STABILITY OF THE ALMOST HERMITIAN CURVATURE FLOW

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ABSTRACT

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The Almost Hermitian Curvature flow was introduced in [7] by Streets and Tian in order to study almost hermitian structures, with a particular interest in symplectic structures. This flow is given by a diffusion-reaction equation. Hence it is natural to ask the following: which almost hermitian structures are dynamically stable? An almost hermitian structure $(\widetilde{\omega}, \widetilde{J})$ is dynamically stable if it is a fixed point of the flow and there exists a neighborhood $\mathcal{N}$ of $(\widetilde{\omega}, \widetilde{J})$ such that for any almost hermitian structure $(\omega(0), J(0)) \in \mathcal{N}$ the solution of the Almost Hermitian Curvature flow starting at $(\omega(0), J(0))$ exists for all time and converges to a fixed point of the flow. We prove that on a closed Kähler-Einstein manifold $(M, \widetilde{\omega}, \widetilde{J})$ such that either $c_1(\widetilde{J}) < 0$ or $(M, \widetilde{\omega}, \widetilde{J})$ is a Calabi-Yau manifold, then the Kähler-Einstein structure $(\widetilde{\omega}, \widetilde{J})$ is dynamically stable.

ACKNOWLEDGMENTS

TABLE OF CONTENTS

Chapter 1

Preliminaries

1.1 Introduction

Let (M, J, g) be a closed almost complex manifold such that J is compatible with the Riemannian metric g, that is for any vector fields X and Y we have $g(X, Y) = g(J(X), J(Y))$. To the metric g we associate the 2-form ω defined by $\omega(X, Y) = g(J(X), Y)$. We call such a pair (ω, J) an *almost hermitian structure.*

The Ricci flow has proven to be a successful tool in studying the Riemannian geometry of manifolds. Therefore it is natural to attempt to use a parabolic flow to understand the almost hermitian geometry of almost complex manifolds. However, the Ricci flow does not, in general, preserve the set of almost hermitian structures. In [7], Streets and Tian introduce the Almost Hermitian Curvature flow (AHCF), which is a weakly-parabolic flow on the space of almost hermitian structures.

AHCF generalizes Kähler Ricci flow in the sense that if the initial structure (ω_0, J_0) is Kähler, then the evolution of $(\omega(t), J(t))$ by AHCF coincides with Kähler Ricci flow. In [8], Streets and Tian construct a parabolic flow on the space of hermitian structures (ω, J) (here J is integrable), called Hermitian Curvature flow (HCF). AHCF also generalizes HCF.

As we will see below AHCF is, in fact, a family of geometric flows. Streets and Tian have a particular interest in one of these flows, called Symplectic Curvature flow (SCF). Given

an almost hermitian structure (ω_0, J_0) such that $d\omega_0 = 0$, under SCF $\omega(t)$ is a closed form as long as the flow exists. Therefore, SCF is a tool which can be used to study symplectic structures. Hence we see that AHCF is a very general family of geometric flows.

The Almost Hermitian Curvature flow is a coupled flow of metrics and almost complex structures. It is written

$$
\begin{aligned}
\frac{\partial}{\partial t}\omega &= -2S + H + Q \\
\frac{\partial}{\partial t}J &= -\mathcal{K} + \mathcal{H}.
\end{aligned}
\tag{1.1}
$$

S is a "Ricci-type" curvature. In particular, $S_{ij} = \omega^{kl}\Omega_{klij}$ and Ω is the curvature of the almost-Chern connection ∇. That is, ∇ is the unique connection satisfying $\nabla\omega = 0$, $\nabla J = 0$ and $T^{1,1} = 0$. $T^{1,1}$ is the $(1,1)$ component of the torsion of ∇. Q is any $(1,1)$ form that is quadratic in the torsion of ∇. $\mathcal{K}^i_j = \omega^{kl}\nabla_k N^i_{lj}$ where N is the Nijenhuis tensor with respect to J. $\mathcal{H}$ is any endomorphism of TM that is quadratic in N and skew commutes with J. The term $H(X,Y) \doteq \frac{1}{2}\left[\omega((-\mathcal{K}+\mathcal{H})(X), J(Y)) + \omega(J(X), (-\mathcal{K}+\mathcal{H})(Y))\right]$ is required in order to maintain the compatibility of ω_t with J_t. Streets and Tian prove short-time existence and uniqueness (see Theorem 1.1 in [7]) of the flow starting at an almost hermitian structure $(\omega(0), J(0))$. Notice that the generality with which the tensors Q and $\mathcal{H}$ are defined implies that (1.1) is in fact a family of geometric flows. This family of geometric flows includes Hermitian Curvature flow, Symplectic Curvature flow and Kähler Ricci flow. Associated to AHCF is the volume-normalized version of the flow (VNAHCF), the volume-normalized version is the one with which we will work.

One natural question to ask is: does M admit a Kähler-Einstein structure? If so, is it detected by VNAHCF? The main result of the paper is the following:

2

Theorem 1. *Let $(M^{2n}, \widetilde{\omega}, \widetilde{J})$ be a closed complex manifold with $(\widetilde{\omega}, \widetilde{J})$ a Kähler-Einstein structure such that either $c_1(\widetilde{J}) < 0$ or $(M, \widetilde{\omega}, \widetilde{J})$ is a Calabi-Yau manifold. Then there exists $\epsilon > 0$ such that if $(\omega(0), J(0))$ is an almost hermitian structure with $\big|(\omega(0) - \widetilde{\omega}, J(0) - \widetilde{J})\big|_{C^\infty} < \epsilon$, then the solution to the volume normalized AHCF starting at $(\omega(0), J(0))$ exists for all time and converges exponentially to a Kähler-Einstein structure (ω_{KE}, J_{KE}).*

Remark 2. Theorem 1 gives evidence that the Almost Hermitian Curvature flow reflects the underlying almost hermitian geometry of M.

Remark 3. In this paper we define a *Calabi-Yau manifold* $(M, \widetilde{\omega}, \widetilde{J})$ to be a compact Kähler manifold with trivial canonical bundle such that $\widetilde{\omega}$ is a Kähler-Einstein metric with $\mathrm{Ric}(\widetilde{\omega}) = 0$.

Remark 4. In the case when $c_1(\widetilde{J}) < 0$, the Kähler-Einstein structure that the flow starts close to is the same one that the flow converges to, in other words $(\widetilde{\omega}, \widetilde{J}) = (\omega_{KE}, J_{KE})$. This is proved in Theorem 13.

In the Calabi-Yau case we cannot guarantee that $(\widetilde{\omega}, \widetilde{J})$ and (ω_{KE}, J_{KE}) are the same Calabi-Yau structure.

The notion of stability in Theorem 1 is often referred to as dynamic stability. Dynamic stability has also been studied in the case of the Hermitian Curvature flow by Streets and Tian ([8]) and for the Ricci flow by Sesum ([6]) and by Guenther, Isenberg, and Knopf ([4]).

The first step in proving Theorem 1 is to show that Kähler-Einstein structures behave like sinks of the linear flow associated to VNAHCF, this is done in Section 1.2. Also in Section 1.2, we derive parabolic estimates for the VNAHCF (see Theorem 12).

Next, in Chapter we prove Theorem 1 in the case when $c_1(\widetilde{J}) < 0$. Finally, in the last chapter we complete the proof of Theorem 1 by showing how to find a Kähler-Einstein

structure (ω_{KE}, J_{KE}) to which the flow exponentially converges in the Calabi-Yau case.

1.2 Linear Stability and Parabolic Estimates

To prove theorem 1 we first show that, on a linear level, any Kähler-Einstein structure $(\widetilde{\omega}, \widetilde{J})$ is a "degenerate sink" with respect to the Almost Hermitian Curvature flow, meaning that the linear operator associated to the non-linear flow is negative semi-definite at Kähler-Einstein structures. For notation sake write VNAHCF:

$$
\begin{aligned}
\frac{\partial}{\partial t}\omega &= \mathcal{F} \\
\frac{\partial}{\partial t}J &= \mathcal{G}.
\end{aligned}
\tag{1.2}
$$

To the operator $(\mathcal{F}, \mathcal{G})$, we have the associated linear operator $(\dot{\mathcal{F}}, \dot{\mathcal{G}})$. In particular, we consider a one-parameter family of compatible, unit volume, almost hermitian structures $(\omega(a), J(a))$ and $(\dot{\mathcal{F}}, \dot{\mathcal{G}}) \doteq \frac{\partial}{\partial a}\big|_{a=0}(\mathcal{F}, \mathcal{G})(\omega(a), J(a))$. Similarly we write $(\dot{\omega}, \dot{J}) \doteq \frac{\partial}{\partial a}\big|_{a=0}(\omega(a), J(a))$.

Definition 5. An almost-hermitian structure (ω, J) is called *static* provided $(\mathcal{F}(\omega, J), \mathcal{G}(\omega, J)) = 0$. Moreover, a static structure (ω, J) is *linearly stable* if the linearization $\mathcal{L} \doteq (\dot{\mathcal{F}}, \dot{\mathcal{G}})_{(\omega, J)}$ is negative semi-definite, that is $\langle \mathcal{L} \cdot, \cdot \rangle_{L^2(g)} \le 0$.

Notice that Kähler-Einstein structures are static under VNAHCF. Next, we prove

Theorem 6. *Let $(M, \widetilde{J})$ be a closed complex manifold with $c_1(\widetilde{J}) \le 0$, then any Kähler-Einstein structure $(\widetilde{\omega}, \widetilde{J})$ on M is linearly stable.*

Proof. Employing the DeTurck trick as in Proposition 5.4 and 5.5 of [7] the weak-ellipticity of $(\mathcal{F}, \mathcal{G})$ follows. Furthermore, computing the linearization of $(\mathcal{F}, \mathcal{G})$ at a Kähler-Einstein

structure, in complex coordinates with respect to $\widetilde{J}$, the linearization is written

$$\dot{\mathcal{F}}_{i\bar{j}} = -2\nabla^*\nabla\dot{\omega}_{i\bar{j}} + 2\dot{\omega}^{k\bar{l}}R_{k\bar{l}i\bar{j}} \tag{1.3}$$

$$\dot{\mathcal{F}}_{ij} = -2\nabla^*\nabla\dot{\omega}_{ij} \tag{1.4}$$

$$\dot{\mathcal{G}}^i_{\bar{j}} = -2\nabla^*\nabla\dot{J}^i_{\bar{j}} + 2\dot{J}^{pq}R^i_{\bar{j}pq}. \tag{1.5}$$

Here ∇^* is the $L^2(\widetilde{g})$ adjoint of ∇ and R denotes the Riemannian curvature of $\widetilde{g}$.

To show that $(\widetilde{\omega}, \widetilde{J})$ is linearly stable we have to deal with the fact that in (1.3) and (1.5) the lower order terms do not have a sign. To see that the linearized operator is negative semi-definite at Kähler-Einstein structures we use a couple of Weitzenböck-Bochner formulas (cf. [2]).

Lemma 7. *Let α and β be a (0,2) and (1,1) form respectively. If $\widetilde{g}$ is a Kähler-Einstein metric, then we have*

$$(\Delta_d\alpha)_{\overline{ij}} = 2\nabla^*\nabla\alpha_{\overline{ij}} + 2\frac{s}{n}\alpha_{\overline{ij}} \tag{1.6}$$

$$(\Delta_d\beta)_{i\bar{j}} = 2\nabla^*\nabla\beta_{i\bar{j}} - 2\beta^{k\bar{l}}R_{k\bar{l}i\bar{j}} + 2\frac{s}{n}\beta_{i\bar{j}}. \tag{1.7}$$

Where Δ_d is the Hodge Laplacian with respect to $\widetilde{g}$ and s is the scalar curvature of $\widetilde{g}$. In addition we will use another Weitzenböck-Bochner formula.

Lemma 8. *Given a $T^{1,0}(M, \widetilde{\omega}, \widetilde{J})$ valued $(0,1)$-form, ϕ and Kähler-Einstein metric $\widetilde{g}$ we have:*

$$(\Delta_{\overline{\partial}}\phi)^i_{\bar{j}} = \nabla^*\nabla\phi^i_{\bar{j}} - \phi^{pq}R^i_{\bar{j}pq} + \frac{s}{n}\phi^i_{\bar{j}}. \tag{1.8}$$

Where $\Delta_{\overline{\partial}}$ represents the complex laplacian $\overline{\partial}\overline{\partial}_g^* + \overline{\partial}^*\overline{\partial}_{\widetilde{g}}$.

Therefore using equations (1.3), (1.4), (1.6) and (1.7), we have that

$$\dot{\mathcal{F}} = -\Delta_d\dot{\omega} + 2\frac{s}{n}\dot{\omega}. \tag{1.9}$$

Similarly, using equations (1.5) and (1.8), we have

$$\dot{\mathcal{G}} = -\Delta_{\overline{\partial}}\dot{J} + 2\frac{s}{n}\dot{J}. \tag{1.10}$$

Combining (1.9) and (1.10) we see that

$$\mathcal{L}(\dot{\omega}, \dot{J}) = \left(-\Delta_d\dot{\omega} + 2\frac{s}{n}\dot{\omega}, -\Delta_{\overline{\partial}}\dot{J} + 2\frac{s}{n}\dot{J}\right). \tag{1.11}$$

Finally since $c_1(\widetilde{J}) \leq 0$ implies that $s \leq 0$; by integrating the theorem follows. $\quad\square$

Notice that if $c_1(\widetilde{J}) < 0$, then the scalar curvature of $\widetilde{g}$ is negative; that is $s < 0$. Hence from (1.11) it follows that if $c_1(\widetilde{J}) < 0$, then $\mathcal{L}$ is strictly negative definite with respect to $L^2(\widetilde{g})$. Let $\lambda = \min\{|\lambda_i| : \lambda_i \text{ is an eigenvalue of } \mathcal{L}\}$. Further let $\mathcal{C}$ denote the space of almost hermitian structures modulo diffeomorphism. Therefore we have proved the following corollary.

