

CONTROLLABILITY OF HYPERBOLIC AND DEGENERATE PARABOLIC  
EQUATIONS IN ONE DIMENSION

By

Jonathan Matthew Bohn

A DISSERTATION

Submitted to  
Michigan State University  
in partial fulfillment of the requirements  
for the degree of

Mathematics — Doctor of Philosophy

2018

## ABSTRACT

### CONTROLLABILITY OF HYPERBOLIC AND DEGENERATE PARABOLIC EQUATIONS IN ONE DIMENSION

By

**Jonathan Matthew Bohn**

In this thesis, we study the controllability problem for two systems of partial differential equations. We will first consider the wave equation with variable coefficients and potential in one dimension,  $u_{tt} - (a(x)u_x)_x + pu = 0$ , with control function  $v(t)$  acting on the boundary. We consider a class of functions corresponding to a special weight function that contains the variable coefficient  $a(x)$ . From here, we derive a global Carleman estimate for this system, and establish the controllability property. We then later extend the class of admissible functions  $a(x)$  for which the controllability property holds true.

We then study the controllability problem for the degenerate heat equation in one dimension. For  $0 \leq \alpha < 1$ , on  $(0, 1) \times (0, T)$ , we consider  $w_t - (x^\alpha w_x)_x = f$ . This equation is degenerate because the diffusion coefficient  $x^\alpha$  is positive in the interior of the domain and vanishes at the boundary. We consider this problem under the Robin boundary conditions. Again, we derive a Carleman estimate for this system, taking into account the new boundary terms that arise from the Robin conditions.

To those who helped me along the way.

## ACKNOWLEDGMENTS

I am deeply grateful to my advisor, Dr. Zhengfang Zhou. I thank him for his insight, encouragement, and time. I enjoyed our meetings and have learned much from him, not only in mathematics, but from his exceptional personality. I am fortunate to have studied under his guidance and support.

I would like to thank the Mathematics Chair Dr. Keith Promislow, Director of Graduate Studies Dr. Jeffrey Schenker, and past supervisors for providing me with the opportunity to study at Michigan State.

I would like to give thank to all the faculty and staff at MSU that helped me along the way. I include special thanks to Dr. Baisheng Yan, Dr. Casim Abbas, Dr. Gabriel Nagy, and Dr. Sheldon Newhouse, each of who went beyond their normal duties in a memorable way for me. I would like to show special appreciation to Ms. Barbara Miller, Ms. Debra Lecato, and Mr. James Chang.

Special thanks to Lianzhang Bao. I again enjoyed our meetings and learned much from our correspondence. In particular, my experience in China was one of the most memorable of my life due to your planning and diligence.

Special thanks to Seonghak Kim. You were the best office mate I could have. I give thanks to Xin Yang and Khaldoun Al-Yasiri, who shared some challenges with me and provided support when I needed it.

To James, Steve, and Logan, you helped me more than you know. Thanks to Nick and Erick for scheduling help and Chris for proofreading.

I would like to thank my family, who has always supported me and provided me with guidance.

Finally, I thank all of the students I taught that helped to keep me sharp. Teaching was one of the most enjoyable aspects for me. I hope that I in turn helped you in your chosen careers.

# TABLE OF CONTENTS

|                                               |            |
|-----------------------------------------------|------------|
| <b>KEY TO SYMBOLS</b>                         | <b>vii</b> |
| <b>Chapter 1 Introduction</b>                 | <b>1</b>   |
| <b>Chapter 2 Preliminaries</b>                | <b>5</b>   |
| 2.1 Regularity of the Wave Equation           | 5          |
| 2.2 Transposition Solutions                   | 10         |
| 2.2.1 Application                             | 14         |
| <b>Chapter 3 Hyperbolic Equations</b>         | <b>19</b>  |
| 3.1 Introduction                              | 19         |
| 3.1.1 Weight Functions                        | 23         |
| 3.2 Carleman Estimates                        | 25         |
| 3.2.1 Carleman Estimate for large $T$         | 34         |
| 3.2.2 Pointwise Carleman Estimate at $-T$     | 40         |
| 3.3 Control                                   | 42         |
| 3.4 Increased Regularity                      | 46         |
| <b>Chapter 4 Degenerate Parabolic Control</b> | <b>51</b>  |
| 4.1 Robin Boundary Conditions                 | 52         |
| 4.2 Carleman Estimate                         | 57         |
| <b>APPENDIX</b>                               | <b>68</b>  |
| <b>BIBLIOGRAPHY</b>                           | <b>73</b>  |

## KEY TO SYMBOLS

1.  $\mathbb{R}^n = n$ -dimensional real Euclidean space, where  $\mathbb{R} = \mathbb{R}^1$ .
2.  $\Omega \subset \mathbb{R}^n =$  open set in  $\mathbb{R}^n$ .
3.  $\partial\Omega =$  boundary of  $\Omega$ .
4.  $\nu = \nu(x) =$  unit outward normal.
5. For  $u : \Omega \rightarrow \mathbb{R}, x \in \Omega$ , then  $u_{x_i} = \frac{\partial u}{\partial x_i} = \lim_{h \rightarrow 0} \frac{u(x+he_i) - u(x)}{h}$ , whenever this limit exists.
6. For  $u : \Omega \rightarrow \mathbb{R}$ , then  $\Delta u = \sum_{i=1}^n u_{x_i x_i}$ , the *Laplacian* of  $u$ .
7.  $AC(\Omega) = \{u : \Omega \subset \mathbb{R} \rightarrow \mathbb{R} | u \text{ is absolutely continuous}\}$ .
8.  $C^k(\Omega) = \{u : \Omega \rightarrow \mathbb{R} | u \text{ is } k - \text{times continuously differentiable}\}$ .
9.  $C_c(\Omega), C_c^k(\Omega)$ , etc., are those functions in  $C(\Omega), C^k(\Omega)$ , etc., with compact support.
10.  $L^p(\Omega) = \{u : \Omega \rightarrow \mathbb{R} | u \text{ is measurable and } \|u\|_{L^p(\Omega)} < \infty\}$ , for  $1 \leq p < \infty$ , where  $\|u\|_{L^p(\Omega)} \doteq \left(\int_{\Omega} |u|^p dx\right)^{\frac{1}{p}}$ .
11.  $L^\infty(\Omega) = \{u : \Omega \rightarrow \mathbb{R} | u \text{ is measurable and } \|u\|_{L^\infty(\Omega)} < \infty\}$ , where  $\|u\|_{L^\infty(\Omega)} \doteq \text{esssup}_{\Omega} |u|$ .
12.  $W^{k,p}(\Omega) = \{u \in L^p(\Omega) | D^\alpha u \in L^p(\Omega), \forall |\alpha| \leq k\}$ , the Sobolev spaces.
13.  $W_0^{k,p}(\Omega)$  is the closure of  $C_c^\infty$  in  $W^{k,p}(\Omega)$ .
14.  $H^k(\Omega) = W^{k,2}(\Omega), H_0^k(\Omega) = W_0^{k,2}(\Omega)$ .
15.  $H^{-1}(\Omega)$  is the dual space to  $H_0^1(\Omega)$ .
16. For  $X$  a real Banach space, with norm  $\|\cdot\|$ , then  $L^p(0, T; X) = \left\{ \mathbf{u} : [0, T] \rightarrow X | \mathbf{u} \text{ is strongly measurable and } \|\mathbf{u}\|_{L^p(0,T;X)} < \infty \right\}$ , for  $1 \leq p < \infty$ , where  $\|\mathbf{u}\|_{L^p(0,T;X)} = \left( \int_0^T \|\mathbf{u}(t)\|^p dt \right)^{\frac{1}{p}}$ .
17.  $C([0, T]; X) = \left\{ \mathbf{u} : [0, T] \rightarrow X | \mathbf{u} \text{ is continuous, and } \|\mathbf{u}\|_{C([0,T];X)} < \infty \right\}$ , where  $\|\mathbf{u}\|_{C([0,T];X)} = \max_{0 \leq t \leq T} \|\mathbf{u}(t)\|$ .
18.  $W^{1,p}(0, T; X) = \{ \mathbf{u} \in L^p(0, T; X) | \mathbf{u}' \text{ exists in the weak sense, and } \mathbf{u}' \in L^p(0, T; X) \}$ .
19. For  $m > 0$ ,  $L_{\leq m}^\infty(\Omega) = \{q \in L^\infty(\Omega), \|q\|_{L^\infty(\Omega)} \leq m\}$ .

20. For  $m > 0$ ,  $L_{\leq m}^{\infty}(\Omega \times (0, T)) = \{p \in L^{\infty}(\Omega \times (0, T)), \|p\|_{L^{\infty}} \leq m\}$ .
21.  $H_a^1(0, 1) = \{u \in L^2(0, 1) | u \in AC([0, 1]) \text{ and } \sqrt{a}u_x \in L^2(0, 1)\}$ .
22.  $H_a^2(0, 1) = \{u \in H_a^1(0, 1) | au_x \in H^1(0, 1)\}$ .
23.  $\mathcal{B}((0, 1) \times (0, T)) = C([0, T]; L^2(0, 1)) \cap L^2(0, T; H_a^1(0, 1))$ .
24. For operator  $(A, D(A))$ ,  
 $\mathcal{H}((0, 1) \times (0, T)) = L^2(0, T; D(A)) \cap H^1(0, T; L^2(0, 1)) \cap C([0, T]; H_a^1(0, 1))$ .



# Chapter 1

## Introduction

In this study, we present some applications of the problem of controllability in partial differential equations. The controllability problem in the context of this work may be described in summary as follows: Consider a evolution system described in terms of partial differential equations and a time  $T > 0$ . Allow ourselves to influence this system through means of a control function. Now, given a set of initial states and final states, the goal is to find such a control function, so that by means of the control, the system is driven from the initial states to the final states in time  $T$ . The natural question arises then; Which systems are controllable?

We will consider this question for two different systems. An answer can be developed depending on what space the initial and final states reside. In Chapter 2, we develop the framework necessary for the solution space to study the hyperbolic system of Chapter 3. Of interest then, let  $\Omega \in \mathbb{R}^n$  be a domain,  $\omega \subset \Omega$ ,  $T > 0$ , and consider the linear wave equation with interior control function  $f$  in  $\Omega \times (0, T)$ ,

$$\begin{cases} u_{tt} - \Delta u = f(x, t)\chi_{\omega}, & (x, t) \in \Omega \times (0, T), \\ u = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (1.1)$$

It is well known that for  $f \in L^2(\Omega)$ ,  $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ , then (1.1) is well posed,

with a unique finite energy solution  $u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ . However, the controllability question is most naturally posed in the larger class  $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ . In Chapter 2, we present the *Transposition Solutions*, arising from this form of the initial states. Then, in Chapter 3, the control function for the hyperbolic equation can be developed in terms of these solutions.

In Chapter 3, we study the variable coefficient wave equation. For control  $v(t)$ , and potential function  $p(x, t)$ , consider

$$\begin{cases} u_{tt} - (a(x)u_x)_x + p(x, t)u = 0, & (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = 0, \quad u(1, t) = v(t), & t \in (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (0, 1). \end{cases} \quad (1.2)$$

By the Hilbert Uniqueness Method [38], the controllability of such a system is equivalent to the *observability* of the adjoint problem. That is, for an appropriately defined subset of the boundary,  $\Gamma(x_0)$ , one can positively show controllability by obtaining an inequality of the form

$$\|(u_0, u_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq C \int_0^T \int_{\Gamma(x_0)} \left| \frac{\partial u}{\partial n} \right|^2 d\Gamma dt. \quad (1.3)$$

A more refined inequality is the *Carleman Estimate*, which contains the necessary information of observability. In Section 3.2 we develop the Carleman Estimate for a certain class of functions  $a(x)$ , assumed to be  $C^2(\Omega)$ , and in the class  $\mathcal{B}$  defined in (3.7). Specifically, for these  $a(x)$ , let  $T > 0$ , and assume that  $x_0 < 0, a_0 > 0$ . There exists  $\beta \in (0, 1)$  such that

$$T > \frac{1}{\beta} \max_{x \in [0, 1]} \frac{(x - x_0)}{\sqrt{a(x)}}. \quad (1.4)$$

Then for any  $m > 0$ , we will show there exists  $\lambda > 0$  independent of  $m$ ,  $s_0 = s_0(m) > 0$  and a positive constant  $M = M(m)$  such that for  $\psi$  and  $\varphi$  as defined as in (3.10), for  $a(x) \in \mathcal{B}(x_0, a_0)$  defined in (3.7), for all  $p(x, t) \in L_{\leq m}^\infty((0, 1) \times (-T, T))$  and for all  $s \geq s_0$ ,

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |w|^2 dx dt \\ & + s \int_0^1 e^{2s\varphi(-T)} (|w_t(-T)|^2 + |w_x(-T)|^2) dx + s^3 \int_0^1 e^{2s\varphi(-T)} |w(-T)|^2 dx \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw + pw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt, \end{aligned} \quad (1.5)$$

for all  $w \in L^2(-T, T; H_0^1(0, 1))$  also satisfying  $w_{tt} - (a(x)w_x)_x + pw \in L^2((0, 1) \times (-T, T))$  and  $w_x(1, t) \in L^2(-T, T)$ .

In Section 3.3, we show the connection between the Carleman Estimate and controllability. We will show that with the Carleman Estimate as above, controllability holds true. In Section 3.4, we expand the class of functions  $a(x)$  for which controllability holds. For  $a(x)$  defined up to the base point  $x_0$ , we extend the admissible class of functions to those  $a(x) \in C^1(\Omega)$ .

In Chapter 4, we pose the same questions for the degenerate parabolic equation. Consider the parabolic equation

$$\begin{cases} u_t - (a(x)u_x)_x = f(x, t)\chi_\omega, & (x, t) \in (0, 1) \times (0, T), \\ u(x, 0) = u_0(x), & x \in (0, 1). \end{cases} \quad (1.6)$$

Then if  $a(0) = 0$ , this system is said to be *degenerate*. In the case  $a(x) = x^\alpha$ , the Carleman Estimate is known to hold under the appropriate Dirichlet or Neumann boundary conditions, for  $0 \leq \alpha < 2$ . In Section 4.1, we recall and apply a recent work necessary for

well-posedness for Section 4.2. In Section 4.2, with this chosen  $a(x)$ , we state and prove the analogous Carleman Estimate with Robin boundary conditions.

# Chapter 2

## Preliminaries

### 2.1 Regularity of the Wave Equation

The goal of the first part of this section is to introduce the relevant existence and uniqueness results for the wave equation with less regular data. It is well known that the hyperbolic system is well posed with suitable regularity on initial conditions, for example, data in  $H_0^1(\Omega) \times L^2(\Omega)$ . However, the system can be studied with less regular initial conditions. The corresponding *transposition solutions* can only be understood in the transposition sense of Lions [39]. In turn, we will apply the transposition solutions to the controllability problem for the wave equation of section 3.1, as in a very natural sense, this is the largest space for which the problem is well-posed.

Consider the system

$$\left\{ \begin{array}{ll} \varphi_{tt} - \Delta \varphi = f, & (x, t) \in \Omega \times (0, T), \\ \varphi = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \varphi(x, 0) = \varphi_0(x), & x \in \Omega, \\ \varphi_t(x, 0) = \varphi_1(x), & x \in \Omega, \end{array} \right. \quad (2.1)$$

where  $f \in L^1(0, T; L^2(\Omega))$ , and the initial conditions  $\varphi_0(x) \in H_0^1(\Omega)$ , and  $\varphi_1(x) \in L^2(\Omega)$ .

