematics (Stigler & Hiebert, 1999, p. 58). Pimm made similar observations, stating that definitions hold a discriminatory power, "[d]efinitions, by definition, place limits around what is being defined" (Pimm, 1993, p. 262). He also suggests that definitions, when presented to students, project a sense of finality thus limiting student engagement in the thought process that Stigler and Hiebert encourage. In this study, the preservice teachers will bring their own known definitions that will be challenged by the five principles used in the task, therefore will have opportunities to engage in definitional reasoning. The problem, in other words, is the real learning that professional mathematicians engaged in to construct definitions is denied to today's students by handing them predefined terms.

Past research indicates that both students' and teachers' actions and comments reveal a lack of understanding concerning both the content of mathematical definitions and the role of mathematical definitions in supporting the learning of mathematics (de Villiers, 1998; Edwards & Ward, 2004, 2008; Linchevsky et al., 1992; NCTM, 2009; Zaslavsky & Shir, 2005). In mathematics, definitions are deemed meta-mathematical and share positions with other mathematical terms such as axiom, postulate, and theorem - all of which serve a function and hold a position of status in mathematics (Pimm, 1993, p. 262). To this end, Pimm suggests that definitions possess a discriminatory power, the power to expand or narrow a student's thought process in the learning of mathematics (Pimm, 1993). This power when paired with his statement on the sense of finality that definitions can hold inhibits the meta-mathematical role that definitions bring to the building of a logical structure in mathematics.

Further research has found that opportunities for preservice teachers to understand the role and nature of definitions are often limited when a mathematical lesson begins with a definition as already determined (de Villiers, 1998). Such learning occurrences limit or inhibit the understanding that mathematical definitions are human inventions and discoveries (NCTM, 2009). Human invention, initiated by intuition, is where the beginnings of experiencing and understanding the meta-mathematical nature of definitions occur. This is accompanied by actions of creating, refining, and negotiating the meanings of the definitions being developed. All of these actions support the logical structure of mathematics. In particular, this lack of opportunity has a significant impact on how teachers understand geometric concepts through the use of either hierarchal or partitional definitions. Partitional definitions represent geometric concepts as exclusive or disjoint leading many students to not see squares are rectangles.

While there is research on definitions and their application in geometry that research is incomplete. Ouvrier-Buffet argues that there still exists a serious lack of research that investigates how definitions are constructed by students (Ouvrier-Buffet, 2006). Studies on definition-construction may inform research on the advanced mathematical thinking of students. For instance, Berger (2005) suggests that such thinking is observable in the context of mathematical proofs when stating "[t]he issue of how an individual makes personal meaning of a mathematical object presented in the form of a definition is particularly relevant to the study of advanced mathematical thinking" (Berger, 2005, p. 1). A few studies have addressed this issue by engaging students in both creating and critiquing their newly formed definitions.

Other studies have suggested more research must be done specifically to focus on preservice teachers' understanding and use of definitional reasoning since definitions are essential to all mathematical levels (Gomes, Ribeiro, Pinto, & Martins, 2013, p. 283). They suggest that geometric definitions be one of those areas of research.

Geometric shapes are related in a hierarchical structure and that situation has implications on their definitions. Current educational policy recognizes that hierarchical definitions build on the idea of the inclusive nature of shapes in terms of subsets (NCTM, 2009). This understanding has direct impact on how a teacher presents ideas connected to hierarchical classification of quadrilaterals. However, future teachers are expected to have expertise in the hierarchical nature of geometric concepts as stated in the Common Core State Standards:

> Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories (Common Core State Standards, 2010, p. 26).

Related recommendations for the fifth grade include:

> Understanding that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category (e.g. all rectangles have four right angles and squares are rectangles, so all squares have four right angles).
> Classifying two-dimensional figures in a hierarchy based on properties (Common Core State Standards, 2010, p. 38).

Furthermore, other educational policy expectations for students from the Standards of Mathematical Practice in the Common Core State Standards states that, "Mathematically proficient students understand and use stated assumptions, definitions, and previously established results in constructing arguments" (Common Core State Standards, 2010, p. 6). The *Professional Standards* includes the expectation for teachers to engage in "deciding when and how to attach mathematical notation and language to students' ideas" (NCTM, 1991, p. 35). Likewise, the Conference Board of Mathematical Sciences also recommends that teachers communicate geometric ideas with both knowledge of technical vocabulary and an understanding of the importance of definition in mathematical understanding (Conference Board of Mathematical Sciences, 2001). This study has been designed to accomplish an identified need that couples past research findings and these policy recommendations. My study hopes to learn more about the process of definition construction with preservice teachers engaged in learning the geometric concepts embedded in defining quadrilaterals.

Specifically, the study may reveal what preservice teachers understand about the nature and structure of definitions for certain quadrilaterals. The study provides opportunities for the preservice teachers to write high quality definitions that emulate the principles of mathematical definitions while engaging in the behaviors of professional mathematicians. In addition, the relationships and classification of quadrilaterals is connected to the tasks in terms of seeking definitions that are minimal and demonstrate necessary and sufficient conditions for the figures. The study provides an opportunity in a definition-construction process that engages them in revising, changing, or adapting their own and current of quadrilaterals into mathematical definitions of high quality.

Significance of Study and Assumptions

It is my hope that this research will contribute to the research previously done on definition construction and enhance what is already known about geometric definitions. Geometric definitions for quadrilaterals present an opportunity to connect the hierarchical structure of the polygons to the principles of mathematical definitions. The role of our future teachers in developing an understanding of the definitions of geometrical objects as well as how these definitions are used to promote reasoning about geometrical objects is a necessary ability for their classroom teaching. This understanding is essential given the fact there are many ways to define the same geometrical shape. The arbitrary nature of mathematical definitions presents a teacher with many challenges when making curricular decisions in the classroom. Definitions provide opportunities for teachers to engage students in the exploration of equivalency or arbitrariness of definitions for concepts. Therefore, teachers need to make decisions as to which definitions to use as well as how to demonstrate the equivalence of several definitions for the same mathematical concept. Didactical considerations are also part of teacher knowledge about definitions, not only from a curricular view but also in terms of the concept images and definitions students bring to the learning situation (Gomes et al., 2013). This notion is especially true for geometric concepts such as quadrilaterals as some students have visual concept images of the shapes while others can see the relationships among the shapes more readily. This starting position in their cognitive structure impacts what and how they will benefit from definition construction.

Future implications from this research may impact the curriculum of teacher training programs as well as the curricular materials available for teaching definition and the use of definition. The results of this study may contribute to the knowledge base on definition-construction in both geometric and other mathematical content areas. It should add to the research base on the mathematical knowledge that elementary/middle school preservice teachers hold concerning the nature and role of mathematical definitions. This study may reveal notions previously held by preservice teachers about definitions that they in turn then bring to the discipline of teaching mathematical definitions.

This dissertation study is an extension of my prior research, which demonstrated that personal concept definitions of the preservice teachers were descriptions of properties without hierarchal structure. It also showed that the role of definition is not completely understood by the preservice teacher. This suggests that engaging preservice teachers in a sequence of tasks that are deliberately focused on the hierarchical structure and quality of definitions could change or at least impact the structure of the personal concept definitions of the participating preservice teachers. While the preservice teachers engage in the tasks and discuss their ideas about the five principles, they may produce high quality definitions and demonstrate shifts in their reasoning. This study may contribute to a small but growing body of research about preservice teachers' understanding of the nature of definitions, with a particular reference to geometric understanding of the definitions of quadrilaterals.

From previous research, it is expected that the elementary preservice teachers will demonstrate confusion when classifying between the definitions of squares and rectangles (Fujita & Jones, 2006; Pickreign, 2007). The assumption is that the strength of their personal figural images and definitions, with particular reference to prototypical mental images for squares and rectangles, will play an important role in the creation of mathematical definitions for these shapes (Fujita & Jones, 2006; Vinner & Hershkowitz, 1983). Hence, teachers and students may benefit from an opportunity to examine their own personal concept images and personal concept definitions in terms of the effects they have on understanding mathematical definitions that exhibit the characteristics understood by the mathematical community.

Overview of Methodology

This study is a qualitative study that is guided by the principles of previous research on definition-construction processes described in detail in Chapter 2. The theoretical framework embraces the understanding that the nature of mathematical definitions includes both the process of defining and the product of producing a definition. The methodology used in this study is a design-based research approach also referred to as "design experimentation" (Sandoval & Bell, 2004).

The sequence of tasks in this study simulates a portion of a definition construction opportunity where definitions are negotiated and refined. This study has two purposes: first a theoretical orientation that contributes to the domain of geometric definition, and second, a pragmatic offering that contributes a supportive structure to enable the learning of geometric definitions. This study hopes to simulate the following: "Design-based research simultaneously pursues the goals of developing effective learning environments and using such environments as natural laboratories to study learning and teaching" (Sandoval & Bell, 2004, p. 200). To this end, each task in this study has two goals: one pedagogical and one for the researcher. For example, the first task in this study asks each participant to draw an image for a square and a rectangle followed by writing a *definition-of* each shape. From a research perspective, this task collects the initial data in the form of *personal figural concepts* for the study and provides evidence of possible prototypical images which previous research has shown has significant impact on *personal concept definitions* (Fujita & Jones, 2007). The definitions provide the *personal concept definitions* that will be analyzed by the participants in subsequent tasks for characteristics of high quality definitions. Thus the *personal concept definitions* and drawings that reflect the *personal concept images* become the basis of the *Situational level* in the defining as a mathematical activity (DMA) framework, which consists of levels of change and movement towards a broader concept image for the object under consideration. The framework is further explained in Chapter 3. From a pedagogical perspective, the teacher is made aware of the concept images of the students and each student also becomes aware of their particular concept images thus supporting research suggestions of Edwards and Ward that students need this awareness as a first step in understanding misunderstandings concerning mathematical definitions (Edwards & Ward, 2005). Chapter 4 and Appendix A elaborates the specific goals of all the tasks for this study.

To accomplish the goals of this study, seven preservice teachers taking required mathematics content courses required in the K – 8 program of studied volunteered. The participants met three times for a length of two hours at each session to complete the tasks. All work and discussion was recorded.

The dialogue and artifacts will be analyzed using the Defining Mathematics Activity framework that is discussed in detail in Chapter 3. The design of this study, a sequence of tasks, creates opportunities for the preservice teachers to refine, negotiate and create high quality definitions as accepted by the mathematical community.

Research Questions

This is a qualitative study; therefore, the research questions are designed in order to produce descriptive information on the reflective thinking processes of the preservice teachers as they are engaged in doing and discussing the tasks. This study explores the extent to which preservice teachers in this study use the five principles of high quality definitions. These five principles are discussed in detail in the methodology section of Chapter Three. More specifically, the questions that guide this study are provided next:

To what extents do the preservice teachers use the five principles to …

Analyze given definitions and negotiate the meaning of definitions? And, refine and create new definitions?

a. What personal concept images and personal concept definitions do the preservice teachers hold for quadrilaterals?

b. When preservice teachers discuss their personal concept definitions, what criteria about the nature of definitions emerge, and what characteristics about the concepts of quadrilaterals are revealed?

c. How do the preservice teachers perceive the five principles in their efforts to construct high quality definitions?

The potential descriptive information gained from the challenge proposed in the tasks is supported in the statements of Arcavi (2003) and Fischbein (1993). Arcavi proposes that mathematical visualization involves 'seeing the unseen' in a generic example such as a drawing while Fischbein argues that geometrical figures are really "*figural concepts*" or entities that are simultaneously spatial representations and concepts (Arcavi, 2003; Fischbein, 1993). Fischbein's theory describes geometrical figures as having a dual nature which allows the drawing or image (shape, location, size) to provide a source for the thinking about the generality or abstractness of the conceptual characteristics (Fischbein, 1993). This theory is addressed in Chapter 3. Fujita and Jones have conjectured that this dual nature may be the source of the difficulties preservice teachers have when working with definitions and hierarchical relationships of quadrilaterals (Fujita & Jones, 2007).

In the tasks of this dissertation, the preservice teachers' *personal figural concepts* will be challenged. In particular, there may be an observable tension between the individual assumptions the preservice teachers hold for certain quadrilaterals and the hierarchical definitions embraced by the mathematical community. The tasks and DMA framework are meant to serve as way to examine and document the paths of participation the preservice teachers take while engaging in the process of defining. Their conversations may provide a record of stages or nuances of movement in reflective thought or where more difficult transitions in definition understanding occur.

Objectives and Outcomes

The major objective of this study is to determine if a sequence of tasks around the five principles, helps preservice create, negotiate, and refine high quality definitions for quadrilaterals, parallelograms, rectangles, and squares. Their discussion results will adjust future tasks as the study progresses to support and scaffold their understanding of the nature of high quality mathematical definitions while critiquing given definitions and then writing their own definitions. At the end of the study it is expected their definitions will exhibit the five principles. The findings of this study could also inform the research on definition-construction, specifically addressed in geometric situations.

An expected outcome from this study is to corroborate earlier research findings that the preservice teachers will hold prototypes for their mental pictures of the quadrilaterals and will write *personal concept definitions* that are descriptive definitions. This study will contribute information to the mathematics education field specifically on how preservice teachers understand mathematical definitions in terms of the five characteristics as well as the role definitions play in mathematical learning.

Overview of Dissertation

This dissertation is organized through the use of chapters. Chapter 1 provides the rationale, purpose of the study, and research questions. Chapter 2 discusses the review of literature to include such topics as: (1) How mathematicians create and revise definitions, (2) Prior research on in service and preservice understanding of mathematical definitions, and (3) The development of definition-construction frameworks. Chapter 3 includes topics such as: (1) The methodology of the design-based research used to create the tasks, (2) The description of the DMA, and (3) a brief summary of a pilot study. Chapter 4 gives the findings of the study that include: (1) The PSTs hold prototype images and write descriptive phrases, (2) The PSTs hold intuitive values for definitions, (3) The PSTs have difficulties defining and using the five principles, (4) The dialogue demonstrates the interplay of the hierarchy and the five principles, (and 5) the definitions demonstrate high quality at the end of the study. Chapter 5 discusses the findings of the study to reach conclusions and implications such as the intuitive values held by the PSTs are aligned with the principles even though the PSTs did not see the alignment, and the five principles are teachable. And Chapter 6 provides both a discussion of the findings and recommendations to suggest the five principles be used in other content areas and other kinds of defining mathematical activities be used in earlier grades.

CHAPTER 2: LITERATURE REVIEW

This chapter begins with a systematic overview of the literature that establishes the nature and role of mathematical definitions in the creation of mathematical concepts. It seems fitting to begin the first section of this review from the perspective of the mathematical community; thus, the review of literature begins by analyzing the nature and role of definition from the perspective of professional mathematicians. The review then shifts to literature that demonstrates what is currently known concerning how teachers and preservice teachers understand the role and nature of mathematical definitions. This understanding illustrates discrepancies among preservice teachers' and in service teachers' perspectives with those of professional mathematicians. The review continues in a third section with studies that demonstrate such discrepancies, indicating the role and nature of one's understanding and interpretation of a mathematical definition is influenced by the concept image. Consequently, the literature review shifts in the to research concerning the role of the concept image on one's understanding of a mathematical definition. At this point in the review, attention is focused on geometric understanding of definitions with a focus on quadrilaterals, as this is the intent of this study.

From my perspective, the nature of mathematical definitions includes the characteristics of mathematical definitions as well as the role that definitions take during the learning of mathematical concepts with a particular focus on how preservice teachers understand mathematical definitions for certain quadrilaterals. The nature of mathematical definitions as established by the five logical principles also support the classification of quadrilaterals. This dual perspective of definition shapes this study.

The Nature and Role of Mathematical Definitions from Expert Mathematicians

For this study, it is important to provide evidence of the nature as well as the role of mathematical definition as it has been established within the mathematical community. This viewpoint helps to establish the theoretical perspective of what a mathematical definition must embody and support. Consequently, the nature and role of mathematical definitions impacts the learning of mathematics.

The nature of mathematical definitions refers in part to the characteristics that distinguish mathematical definitions from everyday definitions. Mathematical definitions have characteristics or properties that are very different from dictionary definitions, which often are simply descriptions of lexical units. Pimm (1993) asked the question "…is a mathematical definition anything other than a *definition-of* a mathematical term?" (Pimm, 1993, p. 262). What makes a mathematical definition more than a description? The answer to this question lies in the precision required of a mathematical definition that is the result of its position as a mathematical construct. In other words, definitions also have a distinct role in the development of mathematics. Definitions like theorems and axioms are used in the process of creating and defining mathematics itself. This process includes the human activity of discovery making definitions man-made (Pimm, 1993). As a result, mathematical definitions are far more precise than definitions used in everyday language (Edwards & Ward, 2008).

The characteristics that contribute to the precision of mathematical definitions as a result of this meta-mathematical process include:

- Giving a name to a new concept where the name should appear only once in the definition.

- Only previously defined concepts should be used.
- Necessary and sufficient conditions are established in the process of defining.
- Conditions should be minimal.
- Definitions are arbitrary that is equivalent forms for the concept may exist (Vinner, 1991; Winicki-Landman & Leikin, 2000; Zazkis & Leikin, 2008).

The characteristics of arbitrary and minimal are not agreed upon by all of those who use mathematical definitions.

For instance, Borasi (1991) discussed these characteristics in terms of functions that mathematical definitions should fulfill. Both the characteristics and role of definitions are seen as functions as the statements below suggest:

1. Mathematical definitions should allow users to discriminate between instances and non-instances of the concept with certainty, consistency, and efficiency. In other words, does a new possibility satisfy all the properties that the definition gives? (Borasi, 1991, p. 288).
2. Does the definition synthesize the mathematical essence of the concept? This means can all the properties of the concept be logically derived from what is provided in the definition (Borasi, 1991, pp. 17-18).

The first statement of Borasi hints at the functional precision of a mathematical definition, which characterizes a more scientific idea of a mathematical concept. This precision seems to be a reasonable characteristic of definition as used in a scientific domain (Morgan, 2005). The notion of consistency, certainty, and efficiency are similar to the characteristics of Leikin and others. However, the second characteristic speaks to the function that definitions play in the axiomatic structure of mathematics as well as the meta-mathematical nature of definitions. This idea suggests that mathematical definitions are more than mere descriptions of concepts. Vinner adds to this notion by stating that definitions in mathematics provide an essential means for the development of mathematical concepts (Vinner, 1991). He claims this is unique to the development of mathematical concepts since in everyday life definitions do not generally development concepts (Vinner, 1991). Tall (1992) then suggests that a characteristic of a mathematical definition in advanced mathematics is its use or, in other words, its ability to provide proof. Alcock and Simpson extend this notion of use with the idea of choice where the choice of a definition influences the deductive process leading to proof (Alcock & Simpson, 2002). These researchers seem to suggest that mathematical definitions embody both a process and a product. My study uses the processes of negotiating, redefining, and creating definitions in order to produce a product of a written definition. This process which includes the idea of choice would suggest that both the role and nature of definitions is best described as a purposeful formulation placing the mathematician in an active role of decision making about the appropriate choice of a definition during the deductive process (Morgan, 2005).

The use of mathematical definitions during the deductive process is central to professional mathematician's approach to the creation of new mathematical ideas as well as the their synthesis into a definition (Parameswaran, 2010). Her study examined the cognitive tools that expert mathematicians, currently engaged in mathematics research, employ as they develop new and deep understandings of abstract mathematical definitions (Parameswaran, 2010, p. 43). While interviewing professional mathematicians about mathematical definitions, certain cognitive processes emerged resulting in the following themes: the role of examples, conflict resolutions, reformulations, and generalizations (Parameswaran, 2010, p. 43). These same cognitive processes are found in my study through task engagement.

According to her findings, expert mathematicians use examples to develop their understanding and creation of mathematical definitions (Parameswaran, 2010, p. 45). As one mathematician noted, examples help to determine what to include as well as the counter examples to exclude. Another mathematician in the study noted that the comprehension of a definition might be a challenge to both the examples and intuitions he has already accumulated about the definition. Another noted that examples help him decide if the new definition is empty. While another noted that examples of definitions provide different routes in order to make connections that refine the mental picture of the object. Perhaps the strongest statement suggests that examples actually allow for the definition to exist: “Definitions cannot stand abstractly without examples…a definition is a collection of all the examples that will conform to that definition” (Parameswaran, 2010, p. 47). Examples help to single out the concept being defined so my study provides the preservice teachers (PSTs) with opportunities to examine their own examples of definitions with other mathematical definitions for the quadrilaterals.

Another important part in the approaches expert mathematicians use in understanding definitions is the use of conflicts. The use of conflicts was declared to be a significant support in the process of understanding a new definition. One claimed that conflicts may be resolved and are supported when the same mathematical object is viewed from several perspectives by saying that the conflict forces a process of re-evaluating one’s understanding. One mathematician stated, “If the definition is equivalent, I note it in the back of my mind that this is an equivalent formulation" (Parameswaran, 2010, p. 46). Evaluating equivalent formulations becomes a necessary part of the process of generalization as it creates cognitive conflict. This conflict is central to this study as the PSTs are asked to rethink and rewrite definitions. Conflicts are opportunities to reformulate concept definitions, which in turn provide avenues for generalization.

Parameswaran summarizes:

> "At times, a definition and a theorem can be interchanged. That is, the theorem yields an equivalent reformulation of the definition, leading to deeper understanding. For example, an equilateral triangle may be defined as a triangle in which all sides have equal length. The theorem stating that a triangle is equilateral if and only if it is equiangular provides an equivalent reformulation of its definition. Thus, a mathematical object is better understood through characterizing properties" (Parameswaran, 2010, p. 49).

In other words, definitions and theorems work together in proof. A well-written definition allows for a mathematician to make generalizations. These generalizations are made possible because a well written mathematical definition is constructed in such a way that the five principles have been utilized.

Building on the concept of definition, Ouvrier-Buffet (2004) notes that a definition is also axiomatic and inscribed in a mathematical theory. Others alluded to this codependence as well by acknowledging that proving and definition creation supported each other. For instance, proving theorems often was seen as a utility that used the definitions in the process thus enhancing their understanding of the definition.

Some mathematicians spoke to pedagogical considerations that must be realized as the abstraction of definitions from examples and counter examples is complex and challenged when the creativity of intuition embraced by examples must become formalized in the writing of mathematical proofs. One mathematician notes, “I wish to suggest that a mathematical object is its defining property” (Parameswaran, 2010, p.50). The cognitive conflict noted by expert mathematicians is an important part of the nature of mathematical definitions and drives the role of definition in mathematics as a process of resolution, reformulation, and generalization. It is this cognitive conflict that challenges students as they encounter mathematical definitions. Students view a mathematical definition through their personal concept image, which includes mental images, learning experiences, and reformulations of the definition thus creating conflicts with the mathematical definition. Vinner calls attention to the necessity for this conflict because exposing students to the flaws in their personal concept images allows them to enter the process of reconstruction of their personal mathematical concept definition (Vinner, 1991, p.79). Edwards and Ward (2010) also suggest that students be made aware of their personal concept images and personal concept definitions in terms of how they conflict with mathematical definitions.

The process of defining can be described using the five principles of mathematical definitions. These five principles denote the dynamic nature of mathematical definitions. These principles depend on conditions that in turn connect to the next and subsequent principles. The fifth characteristic that is stated by Leikin and others that definitions are arbitrary gains support from the expert mathematicians who indicate that equivalent reformulations of definitions do exist and are used to enrich understanding and support the idea that definitions are man-made (Parameswaran, 2010). Definitions imply in their arbitrary nature that choice exists in their use. Acknowledging this choice in the pedagogy, textbooks, and teacher presentation of concepts is found too infrequently.