Corollary 9. *Let $(M, \widetilde{\omega}, \widetilde{J})$ be a closed complex manifold such that $(\widetilde{\omega}, \widetilde{J})$ is a Kähler-Einstein structure and moreover $c_1(\widetilde{J}) < 0$. Let $\psi \in T_{(\widetilde{\omega},\widetilde{J})}\mathcal{C}$, then*

$$\langle \mathcal{L}_{(\widetilde{\omega},\widetilde{J})}\psi, \psi\rangle_{L^2(\widetilde{g})} \leq -\lambda|\psi|^2_{L^2(\widetilde{g})}.$$

Corollary 9 will be crucial to proving Theorem 1 in the case when $c_1(\widetilde{J}) < 0$ (see Chapter).

Fix a Kähler-Einstein structure $(\widetilde{\omega}, \widetilde{J})$ and let $(\omega(t), J(t))$ be a solution of the coupled system (1.2) starting at an initial almost hermitian structure $(\omega(0), J(0))$. We will quantify the amount by which the solution deviates from $(\widetilde{\omega}, \widetilde{J})$ using

$$\rho(t) = (\omega(t) - \widetilde{\omega}, J(t) - \widetilde{J}).$$

Notice that $\rho(t) \in \Lambda^2(M) \times \text{End}(TM)$. Throughout the paper we use the operator norm on $\text{End}(TM)$.

As noted in the proof of Theorem 1.1 in [7], $\mathcal{C}$ is a non-linear manifold. In the following lemma we will see that $\rho(t) \notin T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$, however $\rho(t)$ can be estimated by an element of the tangent space $T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$.

Lemma 10. *Fix t and let $(\omega(t), J(t))$ be an almost hermitian structure. Write $\omega(t) = \widetilde{\omega} + h(t)$ and $J(t) = \widetilde{J} + K(t)$, in other words $\rho(t) = (h(t), K(t))$. If $|\rho(t)|_{C^0} < 1$, then there exists $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ so that*

$$|\psi(t)|_{C^k} \leq |\rho(t)|_{C^k}, \tag{1.12}$$

$$|\rho(t)|_{L^2} \leq |\psi(t)|_{L^2} + C_1|\psi(t)|_{L^2}^2 \tag{1.13}$$

and

$$|\rho(t)|_{C^k} \leq |\psi(t)|_{C^k} + C_2 |\psi(t)|_{C^k}^2 \tag{1.14}$$

where C_1 and C_2 depend on the L^2 and C^k norms of $\rho(t)$ respectively.

Proof. We will begin by studying the tangent space $T_{(\widetilde{\omega},\widetilde{J})}\mathcal{C}$. Let (ω_s, J_s) denote a path of almost hermitian structures such that $(\omega_s, J_s)|_{s=0} = (\widetilde{\omega}, \widetilde{J})$ and let $\frac{\partial}{\partial s}\big|_{s=0}(\omega(s), J(s)) \doteq (\dot\omega, \dot J)$. Given vector fields X and Y, the compatibility condition is written:

$$\omega_s(X, Y) = \omega_s(J_s(X), J_s(Y))$$

and the almost complex condition is written:

$$J_s^2(X) = -X.$$

Hence the linearized compatibility and almost complex conditions are given by:

$$\dot\omega(X, Y) = \dot\omega(\widetilde{J}(X), \widetilde{J}(Y)) + \widetilde{\omega}(\dot J(X), \widetilde{J}(Y)) + \widetilde{\omega}(\widetilde{J}(X), \dot J(Y)) \tag{1.15}$$

$$0 = \dot J \circ \widetilde{J}(X) + \widetilde{J} \circ \dot J(X). \tag{1.16}$$

From (1.16) we see that the tangent space to the space of almost complex structures is given by endomorphisms that skew-commute with $\widetilde{J}$. Equivalently, $\dot J$ can be viewed as a section of $\left[\Lambda^{0,1} \otimes T^{1,0}\right] \oplus \left[\Lambda^{1,0} \otimes T^{0,1}\right]$. Here we use $\widetilde{J}$ to decompose $TM = T^{1,0}M \oplus T^{0,1}M$.

First we will prove that the endomorphism $K(t)$ can be estimated by an element of the tangent space to the space of almost complex structures at $\widetilde{J}$. For the sake of notation we

8

will often write $K(t) = K$.

Using $\widetilde{J}$ we decompose $K = K_{0,1}^{1,0} + K_{0,1}^{0,1} + K_{1,0}^{1,0} + K_{1,0}^{0,1}$ where $K_{0,1}^{1,0} : T^{0,1} \to T^{0,1}$, equivalently

$$K_{0,1}^{1,0} \in \Lambda^{0,1} \otimes T^{1,0}.$$

Take $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})} \mathcal{C}$ and write $\psi(t) = (\psi_1(t), \psi_2(t)) \in \Lambda^2(M) \times \mathrm{End}(TM)$. We define

$$\psi_2(t) \doteq K_{0,1}^{1,0} + K_{1,0}^{0,1}. \tag{1.17}$$

That is, $\psi_2(t)$ is defined to be the projection of K onto $\left[\Lambda^{0,1} \otimes T^{1,0}\right] \oplus \left[\Lambda^{1,0} \otimes T^{0,1}\right]$. Next we will show that $K_{0,1}^{0,1}$ and $K_{1,0}^{1,0}$ are quadratic in $\psi_2(t)$. We will only prove this for $K_{0,1}^{0,1}$ since the same argument applies to $K_{1,0}^{1,0}$.

Using that $J(t)$ is an almost complex structure we see that $K(t)$ satisfies:

$$0 = K \circ \widetilde{J}(X) + \widetilde{J} \circ K(X) + K^2(X). \tag{1.18}$$

Now for K acting on $T^{0,1}$ we will write $K = K_{0,1}^{0,1} + K_{0,1}^{1,0}$. Therefore using (1.18), on $T^{0,1}$ we have

$$0 = -2\sqrt{-1}K_{0,1}^{0,1} + K_{0,1}^{0,1} \circ K_{0,1}^{0,1} + K_{0,1}^{1,0} \circ K_{0,1}^{0,1} + K_{1,0}^{1,0} \circ K_{0,1}^{1,0} + K_{1,0}^{0,1} \circ K_{0,1}^{1,0}$$

and so by type consideration,

$$K_{0,1}^{0,1} = -\frac{\sqrt{-1}}{2} \left(K_{0,1}^{0,1} \circ K_{0,1}^{0,1} + K_{1,0}^{0,1} \circ K_{0,1}^{1,0} \right). \tag{1.19}$$

9

Notice that on $T^{0,1}$, $K^{0,1}_{1,0} \circ K^{1,0}_{0,1} = \psi_2(t)^2$. Hence we are able to write $K^{0,1}_{0,1}$ in terms of $\left(K^{0,1}_{0,1} \right)^2$ and a term that is quadratic in $\psi_2(t)$.

Next, consider the first term on the right-hand side of (1.19), $\left(K^{0,1}_{0,1} \right)^2$. We will use (1.19) to show that $\left(K^{0,1}_{0,1} \right)^2$ can be expressed as $\left(K^{0,1}_{0,1} \right)^4$ plus terms which are quadratic in $\psi_2(t)$. Plugging (1.19) into each factor of $\left(K^{0,1}_{0,1} \right)^2$, we see that

$$\left(K^{0,1}_{0,1} \right)^2 = -\frac{1}{4} \left[\left(K^{0,1}_{0,1} \right)^4 + \left(K^{0,1}_{0,1} \right)^2 \circ \psi_2^2 + \psi_2^2 \circ \left(K^{0,1}_{0,1} \right) + \psi_2^4 \right],$$

which can be substituted into the term $K^{0,1}_{0,1} \circ K^{0,1}_{0,1}$ in (1.19). Iterating this process by successively plugging (1.19) into the highest power term in $K^{0,1}_{0,1}$, we see that $K^{0,1}_{0,1}$ can be expressed as a series. Notice that since $|\rho|_{C^0} < 1$ it follows that $\left| K^{0,1}_{0,1} \right|_{C^0} < 1$, and so this series converges. Therefore

$$K^{0,1}_{0,1} = \psi_2(t)^2 + \text{ [higher-power terms in } \psi_2 \circ \text{ higher-power terms in } K]. \qquad (1.20)$$

Next we will show that the two form $h(t)$ can be estimated by an element of the tangent space to the space of compatible metrics. Notice that for vector fields X and Y,

$$\dot{\omega}(X, Y) - \dot{\omega}(\widetilde{J}(X), \widetilde{J}(Y)) = 2\dot{\omega}^{(2,0)+(0,2)}(X, Y),$$

and so by (1.15)

$$2\dot{\omega}^{(2,0)+(0,2)}(X, Y) = \widetilde{\omega}(\dot{J}(X), \widetilde{J}(Y)) + \widetilde{\omega}(\widetilde{J}(X), \dot{J}(Y)).$$

Using the compatibility of $\omega(t)$ and $J(t)$, we see that $h(t) = \omega(t) - \widetilde{\omega}$ and $K(t) = J(t) - \widetilde{J}$

10

satisfy:

$$h(X, Y) = h(\widetilde{J}(X), \widetilde{J}(Y)) + \widetilde{\omega}(K(X), \widetilde{J}(Y)) + \widetilde{\omega}(\widetilde{J}(X), K(Y))$$

$$+ \widetilde{\omega}(K(X), K(Y)) + h(K(X), \widetilde{J}(Y)) + h(\widetilde{J}(X), K(Y)) \tag{1.21}$$

$$+ h(K(X), K(Y)).$$

We define $\psi_1(t)$ as follows

$$\psi_1^{(1,1)} \doteq h^{(1,1)} \tag{1.22}$$

$$\psi_1^{(2,0)+(0,2)}(X, Y) \doteq \widetilde{\omega}(K(X), \widetilde{J}(Y)) + \widetilde{\omega}(\widetilde{J}(X), K(Y)) \tag{1.23}$$

$$= \widetilde{\omega}(\psi_2(X), \widetilde{J}(Y)) + \widetilde{\omega}(\widetilde{J}(X), \psi_2(Y)). \tag{1.24}$$

The last equality follows from the definition of ψ_2 and the fact that $\widetilde{\omega}$ is of type $(1, 1)$.

Next we will show that $h^{(2,0)+(0,2)} - \psi_1^{(2,0)+(0,2)}$ can be expressed as terms that are quadratic in $\psi(t)$. Combining (1.21), (1.23) and (1.24) we have

$$2\left(h^{(2,0)+(0,2)}(X, Y) - \psi_1^{(2,0)+(0,2)}(X, Y)\right) = h(K(X), \widetilde{J}(Y)) + h(\widetilde{J}(X), K(Y))$$

$$\widetilde{\omega}(K(X), K(Y)) + h(K(X), K(Y)). \tag{1.25}$$

As we proved above in (1.17) and (1.20), K can be written in terms of ψ_2 and hence the terms in the second line of (1.25) are higher-power in ψ_2. Next we consider the term $h(K(X), \widetilde{J}(Y))$. Since the left-hand side of (1.25) is a section of $\Lambda^{(2,0)+(0,2)}$, let $X, Y \in T^{0,1}M$. So for $X, Y \in T^{0,1}M$ we can write the components of $h(K(X), \widetilde{J}(Y))$ as

$$K_{0,1}^{0,1} h^{(0,2)} + K_{0,1}^{1,0} h^{(1,1)}. \tag{1.26}$$

11

From (1.17) and (1.22) we see that the second term in (1.26) is quadratic in ψ. By (1.20) the first term is quadratic in ψ plus higher-power terms in ψ composed with higher-power terms in ρ. Notice that the same argument can be applied to $h(\widetilde{J}(X), K(Y))$. Abusing notation we let $\psi * \psi$ denote terms which are quadratic in ψ plus terms that are higher-power in ψ composed with terms that are higher-power in ρ. Therefore we have

$$h^{(2,0)+(0,2)}(X,Y) - \psi_1^{(2,0)+(0,2)}(X,Y) = \psi * \psi. \tag{1.27}$$

Notice that by the definition of $\psi(t)$, given in (1.17) (1.22) and (1.23), the inequality

$$|\psi(t)|_{C^k} \leq |\rho(t)|_{C^k}$$

follows immediately. Again using the definition of $\psi(t)$ along with (1.20) and (1.27) we see that

$$|\rho(t)|_{C^k} \leq |\psi(t)|_{C^k} + C|\psi(t)|_{C^k}^2$$

where C depends on the C^k norm of $\rho(t)$. Notice that (1.13) follows analogously. $\qquad\square$

In Theorem 6 we proved that the linearization of $(\mathcal{F}, \mathcal{G})$, denoted $\mathcal{L}$, is negative semi-definite on $T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$. The goal is to use the sign on $\mathcal{L}$ to prove exponential decay of $\rho(t)$. However as we observed in the previous lemma, $\rho(t) \notin T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$. To deal with this we will prove exponential decay of $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ which, by (1.14), will prove exponential decay of $\rho(t)$.

Next we show that $\psi(t)$ evolves by a parabolic flow equation and moreover that we have estimates on the non-linear part of the flow.

Lemma 11. *Let $\mathcal{L}$ be the differential operator defined by (1.11). Then for $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ defined by (1.17) (1.22) and (1.24) we have*

1. $\frac{\partial}{\partial t}\psi(t) = \mathcal{L}(\psi(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi(t))$

2. $|A((\widetilde{\omega}, \widetilde{J}), \psi(t))|_{C^k} \leq C(|\psi(t)|_{C^k}|\nabla^2\psi(t)|_{C^k} + |\nabla\psi(t)|^2_{C^k})$

where C depends on the C^k norm of $\rho(t)$.