Then, with these regularity assumptions, the solutions  $\varphi$  of (2.1) satisfy

$$\varphi \in C\left([0, T]; H_0^1(\Omega)\right) \cap C^1\left([0, T]; L^2(\Omega)\right). \quad (2.2)$$

For the proof, see for example [3] or [16]. However, the finite energy property does not apriori show  $L^2$  regularity of  $\frac{\partial \varphi}{\partial \nu}$  by classical trace inequalities. The result is in fact true that  $\frac{\partial \varphi}{\partial \nu} \in L^2(\partial\Omega \times (0, T))$ . This is not straightforward, and requires careful analysis of the resulting terms after using a multiplier technique. For this reason, the regularity for  $\frac{\partial \varphi}{\partial \nu}$  is often referred to as *Hidden Regularity*. The correct multiplier for this particular problem is due to Ho [26] (cf. [36], [37], [38]). The regularity result will be needed for the definition of the transposition solutions to the wave equation in Section 2.2.1. The Hidden Regularity proof below is adapted from [37].

**Theorem 1.** (*Hidden Regularity*). *Let  $\Omega \in \mathbb{R}^n$  be a domain with smooth boundary, and let  $\varphi$  solve*

$$\begin{cases} \varphi_{tt} - \Delta \varphi = f, & (x, t) \in \Omega \times (0, T), \\ \varphi = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \varphi(x, 0) = \varphi_0(x), & x \in \Omega, \\ \varphi_t(x, 0) = \varphi_1(x), & x \in \Omega, \end{cases} \quad (2.3)$$

where  $f \in L^1(0, T; L^2(\Omega))$ , and initial conditions  $\varphi_0(x) \in H_0^1(\Omega)$ , and  $\varphi_1(x) \in L^2(\Omega)$ .

Then  $\frac{\partial \varphi}{\partial \nu} \in L^2(\partial\Omega \times (0, T))$ , and furthermore,

$$\left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0, T))} \leq C \left( \|f\|_{L^1(0, T; L^2(\Omega))} + \|\varphi_0\|_{H_0^1(\Omega)} + \|\varphi_1\|_{L^2(\Omega)} \right), \quad (2.4)$$

for some constant  $C$ .

Note: In the proof, always use the Einstein summation convention when applicable. That is, for functions  $\{f_i\}_{i=1}^m, \{g_i\}_{i=1}^m$ , then

$$f_i g_i = \sum_{i=1}^m f_i g_i. \quad (2.5)$$

*Proof.* By the standard approximation argument, the problem reduces to proving the result for regular functions  $f, \varphi_0$ , and  $\varphi_1$ .

This will use a function multiplier from [26], for which we always assume the Einstein summation notation of (2.5). For  $k = 1, \dots, n$ , let  $h_k(x)$  be functions such that  $h_k \in C^1(\bar{\Omega})$  and  $h_k = \nu_k$  on  $\partial\Omega$ . (Such functions exist because of the assumed regularity of the boundary). Now, we incorporate the multiplier from [26]. Multiply both sides of (2.3) by  $h_k \frac{\partial \varphi}{\partial x_k}$  and integrate to obtain

$$\int_{\Omega} \varphi_{tt} h_k \frac{\partial \varphi}{\partial x_k} dx - \int_{\Omega} \Delta \varphi h_k \frac{\partial \varphi}{\partial x_k} dx = \int_{\Omega} f h_k \frac{\partial \varphi}{\partial x_k} dx. \quad (2.6)$$

We will integrate by parts, considering terms separately. First, look at the second term of (2.6). We have

$$\int_{\Omega} -\Delta \varphi h_k \frac{\partial \varphi}{\partial x_k} dx = \int_{\Omega} \frac{\partial \varphi}{\partial x_i} \frac{\partial}{\partial x_i} \left( h_k \frac{\partial \varphi}{\partial x_k} \right) dx - \int_{\partial\Omega} \frac{\partial \varphi}{\partial \nu} h_k \frac{\partial \varphi}{\partial x_k} dS \quad (2.7)$$

$$= \int_{\Omega} \frac{\partial \varphi}{\partial x_i} h_k \frac{\partial^2 \varphi}{\partial x_i \partial x_k} dx + \int_{\Omega} \frac{\partial \varphi}{\partial x_i} \frac{\partial h_k}{\partial x_i} \frac{\partial \varphi}{\partial x_k} dx - \int_{\partial\Omega} \left( \frac{\partial \varphi}{\partial \nu} \right)^2 dS, \quad (2.8)$$

where we have used the definition of  $h_k = \nu_k$  on the boundary. Now, the first term of (2.8)

becomes

$$\int_{\Omega} \frac{\partial \varphi}{\partial x_i} h_k \frac{\partial^2 \varphi}{\partial x_i \partial x_k} dx = \frac{1}{2} \int_{\Omega} h_k \frac{\partial}{\partial x_k} \left( |\nabla \varphi|^2 \right) dx. \quad (2.9)$$

Upon integration by parts,

$$\int_{\Omega} \frac{\partial \varphi}{\partial x_i} h_k \frac{\partial^2 \varphi}{\partial x_i \partial x_k} dx = \frac{1}{2} \int_{\partial \Omega} h_k \nu_k |\nabla \varphi|^2 dS - \frac{1}{2} \int_{\Omega} \frac{\partial h_k}{\partial x_k} |\nabla \varphi|^2 dx. \quad (2.10)$$

Since  $\varphi = 0$  on  $\partial \Omega$ , we have that  $|\nabla \varphi|^2 = \left( \frac{\partial \varphi}{\partial \nu} \right)^2$ . In combination with (2.8), we discover that the second term from (2.6) can be written as

$$\begin{aligned} & - \int_{\Omega} \Delta \varphi h_k \frac{\partial \varphi}{\partial x_k} dx \\ & = - \frac{1}{2} \int_{\Omega} \frac{\partial h_k}{\partial x_k} |\nabla \varphi|^2 dx + \int_{\Omega} \frac{\partial \varphi}{\partial x_i} \frac{\partial h_k}{\partial x_i} \frac{\partial \varphi}{\partial x_k} dx - \frac{1}{2} \int_{\partial \Omega} \left( \frac{\partial \varphi}{\partial \nu} \right)^2 dS. \end{aligned} \quad (2.11)$$

Now, return to (2.6). Integrating by parts with respect to time gives

$$\begin{aligned} & \int_0^T \int_{\Omega} \left( \varphi_{tt} h_k \frac{\partial \varphi}{\partial x_k} \right) dx dt \\ & = - \int_0^T \int_{\Omega} \varphi_t h_k \frac{\partial \varphi_t}{\partial x_k} dx dt + \int_{\Omega} \varphi_t h_k \frac{\partial \varphi}{\partial x_k} \Big|_0^T dx \\ & = - \frac{1}{2} \int_0^T \int_{\Omega} \frac{\partial h_k}{\partial x_k} (\varphi_t)^2 + \int_{\Omega} \varphi_t h_k \frac{\partial \varphi}{\partial x_k} \Big|_0^T dx. \end{aligned} \quad (2.12)$$

Integrating, and combining (2.6) and (2.12), we arrive at the expression

$$\begin{aligned} & - \frac{1}{2} \int_0^T \int_{\Omega} \frac{\partial h_k}{\partial x_k} (\varphi_t)^2 + \int_{\Omega} \varphi_t h_k \frac{\partial \varphi}{\partial x_k} \Big|_0^T dx \\ & = - \frac{1}{2} \int_0^T \int_{\Omega} \frac{\partial h_k}{\partial x_k} |\nabla \varphi|^2 dx dt + \int_0^T \int_{\Omega} \frac{\partial \varphi}{\partial x_i} \frac{\partial h_k}{\partial x_i} \frac{\partial \varphi}{\partial x_k} dx dt - \frac{1}{2} \int_0^T \int_{\partial \Omega} \left( \frac{\partial \varphi}{\partial \nu} \right)^2 dS dt \end{aligned} \quad (2.13)$$



$$+ \int_0^T \int_{\Omega} f h_k \frac{\partial \varphi}{\partial x_k} dx dt.$$

Now, since  $h \in C^1(\Omega)$ , there exists constants  $c$  and  $\tilde{c}$  such that

$$|h(x)| \leq c, \text{ and } \sum_{i,j=1}^n |\partial_i h_j(x)| \leq \tilde{c}.$$

Solving (2.13) for  $\int_0^T \int_{\partial\Omega} \left( \frac{\partial \varphi}{\partial \nu} \right)^2 dS dt$  and using the above bounds, we arrive at the expression for  $\left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0,T))}$ .

Now, we can recover the desired inequality (2.4). Define the energy

$$E(t) = \frac{1}{2} \int_{\Omega} \varphi_t^2(x, t) + |\nabla \varphi(x, t)|^2 dx, \quad 0 \leq t \leq T. \quad (2.14)$$

Then, by computing the derivative, we easily see

$$\frac{dE}{dt} = \int_{\Omega} \varphi_t (\varphi_{tt} - \Delta \varphi) dx = 0.$$

That is, the energy is independent of  $t$ . In particular, for  $t = 0$ , we use (2.13) and (2.14) to obtain the inequality in terms of the initial data, as desired,

$$\left\| \frac{\partial \varphi}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0,T))} \leq C \left( \|f\|_{L^1(0,T;L^2(\Omega))} + \|\varphi_0\|_{H_0^1(\Omega)} + \|\varphi_1\|_{L^2(\Omega)} \right). \quad (2.15)$$

□

## 2.2 Transposition Solutions

Now, we introduce the notion of transposition solutions. Because of the decreased regularity of the initial conditions, the solution to the wave equation does not exist in the classical sense. Rather, the solution must be defined through an intermediate system. The general method for the transposition solutions can be found in volumes 1 and 2 of [39]. The short introduction for our problem is adapted from [45]. The main idea is that the solution of the less understood equation can be interpreted by a different well understood one.

Consider the system

$$\left\{ \begin{array}{ll} u_{tt} - \Delta u = f, & (x, t) \in \Omega \times (0, T), \\ u = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x), & x \in \Omega, \\ u_t(x, 0) = u_1(x), & x \in \Omega, \end{array} \right. \quad (2.16)$$

where  $(u_0, u_1) \in L^2(\Omega) \times H^{-1}(\Omega)$ , and  $f \in L^2(0, T; H^{-1}(\Omega))$ . For ease of notation, let us set

$$Y = L^2(\Omega) \times H^{-1}(\Omega) \text{ and } D(A) = H_0^1(\Omega) \times L^2(\Omega).$$

Then, for  $(y_1, y_2) \in D(A)$ , we define  $A$  by its action

$$Ay = A(y_1, y_2) = (y_2, -\Delta y_1).$$

Here,  $-\Delta : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ , and so we may also associate the scalar product on  $H^{-1}(\Omega)$

in the usual fashion, defined by

$$(\Delta y_1, \xi)_{H^{-1}(\Omega)} = - \int_{\Omega} \nabla y_1 \cdot \nabla (-\Delta)^{-1} \xi dx. \quad (2.17)$$

In (2.17) above,  $(-\Delta)^{-1} \xi$  is that solution  $w$  of the equation  $-\Delta w = \xi$  in  $\Omega$  (for  $w \in H_0^1(\Omega)$ ). As is standard then, the method of proof for well-posedness will be to construct a semigroup of contractions.

**Theorem 2.** *The operator  $(A, D(A))$  is the generator of a semigroup of contractions on  $Y$ .*

The following short lemma will prove to be useful in the proof of Theorem 2.

**Lemma 1.** *The map  $f \mapsto \|f\|_{H^{-1}(\Omega)} = \langle f, (-\Delta)^{-1} f \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}^{\frac{1}{2}}$  is a norm in  $H^{-1}(\Omega)$  equivalent to the usual norm.*

*Proof.* Since  $(-\Delta)^{-1} : H^{-1}(\Omega) \rightarrow H_0^1(\Omega)$  is an isomorphism, then also the map  $f \mapsto \|(-\Delta)^{-1} f\|_{H_0^1(\Omega)}$  is a norm in  $H^{-1}$  equivalent to the usual norm. Now, let  $f \in H^{-1}(\Omega)$ , and test the equation  $-\Delta ((-\Delta)^{-1} f) = f$  with  $(-\Delta)^{-1} f$ . We get upon integration by parts that

$$\int_{\Omega} |\nabla ((-\Delta)^{-1} f)|^2 dx = \langle f, (-\Delta)^{-1} f \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}. \quad (2.18)$$

By Poincaré's inequality, we have that the left hand side of the inequality is equivalent to the  $H_0^1$  norm. By our first statement,  $f \mapsto \|(-\Delta)^{-1} f\|_{H_0^1(\Omega)}$  in  $H^{-1}$  is also an equivalent norm. Therefore,

$$c \|f\|_{H^{-1}(\Omega)}^2 \leq \int_{\Omega} |\nabla ((-\Delta)^{-1} f)|^2 dx = \langle f, (-\Delta)^{-1} f \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}$$

$$\begin{aligned}
&\leq \|f\|_{H^{-1}(\Omega)} \|(-\Delta)^{-1}f\|_{H_0^1(\Omega)} \\
&\leq C \|f\|_{H^{-1}}^2.
\end{aligned} \tag{2.19}$$

Taking square roots, the lemma is proved.  $\square$

With this lemma, we continue with the proof of the theorem.

*Proof of Theorem 2.* The proof will involve showing the hypotheses for the Hille-Yosida Theorem, which is standard. The statement of the Hille-Yosida Theorem can be found in [16]. First,  $A$  must be shown to be densely defined and closed. It is clear that  $D(A)$  is dense in  $Y$ . Let us show that  $A$  is closed on  $D(A)$ . Let  $y_n = (y_{1n}, y_{2n}) \in D(A)$  be a sequence converging to some  $(y_1, y_2) \in D(A)$ , where  $Ay_n$  converges to some  $(f, g) \in L^2(\Omega) \times H^{-1}(\Omega)$ . By the definition of  $A$ , we have that  $y_2 = f$ . Now, because  $(-\Delta)^{-1}$  is continuous as an operator from  $H^{-1}(\Omega)$  to  $H_0^1(\Omega)$ , we have that  $(-\Delta)^{-1}(-\Delta y_{1n}) = y_{1n} \rightarrow (-\Delta)^{-1}g$ . From here, it follows that  $y_1 \in H_0^1(\Omega)$  with  $g = -\Delta y_1$ .

For the second part of the Hille-Yosida Theorem, it must be shown that every positive  $\lambda$  belongs to the resolvent set and that the resolvent operator defined by  $R_\lambda u \doteq (\lambda I - A)^{-1}u$  satisfies  $\|R_\lambda\| \leq \frac{1}{\lambda}$  for  $\lambda > 0$ . Recall  $\lambda$  belongs to the resolvent set if the operator  $\lambda I - A$  is one-to-one and onto.