The use of examples allows for the reasoning about the inclusion or exclusion; similarly it helps to establish the necessary and sufficient conditions that distinguish how mathematical concepts are developed and consequently defined. This same sentiment is claimed here for students in a study on the understanding of a tangent: "… it is important that learners encounter a variety of examples and counter-examples which emphasize specificity and generality of the cases and are related to necessity and sufficiency of different conditions of the concept" (Winicki-Landman & Leikin, 2000, p. 21). This experience is especially important for geometry and hierarchical thinking both goals of research for this dissertation. The five principles align with the hierarchical structure of quadrilateral relationships in this study.

The conclusions shared in Parameswaran's study demonstrate that the nature of a definition goes beyond a mere description or lexical unit. The nature and role of a mathematical definition outlines the essential process of reasoning and reflecting that results in a particular definition. The end result of definition-construction attaches both meaning while encompassing the idea of the concept. This thought is noted by Morgan who states, "…the concepts the mathematician works with may be more or less intuitive, derived from special cases. The construction of the formal definition and consequent creation of a technical term is, thus purposeful, and creative, aiming not simply to describe or "capture" a pre-existing concept but to shape that concept in a way that lends itself to particular purposes" (Morgan, 2005, p. 862). This study uses the five principles for this to extend the PSTs' personal concept definitions into high quality definitions.

From the professional mathematicians' perspective, the role of mathematical definition goes beyond using correct vocabulary when expressing the meaning of a mathematical object. This situation, however, is well documented as problematic in the understanding and learning of definitions for teachers and preservice teachers.

Beginning with a question about what constitutes a good mathematical definition, Wilson notes that in spite of the importance of using definitions in mathematics, "… there is little agreement on what constitutes a good definition" (Wilson, 1990, p. 33). Two explanations emerge from Poincare who pondered the question "What is a good definition?" (1) One possible explanation is "For the philosopher or the scientist it is a definition, which applies to all the objects defined, and only those; it is the one satisfying the rules of logic. (2) But in teaching it is not that; a good definition is one understood by the scholars [students]" (Poincare, 1914, p. 43). Poincare's idea of a good definition goes beyond the five principles of mathematical definitions by suggesting that a good definition must also embrace didactic considerations. His statements suggest the very important role teachers play in the choice of one over another definition because definitions serve a purpose in shaping and supporting student learning. Winicki-Landman and Leikin support this idea by saying that in every stage of learning, a teacher must consider the definition that is most appropriate (Winicki-Landman & Leikin, 2000). Equivalent definitions may be appropriate at different times in a student's learning trajectory as logical structures are built that support that student's mathematical knowledge (Winicki-Landman & Leikin, 2000, p. 21).

For expert mathematicians, that purpose of a definition's choice is often the formulation of a new definition, the refinement of a definition, or the support that a definition provides in deductive reasoning. The opportunity to engage in definitional reasoning does not occur when definitions are simply presented to a student from a textbook. An inappropriate choice of a definition may limit students' understanding of the concept at hand especially when a student has not had an opportunity to develop several examples of the concept through the process of reasoning.

Research on Preservice and In-service Teacher Understanding of The Nature and Role of Mathematical Definitions

In a study of mathematical knowledge for teaching (MKT), Gomes et al., claim that the choice of a mathematical definition has didactical implications such as the logical path that a definition creates for subsequent study encountered during mathematical pursuits, when students study quadrilaterals (Gomes, Ribeiro, Pinto, & Martins, 2013). Definitions are part of that pursuit. As noted by several researchers, definitions should be a basic unit of mathematical discourse, and teacher knowledge affects and directs that discourse (Gomes et al., 2013; Keiser, 2000; Leikin & Zazkis, 2010). Again, this implies that a teacher's choice concerning the approach to using definitions in the mathematic classrooms has significant impact on student learning and how students learn to reason about mathematical ideas.

Two related studies on the defining of mathematical concepts provided the field with insight on how teachers perceive the nature of mathematical definitions through choices of equivalent definitions. In each study, Leikin and Winicki-Landman, observed teachers' understandings of the arbitrariness of definition (Leikin & Winicki-Landman, 2000; Winicki-Landman & Leikin, 2000). Didactics emerged from the teachers when considering a *definition-of* a concept to present. In a study, they noted "…teachers develop an understanding of what didactic approaches to defining mathematical are appropriate to their students" (Leikin & Zazkis, 2010, p. 455). These didactics are based upon the logical relationships between the different statements in referencing the concept.

The didactic considerations the teachers promoted include the following: the fact that only previously known concepts of the student should be used, the zone of proximal development must be considered, intuitions of the learner must be considered, and the definition must be elegant. The researchers suggest teachers provide students with many examples and counter-examples that focus both on the specificity and generality in order to promote understanding of both necessary and sufficient conditions for the concept being defined. Such an approach may enable the students to develop mathematical thinking about the definitions of the concept that help them to build the logical structures (Winicki-Landman & Leikin, 2000). The hierarchy of geometric figures is dependent on necessary and sufficient conditions and is a part of this logical structure.

In a subsequent study with didactical implications, teachers were presented with a problematic situation concerning the existence of several definitions that co-exist for mathematical concepts or the existence of alternate definitions. The goal was two-fold: (1) To make teachers aware of the arbitrariness or equivalence of several definitions for a concept, and (2) to describe how definitions can be used to develop mathematical learning in students. During this study, teachers were engaged in the *definition-of* absolute value and also with geometric definitions for certain quadrilaterals. Certain teacher strategies emerged as teacher sought relationships between definitions. The researchers identified these as the *properties strategy* and the *sets strategy* (Leikin & Winicki-Landman, 2000, p. 25). In the first strategy, teachers looked for logical relationships provided in the defining properties; while in the second, teachers made decisions based on comparing sets of objects with the provided definition. The teachers used the second strategy with the quadrilaterals. The researchers attributed this spatial tendency was due to the visual nature of the shapes.

The workshops involved in this research project produced some notable teacher preferences about multiple definitions for a concept. In general, teachers preferred definitions that were more precise as they felt this would help to avoid student confusion. In addition, teachers wanted definitions that held the following attributes: intuitiveness; matched students' knowledge and needs; clarity to students or ease in understanding; convenience when applying to problem solving; and enabling mathematical generalization (Leikin & Winicki-Landman, 2000, p. 27). As in the work of Linchevsky et al (1992), minimality was not one of the critical characteristics considered by the teachers.

This study reached several conclusions, including the similarities between teachers' concerns about the nature of mathematical definitions in the learning process of students and those mentioned in other studies. Leikin and Winicki-Landman (2000) suggest these workshops identified two issues that are connected to the characteristic of arbitrariness, defining is more than giving a name; it includes both establishing properties and creating the set of exemplifying objects. Teachers' choices of these strategies demonstrated to them that even the process of defining can be considered arbitrary (Leikin & Winicki-Landman, 2000).

Another study by Ribeiro, Carrillo, & Monteiro, examined teacher knowledge concerning squares and rectangles to discern the difference between the knowledge of knowing a topic versus knowing how to teach the topic well. The researchers elaborate that in the case of the teacher in the study, her limited knowledge, about the difference between squares and rectangle only in terms of the length of the sides hindered what she could teach the student. And thus, limited what the student could learn in terms of the classification of these shapes. The researchers claim the student is limited to a disjunctive classification for these shapes along with being exposed to an incomplete definition (Ribeiro, Carrillo, & Monteiro, 2009). The aim of this study was to bring more awareness to situations of limited knowledge and the impact such has on student learning.

Studies on preservice teachers' knowledge mirror that of in-service teachers' knowledge as the following example exemplifies. In a recent study, Gomes and others feel that, teacher training programs in Portugal seem to have neglected training in the importance of definitions and their role in the learning of mathematical concepts aside (Gomes et al., 2013, p. 283). Another study by Chessler (2012) found that preservice teachers had difficulties understanding the role and nature of definitions along with misunderstandings of the mathematical content in the definitions. Specifically, the content in this study included word usage and equivalent forms. He felt their lack of reasoning behavior became further complicated by weak content knowledge, so the preservice teachers did not understand the meta-mathematical nature of definitions (Chesler, 2012). These preservice teachers required guidance about the process of defining, the concept of definition in mathematics, and the application of definition. The concept of meta-mathematical nature of definitions is specifically used in this dissertation in relationship to the concept of minimal and necessary and sufficient as characteristics of a well-constructed definition.

In a study about prospective secondary mathematics teachers, Zazkis and Leikin (2008) derived from generated examples of definitions of a square that the prospective teachers' pedagogical concerns had a strong influence over their mathematical concerns about the appropriateness of a definition. For example, when the participants considered the characteristic of minimality, statements such as, "You want a student to understand a definition, not struggle with it", was common (Zazkis & Leikin, 2008, p. 144). This finding suggests that preservice teachers felt extra words added clarity. However, a mathematical conclusion and concern found in the study of generated examples (definitions) for a square, was if the prospective secondary teachers had the ability to distinguish between necessary and sufficient conditions or not. The results of this study suggest a caution to those considering having students generate definitions - such activities may produce the notion of right or wrong definitions. This suggests that definition-construction activities must guard against this potential issue (Zazkis & Leikin, 2008, p. 147). Therefore, careful attention should be made when the distinction is established between necessary and sufficient conditions as well as the awareness as to what such conditions offer to a definition.

Leikin and Zazkis later studied the content dependence on prospective teachers' knowledge related to defining (Leikin & Zazkis, 2010). The major findings in this study were those prospective mathematics teachers' understandings of definitions differed depending upon the content area. Specifically, they held richer knowledge of definitions and their use in the content area of geometry. This result could be attributed to the school tradition of learning geometry as a proof and logic based course where implications can be made as a result of given conditions or to poorly understood definitions.

Gomes et al. examined specifically prospective teachers' knowledge about defining the rectangle. The researchers were concerned then with the development of tasks that allow for understanding and growth particularly in geometry for teacher trainees in Portugal. Again, as in the studies of Zazkis and Leikin (2008), the researchers found an absence of the ability of the trainees to establish necessary and sufficient conditions in the definition and a lack of regard for minimality. Similar conclusions were reported by both Govender (2011) as well as from earlier work of Linchevsky et al. (1992) who found the preservice teachers do not understand that definitions in geometry have to be economical (contain no superfluous information) and that they are arbitrary (in the sense that several alternative definitions may exist) (Govender, 2002; Govender & de Villiers, 2004, p. 34; Linchevsky, Vinner, & Karsenty, 1992). This dissertation investigates the use of the five principles of mathematics as a means of obtaining a deeper understanding of these issues.

Research Concerning The Role of Concept Image

Fundamental to the research presented so far is the shared belief that defining involves a growing skill to handle both the figural and conceptual aspects of a geometric figure when engaging in the activity of defining (Mariotti & Fischbein, 1997). This idea is also fundamental to the definition construction processes proposed by Ouvrier-Buffet who states, "… a parallel exists between the construction of a concept and the construction of a definition" (Ouvrier-Buffet, 2003, p. 1). Research on the role of concept image is grounded in Tall and Vinner's perspectives of the formal concept definitions accepted by the mathematics community as "a form of words used to specify that concept" … while the concept image is "the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties …" (Tall & Vinner, 1981, p. 152). These words are best exemplified in the figure of Rosken and Rolka (2007) by which the differentiation of mathematical knowledge with the subjective constructions that play an important role in understanding mathematics by our students (Rosken & Rolka, 2007, p. 184). These subjective constraints become the focus of the personal concept images and their impact of definitional understanding.

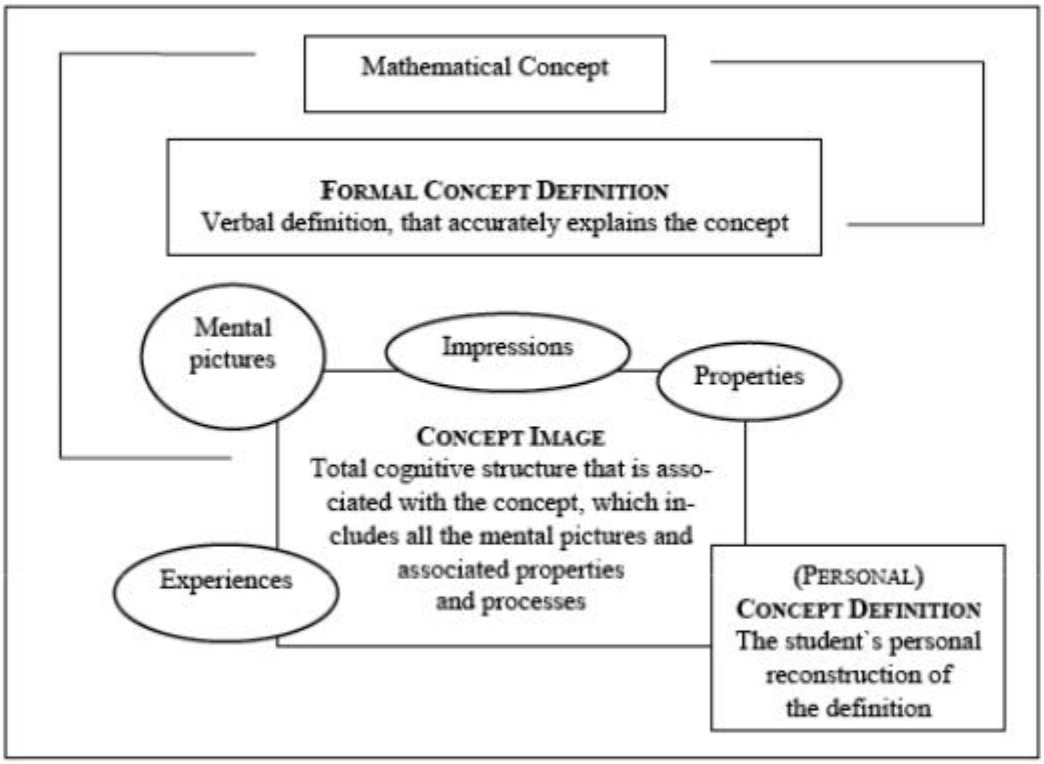


Figure 1. Rosken and Rolka (2007, p. 184) - Depiction of Mathematical Concept Reprinted with Permission

Gomes also noted the disjointed classification of quadrilaterals resulted from many preservice teachers relying on their concept images to give descriptive definitions for the rectangle. Gomes et al. strongly recommend focusing educational efforts for trainee teachers to include work on what is understood as a valid mathematical definition and the bringing together of their concept images and concept definitions in this effort (Gomes et al., 2013).

The research of Fujita and Jones used the perspective of Tall and Vinner as the focus of their work that has extended this understanding of concept image in their work on quadrilaterals. Their early work investigated how preservice teachers use defining when classifying quadrilaterals; they found preservice teachers weak in this area due to a lack of understanding of the parallelogram attributed to the personal concept images the preservice teachers held for this shape (Fujita & Jones, 2006, 2007). The researchers considered the parallelogram an important concept in supporting logical thinking in geometry. Their findings suggest that the ability to define does have an impact on the ability to classify and to write economical definitions in geometry.

Vinner (1991, p.65) notes that definitions pose a problematic situation in the learning of mathematics due to the conflict they impose on one's cognitive processes when engaging in concept. Concept acquisition cannot be acquired simply through the words in a definition. The form of the words in the mathematical definition create the cognitive conflict that either results in understanding or not, yet both the form of the words and the conflict are needed. As stated by Lakatos, "A definitional procedure is a procedure of concept formation" (Lakatos, 1961, p. 54). Concept formation is the aim of definition construction activities. In keeping with Vinner's thoughts, Ouvrier-Buffet (2006) has advocated for definition construction activities that simultaneously develop the concept image while constructing the definition. The aim of this dissertation is to investigate a possible connection between creating high-quality definitions while demonstrating a command of the hierarchal classification of quadrilaterals.

The work of Fujita and Jones is pertinent to this study due to their focus on concept image particularly their work which pertains to preservice teacher understanding of quadrilaterals. One of the aspects of the work of Fujita and Jones was to clarify the role of mental images in both the teaching and learning of geometry. Their research embraces the theory of Fischbein (1993) who notes a geometrical figure embraces both an abstract concept simultaneously with an image. According to this theory, geometrical reasoning involves an interaction and harmony between the abstract conceptual properties and the mental images (Fujita, Jones, & Yamamoto, 2004). Fujita and Jones think of this as *geometric intuition* whereby a person must be able to not only create geometrical figures in their mind, but they must also have the abilities to manipulate such figures, see their properties, and relate these properties to other geometric concepts (Fujita et al., 2004, p. 2). As with Fischbein, this ability needs to be developed with specific and intentional teaching towards that end. The idea of *personal figural concept* was developed by these researchers as a mechanism to analyze the understanding of quadrilateral definitions and classification. A personal figural concept includes both a personal concept image and a personal concept definition and is used in this dissertation.

Students can have their own unique remembered personal concept images and their own personal concept definitions that are constructed through their geometric learning experiences (Fujita & Jones, 2006, p. 130). Fujita and Jones research explored the nature of the gaps between the formal concept definitions of the mathematical community and the *personal figural concepts* held by preservice teachers for quadrilaterals. When the preservice teachers were asked to draw the quadrilaterals and then define them, the majority could draw correct images, but far fewer were able to define the shapes. Furthermore, the definitions offered were often incomplete, only mentioning sides and not angles. In my study, these same issues are investigated with the guidance of the five principles. The five principles support the hierarchical nature of the quadrilaterals and provide a method to find gaps in the PSTs understanding of the shapes relationships.

Their study also revealed the existence of implicit models that influence the individual's ability to define and see quadrilateral relationships. An example of an implicit model comes in the form of a speculative statement of parallelograms where the definition is provided noting it is "a quadrilateral whose opposite sides are parallel to each other" but the implicit property that the adjacent angles are not equal is added by the student and comes from a prototype image of the shape thus excluding rectangles or squares (Fujita & Jones, 2007, p. 11). Implicit properties that are held by a learner influence that learner's abilities to define and classify such as seeing only limited images of leaning parallelograms. Fujita and Jones' research suggests that students' geometrical reasoning processes are strongly influenced by prototypical phenomenon as defined by Hershkowitz. Hershkowitz defines this phenomenon as follows: "Each concept has one of more prototype examples that are attained first and therefore exist in the concept image of most subjects. The prototype examples were usually the subset of examples that [had] the longest list of attributes – all critical attributes of the concept and those specific (non-critical) attributes that had strong visual characteristics" (Hershkowitz, 1990, p. 82). Therefore, this behavior is possibly revealed when a student claims that rectangles do not have all equal sides or that parallelograms do not have equal angles – a result of implicitly held properties due to prototypical images (Fujita & Jones, 2007, p. 12). They suggested further research be done to reveal more about the implicit properties learners hold for quadrilaterals as this affects their abilities to define.

Govender and De Villiers conducted another study related to defining and hierarchical understanding with eighteen secondary preservice teachers (Govender & de Villiers, 2004). The aim of this study was to determine if the use of dynamic geometry software (DGS) package (Sketchpad) could reveal the nature of understanding these preservice teachers held for geometric concepts, and if the use of such software could improve their abilities to define the concepts. Their study had several goals: reveal the nature of understanding already held by the secondary preservice teachers for definitions of quadrilaterals; how and if this nature changed as a result of using Sketchpad; and how the mechanism of the dynamic geometry software influenced the preservice teachers' competence in defining and classifying quadrilaterals (Govender & de Villiers, 2004). According to Govender (2011), dynamic software can serve as an interactive context for making generalizations about geometric objects due to the dragging feature that allows for the elements of the object to change which engages students in switching back and forth from figures to concepts to progress from empirical understanding to theoretical understanding (Govender, 2011, p. 29). Such a mechanism, therefore, provides a means for preservice teachers to engage in the process of defining by testing the properties and relationships of the shapes in a dynamic environment that encourages the development of the high-quality definitions sought in this dissertation.

Vinner indicates the existence of a correlation between formal definition-construction and deeper mathematical understanding by stating. "…the ability to construct a formal definition is for us a possible indication of deep understanding" (Vinner, 1991, p.79). It is the goal of this dissertation to use the mechanism of the five logical principles and a series of tasks to engage preservice teachers in the process of defining in order to produce high-quality mathematical definitions and to deepen their understanding of quadrilaterals. In addition, their abilities to develop an understanding of the hierarchical relationship of the quadrilaterals in the study may also be influenced and deepened.

In summary, research notes the existence of the cognitive conflict that exists between the understanding of a formal mathematical definition and the personal figural concept that is part of the cognitive structure of a student. Preservice teachers and in-service teachers have ideas about what definitions in mathematics should be which do not always align with mathematical requirements. Both in service and preservice teachers demonstrate inadequate understanding of geometric definitions for quadrilaterals. Definition construction activities address the underlying co development of the definition and the concept through methods that simulate the behaviors of professional mathematicians. The five principles in this study provide a means of creating a conflict between the personal figural concepts of the individual and that of the mathematical definitions for quadrilaterals.

CHAPTER 3: METHODOLOGY

Introduction

This chapter describes the research methodology and the procedures used for the collections of data in this qualitative study. The research methodology employed is designed-based with the five logical principles serving as a mechanism to promote growth in the preservice teachers abilities to write high quality mathematical definitions for quadrilaterals. Not only does this design support the purposes of this study, but it also aligns with the objective of the study which is to determine if a sequence of tasks around the five principles, helps preservice create, negotiate, and refine high-quality definitions for quadrilaterals, parallelograms, rectangles, and squares. The focus of the development of the instruments (tasks) is to create situations that require the redefining, negotiating, or creating of definitions of high quality. This chapter provides the following information: (1) background about definition-construction frameworks, (2) the five principles, (3) the DMA or defining mathematics activity framework and how this was used to analyze the data obtained resulting from the dialogue that occurred, (4) the participants, (5) data collection methods, (6) a brief summary of a pilot study, (7) and concluding remarks.

Design-based Methodology

The principles of previous research on definition-construction processes and the proposed frameworks of Ouvrier-Buffet guide this qualitative study. The theoretical framework embraces the understanding that the nature of mathematical definitions includes both the process of defining and the product of producing a definition (Ouvrier-Buffet, 2004a). A designed-based research approach simultaneously pursues the goals of developing effective learning environments and of using such environments as natural laboratories to study learning and teaching (Sandoval & Bell, 2004, p. 200). Another interpretation is that "…design experiments entail both 'engineering' particular forms of learning and systematically studying those forms of learning within the context defined by the means of supporting them" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003, p. 9). Therefore, this study has two purposes: (1) Theory building for definition construction and (2) The development of task to support the learning of definitions for quadrilaterals while using the five principles. Pedagogically, the study may support student understanding of particular geometric definitions in the classroom while theoretically, the results may inform the growing base of knowledge concerning definition-construction. Therefore, this study hopes to support the goals of design-based research by examining preservice teachers engaged in tasks that support definitional reasoning in the creations of high-quality geometric definitions.

Definition-Construction Frameworks

The work of Ouvrier-Buffet (2004) offers a substantial amount of research aimed at describing a framework to measure the definition-construction process. This French researcher notes a serious lack of such a framework yet believes such a framework may help to explain what is meant by "doing mathematics" (Ouvrier-Buffet, 2004b, p. 1). In order for this to occur, Ouvrier-Buffet (2004) such frameworks must go beyond classification and redefining tasks to include activities by which construction of definitions support the resolution of negotiation and creation. Therefore, the goal of the empirical work of this researcher is to extend the existing theoretical models, mainly from Lakatos (1961), in order to establish a theoretical background that is appropriate to every field of mathematics and create appropriate situations of definition construction.

According to the researcher this would establish the following: (1) Such would describe a defining process from both a didactical and mathematical viewpoint, (2) Such would include students' preexisting conceptions on definitions such that an evolution of conceptions occurs (Ouvrier-Buffet, 2004b, p. 2).