Proof. To prove $\frac{\partial}{\partial t}\psi(t) = \mathcal{L}(\psi(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi(t))$ we first study the evolution of $\rho(t)$. Notice that since $(\widetilde{\omega}, \widetilde{J})$ is independent of t,

$$\frac{\partial}{\partial t}\rho(t) = \frac{\partial}{\partial t}(\omega(t), J(t)) = \big(\mathcal{F}(\omega(t), J(t)), \mathcal{G}(\omega(t), J(t))\big). \tag{1.28}$$

Furthermore since $(\widetilde{\omega}, \widetilde{J})$ is a static structure, when we linearize $(\mathcal{F}, \mathcal{G})$ at $(\widetilde{\omega}, \widetilde{J})$ in the direction $\psi(t)$, we have

$$(\mathcal{F}, \mathcal{G}) = \mathcal{L}(\psi(t)) + A((\widetilde{\omega}, \widetilde{J}), \rho(t)). \tag{1.29}$$

Hence from (1.28) and (1.29) it follows that

$$\frac{\partial}{\partial t}\rho(t) = \mathcal{L}(\psi(t)) + A((\widetilde{\omega}, \widetilde{J}), \rho(t)), \tag{1.30}$$

where A represents the error in approximating $(\mathcal{F}, \mathcal{G})$ by the linearization $\mathcal{L}$. As in [8] and [6] we have the following C^k bounds on A:

$$|A|_{C^k} \leq C(|\rho|_{C^k}|\nabla^2\rho|_{C^k} + |\nabla\rho|^2_{C^k}).$$

Therefore by Lemma 10 we have the following bounds on A:

$$|A|_{C^k} \leq C(|\psi|_{C^k}|\nabla^2\psi|_{C^k} + |\nabla\psi|^2_{C^k}). \tag{1.31}$$

Next we will use the definition of $\psi(t)$ and the evolution of $\rho(t)$ to derive an evolution equation for $\psi(t)$. By the definition of $\psi_2(t)$ and the $(1,1)$ part of $\psi_1(t)$ (see (1.17) and (1.22) respectively) we have

$$\frac{\partial}{\partial t}\psi_2(t) = \frac{\partial}{\partial t}\left(K^{1,0}_{0,1} + K^{0,1}_{1,0}\right) \tag{1.32}$$

$$\frac{\partial}{\partial t}\psi_1^{(1,1)}(t) = \frac{\partial}{\partial t}h^{(1,1)}(t). \tag{1.33}$$

For $\frac{\partial}{\partial t}\psi_1^{(2,0)+(0,2)}(t)$, it follows from (1.25) that

$$\frac{\partial}{\partial t}\psi_1^{(2,0)+(0,2)}(t) = \frac{\partial}{\partial t}h^{(2,0)+(0,2)}(t) + \left(\frac{\partial}{\partial t}\rho\right)*\rho, \tag{1.34}$$

since the $(2,0) + (0,2)$ components of $\psi_1(t)$ and $h(t)$ differ by terms that are quadratic in $\rho(t) = (h(t), K(t))$. Notice that by (1.30) and (1.31) we have $\frac{\partial}{\partial t}\rho$ is second order in ψ and hence the final term in (1.34) may be absorbed in the error estimate A. Therefore from (1.32), (1.33), (1.34) and (1.30) it follows that

$$\frac{\partial}{\partial t}\psi(t) = \mathcal{L}(\psi(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi(t)),$$

where A is a different tensor than in (1.30), but we still have $|A|_{C^k} \leq C(|\psi|_{C^k}|\nabla^2\psi|_{C^k} + |\nabla\psi|^2_{C^k})$. $\qquad\square$

Roughly speaking, the following theorem says that given any finite time $T > 0$, by

starting the flow very close to $(\widetilde{\omega}, \widetilde{J})$, the solution $(\omega(t), J(t))$ remains close to $(\widetilde{\omega}, \widetilde{J})$ on the interval $[0, T)$.

Theorem 12. *Given $T > 0$, $\epsilon' > 0$ and an integer $k \geq 0$, there exists $\epsilon = \epsilon(T, \epsilon', k) > 0$ such that if $|\rho(0)|_{C^\infty} < \epsilon$ then, $(\omega(t), J(t))$ exists on $[0, T)$ and moreover $|\rho(t)|_{C^k} < \epsilon'$ on $[0, T)$.*

Proof. First, for ϵ sufficiently small, work of Streets and Tian (see Theorem 1.1 in [7]) shows that there exists $T' > 0$ such that the solution $(\omega(t), J(t))$ exists on $[0, T']$ and moreover $|\rho(t)|_{C^k} < \epsilon'$ on $[0, T')$. Suppose by way of contradiction that there exists a maximal T'' so that for all $\epsilon > 0$ the solution exists and $|\rho(t)|_{C^k} < \epsilon'$ on $[0, T'')$ with $T'' < T$. Fix $\widetilde{T} < T''$. To derive a contradiction we will produce bounds on the C^k norm of $\rho(t)$ on $[0, \widetilde{T}]$ in terms of ϵ, independent of $\widetilde{T}$.

Recall from Lemma 10 that associated to $\rho(t)$ we have $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$. In order to obtain C^k estimates on $\rho(t)$ in terms of ϵ we will produce C^k bounds on $\psi(t)$ and employ (1.14). To this end, we study the evolution of $\psi(t)$. Recall from Lemma 11 part (1) that

$$\frac{\partial}{\partial t}\psi(t) = \mathcal{L}(\psi(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi(t)) \tag{1.35}$$

where $\mathcal{L}$ is negative semi-definite and A represents the error in approximating $(\mathcal{F}, \mathcal{G})$ by $\mathcal{L}$. From part (2) of Lemma 11 we have

$$|A|_{C^k} \leq C(|\psi|_{C^k}|\nabla^2\psi|_{C^k} + |\nabla\psi|^2_{C^k}). \tag{1.36}$$

Notice that C depends on the C^k norm of $\rho(t)$ which we are assuming is bounded by ϵ' for $t \in [0, T'') \supset [0, \widetilde{T}]$.

In the estimates that follow Rm will denote the curvature of the fixed metric $\widetilde{g}$ and ∇ will denote the Levi-Civita connection of $\widetilde{g}$. Moreover we will use the fact that M is compact and hence there exists a constant C such that $|\operatorname{Rm}|_{C^\infty} < C$.

1.2.1 L^2 bounds of ψ in terms of ϵ

The linear stability of Kähler-Einstein structures will allow us to produce L^2 bounds on $\psi(t)$ in terms of ϵ which are independent of $\widetilde{T}$. Indeed, for $t \in [0, \widetilde{T}]$ by (1.35) and using that $\langle \mathcal{L}_{(\widetilde{\omega},\widetilde{J})} \cdot, \cdot \rangle_{L^2(\widetilde{g})} \leq 0$, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} = \int_M \left\langle \frac{\partial}{\partial t}\psi, \psi \right\rangle \leq \int_M A * \psi. \tag{1.37}$$

Now, using the bound on A given in (1.36), we see that

$$\int A * \psi = \int \psi^{*2} * \nabla^2 \psi + \nabla \psi^{*2} * \psi.$$

Using integration by parts on the second term yields

$$\int_M A * \psi \leq \int_M \psi^{*2} * \nabla^2 \psi. \tag{1.38}$$

For $t \in [0, \widetilde{T}]$, by assumption and (1.12), $|\psi(t)|_{C^k} < \epsilon'$ therefore $\int_M \psi^{*2} * \nabla^2 \psi \leq C\epsilon' \int_M |\psi|^2$. Hence combining (1.37) and (1.38) we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} \leq C_1\epsilon' \int_M |\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}}. \tag{1.39}$$

Therefore for any $t \in [0, \widetilde{T}]$,

$$|\psi(t)|_{L^2}^2 \le e^{C_1 \epsilon' T} |\psi_0|_{L^2}^2 \le \epsilon e^{C_1 \epsilon' T}. \tag{1.40}$$

1.2.2 $L^{1,2}$ bounds of ψ in terms of ϵ

Given the L^2 bounds above, we bootstrap to obtain higher-order bounds. Notice that linear stability was only used to start the bootstrapping process. Using (1.3), (1.4), (1.5) and (1.35) we have

$$\frac{1}{2} \frac{\partial}{\partial t} \int_M |\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} = \int_M \left\langle \frac{\partial}{\partial t} \psi, \psi \right\rangle = \int_M \left\langle - \nabla^* \nabla \psi + \mathrm{Rm} * \psi + A, \psi \right\rangle. \tag{1.41}$$

Since M is compact there exists a constant C_2 such that $|\mathrm{Rm}(\widetilde{g})|_{C^\infty} < C_2$. Moreover, we can bound the term associated with A as we did in (1.38) and (1.39) to get

$$\frac{1}{2} \frac{\partial}{\partial t} \int_M |\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} \le - \int |\nabla \psi|^2 dvol_{\widetilde{g}} + C_3 \int |\psi|^2 dvol_{\widetilde{g}}. \tag{1.42}$$

Integrating from 0 to $\widetilde{T}$, we see that

$$\int_0^{\widetilde{T}} \int_M |\nabla \psi|^2 + \frac{1}{2} \int_M |\psi(\widetilde{T})|^2 \le \frac{1}{2} \int_M |\psi_0|^2 + C_3 \int_0^{\widetilde{T}} \int_M |\psi|^2.$$

Now the L^2 bounds from (1.40) imply

$$\int_0^{\widetilde{T}} \int_M |\nabla \psi|^2 \le C_4 T e^{C_1 \epsilon' T} |\psi_0|_{L^2}^2 \le C_4 T e^{C_1 \epsilon' T} \epsilon. \tag{1.43}$$

17

1.2.3 $L^{2,2}$ bounds of ψ in terms of ϵ

Next, we use the $L^{1,2}$ bounds above to produce $L^{2,2}$ bounds. Similar to (1.41),

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\nabla\psi|^2_{\widetilde{g}}dvol_{\widetilde{g}} = \int_M \langle\nabla(-\nabla^*\nabla\psi + \mathrm{Rm}*\psi + A), \nabla\psi\rangle. \tag{1.44}$$

First consider the term $\int\langle\nabla(-\nabla^*\nabla\psi), \nabla\psi\rangle$ above. Commuting covariant derivatives and using integration by parts we get

$$\int_M \langle\nabla(-\nabla^*\nabla\psi), \nabla\psi\rangle = -\int_M |\nabla^2\psi|^2 + \int_M \mathrm{Rm}*\nabla\psi*\nabla\psi. \tag{1.45}$$

Next we obtain estimates on the term $\int\langle\nabla(\mathrm{Rm}*\psi), \nabla\psi\rangle = \int\langle\nabla\,\mathrm{Rm}*\psi + \mathrm{Rm}*\nabla\psi, \nabla\psi\rangle$ from equation (1.44). Since $|\mathrm{Rm}|_{C^\infty} < C_2$, we can use Young's Inequality, to show

$$\int_M \langle\nabla(\mathrm{Rm}*\psi), \nabla\psi\rangle \leq C'\int_M |\nabla\psi|^2 + C''\int_M |\psi|^2. \tag{1.46}$$

Finally, we consider the final term in (1.44), $\int \nabla A * \nabla\psi$. Using the estimates on A from (1.36), we have

$$\int \nabla A * \nabla\psi = \int \nabla\psi^{*2} * \nabla^2\psi + \int \psi * \nabla\psi * \nabla^3\psi.$$

Integration by parts on the last term yields $\int \nabla A * \nabla\psi = \int \nabla\psi^{*2} * \nabla^2\psi + \int \psi * \nabla^2\psi^{*2}$ and hence

$$\int_M \langle\nabla A, \nabla\psi\rangle \leq C'''\int_M |\nabla\psi|^2 + C_7\epsilon'\int_M |\nabla^2\psi|^2 \tag{1.47}$$

since $|\psi|_{C^k} < \epsilon'$ for $t \in [0, \widetilde{T}]$.

Combining (1.44), (1.45), (1.46), and (1.47) we see that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\nabla\psi|^2_{\widetilde{g}} dvol_{\widetilde{g}} \leq -\int_M |\nabla^2\psi|^2 + C_5\int_M |\psi|^2 + C_6\int_M |\nabla\psi|^2 + C_7\epsilon'\int_M |\nabla^2\psi|^2.$$
(1.48)

Hence, we choose ϵ' small enough so that $C_7\epsilon' < \frac{1}{2}$. Integrating (1.48) from 0 to $\widetilde{T}$ we have

$$\int_0^{\widetilde{T}}\int_M |\nabla^2\psi|^2 + \int_M |\nabla\psi(\widetilde{T})|^2 \leq \int_M |\nabla\psi_0|^2 + 2C_5\int_0^{\widetilde{T}}\int_M |\psi|^2 + 2C_6\int_0^{\widetilde{T}}\int_M |\nabla\psi|^2.$$
(1.49)

Therefore, using the L^2 estimate from (1.40) and the $L^{1,2}$ estimate from (1.43) we have

$$\int_0^{\widetilde{T}}\int_M |\nabla^2\psi|^2 \leq C_7 T e^{C_1\epsilon'T}|\psi_0|^2_{L^{1,2}} \leq C_7 T e^{C_1\epsilon'T}\epsilon.$$
(1.50)

Notice that (1.49) also gives bounds on $|\nabla\psi(\widetilde{T})|^2_{L^2}$. Moreover by integrating (1.48) from 0 to t for $t \in [0, \widetilde{T}]$ these bounds hold not just at $\widetilde{T}$ but for any $t \in [0, \widetilde{T}]$. Hence we also have

$$\sup_{[0,\widetilde{T}]} |\nabla\psi|^2_{L^2} \leq C_7 T e^{C_1\epsilon'T}\epsilon.$$

Now since $\frac{\partial}{\partial t}\psi$ is second order in ψ, estimate (1.50) also gives $\int_0^{\widetilde{T}}\int_M |\frac{\partial}{\partial t}\psi|^2 \leq C T e^{C_1\epsilon'T}\epsilon$.
Next we use induction to show that for any p we have both:

$$\int_0^{\widetilde{T}}\int_M |\nabla^p\psi|^2 \leq C(p)T e^{C_1\epsilon'T}\epsilon$$

19

and

$$\sup_{[0,\widetilde{T}]} |\nabla^{p-1}\psi|_{L^2}^2 \leq C(p)Te^{C_1\epsilon'T}\epsilon.$$

1.2.4 $L^{m+1,2}$ bounds on ψ given $L^{m,2}$ bounds

To produce $L^{m+1,2}$ bounds on ψ given $L^{s,2}$ estimates for $s = 1, 2, \ldots, m$ we compute the evolution of the L^2 norm of $\nabla^m\psi$.