Now, let  $\lambda > 0$ . For  $f \in L^2(\Omega), g \in H^{-1}(\Omega)$ , consider the system

$$\begin{cases} \lambda y_1 - y_2 = f, & \text{in } \Omega, \\ \lambda y_2 - \Delta y_1 = g, & \text{in } \Omega. \end{cases} \tag{2.20}$$

Notice then that  $R_\lambda(f, g) = (\lambda I - A)^{-1}(f, g) = (y_1, y_2)$ . By multiplying the top equation

in (2.20) by  $\lambda$  and adding the two equations, we obtain

$$\lambda^2 y_1 - \Delta y_1 = \lambda f + g, \text{ in } \Omega. \quad (2.21)$$

By the standard elliptic theory (see [25] for example), there exists a unique weak solution to (2.21) in  $H_0^1(\Omega)$ . Now, we need to obtain a suitable estimate. By applying  $(-\Delta)^{-1}$  to the first equation of (2.20), we get

$$\lambda(-\Delta)^{-1} y_1 - (-\Delta)^{-1} y_2 = (-\Delta)^{-1} f. \quad (2.22)$$

Next, by testing (2.20) with  $(-\Delta)^{-1} y_2$  and substituting by (2.22), we obtain

$$\begin{aligned} & \lambda \left\langle y_2, (-\Delta)^{-1} y_2 \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \lambda \left\langle -\Delta y_1, (-\Delta)^{-1} y_1 \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ &= \left\langle g, (-\Delta)^{-1} y_2 \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \left\langle -\Delta y_1, (-\Delta)^{-1} f \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}. \end{aligned} \quad (2.23)$$

Using Lemma 1, the first term on the right hand side of (2.23) can be estimated,

$$\left\langle g, (-\Delta)^{-1} y_2 \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \leq \|g\|_{H^{-1}(\Omega)} \|(-\Delta)^{-1} y_2\|_{H_0^1(\Omega)} \quad (2.24)$$

$$\leq C \|g\|_{H^{-1}(\Omega)} \|y_2\|_{H^{-1}(\Omega)}. \quad (2.25)$$

Also, we can see by an application of Green's formula that

$$\begin{aligned} \left\langle -\Delta y_1, (-\Delta)^{-1} y_1 \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} &= \int_{\Omega} \left( (-\Delta y_1) ((-\Delta)^{-1} y_1) \right) dx \\ &= \int_{\Omega} y_1^2 dx. \end{aligned} \quad (2.26)$$

The same is true replacing  $y_1$  with  $y_2$ . Also,

$$\begin{aligned} \left\langle -\Delta y_1, (-\Delta)^{-1} f \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} &= \int_{\Omega} \left( (-\Delta y_1) ((-\Delta)^{-1} f) \right) dx \\ &= \int_{\Omega} y_1 f dx. \end{aligned} \quad (2.27)$$

From (2.23), (2.24), (2.26), and (2.27), we finally obtain

$$\lambda \|y_1\|_{L^2(\Omega)}^2 + \lambda \|y_2\|_{H^{-1}(\Omega)}^2 \leq C \left( \|f\|_{L^2(\Omega)} \|y_1\|_{L^2(\Omega)} + \|g\|_{H^{-1}(\Omega)} \|y_2\|_{H^{-1}(\Omega)} \right). \quad (2.28)$$

This bound is valid for all  $f \in L^2(\Omega)$ ,  $g \in H^{-1}(\Omega)$ , and so we can obtain the required estimate by taking the norm on  $L^2(\Omega) \times H^{-1}(\Omega)$  as  $\|(y_1, y_2)\|_{L^2(\Omega) \times H^{-1}(\Omega)} = \|y_1\|_{L^2(\Omega)}^2 + \|y_2\|_{H^{-1}(\Omega)}^2$ . By the Hille-Yosida Theorem,  $A$  is the generator of a semigroup of contractions on  $Y$ .  $\square$

### 2.2.1 Application

Here we apply the method to our specific problem of interest. We will use the transposition solutions defined in this section to the problem outlined in Chapter 3.

Consider the problem

$$\begin{cases} z_{tt} - \Delta z = f, & (x, t) \in \Omega \times (0, T), \\ z = u, & (x, t) \in \partial\Omega \times (0, T), \\ z(x, 0) = z_0(x), & x \in \Omega, \\ z_t(x, 0) = z_1(x), & x \in \Omega, \end{cases} \quad (2.29)$$

where  $f \in L^1(0, T; H^{-1}(\Omega))$ , the control function  $u(x, t) \in L^2(\partial\Omega \times (0, T))$ , and the initial

conditions  $z_0(x) \in L^2(\Omega)$ ,  $z_1(x) \in H^{-1}(\Omega)$ .

By Theorem 2, we know that problem (2.29) has a solution when  $u$  is identically 0. But, to study the control problem, one must make sense of what it means for  $u$  to control the boundary. This is made precise in the method of transposition. We now introduce the notion of solutions for (2.29). This will be the definition used when proving existence and uniqueness.

**Definition 2.2.1.** (Transposition Solution) A function  $z = z(x, t) \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$  is a *transposition solution* if

$$\begin{aligned} \int_0^T \int_{\Omega} f y dx dt &= \int_0^T \int_{\Omega} z \varphi dx dt + \left\langle \frac{\partial z}{\partial t}(T), y^T \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \\ &\quad - \left\langle \frac{\partial z}{\partial t}(0), y(0) \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} - \int_{\Omega} z(T) \gamma^T dx + \int_{\Omega} z(0) \frac{\partial y}{\partial t}(0) dx \\ &\quad + \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu} u ds dt, \end{aligned} \tag{2.30}$$

for all  $(\varphi, y^T, \gamma^T) \in L^1(0, T; L^2(\Omega)) \times H_0^1(\Omega) \times L^2(\Omega)$ , where  $y$  solves the problem

$$\begin{cases} y_{tt} - \Delta y = \varphi, & (x, t) \in \Omega \times (0, T), \\ y = 0, & (x, t) \in \partial\Omega \times (0, T), \\ y(x, T) = y^T, & x \in \Omega, \\ y_t(x, T) = \gamma^T, & x \in \Omega. \end{cases} \tag{2.31}$$

Now, we introduce the main existence/uniqueness theorem, in the sense of transposition, due to [38].

**Theorem 3.** (*Existence and Uniqueness of Transposition Solutions*). For every set of functions  $(f, u, z_0, z_1) \in L^1(0, T; H^{-1}(\Omega)) \times L^2(\partial\Omega \times (0, T)) \times L^2(\Omega) \times H^{-1}(\Omega)$ , the equation

(2.29) has a unique transposition solution  $z$ . The map to  $z$  is continuous as a map into  $C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ .

*Proof.* By Theorem 2, because of the semigroup theory, given initial data  $(f, z_0, z_1) \in L^1(0, T; H^{-1}(\Omega)) \times L^2(\Omega) \times H^{-1}(\Omega)$ , there is a unique solution  $z = z(f, 0, z_0, z_1)$  to system (2.29). The map to  $z$  is linear and continuous into  $C([0, T]; L^2(\Omega)) \cap C^1(0, T; H^{-1}(\Omega))$ . Thus, without loss of generality, by linearity we can consider the specific case where  $(f, z_0, z_1) = (0, 0, 0)$ . For, if  $z$  solves (2.29) for  $u \equiv 0$ , and  $\tilde{z}$  solves (2.29) for  $(f, z_0, z_1) = (0, 0, 0)$ , then  $w = z + \tilde{z}$  solves (2.29).

Now, let  $\Lambda$  be defined by  $\Lambda(\varphi) = \frac{\partial y}{\partial \nu}$ , where  $y$  solves

$$\begin{cases} y_{tt} - \Delta y = \varphi, & (x, t) \in \Omega \times (0, T), \\ y = 0, & (x, t) \in \partial\Omega \times (0, T), \\ y(x, T) = 0, & x \in \Omega, \\ y_t(x, T) = 0, & x \in \Omega. \end{cases} \quad (2.32)$$

By the Hidden Regularity estimate from Theorem 1, we have that  $\frac{\partial y}{\partial \nu} \in L^2(\partial\Omega \times (0, T))$ . Thus  $\Lambda$  defines a map  $\Lambda : L^2(\Omega \times (0, T)) \rightarrow L^2(\partial\Omega \times (0, T))$ . Now, set  $z = \Lambda^* u$ , where  $\Lambda^*$  is defined in the sense of  $\langle u, \Lambda\varphi \rangle_{L^2(\partial\Omega)} = \int_{\partial\Omega} u \frac{\partial y}{\partial \nu} ds = \langle \Lambda^* u, \varphi \rangle_{L^2(\Omega)}$ . Then,  $z \in L^2(\Omega \times (0, T))$ . By multiplying (2.29) by  $y$  and integrating by parts in space in time, we arrive at

$$-\left\langle \frac{\partial z}{\partial t}(0), y(0) \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \int_{\Omega} z(0) \frac{\partial y}{\partial t}(0) dx + \int_0^T \int_{\partial\Omega} \frac{\partial y}{\partial \nu} u ds dt = 0. \quad (2.33)$$

That is,  $z$  solves (2.29) in the transposition sense (uniquely) with zero initial data, in the case  $y^T = 0, \gamma^T = 0$ .



Now, we must show that the solution  $z$  actually does belong to  $z \in C([0, T]; L^2(\Omega)) \cap C^1([0, T]; H^{-1}(\Omega))$ . Let  $(u_n)_{n=1}^\infty$  be a regular approximating sequence. Let  $\tau \in [0, T]$ , and  $(y^\tau, \gamma^\tau) \in H_0^1(\Omega) \times L^2(\Omega)$ . Then there is a solution  $y \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$  to

$$\begin{cases} y_{tt} - \Delta y = 0, & (x, t) \in \Omega \times (0, T), \\ y = 0, & (x, t) \in \partial\Omega \times (0, T), \\ y(x, \tau) = y^\tau, & x \in \Omega, \\ y_t(x, \tau) = \gamma^\tau, & x \in \Omega. \end{cases} \quad (2.34)$$

Since the initial data has enough regularity, the Hidden Regularity estimate of Theorem 1 applies, and we have

$$\left\| \frac{\partial y(y^\tau, \gamma^\tau)}{\partial \nu} \right\|_{L^2(\partial\Omega \times (0, T))} \leq C \left( \|y^\tau\|_{H_0^1(\Omega)} + \|\gamma^\tau\|_{L^2(\Omega)} \right). \quad (2.35)$$

Now, recall definition 2.2.1 of the transposition solution. Take  $\gamma^\tau = 0$  and  $y^\tau = 0$  respectively. From (2.34), the two equations are satisfied by  $z_n \doteq z(0, u_n, 0, 0)$ :

$$\left\langle \frac{\partial z_n}{\partial t}(\tau), y^\tau \right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} = \int_\Omega \frac{\partial z_n}{\partial t}(\tau) y^\tau dx = - \int_0^\tau \int_{\partial\Omega} \frac{\partial y(y^\tau, 0)}{\partial \nu} u_n ds dt \quad (2.36)$$

and

$$\int_\Omega z_n(\tau) \gamma^\tau dx = \int_0^\tau \int_{\partial\Omega} \frac{\partial y(0, \gamma^\tau)}{\partial \nu} u_n ds dt. \quad (2.37)$$

From here, we can easily obtain the Cauchy estimates. By using Hölder's inequality and (2.35),

$$\|z_n - z_m\|_{C([0, T]; L^2(\Omega))} = \sup_{\tau \in [0, T]} \sup_{\|\gamma^\tau\|_{L^2(\Omega)}=1} \left| \int_0^\tau \int_{\partial\Omega} \frac{\partial y(0, \gamma^\tau)}{\partial \nu} (u_n - u_m) ds dt \right|$$

$$\leq C \|u_n - u_m\|_{L^2(\partial\Omega \times (0,T))}, \quad (2.38)$$

and

$$\begin{aligned} \|(z_n)_t - (z_m)_t\|_{C([0,T];H^{-1}(\Omega))} &= \sup_{\tau \in [0,T]} \sup_{\|y^\tau\|_{H_0^1(\Omega)}=1} \left| \int_0^\tau \int_{\partial\Omega} \frac{\partial y(y^\tau, 0)}{\partial \nu} (u_n - u_m) ds dt \right| \\ &\leq C \|u_n - u_m\|_{L^2(\partial\Omega \times (0,T))}. \end{aligned} \quad (2.39)$$

Taking limits, the theorem is proved. □

# Chapter 3

## Hyperbolic Equations

### 3.1 Introduction

In this section, we consider global Carleman type estimates for the one-dimensional wave equation with variable coefficients and a potential term. Specifically, the system of study is the following:

$$\left\{ \begin{array}{ll} u_{tt} - (a(x)u_x)_x + p(x, t)u = 0, & (x, t) \in (0, 1) \times (0, T), \\ u(0, t) = 0, \quad u(1, t) = v(t), & t \in (0, T), \\ u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), & x \in (0, 1). \end{array} \right. \quad (3.1)$$

The end time is fixed with  $T > 0$ . Also, we assume that  $a(x) \in C^2([0, 1])$  with  $a(x) \geq a_0 > 0$  in  $[0, 1]$ , the potential  $p(x) \in L^\infty((0, 1) \times (0, T))$ , and the initial conditions  $u_0 \in L^2(0, 1)$  and  $u_1 \in H^{-1}(0, 1)$ . Here,  $v = v(t) \in L^2(0, T)$  is called the *control function* and  $u = u(x, t)$  is the associated state.

From Theorem 3, for  $(u_0, u_1) \in L^2(0, 1) \times H^{-1}(0, 1)$ , there is a unique transposition solution  $u(x, t)$  to (3.1), such that

$$u \in C^0([0, T]; L^2(0, 1)) \cap C^1([0, T]; H^{-1}(0, 1)). \quad (3.2)$$

The *exact controllability problem* for (3.1) at time  $T > 0$  is then the following: Given

initial conditions  $(u_0, u_1)$  and final conditions  $(z_0, z_1)$ , find a control function  $v \in L^2(0, 1)$  such that the corresponding solution  $u$  satisfies

$$u(x, T) = z_0, u_t(x, T) = z_1, \text{ for } x \in (0, 1). \quad (3.3)$$

In other words, through the control function  $v(t)$ , is it possible to drive a set of given initial conditions to a prescribed final state? Because of linearity and time reversibility of the operator of study, exact controllability of (3.1) at time  $t = T$  is equivalent to *null controllability* at  $T$ . That is, under the same initial conditions, the corresponding solution  $u$  satisfies

$$u(x, T) = 0, u_t(x, T) = 0, \text{ for } x \in (0, 1). \quad (3.4)$$

The controllability problem for the wave equation has been studied for both one dimension and higher dimensions. It is known that (3.1) is null controllable for a large enough time  $T$  depending on  $a$ , although the majority of previous works focus mainly on the case  $a \equiv 1$ . By the Hilbert Uniqueness Method [38], the null controllability of (3.1) is equivalent to the *observability* of the adjoint problem under certain geometric conditions from [4] on subsets  $\Gamma(x_0)$  of the boundary  $\Gamma$ . These geometric conditions are taken into account because of possible Gaussian beam solutions from [41], [43], [44], that can propagate along curves. Thus, for  $x_0 \in \mathbb{R}^n$ , if one considers subsets of the form

$$\Gamma(x_0) = \{x \in \Gamma : (x - x_0) \cdot n(x) > 0\},$$

then the boundary observability inequality for the controllability problem is

$$\|(u_0, u_1)\|_{L^2(\Omega) \times H^{-1}(\Omega)}^2 \leq C \int_0^T \int_{\Gamma(x_0)} \left| \frac{\partial u}{\partial n} \right|^2 d\Gamma dt. \quad (3.5)$$

Previous authors have developed a variety of tools to derive this inequality. Multiplier methods, [32], [38], microlocal analysis, [4], finite difference, [56], and Carleman estimates [5], [6], [13], [22], [24], [27], [29], [49], among other techniques have been developed. In addition, for particular cases, sharp results about the observability constant  $C$ , and the time  $T$  are known; see for example [12], [15], [50], [52], and [53]. The problem has been studied in the context of unique continuation in [2], [14], [28], [35], [30], [31], [47], and [48]. Controllability results for a subset of functions  $a(x)$  for higher dimensions was considered in [40]. A survey paper for these results can be found in [54].