From her work, two kinds of situations of definitions construction tasks were proposed. One type is a classification task in which characteristics of a concept are examined and refined through the use of examples and counterexamples in order to arrive at a definition of the object. Here the student reflect on the different offerings as well as their own definitions and work towards consensus towards a common definition that embodies minimal and inclusive properties. A second type of SDC is called a problem-situation, a more open-ended approach, where the starting point for the discussion was not provided. This type encouraged the students to explain a method of constructing the definition. In each type, definitional reasoning is at the center and students engage in the construction of mathematical definitions to resolve the problem at hand. In my study, the students will resolve any conflicts between their personal concept images and definitions with the five principles as they negotiate the writing of high quality definitions.

The Five Principles

In my study, the new conceptual understanding is the use the five principles adapted from Zazkis and Leikin (2010) that create a high quality definition. For this study, the five principles are:

- Defining is giving a name: the statement used as a definition presents the name of the concept and this term (name) appears only once in the statement.

- In defining a new concept, only previously defined concepts may be used.
- A definition establishes necessary and sufficient conditions.
- The set of necessary and sufficient conditions must be minimal.
- Mathematical definitions are *arbitrary* that is several different and correct definitions may exist for a concept.

Each task in the study incorporates the usage of the five principles in the process of constructing, negotiating, and refining definitions. The preservice teachers are provided with opportunities to interpret and use the five principles in the content area of defining quadrilaterals.

Defining As A Mathematical Activity Framework – DMA

This design-based research study is grounded in a defining mathematical activity framework called the DMA. Not only does this framework provide the ideas for the tasks in the study but it also provides a way to analyze the data and dialogue through the five logical principles. The DMA framework is an emerging framework that embraces the perspective of an *advancing mathematical activity* that describes situations where students are actively engaged in the acquisition of mathematical knowledge. This framework merges two avenues of research, that of the theory of Realistic Mathematics Education and the research on concept image and concept definition (Zandieh & Rasmussen, 2010). The framework captures activity that can be analyzed both through horizontal movements of thinking as well as vertical movements. Horizontal movements include such actions as seeking patterns, making conjectures, sorting and organizing while vertical movement includes the generalization or formalizing of abstract concepts (Rasmussen, Zandieh, King, & Teppo, 2005). Specifically, the framework acknowledges defining as an *advancing mathematical activity* where students can move along a spectrum from informal to formal reasoning while they attempt to align their concept images with a mathematical definition. This is accomplished as the students move through a series of definitions that the students create and contrast with their concept images with the ultimate purpose of new conceptual understanding (Rasmussen & Zandieh, 2000, p. 72).

The DMA framework uses both the work of Gravemeijer (1994) and Freudenthal (1973). These works view mathematical activity as a human activity that can be either descriptive or constructive (Freudenthal, 1973; Gravemeijer, 2004). In the work of de Villiers (1998), descriptive defining uses a few properties to define a known object; whereas, constructive defining to Freudenthal (1973) means to define new ideas from old or known ideas (de Villiers, 1998; Freudenthal, 1973). Constructive defining falls under the heuristics of Realistic Mathematics Education, specifically the notions of emergent modeling where "the model" and the conception of what is being modeled co-evolve (Gravemeijer, 2004, p. 11). Here the model refers to more than one concept. This heuristic model allows for the creation of a sequence of activities that move the student from a perspective of *definition-of* a student's previous activity to a perspective of *definition-for* further mathematical reasoning (Zandieh & Rasmussen, 2010, p. 58). For my study, the student will work towards a *definition – for* each quadrilateral from the personal concept *definition – of* the shape.

The activities are thought of as an organizing activity. As a clarification of the organizing activity, Zandieh and Rasmussen (2010) note that models are “student-generated ways of organizing their activity with observable and mental tools” (Zandieh & Rasmussen, 2010, p. 58). According to the researchers, observable tools include things in the student’s environment such as graphs, diagram, explicitly stated definitions, physical and objects while mental tools include the ways students think and reason (Zandieh & Rasmussen, 2010, p. 58).

The emergent model can be further described as layers of activity that occur during the transition from *definition-of* to *definition-for* as a means of capturing the conceptual transition (Zandieh & Rasmussen, 2010). The layers of activity are described as follows:

- *Situational level* - This level captures the starting point of an activity where students may begin their understanding by using physical models related to their understanding of some real world setting for the activity. The level is noted as being embedded in and experienced through a real life setting. This level could include mental images associated with real world experiences such as using cubes to represent people in a room. In this study, both the *personal concept images* and *personal concept definitions* as defined by Fujita and Jones as *personal figural concepts* will be used at the foundational beginning for each PST to provide the contrast level as the study progresses (Fujita & Jones, 2007).
- *Referential level* - The level in the activity process moves the focus to models that refer to the activity (sometimes mental) of the first level. This level might capture the use of both concept images and definitions from the *situational level.* In this

study, participants are asked to refine and reformulate given definitions using the intervention.

- *General level* – At this level the transition begins to change from the *definition-of* to *definition-for* whereby the personal concept definition from the beginning original task is not longer referenced in the reasoning. Attention is focused on mathematic relationships so for this study, necessary and sufficient conditions from the intervention play a role.
- *Formal level* – At this level new mathematical realities emerge that are independent of any of the previous levels (Rasmussen & Zandieh, 2000; Zandieh & Rasmussen, 2010). For this study, the *formal level* is beyond the scope of this study because at this level researchers would examine the use of the five principles in another mathematical context.

The DMA framework provides a way to understand the role that defining can have as students transition from informal to more formal ways of reasoning by framing this reasoning around the creation and use of concept images and concept definitions (Zandieh & Rasmussen, 2010). The researchers investigated how students began to understand a new mathematical idea - a *definition-for* triangles on the sphere based on the students' former understanding of triangles on the plane as students worked through the tasks they moved from a *definition-of* to a *definition-for.* This movements through levels in this study resulted in the emerging DMA framework. For my dissertation, the levels have been adapted to the geometric focus of the study. The level descriptions for this study, adapted from Zandieh and Rasmussen (2010), are given in next.

- Situational Level – Makes a general comment about the given definition without embracing the five principles. They are unable to release their personal concept images.
- Referential Level – Makes a comment that indicates the beginnings of understanding of the principle in the given high-quality definition. They begin to explain the principle in the high-quality definition yet may reference their personal concept image/definition.
- General Level - Makes a comment that indicates strong understanding and ownership of the principle. Comments suggest embracing of the high-quality definition with no reference to personal concept images.

Two studies that have used the DMA framework will be discussed next as they offer evidence of the usefulness of the framework in analyzing definition-construction activities. In the first study, post-secondary students were using both their concept images and definitions about parallel lines from the plane to a sphere thus creating new definitions (Whitney, Kartal, & Zawojewski, 2012). This fourteen-week study involved the recording of student work and discourse through activities that elicited student concept images and planar definitions for parallel lines and how these ideas were transferred to a spherical axiomatic understanding. The researchers used stages to understand the data collected. First, they identified places in the transcripts that gave evidence of students engaged in creating or defining the idea of parallel. Secondly, they described each episode with respect to a level in the DMA framework. A final stage classified the evolving definitions into "types" based on how parallel notions in the plane were used in the generation of spherical definitions (Whitney et al., 2012). For example, at the beginning of the lessons, students focused on their concept definitions as the basis for forming a definition on the sphere and would argue notions of the necessity of lines being equidistant. In addition, the students also were challenged to reconceptualize their planar ideas of transversal and corresponding angles. Referential activity occurred as the students used a concept definition to create a concept image on the sphere or when they moved back and forth from the *situational* understanding of parallel lines. When students were able create the definition totally in the context of the sphere, the *general level* of the DMA was reached. In this study, the *formal level* of the DMA was not addressed, as the students did not have an opportunity to use the new *definition-for* parallel on a sphere in other contexts.

The second study, a teaching experiment, involved the engagement of two post-secondary students in tasks designed to reinvent the formal *definition-of* limit (Swinyard, 2011). The results of this study suggest that students have the potential to reinvent a coherent *definition-of* limit and gave evidence of the types of reasoning used during that reinvention. The key to this potential ability is supported by intentional and planned guidance of the researcher or teacher that is not necessarily at odds with the reasoning of the student. In this study, Swinyard (2011) felt it necessary to guide each student's discussions around graphical representations of the limit, yet this guidance was carefully executed. He notes, "Directing them to center their discussions around graphical representations, shifting their focus to reinventing a *definition-of* limit at infinity, and purposely engaging them in conversation designed to elicit a shift to a *y*-first perspective were all substantive interventions on my part as the researcher. The key was that Amy and Mike took ownership of the iterative process of constructing a precise *definition-of* limit, and in so doing, developed sophisticated understanding of what is a complex mathematical idea" (Swinyard, 2011, p. 112). In particular, Swinyard (2011) uses as an example how Amy chose words of *"arbitrarily close"* or "as close to 4 as you want" as a means of describing mathematically the physical process she was trying to describe that a limit's existence depends not on satisfying every *definition-of* closeness but on any arbitrary definition (Swinyard, 2011, p. 108). In essence, Swinyard poses that this adoption of an *arbitrary closeness perspective* gave rise to an acceptance of the need for mathematical rigor in describing the physical process that led to the mathematical *definition-of* limit.

Both of these studies provide information to this designed-based study. First, the design of the tasks must support scaffolding around the understanding of the five principles. Secondly, in order to facilitate the tasks, it is important for me to understand how the preservice teachers understand the five principles. Careful consideration must be given to the *personal figural concepts* the preservice teachers hold and how the characteristics of these concepts play out in the tasks. The tasks, with explanations as to their purpose, are found in Appendix A at the end of this dissertation. Each task is designed to understand what the preservice teachers understand about the properties and relationships of the quadrilaterals and to determine how this understanding evolves and changes. For the first two tasks, data is gathered on the beginning understandings of the quadrilaterals as well as the PSTs' understandings of the five principles. The second set of two tasks challenge the PSTs to examine the meaning and use of the principles in high quality definitions to allow for further clarification of the meaning of the five principles. The last set of two tasks promotes growth in the use of the five principles and the hierarchy of the quadrilaterals as the PSTs are asked to redefine given definitions and create new definitions for each quadrilateral.

Participants

Seven preservice, elementary/middle school college majors, participated as volunteers for this study. The participants are enrolled in a Midwest community college, population 10,000, that supports educational transfer programs to nearby four-year universities. Details about the current educational background of each preservice teacher (PST) are provided in Chapter 4. The participants were asked to meet three times four two hours to engage with two tasks at each session. The sessions occurred two weeks into the start of the second semester of the academic year. Their discussion and artifacts were recorded using audio recording devices and scribing pens. All the volunteers signed a consent form as part of the IRB requirements of the university. Participants were compensated for their time.

Data Collection Methods and Unit of Analysis

The analysis of the dialogue serves as the main unit of analysis. Anything they PSTs write or draw also becomes crucial evidence of the definitional reasoning that occurs during the execution of the tasks. The discussion that occurs during each task serves as an indicator of changes or shifts in the levels of thinking about definitions through the five principles. The tasks bring about opportunities to discuss how to refine definitions that do not exhibit the five principles or to create new high-quality definitions. The dialogue then serves as a key form of documenting changes or shifts in thinking about the definitions that can be categorized by the DMA levels.

Pilot Study Results

Prior to executing the study, a pilot was conducted with two preservice teacher volunteers from the same community college with the same declared educational major. The purpose of the pilot was to evaluate the effectiveness of the tasks in producing engaging dialogue about the quality of definitions. The task plan for the pilot is given next:

Table 1. Intent of Pilot Study Tasks

Day	Task	Purpose/Focus
1	1	Determine PCIs/PCDs
	2	Determine PSTs interpretation of the five principles
2	3	Rewriting of definitions to examine five principle understanding
3	4	Application of principles 1 to 4 by creating definitions

The pilot also was informative in determining the timing of each session, the quality of the recording devices, and my ability to transcribe useful data.

The findings of the pilot suggested the following:

- Their personal concept images were prototypes even after declaring drawing the square would be sufficient enough to represent all four quadrilaterals.
- Their definitions lacked sentence structure and were descriptive phrases.
- They had difficulty translating the meaning of the five principles.
- They could not effectively use the five principles.

- Four tasks were not enough when addressing necessary, sufficient, minimal and arbitrary.

A change for the study was reevaluating the tasks, increasing the number of tasks from four to six and adding a third session. The major task change involved providing the PSTs in the study with a task that asked them to identify the five principles in two high quality definitions. This new task replaced the old pilot task given below:

Use the five principles to analyze the given definitions of rectangles then rewrite the definitions to demonstrate the five principles.

- Four sides, two opposite sides that are the same length and the other two are the same length but shorter.
- All sides and angles have the same length/degrees.
- A parallelogram with four right angles.
- Pairs of opposite sides parallel and equal length and all right angles.

Initial Task 3 Pilot Study Using the Five Principles

This old task proved to be ineffective in that the PSTs in the pilot did not have a meaningful understanding of the five principles to write new definitions using the principles.

More time was also allowed for the PSTs in the study to discuss the similarities and differences in their drawings and definitions of the shapes. This discussion became the focus of the first session in the study.

The pilot study did not allow for a robust discussion of the PSTs personal concept images or definitions. Using the advice of Edwards and Ward (2005, 2008) the time allowed for this task was increased as the dialogue produces the awareness the PSTs need about their starting or situational levels. Vinner also suggests that both students and teachers note the interplay between *personal concept images and definitions* when embarking on learning definitions and reminds mathematics educators of the "cognitive power that [a] definitions has on the student's mathematical thinking" (Vinner, 1991, p. 80). These changes to time allotted this task were in keeping with the research goals for the study.

Summary

The personal figural concepts serve two purposes in this dissertation: (1) Their strength can often hinder progress in definition-construction; (2) They provide a point of contrast for the PSTs who with the support of the five principles can shift from a definition-of understanding to a definition-for understanding of the quadrilaterals in this study. Dialogue offers a way to determine how personal concept images and definitions affect the definitional reasoning of the PSTs. The situational level since it can be used as an effective way for the PSTs to contrast their concept images with mathematical concept definitions. The five principles are aligned with the hierarchy of the quadrilaterals in the study. When Zandieh and Rasmussen (2010) discuss a new mathematical reality students are expected to reach, this study sees the new mathematical reality as two-fold. It means embracing both the use of the five principles in writing definitions and enhanced understanding of the hierarchy as well.

CHAPTER 4: FINDINGS

Introduction

Chapter Four presents the findings from this study. These findings provide data and evidence to answer each research question but in particular, the overarching question concerning how the PSTS use the five principles to analyze, negotiate, refine and create new definitions is supported. In this chapter, the first section gives academic background of the seven PSTs who volunteered for this study. The remaining sections report the findings that emerged from the dialogue and how this dialogue informed the remaining meetings of the PSTs and the tasks they would discuss. Two tasks were completed for each of the three meetings over a period six weeks. The dynamics of each session changed in terms of PST involvement. The first meeting was conducted with two groups while the second meeting was one large group of six. The final meeting was enacted using pairs of the PSTs. PST 2 made special arrangements to meet one on one with me due to scheduling issues at work for sessions two and three. After the section on PST background, an overview of the tasks with their intents is provided. Specific details about the sequence of tasks enactments, PST dialogues, and findings for each session with a focus on informing the research questions follow. The findings demonstrate the evolutional thinking about definitions as a way to reason about quadrilaterals as well as how the use of the definitions of quadrilaterals supported their emerging understanding concerning the nature mathematical definitions.

Key findings that are emerged are: the PSTs personal concept images are prototypical; the PSTs personal concept definitions demonstrated both a lack of sentence structure and hierarchical reference for the quadrilaterals; the PSTs hold didactical concerns about how definitions for quadrilaterals should be learned; the PSTs voiced intuitive notions about the features of the geometric definitions called emergent criteria; PSTs had difficulty communicating the meaning of the five principles; and the PSTs demonstrated that the five principles had become part of their total cognitive structure by writing high quality definitions.

These findings are discussed as they emerged and how one finding impacted another finding as the design-based study, introduced in Chapter 3, progressed. In other words, the specific findings from one session were connected to outcomes later in the study. One example of this comes from the lack of sentence structure which made a robust understanding of principle one, “Defining is given a name”, difficult for the PSTs to interpret in their own words. When the object being named or defined (noun) was left out of their definitions principle one was hard for them to articulate.

The chapter ends with concluding summaries of each PST’s growth in terms of DMA levels, and notes that the difficulties students have with mathematical definitions goes beyond just the understanding the content but also includes difficulty understanding the nature and use of definitions.

The Background of the Preservice Teachers in the Study and Tasks Overview

The participants’ current status in the required mathematics content course for the program is given in the table. All seven were enrolled in the education program at a mid-sized Midwestern community college. The mathematics content course requirements include a course in Number Concepts as the prerequisite for two other courses to be taken separately or at the same time without regards to order. Each course is a fifteen-week course where the students meet twice a week. The data was collected as the Winter Term had just begun.

Table 2. Current Status for Mathematical Content Courses Per PST

Name	Number Concepts for K-8 Teachers	Geometry for K-8 Teachers	Statistics and Probability for K – 8 Teachers
PST 1	Completed	Taking now	N/A
PST 2	Completed	Completed (twice)	Completed
PST 3	Completed	Not completed	Taking now
PST 4	Completed	Not completed	Taking now
PST 5	Completed	Completed	Taking now
PST 6	Completed	Completed	Taking now
PST 7	Completed	Taking now	Taking now

Each PST had completed the number concepts course. PST 2 was the only participant who had completed all three required courses and had taken the geometry content course twice for a grade improvement. PST 1 and PST 7 were currently taking the geometry course and PST 7 was also enrolled in probability and statistics. For PST 1 and PST 7, being currently in the geometry course meant they were wrapping up a unit on quadrilaterals thus had a fresh perspective on the shapes. PST 3 and PST 4 had not taken the required geometry course but were taking probability and statistics. These two PSTs remarked they were working from geometry experiences from high school. For courses currently being taken, the PSTs had only had about three weeks of instruction since the data collection started three weeks into the current college term. This information on course completion will be revisited both in terms of how the knowledge from a content course was communicated in the dialogue and how each PST performed for a given task throughout the sessions through contributions to the dialogues.

A table giving an overview of the tasks used in the study is given next. The findings that came from the dialogue of Task 1 and Task 2, and in the case of Task 1, the artifacts, determined the design and the how the next session would be conducted. This table gives the overall picture of what occurred for this study.

Table 3. Overview of Tasks for Study

Day	Task	Organization	Purpose/Focus	Analysis
1	1	Two Groups	Determine PCIs/PCDs	Coded PCIs/PCDs and initial values
	2	Two Groups	Determine PSTs interpretation of the five principles	Made researcher comments on their interpretations
2	3	One Group	Examine PSTs identification of the five principles in a high quality definition	Coded using DMA level
	4	One Group	PSTs rewrite their PCDs	Examined for the presence of the five principles and DMA level
3	5	Pairs	Application of the five principles through rewriting definitions	Made researcher comments on their revisions and on their dialogue
	6	Pairs	Application of principles 1 to 4 by creating definitions	Examined for DMA level in written definition and dialogue

Preservice Teachers' Concept Images

Task 1 provided the baseline data for the study in terms of the personal concept images and definitions PSTs hold, thereby, supplying partial evidence to answer the first research question, "What personal concept images and personal concept definitions do the preservice teachers hold for quadrilaterals?"

Their personal concept images and definitions from this task provide a window into prior learning experiences the PSTs had concerning the quadrilaterals learning the quadrilaterals. Of equal importance was the dialogue about similarities and differences that initiated the PSTs awareness about the perceived quality of their images and written definitions. This dialogue also produced some unexpected findings such as *initial values* the PSTs held for definitions of these quadrilaterals.

The drawings from Task 1 are a concrete way for the PSTs to demonstrate their personal concept image, i.e. how their total cognitive structure for a polygon is expressed. Discussion of drawings helped the researcher interpret what the PSTs valued by what they drew. Their discussion about comparing the drawings and personal concept definitions revealed qualities they value in a definition as well as what and how they understood the concepts of the quadrilaterals. For some the quadrilaterals shared properties and were related in a hierarchical structure but for some the quadrilaterals were separate entities as demonstrated in the discussion.

The findings for Task 1 were that the drawings were prototypical as in the pilot. The personal concept definitions lacked sentence structure and were descriptions of the shapes drawn with a list of characteristics that reference side or angle properties.

On the paper provided, please do the following:

Draw an image for a quadrilateral, parallelogram, rectangle and square.
Now write a definition-for each shape.

In your group, discuss how your images for each shape are the same and how they are different.

Repeat the conversation for each definition your created.

Was it easier to determine how the images and definitions are alike or was it easier to determine their differences? Why do you think this was so?

If your definitions were given to someone outside a mathematics classroom, do you feel he/she would have confidence in their understanding of each shape?

Task 1. Drawing Images and Writing Definitions

Sample drawings of the concept images for the PSTs are given in Figure 2 followed by their personal concept definitions for the four quadrilaterals in the study. Also reported is some sample dialogue that occurred during a discussion of the similarities and differences for the images and definitions where criteria or intuitive ideas about the nature of definitions emerged.

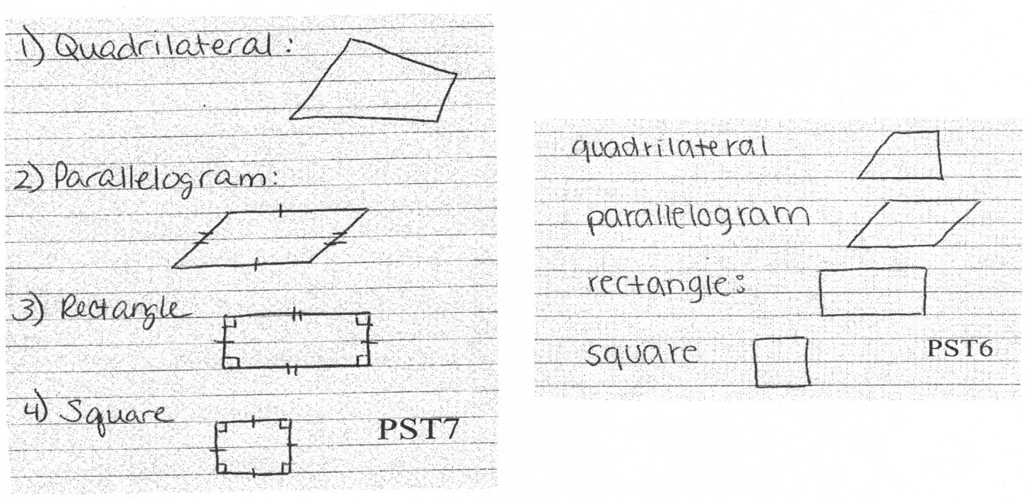


Figure 2. Representative Drawings by PST 7 and PST 6

The drawings produced by all seven participants were similar to those shown and were of a prototypical nature. Orientation of the shapes often seen in research of this type was identical in all seven participants. Of the seven sets collected, PST 1 and PST 7 included markings on their images indicating congruence and/or parallelism of sides and/or angles while the others did not mark their drawings. In the dialogue, the PSTs noted such markings added clarity helping them remember important qualities of a shape. The drawings of the quadrilaterals produced by five of the PSTs were prototypes for but two others, PST 1 and PST 2, drew isosceles trapezoids where PST 2 stated, "these really are a kind of quadrilateral". Here kind of quadrilateral implies some understanding of the relationship of trapezoids to quadrilaterals. PST 5 responded to PST 2, "But I only drew a quadrilateral because it is any four-sided figure with four angles and then I remembered it had to be a closed shape." Another comment from PST 1 indicates her prior knowledge when she reminds the others that if you start with the word polygon, you don't need to say closed shape since polygon implies that property. To this PST 6 adds, "even parallelograms and squares are quadrilaterals." From this comment, PST 6 demonstrates that she has an understanding of the relationship of the shapes.