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\nabla^m\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} = \int_M \langle \nabla^m(-\nabla^*\nabla\psi + \mathrm{Rm}*\psi + A), \nabla^m\psi\rangle. \qquad (1.51)$$

First we consider the term $\int \langle \nabla^m(-\nabla^*\nabla\psi), \nabla^m\psi\rangle$ above. Similar to, (1.45) commuting the covariant derivatives and using integration by parts yields

$$\int_M \langle \nabla^m(-\nabla^*\nabla\psi), \nabla^m\psi\rangle = -\int_M |\nabla^{m+1}\psi|^2 + \int_M \sum_{j=0}^{m-1} \nabla^j\mathrm{Rm}*\nabla^{m-j}\psi*\nabla^m\psi.$$

Furthermore, using that $|\mathrm{Rm}|_{C^\infty} < C_2$ and employing Young's Inequality on each of the final $m-1$ terms on the right-hand side we have that there exists a constant C' such that:

$$\int_M \langle \nabla^m(-\nabla^*\nabla\psi), \nabla^m\psi\rangle \leq -\int_M |\nabla^{m+1}\psi|^2 + C'\sum_{j=0}^m |\nabla^j\psi|_{L^2}^2. \qquad (1.52)$$

Next we study the term $\int \langle \nabla^m(\mathrm{Rm}*\psi), \nabla^m\psi\rangle$ from equation (1.51). Again using that $|\mathrm{Rm}|_{C^\infty} < C_2$, by Young's Inequality we see that

$$\int_M \langle \nabla^m(\mathrm{Rm}*\psi), \nabla^m\psi\rangle = \int_M \nabla^m\psi*\nabla^m\psi + \cdots + \int_M \psi*\psi$$

and hence there exists a constant C'' such that

$$\int_M \langle \nabla^m(\mathrm{Rm} * \psi), \nabla^m \psi \rangle \leq C'' \sum_{j=0}^m |\nabla^j \psi|_{L^2}^2. \tag{1.53}$$

Finally consider the term $\int \langle \nabla^m A, \nabla^m \psi \rangle$ from (1.51). Again we use the estimates on A from (1.36). Here we have

$$\int_M \langle \nabla^m A, \nabla^m \psi \rangle \leq \int_M \sum_{j=0}^m \nabla^{j+2}\psi * \nabla^{m-j}\psi * \nabla^m \psi + \int_M \sum_{j=0}^m \nabla^{j+1}\psi * \nabla^{m+1-j}\psi * \nabla^m \psi.$$

We will now show how to estimate the highest order terms in the right-hand side of the above inequality. First we rewrite the right-hand side as $\int \nabla^{m+2}\psi * \nabla^m \psi * \psi + \nabla^{m+1}\psi * \nabla^m \psi * \nabla\psi +$ lower order terms. Integration by parts on the first term yields

$$\int \langle \nabla^m A, \nabla^m \psi \rangle \leq \int \nabla^{m+1}\psi * \nabla^{m+1}\psi * \psi + \nabla^{m+1}\psi * \nabla^m \psi * \nabla\psi + \text{lower order terms}.$$

$$\tag{1.54}$$

Next we use Young's Inequality on the second term on the right-hand side. In particular, Young's Inequality is written $ab \leq \eta a^2 + C(\eta)b^2$ where $\eta > 0$ can be taken arbitrarily small at the expense of making $C(\eta)$ large. Hence by Young's Inequality,

$$\int \nabla^{m+1}\psi * \nabla^m \psi * \nabla\psi \leq \eta \int |\nabla^{m+1}\psi|^2 + C(\eta) \int |\nabla^m \psi|^2 |\nabla\psi|^2. \tag{1.55}$$

Therefore combining (1.54) and (1.55) and using that $|\psi|_{C^k} < \epsilon'$ for $t \in [0, \widetilde{T}]$ we get $\int_M \langle \nabla^m A, \nabla^m \psi \rangle \leq (C_8 \epsilon' + \eta) \int_M |\nabla^{m+1}\psi|^2 + C_9 \int_M |\nabla^m \psi|^2 + \text{lower order terms. Hence}$

21

we choose $\eta = \frac{1}{4}$ and ϵ' sufficiently small so that

$$C_8 \epsilon' < \frac{1}{4}. \qquad (1.56)$$

And so,

$$\int_M \langle \nabla^m A, \nabla^m \psi \rangle \leq \frac{1}{2} \int_M |\nabla^{m+1}\psi|^2 + C_9 \int_M |\nabla^m \psi|^2 + \text{lower order terms.} \qquad (1.57)$$

We now have estimates for each term in the evolution of the L^2 norm of $\nabla^m \psi$ given in (1.51). In particular combining (1.51) (1.52), (1.53), and (1.57) we have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\nabla^m \psi|^2_{\widetilde{g}} dvol_{\widetilde{g}} \leq -\frac{1}{2}\int_M |\nabla^{m+1}\psi|^2 + C_{10}\sum_{j=0}^m |\nabla^j \psi|^2_{L^2}. \qquad (1.58)$$

Integrating from 0 to $\widetilde{T}$ we get

$$\int_0^{\widetilde{T}}\int_M |\nabla^{m+1}\psi|^2 + \int_M |\nabla^m \psi(\widetilde{T})|^2 \leq \int_M |\nabla^m \psi(0)|^2 + 2C_{10}\int_0^{\widetilde{T}}\sum_{j=0}^m |\nabla^j \psi|^2_{L^2}. \qquad (1.59)$$

Now we can employ the $L^{s,2}$ estimates for $s = 1, \ldots, m$ to get $L^{m+1,2}$ bounds. In particular,

$$\int_0^{\widetilde{T}}\int_M |\nabla^{m+1}\psi|^2 \leq C_{11}\widetilde{T}e^{C_1 \epsilon' T}|\psi_0|^2_{L^{m,2}} \leq C_{11}\widetilde{T}e^{C_1 \epsilon' T}\epsilon.$$

Notice that (1.59) also gives bounds on $|\nabla^m \psi(\widetilde{T})|^2_{L^2}$. Moreover by integrating (1.58) from 0 to t for any $t \in [0, \widetilde{T}]$, the bound on $|\nabla^m \psi|^2_{L^2}$ holds not just at $\widetilde{T}$ but for any $t \in [0, \widetilde{T}]$.

Hence we also have

$$\sup_{[0,\widetilde{T}]} |\nabla^m \psi|^2_{L^2} \le C_{11} T e^{C_1 \epsilon' T} |\psi_0|^2_{L^{m,2}} \le C_{11} T e^{C_1 \epsilon' T} \epsilon.$$

This proves that for any p

$$\int_0^{\widetilde{T}} \int_M |\nabla^p \psi|^2 \le C(p) T e^{C_1 \epsilon' T} |\psi_0|^2_{L^{p-1,2}} \le C(p) T e^{C_1 \epsilon' T} \epsilon \qquad (1.60)$$

and

$$\sup_{[0,\widetilde{T}]} |\nabla^{p-1} \psi|^2_{L^2} \le C(p) T e^{C_1 \epsilon' T} |\psi_0|^2_{L^{p-1,2}} \le C(p) T e^{C_1 \epsilon' T} \epsilon. \qquad (1.61)$$

Furthermore, since $\frac{\partial}{\partial t} \psi$ is second order in ψ, (1.60) also implies that $\int_0^{\widetilde{T}} \int_M |\frac{\partial^q}{\partial t^q} \nabla^r \psi|^2 \le C\epsilon$ for any $q, r > 0$, where C is independent of $\widetilde{T}$.

Now use the Sobolev Embedding Theorem, with respect to $\widetilde{g}$, to obtain C^k bounds on ψ in terms of ϵ. And hence by (1.14) we have C^k bounds on ρ in terms of ϵ. In [7], Theorem 1.9, Streets and Tian prove that if there is a finite time singularity τ of the flow, then $\lim_{t \to \tau} \sup\{|\operatorname{Rm}|_{C^0}, |DT|_{C^0}, |T|^2_{C^0}\} = \infty$. Here D denotes the Levi-Citia connection and Rm is the curvature of D. Therefore, the fact that the estimates above are independent of $\widetilde{T}$ implies that the solution exists on $[0, T'']$. Again, using the short-time existence result of Streets and Tian ([7]), the solution can be extended past time T''. Moreover, for ϵ sufficiently small, we maintain the C^k estimates on $\rho(t)$ past time T''. This contradicts the maximality of T''. $\qquad \square$

Chapter 2

Dynamic Stability when $c_1(\widetilde{J}) < 0$

In this chapter we prove that when $c_1(\widetilde{J}) < 0$, VNAHCF converges exponentially to the Kähler-Einstein structure $(\widetilde{\omega}, \widetilde{J})$. As above we let $\rho(t) = (\omega(t) - \widetilde{\omega}, J(t) - \widetilde{J})$.

Theorem 13. *Let $(M^{2n}, \widetilde{\omega}, \widetilde{J})$ be a closed complex manifold where $(\widetilde{\omega}, \widetilde{J})$ is a Kähler-Einstein structure such that $c_1(\widetilde{J}) < 0$. Given a positive integer k, there exists $\epsilon = \epsilon(k) > 0$ such that if $(\omega(0), J(0))$ is an almost hermitian structure with $\left|\rho(0)\right|_{C^\infty} < \epsilon$, then the solution to the volume-normalized AHCF starting at $(\omega(0), J(0))$ exists for all time and converges exponentially in C^k to $(\widetilde{\omega}, \widetilde{J})$.*

Proof. We prove Theorem 13 using two lemmas. As in Chapter 2 we have to deal with the non-linearity of the space of almost hermitian structures. To prove Theorem 13 we will show that there exists ϵ so that if $\left|\rho(0)\right|_{C^\infty} < \epsilon$, then $\psi(t)$ exponentially decays in C^k. Finally employing (1.14) exponential C^k decay of $\rho(t)$ will follow from exponential C^k decay of $\psi(t)$.

Lemma 14. *Given $\delta > 0$ and an integer $k \geq 0$, there exists $\epsilon_1 = \epsilon_1(\delta, k) > 0$ such that if $\left|\rho(0)\right|_{C^\infty} < \epsilon_1$ then $\left|\psi(t)\right|_{C^k} < \delta$ for all $t \geq 0$ and moreover $|\psi(t)|_{L^2}^2 \leq Ce^{-\lambda t}$ for all $t \geq 0$.*

Proof. As in Section 1.2, let $\lambda = \min\{|\lambda_i| : \lambda_i \text{ is an eigenvalue of } \mathcal{L}\}$. Further let $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ be the element of the tangent space, from Lemma 10, associated to $\rho(t)$. Recall that

from Corollary 9 that $c_1(\widetilde{J}) < 0$ implies that

$$\int_M \langle \mathcal{L}_{(\widetilde{\omega},\widetilde{J})}\psi, \psi \rangle_{\widetilde{g}} \le -\lambda|\psi|^2_{L^2(\widetilde{g})}. \tag{2.1}$$

And by (1.35),

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\psi|^2_{\widetilde{g}}dvol_{\widetilde{g}} = \int_M \left\langle \frac{\partial}{\partial t}\psi, \psi \right\rangle = \int_M \left\langle \mathcal{L}\psi + A, \psi \right\rangle. \tag{2.2}$$

Then for any t for which $|\psi(t)|_{C^2} < \delta$, we can employ the bound

$$\int \langle A, \psi \rangle \le C_1\delta|\psi|^2_{L^2} \tag{2.3}$$

derived in (1.38) and (1.39). Combining (2.1), (2.2) and (2.3) yields

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\psi|^2_{\widetilde{g}}dvol_{\widetilde{g}} \le -\lambda|\psi|^2_{L^2} + C_1\delta|\psi|^2_{L^2}.$$

Here we choose δ so that

$$C_1\delta < \frac{1}{2}\lambda. \tag{2.4}$$

Integrating from 0 to t yields L^2 exponential decay of $\psi(t)$. In particular,

$$|\psi(t)|^2_{L^2} \le e^{-\lambda t}|\psi(0)|^2_{L^2} \tag{2.5}$$

for any t for which $|\psi(t)|_{C^2} < \delta$. Therefore to complete the lemma, we will show that

25

given $k \geq 2$, there exists $\epsilon_1(k, \delta) > 0$ such that if $|\rho(0)|_{C^\infty} < \epsilon_1$, then $|\psi(t)|_{C^k} < \delta$ for all $t \in [0, \infty)$. Notice again that from (1.12) it follows that $|\rho(0)|_{C^\infty} < \epsilon_1$ implies $|\psi(0)|_{C^\infty} < \epsilon_1$.

By Theorem 12 we know that given $T > 0$, $k \geq 0$ and $\epsilon' > 0$, there exists $\epsilon > 0$ such that $|\rho(0)|_{C^\infty} < \epsilon$ implies that $|\psi(t)|_{C^k} < \epsilon'$ on $[0, T)$. We apply Theorem 12 with $\epsilon' = \delta$ and δ sufficiently small so that (2.4) holds. Let ϵ_2 denote a constant that is small enough so that if $|\psi(0)|_{C^\infty} < \epsilon_2$ then $|\psi(t)|_{C^k} < \delta$ on $[0, T)$ and assume that

$$|\psi(0)|_{C^\infty} < \epsilon_2. \tag{2.6}$$

Given T, let $t_0 < t < T$, then integrating (1.42) from t_0 to t we have

$$\int_{t_0}^t |\nabla \psi|_{L^2}^2 \leq \frac{1}{2} |\psi(t_0)|_{L^2}^2 + C_3 \int_{t_0}^t |\psi(s)|_{L^2}^2. \tag{2.7}$$

Furthermore since (2.5) holds on $[0, T)$,

$$\int_{t_0}^t |\psi(s)|_{L^2}^2 \leq \int_{t_0}^t e^{-\lambda s} |\psi(0)|_{L^2}^2 = \frac{1}{\lambda} e^{-\lambda t_0} |\psi(0)|_{L^2}^2. \tag{2.8}$$

Therefore combining (2.7) and (2.8) yields

$$\int_{t_0}^t |\nabla \psi|_{L^2}^2 \leq \frac{1}{2} |\psi(t_0)|_{L^2}^2 + \frac{C_3}{\lambda} e^{-\lambda t_0} |\psi(0)|_{L^2}^2 \leq \left(\frac{1}{2} + \frac{C_3}{\lambda} \right) e^{-\lambda t_0} |\psi(0)|_{L^2}^2. \tag{2.9}$$

The last inequality is again by (2.5). The key observation here is that the $L^{1,2}$ estimate in (2.9) is independent of t.