We note here that we consider control on the boundary. One could also consider a control function as acting in the interior of the domain (as will be done in Chapter 4). In some sense, the boundary control is connected to internal control in the limit, as in [17], [18].

Carleman estimates (developed in section 3.2), the focus of this chapter, offer more information and utility than just the existence of the observability inequality. They give, for example, the advantage of being able to easily deal with  $L^\infty$  potential terms, while requiring less regularity than the microlocal analysis technique. This flexibility means that studying the effects that the variable coefficient  $a(x)$  introduces is of interest.

Once the Carleman estimate is proved, a variational technique adapted from [27] naturally produces the control function required. Thus, the question becomes: For which functions  $a(x)$  does the Carleman estimate hold?

In the recent work [13], the authors study the same control problem (3.1) for the one

dimensional case. They prove Carleman estimates and controllability for (3.1), but with a restrictive requirement on the function  $a(x)$ . Specifically, they require  $a(x)$  to belong to the family

$$\mathcal{A}(x_0, a_0) = \left\{ a \in C^3([0, 1]) : a(x) \geq a_0 > 0, \right. \\ \left. - \min_{[0,1]} (a(x) + (x - x_0)a_x(x)) < \min_{[0,1]} \left( a(x) + \frac{1}{2}(x - x_0)a_x(x) \right) \right\}, \quad (3.6)$$

where  $x_0 < 0$  and  $a_0 > 0$  is a positive constant. It can be easily seen that all constant functions are in this family. Also, since  $x_0 < 0$ ,  $\mathcal{A}$  contains all strictly positive, monotonically increasing functions. Furthermore, if the derivative is small,  $|a_x| \ll 1$ , the functions will belong to this family. However, if the function decreases too rapidly in  $(0,1)$ , this condition may fail to be true.

In Section 3.2, we consider a different condition on  $a(x)$ , providing more functions for which controllability of (3.1) will hold, as well as an computationally easier condition to check. Let us take  $x_0 < 0$  and  $a_0 > 0$  a positive constant. We introduce the family

$$\mathcal{B}(x_0, a_0) = \left\{ a \in C^2([0, 1]) : a(x) \geq a_0 > 0, \quad 2 - \frac{a_x(x)(x - x_0)}{a(x)} > 0 \right\}. \quad (3.7)$$

It will be seen that this is a natural choice of family, given the weight function that will be chosen in Section 3.1.1. Besides the decreased regularity needed on  $a(x)$ , notice that  $\mathcal{B}(x_0, a_0)$  also contains all constant functions, as well as any strictly positive, monotonically decreasing function. From now on, we shall always assume that the function  $a(x)$  belongs to  $\mathcal{B}(x_0, a_0)$ . Also, for the remainder of the section, by a time translation we will use the time interval  $(-T, T)$  to make the notation and proofs easier. With this assumption, the main

result of this section is the existence of a control function  $v(t)$  so that the controllability for (3.1) holds true.

**Theorem 4.** *Given initial conditions  $u_0(x) \in L^2(0, 1)$ ,  $u_1(x) \in H^{-1}(0, 1)$ , for  $T$  large enough, there exists a control function  $v(t) \in L^2(\{1\} \times (-T, T))$  such that the solution  $u(x, t)$  of*

$$\begin{cases} u_{tt} - (a(x)u_x)_x + p(x, t)u = 0, & (x, t) \in (0, 1) \times (-T, T), \\ u(0, t) = 0, \quad u(1, t) = v(t), & t \in (-T, T), \\ u(x, -T) = u_0(x), \quad u_t(x, -T) = u_1(x), & x \in (0, 1), \end{cases} \quad (3.8)$$

satisfies  $u(x, T) = u_t(x, T) = 0$ .

The rest of this chapter is organized as follows. In Section 3.1.1, we outline the particular weight functions used for the Carleman estimate. In Section 3.2, we state and prove various Carleman estimates. First, the estimate is proved in the special case that the functions vanish at the time endpoints, and then this restriction is eventually removed. Here we give a precise time for the  $T$  condition necessary in Theorem 4. In Section 3.3, after some preliminary definitions, we use the results of Section 3.2 to prove Theorem 4. This theorem will then be extended in Section 3.4, where we prove an increased regularity result.

### 3.1.1 Weight Functions

Carleman Estimates give weighted bounds on functions and their derivatives. As in [13], often the weight functions are defined in the following fashion: Let  $x_0 \notin \overline{\Omega} \in \mathbb{R}^n$  and

$\beta \in (0, 1)$ . Define for  $(x, t) \in \Omega \times (-T, T)$ ,

$$\tilde{\psi}(x, t) = |x - x_0|^2 - \beta t^2 + C_0, \text{ and for } \lambda > 0, \tilde{\varphi}(x, t) = e^{\lambda \tilde{\psi}(x, t)}, \quad (3.9)$$

where  $C_0 > 0$  is chosen such that  $\tilde{\psi} \geq 1$  in  $\Omega \times (-T, T)$ .

However, since the system of study (3.1) contains the function  $a(x)$ , it is reasonable that this should be incorporated into the weight function. Thus, in this system, we consider a special test function for the one-dimensional case. Fix  $x_0 < 0$ , and then define the weight function

$$\psi(x, t) = \int_0^x \frac{s - x_0}{a(s)} ds - \beta t^2 + C_0, \text{ and for } \lambda > 0, \varphi(x, t) = e^{\lambda \psi(x, t)}, \quad (3.10)$$

where  $C_0 > 0$  is chosen such that  $\psi \geq 1$  in  $(0, 1) \times (-T, T)$ . Note that by the choice of  $a(x) \in \mathcal{B}(x_0, a_0)$  from (3.7), then (3.10) is well defined. Here  $\varphi(x, t)$  depends on  $\beta$  and  $\lambda$ , but for simplicity of notation these dependencies are omitted. Also, as in [5] we define for  $m > 0$ , the spaces

$$L_{\leq m}^\infty(0, 1) = \{q \in L^\infty(0, 1), \|q\|_{L^\infty(0, 1)} \leq m\},$$

$$L_{\leq m}^\infty((0, 1) \times (-T, T)) = \{p \in L^\infty((0, 1) \times (-T, T)), \|p\|_{L^\infty} \leq m\}.$$

We will also use the following notation when appropriate:

$$Lu := u_{tt} - (a(x)u_x)_x. \quad (3.11)$$



## 3.2 Carleman Estimates

The method of deriving the Carleman Estimate comes from incorporating the weight function into a series of integration by parts. This type of argument was used in the method from [22]. Also, see [5], [27], [53]. The novelty comes from the new weight function, where  $\psi$  now has  $a(x)$  in its definition. First, a new state  $w$  is considered from the original state and the weight function. From here, the estimate can be derived for  $w$ . The main result is the global Carleman estimate.

**Theorem 5.** (*Carleman Estimate*). *Assume that  $x_0 < 0, a_0 > 0$ . There exists  $\beta \in (0, 1)$  such that*

$$T > \frac{1}{\beta} \max_{x \in [0, 1]} \frac{(x - x_0)}{\sqrt{a(x)}}. \quad (3.12)$$

*Then for any  $m > 0$ , there exists  $\lambda > 0$  independent of  $m, s_0 = s_0(m) > 0$  and a positive constant  $M = M(m)$  such that for  $\psi$  and  $\varphi$  as defined as in (3.10), for  $a(x) \in \mathcal{B}(x_0, a_0)$ , for all  $p(x, t) \in L_{\leq m}^\infty((0, 1) \times (-T, T))$  and for all  $s \geq s_0$ ,*

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |w|^2 dx dt \\ & + s \int_0^1 e^{2s\varphi(-T)} (|w_t(-T)|^2 + |w_x(-T)|^2) dx + s^3 \int_0^1 e^{2s\varphi(-T)} |w(-T)|^2 dx \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw + pw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt, \end{aligned} \quad (3.13)$$

*for all  $w \in L^2(-T, T; H_0^1(0, 1))$  also satisfying  $w_{tt} - (a(x)w_x)_x + pw \in L^2((0, 1) \times (-T, T))$  and  $w_x(1, t) \in L^2(-T, T)$ .*

Before showing Theorem 5, we first prove a Carleman estimate in the special case where the function and its derivative vanishes at the endpoints, so that  $v(\cdot, \pm T) = v_t(\cdot, \pm T) = 0$ .

Note here that  $T$  can be arbitrary.

**Theorem 6.** *Let  $\varphi(x, t)$  be defined as in (3.10). Also, let  $a(x) \in \mathcal{B}(x_0, a_0)$  as defined in (3.7). Then, there exists positive constants  $s_0$ , and  $M$ , depending only on  $x_0, a_0, \|a\|$ , and  $T$  such that, for all  $s \geq s_0$ ,*

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|v_t|^2 + |v_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lv|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |v_x(1, t)|^2 dt \end{aligned}$$

for any  $v \in L^2(-T, T; H_0^1(0, 1))$  also satisfying  $Lv \in L^2((0, 1) \times (-T, T))$ ,  $v_x(1, t) \in L^2(-T, T)$ , and  $v(\cdot, \pm T) = v_t(\cdot, \pm T) = 0$ .

*Remark:* All computations can be done for smooth functions  $v$ . By classical results using density arguments, the result will hold true for all  $v(x, t)$  satisfying the conditions of Theorem 6.

*Proof.* As noted previously, the proof follows similar arguments to that in [13]. We indicate in the proof especially where the new admissibility condition of (3.7) and weight function of (3.10) are used.

Consider the function

$$w = e^{s\varphi} v, \tag{3.14}$$

and define  $Pw$  as the conjugation

$$Pw = e^{s\varphi} L(e^{-s\varphi} w) = e^{s\varphi} ((e^{-s\varphi} w)_{tt} - (a(e^{-s\varphi} w)_x)_x). \tag{3.15}$$

By the admissibility condition (3.7), we can find constants  $\alpha, \beta$  such that

$$0 < \frac{4\beta}{2\beta + 1} < \alpha < \frac{2 - a_x \psi_x}{2\beta + 1}. \quad (3.16)$$

Then, we can decompose the function as  $Pw = P_1w + P_2w + Rw$ , where

$$\begin{aligned} P_1w &= w_{tt} - (aw_x)_x + s^2\lambda^2\varphi^2w(|\psi_t|^2 - a|\psi_x|^2), \\ P_2w &= (\alpha - 1)s\lambda\varphi w(\psi_{tt} - (a\psi_x)_x) - s\lambda^2\varphi w(|\psi_t|^2 - a|\psi_x|^2) \\ &\quad - 2s\lambda\varphi(\psi_tw_t - a\psi_xw_x), \\ Rw &= -\alpha s\lambda\varphi w(\psi_{tt} - (a\psi_x)_x). \end{aligned} \quad (3.17)$$

Let us note here that by the choice of  $\psi$  from (3.10), the term  $(\psi_{tt} - (a\psi_x)_x) = -2\beta - 1$  is constant. This will simplify some of the expressions below. However, we keep this expression in its generic form, so that one may use different possible weight functions not considered in this work. We will first estimate the integral

$$I = \int_{-T}^T \int_0^1 (P_1w)(P_2w) dx dt = \sum_{i,j=1}^3 I_{ij}. \quad (3.18)$$

There are nine integrals to compute. Throughout, we will repeatedly use the assumption that  $w(\cdot, \pm T) = w_t(\cdot, \pm T) = 0$ . Then, integration by parts in time, space, or both gives the following explicit representations. Because of the complexity of the expressions, only the final representations will be provided, without intermediate computation.

$$I_{11} = (\alpha - 1)s\lambda \int_{-T}^T \int_0^1 w_{tt}\varphi w(\psi_{tt} - (a\psi_x)_x) dx dt$$

$$\begin{aligned}
&= (1 - \alpha)s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (\psi_{tt} - (a\psi_x)_x) dx dt \\
&- \frac{(1 - \alpha)}{2} s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 \psi_{tt} (\psi_{tt} - (a\psi_x)_x) dx dt \\
&- \frac{(1 - \alpha)}{2} s\lambda^3 \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_t|^2 (\psi_{tt} - (a\psi_x)_x) dx dt.
\end{aligned}$$

$$\begin{aligned}
I_{12} &= -s\lambda^2 \int_{-T}^T \int_0^1 w_{tt} \varphi w (|\psi_t|^2 - a|\psi_x|^2) dx dt \\
&= s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w_t|^2 (|\psi_t|^2 - a|\psi_x|^2) dx dt - s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_{tt}|^2 dx dt \\
&- \frac{5s\lambda^3}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_t|^2 \psi_{tt} dx dt + \frac{s\lambda^3}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 a |\psi_x|^2 \psi_{tt} dx dt \\
&- \frac{s\lambda^4}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 |\psi_t|^2 (|\psi_t|^2 - a|\psi_x|^2) dx dt.
\end{aligned}$$

$$\begin{aligned}
I_{13} &= -2s\lambda \int_{-T}^T \int_0^1 w_{tt} \varphi (\psi_t w_t - a\psi_x w_x) dx dt \\
&= s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 \psi_{tt} dx dt + s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w_t|^2 |\psi_t|^2 dx dt \\
&+ s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (a\psi_x)_x dx dt + s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w_t|^2 a |\psi_x|^2 dx dt \\
&- 2s\lambda^2 \int_{-T}^T \int_0^1 \varphi a \psi_x \psi_t w_x w_t dx dt.
\end{aligned}$$