In summary, their drawings were noted as prototypical and provided the first necessary piece of data for the Situational Level of the DMA framework or the significant glimpse into the concept images they all held. The drawings they provided influenced the personal concept definitions they produced. As is noted in patterns in the next section, these personal concept definitions or the student's reconstruction of the formal definition, were often descriptions of their drawings yet some of the dialogue excerpts demonstrated the potential for the use if hierarchical relationships.

Preservice Teachers' Personal Concept Definitions

The personal concept definitions were coded using a modified coding scheme from the pilot study and as provided in Chapter 3. The following elements make up the coding for this study: *Image Description, Technical Structure – Phrase or sentence,* and *Hierarchical Reference* or *Inclusive Reference.* Image description refers to the selection of words used, such as was the written form only a description of the drawing that included visible characteristics of the drawing or were non-visible properties listed as well. Technical structure references the form of the definition. Distinctions were made for a form of words written as a phrase or group of phrases or as a complete sentence where the noun identified what concept was being defined followed by a verb and clarifying characteristics. Though the researcher expected to see complete sentences, the PSTs were comfortable indicating a definition by simply referencing the shape name followed by a colon then description of the image. The result of this meant the noun or shape name was disconnected from the descriptive phrases and no verb was present. The effects of the lack of sentence structure impacted the PSTs abilities to understand and interpret the meaning of principle one in Task 2 are discussed later.

Hierarchical or inclusive reference referred to type of relationship to another geometric polygon or non-geometric figure noted in the definitions. The definition was considered as hierarchical if the classification was related to a geometric shape but considered as inclusive if it hinted at being related to an undefined geometric construct like shape or figure.

The personal concept definitions took various forms such as phrases that lacked the sentence structure of containing a noun and a verb thus the name of the object was not present. The sample of definitions is provided in Table 3 for PST1 and PST 4. Appendix B contains all of the coded personal concept definitions. These two PSTs were selected to demonstrate the contrast that I had noticed in their definitions. PST 1 started each definition by not only naming the concept being defined but she also embedded each definition in the starting position of quadrilateral. When she was asked to explain this strategy for her definitions, “We were taught to always start each definition with the word quadrilateral.” Though this might be considered potentially a limiting factor when she is asked to write minimal definitions that use necessary and sufficient conditions, she was able to connect principle two’s previously defined conditions with principle three’s necessary and sufficient conditions, therefore, her final definitions demonstrated the fourth principle of minimal. Her final definitions are discussed in detail later in this paper.

PST 4 had not taken the geometry course and wrote definitions that reflected descriptions of strong prototype images of the shapes. PST 1 is coded with hierarchical reference for use of the geometric term polygon but PST 4 is coded as inclusive in her first definitions due to her use of shape. She reminded her group that she had not had a geometry course since high school and later she admits learning the shapes as separate concepts. Throughout the study, she reasoned with her strong prototypical concept images impeding changes in her thinking about definitions.

To investigate the nature of their personal concept definitions each definition for each polygon per PST was coded. The complete distribution is found in Appendix B.

Table 4. Sample Coding of Personal Concept Definitions for PST 1 and PST 4

Subject	Definitions	Codes
PST 1	A quadrilateral is a 4-sided polygon	Sentence Hierarchical Reference
	A parallelogram is a quadrilateral with opposite sides parallel to each other and opposite angles congruent.	Image Description Sentence Hierarchical reference
	A rectangle is a quadrilateral with opposite sides parallel to each other and four 90° angles	Image Description Sentence Hierarchical reference
	A square is a quadrilateral with all sides and angles congruent.	Image Description Sentence Hierarchical reference
PST 4	Any four sided shape – closed, straight lines	Image Description Phrases Inclusive reference
	Parallelogram: "rectangle on a slant", 4 sides – two sets of equal sides-parallel, top/bottom parallel, Left' right parallel.	Image Description Phrases
	Rectangle – Two sets of parallel sides – 2 sets of equal sides	Image Description Phrases
	Square – four equal sides, 4 equal angles, 2 sets of parallel side lengths	Image Description Phrases

The next table highlights three important features selected because of their impact on the principles of naming, necessary and sufficient conditions and minimal.

When all seven PSTs are examined, image description occurred 100 % of the time for all four of the quadrilaterals. This was expected due to findings from prior research indicating these personal concept definitions often contain long list of properties (Vinner & Hershkowitz, 1983).

Table 5. Summary of Codes Across the Polygons for all PSTs

Polygon	Image Description # (%)	Structure Sentence # (%)	Hierarchical Reference # (%)
Quadrilateral n = 7	7 (100%)	4 (57%)	2 (29%)
Parallelogram n = 7	7 (100%)	2 (29%)	3 (43%)
Rectangle n = 7	7 (100%)	2 (29%)	3 (43%)
Square n = 7	7 (100%)	2 (29%)	3 (43%)

This tendency to include all such properties for a shape ultimately blocks their ability to write a minimal definition as is seen later in the findings. Another finding that impeded the use of the five principles was the lack of sentence structure. This suggested the PSTs were likely to have issues with principle one and they did in Task 2. Hierarchical reference to a previously defined polygon occurs less than 50% of the time for all the polygons and did affect the principles of necessary and sufficient as well as minimal as is reported in the findings for second day.

PST 7 consistently wrote complete sentences where each subsequent definition depended on the previous definition demonstrating clear hierarchical referencing. Though PST 1 wrote complete sentences and started each definition by beginning with the term quadrilateral thus giving each a hierarchical referencing but each definition contained added description of the shape. PST 5 and 6 were consistent in using inclusive referencing while PST 3 and 4 only used inclusive referencing for the quadrilateral. Otherwise, PST 3 and 4 used both image description and phrases in three out of their four definitions. The image description included necessary but also redundant characteristics in the list of descriptive phrases as indicated in the research of Hershkowitz and Vinner (1983).

To summarize, image description is a strong trait in this table as well as the lack of sentence structure. Hierarchical understanding in their written definitions occurred only 43% of the time for parallelograms, rectangles, and squares and is hard to accomplish with out the technical structure of a noun and verb provided in a sentence. Each PST's personal concept definition provides a window into how each PST reconstructs the formal mathematical definition. Not only do these personal concept definitions show an understanding of their mathematical content for each polygon in the study, they also suggest how a PST understands the structure of defining mathematical content. From this table, description is important to the PSTs when defining but considerations of sentence structure that explicitly names the concept being defined is not. Not attending to the hierarchical nature of the polygons in the study as a part of their definitions suggests these PSTs do not have a consistent understanding of how these polygons are related. The next section reveals initial values called emergent criteria in this study that accompanied this discussion.

Emergent Criteria

As a final part of Task 1, the PSTs discussed the similarities and differences in their drawings and definitions. The purpose of this portion of the task was to allow the PSTs time to reflect and share their ideas about what they had produced and examine their own concept images or drawings and the personal concept definitions. The dialogue that occurred included statements about how they valued the listing of properties of the shapes in their definitions and how the use of markings made their drawings show these properties. Such dialogue did occur but other significant thoughts emerged and are provided in Table 6. These findings are examined and discussed in the remainder of this section.

Table 6. Emergent Criteria with Examples

Emergent Criteria	Examples from Dialogue	Researcher Comments
Previously known concept	I wrote, a quadrilateral is a four sided polygon but that is assuming the person knows what a polygon is.	PST 1 indicates the term polygon must be understood
Prior Knowledge	With prior knowledge of what a polygon is then you could understand what a quadrilateral came from so having prior knowledge you don't have to explain it further.	PST 5 and PST 6 indicates the value of knowing the meaning of polygon while indicating less to explain hints at minimal
Hierarchical	Starting with a quadrilateral was drilled into us in geometry class like this is a quadrilateral with these characteristics. But a square can be a parallelogram and they can both be quadrilaterals.	PST 1 using classroom learning but acknowledging the polygons are related to the quadrilateral PST 6 statement indicates some hierarchical understanding
Minimal	For my definitions, I went on the previous shape so I defined my parallelogram based on the quadrilateral and a square based on a rectangle – I just did not want to write all the information again.	PST 7 shows understanding of the relationships of the shapes and the hierarchy that can make writing the definition more minimal

Table 6 (cont'd)

Didactics	If you teach them the definition of a square and write down a quadrilateral with the criteria list are they going to know it? I feel with kids you have to be more specific.	PST 2 is indicating her concern for how her preschool students learn through repetition
Arbitrariness	All of us left out some minor detail but all together our definitions could lead someone to the right conclusion.	PST 5 realizes their definitions are almost equivalent but not quite if used together than a better and clearer idea emerges
Necessary and Sufficient	So when defining a parallelogram, say it is a quadrilateral and add with parallel sides since we know a quadrilateral is closed and has four straight sides. But you need more to say a parallelogram is a rectangle.	PST 7 indicates adding what is necessary to define a parallelogram from a quadrilateral but adds more is needed to define a rectangle

Using the dialogue from this first day the PSTs' comments were coded for and the prevalence of coded excerpts of the emergent criteria is given next. The highlights from Table 7 suggest that the PSTs value previously known concepts, prior knowledge, and hierarchical considerations. Even though the hierarchical concern has a high occurrence in verbal discussion, this criterion is not seen in their written definitions. Throughout the conversation, PST 7 voiced strong opinions about the hierarchical nature of her definitions. The point she stressed was how a definition could be written more compactly if each subsequent definition was built on the previous.

To engage more dialogue, I asked, "can you talk more about the implications then of the words you use in a definition?" This question brought responses from PST 1, 2, 3, and 6. PST 1 voiced agreement when she added, "if you know the definition of a

Table 7. Prevalence of Emergent Criteria in Responses to Task 1

Criteria Code	Number of Comments	Percent of Total
Previously known concept	10	28%
Prior Knowledge	3	9%
Hierarchical	10	28%
Necessary, Sufficient, Minimal	8	23%
Didactics	2	6%
Arbitrariness	2	6%
Total	35	100%

polygon as a closed figure then you could just say that a polygon is a four-sided polygon – you don't need to say closed again." In spite of this comment, this criterion was missing in the written many of the PSTs' definitions. At this early stage in writing and discussing their personal concept definitions, the tendency to include unnecessary properties coupled with weak concept and relational understanding of the polygons in the study contributed to this missing use of hierarchy.

PST 3 and PST 6 were concerned with the didactics for children, as they need detail in teachers' explanations of definitions. PST 2 agreed from her work with preschoolers, "you need to be repetitive with kids." Poincare (1914) noted as well that a good definition is one that is understood by the students. However, he also wrote that a good definition must apply to all the objects being defined and only to them (Poincare, 1914). Here the PSTs seem to be confusing a mathematical definition's purpose and classroom practice. This also indicates the PSTs' lack of understanding about the role of mathematical definitions as a platform for reasoning about the relationships embedded in the shapes hierarchical structure.

The conversation about detail took two directions as PST 6 agreed that with prior knowledge you would not have to explain things further. PST 5 waivered when she noted that if you did know about the meaning of a polygon then the quadrilateral could be defined from there but she still felt detail was necessary to mention parallel sides again when defining a rectangle as her definitions indicated. The emergent criteria of prior knowledge, previously known concept, and hierarchy often overlapped as in this sample of dialogue. PST 7 added that knowing the previously defined term of parallelogram meant you did not have to repeat all the characteristics when defining a rectangle. Not only does this statement acknowledge the hierarchical relationships of the parallelogram and the rectangle, but also it suggests how the previously defined concept minimizes the need to restate the properties.

The criteria that the PSTs consistently held for definitions of these quadrilaterals were previously defined concept, which implies some prior knowledge, hierarchical considerations, and writing concise or minimal definitions. However, the predominance of these criteria rests on the control PST 7 who contributed 30% of the dialogue excerpts.

The nature of the mathematical definitions in terms of the interplay of the five principles were beginning to emerge as this next section of discussion shows, here the criteria can overlap or one criterion implies another. The following example shows the overlap of the implications of knowing previously defined terms on determining what is necessary and sufficient.

PST 5 – (When defining a parallelogram) – Since we said it's a quadrilateral you already know it has four sides so there can only be two sets of sides – as long as you know a quadrilateral is four-sided.

In this next example, prior knowledge is linked to minimal as criteria.

PST 1 – In class we wrote definitions from the term quadrilateral and did not say four-sided closed shape each time, We used it to be shorter, but you need prior knowledge.

These excerpts suggest they note the consequences of one principle on another whether that is previously defined concept, necessary and sufficient or minimal, but again this was not evident in their written definitions. Other than PST 7, the written definitions continued to list the properties. However, the five principles do overlap supporting the dynamic nature of mathematical definitions through the interconnections of the five principles. Their dialogue indicates they can verbalize these dynamic nature ideas.

Therefore, the findings for the discussion for similarities and differences in their personal concept definitions produced not only discussion about how they compared and viewed the concept images and personal concept definitions but also provided evidence on criteria PSTs hold about the nature of definitions emerge. The dialogue demonstrated that the PSTs held intuitive ideas about the nature of mathematical definitions and the three that often emerged are aligned with the five logical principles. Specifically, prior knowledge or the understandings of previously defined terms are stated in the second principle, "In defining a new concept, only previously defined concepts must be used." The PSTs addressed this 37% of the time in their dialogue. The hierarchical nature of mathematical definitions and specifically noted for these quadrilaterals was a singularly strong criterion for PST 7 as she used not only previously defined terms to build her definitions but defended her position with thinking about what was necessary to say to make her definitions not so repetitive in at most three of her definitions. PST 7's dialogue indicates the logical connections of the five principles to the criteria she is embracing.

Interpretations of the Five Principles

This section outlines the PSTs' interpretations of the five principles. Though the PSTs indicated emergent criteria that aligned with the principles, this connection is not seen in what they discuss or write. The interpretations they produced become a second set of personal concept definitions for them. The interpretations of the five principles were attempts at translating the abstract principles into their own words and indicated difficulties in the translations.

This second sheet of paper gives five logical principles that should be fulfilled when defining a mathematical concept according to mathematicians.

Please discuss your interpretations of the meanings of these principles and record your ideas on the paper provided.

Defining is giving a name; the statement used as a definition presents the name of the concept and this term (name) appears only once in the statement.

In defining a new concept, only previously defined concepts may be used.

A definition establishes necessary and sufficient conditions.

The set of necessary and sufficient conditions must be minimal.

Mathematical definitions are *arbitrary* that is several different and correct definitions may exist for a concept.

As this task ends, determine how you will agree on an interpretation of these five principles. Your understanding of the five principles will be important in the rest of the tasks. Therefore, it is important that a list of understandings be created as we move forward.

Task 2. Interpreting the Five Principles

This task's purpose is to introduce the five logical principles, which encapsulate the nature of mathematical definitions, to determine how the PSTs verbalize their understanding of the five principles thus creating a set of personal concept definitions for the principles. The PSTs beginning perceptions start with this task and the evolution of these perceptions are reported in the dialogue of Task 3. As found in the pilot study, lack of clarity in understanding the meaning of the five principles and difficulties knowing the meanings of certain words used in each principle emerged in the discussion.

The findings demonstrate a surface level comprehension of the mathematical *terms* used in each principle. Certain words such as *necessary, sufficient*, *minimal, arbitrary*, and even the word *concept* have specific meanings in the context of mathematics. These meanings are broader in everyday usage. The word usage situation made it difficult for the PSTs to interpret the principles through their own cognitive structures especially since this is the first time these formal ideas were presented. The interpretations were personal concept definitions for the principles.

Table 8. Summaries of Group Interpretations of the Principles

Principle	Interpretation of Group	Researcher Notes
1. Defining is giving a name; the statement used as a definition presents the name of the concept and this term (name) appears only once in the statement.	Group One – Defining is observation based on common characteristics of (name) Group Two – Defining is giving a definition to a term's characteristics that a person's senses interpret, and labeling that definition with a name that can be clearly recalled.	Group One – Here the word observation suggests preparing to define not defining itself Group Two – Defining characteristics instead of defining the whole concept
2. In defining a new concept, only previously defined concepts may be used.	Group One – When building a new concept, you must simplify previously defined concepts in order to build a new understanding.	Group One - Simplifying previous concepts suggests didactics but is a strange wording

Table 8. (cont'd)

	Group Two – Defining a new concept is not solely limited to previously defined concepts. However, if other concepts are used they must be defined.	Group Two – Seems to suggest a focus on defining concepts and not the whole
3. A definition establishes necessary and sufficient conditions.	Group One – A definition only includes simple and necessary information.	Group One – Use of the word simple not clear but agree to necessary information (facts or conditions) not clear what they mean here and without regard to the term sufficient
	Group Two – A definition needs to be specific, general, and concise that can be expanded upon if needed.	Group Two – Offering contracting ideas in use of specific, general, and concise
4. The set of necessary and sufficient conditions must be minimal.	Group One – The description given is explained in terms that one already understands.	Group One –Relying more on the notion of previously defined terms – no understanding of the mathematical meaning of necessary and sufficient.
	Group Two – none written	Group Two – no response due to time
5. Mathematical definitions are arbitrary that is several different and correct definitions may exist for a concept.	Group One – Though a concept may be interpreted differently, they may all still be correct.	Group One – Use of word interpreted moves away from how a mathematical definition singles out meaning.
	Group Two – none written	Group Two – no response due to time.

The dialogue did produce some insights into their grappling with the principles and gave direction for a new task for the second session. Though neither group could verbalize the act of defining as “naming a concept” these selections of dialogue demonstrate how they engaged aspects of their cognitive structures in the process. For one group, their sensory and perceptual aspects of the cognitive structure play a role. The next section of dialogue illustrates how group two, PST 3, 4 and 7 tried to verbalize the process of defining or giving meaning to a commonly known physical object of a table. They begin by thinking about the common characteristics of any table as a member of the furniture group.

> *PST 7 – Let’s think about defining a table, it is furniture with common characteristics like four legs.*
> *PST 3 – Yes, but are we talking about, what kind of table?*
> *PST 4 – Everyone knows what a table is but it is hard to define.*

This conversation continued for half of the planned time allotted to discussing all the principles. After discussion about the physical characteristics of a table and its various uses in different situations, PST 7 took the lead and brought closure.

> *PST 7 – I get what you mean so the word table brings ideas to people but we have more to discuss. How about this, defining is giving a definition to a term’s characteristics that a person senses than interpret it with a name that can be clearly recalled.*

The group agreed to this as a consensus for the three of them on this principle. The references made to the sensory aspects of defining along with prior experiences of a concept provided evidence for how the total cognitive structure plays a role in defining. By using the sensory object familiar to all of them like the table, they then anchored their discussion in the table’s characteristics as they tried to verbalize the idea of giving a name to something so others could interpret it’s meaning.

The discussion of principle two, already identified as a emergent criterion in the first task, was met with mixed interpretations as seen in the Table 7 under researcher notes. Their comments indicate confusion between *term, word,* and *concept* for the

PSTs. This same confusion was noted in their attempts at principle 3 and 4 for their understanding of *necessary, sufficient*, and *minimal*.

The enactment of Task 2 prompted a change for the third task. Their interpretations of the five principles indicated they did not understand what each represented. A lack of understanding about the intent and meaning of principle 1, was significant and needed attention in Task 3.   As a result, the new task provided two examples of high quality definitions and asks the PSTs to identify the principles found in the definition. The researcher also decided to conduct the session's dialogue as one group to encourage more sharing and practice in building consensus.

Preservice Teachers' Interpretations of the Five Principles in Use

This section describes discussion and outcomes of the second day where PSTs were asked to identify the principles in two mathematical definitions. Two tasks were completed and all of the PSTs except PST 2 worked as one large group. The findings demonstrated the difficulties the PSTs had while attempting to identify and understand the five principles in use in a high quality definition.

For this task, you will work together to identify where the five principles are exhibited in these definitions. You are to arrive at consensus on the identification.

A rectangle is a parallelogram with at least one right angle.

Principle 1 –
Principle 2 –
Principle 3 –
Principle 4 –

Principle 5 - Explain how the given definition is arbitrary and compares to the one above.

A rectangle is a quadrilateral with three right angles.

Task 3. Identifying the Five Principles

The purpose of the task is to help clarify and offer another opportunity for the PSTs to interpret the five principles. This task was developed due to the unsatisfactory results of Task 2 from the first session. This task places the five principles in a text that is not as theoretical as that of Task 2. Therefore, Task 3 is designed to focus the preservice teachers on applying the five principles hoping to improve the verbalization of the five principles from Task 2.

The findings suggest identifying the principles in use is difficult. Their inabilities to rewrite the meaning of the five principles continue in two ways. First, the mathematical contextual use of terms like necessary, sufficient, minimal and arbitrary are still interpreted in an everyday usage way. Secondly, new word usage in the definitions themselves such as the phrase 'at least one right angle' is taken literally and not in the context of mathematics usage.

To completely analyze this dialogue, I looked at the data first for DMA level then secondly for principle understanding in the dialogue excerpts. The DMA levels as adapted from other studies and defined for this study are found in Chapter 3 the numbers indicate the times a comment suggested a particular DMA level in the dialogue.

The responses at a General level represent the prevalence of PST 7's comments that reflected her hierarchical concerns in defining. In the dialogue, 56% of the excerpts accounted for the other PSTs who made references to the concept images or their personal concept definitions when discussing at a Situational level.

A second pass of this dialogue examined the clarity expressed about a principle when it was mentioned. Table 10 gives the times each principle was mentioned. It should be noted that some of the comments address more than one principle and are listed more

Table 9. Occurrences of DMA Levels

Level	Number of Occurrences	Percent of Total
Situational	15	38%
Situational<>Referential	7	18%
Referential	1	3%
Referential<->General	4	10%
General	12	31%
Total	39	100%

than once. The PST who made the comment was also recorded. PST 7 began the conversation while PST 4 offered comments of agreement or affirmation using her static images of the polygons to defend her statements. During this time, PST 3 exhibited a negative attitude about high quality definitions putting too many demands on remembering what other terms represent and mean.

Table 10. Frequencies of Discussion Comments per Principle

Principle #	Description	Count	Percent
1	Naming	15	19.5%
2	Previously defined	18	23.3%
3	Necessary/Sufficient	22	28.6%
4	Minimal	17	22.1%
5	Arbitrary	5	6.5%
	Total	77	100%

The percentage for principle 3 was the highest and occurred when the PSTs were making sense of 'at least one right angle' and the notion of a sufficient definition. For this principle, PST 2 felt if you were defining a rectangle through a parallelogram then that was necessary but the right angle also was important in distinguishing the rectangle. As she noted, "I see the rectangle in the parallelogram circle in my head but it needs the right angles to make it a rectangle." This statement indicates that she is leaving her concept image of the rectangle and parallelogram as separate entities from Task 1 and using a visual of a Venn diagram to see a relationship between the two polygons. Also her statement about the right angles indicates she understands the necessary condition of right angles in order for the parallelogram to be a rectangle. By the end of the study, PST 2 no longer referenced her Venn diagram but used parallelogram as a concept and not a word.