Next, to obtain a similar $L^{2,2}$ estimate we integrate (1.48) from t_0 to t.

$$\int_{t_0}^{t} |\nabla^2 \psi|_{L^2}^2 + |\nabla \psi(t)|_{L^2}^2 \leq |\nabla \psi(t_0)|_{L^2}^2 + C_5 \int_{t_0}^{t} |\psi(t)|_{L^2}^2 + C_6 \int_{t_0}^{t} |\nabla \psi(t)|_{L^2}^2. \qquad (2.10)$$

Bounding the last two terms of (2.10) using (2.8) and (2.9) yields

$$\int_{t_0}^{t} |\nabla^2 \psi|_{L^2}^2 + |\nabla \psi(t)|_{L^2}^2 \leq |\nabla \psi(t_0)|_{L^2}^2 + Ce^{-\lambda t_0} |\psi(0)|_{L^2}^2.$$

Again the key observation is that the estimate above is independent of t.

Using the same inductive argument as in the proof of Theorem 12 shows that for any p,

$$\int_{t_0}^{t} |\nabla^p \psi|_{L^2}^2 + |\nabla^{p-1} \psi(t)|_{L^2}^2 \leq C_1(p)|\psi(t_0)|_{L^{p-1,2}}^2 + C_2(p)e^{-\lambda t_0} \qquad (2.11)$$

where $C_1(p)$ and $C_2(p)$ are independent of t. Notice that there exists a constant C, such that $|\psi(t_0)|_{L^{p-1,2}}^2 \leq C|\psi(t_0)|_{C^{p-1}}$. Hence by (2.11) we have $|\nabla^{p-1} \psi(t)|_{L^2}^2 \leq C_1'(p)|\psi(t_0)|_{C^{p-1}} + C_2(p)e^{-\lambda t_0}$. Therefore applying the Sobolev Embedding Theorem we have

$$|\psi(t)|_{C^k} \leq C_1(k)|\psi(t_0)|_{C^{p-1}} + C_2(k)e^{-\lambda t_0} \qquad (2.12)$$

where $C_1(k)$ and $C_2(k)$ are independent of t.

Since (2.12) is independent of t, to prove that $|\psi(t)|_{C^k} < \delta$ for all $t \in [0, \infty)$, it suffices to show that there exists a constant ϵ_1 with $0 < \epsilon_1 \leq \epsilon_2$ and such that $|\rho(0)|_{C^\infty} < \epsilon_1$ implies that the right-hand side of (2.12) is bounded above by δ.

First we bound the second term on the right-hand side of (2.12). Notice that, given $\delta > 0$ small enough so that we have (2.4), the argument above which led to inequality (2.12)

holds under the assumption (2.6). Furthermore, notice that the estimate in (2.12) holds for $t_0 < T$, independent of T. Therefore we take T to be sufficiently large so that $T > t_0$ and

$$C_2(k)e^{-\lambda t_0} < \frac{1}{2}\delta. \tag{2.13}$$

To bound the first term in (2.12) we again use Theorem 12 with $\epsilon' = \frac{1}{2C_1(k)}\delta$ and $T > t_0$. Hence, by Theorem 12 there exists $\epsilon_3 > 0$ such that $|\rho(0)|_{C^\infty} < \epsilon_3$ implies that

$$|\psi|_{C^{p-1}} < \epsilon' = \frac{1}{2C_1(k)}\delta \tag{2.14}$$

for $t \in [0, T) \supset [0, t_0]$.

Finally, choose $\epsilon_1 = min\{\epsilon_2, \epsilon_3\}$. Hence combining (2.12), (2.13) and (2.14) proves that if $|\psi(0)|_{C^\infty} < \epsilon_1$, then (2.12) holds independent of t. Therefore it follows that $|\psi(t)|_{C^k} < \delta$ for all $t \geq 0$ and moreover the L^2 decay estimate in (2.5) holds for all $t \geq 0$. $\qquad \square$

To finish the proof of Theorem 13, we show that the L^2 decay estimate above and parabolic theory can be used to prove C^k decay of $\psi(t)$.

Lemma 15. *Given an integer $k \geq 2$, there exists $\delta = \delta(k) > 0$ such that if both $|\psi(t)|_{C^k} < \delta$ for all $t \in [0, \infty)$ and $|\psi(t)|_{L^2}^2 \leq Ce^{-\lambda t}$ then $|\psi(t)|_{C^k} \leq C(k)e^{-\lambda t}$.*

Proof. We begin the proof by deriving an $L^{1,2}$ exponential decay estimate. The same argument that was used to derive (1.41) shows that there exists a constant C_1 such that

$$\frac{\partial}{\partial t}|\psi|_{L^2}^2 \leq -|\nabla\psi|_{L^2}^2 + C_1|\psi|_{L^2}^2,$$

where C_1 depends on both $(\widetilde{\omega}, \widetilde{J})$ and $|\psi(t)|_{C^2}$; but by assumption $|\psi|_{C^2} < \delta$ for all $t \geq 0$.

Integrating from t to ∞ yields,

$$\int_t^\infty |\nabla \psi|_{L^2}^2 \le |\psi(t)|_{L^2}^2 + C_1 \int_t^\infty |\psi|_{L^2}^2 \le Ce^{-\lambda t}. \tag{2.15}$$

The last inequality follows from the assumed L^2 exponential decay estimate.

Next, for a fixed t, let $\theta(s)$ be a smooth function which is 0 for $s \in \left[t - 1, t - \frac{1}{2}\right]$, monotonically increasing from 0 to 1 for $s \in \left[t - \frac{1}{2}, t\right]$ and 1 for $s \ge t$. As we shall see below, $\theta(s)$ will be used to deal with boundary terms which arise in the parabolic estimates that follow. The same argument that was used to produce (1.48) shows that there exist constants such that

$$\frac{1}{2}\frac{\partial}{\partial t}\int_M |\nabla \psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} \le -\int_M |\nabla^2 \psi|^2 + C_2 \int_M |\psi|^2 + C_3 \int_M |\nabla \psi|^2 + C_4 \delta \int_M |\nabla^2 \psi|^2, \tag{2.16}$$

again these constants depend on both $(\widetilde{\omega}, \widetilde{J})$ and $|\psi(t)|_{C^2}$. Now we choose δ sufficiently small so that $C_4 \delta < \frac{1}{2}$. Hence using (2.16) and that both $\theta(s)$ and its derivative are uniformly bounded,

$$\frac{\partial}{\partial s}\left(\theta(s)|\nabla \psi(s)|_{L^2}^2\right) \le C_5|\nabla \psi(s)|_{L^2}^2 + C_6|\psi(s)|_{L^2}^2. \tag{2.17}$$

We integrate (2.17) in s from $t - \frac{1}{2}$ to t for $t \ge 1$. Using that $\theta\left(t - \frac{1}{2}\right) = 0$ and $\theta(t) = 1$ we

get

$$|\nabla\psi(t)|_{L^2}^2 \le C_5 \int_{t-\frac{1}{2}}^t |\nabla\psi(s)|_{L^2}^2 + C_6 \int_{t-\frac{1}{2}}^t |\psi(s)|_{L^2}^2$$

$$\le C_5 \int_{t-\frac{1}{2}}^\infty |\nabla\psi(s)|_{L^2}^2 + C_6 \int_{t-\frac{1}{2}}^\infty |\psi(s)|_{L^2}^2. \tag{2.18}$$

Hence using the L^2 decay assumption and (2.15) it follows from (2.18) that

$$|\nabla\psi(t)|_{L^2}^2 \le Ce^{-\lambda t}. \tag{2.19}$$

This proves exponential $L^{1,2}$ decay.

Next we prove $L^{2,2}$ decay. By (2.16) with δ small enough so that $C_4\delta < \frac{1}{2}$,

$$\frac{\partial}{\partial t}|\nabla\psi|_{L^2}^2 \le -|\nabla^2\psi|_{L^2}^2 + 2C_3|\nabla\psi|_{L^2}^2 + 2C_2|\psi|_{L^2}^2.$$

Integrating from t to ∞ yields

$$\int_t^\infty |\nabla^2\psi|_{L^2}^2 \le |\nabla\psi(t)|_{L^2}^2 + 2C_3 \int_t^\infty |\nabla\psi|_{L^2}^2 + 2C_2 \int_t^\infty |\psi|_{L^2}^2 \le Ce^{-\lambda t}. \tag{2.20}$$

Where (2.19), (2.15), and the L^2 decay assumption were used in the first, second, and third term on the right-hand side respectively.

Now, as in (1.56) we choose δ small enough so that (1.58) holds for $m = 2$. Therefore there exists a constant C_6 such that

$$\frac{1}{2}\frac{\partial}{\partial t} \int_M |\nabla^2\psi|_{\widetilde{g}}^2 dvol_{\widetilde{g}} \le -\frac{1}{2} \int_M |\nabla^3\psi|^2 + C_6 \sum_{j=0}^2 |\nabla^j\psi|_{L^2}^2.$$

30

Hence, there exists a constant C_7 so that

$$\frac{\partial}{\partial s}\left(\theta(s)|\nabla^2\psi(s)|_{L^2}^2\right) \leq C_7|\nabla^2\psi(s)|_{L^2}^2 + C_7|\nabla\psi(s)|_{L^2}^2 + C_7|\psi(s)|_{L^2}^2. \tag{2.21}$$

We integrate (2.21) in s from $t-\frac{1}{2}$ to t for $t \geq 1$. Using that $\theta\left(t-\frac{1}{2}\right) = 0$ and $\theta(t) = 1$ we get

$$|\nabla^2\psi(t)|_{L^2}^2 \leq C_7 \int_{t-\frac{1}{2}}^{t} \sum_{j=0}^{2} |\nabla^j\psi(s)|_{L^2}^2 \leq C_7 \int_{t-\frac{1}{2}}^{\infty} \sum_{j=0}^{2} |\nabla^j\psi(s)|_{L^2}^2.$$

Hence using (2.20), (2.15) and the assumed L^2 decay we get

$$|\nabla^2\psi(t)|_{L^2}^2 \leq Ce^{-\lambda t}.$$

This gives exponential $L^{2,2}$ decay.

Continuing in this way we get

$$|\psi(t)|_{L^{p,2}}^2 \leq C(p)e^{-\lambda t}.$$

Furthermore by the Sobolev Embedding Theorem we get $|\psi(t)|_{C^k} \leq C(k)e^{-\lambda t}$. $\qquad\square$

By Lemma 15 we know that given $k \geq 2$, there exists $\delta > 0$ so that if both $|\psi(t)|_{C^k} < \delta$ and $|\psi(t)|_{L^2}^2 \leq Ce^{-\lambda t}$ hold for all $t \geq 0$, then $|\psi(t)|_{C^k} \leq C(k)e^{-\lambda t}$ for all $t \geq 0$. Furthermore by Lemma 14 we know that there exists $\epsilon_1 > 0$ such that if $|\psi(0)|_{C^\infty} < \epsilon_1$, then both $|\psi(t)|_{C^k} < \delta$ and $|\psi(t)|_{L^2}^2 \leq Ce^{-\lambda t}$ hold for all $t \geq 0$. Hence let δ be determined by Lemma 15. To finish the proof of Theorem 13, we apply Lemma 14 with $\epsilon = \epsilon_1$ and note that by (1.14) exponential decay of $\rho(t)$ follows from exponential decay of $\psi(t)$. $\qquad\square$

Chapter 3

Dynamic Stability in the Calabi-Yau Case

3.1 The Kernel of $\mathcal{L}$ and the Space of Calabi-Yau Structures

In Chapter we proved Theorem 1 when $c_1(\widetilde{J}) < 0$ by using that, in this case, the linearization $\mathcal{L}$ is negative definite. However in the Calabi-Yau case, the kernel of $\mathcal{L}$ is non-trivial and so the non-linear part of the flow is no longer controlled by the linear part. In this chapter we will show that in the Calabi-Yau case we can find a Calabi-Yau structure to which the flow exponentially converges.

In order to find a Calabi-Yau structure to which the flow exponentially converges we will construct a sequence $\{(\omega_j, J_j)\}$ of successively better Calabi-Yau structures; in the sense that the solution $(\omega(t), J(t))$ to the VNAHCF converges exponentially on larger and larger intervals. Moreover we will prove that each of these Calabi-Yau structures is contained in a fixed neighborhood of the original Calabi-Yau structure (see Theorem 20 part (2)). This will allow us to extract a limit (ω_{KE}, J_{KE}) to which $\{(\omega_j, J_j)\}$ subconverges. One could imagine that if this sequence failed to converge that we would only be able to conclude that the solution becomes asymptotic to the space of Calabi-Yau structures.

In order to choose a new Calabi-Yau structure we will use Koiso's Theorem. Before stating Koiso's Theorem we need a definition.

Definition 16. Let $\mathcal{AC}$ denote the space of almost complex structures on M modulo diffeomorphism. A complex structure J is *unobstructed* if for any $\dot{J} \in T_J \mathcal{AC}$ such that $\dot{N}(\dot{J}) = 0$, there exists a path of complex structures $J(a)$ such that $J(0) = J$ and $\frac{\partial}{\partial a}\big|_{a=0} J(a) = \dot{J}$. Again, N denotes the Nijenhuis tensor.

Theorem 17. *(Koiso [5]) Let (ω, J) be a Kähler-Einstein structure on M. Assume that:*

  1. *the first Chern class of J is zero;*

  2. *J is unobstructed.*

Then the space of Kähler-Einstein structures, modulo diffeomorphism, around (ω, J) is a manifold.

In order to make use of Koiso's Theorem we employ a theorem of Tian and Todorov.

Theorem 18. *(Tian [9] and Todorov [10]) Let (M, J) be a closed Calabi-Yau manifold. Then J is unobstructed.*

Next we will describe the tangent space of Calabi-Yau structures at $(\widetilde{\omega}, \widetilde{J})$. Using the above two theorems we prove that the kernel of $\mathcal{L}$ is isomorphic to the tangent space of Calabi-Yau structures at $(\widetilde{\omega}, \widetilde{J})$. Let $\mathcal{U}$ denote the space of Calabi-Yau structures near $(\widetilde{\omega}, \widetilde{J})$ modulo diffeomorphism.