Making use of the fact that  $\psi_{tt} - (a\psi_x)_x$  is constant,

$$\begin{aligned}
I_{21} &= (1 - \alpha)s\lambda \int_{-T}^T \int_0^1 (aw_x)_x \varphi w (\psi_{tt} - (a\psi_x)_x) dx dt \\
&= -(1 - \alpha)s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\psi_{tt} - (a\psi_x)_x) dx dt \\
&+ \frac{(1 - \alpha)}{2} s\lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 (a\psi_x)_x (\psi_{tt} - (a\psi_x)_x) dx dt \\
&+ \frac{(1 - \alpha)}{2} s\lambda^3 \int_{-T}^T \int_0^1 \varphi a |w|^2 |\psi_x|^2 (\psi_{tt} - (a\psi_x)_x) dx dt
\end{aligned}$$

$$\begin{aligned}
& - (1 - \alpha) s \lambda^2 \int_{-T}^T \int_0^1 \varphi |w|^2 a \psi_x (a \psi_x)_{xx} dx dt \\
& - \frac{(1 - \alpha)}{2} s \lambda \int_{-T}^T \int_0^1 \varphi |w|^2 (a_x (a \psi_x)_{xx} + a (a \psi_x)_{xxx}) dx dt.
\end{aligned}$$

$$\begin{aligned}
I_{22} &= s \lambda^2 \int_{-T}^T \int_0^1 (a w_x)_x \varphi w (|\psi_t|^2 - a |\psi_x|^2) dx dt \\
&= -s \lambda^2 \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (|\psi_t|^2 - a |\psi_x|^2) dx dt \\
&\quad - \frac{s \lambda^2}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 \left( (|a_x|^2 + a a_{xx}) |\psi_x|^2 + 4 a a_x \psi_x \psi_{xx} + 2 a (a \psi_x)_x \psi_{xx} \right) dx dt \\
&\quad + \frac{s \lambda^3}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 (a \psi_x)_x (|\psi_t|^2 - a |\psi_x|^2) dx dt \\
&\quad + \frac{s \lambda^4}{2} \int_{-T}^T \int_0^1 \varphi |w|^2 a |\psi_x|^2 (|\psi_t|^2 - a |\psi_x|^2) dx dt \\
&\quad - s \lambda^3 \int_{-T}^T \int_0^1 \varphi |w|^2 a \psi_x (a_x |\psi_x|^2 + 2 a \psi_x \psi_{xx}) dx dt.
\end{aligned}$$

$$\begin{aligned}
I_{23} &= 2s \lambda \int_{-T}^T \int_0^1 (a w_x)_x \varphi (\psi_t w_t - a \psi_x w_x) dx dt \\
&= s \lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\psi_{tt} + a \psi_{xx}) dx dt \\
&\quad + s \lambda^2 \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (|\psi_t|^2 + a |\psi_x|^2) dx dt - 2s \lambda^2 \int_{-T}^T \int_0^1 \varphi a \psi_x \psi_t w_x w_t dx dt \\
&\quad - 2s \lambda \int_{-T}^T [a(1)^2 |w_x(1, t)|^2 \varphi(1, t) \psi_x(1, t) - a(0)^2 |w_x(0, t)|^2 \varphi(0, t) \psi_x(0, t)] dt.
\end{aligned}$$

$$I_{31} = (\alpha - 1) s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a |\psi_x|^2) (\psi_{tt} - (a \psi_x)_x) dx dt.$$

$$I_{32} = -s^3 \lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a |\psi_x|^2)^2 dx dt.$$

$$\begin{aligned}
I_{33} &= -2s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 w (|\psi_t|^2 - a|\psi_x|^2) (\psi_t w_t - a\psi_x w_x) dx dt \\
&= s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad + 2s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 \left( |\psi_t|^2 \psi_{tt} + \frac{1}{2} a a_x \psi_x |\psi_x|^2 + a^2 |\psi_x|^2 \psi_{xx} \right) dx dt \\
&\quad + 3s^3\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt.
\end{aligned}$$

Collecting all the terms and simplifying, we find that

$$\begin{aligned}
I &= \int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt \\
&= s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (2\psi_{tt} - \alpha(\psi_{tt} - (a\psi_x)_x)) dx dt \\
&\quad + s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\alpha(\psi_{tt} - (a\psi_x)_x) + 2(a\psi_x)_x - a_x \psi_x) dx dt \\
&\quad + 2s\lambda^2 \int_{-T}^T \int_0^1 \varphi (|w_t|^2 |\psi_t|^2 - 2a\psi_x \psi_t w_x w_t + a^2 |w_x|^2 |\psi_x|^2) dx dt \\
&\quad + 2s^3\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt \\
&\quad + s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (2|\psi_t|^2 \psi_{tt} + a a_x \psi_x |\psi_x|^2 + 2a^2 |\psi_x|^2 \psi_{xx}) dx dt \\
&\quad + \alpha s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt \\
&\quad - 2s\lambda \int_{-T}^T \left( a(1)^2 |w_x(1, t)|^2 \varphi(1, t) \psi_x(1, t) - a(0)^2 |w_x(0, t)|^2 \varphi(0, t) \psi_x(0, t) \right) dt \\
&\quad + X,
\end{aligned}$$

where  $X$  is the sum of the remaining terms, satisfying

$$|X| \leq Ms\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt.$$

Now, we will take advantage of the new class of admissible functions of  $\mathcal{B}(x_0, a_0)$  in estimating  $I$ . First, let us notice that the third term above is a perfect square, and so we have

$$2s\lambda^2 \int_{-T}^T \int_0^1 \varphi \left( |w_t|^2 |\psi_t|^2 - 2a\psi_x \psi_t w_x w_t + a^2 |w_x|^2 |\psi_x|^2 \right) dx dt \geq 0.$$

Consider next the terms of order  $s\lambda$ . Let us briefly mention here that estimation of these  $s\lambda$  terms is how the authors from [13] arrived at the class  $\mathcal{A}$  defined in (3.6). For us, since  $\alpha$  and  $\beta$  were chosen such that  $\frac{4\beta}{2\beta+1} < \alpha$  by (3.16), we obtain by computation that the integrand of the first term in  $I$  satisfies

$$2\psi_{tt} - \alpha(\psi_{tt} - (a\psi_x)_x) > 0.$$

Similarly, since  $\alpha < \frac{2-a_x\psi_x}{2\beta+1}$  by (3.16), we obtain that the integrand of the second term satisfies

$$\alpha(\psi_{tt} - (a\psi_x)_x) + 2(a\psi_x)_x - a_x\psi_x > 0.$$

Thus, the order  $s\lambda$  terms can be estimated as follows:

$$\begin{aligned} & s\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 (2\psi_{tt} - \alpha(\psi_{tt} - (a\psi_x)_x)) dx dt \\ & + s\lambda \int_{-T}^T \int_0^1 \varphi a |w_x|^2 (\alpha(\psi_{tt} - (a\psi_x)_x) + 2(a\psi_x)_x - a_x\psi_x) dx dt \\ & \geq Ms\lambda \int_{-T}^T \int_0^1 \varphi |w_t|^2 dx dt + Ms\lambda \int_{-T}^T \int_0^1 \varphi |w_x|^2 dx dt. \end{aligned}$$

It remains to estimate the higher order terms. We can rewrite these terms in the following

fashion:

$$\begin{aligned}
& 2s^3\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2)^2 dx dt \\
& + s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (2|\psi_t|^2 \psi_{tt} + a a_x \psi_x |\psi_x|^2 + 2a^2 |\psi_x|^2 \psi_{xx}) dx dt \\
& + \alpha s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 (|\psi_t|^2 - a|\psi_x|^2) (\psi_{tt} - (a\psi_x)_x) dx dt \\
& = s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 F_\lambda(x, Y(x, t)) dx dt,
\end{aligned}$$

where we have let  $Y = |\psi_t|^2 - a|\psi_x|^2$  and

$$\begin{aligned}
F_\lambda &= 2\lambda Y^2 + (2\psi_{tt} + \alpha(\psi_{tt} - (a\psi_x)_x))Y + a|\psi_x|^2(2\psi_{tt} + a_x\psi_x + 2a\psi_{xx}) \\
&= 2\lambda Y^2 - (4\beta + \alpha(2\beta + 1))Y - a(x)|\psi_x|^2(4\beta + a_x\psi_x - 2).
\end{aligned}$$

This is a quadratic polynomial in  $Y$ . As such, since  $\lambda > 0$ , we can estimate by the critical point to get  $F_\lambda(x, Y) \geq -\frac{(4\beta + \alpha(2\beta + 1))^2}{8\lambda} - a(x)|\psi_x|^2(4\beta + a_x\psi_x - 2)$ . Once again, by (3.16), we have  $4\beta + a_x\psi_x - 2 < 0$ . Thus, for  $\lambda$  large enough the function  $F_\lambda$  is nonnegative, and we can conclude

$$s^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 F_\lambda(x) dx dt \geq Ms^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt.$$

Combining all of the estimates so far gives

$$\begin{aligned}
I &= \int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt \\
&\geq Ms\lambda \int_{-T}^T \int_0^1 \varphi(|w_t|^2 + |w_x|^2) dx dt + Ms^3\lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt
\end{aligned} \tag{3.19}$$



$$-Ms\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt - Ms\lambda \int_{-T}^T |w_x(1, t)|^2 dt.$$

Let us use (3.19) to finally obtain the desired estimate. Recall the decomposition (3.17) of  $Pw$  to write

$$\int_{-T}^T \int_0^1 (|P_1 w|^2 + |P_2 w|^2 + 2(P_1 w)(P_2 w)) dx dt = \int_{-T}^T \int_0^1 |Pw - Rw|^2 dx dt.$$

By the Cauchy estimate, we have that

$$\int_{-T}^T \int_0^1 |Pw - Rw|^2 dx dt \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms^2 \lambda^2 \int_{-T}^T \int_0^1 \varphi^2 |w|^2 dx dt.$$

Thus, we obtain

$$\int_{-T}^T \int_0^1 (P_1 w)(P_2 w) dx dt \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms^2 \lambda^2 \int_{-T}^T \int_0^1 \varphi^2 |w|^2 dx dt. \quad (3.20)$$

Combining (3.19) and (3.20) gives

$$\begin{aligned} & s\lambda \int_{-T}^T \int_0^1 \varphi(|w_t|^2 + |w_x|^2) dx dt + s^3 \lambda^3 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 |Pw|^2 dx dt + Ms\lambda \int_{-T}^T |w_x(1, t)|^2 dt \\ & + Ms\lambda^4 \int_{-T}^T \int_0^1 \varphi^3 |w|^2 dx dt + Ms^2 \lambda^2 \int_{-T}^T \int_0^1 \varphi^2 |w|^2 dx dt. \end{aligned}$$

When  $s$  is large, the last two terms on the right hand side can be absorbed to the left.

Finally, we can use  $w = ve^{s\varphi}$ ,  $Pw = e^{s\varphi}Lv$  to rewrite the equation in terms of  $v$ , so that

Theorem 6 holds. We obtain

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|v_t|^2 + |v_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lv|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |v_x(1, t)|^2 dt \end{aligned}$$

□

### 3.2.1 Carleman Estimate for large $T$

Theorem 6 was valid for functions that vanished at the endpoints. If the time  $T$  is large enough, this condition can be removed. The argument will combine the previous Carleman inequality along with an energy estimate. The major ideas of the proof can be found in [5], although some modification is needed.

First, we recall the weighted Poincaré inequality from [5], as applied to this case:

**Lemma 2.** (*Poincaré*): *Let  $\phi \in C^2([0, 1])$  and assume that*

$$\inf_{[0,1]} |\phi_x| \geq \delta > 0.$$

*Then, there exists some constants  $s_0 > 0$  and  $M > 0$  such that for all  $s \geq s_0$ , and for all  $z \in H_0^1([0, 1])$ ,*

$$s^2 \int_0^1 e^{2s\phi} |z|^2 dx \leq M \int_0^1 e^{2s\phi} |z_x|^2 dx. \quad (3.21)$$

For the following, we will need to use a slightly modified form of this inequality, with

$a(x) \in \mathcal{B}(x_0, a_0)$ , which is immediately seen.

$$s^2 \int_0^1 e^{2s\phi} |z|^2 dx \leq M \int_0^1 e^{2s\phi} a(x) |z_x|^2 dx. \quad (3.22)$$

Note that by direct computation, our weight function  $\varphi(x, t)$  defined in (3.10) indeed satisfies the hypothesis for Lemma 2. Then, we can state the Carleman estimate for large enough  $T$ .

**Theorem 7.** *Let  $\varphi(x, t)$  be defined as in (3.10). Also, let  $\beta$  satisfy (3.16) and  $T$  be such that  $T > \frac{1}{\beta} \max_{x \in [0, 1]} \frac{(x - x_0)}{\sqrt{a(x)}}$ . Then there exists some  $s_0 > 0$  and a positive constant  $M$ , such that for all  $s \geq s_0$ ,*

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt, \end{aligned} \quad (3.23)$$

for all  $w \in L^2((-T, T); H_0^1(0, 1))$  satisfying  $Lw \in L^2((0, 1) \times (-T, T))$  and  $w_x(1, t) \in L^2(-T, T)$ .

*Proof.* By the assumption on  $T$ , there exists some  $\eta \in (0, T)$ , and  $\epsilon > 0$  such that

$$(1 - \epsilon)(T - \eta)\beta \geq \max_{x \in [0, 1]} \frac{x - x_0}{\sqrt{a(x)}}. \quad (3.24)$$

To facilitate work near the time endpoints, we introduce a cutoff function  $\chi \in C_c^\infty(\mathbb{R})$  such that  $0 \leq \chi \leq 1$  and

$$\chi(t) = \begin{cases} 1, & \text{if } |t| \leq T - \eta, \\ 0, & \text{if } |t| \geq T. \end{cases}$$

Set  $v = \chi w$ . This function satisfies the hypotheses  $v(\cdot, \pm T) = v_t(\cdot, \pm T) = 0$  of Theorem 6, and so the Carleman estimate from there applies:

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|v_t|^2 + |v_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lv|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |v_x(1, t)|^2 dt. \end{aligned}$$

Now, since  $Lv = \chi Lw + \chi_{tt}w + 2\chi_t w_t$ , it follows that

$$\begin{aligned} & s \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} |w|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt \\ & + M \int_{-T}^{-T+\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w|^2) dx dt + M \int_{T-\eta}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w|^2) dx dt. \end{aligned} \tag{3.25}$$

Denote by  $E_s = E_s(t)$  the energy correlated with  $L$ ,

$$E_s(t) = \frac{1}{2} \int_0^1 e^{2s\varphi} (|w_t|^2 + a(x)|w_x|^2) dx, \text{ for all } t \in (-T, T).$$

Then, taking derivatives,

$$\frac{dE_s}{dt} = s \int_0^1 e^{2s\varphi} \varphi_t (|w_t|^2 + a|w_x|^2) dx + \int_0^1 e^{2s\varphi} (w_t w_{tt} + a w_{xt} w_x) dx.$$

Through integration by parts,

$$\begin{aligned} & \frac{dE_s}{dt} - s \int_0^1 e^{2s\varphi} \varphi_t (|w_t|^2 + a|w_x|^2) dx + 2s \int_0^1 e^{2s\varphi} a \varphi_x w_t w_x dx \\ & = \int_0^1 e^{2s\varphi} w_t Lw dx. \end{aligned} \tag{3.26}$$

We must now estimate these terms. Let us first restrict the value of  $t$  to work in one specific time interval.