Previously defined terms was the next highest but this reflects their concern earlier demonstrated in the emergent criteria and it appears to be an easier principle to interpret as a result. The small percent for principle five reflects the time issue as the session was nearing the end. Principle one was discussed at the beginning when PST 7 took the lead and once she had spoken, the others accepted the name of rectangle and as such the issue was not addressed again. In the group of six, PST 7 spoke 30% of the time exhibiting guiding understanding. PST 5 argued for clarity in wording, joined by PST 6 noting that even though the word rectangle was mentioned only once so was the word parallelogram. This literal interpretation of Principle 1's reference to the name being used once shows the lack of understanding about the use of once in mathematical context. PST 3 and 4 would speak in agreement where PST 4 would say 'good idea' both following the group. PST 3 voiced a dislike of the definition saying if she had to always think about all the properties of a parallelogram each time it was too much thinking. Both PST 3 and PST 4 continued this behavior showing little change in understanding of either the content of the quadrilaterals or the nature of mathematical definitions and neither of these PSTs had taken the required geometry course.

The excerpts indicating the mention of a principle and the level of understanding in terms of use of a principle demonstrated interesting contrasts to their earlier discussion about the similarities and differences in their images and definitions.

Table 11. Correct Usage Frequencies and Percents of Each Principle

Principle #	Count of Use		% Use	
(n = 77)	Correct	Incorrect	Correct	Incorrect
1	12	3	80	20
2	10	8	56	44
3	9	13	41	59
4	10	7	59	41
5	2	3	40	60

This table demonstrates that in spite of 28.6% of the excerpts addressed necessary and sufficient from Table 10, only 41% were coded as indicating correct understanding the principle in usage in Table 11. Their emergent criteria from Task 1 also indicated they mentioned the necessary and sufficient concern 23% of the time. Though they noted it as a criteria, their understanding of its meaning is less than 50%. Again contrary to the results from Task 1, where the PSTs voiced a concern for previously defined terms, only 56% demonstrated such understanding in this definition. Though they voiced that previously defined terms and prior knowledge were important, using the previously defined terms as windows to the consequences of relationships in the quadrilaterals was not clearly stated.

The 80% clarity for the first principle is solely attributed to the statements made by PST 2 and 7. PST 6 indicated a lack of clarity for the principle when she literally counted the number of times a word appeared in the definition and since both rectangle and parallelogram appeared once she was not certain what was being named and defined. Likewise the split in percents for principle five suggests the PSTs struggle with the relationship of rectangles to the quadrilateral as well as the consequences of needing more information about the angles at this point. It took PST 7 to again convince the others that if you had a quadrilateral and three right angles then the fourth angle was also right thus at most you could only assume a rectangle since the sides were not mentioned and you could not assume they were equal.

PST 3, 4, and 6 clung to the thought that this could mean a square as well. From Task 1, all of these PSTs drawings lacked markings indicating congruence or parallelism of sides so the visual link was missing even though their personal concept definitions attempted to list several properties for each shape. For these PSTs, their lack of a robust understanding of the properties of a parallelogram and what these properties offer the rectangle and square shows. They attempt to assert unneeded properties into the conversation. PST 4 remembers learning these shapes as separate items and PST 3 objected to the requirement of thinking about the properties of the parallelogram as related to the rectangle in this definition. PST 6 also defined a parallelogram in terms of the rectangle in her rewrite. All of these instances demonstrate not only the strength of their concept images but the important role learning experiences play in definitional understanding of the quadrilaterals.

In summary, their dialogue indicated a lack of clarity as to the meaning of each principle in a high quality definition. Though PST 7 and PST 2 had stronger understanding, the other PSTs needed more work to gain clarity. PST 5 and 6 made interesting comments that suggest untangling these principles is not an easy endeavor as one principle implies the other. In particular, PST 5 talked about minimal information to describe a rectangle through a parallelogram with at least one right angle noting more information was not needed as the necessary and sufficient information was implied in both conditions. The overlap of necessary and sufficient with the minimal or principle 4 in her discussion demonstrates how the principles work together in a dynamic fashion. However, this task was done as a large group, the next task offered insight into their dialogues about redefining with the use of the principles.

Writing Definitions after Discussing the Principles

In this section, the findings of the second task for day two are discussed. Basically, there was not much change from their first attempts at writing definitions from the first session to writing new definitions at the end of the second session. Sample definitions are presented that indicate little embracing and use of the five principles occurred. A complete list of the definitions for all four quadrilaterals for each PST is found in Appendix C.

The purpose of Task 4 is to further examine an understanding of the five principles in asking the PSTs to rewrite definitions for the four geometric polygons in the study. The artifacts are analyzed against their initial personal concept definitions to see if

In this task, please write a second set of definitions for the following geometric objects. Record your definitions in the Livescribe books.

Quadrilateral

Parallelogram

Rectangle

Square

Task 4. A Second Writing of Definitions for the Quadrilaterals

change has occurred per principle.

The findings still demonstrated the elements of strong concept images of their prototypical drawings and their personal concept definitions that were descriptive phrases listing properties that were needed or not needed. Attitudes about personal preferences prevailed in their definitions thus resulting in definitions similar to those written during first day's session in Task 1. Their first definition was compared to this new attempt. Evidence of the existence of a principle is marked yes (Y) or no (N). Some illustrative responses where change or no changes are given below and the entire set of comparisons are found in Table 16 in Appendix B.

Though nicely written as complete sentences, the definitions show little change or use of the five principles. The DMA for PST 1's quadrilateral definition does acknowledge in the ending version that **any** four-sided polygon would be defined as a quadrilateral perhaps suggesting broader thinking. However, this observation was not consistent in her other definitions. This PST demonstrates her commitment to how she learned to write definitions in class, by starting each with the term quadrilateral. Her parallelogram definition did not change; therefore they did not meet the principle of arbitrary.

Table 12. Comparing Definitions for Quadrilateral and Parallelogram for PST 1

Point of creation (Shape)	Quoted definition	Presence of the principle in the definition					DMA Level
		1	2	3	4	5	
Starting definition	A quadrilateral is a 4-sided polygon.	Y	Y	Y	Y		Referential
Ending definition	A quadrilateral is any four-sided polygon.	Y	Y	Y	Y	Y	General
Starting definition	A parallelogram is a quadrilateral with opposite sides parallel to each other and opposite angles congruent.	Y	Y	Y	Y		Situational
Ending definition	A parallelogram is a quadrilateral with opposite sides parallel to each other and opposite angles congruent.	Y	Y	Y	Y	N	Situational

Table 13. Comparing Definitions for Rectangle for PST 6

Starting definition	A closed, 4-sided figure with two sets of equal side lengths and angles.	N	Y	N	N		Situational
Ending definition	A parallelogram is a rectangle with 2 sets of equal angles.	Y	N	N	N	N	Situational

For the parallelogram PST 6 did write a complete sentence thus demonstrating the use of principle one but has sacrificed accuracy in her ending definition when attempting to be minimal calling a parallelogram a rectangle. Her use of two sets of equal angles does not insure a rectangle here. During her contributions to the dialogue, she did acknowledge however that rectangles were parallelograms. This error in classification left her at the situational level.

PST 4 demonstrates the use of her concept image. According to her information, she had not taken the required geometry course so what she offers is coming from both her experiences and resulting mental images. In neither definition did she use a complete sentence to indicate the concept being defined, though she used previously defined concepts. To her, the necessary and sufficient conditions along with the minimal principle cannot be met in this list of properties for a rectangle. She maintains the Situational level the study.

Table 14. Comparing Definitions for Rectangle for PST 4

Starting definition	"rectangle on a slant" 4 sides, 2 sets of 2 equal sides – parallel.	N	Y	N	N		Situational
Ending definition	Four sided shape with two shorter sides and two long sides that are slanted.	N	Y	N	N	N	Situational

Across all of the PSTs, the findings in Appendix C showed change in 43% of the starting and ending definitions in the use of the five principles. PST 1 embraced the five principles in 3 of her 4 definitions as well as PST 5 and PST 6. PST 3 showed change in 2 of her 4 definitions. Even PST 7 who indicated that writing the definitions in terms of the previously defined concept had issues writing a minimal definition stating it just felt better to say four right angles in her definition of the rectangle; yet she led the conversation explaining the meaning of at least one right angle. She only succeeded 1 in 4 times. Throughout this dialogue she indicated her understanding of the implications of having one right angle in a parallelogram. However, the growth demonstrated in this analysis provided evidence of the PSTs' beginning perceptions and abilities to use the five principles for the PSTs. The lack of sentence structure meant that principle one was still not clear in their usage, listing of properties meant the impact of previously defined terms as concepts that embodied necessary and sufficient conditions needed more discussion to enable a use of writing both a minimal and arbitrary definition. Though PST 2 was beginning to embrace the term parallelogram as a concept and PST 7 already could understand this, PST 3 and PST 4 were committed to their concept images.

Thus, in summary, these tasks showed little growth in the use of the five principles. Though the dialogue offered glimpses of hope that PST 2, 5 and PST 7, understood them the end products did not validate this finding. Though this set of tasks produced little evidence of change, the robust dialogue did offer some new and useful information such as the need to realize that certain words in mathematics carry significant conditions that students do not understand in the way mathematicians do. The two tasks for the last meeting were changed to engage the PSTS in rewriting definitions where one or more of the principles were missing. The last task would enforce the use of principle one for the polygons as well as the principle of minimal.

Negotiating the Writing of High Quality Definitions

During the last session, the PSTs worked in pairs but again PST 2 met with the researcher at a separate time. The findings showed growth in their perceptions and usage of the principles. Sentence structure was corrected and through a robust dialogue of the relationships of the quadrilaterals, the PSTs wrote definitions that focused on the minimal principle as well.

Working with a partner, you are to consider the following definitions. For each definition, you are to rewrite them with a focus on both principle 1 and 5.

1. A square is a quadrilateral with all sides and angles congruent.

2. A rectangle with shorter, straight, equal, closed line sides

3. Four sides shape with two short sides, two long sides, four right angles.

Task 5. The Rewriting of Definitions Using the Five Principles

The purpose of Task 5 is to examine the PSTs negotiation of the writing of high quality definitions. The definitions in the task came from the personal concept definitions of Task 1. With a focus on the fourth principle of minimal, the arguments about what is necessary and sufficient in each case occurred and not only about the relationships of the quadrilaterals but also their thinking about the dynamic connections of these five principles. This task requires the use of all the principles with attention to naming the concept being defined and making their rewrites as minimal as possible.

Selected supportive examples of ensuing dialogue are discussed next that demonstrate these principles. For example, PST 2 works on the second definition that does not name the concept being defined instead listing properties.

> *PST 2 – Is it comparing a rectangle to a square? It is not a complete sentence; it wants me to say something is, a ______. No, they are not defining a rectangle; I think they are trying to define a square because of the equal.*

PST 2 determines they are defining a square and writes, "A square is a rectangle with all sides and angles congruent." She then goes on to evaluate the minimal principle.

PST 2 – Oh, if I have the rectangle I already have the right angles so I don't need to say anything about the angles again. So a square is a rectangle with congruent sides would be the new definition.

PST 2 was able to reason at this stage about the rectangle without referencing her visual of circles that she used before in other tasks. Shared properties between the rectangle and square through her understanding helped her rewrite this definition. PST 2 demonstrated growth in the area of understanding the relationships of the quadrilaterals and how knowing this supports the principle of minimal. She indicates an understanding of necessary and sufficient conditions as well noting how the rectangle as a concept provides the right angles for the square.

Another pair grappled with the third definition in the task, "Four sides shape with two short sides, two long sides, four right angles." PST 4 originally wrote this definition but she gave no indication of recognizing it. In the following, PST 7 leads the thought process and PST 4 acknowledges in agreement with affirmative statements like "good idea." PST 4 continues to rely on her concept image of a rectangle as a slanted parallelogram. She struggled to make relational connections between and among the shapes.

PST 7 – This is defining a rectangle because it mentions different side lengths and the four right angles, so it is not a square.

PST 4 – Yes, I agree.

PST 7 – Let's write, A rectangle is a quadrilateral with parallel and congruent sides and angles.

PST 4 – Sounds good to me.

PST 7- I think this is not minimal enough, how about, A rectangle is a parallelogram with congruent angles."

PST 4 – Sounds like that works.

Though this dialogue shows the understanding of PST 7, it does show her concern about not mentioning all the angles. This was her concern earlier in Task 3. Even though she led the discussion of at least one right angle she noted she just felt better saying something about all four angles. PST 4 gives little insight into her growth in using the five principles and assumes the role of agreeing perhaps to cover up her lack of understanding. PST 4 has not had the required geometry content course where she will engage in classification tasks and defining quadrilaterals, which could change her concept image.

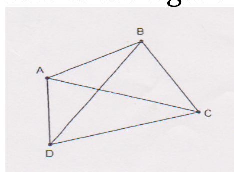
These findings show growth did occur but not for all in the study. Perhaps this finding is related to prior learning experiences with the shapes. PST 7 could envision a hierarchy chart showing the shapes as connected to the previous and used this in her definitions; PST 4 relied on her concept images and stated that she had learned the shapes as separate entities and she had not had the required content course. Her thoughts reflected her high school experience some time in the past. In terms of the research question, PST 1, 2, 5, 6, and 7 developed useful perceptions of the principles or the relationships of the quadrilaterals.

Writing Definitions Based on Hierarchy

The findings in this section show that the PSTs were able to write high quality definitions using the five principles. The purpose of Task 6 is to determine if high quality definitions can be written for a changing quadrilateral ABCD. Another purpose is to note changes in the DMA levels if they occur by analyzing both from the dialogue and the finished products.

This task asks you to write high quality definitions for a changing quadrilateral.
Draw a picture of the shape.
Provide as many true statements about the shape.
Provide a written definition of the shape using the five principles.

This is the figure you will refer to in this task.



In each portion of the task, quadrilateral ***ABCD*** will change.

1. In this case assume ***ABCD*** is a quadrilateral.
2. In this case assume ***ABCD*** is a parallelogram.
3. In this case assume ***ABCD*** is a rectangle.
4. In this case assume ***ABCD*** is a square.

Task 6. Writing High Quality Definitions Using the Five Principles

A sample of some true statements is given in the dialogue excerpts. This excerpt illustrates the growth PST 1 had acquired concerning the relationships and hierarchy of the quadrilaterals. PST 3 indicated that she agreed.

PST 1- Well, if it is quadrilateral ABCD then it has four sides and four corners and is a polygon. When it changes to a parallelogram then it has two set of opposite and congruent sides and the opposite angles are congruent. But a rectangle means the same as the parallelogram but all angles are congruent. For ABCD to be a square the all sides and angles are congruent.

PST 3 – I agree. So let's write definitions now and compare to make them show the five principles.

PST 1 demonstrates not only an ability to delineate the characteristics of the shapes but also the understanding of the relationship of the shapes as the characteristics mean implications for other shapes. During the first session this PST shared that she had learned to write all of her definitions by starting them with the term quadrilateral. This finding is significant for this task because she sees the similarities in the polygons and also notes their differences. Her statements also indicate that she embraces hierarchical relationships, which may impact how she handles both necessary and sufficient conditions and the principle of minimal. She denotes the quadrilateral has four sides and four angles and adds how the side and angle relationships change as the quadrilateral becomes the parallelogram. The relationship knowledge continues as she then talks about the rectangle's relationship to both the parallelogram and the square. PST 3 unfortunately only acknowledges her statements and agrees, offering no window into her true understanding. Her agreement does not necessarily demonstrate an understanding on her part. The resulting definitions from PST 1 and 3 are presented in Figure 10.

1. A quadrilateral is a four-sided polygon.
2. A parallelogram is a quadrilateral with two sets of parallel and congruent sides and opposite angles congruent.
3. A rectangle is a parallelogram with one right angle.
4. A square is a rectangle with all sides congruent.

Task 6. Final Definitions for PST 1 and PST 3

They then commented that the definitions were complete sentences and the way they were written demonstrated that each definition was built on the previous. In summary, PST 1 and PST 3 did write high quality definitions.

1. A quadrilateral is a closed figure with four straight sides whose interior angles add up to 360°s.
2. A parallelogram is a quadrilateral with opposite sides parallel and congruent and with opposite congruent angles.
3. A rectangle is a parallelogram with right angles.
4. A square is a rectangle with equal sides.

Task 6. Final Definitions for PST 4 and PST 7

The results for PST 4 and PST 7 are given in Figure 11 and demonstrate change but not at the same level of use of the principles of minimal as PST 1 and PST 3. Though these definitions represent the influence of PST 7 who stated in her discussion that using the previous term to define the next made the writing shorter, the influence of

PST 4 who wrote phrases with lists of properties is evident as well due to the listing of properties. Her comfort level is expressed in words like I agree or this is good especially for the first two definitions. Prior to submitting their work, PST 7 admitted that using the term polygon made more sense to her than the word figure in the first definition for the pair but the change was not made. The artifacts produced demonstrated that they could also produce high quality definitions but not all the principles were evident. For their first and third definition, minimal was not evident.

1. A quadrilateral is a four-sided polygon with the sum of all angles being 360°s.
2. A parallelogram is a quadrilateral with two sets of parallel sides and opposite angles congruent.
3. A rectangle is a parallelogram with a ninety-degree angle.
4. A square is a rectangle with congruent sides.

Task 6. Final Definitions for PST 5 and PST 6

The final definitions for PST 5 and PST 6 are discussed next and are in Figure 12. This pair also negotiated the writing of definitions that exhibit the principles of high quality definitions in these products. PST 5 had voiced a dislike for such short definitions but later added this kind of definition helps you think about relationships. PST 6 noted this seemed to be a good way to learn how the shapes were related. She added that if she had learned her definitions through the use of the five principles then answering true and false questions on exams would have been easier.

The final definitions for PST 2 are in the last figure.

1. A polygon with four sides is a quadrilateral.
2. A parallelogram is a quadrilateral with opposite angles congruent.
3. A rectangle is a parallelogram with one right angle.
4. A square is a rectangle with congruent sides, or a square is a regular polygon.

Task 6. Final Definitions for PST 2

PST 2 had taken the geometry course twice and admitted completing each time without a clear understanding of the meaning of a parallelogram. She did say that after doing the tasks in the study, she felt that she finally owned an understanding of a parallelogram. This was an interesting comment as I had observed her often draw and reference Venn diagrams and her self-drawn hierarchy chart that she did not use for this task. She also wrote two definitions for the square and used the idea of regular polygon. This is likely due to her prior learning from the geometry class.

To summarize, this task, that asked the PSTs to create definitions, did demonstrate that all PSTs working as pairs were using the five principles. The final products showed the use but more importantly, the discussion of each pair, or from PST 2 individually, indicated their thought process was about using and embracing the principles. The dialogue about the principles was justified through their hierarchical understanding of the quadrilaterals. The change or growth in the DMA level of each PST was different as was their use of the five principles. In each case, the tasks enabled the PSTs to add yet another learning experience to their concept image or total cognitive structure contributing to their abilities to change their definitions.

Overall, this chapter gives several findings. First, the preservice teachers personal concept images are prototypical. Though this finding has substantial prior research backing, the consequences are seen in this study as conflicting with the PSTs use of the five principles. The visual images controlled thinking thus inhibiting growth in the understanding of the hierarchy of the shapes. Another issue produced from the prototypical images was the effects such had on writing the personal concept definitions. The lack of sentence structure was noted as well. Many wrote a list of properties about the shape being defined and often the listings consisted of redundant characteristics. The lack of sentence structure affected the acceptance and understanding of principle 1. As a result, the phrases produced did not allow for their definitions to be grounded in a concept. Previously defined concepts, necessary and sufficient conditions lost their connection to the object being named as well when the object was not named.

The PSTs did demonstrate initial values they held for definitions. These were a significant part of their total cognitive structure throughout the study as certain dialogue excerpts demonstrated. Prior knowledge and previously defined terms were important to the PSTs and in the dialogue this concern was discussed as they voiced this would make a difference in how a student would understand a definition. These initial values were labeled in the study as emergent criteria, labels I provided through a mathematical perspective. My labels and their initial values did not always align as was indicated when the PSTs tried to communicate their understanding of the principles. However, their abilities to make sense of the meanings of the five principles did improve when they were provided with examples of high quality definitions. Such was the case as well as the PSTs expressed difficulties with the stipulated meanings of mathematical terms or phrases such as necessary, sufficient, and 'at least'. As the PSTs discussed the high quality definitions, these mathematical words were seen in a different way. Their didactical concerns provided them a space to comfortably share their learning experiences with the polygons.

As the study ended, the PSTs were asked to create definitions using a static image of a quadrilateral and build subsequent definitions from the previous definition. The main focus was to see if they had grown in their understanding of the relationship of the quadrilaterals to one another so they could make use of necessary and sufficient and minimal. Though the results varied, the change in their final definitions showed that the five principles had become a part of their learning experience.

The next chapter interprets these findings in light of the research questions. Connecting the findings with the literature provides insight into definition construction as applied to preservice teacher understanding of mathematical definitions. The next chapter will also summarize what the findings did not answer thus providing opportunities for future research on preservice teacher understanding about the content and nature of mathematical definitions.

CHAPTER 5: DISCUSSION

Introduction

This chapter discusses the findings presented in Chapter 4 that included the mathematical content understanding of the PSTs, the emergent criteria the PSTs had for definitions, the PSTs interpretations and usage of the five principles. The discussion will demonstrate that these findings do not exist as separate entities but overlap.

The discussion begins with the findings that address mathematical content understanding, which in turn affects the understanding of the nature of mathematics in the PSTs. The research questions also serve as guideposts to demonstrate the overlap of aspects of the findings as consequential to one another. The discussions are subdivided into sections about mathematical content understanding, emergent criteria, and preservice teachers' perceptions and use of the five principles, and a summary indicating the extent of usage of the five principles. Throughout the research questions are referenced in each section. This chapter concludes with a discussion of the overlap of the findings.

The following specific findings were observed and given in Chapter 4:

(1) The PSTs produced drawings representing their personal concept images that were prototypical and specialized to the individual shapes.

(2) Their personal concept definitions lacked sentences structure where the noun and verb were not present.

(3) The final personal concept definitions demonstrated hierarchical understanding.

(4) Didactical concerns brought to light how they had learned the shapes.

(5) Emergent criteria indicated the PSTs had intuitive values for these definitions.

(6) The PSTs had misunderstandings for the meaning of certain mathematical terms and had difficulties communicating the meaning of the five principles.

(7) The PSTs had trouble aligning their emergent criteria with the five principles.

(8) The PSTs could verbalize the connections of the five principles in their dialogue.

The discussion addresses areas of overlap, through the interplay of reasoning structures, therefore, providing answers and evidence for more than one research question. By the word overlap, I mean there were consequences of one finding having impact on another. An example of this overlap was demonstrated when PST 2 in the study and a PST 1 in the pilot said that one image such as a square could be drawn to represent all four shapes. However, in spite of PST 2 stating this criterion of hierarchy, it was not evident in her personal concept definitions. In my study the idea that verbalizing the square could be drawn to represent all the quadrilaterals, the hierarchical understanding did not appear in their initial definitions of the quadrilaterals but did in the end. For each PST, the visual image overshadowed the written definition in terms of the descriptive properties.

Mathematical Content Understanding

This section discusses mathematical content understanding of the PSTs through their drawings and personal concept definitions of the shapes. The dialogue about their products also revealed concepts the PSTs held for the quadrilaterals to include the relationships of the shapes. The artifacts produced by the PSTs were similar to outcomes of previous research on concept images including my pilot. In my study, the drawings - concrete evidence of the mental images formed by the learning experiences of the PSTs as part of their total cognitive structure - were prototypical especially in the orientation of the figures drawn. A few of the drawings included the mathematical symbolism indicating congruence of either sides and/or angles. These drawings exemplified the prototype phenomenon from the work of Hershkowitz (1990) where non-critical aspects such as orientation affect the visual characteristics of the figure. In his work, de Villiers (1994) also found that this visual visualization, consequently, impedes the ability for a student to understand relationships among the geometric shapes. My study noted this as well as indicated throughout Chapter 4. Fischbein et al. (1985) found that these images, called figural concepts, also impede geometric thinking especially logical deduction about relationships. My study found this to be true when PSTs examined their personal concept definitions. Some definitions used previously defined terms in their definitions as necessary and sufficient conditions thus implying hierarchy. Yet, some definitions were written as descriptive phrases and showed no level of hierarchy of logical relationships among the shapes.