Lemma 19. *Let $(M, \widetilde{\omega}, \widetilde{J})$ be a closed Calabi-Yau manifold, then $T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{U} \cong \operatorname{Ker} \mathcal{L}$.*

Proof. Let $(\omega(a), J(a))$ be a one-parameter family of unit volume almost hermitian structures and write $\frac{\partial}{\partial a}\big|_{a=0}(\omega(a), J(a)) = (\dot{\omega}, \dot{J})$. First since Calabi-Yau structures are static under

33

the system (1.2), we have $T_{(\widetilde{\omega},\widetilde{J})}\mathcal{U} \subseteq \operatorname{Ker}\mathcal{L}$. To prove $\operatorname{Ker}\mathcal{L} \subseteq T_{(\widetilde{\omega},\widetilde{J})}\mathcal{U}$, let $(\dot{\omega}, \dot{J}) \in \operatorname{Ker}\mathcal{L}$.

We make the following claim. *If $\dot{J} \in \ker\dot{\mathcal{G}}$ then $\dot{J} \in \ker\dot{N}$.* To see this, first notice that from (1.10) and using that the scalar curvature $s_{\widetilde{g}} = 0$, if $\dot{J} \in \ker\dot{\mathcal{G}}$, then $\Delta_{\overline{\partial}}\dot{J} = 0$. By integrating we see that $\overline{\partial}\dot{J} = 0$. On the other hand, in coordinates, the Nijenhuis tensor is written

$$N^i_{jk} = J^p_j \partial_p J^i_k - J^p_k \partial_p J^i_j - J^i_p \partial_j J^p_k + J^i_p \partial_k J^p_j.$$

Hence,

$$\dot{N}^i_{jk} = \dot{J}^p_j \partial_p J^i_k + J^p_j \partial_p \dot{J}^i_k - \dot{J}^p_k \partial_p J^i_j - J^p_k \partial_p \dot{J}^i_j - \dot{J}^i_p \partial_j J^p_k - J^i_p \partial_j \dot{J}^p_k + \dot{J}^i_p \partial_k J^p_j + J^i_p \partial_k \dot{J}^p_j.$$

Now each of the terms above of the form $\dot{J} * \partial J$ can be written as $\dot{J} * \partial J = \dot{J}(\nabla J + \Gamma * J)$. So using normal, complex coordinates (with respect to the Calabi-Yau structure $(\widetilde{g}, \widetilde{J})$), at a point $p \in M$, we have that each of these terms vanish. Here we also made use of the fact that when $(\widetilde{g}, \widetilde{J})$ is Kähler the Chern connection coincides with the Levi-Civita connection and so $\widetilde{J}$ is parallel with respect to the connection. Next, since $\dot{J} \in \Lambda^{0,1} \otimes T^{1,0}$ in these normal, complex coordinates at $p \in M$, we have,

$$\dot{N}^i_{\overline{j}\overline{k}} = J^{\overline{p}}_{\overline{j}} \partial_{\overline{p}} \dot{J}^i_{\overline{k}} - J^{\overline{p}}_{\overline{k}} \partial_{\overline{p}} \dot{J}^i_{\overline{j}} - \dot{J}^i_p \partial_{\overline{j}} J^p_{\overline{k}} + \dot{J}^i_p \partial_{\overline{k}} J^p_{\overline{j}} = 0.$$

This proves the claim.

By Theorem 18 there exists a path of complex structures $J(a)$ where $J(0) = J$ and $\frac{d}{da}J\big|_{a=0} = \dot{J}$. Next, using (1.9) we have that $\dot{\omega} \in \ker\dot{\mathcal{F}}$ implies that $\Delta_d\dot{\omega} = 0$, that is $\dot{\omega}$ is harmonic. Therefore, by the Calabi-Yau Theorem ([1], [11], [12], also see Theorem 2.29 in

[3]) $\omega(a)$ is a variation through Kähler metrics such that $[\omega(a)] = [\omega_{KE}(a)]$, where $\omega_{KE}(a)$ is Ricci-flat. Moreover by the Hodge Decomposition Theorem there is a unique harmonic representative in each cohomology class. Hence $\omega(a) = \omega_{KE}(a)$ and we have that $\dot{\omega}$ arises as a variation through Calabi-Yau metrics. $\qquad\square$

Notice that $\Lambda^2(M) \times \mathrm{End}(TM)$ is an affine space which can be viewed as a vector space by taking $(\widetilde{\omega}, \widetilde{J})$ to be the origin. Throughout this chapter we will view $\Lambda^2(M) \times \mathrm{End}(TM)$ as a vector space. Let $\pi_0 : \Lambda^2(M) \times \mathrm{End}(TM) \to \mathrm{Ker}\,\mathcal{L}$ be the projection onto the kernel of $\mathcal{L}$.

Let $(\widetilde{\omega}, \widetilde{J})$ denote the Calabi-Yau structure from Theorem 1. Roughly speaking, we will next prove that there exists a better Calabi-Yau structure (ω_I, J_I); in the sense that the solution $(\omega(t), J(t))$ to VNAHCF exponentially converges to (ω_I, J_I) on an interval I (see Theorem 20 and Lemma 21). Throughout this chapter $\rho_I(t) \doteq (\omega(t) - \omega_I, J(t) - J_I)$ will quantify the distance the solution is from this new Calabi-Yau structure. As above let $\rho(t) = (\omega(t) - \widetilde{\omega}, J(t) - \widetilde{J})$. Notice that we may view both $\rho(t)$ and $\rho_I(t)$ as elements of $\Lambda^2(M) \times \mathrm{End}(TM)$.

As in Section 1.2 we have to deal with the non-linearity of the space of almost hermitian structures modulo diffeomorphism denoted $\mathcal{C}$. Notice that we may write $\rho_I(t) = \rho(t) - \widetilde{\rho}_I$ where $\widetilde{\rho}_I \doteq (\omega_I - \widetilde{\omega}, J_I - \widetilde{J})$. From Lemma 10 associated to $\rho(t)$ we have $\psi(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ and analogously associated to $\widetilde{\rho}_I$ we have $\widetilde{\psi}_I \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$. Hence associated to $\rho_I(t)$ we have $\psi_I(t) \in T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ defined by

$$\psi_I(t) \doteq \psi(t) - \widetilde{\psi}_I.$$

Moreover, employing the same argument as in the proof of Lemma 10 we have

$$|\psi_I(t)|_{C^k} \le |\rho_I(t)|_{C^k}, \tag{3.1}$$

$$|\rho_I(t)|_{L^2} \le |\psi_I(t)|_{L^2} + C|\psi_I(t)|_{L^2}^2 \tag{3.2}$$

and

$$|\rho_I(t)|_{C^k} \le |\psi_I(t)|_{C^k} + C|\psi_I(t)|_{C^k}^2. \tag{3.3}$$

Similarly by the proof of Lemma 11 we have

$$\frac{\partial}{\partial t}\psi_I(t) = \mathcal{L}(\psi_I(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi_I(t)) \tag{3.4}$$

where

$$|A((\widetilde{\omega}, \widetilde{J}), \psi_I(t))|_{C^k} \le C(|\psi_I(t)|_{C^k}|\nabla^2\psi_I(t)|_{C^k} + |\nabla\psi_I(t)|_{C^k}^2). \tag{3.5}$$

3.2 A New Calabi-Yau Structure

Next we will use the identification of the kernel of $\mathcal{L}$ and the tangent space of Calabi-Yau structures at $(\widetilde{\omega}, \widetilde{J})$, from Lemma 19, to find a new Calabi-Yau structure denoted (ω_I, J_I) such that $|\pi_0(\psi_I(t))|_{L^2}$ is small relative to $|\psi_I(t)|_{L^2}$. Furthermore we will show that the new Calabi-Yau structure is contained in a fixed neighborhood of the original Calabi-Yau

structure $(\widetilde{\omega}, \widetilde{J})$.

Theorem 20. *Given t_0 and $T > 0$, let $I = [t_0, t_0 + T]$. There exists $\delta(T, \widetilde{g})$ such that if $\sup_I |\psi(t)|_{C^k} < \delta$ with $k \geq 2$, then there exists a Calabi-Yau structure (ω_I, J_I) with the following properties:*

1. $|\pi_0(\psi_I)|^2_{L^2(\widetilde{g})} \leq \frac{1}{4}|\psi_I|^2_{L^2(\widetilde{g})}$ *on I*

2. $|(\omega_I - \widetilde{\omega}, J_I - \widetilde{J})|_{C^k} \leq C \sup_I |\psi|_{C^k}$.

Proof. First by Theorem 17 we know that $\mathcal{U}$ has a manifold structure near $(\widetilde{\omega}, \widetilde{J})$ and moreover by Lemma 19 we have $\operatorname{Ker} \mathcal{L} \cong T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{U}$.

By identifying $\operatorname{Ker} \mathcal{L}$ and $T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{U}$ we will view $(\widetilde{\omega}, \widetilde{J})$ as the origin of $\operatorname{Ker} \mathcal{L}$. Let $\Phi = \Phi_{(\widetilde{\omega}, \widetilde{J})} : \operatorname{Ker} \mathcal{L} \to \mathcal{U}$ denote the exponential map at $(\widetilde{\omega}, \widetilde{J})$. Now since $\mathcal{D}_{(\widetilde{\omega}, \widetilde{J})}\Phi$ is the identity map, the inverse function theorem may be applied to Φ. By the inverse function theorem there exists a neighborhood $V \subset \operatorname{Ker} \mathcal{L}$ of $(\widetilde{\omega}, \widetilde{J})$ on which the exponential map is invertible.

Let δ_1 be small enough so that

$$|\pi_0(\psi(t_0))|_{C^k} < \delta_1 \text{ implies that } \pi_0(\psi(t_0)) \in V. \tag{3.6}$$

Notice that if $\sup_I |\psi|_{C^k} < \delta_1$ then since $t_0 \in I$, it is clear that $|\pi_0(\psi(t_0))|_{C^k} < \delta_1$. Hence by the inverse function theorem there is a Calabi-Yau structure $(\omega_I, J_I) \in \mathcal{U}$ such that

$$(\Phi|_V)^{-1}\big((\omega_I, J_I)\big) = \pi_0(\psi(t_0)). \tag{3.7}$$

Applying Φ to each side of (3.7), it follows from the inverse function theorem that there

37

exists a constant C so that

$$|(\omega_I - \widetilde{\omega}, J_I - \widetilde{J})|_{C^k} \leq C|\pi_0(\psi(t_0))|_{C^k}$$

$$\leq C \sup_I |\psi|_{C^k}.$$

This proves (2).

Next, using that $(\Phi|_V)^{-1}((\omega_I, J_I)) = \pi_0((\omega_I - \widetilde{\omega}, J_I - \widetilde{J}))$, from (3.7) we have

$$\pi_0(\psi_I(t_0)) = 0. \tag{3.8}$$

In other words, there exists a Calabi-Yau structure (ω_I, J_I) such that at time t_0, $\psi_I(t)$ is orthogonal, with respect to $L^2(\widetilde{g})$, to $\operatorname{Ker} \mathcal{L}$.

To prove (1) we will carefully study the evolution of $\psi_I(t)$ starting at $t = t_0$ in order to get L^2 estimates on $\pi_0(\psi_I)$. First let $||\psi_I||_{M \times I} \doteq \int_I |\psi_I|_{L^2(\widetilde{g})}$ denote the L^2 norm on $M \times I$. Let $\{B_i\}$ be an orthonormal basis, with respect to $L^2(\widetilde{g})$, of $T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ determined by the eigenspace decomposition of $\mathcal{L}$. Then there exist constants c_i so that $\{c_i B_i e^{\lambda_i t}\}$ is an orthonormal basis, with respect to $||\cdot||_{M \times I}$, of $\operatorname{Ker}\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\big|_{M \times I}$ where λ_i is the eigenvalue associated to B_i.

We let $\pi^I(\psi_I(t))$ denote the projection of $\psi_I(t)$ onto $\operatorname{Ker}\left(\frac{\partial}{\partial t} - \mathcal{L}\right)\big|_{M \times I}$. In other words,

$$\frac{\partial}{\partial t}\pi^I(\psi_I(t)) = \mathcal{L}(\pi^I(\psi_I(t))). \tag{3.9}$$

From (3.8), we have $\pi^I(\psi_I(t_0)) = \sum_{\lambda_i \neq 0} k_i B_i$. It then follows that

$$\pi^I(\psi_I(t)) = \sum_{\lambda_i \neq 0} k_i B_i e^{\lambda_i(t-t_0)}. \tag{3.10}$$

We write

$$\psi_I(t) = \pi^I(\psi_I(t)) + \xi_I(t). \tag{3.11}$$

Since $\pi^I\big(\psi_I(t)\big)$ is orthogonal to $\mathrm{Ker}\,\mathcal{L}$ on I, it follows that for $t \in I$,

$$|\pi_0(\psi_I(t))| \leq |\xi_I(t)|. \tag{3.12}$$

Therefore to obtain estimates on $\pi_0\big(\psi_I(t)\big)$ we compute the evolution of $\xi_I(t)$. Moreover, from (3.10) and (3.11), since $\lambda_i < 0$ is bounded away from 0 for all i, we have that $\xi_I(t)$ converges exponentially to $\psi_I(t)$. Therefore there is a uniform constant C so that on I,

$$|\xi_I(t)| \leq C|\psi_I(t)|. \tag{3.13}$$

To compute the evolution of $\xi_I(t)$ we compare two evolution equations for $\psi_I(t)$. From (3.4), $\psi_I(t)$ satisfies $\frac{\partial}{\partial t}\psi_I(t) = \mathcal{L}(\psi_I(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi_I)$ and hence,

$$\frac{\partial}{\partial t}\psi_I(t) = \mathcal{L}(\pi^I(\psi_I(t))) + \mathcal{L}(\xi_I(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi_I(t)). \tag{3.14}$$

Furthermore, $\pi^I(\psi_I(t))$ satisfies (3.9) and so

$$\frac{\partial}{\partial t}\psi_I(t) = \frac{\partial}{\partial t}\left(\pi^I(\psi_I(t)) + \xi_I(t)\right) = \mathcal{L}(\pi^I(\psi_I(t))) + \frac{\partial}{\partial t}\xi_I(t). \qquad (3.15)$$

Combining equations (3.14) and (3.15) we have that $\xi_I(t)$ evolves by

$$\frac{\partial}{\partial t}\xi_I(t) = \mathcal{L}(\xi_I(t)) + A((\widetilde{\omega}, \widetilde{J}), \psi_I(t)). \qquad (3.16)$$