**Estimate for  $t \in (T - \eta, T)$ :**

We claim that

$$\frac{dE_s}{dt} - s \int_0^1 e^{2s\varphi} (\varphi_t + \sqrt{a}\varphi_x) (|w_t|^2 + a|w_x|^2) dx \leq \int_0^1 e^{2s\varphi} w_t L w dx. \quad (3.27)$$

To see this, first note that since  $a(x)$  and  $\varphi_x$  are strictly positive, we have

$$2s\varphi_x w_t w_x a \geq -s\varphi_x \sqrt{a} (|w_t|^2 + a|w_x|^2).$$

So, from (3.26) above, this gives

$$\begin{aligned} \frac{dE_s}{dt} - s \int_0^1 e^{2s\varphi} \varphi_t (|w_t|^2 + a|w_x|^2) dx - s \int_0^1 e^{2s\varphi} \varphi_x \sqrt{a} (|w_t|^2 + a|w_x|^2) dx \\ \leq \int_0^1 e^{2s\varphi} w_t L w dx, \end{aligned}$$

from which

$$\frac{dE_s}{dt} - s \int_0^1 e^{2s\varphi} (\varphi_t + \sqrt{a}\varphi_x) (|w_t|^2 + a|w_x|^2) dx \leq \int_0^1 e^{2s\varphi} w_t L w dx. \quad (3.28)$$

Due to (3.24), the definition of  $\varphi$  from (3.10), and the restriction of  $t \in (T - \eta, T)$ , we notice that

$$-(\varphi_t + \sqrt{a}\varphi_x) \geq \left( 2\beta(T - \eta) - \sqrt{a} \frac{x - x_0}{a(x)} \right) e^{\lambda\psi} \geq c > 0.$$

Thus, we arrive at

$$\frac{dE_s}{dt} + sc \int_0^1 e^{2s\varphi} (|w_t|^2 + a|w_x|^2) dx \leq \int_0^1 e^{2s\varphi} w_t L w dx. \quad (3.29)$$

The right hand side can also be bounded,

$$\left| \int_0^1 e^{2s\varphi} w_t L w dx \right| \leq \frac{sc}{2} \int_0^1 e^{2s\varphi} |w_t|^2 dx + \frac{1}{2sc} \int_0^1 |Lw|^2 dx.$$

By absorbing the first term above to the left hand side of (3.29), and using the definition of the energy, we have

$$\frac{dE_s}{dt} + \frac{1}{2} sc E_s \leq \frac{1}{2sc} \int_0^1 e^{2s\varphi} |Lw|^2 dx.$$

Then, by the Gronwall Lemma, for all  $t \in (T - \eta, T)$ ,

$$\begin{aligned} E_s(t) &\leq E_s(T - \eta) e^{sc(T - \eta - t)} + \frac{1}{2sc} \int_{T - \eta}^t e^{sc(\tau - t)} \int_0^1 e^{2s\varphi(\tau)} |Lw(\tau)|^2 dx d\tau \\ &\leq E_s(T - \eta) e^{sc(T - \eta - t)} + \frac{1}{2sc} \int_{T - \eta}^T \int_0^1 e^{2s\varphi(\tau)} |Lw(\tau)|^2 dx d\tau. \end{aligned} \quad (3.30)$$

Integration in the  $t$  variable gives

$$\begin{aligned} \int_{T - \eta}^T E_s(t) dt &\leq E_s(T - \eta) \int_{T - \eta}^T e^{sc(T - \eta - t)} dt + \frac{\eta}{2sc} \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt \\ &\leq \frac{M}{s} E_s(T - \eta) + \frac{M}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt. \end{aligned} \quad (3.31)$$

It remains then to get a pointwise bound on  $E_s(T - \eta)$ , in terms of  $E_s(\tau)$ , where  $\tau \in (-T + \eta, T - \eta)$ . To do this, we recall (3.26), and integrate both sides in  $t$  from  $\tau$  to  $T - \eta$ ,

producing

$$\begin{aligned} E_s(T - \eta) - E_s(\tau) &= s \int_{\tau}^{T-\eta} \int_0^1 e^{2s\varphi} \varphi_t (|w_t|^2 + a|w_x|^2) \\ &\quad - 2s \int_{\tau}^{T-\eta} \int_0^1 e^{2s\varphi} a \varphi_x w_t w_x dx dt + \int_{\tau}^{T-\eta} \int_0^1 e^{2s\varphi} w_t L w dx dt. \end{aligned}$$

Here we use Cauchy-Schwarz to estimate. Since the resulting integrands are positive, we may integrate over the larger time domain to arrive at

$$E_s(T - \eta) - E_s(\tau) \leq Ms \int_{-T+\eta}^{T-\eta} E_s(t) dt + \frac{M}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt.$$

Once more we integrate, this time in  $\tau$  between  $-T + \eta$  and  $T - \eta$ . Since  $s$  is large enough, the second term on the left can be absorbed to obtain

$$E_s(T - \eta) \leq Ms \int_{-T+\eta}^{T-\eta} E_s(t) dt + \frac{M}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt. \quad (3.32)$$

Using the relation (3.32) on the right hand side of (3.31), the fact that  $s$  is large, and positivity of the integrands gives

$$\begin{aligned} &\int_{T-\eta}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + a|w_x|^2) dx dt \\ &\leq M \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + \frac{M}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt. \end{aligned} \quad (3.33)$$

By the Weighted Poincaré estimate from Lemma 2, we finally get

$$s \int_{T-\eta}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + a|w_x|^2 + s^2|w|^2) dx dt$$

$$\leq Ms \int_{-T+\eta}^{T-\eta} E_s(t) dt + M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt. \quad (3.34)$$

**Estimate for  $t \in (-T, -T + \eta)$**

By the change of variable  $t \rightarrow -t$ , we obtain similar estimates on  $(-T, -T + \eta)$ . By the same procedure, we obtain

$$\begin{aligned} & \int_{-T}^{-T+\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + a|w_x|^2) dx dt \\ & \leq M \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + \frac{M}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt, \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} & s \int_{-T}^{-T+\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + a|w_x|^2 + s^2|w|^2) dx dt \\ & \leq Ms \int_{-T+\eta}^{T-\eta} E_s(t) dt + M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt. \end{aligned} \quad (3.36)$$

To conclude, combining estimates (3.25), (3.33), and (3.35), for  $s$  large enough, we can say

$$\begin{aligned} & s \int_{-T+\eta}^{T-\eta} \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2 + s^2|w|^2) dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt. \end{aligned} \quad (3.37)$$

□

### 3.2.2 Pointwise Carleman Estimate at $-T$

Following similar steps as in Theorem 7, we obtain an additional estimate at time  $-T$ .



**Corollary 1.** *Under the conditions of Theorem 7, we obtain*

$$\begin{aligned} & s \int_0^1 e^{2s\varphi(-T)} (|w_t(-T)|^2 + |w_x(-T)|^2) dx + s^3 \int_0^1 e^{2s\varphi(-T)} |w(-T)|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt, \end{aligned} \quad (3.38)$$

for all  $w \in L^2((-T, T); H_0^1(0, 1))$  satisfying  $Lw \in L^2((0, 1) \times (-T, T))$  and  $w_x(1, t) \in L^2(-T, T)$ .

*Proof.* By the change of variable  $t \rightarrow -t$ , we obtain similar estimates to (3.30) and (3.32). Evaluating these at  $t = -T$ , we get an estimate for  $E_s(-T)$ . Then, by combining the Poincaré inequality and Theorem 7, we obtain the desired estimate (3.38).  $\square$

With the Carleman estimates for the operator  $Lw$ , we can obtain the Carleman estimate for the operator  $L + p$  for bounded  $p$ . With the results proved so far, we can finally complete the proof of the first theorem.

*Proof. (Theorem 5).* Notice that  $p(x, t) \in L_{\leq m}^\infty((0, 1) \times (-T, T))$  so that

$$|Lw|^2 \leq 2|Lw + pw|^2 + 2\|p\|_{L^\infty((0, 1) \times (-T, T))}^2 |w|^2 \leq 2|Lw + pw|^2 + 2m^2 |w|^2.$$

Taking  $s_0$  large enough, the term

$$2Mm^2 \int_{-T}^T \int_0^1 e^{2s\varphi} |w|^2 dx dt$$

can be absorbed by the left hand side of (3.23). Then combining (3.23) and (3.38), we obtain

the result of (3.13),

$$\begin{aligned}
& s \int_{-T}^T \int_0^1 e^{2s\varphi} (|w_t|^2 + |w_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |w|^2 dx dt \\
& + s \int_0^1 e^{2s\varphi(-T)} (|w_t(-T)|^2 + |w_x(-T)|^2) dx + s^3 \int_0^1 e^{2s\varphi(-T)} |w(-T)|^2 dx \\
& \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lw + pw|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |w_x(1, t)|^2 dt.
\end{aligned}$$

□

### 3.3 Control

For the remainder of the section, assume that the time condition (3.12) holds true. We now would like to use the Carleman estimates to prove controllability for the original system (3.1). The technique used is based on [5] and [22]. We restate here the main theorem to prove.

**Theorem 8.** *Let  $u_0(x) \in L^2(0, 1)$ ,  $u_1(x) \in H^{-1}(0, 1)$  be initial conditions. Then there exists a control  $v(t) \in L^2(\{1\} \times (-T, T))$  such that the solution  $u(x, t)$  of*

$$\begin{cases} u_{tt} - (a(x)u_x)_x + p(x, t)u = 0, & (x, t) \in (0, 1) \times (-T, T), \\ u(0, t) = 0, \quad u(1, t) = v(t), & t \in (-T, T), \\ u(x, -T) = u_0(x), \quad u_t(x, -T) = u_1(x), & x \in (0, 1), \end{cases} \quad (3.39)$$

satisfies  $u(x, T) = u_t(x, T) = 0$ .

The control will arise as part of the minimizer of some functional  $\mathcal{F}_{s,p}$  to be specified in the next theorem. Let us first classify where this functional should be defined. For a

potential  $p \in L^\infty((0, 1) \times (-T, T))$ , define the space

$$\mathcal{T}(p) = \left\{ u \in L^2(-T, T; H_0^1(0, 1)), \text{ where } Lu + pu \in L^2((0, 1) \times (-T, T)) \right. \\ \left. \text{and } u_x(1, t) \in L^2((-T, T)) \right\}. \quad (3.40)$$

One can note that this is the space of functions that are well-defined for the Carleman estimate (3.13). Also, since  $pu \in L^2((0, 1) \times (-T, T))$  whenever  $p \in L^\infty((0, 1) \times (-T, T))$  and  $u \in L^2(-T, T; H_0^1(0, 1))$ , we see that this space does not depend on  $p$ . Let us then drop the dependency and label in the future  $\mathcal{T}(p) \equiv \mathcal{T}$ . We can then define the norm on  $\mathcal{T}$  as follows :

$$\|z\|_{s,p}^2 = \frac{1}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} |z_{tt} - (a(x)z_x)_x + pz|^2 dx dt + \int_{-T}^T e^{2s\varphi} |z_x(1, t)|^2 dt. \quad (3.41)$$

Because of the Carleman estimate (3.13), this is a norm for  $s \geq s_0$ . As this quantity is the (scaled) right hand side of the estimate (3.13), we see that if  $\|z\|_{s,p} = 0$ , then  $z = 0$  identically. Furthermore,  $e^{2s\varphi}$  is bounded from above and below on this domain by positive constants depending on  $s$ , so that  $\|\cdot\|_{s,p}$  is a norm for all  $s > 0$ .

Next, we define the natural norms to be used for the initial values. On  $H_0^1(0, 1) \times L^2(0, 1)$ , introduce  $\|\cdot\|_{-T,s}$  defined by

$$\|(z_0, z_1)\|_{-T,s}^2 = \int_0^1 e^{2s\varphi(-T)} (|(z_0)_x|^2 + |z_1|^2) dx. \quad (3.42)$$

One can see that by the estimate (3.13), and by bounding functions depending on  $s$ , for

all  $s > 0$ , there is a constant  $C$  depending on  $p$  and  $s$  such that

$$\|(z(-T), z_t(-T))\|_{-T,s} \leq C(s, p) \|z\|_{s,p} \quad (3.43)$$

for those functions with the regularity needed for the Carleman estimate. Also, on the dual space  $L^2(0, 1) \times H^{-1}(0, 1)$ , take as a family of norms

$$\|(u_0, u_1)\|_{-T,s,*}^2 = \int_0^1 e^{-2s\varphi(-T)} \left( |u_0|^2 + \left| \frac{d}{dx} (-\Delta_d)^{-1} u_1 \right|^2 \right) dx. \quad (3.44)$$

Here, the notation  $(-\Delta_d)^{-1} u_1$  is the one dimensional version of the definition as in (2.17). Now, we can define the correct functional and produce the desired control function as minimizer of that functional.

**Theorem 9.** *Assume that the time condition from Theorem 7 holds. Then the functional*

$$\begin{aligned} \mathcal{F}_{s,p}(z) = & \frac{1}{2s} \int_{-T}^T \int_0^1 e^{2s\varphi} |z_{tt} - (a(x)z_x)_x + pz|^2 dx dt + \frac{1}{2} \int_{-T}^T e^{2s\varphi} |z_x(1, t)|^2 dt \\ & + \int_0^1 u_0 z_t(-T) - \langle u_1, z(-T) \rangle_{H^{-1} \times H_0^1} dx \end{aligned} \quad (3.45)$$

is continuous, convex, and coercive on  $(\mathcal{T}, \|\cdot\|_{s,p})$ , and thus has a unique minimizer  $Z[s, p] = Z \in \mathcal{T}$ .

*Proof.* Fix  $s > 0$  and  $p \in L^\infty((0, 1) \times (-T, T))$ . Convexity is straightforward to check. Then, by the estimate (3.43), for  $(u_0, u_1) \in L^2(0, 1) \times H^{-1}(0, 1)$ , the functional  $\mathcal{F}_{s,p}$  is defined and continuous on  $\mathcal{T}$ . The same estimate also implies the coercivity condition, since we have the

inequality

$$\begin{aligned}\mathcal{F}_{s,p}(z) &\geq \frac{1}{2} \|z\|_{s,p}^2 - \|(z(-T), z_t(-T))\|_{-T,s} \|(u_0, u_1)\|_{-T,s,*} \\ &\geq \frac{1}{2} \|z\|_{s,p}^2 - C(s,p) \|z\|_{s,p} \|(u_0, u_1)\|_{-T,s,*}.\end{aligned}$$

Thus, there is a unique minimizer  $Z[s, p]$ .  $\square$

We now use this minimizer to prove the existence of the control.

*Proof.* (*Theorem 8*). Fix  $s > 0$  and  $p$ . Let  $Z = Z[s, p]$  be that minimizer from Theorem 9.

We claim that by taking

$$U[s, p] = \frac{1}{s} e^{2s\varphi} (Z_{tt} - (a(x)Z_x)_x + pZ) \text{ and } V[s, p] = \frac{1}{a(1)} e^{2s\varphi} Z_x(1, t),$$

then  $U[s, p]$  solves (3.39) with the corresponding control  $V[s, p]$ , and also satisfies  $U(x, T) = U_t(x, T) = 0$ . We will check the control is as claimed. The Euler-Lagrange equation can be computed directly.

$$\begin{aligned}&\frac{1}{s} \int_{-T}^T \int_0^1 e^{2s\varphi} (z_{tt} - (a(x)z_x)_x + pz) (Z_{tt} - (a(x)Z_x)_x + pZ) dx dt \\ &+ \int_{-T}^T e^{2s\varphi} (z_x(1, t)) (Z_x(1, t)) dt + \int_0^1 u_0 z_t(-T) - \langle u_1, z(-T) \rangle_{H^{-1} \times H_0^1} dx = 0.\end{aligned}\tag{3.46}$$

Let  $z(x, t) \in C_C^\infty((0, 1) \times (-T, T))$ . Then, after integration by parts, we can check directly by computation that with  $U = \frac{1}{s} e^{2s\varphi} (Z_{tt} - (a(x)Z_x)_x + pZ)$ , we have  $U_{tt} - (a(x)U_x)_x + pU = 0$ .