The complexity involved in the process of defining was noted in the results of the PSTs initial personal concept definitions. As illustrated in Chapter 4, their definitions were descriptions of how they drew the shapes, and their definitions showed lack of sentence structure. For the few complete sentences, these too gave lists of properties and redundant characteristics of the quadrilaterals being defined. The opportunities for hierarchical linkages were not used. Inclusive ideas were only implied. One exception was PST 7. She did use sentence structure that was highly reflective of her hierarchical knowledge of the shapes. This correlation provided her the opportunity to understand the meaning of principle one when she recognized it in a high quality definition, as well as the impact of using necessary and sufficient conditions and previously defined terms.

For the other PSTs, the strength of their visual, prototypical images along with the lack of sentence structure that grounds the object being defined impeded their abilities to connect the shapes through relationships and hierarchy thus impacting the other principles as well. As Morgan (2005) noted, that technical elements or linguistic considerations must be maintained in the writing of definitions. These elements include the existence of the term, a verb, mention of a classification for the term, and then a list of distinguishing characteristics. My study demonstrates the consequences of a lack of sentence structure. Without a term, principle one is violated. Leaving out classification affects both previously defined concepts, and necessary and sufficient conditions. Without a clear understanding of what concepts are necessary and sufficient for defining a concept, the principle of minimal is not possible. This lack of attention to sentences structure then impedes the effective use of the five principles when writing definitions for the shapes.

Most of the defining issues I observed were linked to the prototypical concept images of the PSTs and the lack of sentence structure. The PSTs who wrote descriptive phrases but never used the specific term did not connect their list of defining characteristics to that term.

The consequences of prototypical and visual image along with phrase centered descriptions of the images reveals how the PSTs understood the characteristics of the quadrilaterals. For the PSTs in my study, seeing the shapes as visual images and without any suggestion of shared properties and relationships through hierarchy, served as impediments to embracing the quadrilaterals as concepts that are related in a hierarchical sense. This prototypical visual image affected the PSTs abilities to visualize the general aspects of the shapes. From Chapter 4, PST 4 only sees the parallelogram as a slanted figure with two sets of sides with different lengths. The word parallelogram comes to the forefront of her thinking in contrast to a PST who could see the parallelogram as a concept containing general aspects shared with other quadrilaterals in the study. As Fischbein (1993) wrote, a geometrical figure is a "figural concept" encompassing not only conceptual aspects that include both generalities and abstractness, but it is also figural that includes its shape, size, and orientation. The personal concept images and definitions in my study confirmed Fischbein's statements. When this type of "figural concept" thinking controls the total cognitive structure, the five principles are difficult to use in writing definitions because hierarchical thinking is not taking place.

The research of Fujita and Jones (2006) as well as the research of Okazaki and Fujita (2007) found their preservice teachers' face difficulties classifying quadrilaterals through the use of hierarchical thinking. Specifically, Fujita and Jones found only a few of the preservice teachers in their study showed that they had a sophisticated enough knowledge of parallelograms properties to think hierarchically (Fujita & Jones, 2006, p. 29). As an exception to this finding from Fujita and Jones, my PSTs did acknowledge the importance of knowing what a parallelogram was in a conceptual or abstract way. PST 2, who took the geometry content course twice, could at first see the parallelogram as a part of a Venn diagram, but by the end of the study, she moved beyond that visual to understanding the parallelogram as a concept. Her final written definitions demonstrated her ability to think logically about the relationship of the shapes. However, the PSTs, who did not have a robust understanding of the relationships of the shapes in the study, had difficulty understanding or using the five principles as a means for writing high quality definitions.   As discussed in Chapter 4, PST 2 was able to develop an understanding of the relationship of the shapes once she comprehended the concept of parallelogram. Her final definitions demonstrated the use of hierarchy.

As seen in an example mentioned at the beginning of this chapter, simply claiming that one shape, such as the square, could be drawn to represent all the quadrilaterals did not produce definitions demonstrating this hierarchy. For the PSTs who made this claim, their need to write very explicit definitions for each shape without hierarchy could bear more examining.

Definitions play a central role in classification of mathematical objects and while pure mathematicians understand this as a "so-called classical model in which a category is specified by a fixed definition … category membership is completely determined by a set of necessary and sufficient condition" (Alcock & Simpson, 2011, p. 92) students do not understand this way of classifying.   Students focus on differences over similarities when engaging in the normal human cognitive behavior of categorization. This was the case in my study as the PSTs focused on discussing the characteristics of the shapes and not how the shapes were related because of the characteristics the shapes shared.

The students' claim that they could draw a square to represent all four shapes appeared to be visually based and not based on definitions. This could be due to the fact that the shapes are viewed through the concrete images each PST drew or it could reflect a lack of understanding concerning what a definition in mathematics should be. But as studies such as Edwards and Ward (2008) noted even in the presence of correct mathematical definitions, students when engaged in activities involving reasoning reverted back to their own concept images and definitions for the mathematical object under consideration. For my study and the pilot, this claim made by the PSTs might be a result of seeing some type of hierarchy chart for the shapes, but the definitions were written separately because of as remembering learning them that way in the classroom.

In my study, an interesting related finding emerged as the PSTs reflected on how they had learned about these quadrilaterals and discussed the differences and similarities found in their drawings and personal concept definitions. They begin to question each other as to how they learned the shapes. Two pathways of learning, as separate shapes or connected by hierarchy charts or Venn diagrams, were mentioned and each different way provided more evidence about the perspectives one acquires as a result of prior learning experiences. Edwards and Ward (2010) alluded to this in the work with mathematics majors. As they suggest, it is beneficial to spend time working with students by making them aware of Vinner's model for concept image since it makes students aware of their own thinking and why they think this way. In my case, student discussion of their drawings and personal concept definitions brought up a natural conversation about their own learning experiences. Those who were taught through a hierarchical chart or through the use of Venn diagrams, produced definitions that reflected more use of the five principles at the end of the study compared to the beginning. While those who learned the shapes are separate entities, used phrases describing the shapes properties. In other words, these discussions became another window for understanding the impact of past experiences as part of the total cognitive structure of the present. Edwards and Ward (2010) referred to similar occurrences from student conversations on definitions in their work. The discussion of student learning experiences in my study had an impact on both their current understanding of quadrilaterals and later on their understanding and use of the five principles.

To summarize, my study has shown that how a PST learns about the quadrilaterals affects the ways that PST perceives both the image of the shape and the personal concept definition the PST writes. In Chapter 4, for example, the PST who learned each shape separately reasoned through her prototypical visual images and wrote descriptive phrases for her definitions. However, for PST 7 who had learned the shapes through a hierarchy chart, through her drawings were marked prototypes, her definitions were at a General level of quality in the DMA framework.

Emergent Criteria

As the PSTs were discussing the similarities and differences in their drawings and definitions, another phenomenon was presented – intuitive values. As a result of this dialogue, intuitive values about definitions were revealed thus providing evidence that preservice teachers did have ideas about the nature of mathematical definitions. Other researchers have examined other emergent criteria. For example, Zaslavsky and Shir (2005) in a research study designed to determine students' conceptions of mathematical definitions found that students were concerned with the clarity of a definition in terms of its communication possibly. Not only, did they want to feel that a definition was complete to them, they wanted to honor the personal preferences of that definition writer. A disconnect is observed between the thinking of mathematician and that of a student. The mathematician values definitions based on objective characteristics such as the five principles, while the student values definitions that reflect their prior experience and personal concept images. The PSTs in my study did not indicate using the five principles in their prior learning of geometric definitions.

Students in their study also noted that definitions should possess clarity in the conditions given and have no superfluous information but did not agree on the minimal condition. Interestingly, their students did not accept definitions that were procedural. They felt definitions should be based on known concepts as long as the concepts were not too basic (Zaslavsky & Shir, 2005). One PST, in my study, who had not taken the geometry content course, complained about too many basic previously known concepts required the user of the definition to "think too hard" to remember all of the properties implied by the concept named. In my study, the PSTs were mixed in accepting the idea that some previously known concepts could make writing a definition more minimal. PST 7 used the previously defined terms to make her definitions more concise but PST 2 and PST 6 worried about needing to reinforce repetition in definitions. Zaslavsky and Shirs' students had a different sense of the both the meaning of previously defined concepts and the use of necessary and sufficient. Their students were concerned about creating correct definitions. The concerns from both studies demonstrate a variety of interpretations of previously defined terms that suggest the interpretation of principle 2 would be seen differently by the PSTs as indicated in the groups discussion in Chapter 4.

The emergent criteria, provided in Chapter 4 in Table 6, also occurred naturally while the PSTs discussed their images and definitions. The emergent criteria was noted as: previously known concept, prior knowledge, hierarchical, minimal, didactics, arbitrariness, and necessary and sufficient. Three of the emergent criteria most prominent were: hierarchical considerations, previously know concept, and necessary and sufficient.

A clarification is important here; the descriptions given to the criteria derived from the excerpts in the PSTs dialogues were my labels. Only the label previously known terms or priori knowledge came directly from the PSTs words. The other labels used for the emergent criteria were my labels coming from a mathematical perspective of the five principles and my interpretation of how the PSTs' excerpts fit with one label or another. In Chapter 4, I discussed how I used excerpts from the dialogue as examples of the different emergent criteria I had labeled. For example, this statement was considered the criterion of previously defined concept, "I wrote, a quadrilateral is a four-sided polygon but that is assuming the person knows what a polygon is".

One interesting finding was that the emergent criteria mentioned by the PSTs were not always present in the personal concept definitions produced. The three predominant criteria closely with the principles, but such connections were not made by the PSTs moving forward in the tasks of my study. One possible conjecture as to why this occurred is the difficulties connected to the abstract mathematical language used in the five principles and in relating that language to a concrete definition. In other words, the PSTs did not recognize when their intuitive values represented some of these principles. Experiences, including how one learned the definitions of the shapes, also appeared to affect the use or recognition of the principles in a definition. There were some exceptions to this as noted in Chapter 4. The emergent criterion between previously known concept and hierarchy is seen as contradictory for PST 1 who could use complete sentences, knew that starting with polygon for her quadrilateral definition made her definition minimal, but did not see the same potential in her other definitions. That is, because she started each definitions by using the term quadrilateral thus forcing a description inherently lacking in the use hierarchy. As in the work of Zaslavsky and Shir (2005), students' may express their conceptions of a mathematical definition in the moment, but this same conception can later change (Zaslavsky & Shir, 2005, p. 338). Therefore, it does not seem appropriate to make claims that possessing intuitive or personal criteria for definitions manifests itself in the production of definitions that are of high quality in my study. The PSTs need opportunities to understand how their personal emergent criteria and the five principles are connected.

Preservice Teachers' Perceptions and Use of the Five Principles

The PSTs' perceptions of the five principles are discussed as an evolutionary process as the PSTs' initial efforts to make meaning of the principles changed throughout the sessions of the study. These changes were due partially to the tasks that asked the PSTs to use the principles. The PSTs initial perceptions of the principles are discussed in related ways because the PSTs process of translating the five principles from words into personal knowledge was sometimes negatively impacted by the stipulated terminology of mathematics, such as 'at least one' as well as their visual images of the shapes and lack of property and relationship understanding in the quadrilaterals.

Their translations for both the pilot and the study lacked clarity to the point where the fifth principle was simply a restatement of "definitions are arbitrary" to a statement that "concepts may be interpreted differently and still are correct". Their efforts to understand the five principles, in essence mathematical definitions, meant the PSTs would have to form personal concept definitions for each of the principles. The five principles in the precise mathematical form as presented in my study were not a part of their total cognitive structure. In both the pilot and the study, the PSTs' interpretations did not come easily in their discussions of the initial meanings. The dialogue suggested the PSTs were meeting the principles for the first time. Forming personal concept images was made more difficult because of the stipulated mathematical language in each principle. The language of mathematicians is not language that the PSTs have had previous experience using. Their interpretations and dialogue excerpts are given in Chapter 4 attest to this situation. In the pilot, “Definitions are arbitrary” produced from the PSTs, “sounds good as it is.” In the study, the response was, “Though a concept may be interpreted differently, they may all still be correct”. In each case, the researcher did discuss that arbitrary meant several different definitions might exist for a concept.

Already mentioned was the lack of sentence structure where the shape being named was not mentioned. Consequently, the first principle, “Defining is giving a name; the statement used as a definition presents the name of the concept and this term (name) appears only once in the statement” was not equated with the need for a noun as the name of the concept. Here was the first place where concept, name, and term present themselves in a mathematical sense and needed recognition of their interrelationship. For example, the term *parallelogram* is the name and the term or word but it is also a mathematical concept embodying abstract and concrete figural aspects. It was only PST 7 who recognized this principle later when discussing the high quality definition, “A rectangle is a parallelogram with at least one right angle”. For her the noun was rectangle and, therefore, the concept being named. However, when literal interpretation of the phrase “the name appears only once” was identified as a count, I realized the difficulties of the stipulated language. This fact is illustrated when PST 6 literally interpreted once as a count since she said in essence that it could be defining a parallelogram as that word appeared only once in the definition as well as the rectangle. As the dialogue progressed, this issue was resolved and PST 7 convinced the others that the parallelogram was indeed a previously defined concept that supported the defining of the rectangle.

The PSTs' initial perceptions were stymied by the mathematical meanings of words such as necessary, sufficient, and at least one right angle. These difficulties are noted in excerpts found in Chapter 4. These difficulties demonstrate the expectation that understanding is immediate and is not accumulated from experiences in their mathematical learning. Even principle two, "In defining a new concept, only previously defined concepts may be used," a criterion the PSTs valued came across as a simplification of the previous concepts. Linchevsky et al (1992) found in their study that the preservice teachers confused properties and definitions thinking that properties implied definitions and vice versa. De Villiers (1998) attributes this to giving students definitions as finished products with no participation in the development of a definition. I feel my study bears this out and consequently, my PSTs could only understand these mathematical terms through their everyday experiences not in the stipulated ways expected by mathematicians.

Their perceptions did change while they were identifying the principles in a high quality definition. Not only did the PSTs begin to identify the principles, they also claimed that the principles were sometimes hard to separate. The dynamic nature of mathematical definitions was discussed as they worked to identify each as a separate principle. This result also occurred in the pilot study. When the PSTs moved beyond understanding the parallelogram as a word for a shape and realized the potential the concept of the parallelogram offered a definition, connections were made between the rectangle and the parallelogram in particular. I interpreted this connection as emergent criteria for hierarchy, previously defined terms and necessary and sufficient in conjunction with the five principles. Using the parallelogram as a previously defined concept with unique side and angle properties for defining a rectangle helped with the transition for understanding how one right angle forced the parallelogram to be a rectangle. When the PSTs were challenged further to consider that equivalence of defining a rectangle as a quadrilateral with three right angles, they relied on the fact that the sum of the angles had to be 360 ° so the fourth angle was also 90 °. This conclusion was made through large group consensus only after PST 5 stated the implications of three angles equally 270 ° forced the fourth to be 90 ° since the sum in any quadrilateral was 360 ° adding that forced the opposite sides to be parallel as well.

However, my study showed this connection should not be taken as proof that they understood and could use the five principles. A concluding task during day 2 asked for the writing of another set of definitions for each shape. This was done individually to determine any effects of discussing the principles on the second set. PST 1's results are given from a portion of Table 18. In this table, the color shows where each PST's first and second written definitions fell on the DMA scale. The colors represent the range of the DMA spread after the first introduction with the five principles.

Table 18. Movement in DMA Levels for PST 1

	Situational		Referential		General
PST1					
quadrilateral		**1st**			**2nd**
parallelogram	**1st** **2nd**				
rectangle	**1st**			2nd	
square	**1st**		**2nd**		

PST 1 did show the greatest movement in her definition for quadrilateral ending at the General level. Her parallelogram definition did not change. Her definition for rectangle indicates shifts (green represents a horizontal shift between the Situational and Referential) and (purple represents a shift between the Referential and General levels) between levels ending at the Referential level. Her second square definition exhibited more Referential characteristics because of starting her definition based as a rectangle than her beginning definition but it was not minimal since she repeated all angle as congruent.

The results overall demonstrated little change in how they wrote the second set of definitions. The results can be seen in Appendix C. Three conclusions from are stated here: (1) Definition construction processes take time and is evolutionary, (2) When asking the PSTs to immediately use the five principles, they revert back to their personal concept images and, (3) The dialogue provided a place for the PSTs to discuss and make comparison and decisions that were directed by the principles while allowing them to think about the quadrilaterals in the study and the nature of mathematical definitions, but this did not immediately transfer to their individual results.

The missing elements in the second set of definitions listed in Appendix B still showed lack of sentence structure, lack of using previously defined terms in support of hierarchy thus impeding evidence of necessary and sufficient conditions and the inability to produce arbitrary definitions. A positive take away comes from the opportunities the rich discussion offered the PSTs in sharing how they understood the relationships of the quadrilaterals with the five principles. Future research needs to examine these three findings in order to determine other ways to incorporate the five principles into the total cognitive structure of students as a part of their experiential knowledge. In brief, the understanding and use of definition cannot be separated from the perspective of the user or the prior experiences of that user.

The Extent of Usage of the Five Principles by the Preservice Teachers

Though the first attempts to write high quality definitions immediately after a discussion of their abstract meaning of the five principles did not produce high quality definitions for all the PSTs, change in the quality of their definitions did eventually occur. As found in Chapter 4, some of the PSTs were able to embrace the five principles in a way that they wrote high quality definitions by the end of the study. From Figure 12, PST 2 wrote for her final square definitions, "A square is a rectangle with congruent sides, or a square is a regular polygon". However, not all of the PSTs embraced or used all of the five principles in the same way. The PSTs who could move beyond reasoning through the visual images used the five principles in the end as was true for PST 2. The discussions of the five principles over the sessions also produced opportunities for the PSTs to discuss their perceptions of quadrilaterals, yet not all understood the hierarchy in the same way. As Edwards and Ward (2008) noted, it is not easy to say students belong to those who completely understand and those who don't. But for some, the nature of definitions is a "teachable" concept (Edwards & Ward, 2008, p. 228). My study shows students at different levels of understanding of quadrilaterals could use the five principles in such a way that different categories of understanding were observable. My results also suggest that the five principles can be use as a teaching concept in conjunction with the hierarchical connections embedded in the quadrilaterals. PSTs can build and enhance their personal concept images to reflect a usage of the five principles after focused guided experiences.

Discussion of Overlap

Future research studies aimed at investigating students engaging in definitional reasoning must first create a conflict between what is known and what is an unknown for the student. For this study, the known was the personal concept image and definition while the unknown was the high quality definitions or new mathematical reality of Zandieh and Rasmussen (2010). Throughout the study, this conflict manifested itself as interplay of multifaceted reasoning structures. The reasoning structures were:

(1) Using the five principles as a mathematical perspective through which to reflect, evaluate and write and definitions.

(2) The dialogue about the writing of personal concept definitions for quadrilaterals that did or did not align with the principles or hierarchy.

(3) The engaging of the personal concept image to show understanding of properties, relationships and the hierarchy of the quadrilaterals.

Each PST, owned as part of their total cognitive structure, personal concept images and definitions for each shape so each experienced the interplay in individual ways. This interplay was influenced by their prior learning experiences with the quadrilaterals as well as their personal emergent criteria held for definitions. The study challenged what the PSTs knew about quadrilaterals and definitions by asking them to use the five principles when writing a new definition. At times their total cognitive structure was a hindrance and at times it promoted growth. The dialogues demonstrated the conflicts and the interplay while the final definitions produced demonstrated the use of the principles and hierarchical connections.

In other words, just the acceptance or the adaptation of the five principles did not give the complete picture of the conflict. Overall, this conflict occurred as each PSTs understanding of the quadrilaterals properties, relationships, and the hierarchy were reconciled with the five principles. For each PST, their personal concept image was challenged. The conflicts were individualistic in nature and were not linear paths of change or growth in DMA levels. Figure 3 demonstrates the overlapping reasoning structures.

The PSTs started with their visual image as the impetus to often describe the image thus creating a personal concept definition, then grappled with the five principles and their understanding of the hierarchy to change their description into a mathematical definition. Their final definitions did or did not exemplify understanding the concept, hierarchy of the quadrilaterals or the five principles as this seemed to be dependent on their individual total cognitive structure. Therefore, the evolution of change was not demonstrated as growth from *definition-of* to *definition – for* in all seven situations.

At times, describing the visual image was the beginning and end of any change such as the case of PST 4. For PST 5, efforts to describe the visual image dominated her attempts to accept both the hierarchy as coordinating with the five principles. Yet in the end, she embraced the five principles as a means of forcing deep thinking and accepted the five principles as the best way to write definitions. PST 2's conflict was her incomplete understanding of the parallelogram in her hierarchal structure for the quadrilaterals. She was able to overcome her dependence on her circles and Venn diagrams to know the parallelogram as a concept whose properties were also shared with the rectangle.

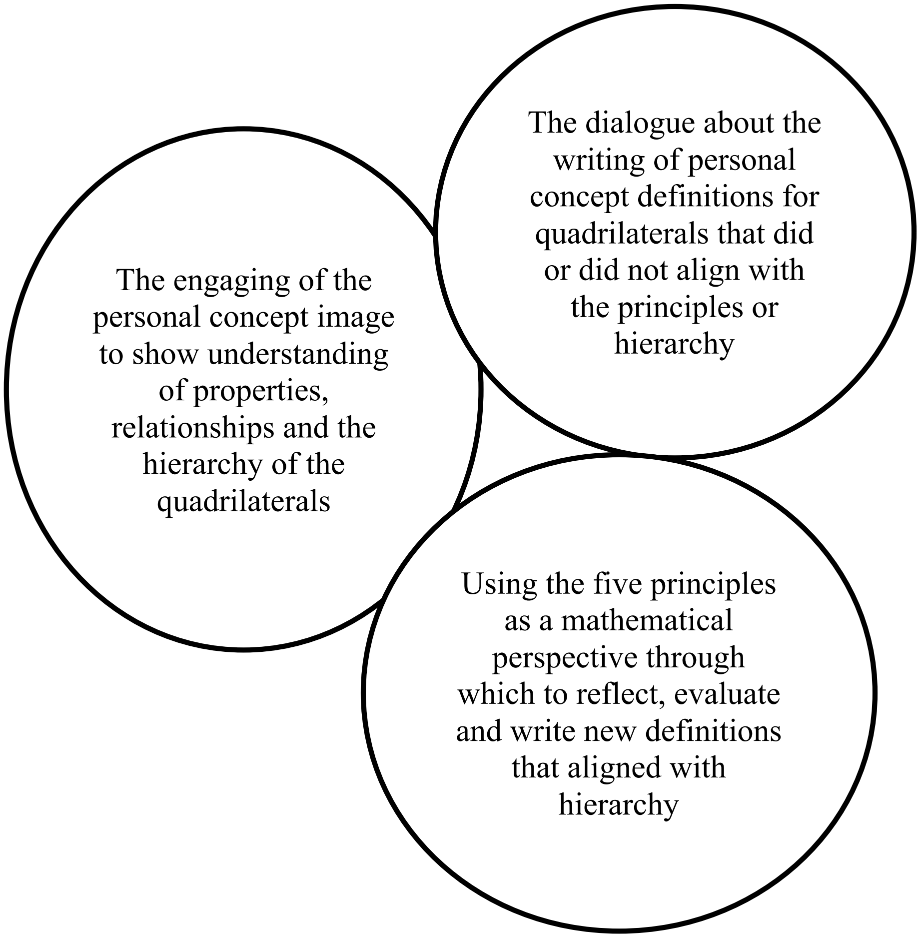


Figure 3. Interplay and Overlap of Reasoning Structures

PST 7 could write definitions that demonstrated her ability to link the five principles with the hierarchical structure of the quadrilaterals, thus her evolution was smaller than the others. PST 6's concern for previously defined terms became meaningful only after she resolved how the quadrilaterals were related through properties they did or did not share. Once this was established, she could agree that this process is a productive way to learn not only the definitions of the quadrilaterals but the support the five principles offered. PST 1's change was smaller as well but she had to allow the learning experience of the five principles to take over her initial classroom learning experience that required each definition to start with the words “a quadrilateral is”. Though she often despaired about the tasks, PST 3 could accept in her dialogue shared with the others ideas that demonstrated her understanding of the relationships of the quadrilaterals. Here conflict seemed to be the acceptance of the benefits of thinking deeply about the geometry. A visual of DMA shifts for each PST is found in Appendix D. The interplay continued throughout the entire study, challenging each PST in ways unique to their total cognitive structure. The dialogue allowed for that interplay to be recorded and analyzed through the DMA framework providing a means to record some growth or change or new mathematical reality for the definitions of the quadrilaterals.