Recall that $\mathcal{L}$ is negative semi-definite; and so by (3.16), on I we have

$$\frac{\partial}{\partial t}|\xi_I(t)|^2_{L^2(\widetilde{g})} = 2\int_M \left\langle \frac{\partial}{\partial t}\xi_I(t), \xi_I(t)\right\rangle_{\widetilde{g}} dvol_{\widetilde{g}} \leq 2\int_M A\left((\widetilde{\omega}, \widetilde{J}), \psi_I(t)\right) * \xi_I(t).$$

Now using the bounds on A from (3.5), the same computation as (1.37) shows that

$$\frac{\partial}{\partial t}|\xi_I(t)|^2_{L^2(\widetilde{g})} \leq C_1 \int_M |\nabla^2\psi_I||\psi_I||\xi_I|.$$

Hence by (3.13),

$$\frac{\partial}{\partial t}|\xi_I(t)|^2_{L^2(\widetilde{g})} \leq C_2 \int_M |\nabla^2\psi_I||\psi_I|^2. \qquad (3.17)$$

Next we assume $\sup_I |\psi(t)|_{C^k} < \delta$ with $k \geq 2$ and $\delta \leq \delta_1$ where δ_1 is from (3.6). Using part (2) of Theorem 20 and the triangle inequality, from (3.17) it follows that on I

$$\frac{\partial}{\partial t}|\xi_I(t)|^2_{L^2} \leq C_3\delta|\psi_I(t)|^2_{L^2}.$$

40

Now since $\xi_I(t_0) = 0$,

$$|\xi_I(t)|^2_{L^2(\widetilde{g})} = \int_{t_0}^t \frac{\partial}{\partial s}|\xi_I(s)|^2_{L^2(\widetilde{g})} ds \le C_3 \delta \int_{t_0}^t |\psi_I(s)|^2_{L^2(\widetilde{g})} ds. \qquad (3.18)$$

Notice that since $\frac{\partial}{\partial t}\psi_I(t) = \frac{\partial}{\partial t}\psi(t)$ is second order in $\psi(t)$ and $\sup_I |\psi(t)|_{C^k} < \delta$ with $k \ge 2$, $\frac{\partial}{\partial t}\psi_I(t)$ is uniformly bounded in terms of δ and hence each $\psi_I(s)$ for $s \in I$ is uniformly equivalent. Therefore

$$|\pi_0(\psi_I(t))|^2_{L^2(\widetilde{g})} \le |\xi_I(t)|^2_{L^2(\widetilde{g})} \le C_4 \delta \int_{t_0}^t |\psi_I(t)|^2_{L^2(\widetilde{g})} ds = C_5 \delta(t - t_0)|\psi_I(t)|^2_{L^2(\widetilde{g})},$$

where the first inequality follows from (3.12) and the second is from (3.18). To finish the proof we choose δ small enough so that both $C_5 T \delta < \frac{1}{4}$ and $\delta \le \delta_1$ hold. $\qquad \square$

We will now use part (1) of Theorem 20 to prove L^2 exponential decay of $\psi_I(t)$ on I. Notice that by (3.3) this implies exponential decay of $\rho_I(t) = (\omega(t) - \omega_I, J(t) - J_I)$ on I.

Lemma 21. *Let I and (ω_I, J_I) be as in Theorem 20. There exists $\beta > 0$ such that if $|\psi|_{C^2} < \beta$ on I, then*

$$\sup_{\left[t_0 + \frac{1}{2}T, t_0 + T\right]} \int_M |\psi_I|^2 dvol_{\widetilde{g}} \le e^{-\frac{T\lambda}{2}} \sup_{\left[t_0, t_0 + \frac{1}{2}T\right]} \int_M |\psi_I|^2 dvol_{\widetilde{g}}$$

where $\lambda = min\{|\lambda_i| : \lambda_i$ is a non-zero eigenvalue of $\mathcal{L}\} > 0$.

Proof. We compute the evolution of $|\psi_I|^2_{L^2}$ and as in (3.4) we have

$$\frac{\partial}{\partial t} \int_M |\psi_I|^2 dvol_{\widetilde{g}} = 2 \int_M \langle \mathcal{L}(\psi_I), \psi_I \rangle dvol_{\widetilde{g}} + \int_M A\left((\widetilde{\omega}, \widetilde{J}), \psi_I(t)\right) * \psi_I dvol_{\widetilde{g}}. \qquad (3.19)$$

41

Recall that by the definition of π_0, $\psi_I - \pi_0(\psi_I)$ is the component of ψ_I orthogonal to the kernel of $\mathcal{L}$. Hence by the definition of λ,

$$2 \int_M \langle \mathcal{L}(\psi_I), \psi_I \rangle dvol_{\widetilde{g}} \leq -2\lambda |\psi_I - \pi_0(\psi_I)|_{L^2}^2 \tag{3.20}$$

$$\leq -2\lambda \left(|\psi_I|_{L^2}^2 - |\pi_0(\psi_I)|_{L^2}^2 \right). \tag{3.21}$$

Let δ be the constant from Theorem 20. By Theorem 20 part (1), if $\sup_I |\psi(t)|_{C^2} < \beta_1$ with $\beta_1 \leq \delta$, then from (3.20) and (3.21) it follows that

$$2 \int_M \langle \mathcal{L}(\psi_I), \psi_I \rangle dvol_{\widetilde{g}} \leq -\frac{3}{2} \lambda |\psi_I|_{L^2}^2. \tag{3.22}$$

Next consider the term $\int A * \psi_I$ from (3.19). We use the estimate on A from (3.5) to bound $\int A * \psi$. Notice that if $|\psi_I|_{C^2} < \beta_3$, then as in (1.38),

$$\int_M A \left((\widetilde{\omega}, \widetilde{J}), \psi_I(t) \right) * \psi_I dvol_{\widetilde{g}} \leq C \beta_3 |\psi_I|_{L^2}^2. \tag{3.23}$$

Now we choose β_3 small enough so that

$$C\beta_3 < \frac{\lambda}{2}. \tag{3.24}$$

Let β_2 be sufficiently small so that $|\psi|_{C^2} < \beta_2$ on I implies that $|\psi_I|_{C^2} < \beta_3$ on I. Notice that this can be done using the triangle inequality and part (2) of Theorem 20.

Finally we choose $\beta = min\{\beta_1, \beta_2\}$. Combining (3.19), (3.22), (3.23) and (3.24) it follows that if $|\psi|_{C^2} < \beta$ on I, then

$$\frac{\partial}{\partial t} \int_M |\psi_I|^2 dvol_{\widetilde{g}} \le -\lambda \int_M |\psi_I|^2 dvol_{\widetilde{g}}. \qquad (3.25)$$

Integrating from t_0 to t gives

$$|\psi_I(t)|_{L^2}^2 \le e^{-\lambda(t-t_0)} |\psi_I(t_0)|_{L^2}^2. \qquad (3.26)$$

Finally since (3.25) implies that $|\psi_I(t)|_{L^2}^2$ is decreasing, plugging $t_0 + \frac{1}{2}T$ into (3.26) proves the lemma. $\qquad \square$

This gives exponential L^2 decay of $\psi_I(t)$ on I. Next we prove a general result about parabolic flows (cf. Lemma 8.8 in [8]). The following lemma says roughly that exponential decay at a later time implies exponential decay at an earlier time.

Lemma 22. *There exists $\nu > 0$ so that if κ solves the parabolic flow equation $\frac{\partial}{\partial t}\kappa = \mathcal{L}(\kappa) + A(\kappa)$ and $|\kappa(t)|_{C^k} < \nu$ for all $t \in [0, t_0 + T]$, then*

$$\sup_{\left[t_0 + \frac{1}{2}T, t_0 + T\right]} \int_M |\kappa|^2 \le e^{-\frac{T\lambda}{2}} \sup_{\left[t_0, t_0 + \frac{1}{2}T\right]} \int_M |\kappa|^2 \qquad (3.27)$$

implies that

$$\sup_{\left[t_0, t_0 + \frac{1}{2}T\right]} \int_M |\kappa|^2 \le e^{-\frac{T\lambda}{2}} \sup_{\left[t_0 - \frac{1}{2}T, t_0\right]} \int_M |\kappa|^2. \qquad (3.28)$$

Proof. Suppose, by way of contradiction, that the lemma fails to hold. Then there is a sequence $\nu_i \to 0$ with $\kappa_i(t)$ solving $\frac{\partial}{\partial t}\kappa_i = \mathcal{L}(\kappa_i) + A(\kappa_i)$ and $|\kappa_i|_{C^k} < \nu_i$ on $[0, t_0 + T]$ and

moreover (3.27) holds but (3.28) does not. Parabolically rescale the solution κ_i; that is let $\widetilde{\kappa}(t)_i = \nu_i^{-1}\kappa_i(\nu_i t)$. Now for all i, $|\widetilde{\kappa}_i|_{C^k} < 1$ on $[0, \nu_i^{-1}(t_0 + T)] \supset [0, t_0 + T]$ and so by compactness we get a convergent subsequence $\widetilde{\kappa}(t)_i \to \widetilde{\kappa}(t)_\infty$ on $[0, t_0 + T]$ as $i \to \infty$. Now since $\widetilde{\kappa}_i$ solves $\frac{\partial}{\partial t}\widetilde{\kappa}_i = \mathcal{L}(\nu_i\widetilde{\kappa}_i) + A(\nu_i\widetilde{\kappa}_i)$ and $A(\kappa)$ is quadratic in κ this implies that $\widetilde{\kappa}_\infty(t)$ solves the linear equation

$$\frac{\partial}{\partial t}\widetilde{\kappa}_\infty = \mathcal{L}(\widetilde{\kappa}_\infty). \tag{3.29}$$

Furthermore since (3.27) and (3.28) are scale invariant it follows that for $\widetilde{\kappa}_\infty$ (3.27) holds but (3.28) does not. This is a contradiction.

To see the contradiction, first notice that (3.29) implies that

$$\frac{\partial}{\partial t}|\widetilde{\kappa}_\infty|_{L^2}^2 \le 0 \tag{3.30}$$

as $\mathcal{L}$ is negative semi-definite. It then follows that

$$|\widetilde{\kappa}_\infty(t_0 + \tfrac{1}{2}T)|_{L^2}^2 = \sup_{\left[t_0+\frac{1}{2}T, t_0+T\right]} |\widetilde{\kappa}_\infty|_{L^2}^2 \tag{3.31}$$

$$\le e^{-\frac{T\lambda}{2}} \sup_{\left[t_0, t_0+\frac{1}{2}T\right]} |\widetilde{\kappa}_\infty|_{L^2}^2 \tag{3.32}$$

$$= e^{-\frac{T\lambda}{2}} |\widetilde{\kappa}_\infty(t_0)|_{L^2}^2, \tag{3.33}$$

where the inequality follows from (3.27). As above, let $\{B_i\}$ be an orthonormal basis, with respect to $L^2(\widetilde{g})$, of $T_{(\widetilde{\omega}, \widetilde{J})}\mathcal{C}$ determined by the eigenspace decomposition of $\mathcal{L}$. We can now

44

write $\widetilde{\kappa}_\infty(t)$ with respect to this basis,

$$\widetilde{\kappa}_\infty(t) = \widetilde{\kappa}_\infty(t_0) \left(\sum_i B_i e^{\lambda_i(t-t_0)} \right). \qquad (3.34)$$

Notice that at time $t = t_0 + \frac{1}{2}T$ we have

$$|\widetilde{\kappa}_\infty(t_0 + \tfrac{1}{2}T)|^2_{L^2} = |\widetilde{\kappa}_\infty(t_0)|^2_{L^2} \left(\sum_i e^{T\lambda_i} \right). \qquad (3.35)$$

By combining (3.31), (3.32), (3.33) and (3.35), it follows that

$$\sum_i e^{T\lambda_i} \le e^{-\frac{T\lambda}{2}}. \qquad (3.36)$$

From (3.34) it follows that

$$|\widetilde{\kappa}_\infty(t_0 - \tfrac{1}{2}T)|^2_{L^2} = |\widetilde{\kappa}_\infty(t_0)|^2_{L^2} \left(\sum_i e^{-T\lambda_i} \right). \qquad (3.37)$$

Finally using (3.37) and the concavity of $f(x) = \frac{1}{x}$ we see that

$$
\begin{aligned}
|\widetilde{\kappa}_\infty(t_0)|^2_{L^2} &= |\widetilde{\kappa}_\infty(t_0 - \tfrac{1}{2}T)|^2_{L^2} \frac{1}{\left(\sum_i e^{-T\lambda_i} \right)} \\
&\le |\widetilde{\kappa}_\infty(t_0 - \tfrac{1}{2}T)|^2_{L^2} \left(\sum_i e^{T\lambda_i} \right) \\
&\le e^{-\frac{T\lambda}{2}} |\widetilde{\kappa}_\infty(t_0 - \tfrac{1}{2}T)|^2_{L^2}.
\end{aligned}
\qquad (3.38)
$$

The last inequality in (3.38) follows from (3.36). Notice that the above inequality along with (3.30) imply that (3.28) holds. This is a contradiction. $\qquad \square$

Corollary 23. *Given $T > 0$ and $j \geq 1$ there exists $\alpha = \alpha(T, j) > 0$ such that if $|\psi(t)|_{C^2} < \alpha$*

on $[0, (j+1)T]$ then there exists a Calabi-Yau structure (ω_j, J_j) so that $\rho_j(t) = (\omega(t) - \omega_j, J(t) - J_j)$ satisfies the following exponential decay estimate:

$$|\rho_j(t)|_{L^2}^2 \leq C e^{-\frac{\lambda t}{2}}$$

for $t \in [0, (j+1)T]$ and a constant C independent of j.

Proof. First notice that by (3.2) it suffices to prove exponential decay of $\psi_j(t)$, the tangent vector associated to $\rho_j(t)$. We will prove exponential decay of $\psi_j(t)$ using the previous two lemmas and Theorem 20.