Next, let us check the initial and boundary conditions of  $U$ . Choose now  $z(x, t)$  such that  $z, z_x$  vanishes on the boundary. Then, noting that  $U$  is a solution to (3.39), the first term

in (3.46) gives, after integration by parts,

$$\int_{-T}^T \int_0^1 U(z_{tt} - (a(x)z_x)_x + pz) dx dt = \int_0^1 (z_t U - z U_t)|_{-T}^T dx. \quad (3.47)$$

But then, using (3.47) in (3.46), what remains is just

$$\int_0^1 (z_t U - z U_t)|_{-T}^T + \int_0^1 u_0 z_t(-T) - \langle u_1, z(-T) \rangle_{H^{-1} \times H_0^1} dx = 0. \quad (3.48)$$

Since  $z$  was arbitrary, we obtain the boundary/initial conditions  $U(T) = U_t(T) = 0$ , and also  $U(-T) = u_0$ ,  $U_t(-T) = u_1$ . Finally, using these conditions and (3.48) in (3.46), we see that

$$\begin{aligned} & \int_{-T}^T a(1)(z(1,t)U_x(1,t) - z_x(1,t)U(1,t)) dt \\ & - \int_{-T}^T a(0)(z(0,t)U_x(0,t) - z_x(0,t)U(0,t)) dt \\ & + \int_{-T}^T e^{2s\varphi}(z_x(1,t))(Z_x(1,t)) dt = 0. \end{aligned} \quad (3.49)$$

Thus,  $U|_{\{1\} \times (-T,T)} = \frac{1}{a(1)} e^{2s\varphi} Z_x(1,t)$ , and  $U|_{\{0\} \times (-T,T)} = 0$ , as claimed.  $\square$

### 3.4 Increased Regularity

In proving the Carleman estimate, there are two main considerations. First, one must choose the weight function  $\psi(x,t)$ . Then, one must prove that through a specified time condition determined by  $\psi$ , this weight function will actually satisfy the inequalities. This is determined through the integration by parts. In turn, when using “direct” computation the weight function  $\psi(x,t)$  determines the class of functions that will be allowed.

We saw that the choice of  $\psi$  from (3.10) determined the family  $\mathcal{B}$  from (3.7). In particular, this choice of weight function also determines the regularity needed on  $a(x)$ . In this section, our main goal is to show that the Carleman estimate still holds true with only  $C^1$  regularity. This in fact is true with just the knowledge that the estimate holds with the “standard” weight  $\tilde{\psi}(x, t)$  from (3.9), assuming that  $a(x)$  is defined up to the base point  $x_0$ . Note this is used in the new time condition.

**Theorem 10.** *Assume that  $x_0 < 0, a_0 > 0$ , and let  $\beta \in (0, 1)$  be such that*

$$T > \frac{1}{\beta} \max_{x \in [0, 1]} \int_{x_0}^x \frac{1}{\sqrt{a(s)}} ds. \quad (3.50)$$

*Then for any  $m > 0$ , there exists  $\lambda > 0$  independent of  $m, s_0 = s_0(m) > 0$  and a positive constant  $M = M(m)$  such that for  $\varphi$  defined as in (3.10), for  $a(x) \in C^1[0, 1]$  with  $a(x) \geq a_0 > 0$ , for all  $p \in L_{\leq m}^\infty((0, 1) \times (-T, T))$  and for all  $s \geq s_0$ ,*

$$\begin{aligned} & s \int_{-T}^T \int_0^1 e^{2s\varphi} (|u_t|^2 + a(x)|u_x|^2) dx dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |u|^2 dx dt \\ & \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |Lu + pu|^2 dx dt + Ms \int_{-T}^T e^{2s\varphi} |u_x(1, t)|^2 dt, \end{aligned} \quad (3.51)$$

*for all  $u \in L^2(-T, T; H_0^1(0, 1))$  also satisfying  $u_{tt} - (a(x)u_x)_x + pu \in L^2((0, 1) \times (-T, T))$  and  $u_x(1, t) \in L^2(-T, T)$ .*

As before, note that all computations in the proof can be done for smooth functions  $u$ . By classical results using density arguments, the result will hold true for  $u$  satisfying the conditions of Theorem 10.

**Proof of Theorem 10:** Let us introduce a change of variable. For a given function  $b(x)$ ,

let

$$y(x) = \int_0^x b(s)ds =: B(x). \quad (3.52)$$

Without loss of generality, we can scale  $y$  by a constant, so that  $y(0) = 0, y(1) = 1$ , and  $y'(x) = b(x)$ . In the particular case of

$$b = \frac{1}{\sqrt{a(x)}}, \quad v(y, t) = u(x, t), \quad (3.53)$$

then we have

$$v_y = u_x(B^{-1}(y))' = u_x \frac{1}{B'(B^{-1}(y))} = u_x \frac{1}{b(B^{-1}(y))} = u_x \frac{1}{b(x)}, \quad (3.54)$$

so that

$$a(x)u_x = a(x)b(x)v_y = \sqrt{a(x)}v_y. \quad (3.55)$$

Taking one more derivative gives

$$\begin{aligned} (a(x)u_x)_x &= v_{yy}y'(x)\sqrt{a(x)} + \frac{1}{2\sqrt{a(x)}}a'(x)v_y \\ &= v_{yy} + \frac{1}{2\sqrt{a(x)}}a'(x)v_y. \end{aligned} \quad (3.56)$$

Consider now the Carleman estimate from Theorem 5 with “standard” weight as in (3.9), defined by

$$\psi = |y - y_0|^2 - \beta t^2 + C, \quad \varphi = e^{\lambda\psi}, \quad (3.57)$$

where  $y_0 < 0$  and  $C$  is chosen so  $\psi \geq 1$ . Because  $\frac{a'}{\sqrt{a}} \in L^\infty$ , and  $p \in L^\infty$ , then denoting



$\square v = v_{tt} - v_{xx}$ , we have that

$$\begin{aligned}
& s \int_{-T}^T \int_0^1 e^{2s\varphi} (|v_t|^2 + |v_y|^2) dy dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dy dt \\
& \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |\square v|^2 dy dt + Ms \int_{-T}^T |v_y(1, t)|^2 dt \\
& \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |v_{tt} - v_{yy} + pv + \frac{1}{2} \frac{a'}{\sqrt{a}} v_y|^2 dy dt \\
& + M \left| \frac{a'}{\sqrt{a}} \right|_{L^\infty} \int_{-T}^T \int_0^1 e^{2s\varphi} |v_y|^2 dy dt + M |p|_{L^\infty} \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dy dt \\
& + Ms \int_{-T}^T e^{2s\varphi} |v_y(1, t)|^2 dt.
\end{aligned}$$

For  $s$  large enough, the terms can be absorbed, and we obtain

$$\begin{aligned}
& s \int_{-T}^T \int_0^1 e^{2s\varphi} (|v_t|^2 + |v_y|^2) dy dt + s^3 \int_{-T}^T \int_0^1 e^{2s\varphi} |v|^2 dy dt \\
& \leq M \int_{-T}^T \int_0^1 e^{2s\varphi} |\square v + pv + \frac{a'}{2\sqrt{a}} v_y|^2 dy dt + Ms \int_{-T}^T e^{2s\varphi} |v_y(1, t)|^2 dt.
\end{aligned}$$

Now, we go back to the  $u$  variable. The space portion of the weight function from (3.57)

changes to

$$\begin{aligned}
|y - y_0|^2 &= |B(x) - y_0|^2 = \left| \int_0^x \frac{1}{\sqrt{a(s)}} ds - \int_0^{x_0} \frac{1}{\sqrt{a(s)}} ds \right|^2 \\
&= \left| \int_{x_0}^x \frac{1}{\sqrt{a(s)}} ds \right|^2.
\end{aligned} \tag{3.58}$$

Let  $Q = |B(x) - y_0|^2 - \beta t^2 + C$ . Then  $v_y = u_x \sqrt{a}$ , and we have

$$\begin{aligned}
& s \int_{-T}^T \int_0^1 e^{2sQ} (|u_t|^2 + a(x)|u_x|^2) \frac{dx}{\sqrt{a}} dt + s^3 \int_{-T}^T \int_0^1 e^{2sQ} |u|^2 \frac{dx}{\sqrt{a}} dt \\
& \leq M \int_{-T}^T \int_0^1 \frac{e^{2sQ}}{\sqrt{a}} [(L+p)u]^2 dx dt + Ms \int_{-T}^T e^{2sQ} a(x) |u_x(1, t)|^2 dt.
\end{aligned}$$

After scaling again, we attain the estimate (3.13).

□

# Chapter 4

## Degenerate Parabolic Control

In the previous chapters, we have studied the control problem for the hyperbolic equation. Here, the variable coefficient  $a(x)$  was assumed to be strictly positive. We now consider one such analogue for the parabolic problem.

Let  $\Omega = (0, 1)$ . Let  $\alpha \in [0, 2)$ , and consider the function  $a(x) = x^\alpha$ . Then,  $a(x)$  vanishes at  $x = 0$ . Let  $h(x, t) \in L^2((0, 1) \times (0, T))$  be a control function, and let  $u_0$  be an initial state in  $\Omega$ . Then, the basic form for the interior degenerate parabolic system with control function  $h$  is

$$\left\{ \begin{array}{ll} u_t - (au_x)_x = h\chi_\omega, & (x, t) \in \Omega \times (0, T), \\ u(1, t) = 0, & t \in (0, T), \\ \left\{ \begin{array}{ll} u(0, t) = 0, & \text{for } 0 \leq \alpha < 1, t \in (0, T), \\ (au_x)(0, t) = 0, & \text{for } 1 \leq \alpha < 2, t \in (0, T), \end{array} \right. & \\ u(x, 0) = u_0, & x \in \Omega. \end{array} \right. \quad (4.1)$$

In this chapter, we study the controllability properties for the degenerate heat equation. We recall the criterion of null controllability for (4.1): Given  $T > 0$ , and initial state  $u_0$ , the goal is to find a control function  $h(x, t)$  such that the solution  $u(x, t)$  to (4.1) satisfies  $u(T) = 0$ .

The properties of the degenerate heat equation of the form (4.1) are less well known than

nondegenerate parabolic equations. An early study was done in [8], where a Carleman-type estimate was established. It is shown in [1], [8], [10], that null controllability holds in the case  $0 \leq \alpha < 2$ , and in [11] that null controllability does not hold in the case  $\alpha \geq 2$ . The study has been expanded to contain multiplicative control [7] and approximate control for the nonlinear degenerate Cauchy-Neumann case [20]. Also, under certain hypothesis on the degeneracy, control in higher dimensions was considered in the recent work [9].

## 4.1 Robin Boundary Conditions

The goal of this section is to show that the Carleman estimates are also true for the degenerate parabolic equation with Robin boundary conditions.

Consider the degenerate parabolic system with Robin boundary conditions,

$$\left\{ \begin{array}{ll} u_t - (a(x)u_x)_x = p(x, t)u + f(x, t, u), & (x, t) \in (0, 1) \times (0, T), \\ \left\{ \begin{array}{ll} \beta_0 u(0, t) + \beta_1 a(0)u_x(0, t) = 0, & t \in (0, T), \\ \gamma_0 u(1, t) + \gamma_1 a(1)u_x(1, t) = 0, & t \in (0, T), \end{array} \right. & \\ u(x, 0) = u_0(x), & x \in (0, 1). \end{array} \right. \quad (4.2)$$

As in [7] and [21], we define the spaces appropriate for the degenerate parabolic equation.

For  $a(x) \in C^0[0, 1]$ , define

$$H_a^1(0, 1) \doteq \{u \in L^2(0, 1) | u \in AC([0, 1]) \text{ and } \sqrt{a}u_x \in L^2(0, 1)\}, \quad (4.3)$$

$$H_a^2(0, 1) \doteq \{u \in H_a^1(0, 1) | au_x \in H^1(0, 1)\}.$$

Then, for  $0 \leq \alpha < 1$ , it can be shown that these are Hilbert spaces under the respective natural norms of

$$\|u\|_{1,a}^2 \doteq \|u\|_{L^2(0,1)}^2 + |u|_{1,a}^2, \quad (4.4)$$

and

$$\|u\|_{2,a}^2 \doteq \|u\|_{1,a}^2 + \|(au_x)_x\|_{L^2(0,1)}^2, \quad (4.5)$$

where  $|u|_{1,a}^2 \doteq \|\sqrt{a}u_x\|_{L^2(0,1)}^2$  is a seminorm.

Furthermore, as in [7], consider the operator  $(A, D(A))$  defined by

$$\left\{ \begin{array}{l} D(A) = \left\{ u \in H_a^2(0,1) \mid \begin{cases} \beta_0 u(0) + \beta_1 a(0)u_x(0) = 0, \\ \gamma_0 u(1) + \gamma_1 a(1)u_x(1) = 0, \end{cases} \right. \\ Au = (au_x)_x + pu, \quad \forall u \in D(A), \end{array} \right. \quad (4.6)$$

for  $p \in L^\infty((0,1))$ . It is shown in [7] that  $A$  is a closed self-adjoint dissipative operator with dense domain in  $L^2(0,1)$ . Then  $A$  is the infinitesimal generator of a  $C_0$ -semigroup of contractions in  $L^2(0,1)$ , which will give rise to the uniqueness/existence theorem.

As the stated goal of this section is to show the Carleman estimate holds true for the original system with the Robin boundary conditions, we shall not need to take advantage of the full nonlinear term in which [21] provides well-posedness in more generality. See [21], Definition 1.1 for the full problem formulation. The important conditions necessary and applicable to this section are the degeneracy condition on  $a(x)$  and the sign condition for the boundary condition coefficients.

1.  $a(x) \in C^0([0,1])$  satisfies  $a(0) = 0$ .
2.  $a(x) \in C^1(0,1)$  is such that  $\frac{1}{a(x)} \in L^1(0,1)$ .

3. (Sign condition) : The constants  $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{R}$  satisfy  $\beta_0^2 + \beta_1^2 > 0, \gamma_0^2 + \gamma_1^2 > 0$ , and the sign conditions  $\beta_0\beta_1 \leq 0$ , and  $\gamma_0\gamma_1 \geq 0$ .

The first two conditions combined are called the weak degenerate condition, corresponding to the diffusion coefficient  $a(x)$ . Notice that the function  $a(x) = x^\alpha$  does indeed satisfy the first two conditions, when  $\alpha \in [0, 1)$ . Since the function degenerates at 0 in (4.2), this equality is in the limit sense. For completeness, and to make sense of the solution space later, we include the conditions on  $f = f(x, t, u)$  from [21] here. Define  $Q_T = (0, 1) \times (0, T)$ . Then,

4.  $f : Q_T \times \mathbb{R} \rightarrow \mathbb{R}$  is such that

- $(x, t) \mapsto f(x, t, u)$  is measurable, for all  $u \in \mathbb{R}$ ,
- $u \mapsto f(x, t, u)$  is continuous, for almost every  $(x, t) \in (0, 1) \times (0, T)$ ,
- $t \mapsto f(x, t, u)$  is locally absolutely continuous for a.e  $x \in (0, 1)$ , and for every  $u \in \mathbb{R}$ .
- There exists constants  $\kappa \geq 0, \mathcal{O} \in [1, 4)$ , and  $\nu \geq 0$  such that

$$|f(x, t, u) - f(x, t, v)| \leq \nu \left(1 + |u|^{\mathcal{O}-1} + |v|^{\mathcal{O}-1}\right) |u - v|, \quad (4.7)$$

$$(f(x, t, u) - f(x, t, v)) (u - v) \leq \nu(u - v)^2, \quad (4.8)$$

$$|f(x, t, u)| \leq \kappa |u|^{\mathcal{O}}, \quad (4.9)$$

$$f_t(x, t, u)u \geq -\nu u^2, \quad (4.10)$$

for a.e.  $(x, t) \in Q_T, \forall u, v \in \mathbb{R}$ .