Overall, there was change for the PSTs in the use of the five principles though not all reached the General level in the DMA. Figure 4 best demonstrates the change as some PSTs did incorporate to some degree the five principles in the experiences.

Summary

To summarize this chapter, the major discussion points were the following:

(1) The PSTs exhibited an inadequate understanding of the properties and relationships of the quadrilaterals that affected their understanding of the hierarchical nature of the shapes.

(2) The PSTs did not understand the nature of mathematical definitions through the principles yet held intuitive personal criteria associated with the five principles for the definitions.

(3) Change occurred in their hierarchical content knowledge impacting an understanding of previously defined concepts along with necessary and sufficient conditions.

(4) The existence of the interplay and overlap of reasoning structures, and

(5) This type of definition construction activity takes time.

The PSTs' lack of understanding about the properties and relationships of the quadrilaterals was expressed through their personal concept images and definitions.

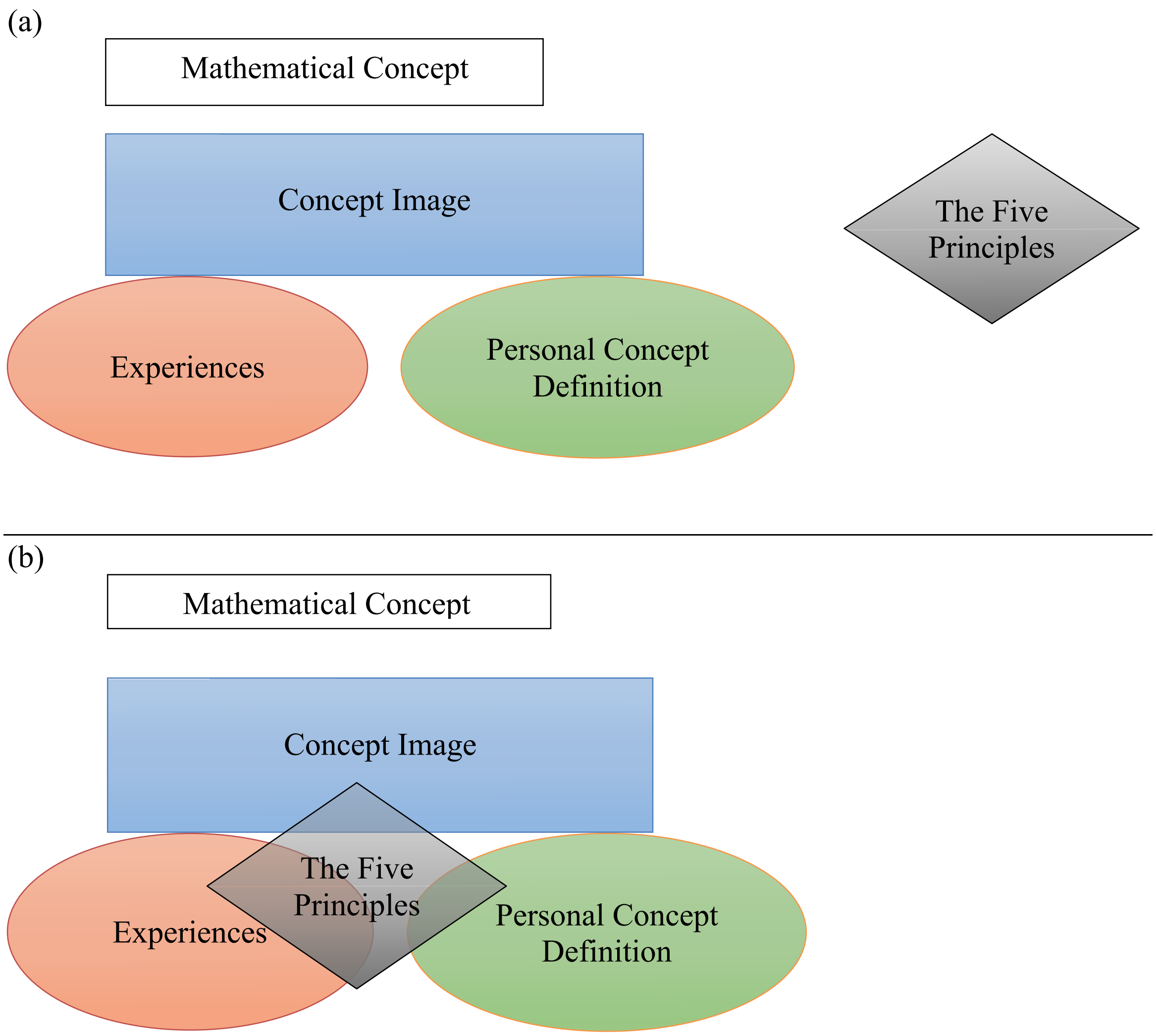


Figure 4. The Five Principles and the Concept Image at the (a) Beginning and (b) End of Tasks

Their concept images were prototypical, and their definitions lacked hierarchical referencing and sentence structure. The full meaning of the concepts embedded in the quadrilaterals was affected due to the stipulated mathematical meanings of certain mathematical terms.

As noted in many previous research studies on students' understandings of the nature of mathematical definitions, my PSTs had difficulties with the abstract principles in spite of the fact they did hold personal and similar criteria for definitions. The PSTs could see the principles in use in an example of a high quality definition, but using just the abstract principles alone impeded progress in understanding the nature of mathematical definitions. The PSTs did discover that the principles were connected to one another and did align with the hierarchy embedded in the quadrilaterals.

For my study, this research design served as a teachable concept where varying degrees of change occurred. However, as noted in the work of Fujita and Jones (2006), an understanding of the concept of parallelogram is the key to promoting logical reasoning. In Chapter 3, the redesigned task 3 for Day 2's work encouraged this reasoning. This was the pivotal point for the PST 2 and PST 6. Therefore, the concept of parallelogram provided the necessary link to the rectangle in my study.

Definition construction can work as a way to engage students in the work of mathematicians. The process takes time and can be very individualistic. Using the five principles as a way to engage PSTs in the understanding of the nature of mathematical definitions gives them an opportunity to expand their total cognitive structure to prepare them to be more effective teachers of mathematical definitions.

CHAPTER 6: IMPLICATIONS FOR FURTHER RESEARCH, LIMITATIONS, AND SUMMARY

Introduction

This study investigated seven preservice teachers abilities to create, negotiate, and write high quality definitions for quadrilaterals. The definitions produced evidence of the five principles of mathematical definitions as a part of the PSTs'concept image or total cognitive structure. Though previous research has documented preservice teachers' understandings of aspects of geometric definitions, the use of the five logical principles as a means of attaining an understanding of the content and nature of definitions for quadrilaterals not been fully investigated.

Seven preservice teachers intending to earn teaching certification for grade K – 8 were engaged in definition construction tasks. The tasks were designed to follow and broadly examine the presence of understanding about the nature of mathematical definitions through the use of the five logical principles. Written work and dialogue was analyzed in multiple ways. Concept images as captured in drawings of the four polygons in the study were coded for similar characterizes held by the PSTs. Dialogue was coded in order to determine what elements the PSTs valued in their written personal concept definitions for the quadrilaterals. This dialogue revealed intuitive values that the PSTs held for definitions. These intuitive values, labeled in the findings emergent criteria, often paralleled the five principles and were the following: previously known concepts and prior knowledge, hierarchy, didactics, necessary and sufficient, minimal, and arbitrariness.

Using the dialogue, tasks were designed to build upon one another and extend the PSTs initial understandings and align these to the five logical principles. The PSTs were asked to write definitions throughout the third session that embodied the five principles. Dialogue was also coded using the DMA framework as a means of noting change in levels of usage of the principles, which would signify that the five principles were becoming a part of their concept image for definitions with the outcome task to write high quality definitions.

This chapter is organized with the following sections: implications for further research on definition construction, implications for teacher education, the relationship for the study's results to theory, and limitations of the study. This chapter ends with statements that focus on the outcomes of the final task.

Further Research on Definition Construction

In keeping with the concerns of Ouvrier-Buffet (2003), more research needs to be done on definition construction. As this study found, concept image is a complex cognitive structure that has strong impact on the decisions made by preservice teachers when writing definitions. At the same time, future teachers need to be flexible in the understanding of mathematical definitions when making curricular choices. A personal concept image that embraces the five principles could aid in teachers' choices of what definition construction practices to use in classroom practice. An understanding of the equivalence of mathematical definitions makes their decision process stronger and more robust.

Future research should investigate the integration of the criteria students bring to the process in order to build definitional reasoning around that focus. However, this research must acknowledge that the five principles won't automatically align with the PSTs' criteria as this study determined. In this study even though the PSTs acknowledged the importance of previously defined terms or prior knowledge, not all of them recognized the effects the previously known concept parallelogram had on the definition of rectangle. The connections embedded in previously defined terms or prior knowledge did not seem to affect their understanding of such implications on hierarchy. First and foremost, students must see a need to value high quality definitions in order to understand the nature of such definitions. The DMA framework can continue to be used as a way to document change and growth in both definitional reasoning strategies and the construction of high quality definitions.

Definition construction takes time and the tasks must not only challenge the personal concept image and definition but also must be designed to support the desired change. There is a delicate balance of challenge, guidance, and purpose for doing definition construction that must be intertwined in designed activities that take each student from his or her personal concept image to a concept image that embraces the five principles. As documented in Chapter 4, not all the PSTs embraced the five principles and moved in DMA levels especially for PST 4. The end result is still very dependent on the individual student and on where each one of them starts the process. For example, PST 4 showed evidence in both her drawings and dialogue that her static personal concept image controlled her reasoning.

Yet, the process of definition construction through the use of the five principles becomes stymied if the technical structure of a sentence is ignored. A definition cannot work if the first principle is ignored. In keeping with the recommendations of Morgan (2005), definition construction must consider the correct technical aspects of simply constructing a sentence. In my study, in spite of the declaration from the PSTs that hierarchy was important in writing definitions, their technical structure did not take advantage of the power of the hierarchy of the quadrilaterals to come through. As documented in Chapter 4, PST 1 was able to write definitions in starting positions other than the quadrilateral by the end of the study, but PST 4 did not as she wrote phrases that did not name the concept being defined. This lack of sentences structure impeded her use of principle one. If the purpose of mathematical definitions is to single out a concept, then that concept must be clearly stated and named.

Just as the need for sentence structure must be remembered when designing definition construction activities, the strength of the personal concept image must also be taken into consideration. Edwards and Ward (2004) also note this when reporting how students revert back to their concept image when the mathematical definition conflicts with the concept image, thus ignoring and avoiding the use of the mathematical definition in arguments and proof.

My study demonstrated that these seven teachers did embrace some of the five principles of mathematical definitions. As noted in Chapter 4, PST 7 found this less challenging than PST 5, who did not at first like such minimal definitions. In the end, both of these PSTs used the five principles in writing the final definitions. PST 6 and PST 5 both shared closing thoughts. PST 6 wished she had learned the definitions in this way in class as she felt it would have made reasoning about true and false questions easier. PST 5 was not a fan of such minimal definitions but in the end respected their precise and accurate brevity in sharing with the group. She also indicated how these definitions make you think hard but that way you see connections. PST 2 had repeated the geometry course twice and realized that her incomplete understanding of the parallelogram was holding her thought process back. She was able to not rely on her Venn diagram near the end and knew the parallelogram as the link to the rectangle and square in her dialogue and definitions.

PST 3 and PST 4 had problems accepting hierarchy. PST 4 had learned the shapes as separate entities and PST 3 disliked the thought process that was required in definitions that demonstrated the use of hierarchy. Neither of these PSTs had taken the required geometry content course in the program while the other PSTs had taken the course in a prior semester or were currently in the course. The hierarchy of the shapes is stressed in the course and many opportunities will be provided these two PSTs as they continue the mathematical course requirements.

Ongoing research could investigate further the reasons that some of the principles were easier to embrace than others. This study showed that even though the PSTs' emergent criteria were voiced, the immediate connection was not recognized but did occur as the study progressed. This could have been the result of lack of understanding of the meaning of the principles or lack of understanding of the quadrilateral relationships or a combination of both.

Future research on definition construction might consider studying younger students, perhaps as young as those in elementary grades, to determine if they have intuitive values for definitions. If preservice teachers hold emergent criteria for definitions, then what might younger students think about definitions? This question was addressed by Kobiela et al. (2018) whose work engaged second graders in a geometric sorting activity. They propose this type of sorting activity could be used in all grade levels to support definitional reasoning. The heart of the lesson focuses on key elements of definitional reasoning: identifying then referring to the mathematical properties of the object being define thus building the precise nature of mathematical language; classifying or constructing examples and nonexamples of the object with the goal of explaining why the example is or is not part of the group and classification build the developing definition; and making sense of the necessary and sufficient properties of the object being defined (Kobiela, Jackson, Savard, & Shahan, 2018, pp. 252-253).

They go on to recommend that the sorting activity should connect and build on students' current thinking about the object. In this example, they built on students' ideas about the properties of a triangle. As the students move toward the definition, the teacher must maintain the focus on arriving at a consensus for the developing definitions thus, teacher planning must be intentional (Kobiela et al., 2018, p. 257). Similar definition construction activities could be beneficial in the other grades as well. While focusing on defining content, the students' concept images are also developed.

Future research could also be conducted with in-service teachers using the mechanism of the five principles. Earlier research of Leikin and Winiki-Landman (2000) conducted workshops with in-service teachers. Findings noted the equivalence and non-equivalence of mathematical statements in terms of the arbitrariness of definitions. In their study, the teachers voiced considerations such as didactics, prior student knowledge, applications to problem solving, and enabling mathematical generalizations (Leikin & Winicki-Landman, 2000, p. 28). Also noted were two themes that emerged: defining as giving a name, and defining through properties or sets of objects (Leikin & Winicki-Landman, 2000, p. 28). These are connected. When giving a name to a concept, the concept implies properties and relationships to sets of objects. In my study, PST 4 could only see the parallelogram as a static slanted rectangle and could not connect the parallelogram's properties to the rectangle's properties. These results found by Leikin and Winiki-Landman (2000) could be expanded to focus on delineating more of the five principles that the in-service teachers unveiled. More work investigating how their experiences with alternate definition choices and this connection to the five principles might produce interesting information.

Implications for Teacher Education

In keeping with the research recommendations of Edwards and Ward (2004, 2008), all mathematics students should study the nature of mathematical definitions at some point. One example in my study demonstrates this need as Task 3 caused the PSTs to reflect on their thinking about the arbitrary nature of definitions through the comparison of two high quality definitions. To do this, Edwards and Ward (2004, 2008) recommend that students study Vinner's model of concept image to gain insight into how personal concept images and definitions do or do not agree with the stipulated mathematical definition required for advanced mathematical thinking. This finding suggests the need to specifically address the relationships among the concept image, mathematical definition, and the personal concept definition in classroom instruction.

However, Edwards and Ward (2004, 2008) also state that students, and for my study PSTs, must be made explicitly aware of the concept image as an integral part of the learning process. By making students explicitly aware of their concept images a more significant goal can be achieved that is its prominence in the learning process. Students reflect on what they know, how they learn, and how the participating in such activities as definitional reasoning challenges them to change, expand, and enhance their concept image of quadrilaterals. Using the concept image invites active student participation into the learning experience, a process that is continuous.

The results of my study demonstrated changes in the concept images of PST 1, 2, 5, 6, and 7 specifically as shifts in their DMA levels were documented in their definitions. Of significance was the dialogue they shared about how they learned, a personal sharing of their concept images through didactical concerns. As future teachers these PSTs have noticed the impact on their concept images through how they learned the definitions themselves. This emphasis on making students aware of the significance using the concept image in the learning process is important for teacher education and may adjust for situations where the concept image is not fully developed. Future teachers must allow their own students to come to an awareness of their own concept images as well.

My study demonstrated that PSTs do hold criteria for the definitions of these quadrilaterals. The criteria discussed in Chapter 4 and Chapter 5 as emergent criteria often demonstrated the alignment with the principles. If preservice teachers are made aware of their emergent criteria and this alignment, their views about the nature of definitions may change. As this study demonstrated, an appropriate place for evaluation of their criteria and the five principles could be in a geometry content course since the hierarchical nature of the polygons and the five principles do align in a more concrete way than other content areas.

The hierarchical nature of the polygons naturally lends itself to the principle of necessary and sufficient as well as minimal. In this study, some PSTs did not fully embrace the word *condition* replacing it with *term* when interpreting the first principle.

With a focus on hierarchical relationships, seeing a parallelogram as a concept instead of just a term in a definition could not only enhance both their geometric understanding of relationships but also demonstrate how minimal definitions could be written depending on where the definition begins. In this case, the PSTs could see that aspects of the nature of mathematical definitions are already a significant part of their understanding.

My study did demonstrate that for some PSTs the understanding of the five principles did emerge over the three sessions of engaging in tasks. These tasks simulated activities similar to those of mathematicians as they go about the work of negotiating meaning for a concept being defined. Therefore, creating these opportunities similar to these in teacher education course work in a specific content area would provide situations for preservice teacher to practice the work of mathematicians and experience the nature of mathematics. There are many concepts in mathematics that could be considered for these activities, such as for example slope. Prior research has indicated that slope is most often remembered by students as the algebraic formula $(y_1 - y_2)/(x_1 - x_2)$ or the geometric ratio of rise over run (Stump, 1999, 2001). These representations are two different definitions for the concept of constant rate of change for a line. The question here becomes how do students and/or teachers navigate the arbitrariness of these representation? Where else might these different representations be found in mathematics? Research has demonstrated that both teachers and students understanding and concept images are fragmented (Nagle & Moore-Russo, 2013; Stump, 1999). My study demonstrated how dialogue about one's concept image begins the process of understanding. Resolving the mathematical representations of slope with one's concept images includes resolving the distinction between ratio and rate, real world applications, physical properties such as steepness, covariation in the words rate of change, tangent, and derivative to name a few. Situations of conflicts such as these provide opportunities to build both an understanding of the use of mathematical terms in context and how one representation is connected to another. Nagle and Moore-Russo put this work's focus on investigating the three-way relationship between personal concept images, instructional materials, and enacted lessons (Nagle & Moore-Russo, 2013, p. 15). Here again, as in my study, the linking of concept image to mathematical definitions is central.

Relationship of Findings to Theory

This study used the DMA framework as to analyze the findings particularly the dialogue that revealed the thoughts of the PSTs while engaging in the tasks. My framework was adapted from the research of Zandieh and Rasmussen (2010); whereby, they created a framework to document the progression of student reasoning from the informal to formal. Their framework is a result of the integration of the work of Vinner (1989) concerning the difference between concept image and concept definitions and the work of Freudenthal (1973) concerning instructional design known as Realistic Mathematics Instruction. The goal of the DMA framework was to simulate the emergent model heuristic of Freudenthal's work that embraced the concept of *models-of* into becoming more sophisticated *models-for* mathematical activity. Zandieh and Rasmussen then conjectured that students' initial *definitions-of* could serve as stepping-stones for reaching *definitions-for* further mathematical reasoning. As I used their framework, detailed in Chapter 2, I specifically used the five principles of mathematical definitions to guide this pathway to more formal reasoning. In my study, I started with the personal concept images and definitions of the PSTs at a Situational level and asked them to discuss the drawings and definitions to look for commonalities and differences. The dialogue demonstrated how they focused on these artifacts yet yielded other information about what they valued in definitions, such as their emergent criteria. The dialogue also indicated difficulties understanding mathematical language in context. This bridge needed to be discussed, and it was, if their reasoning was to move them towards the Referential level. The movement from *definition–of* to *definition–for* was assisted by the use of the five principles.

In using the five principles as a guiding structure, the PSTs were focused both on their own personal concept images and definitions but under the scrutiny of the expectations of formal mathematicians for mathematical definitions. These definitions were referred to as high quality in this study. According to both Zandieh and Rasmussen (2010), defining is more than just creating a definition. It includes the aspects of formulating, negotiating and revising a definition (Zandieh & Rasmussen, 2010, p. 59). Throughout the study, the PSTs were asked to negotiate the appropriateness of their definitions or others based on the five principles thus allowing them to revise definitions to make them more high quality.

My study attempted to maintain this understanding about the defining process as it aligns with the nature of mathematical definitions. My PSTs began by writing their personal concept definitions for four quadrilaterals specifically selected for their hierarchical nature, and their relationships to the dynamic connections embodied in the five principles. Their personal concept images and definitions initiated the Situational level for my study. Through the dialogues, it became evident that the PSTs held emergent criteria for definitions that could become nurtured and extended, while the

PSTs revised both their personal concept definitions and those in the study. For some, the criteria they held was meshed with the five principles, and by the end of the study, some could write *definitions-for* a particular quadrilateral as noted in Chapter 4 for PST 1.

My study seems to shed some light on what Linchevsky et al. (1992) wrote in their study about the implications one might draw from someone at a high Van Hiele level. They asked if it would be correct to assume that someone at a higher van Hiele level would be able to write and embrace mathematical definitions? An answer to this, though small, lies in my study results. According to my study, this is not the case. At the beginning of my study PST 2 suggested she could draw one shape for all four, yet wrote definitions that were phrases and full of description. By the end of the study, she demonstrated hierarchical thinking in her dialogue excerpts and final written definitions. From my study, I could say that the personal concept image needs more research before one can say definitively that a high Van Hiele level equates to the ability to write high quality definitions.

Other research studies have used the DMA framework to understand how students understand formal definitions. Swinyard's work with the student's reinvention of the formal definition of limit demonstrated how students can reinvent a coherent definition through their reasoning (Swinyard, 2011). In another study, Whitney, Kartel, and Zawojewsik investigated how students use their concept images and definitions to create new definitions for axioms in spherical geometry (Whitney, Kartal, & Zawojewski, 2012). They studied the conversations of the students while using the DMA to document the activity of creating and negotiating the new definitions concluding their study was another activity in definition construction that advanced the students' thinking. I also found that the DMA framework was an appropriate way to follow the evolution of student thinking. Though my study was not examining the creation of definitions from a one-dimensional space to a two-dimensional space as Whitney et al. (2012), I did examine the creation of new definitions through the use of the five logical principles. Their study added new definitions to a student's cognitive structure while mine added the five principles to the total cognitive structure of my PSTs.