Let δ, β and ν be the small constants from Theorem 20, Lemma 21 and Lemma 22 respectively. In order to apply the above lemmas and Theorem 20 we let $\alpha = min\{\delta, \beta, \nu\}$ and assume that $|\psi|_{C^2} < \alpha$ on $[0, (J+1)T]$. Employing Theorem 20 (with $t_0 = jT$) and Lemma 21, there exists a Calabi-Yau structure, denoted (ω_j, J_j), such that

$$\sup_{\left[\left(j+\frac{1}{2}\right)T, (j+1)T\right]} |\psi_j|_{L^2}^2 \leq e^{-\frac{T\lambda}{2}} \sup_{\left[jT, \left(j+\frac{1}{2}\right)T\right]} |\psi_j|_{L^2}^2. \tag{3.39}$$

Now, Lemma 22 says that exponential decay at a later time implies exponential decay at an earlier time. In particular, from Lemma 22 and (3.39) it follows that for any $k \in \{\frac{n}{2} : n \in \mathbb{Z} \text{ and } 1 \leq n \leq 2j+1\}$,

$$\sup_{\left[kT, \left(k+\frac{1}{2}\right)T\right]} |\psi_j|_{L^2}^2 \leq e^{-\frac{T\lambda}{2}} \sup_{\left[\left(k-\frac{1}{2}\right)T, kT\right]} |\psi_j|_{L^2}^2. \tag{3.40}$$

Applying (3.40) with $k = \frac{1}{2}$ implies that for any $t \in \left[\frac{1}{2}T, T\right]$,

$$|\psi_j(t)|^2_{L^2} \le e^{-\frac{T\lambda}{2}} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|^2_{L^2} \le e^{-\frac{\lambda t}{2}} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|^2_{L^2}. \tag{3.41}$$

Next, using (3.40) with $k = \frac{1}{2}$ and $k = 1$ yields

$$\sup_{\left[T, \frac{3}{2}T\right]} |\psi_j|^2_{L^2} \le e^{-\frac{T\lambda}{2}} \sup_{\left[\frac{1}{2}T, T\right]} |\psi_j|^2_{L^2} \le e^{-T\lambda} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|^2_{L^2}.$$

Therefore it follows that for $t \in \left[T, \frac{3}{2}T\right]$,

$$|\psi_j(t)|^2_{L^2} \le e^{-T\lambda} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|^2_{L^2} \le e^{-\frac{\lambda t}{2}} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|^2_{L^2}. \tag{3.42}$$

Combining (3.41) and (3.42) yields L^2 exponential decay of $\psi_j(t)$ on $\left[\frac{1}{2}T, \frac{3}{2}T\right]$. Iterating this argument, we see that for $t \in \left[\frac{1}{2}T, (j+1)T\right]$,

$$|\psi_j(t)|^2_{L^2} \le e^{-\frac{\lambda t}{2}} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|^2_{L^2} \le C e^{-\frac{\lambda t}{2}}.$$

Notice that C is independent of j. Indeed by assumption $|\psi(t)|_{C^2} < \alpha$ on $[0, (j+1)T]$. Hence part (2) of Theorem 20 and the triangle inequality imply that $|\psi_j(t)|_{C^2} < C$ on $[0, (j+1)T]$, where C is independent of j. $\qquad\square$

3.3 Finding the Limit Calabi-Yau Structure

In this section we will show that there is a sequence of Calabi-Yau structures $\{(\omega_k, J_k)\}$, indexed by k, which subconverges as $k \to \infty$ to a limit Calabi-Yau structure, which we will denote (ω_{KE}, J_{KE}). By the choice of the Calabi-Yau structures in $\{(\omega_k, J_k)\}$, it will follow that the solution $(\omega(t), J(t))$ of the system (1.2) exponentially converges to (ω_{KE}, J_{KE}) as $t \to \infty$.

Notice that the decay estimate from Corollary 23 may fail to hold for intervals beyond I_j. In order to maintain exponential decay we want to choose another Calabi-Yau structure (ω_{j+1}, J_{j+1}) to which the solution exponentially converges. To ensure that we can continue this process we need to prove that $|\psi(t)|_{C^2}$ is small for all time so that Corollary 23 may be applied on any interval. This is the purpose of the following theorem. As a corollary we will prove the existence of a Calabi-Yau structure, denoted (ω_{KE}, J_{KE}), to which the flow exponentially converges.

Theorem 24. *Let $(M, \widetilde{\omega}, \widetilde{J})$ be a closed complex manifold with $(\widetilde{\omega}, \widetilde{J})$ a Calabi-Yau structure. Given $\epsilon' > 0$ and $k \geq 0$ there exists $\epsilon > 0$ so that $|\rho(0)|_{C^\infty} < \epsilon$ implies that $|\psi(t)|_{C^k} < \epsilon'$ on $[0, \infty)$.*

Proof. We will employ Theorem 12. To do this we make explicit T, ϵ, and ϵ'.

1. Let T be large enough so that

$$\frac{T^2 C_3(k+2)}{e^{T\lambda} - 1} + \frac{1}{e^{T\lambda}} < \frac{1}{2}.$$

Where $C_3(k+2)$ is a constant depending only on k and $(\widetilde{\omega}, \widetilde{J})$.

2. Choose

$$\epsilon' = \alpha.$$

Where α is the constant from Corollary 23.

3. Choose ϵ sufficiently small so that $(\omega(t), J(t))$ exists on $[0, 3T]$ and

$$\sup_{[0,2T]} |\psi(t)|_{C^k} < \frac{\epsilon'}{e^{T\lambda}}.$$

We want to prove that $|\psi(t)|_{C^k} < \epsilon'$ on $[0, \infty)$. Suppose by way of contradiction there

is a finite maximal time T' such that $|\psi(t)|_{C^k} < \epsilon'$ on $[0, T')$ with $k \geq 2$. Write $[0, T') =$

$[0, T] \cup [T, 2T] \cup \cdots \cup [NT, T')$ and let $[jT, (j+1)T] = I_j$. Also let (ω_j, J_j) denote the

Calabi-Yau structure, from Corollary 23, to which $(\omega(t), J(t))$ exponentially converges on

I_j. Using (3.39) and applying Lemma 22 iteratively we have

$$\sup_{I_{j-1} \cup I_j} |\psi_j|_{L^2}^2 \leq e^{-\lambda T(j-1)} \sup_{[0, \frac{T}{2}]} |\psi_j|_{L^2}^2. \tag{3.43}$$

We again use a parabolic regularity argument to prove that from (3.43) we can obtain a

C^{k+2} decay estimate.

Lemma 25. *There exists a constant $\alpha > 0$ so that if both $|\psi|_{C^2} < \alpha$ on I_j and $\sup_{I_{j-1} \cup I_j} |\psi_j|_{L^2}^2 \leq$*

$e^{-\lambda T(j-1)} \sup_{[0, \frac{T}{2}]} |\psi_j|_{L^2}^2$, *then there exists a constant $C(k+2)$ such that*

$$\sup_{I_j} |\psi_j|_{C^{k+2}} < C(k+2) T e^{-T\lambda(j-1)} \sup_{[0, \frac{1}{2}T]} |\psi_j|_{L^2}^2.$$

49

Proof. The proof of Lemma 25 uses essentially the same argument as the proof of Lemma 15 hence we omit some of the details.

From (1.42) we have

$$\frac{\partial}{\partial t}|\psi_j|^2_{L^2} \leq -|\nabla\psi_j|^2_{L^2} + C_1|\psi_j|^2_{L^2}.$$

Fix $t \in I_j$ and integrate from $(j-1)T$ to t;

$$\int_{(j-1)T}^{t}|\nabla\psi_j|^2_{L^2} \leq |\psi_j((j-1)T)|^2_{L^2} + C_1\int_{(j-1)T}^{t}|\psi_j|^2_{L^2}$$
$$\leq CTe^{-T\lambda(j-1)}\sup_{\left[0,\frac{1}{2}T\right]}|\psi_j|^2_{L^2}, \tag{3.44}$$

where the second inequality follows from the L^2 exponential decay assumption.

Next let $\theta(s)$ be a smooth function which is 0 for $s \leq (j-1)T$, monotonically increasing from 0 to 1 for $s \in [(j-1)T, jT]$ and equal to 1 for $s \geq jT$. As in Lemma 25, $\theta(s)$ will be used to deal with the boundary terms that arise in the estimates below. Now from (1.48) we have

$$\frac{\partial}{\partial t}|\nabla\psi_j|^2_{L^2} \leq C_5|\psi|^2_{L^2} + C_6|\nabla\psi|^2_{L^2}$$

and since $\theta(s)$ and its derivative are uniformly bounded, it follows that

$$\frac{\partial}{\partial s}\left(\theta(s)|\nabla\psi_j(s)|^2_{L^2}\right) \leq C_7|\psi_j|^2_{L^2} + C_8|\nabla\psi_j|^2_{L^2}.$$

We now integrate from $(j-1)T$ to $t \in I_j$ and use that $\theta((j-1)T) = 0$ and $\theta(t) = 1$,

$$|\nabla \psi_j(t)|_{L^2}^2 \le C_7 \int_{(j-1)T}^t |\psi_j|_{L^2}^2 + C_8 \int_{(j-1)T}^t |\nabla \psi_j|_{L^2}^2 \qquad (3.45)$$

$$\le CTe^{-T\lambda(j-1)} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|_{L^2}^2, \qquad (3.46)$$

where the first and second terms on the right-hand side of (3.45) were bounded using the L^2 exponential decay assumption and (3.44) respectively. Notice that (3.45) and (3.46) yield the desired $L^{1,2}$ exponential decay estimate.

Continuing in this way, on I_j we get

$$|\psi_j(t)|_{L^{p,2}}^2 \le C(p)Te^{-T\lambda(j-1)} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|_{L^2}^2,$$

moreover by the Sobolev Embedding Theorem, for any $t \in I_j$

$$|\psi_j(t)|_{C^{k+2}} \le C(k+2)Te^{-T\lambda(j-1)} \sup_{\left[0, \frac{1}{2}T\right]} |\psi_j|_{L^2}^2.$$

$\square$

From Lemma 25 it follows that

$$\sup_{I_j} \left| \frac{\partial}{\partial t} \psi \right|_{C^k} \le C_2(k+2)Te^{-T\lambda(j-1)} \epsilon'$$

since $\left| \frac{\partial}{\partial t} \psi \right|_{C^k} = \left| \frac{\partial}{\partial t} \psi_j \right|_{C^k} \le C \sup_{I_j} |\psi_j|_{C^{k+2}}$. Hence, for $j \ge 2$ and any $t \in I_j$, by

integrating we see that

$$|\psi(t)|_{C^k} \leq T \sup_{I_j} \left| \frac{\partial}{\partial t} \psi \right|_{C^k} + \sup_{I_{j-1}} |\psi|_{C^k}$$

$$\leq T \sum_{l=2}^{j} \sup_{I_l} \left| \frac{\partial}{\partial t} \psi \right|_{C^k} + \sup_{I_0 \cup I_1} |\psi|_{C^k}$$

$$< \epsilon' C_3 (k+2) T^2 \left(\frac{1}{e^{\lambda T}} + \frac{1}{e^{2\lambda T}} + \cdots + \frac{1}{e^{(j-1)\lambda T}} \right) + \sup_{I_0 \cup I_1} |\psi|_{C^k}$$

$$\leq \epsilon' \frac{C_3 (k+2) T^2}{e^{\lambda T} - 1} + \sup_{I_0 \cup I_1} |\psi|_{C^k}$$

$$\leq \epsilon' \frac{C_3 (k+2) T^2}{e^{\lambda T} - 1} + \frac{\epsilon'}{e^{T\lambda}}$$

$$< \frac{\epsilon'}{2}.$$

Where the final inequality is from our choice of T and ϵ. The key observation here is that the above inequality is independent of both $j \geq 2$ and $t \in I_j$. Hence the above inequality holds for $j = N$ which contradicts the maximality of T'. Therefore $T' = \infty$. $\square$

The important thing to notice about Theorem 24 is that it allows us to find a Calabi-Yau structure (ω_{KE}, J_{KE}) to which the flow converges.

Corollary 26. *Under the assumptions of Theorem 1 with $(M, \widetilde{\omega}, \widetilde{J})$ a Calabi-Yau manifold, there exists a Calabi-Yau structure (ω_{KE}, J_{KE}) to which the flow exponentially converges.*

Proof. By Theorem 24 we know that given $\epsilon' > 0$ and $k \geq 0$, there exists $\epsilon > 0$ such that if $|\rho(0)|_{C^\infty} < \epsilon$, then $|\psi(t)|_{C^k} < \epsilon'$ for all $t \geq 0$. Let $\epsilon' = \alpha$, where α is the small constant from Corollary 23. Recall that $\rho_j(t) = (\omega(t) - \omega_j, J(t) - J_j)$. Now since $|\psi(t)|_{C^k} < \alpha$ for all $t \geq 0$, by Corollary 23 there exists a sequence of Calabi-Yau structures $\{(\omega_j, J_j)\}$ such that

$\rho_j(t)$ exponentially decays in L^2 for all $t \in [0, (j+1)T]$ and for each j. Specifically,

$$|\rho_j(t)|_{L^2} \le Ce^{-\frac{\lambda t}{2}} \tag{3.47}$$

for $t \in [0, (j+1)T]$ and for each j.

Next, by part (2) of Theorem 20, each of these Calabi-Yau structures (ω_j, J_j) is contained in a fixed neighborhood of $(\widetilde{\omega}, \widetilde{J})$, in particular $|(\omega_j - \widetilde{\omega}, J_j - \widetilde{J})|_{C^k} \le C\epsilon'$. Therefore as $j \to \infty$ we have a convergent subsequence $(\omega_j, J_j) \to (\omega_{KE}, J_{KE})$. And hence by (3.47) we have L^2 exponential convergence of $(\omega(t), J(t))$ to (ω_{KE}, J_{KE}) for all $t \ge 0$. Finally applying the parabolic regularity argument of Lemma 15 it follows that the exponential convergence of $(\omega(t), J(t))$ to (ω_{KE}, J_{KE}) is C^k convergence, that is

$$|(\omega(t) - \omega_{KE}, J(t) - J_{KE})|_{C^k} \le Ce^{-\frac{\lambda t}{2}}. \tag{3.48}$$

In other words, $(\omega(t), J(t))$ is contained in a ball of radius $Ce^{-\frac{\lambda t}{2}}$ of the limit Calabi-Yau structure (ω_{KE}, J_{KE}) for all $t \ge 0$. This gives us the desired exponential decay estimate. $\square$

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