My study demonstrated that using the DMA with the five principles is one productive way to conduct and analyze a definition construction process as I documented change in their final products. My findings may suggest that through the design and implementation of tasks that invites dialogue about the creation, negotiation, and refinement of mathematical definitions, change can occur in the PSTs' abilities to write and understand both the content and nature of mathematical definitions.

Limitations

There are limitations to this work that restrict generalization or require caution going forward. Not only was this a small study using only seven preservice teachers, but the amount of time allocated to understanding the five principles in use was also too restrictive. It became apparent at the end of the third session that more individual work could have offered more precise insight through personal dialogue into individual growth. Though large group dialogue was productive for consensus building as the PSTs had to resolve their personal concept images or definitions with the other's ideas, individual results were not as precise as they might have been and may have been better served with individual interviews. Future research should examine ways to determine how an individual is growing and changing in terms of his/her abilities to write high quality definitions as this cannot be determined by only examining group discussion and written products.

Another limitation is the need to conduct more work with equivalent or arbitrary definitions. My study only had the PSTs argue the arbitrariness of two high quality definitions. Making a more robust task where several definitions were examined for arbitrariness could give more trustworthy results and still allow for examination of all the principles. Once the PSTs left the study, there needed to be a way to see if any of this work in definition construction actually transferred to work done in their current geometry content course or in a geometry course they may take in a future semester. If this transfer of the five principles, now part of their total cognitive structure, could be used in other mathematical content areas, one could conclude the formal level of the DMA had been reached. Though PST 2, PST 5, PST 6 and PST 7 showed the five principles had become part of the new concept image, this was not apparent for PST 3 and PST 4. Perhaps, as these two PSTs take the geometry course in the future, this situation changes.

However, as this study determined, this is a process called construction and two hours of focused work over a three-week span is not enough. Definition construction takes time, and if the process is to become a part of curricula, this must be taken into consideration.

The five principles align nicely with the hierarchical structure of these quadrilaterals. The concern for future research is how will a similar task structure work in other content areas where this alignment is not as obvious. Could the tasks be redesigned to accommodate for this difference? I see this as an issue in future works with slope representations as indicated earlier in this chapter. In particular, how can these representations be seen as arbitrary? In other words, could the nature of mathematical definitions be addressed in content areas where the hierarchy of the concepts is not as straight forward? Has this experience from the participation in this study provided a new or different way for these preservice teachers to teach definitions in their future classrooms?

Summary and Conclusion

My study did demonstrate that definition construction with the mechanism of the five principles as a design element for tasks, did change the concept image of the PSTs as the five principles became a way for them to reason about definitions. This change in cognitive structure affected not only their understanding of the nature of definitions, but also how they understood the hierarchy of quadrilaterals. The study also demonstrated that PSTs do value and embrace certain criteria for definitions as part of their initial ideas about the nature of mathematical definitions. Many of these criteria align with the five principles of the mechanism. Their final definitions demonstrated this alignment and also demonstrated the use of the principles.

Definition construction is a complex and dynamic process that requires deliberate and focused work that integrates the five principles with an individual's personal concept image. The dialogue produced provides a valuable way to determine how the total cognitive structure or concept image impedes change, or helps students, navigate this complex and dynamic process. Personal concept images and definitions add to the complexity of attempting a definition construction activity; I was reminded of this when I asked a colleague what do you think of when I say square? Her answer was, "Yellow. It was always yellow in the preschool books I read my grand children." This statement reminds us of the research of Hershkowitz and Vinner (1987) who indicated that when a mathematical definition is given it is the concept image that comes to mind. Concept images of this type are complex and formed over years of experiences that result in strong and lasting mental images.

APPENDICES

Appendix A

Table 15. Tasks Background

What Students See	Expectations for the Researcher	Purpose
Task 1 On the paper provided, please do the following: • Draw an image for a quadrilateral, parallelogram, rectangle and square. • Now write a definition-for each shape. In your group, discuss how your images for each shape are the same and how they are different. Repeat the conversation for each definition your created. Was it easier to determine how the images and definitions are alike or was it easier to determine their differences? Why do you think this was so? If your definitions were given to someone outside a mathematics classroom, do you feel he/she would have confidence in their	This task provides the researcher with the initial data of the *personal concept definitions* (pcd) and images of the PSTs. For the framework, this is the data for the *Situational level*. Expectations include images of the shapes that are prototypical and pcds that lack evidence of hierarchical classification. The pcds will likely be descriptions of the shape draw with reference to sides, angles, with some reference to congruence or parallelism. And may not be complete sentences. Research indicates a focus on only one property such as side lengths affects MKT, PCK, and the ability to see shapes in a hierarchical and inclusive sense.	The purpose of Task 1 is to gather insight into the concept images and *personal concept definitions* that the students bring from their prior experience and understanding. Discussion of artifacts will help the researcher interpret what the students provide. It is a beginning stage of early discussion concerning what is drawn and why. Discussion may include thoughts about the orientation, shape, size, common features of each drawing and definition. This discussion will also indicate what the preservice teachers consider important in their definitions and drawings. Note – Research from Edwards and Ward suggests a possible value in having students become aware of their own concept images. Students can use this awareness to understand the strong effects concept images hold during reasoning. This same conclusion provides the researcher with insights from the dialogue as to what students use from their cognitive structure of their concept image and guides the

Table 15. (Cont'd)

understanding of each shape? At the end of this task, please return your paper to the researcher.		researcher in facilitating a discussion.
Task 2 This second sheet of paper gives five logical principles that should be fulfilled when defining a mathematical concept according to mathematicians. Please discuss your interpretations of the meanings of these principles and record your ideas on the paper provided. • Defining is giving a name; the statement used as a definition presents the name of the concept and this term (name) appears only once in the statement. • In defining a new concept, only previously defined concepts may be used. • A definition establishes necessary and sufficient conditions. • The set of necessary and sufficient	Task 2 introduces the five principles that are the foundation for reasoning in subsequent tasks where the PSTs will rewrite definitions to exhibit the five principles. The researcher expects the group will engage in conversation that exhibits their need to clarify the meanings of the five principles. At the end of this task, the researcher will have evidence of how the PSTs interpret the five principles.	The purpose of Task 2 is to introduce the five logical principles, which emulate the nature of mathematical definitions and to determine how the PSTs verbalize their understanding of the five principles thus creating a set of Pcds for the principles. Another purpose is to have the students engage in reaching consensus.

Table 15. (Cont'd)

• conditions must be minimal. • Mathematical definitions are *arbitrary* that is several different and correct definitions may exist for a concept. As this task ends, determine how you will agree on an interpretation of these five principles. Your understanding of the five principles will be important in the rest of the tasks. Therefore, it is important that a list of understandings be created as we move forward. Expect me to ask you questions as you engage in your discussion. Anything you draw, say, or write will be collected for data.		
Task 3 For this task, you will work together to identify where the five principles are exhibited in these definitions. You are to arrive at consensus on the identification. ***A rectangle is a parallelogram with at least one right angle.*** Principle 1 – Principle 2 – Principle 3 –	This task provides an opportunity for the preservice teachers to identify the five logical principles in high-quality definitions. Verbalizing the abstract principles is challenging but identification may support their understanding. The conversation may demonstrate reasoning from their *personal concept definitions* through with the five principles as they see the five principles in the given definitions. The framework sees this focus	The purpose for this task is two-fold. Students are challenged to see in the definitions the five principles. While looking at the meaning of previously defined concepts, the hierarchical nature of the shapes comes to the forefront. In essence, this is an opportunity to demonstrate the hierarchical nature of a definition, necessary and sufficient properties, as well as the inclusive nature.

Table 15. (Cont'd)

Principle 4 – Principle 5 – Explain how the given definition is arbitrary and compares to the one above. ***A rectangle is a quadrilateral with three right angles.***	on the five principles as a shift toward a mathematical concept definition and the beginning of the replacement of their pcds. The expectation is thus the beginning of a shift from *definition-of* to *definition-for* in this task that begins to place them in the *situational level.*	The conversation may focus on the fact that these definitions go beyond just a description of the shape in question. This is also an opportunity to show the arbitrary nature of definitions.
Task 5 Working with a partner, you are to consider the following definitions. For each definitions you are to rewrite them with a focus on both principle 1 and 5. 1. A square is a quadrilateral with all sides and angles congruent. 2. A rectangle with shorter, straight, equal, closed line sides 3. Four sides shape with two short sides, two long sides, four right angles.	This task serves as the post-test for the researcher and may provide evidence that the use of the five principles shows changes in the PSTs' abilities to write high-quality definitions which according to the DMA signals the embracing of a new mathematical reality. The researcher expects that the different starting positions of each definition will encourage the use of the five principles and hierarchy. The researcher expects to glean from their conversations about the definitions they create the transition from *definition-of* to *definition-for* at the end of this task. If this occurs, then there is evidence that the PSTs have developed a richer concept image and a high quality mathematical concept	One purpose of this task is as an assessment of the overall goal of helping students create and write high-quality definitions. However, the learning goal or objective is for students to see both the arbitrary nature of definitions and the inclusive and hierarchical nature of geometric definitions as they write the required definitions. The result of this task may or may not demonstrate that the students can create high-quality definitions without referencing with their initial drawings of the shapes or their *personal concept definitions*.

Table 15. (Cont'd)

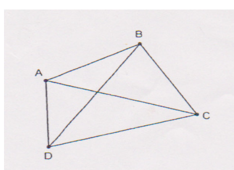
	definition-for rectangles and squares. If this occurs, there may be a shift from the *referential level* to the *general* or even the *formal levels* of the DMA. According to the DMA framework, the *formal level* may also be seen if this type of reasoning (not using concept images and associated pcds in the development of high-quality definitions) is used in other mathematical contexts.	
Task 6 This task asks you to write high quality definitions for a changing quadrilateral. Draw a picture of the shape. Provide as many true statements about the shape. Provide a written definition of the shape using the five principles.  This is the figure you will refer to in this task.	This task provides the researcher with evidence that the PSTs can use all the principles to write high quality definitions. Their conversations may indicate shifts in reasoning as defined in the DMA framework. The dialogue may also reveal deeper understandings about the hierarchy of the shapes.	The main purpose is to use this task as one that is specifically focused on writing high quality definitions. The dialogue will indicate how they reasons through the principles and if their concept images have assimilated the new mathematical reality.

Table 15. (Cont'd)

In each portion of the task, quadrilateral ***ABCD*** will change. 1. In this case assume ***ABCD*** is a quadrilateral. 2. In this case assume ***ABCD*** is a parallelogram. 3. In this case assume ***ABCD*** is a rectangle. 4. In this case assume ***ABCD*** is a square.		

Appendix B

Table 16. Coded Personal Concept Definitions

Coding Criteria - Image Description – Names characteristics Technical Structure – Phrases/s or Sentence Hierarchical/Inclusive Reference – Geometric/Non geometric	
Quadrilateral	
PST 1 – A quadrilateral is a 4-sided polygon.	Sentence, Hierarchical Reference
PST 2 – Quadrilateral: A shape with 4 sides lengths and 4 angles that equal a sum of 360 °s.	Image Description, Phrases, Inclusive Reference
PST 3 – Any 4 sided shape closed.	Image Description, Phrase, Inclusive Reference
PST 4 – Any four-sided shape – closed, straight lines.	Image description, Phrases, Inclusive Reference
PST 5 – A quadrilateral is any 4-sided figure with 4 angles and needs to be a closed shape.	Sentence, Image description, Inclusive Reference
PST 6 – Quadrilateral: a 4-sided figure with the sum of angles equal 360 °s also a closed figure.	Image description, Sentence, Inclusive Reference
PST 7 – A quadrilateral is a shape with four straight sides that are closed.	Sentence, Image Description, Hierarchical Reference

Parallelogram	
PST 1 – A parallelogram is a quadrilateral with opposite sides parallel to each other and opposite angles congruent.	Image Description, Sentence, Hierarchical Reference
PST 2- Parallelogram: A quadrilateral with opposite sides parallel and congruent, and opposite angle congruent	Image Description, Phrase, Hierarchical Reference
PST 3 – Parallelogram: 4 sides, slanted sides, closed, parallel	Image Description, Phrases
PST 4 – Parallelogram: "rectangle on a slant", 4 sides – 2 sets of equal sides-parallel, top/bottom parallel, Left/right parallel.	Image Description, Phrases
PST 5 – Parallelogram: a closed 4-sided figure with 2 sets of opposite congruent sides and angles.	Image Description, Phrases, Inclusive Reference
PST 6 – Parallelogram: A 4-sided figure with 2 sets of equal sides and	Image Description, Phrases, Inclusive Reference
corresponding angles (closed figure)	
PST 7 – A parallelogram is a quadrilateral	Sentence, Hierarchical Reference

Table 16. (cont'd)

Rectangle	
PST 1 – A rectangle is a quadrilateral with opposite sides parallel to each other and 4 90-° angles.	Image Description, Sentence, Hierarchical Reference
PST 2 – Rectangle: a quadrilateral with differing side lengths but 4 equal angles, each 90 °s.	Image Description, Phrase, Hierarchical Reference
PST 3 – Rectangle – 4 sides top and bottom equal sides equal 90 °s.	Image Description, Phrases
PST 4 – Rectangle – Two sets of parallel sides – 2 sets of equal sides	Image Description, Phrases
PST 5 – A closed 4-sided figure with 2 sets of opposite congruent sides and angles.	Image Description, Phrases, Inclusive Reference
PST 6 – Rectangle: a 4-sided figure with all angles 90 °s and 2 sets of equal sides (closed figure).	Image Description, Phrases, Inclusive Reference
PST 7 – A rectangle is a parallelogram with 90-° angles.	Sentence, Hierarchical Reference

Square	
PST 1 – A square is a quadrilateral with all sides and angles congruent.	Image Description, Sentence, Hierarchical Reference
PST 2 – Square: A quadrilateral with equal side lengths and equal angle measurements.	Image Description, Phrase, Hierarchical Reference
PST 3 – Square – 4 equal sides 90 °s	Image Description, Phrase
PST 4 – Square – four equal sides, 4 equal angles, 2 sets of 2 parallel sides.	Image Description, Phrases
PST 5 – A closed 4-sided shape with four 90-° angles and all congruent side lengths.	Image Description, Phrases, Inclusive Reference
PST 6 - Square: A 4 sided figure with all equal sides and all 90 ° angles (closed figure)	Image Description, Phrases, Inclusive Reference
PST 7 – A square is a rectangle with four equal sides.	Sentence, Hierarchical Reference

Appendix C

Table 17. Contrasting Task 1 to Task 2 to Day 1 PCDs

Quadrilateral Definitions

PST	Point of creation	Quoted definition	Presence of the principle in the definition					Beginning level Ending level
			1	2	3	4	5	
PST1	Starting definition	A quadrilateral is a 4 sided polygon.	Y	Y	Y	Y		Referential
	Ending definition	A quadrilateral is any four sided polygon.	Y	Y	Y	Y	Y	General
PST2	Starting definition	A shape w/ 4 side lengths & 4 angle measures equaling 360^{o}	N	Y	N	N		Situational
	Ending definition	Quadrilateral: A shape with 4 side lengths & 4 angles that equal a sum of 360^{o}	N	Y	Y	N	N	Situational
PST3	Starting definition	Any shape with 4 closed sides	N	Y	N	N		Situational
	Ending definition	Any shape with 4 closed ~~sides~~ straight line sides.	N	Y	N	N	N	Situational
PST4	Starting definition	Any four sided shape – closed, straight lines.	N	Y	Y	N		Situational
	Ending definition	Any shape with four, straight, closed sides.	N	Y	Y	N	N	Situational
PST5	Starting definition	A quadrilateral is any 4 sided figure with four angles. Also needs to be a closed figure.	Y	Y	Y	N		Situational
	Ending definition	A closed 4-sided figure with four angles.	N	Y	Y	N	N	Situational
PST6	Starting definition	A closed, 4 sided figure the sum of all angles equal to 360 °s.	N	Y	Y	Y		Situational
	Ending definition	A quadrilateral is a polygon with 4 sides.	Y	Y	Y	Y	Y	General
PST7	Starting definition	Quadrilateral is a shape with four straight sides that are closed.	Y	Y	N	Y		Situational
	Ending definition	A quadrilateral is a closed shape with 4 straight sides.	Y	Y	Y	Y	Y	General

Table 17. (cont'd)

Parallelogram Definitions

PST	Point of creation	Quoted definition	Presence of the principle in the definition 1	2	3	4	5	Beginning level Ending level
PST1	Starting definition	A parallelogram is a quadrilateral with opposite sides parallel to each other and opposite angles congruent.	Y	Y	Y	N		Situational
	Ending definition	A parallelogram is a quadrilateral with opposite sides parallel to each other and opposite angles congruent.	Y	Y	Y	N	N	Situational
PST2	Starting definition	A quadrilateral with opposite sides parallel & congruent, and opposite angles congruent	N	Y	Y	N		Situational
	Ending definition	Parallelogram: A quadrilateral with opposite sides parallel & congruent, and opposite angles congruent.	N	Y	Y	N	N	Situational
PST3	Starting definition	A quadrilateral with 4 closed, slanted, parallel sides	N	Y	N	N		Situational
	Ending definition	A quadrilateral with closed, slanted line sides	N	Y	N	N	N	Situational
PST4	Starting definition	"rectangle on a slant" 4 sides, 2 sets of 2 equal sides – parallel.	N	Y	N	N		Situational
	Ending definition	Four sided shape with two shorter sides and two long sides that are slanted.	N	Y	N	N	N	Situational
PST5	Starting definition	A closed 4-sided figure with two sets of opposite congruent sides and angles.	N	Y	Y	N		Situational
	Ending definition	A quadrilateral with 2 sets of opposite congruent sides and angles.	N	Y	Y	Y	Y	Situational
PST6	Starting definition	A closed, 4 sided figure with two sets of equal side lengths and angles.	N	Y	N	N		Situational
	Ending definition	A parallelogram is a rectangle with 2 sets of equal angles.	Y	Y	N	Y	N	Situational
PST7	Starting definition	Parallelogram is a quadrilateral with two sets of opposite parallel sides.	Y	Y	Y	Y		Referential
	Ending definition	A parallelogram is a quadrilateral with 2 sets of parallel sides.	Y	Y	Y	Y	N	Referential

Table 17. (cont'd)

Rectangle Definitions

PST	Point of creation	Quoted definition	Presence of the principle in the definition					Beginning level Ending level
			1	2	3	4	5	
PST1	Starting definition	A rectangle is a quadrilateral with opposite sides parallel to each other and 4 90^{O} angles.	Y	Y	Y	N		Situational
	Ending definition	A rectangle is a ~~quadrilateral~~ parallelogram with ~~opposite sides parallel to each other and~~ four right angles.	Y	Y	Y	N	Y	General
PST2	Starting definition	A quadrilateral with differing side lengths but 4 equal angles, each 90^{o}.	N	Y	N	N		Situational
	Ending definition	Rectangle: A quadrilateral with differing side lengths but 4 equal angles, each 90^{o}.	N	Y	N	N	N	Situational
PST3	Starting definition	A quadrilateral with 4 sides, 90^{o} corners. Top & bottom sides are equal, sides are equal	N	Y	Y	N		Situational
	Ending definition	A parallelogram with straight closed line sides & at least one 90^{o} angle	N	Y	Y	Y	Y	Situational
PST4	Starting definition	Two sets of parallel sides, 2 sets of equal sides. Top / bottom parallel Left /right parallel	N	Y	N	N		Situational
	Ending definition	Four sided shape with two short sides, two long sides; four right angles.	N	Y	N	N	N	Situational
PST5	Starting definition	A closed 4-sided figure with 2 sets of opposite congruent sides and 4 90 ° angles.	Y	Y	Y	N		Situational
	Ending definition	A parallelogram with all angles equal to 90 °s.	N	Y	Y	N	Y	Referential
PST6	Starting definition	A 4 sided closed figure with two sets of equal sides and all angles 90 °s.	N	Y	Y	N		Situational
	Ending definition	A rectangle is a square with parallel sides.	Y	Y	N	Y	Y	Situational
PST7	Starting definition	Rectangle is a parallelogram with 90 ° angles.	Y	Y	Y	Y		Situational
	Ending definition	A rectangle is a parallelogram with right angles.	Y	Y	Y	N	N	Referential

Table 17. (cont'd)

Square Definitions

PST	Point of creation	Quoted definition	Presence of the principle in the definition					Beginning level Ending level
			1	2	3	4	5	
PST1	Starting definition	A square is a quadrilateral with all sides and angles congruent.	Y	Y	Y	N		Situational
	Ending definition	A square is a ~~quadrilateral~~ rectangle with all sides and angles congruent.	Y	Y	Y	N	Y	Situational
PST2	Starting definition	A quadrilateral with equal side lengths & equal angle measures.	N	Y	Y	N		Situational
	Ending definition	Square: A quadrilateral with equal side lengths & equal angle measures.	N	Y	Y	N	N	Situational
PST3	Starting definition	A quadrilateral with 4 equal sides & 90^o corners	N	Y	Y	N		Situational
	Ending definition	A rectangle with shorter, straight, closed line sides	N	Y	N	N	Y	Situational
PST4	Starting definition	Four equal sides, 4 equal angles, 2 sets of 2 ~~equal~~ parallel sides.	N	Y	Y	N		Situational
	Ending definition	Shape with four equal sides with four right angles.	N	Y	Y	N	N	Situational
PST5	Starting definition	A closed 4-sided shape with 4 90 ° angles and all congruent side lengths.	N	Y	N	N		Situational
	Ending definition	A rectangle with all equal side lengths.	N	Y	Y	N	Y	Situational
PST6	Starting definition	A 4 sided closed figure with all equal sides and four 90 ° angles.	N	Y	Y	N		Situational/Referential
	Ending definition	A square is a rectangle with equal sides.	Y	Y	Y	Y	Y	General
PST7	Starting definition	Square is a rectangle with four equal sides.	Y	Y	Y	N		Referential
	Ending definition	A square is a rectangle with congruent sides.	Y	Y	Y	Y	N	General

APPENDIX D

Table 18. Movement in DMA Levels

	Situational		Referential		General
PST1					
quadrilateral		1st			2nd
parallelogram	1st 2nd				
rectangle	1st			2nd	
square	1st		2nd		
PST2					
quadrilateral	1st 2nd				
parallelogram	1st 2nd				
rectangle	1st				2nd
square	1st				2nd
PST3					
quadrilateral	1st 2nd				
parallelogram	1st 2nd				
rectangle	1st 2nd				
square	1st 2nd				
PST4					
quadrilateral	1st 2nd				
parallelogram	1st 2nd				
rectangle	1st 2nd				
square	1st 2nd				
PST5					
quadrilateral	1st 2nd				
parallelogram	1st 2nd				
rectangle	1st			2nd	
square	1st 2nd				
PST6					
quadrilateral	1st 2nd				
parallelogram		1st			2nd
rectangle	1st 2nd				
square		1st			2nd
PST7					
quadrilateral	1st			2nd	
parallelogram		1st			2nd
rectangle			1st 2nd		
square			1st 2nd		

Appendix E

Permission to Use Graphic

From: Kohrman, Darlene Estherkohrmand@msu.edu

Date: Mon, Oct 1, 2018, 5:18 PM

Subject: Permission to use graphic in The Montana News Enthusiast

To: Bogar, Leslie; Erickson, David; Sriraman, Bharath

Dear Leslie,

I am a PhD. candidate at MSU in the PRIME program. I am finishing my dissertation on definition construction and would like to use a graphic found on p. 184 in an article published in your Montana News Enthusiast.

Rosken, Bettina, & Rolka, Katrin. (2007). Integrating Intuition: The Role of Concept Image and Concept Definition for Students' Learning of Integral Calculus. *TMME Monograph*(3), 181-204.

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