Now we must define the Banach spaces in time for the well-posedness result. As in [7] and [21], given  $T > 0$ , define

$$\mathcal{B}(Q_T) \doteq C\left([0, T]; L^2(0, 1)\right) \cap L^2\left(0, T; H_a^1(0, 1)\right). \quad (4.11)$$

This has the norm

$$\|u\|_{\mathcal{B}(Q_T)}^2 = \sup_{t \in [0, T]} \|u(\cdot, t)\|^2 + 2 \int_0^T \int_0^1 a(x) u_x^2 dx dt. \quad (4.12)$$

Also, define

$$\mathcal{H}(Q_T) \doteq L^2(0, T; D(A)) \cap H^1\left(0, T; L^2(0, 1)\right) \cap C\left([0, T]; H_a^1(0, 1)\right). \quad (4.13)$$

This stronger space has norm

$$\|u\|_{\mathcal{H}(Q_T)}^2 \doteq \sup_{t \in [0, T]} \left( \|u\|^2 + \|\sqrt{a}u_x\|^2 \right) + \int_0^T \left( \|u_t\|^2 + \|(au_x)_x\|^2 \right) dt. \quad (4.14)$$

Finally, we provide for completeness the two notions from [21] of solutions of this problem, providing the necessary well-posedness result. These are the strict solutions and strong solutions, corresponding to the respective solution spaces.

**Definition 4.1.1.** For  $u_0 \in H_a^1(0, 1)$ ,  $u$  is a strict solution of (4.2) if  $u \in \mathcal{H}(Q_T)$  and

$$\left\{ \begin{array}{ll} u_t - (a(x)u_x)_x = p(x, t)u + f(x, t, u(x, t)), & \text{a.e. } (x, t) \in (0, 1) \times (0, T), \\ \beta_0 u(0, t) + \beta_1 a(0)u_x(0, t) = 0, & \text{a.e. } t \in (0, T), \\ \gamma_0 u(1, t) + \gamma_1 a(1)u_x(1, t) = 0, & \text{a.e. } t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, 1). \end{array} \right. \quad (4.15)$$

**Definition 4.1.2.** Let  $u_0 \in L^2(0, 1)$ . Then,  $u$  is a strong solution of (4.2), if  $u \in \mathcal{B}(Q_T)$ ,  $u(\cdot, 0) = u_0$  and there exists a sequence  $\{u_k\}_{k \in \mathbb{N}}$  in  $\mathcal{H}(Q_T)$  such that as  $k \rightarrow \infty$ ,  $u_k \rightarrow u$  in  $\mathcal{B}(Q_T)$ , and for every  $k \in \mathbb{N}$ ,  $u_k$  is the strict solution of

$$\left\{ \begin{array}{ll} u_{kt} - (a(x)u_{kx})_x = p(x, t)u_k + f(x, t, u_k(x, t)), & \text{a.e. } (x, t) \in (0, 1) \times (0, T), \\ \beta_0 u_k(0, t) + \beta_1 a(0)u_{kx}(0, t) = 0, & \text{a.e. } t \in (0, T), \\ \gamma_0 u_k(1, t) + \gamma_1 a(1)u_{kx}(1, t) = 0, & \text{a.e. } t \in (0, T), \\ u_k(x, 0) = u_{k0}(x), & x \in (0, 1). \end{array} \right. \quad (4.16)$$

With all conditions as defined above, the well-posedness results are the major results of [21]. The two cases are as follows.

**Theorem 11.** For all  $u_0 \in H_a^1(0, 1)$ , there exists a unique strict solution  $u \in \mathcal{H}(Q_T)$  in the sense of Definition 4.1.1 to (4.2).

**Theorem 12.** For each  $u_0 \in L^2(0, 1)$ , there exists a unique strong solution  $u \in \mathcal{B}(Q_T)$  in the sense of Definition 4.1.2 to (4.2).



## 4.2 Carleman Estimate

With the well-posedness result, we can now prove the Carleman estimate associated to the degenerate parabolic equation with Robin boundary conditions, extending the work of [8].

**Theorem 13.** *Let  $0 \leq \alpha < 1$ , and let  $T > 0$ . Let  $a(x) = x^\alpha$ . Consider the adjoint degenerate parabolic problem with Robin boundary conditions,*

$$\left\{ \begin{array}{ll} w_t + (a(x)w_x)_x = f(x, t), & (x, t) \in (0, 1) \times (0, T), \\ \beta_0 w(x, t) + \beta_1 a(x)w_x(x, t) = 0, & x = 0, \ t \in (0, T), \\ w(1, t) = 0, & t \in (0, T), \\ w(x, T) = w_T(x), & x \in (0, 1), \end{array} \right. \quad (4.17)$$

where  $\beta_0$  and  $\beta_1$  satisfy the sign conditions  $\beta_0^2 + \beta_1^2 > 0$ , and  $\beta_0\beta_1 \leq 0$ . Then, there exists a function  $\sigma(x, t) : [0, 1] \times (0, T) \rightarrow \mathbb{R}$  of the form  $\sigma(x, t) = p(x)\theta(t)$ , where  $p(x) > 0$  for all  $x \in [0, 1]$ , and  $\theta(t) \rightarrow \infty$  as  $t \rightarrow 0^+, T^-$ , and two positive constants  $C$  and  $R_0 = R_0(C, T)$ , such that for all  $w_T \in L^2(0, 1)$ ,  $f \in L^2((0, 1) \times (0, T))$ , the solution  $w$  to (4.17) satisfies, for all  $R > R_0$ , the Carleman Estimate

$$\begin{aligned} \iint_{Q_T} \left( R\theta x^\alpha w_x^2 + R^3\theta^3 x^{2-\alpha} w^2 \right) e^{-2R\sigma} dx dt & \quad (4.18) \\ \leq C \iint_{Q_T} e^{-2R\sigma} f^2 dx dt + C \int_0^T \left\{ R\theta e^{-2R\sigma} w_x^2 \right\}_{x=1} dt. \end{aligned}$$

*Proof.* Consider the Robin boundary conditions

$$\begin{cases} \beta_0 w(x, t) + \beta_1 a(x) w_x(x, t) = 0, & x = 0, \\ w(1, t) = 0. \end{cases} \quad (4.19)$$

For simplicity, by scaling let us assume that  $\beta_1 = 1$ . Then, call  $\beta_0 = \beta$ . According to the sign condition 3, we have that  $\beta < 0$ . Similar to the procedure for the hyperbolic equation, as in (3.14), we make the change of variables  $z = e^{-R\sigma(x,t)} w(x, t)$ . Then the space derivative satisfies

$$\begin{aligned} z_x &= e^{-R\sigma(x,t)} w(x, t) (-R\sigma_x) + w_x e^{-R\sigma(x,t)} \\ &= z(-R\sigma_x) + w_x e^{-R\sigma(x,t)}. \end{aligned} \quad (4.20)$$

The boundary conditions for  $z(x, t)$  satisfy

$$\begin{aligned} \beta z(x, t) + a(x) z_x(x, t) & \quad (4.21) \\ &= -a(x) z R\sigma_x + a(x) w_x e^{-R\sigma} + \beta e^{-R\sigma} w \\ &= -a(x) e^{-R\sigma} w R\sigma_x + a(x) w_x e^{-R\sigma} + \beta e^{-R\sigma} w \\ &= -a(x) e^{-R\sigma} w R\sigma_x \\ &= -a(x) z R\sigma_x(x, t). \end{aligned}$$

Then we also have the relation for the  $z$  derivative,

$$z_x = e^{-R\sigma(x,t)} w(x, t) (-R\sigma_x) + w_x e^{-R\sigma(x,t)} \quad (4.22)$$

$$= z(-R\sigma_x) + w_x e^{-R\sigma(x,t)}.$$

The same method as from Chapter 3, (3.15) can be utilized. That is, we decompose the equation into 2 parts, and compute the product. Define the relations

$$P_1 z = R\sigma_t z + R^2 a \sigma_x^2 z + (az_x)_x. \quad (4.23)$$

$$P_2 z = z_t + R(a\sigma_x)_x z + 2Ra\sigma_x z_x. \quad (4.24)$$

Then, we have

$$P_1 z + P_2 z = f e^{-R\sigma}. \quad (4.25)$$

From [8] (Lemma 3.1), by computing  $\langle P_1 z, P_2 z \rangle$ , the following identity holds from the decomposition,

$$\begin{aligned} \langle P_1 z, P_2 z \rangle &= \int_0^T \left[ az_x z_t + R^2 a \sigma_t \sigma_x z^2 + R^3 a^2 \sigma_x^3 z^2 \right. \\ &\quad \left. + R\sigma_x a^2 z_x^2 + Ra(a\sigma_x)_x z z_x \right]_0^1 dt \\ &\quad - \frac{1}{2} R \int \int_{Q_T} \theta_{tt} p z^2 - 2R^2 \int \int_{Q_T} \theta \theta_t x^\alpha p_x^2 z^2 \\ &\quad - R^3 \int \int_{Q_T} \theta^3 x^{2\alpha-1} (2xp_{xx} + \alpha p_x) p_x^2 z^2 \\ &\quad - R \int \int_{Q_T} \theta x^{2\alpha-1} (2xp_{xx} + \alpha p_x) z_x^2 - R \int \int_{Q_T} \theta x^\alpha (x^\alpha p_x)_{xx} z z_x. \end{aligned} \quad (4.26)$$

This splits the product into two parts, the boundary terms, and the space-time terms. The interior terms will be evaluated second. The boundary terms now obey the condition

of (4.21). The boundary terms are

$$\begin{aligned} \text{Boundary} = & \int_0^T [az_x z_t]_0^1 dt + R^2 \int_0^T [a\sigma_t \sigma_x z^2]_0^1 dt + R^3 \int_0^T [a^2 \sigma_x^3 z^2]_0^1 dt \\ & + R \int_0^T [\sigma_x a^2 z_x^2]_0^1 dt + R \int_0^T [a(a\sigma_x)_x z z_x]_0^1 dt. \end{aligned} \quad (4.27)$$

To avoid confusion, let us evaluate each of these terms separately. Let us here also define the function  $p(x)$  by

$$p(x) = \frac{2 - x^{2-\alpha}}{(2 - \alpha)^2}. \quad (4.28)$$

Note that  $p(x) > 0$ . Then the derivative satisfies

$$p_x = \frac{-x^{1-\alpha}}{(2 - \alpha)}. \quad (4.29)$$

Using (4.21), the first term from (4.27) gives

$$\begin{aligned} \int_0^T [az_x z_t]_0^1 dt &= \int_0^T [(-\beta z - aR\theta p_x z)z_t]_0^1 dt \\ &= \int_0^T \{(\beta z + aR\theta p_x z)z_t\}_{x=0} dt \\ &= \frac{1}{2} \int_0^T \left\{ \beta(z^2)_t + aR\theta p_x (z^2)_t \right\}_{x=0} dt \\ &= \int_0^T \left\{ \frac{R}{2 - \alpha} \theta_t x z^2 \right\}_{x=0} dt, \end{aligned} \quad (4.30)$$

where we have used the decay in the limit from the definition of  $z$ . The second term of (4.27) gives

$$R^2 \int_0^T [a\sigma_t \sigma_x z^2]_0^1 dt = -R^2 \int_0^T \left\{ a\sigma_t \sigma_x z^2 \right\}_{x=0} dt \quad (4.31)$$

$$= R^2 \int_0^T \left\{ \theta \theta_t \frac{2}{(2-\alpha)^3} x z^2 \right\}_{x=0} dt - R^2 \int_0^T \left\{ \theta \theta_t \frac{x^{2-\alpha}}{(2-\alpha)^3} x z^2 \right\}_{x=0} dt.$$

The third term from (4.27) gives

$$\begin{aligned} R^3 \int_0^T \left[ a^2 \sigma_x^3 z^2 \right]_0^1 dt &= -R^3 \int_0^T \left\{ a^2 \sigma_x^3 z^2 \right\}_{x=0} dt \\ &= R^3 \int_0^T \left\{ \frac{x^{2-\alpha}}{(2-\alpha)^3} \theta^3 x z^2 \right\}_{x=0} dt. \end{aligned} \quad (4.32)$$

The fourth term from (4.27) gives

$$\begin{aligned} R \int_0^T \left[ \sigma_x a^2 z_x^2 \right]_0^1 dt & \\ &= R \int_0^T \left\{ \sigma_x a^2 z_x^2 \right\}_{x=1} dt - R \int_0^T \left\{ \sigma_x a^2 z_x^2 \right\}_{x=0} dt \\ &= -R \int_0^T \left\{ \frac{1}{(2-\alpha)} \theta z_x^2 \right\}_{x=1} dt - R \int_0^T \left\{ \theta a^2 p_x z_x^2 \right\}_{x=0} dt. \end{aligned} \quad (4.33)$$

Now, once again using the Robin boundary conditions from (4.21), we continue the equality from (4.33),

$$\begin{aligned} R \int_0^T \left[ \sigma_x a^2 z_x^2 \right]_0^1 dt & \\ &= -R \int_0^T \left\{ \frac{\theta z_x^2}{(2-\alpha)} \right\}_{x=1} dt + \frac{R}{(2-\alpha)} \int_0^T \left\{ \theta x^{1-\alpha} (-\beta z - a R \theta p_x z)^2 \right\}_{x=0} dt \\ &= -R \int_0^T \left\{ \frac{\theta z_x^2}{(2-\alpha)} \right\}_{x=1} dt + \frac{R}{(2-\alpha)} \int_0^T \left\{ \beta^2 \theta x^{1-\alpha} z^2 \right\}_{x=0} dt \\ &\quad + \frac{2R}{(2-\alpha)} \int_0^T \left\{ x^{1-\alpha} \beta R \theta (a p_x) z^2 \right\}_{x=0} dt + \frac{R}{(2-\alpha)} \int_0^T \left\{ x^{1-\alpha} a^2 R^2 \theta^3 p_x^2 z^2 \right\}_{x=0} dt \\ &= -R \int_0^T \left\{ \frac{\theta z_x^2}{(2-\alpha)} \right\}_{x=1} dt + \frac{R}{(2-\alpha)} \int_0^T \left\{ \beta^2 \theta x^{1-\alpha} z^2 \right\}_{x=0} dt \\ &\quad - \frac{2R^2}{(2-\alpha)^2} \int_0^T \left\{ x^{2-\alpha} \beta \theta^2 z^2 \right\}_{x=0} dt + \frac{R^3}{(2-\alpha)^3} \int_0^T \left\{ x^{3-\alpha} \theta^3 z^2 \right\}_{x=0} dt. \end{aligned} \quad (4.34)$$

The final term from (4.27) gives

$$\begin{aligned}
& R \int_0^T [a(a\sigma_x)_x z z_x]_0^1 dt \\
&= -R \int_0^T \left\{ \theta \left( \frac{-1}{2-\alpha} \right) z (-\beta z - aR\theta p_x z) \right\}_{x=0} dt \\
&= -\frac{R}{(2-\alpha)} \int_0^T \left\{ \beta \theta z^2 \right\}_{x=0} + \frac{R^2}{(2-\alpha)^2} \int_0^T \left\{ \theta^2 x z^2 \right\}_{x=0} dt.
\end{aligned} \tag{4.36}$$

Combining all of the results from (4.30) - (4.36), we arrive at the expression for the boundary terms:

$$\begin{aligned}
\text{Boundary} &= \int_0^T \left\{ \frac{R}{2-\alpha} \theta_t x z^2 \right\}_{x=0} dt \\
&+ R^2 \int_0^T \left\{ \theta \theta_t \frac{2}{(2-\alpha)^3} x z^2 \right\}_{x=0} dt - R^2 \int_0^T \left\{ \theta \theta_t \frac{x^{2-\alpha}}{(2-\alpha)^3} x z^2 \right\}_{x=0} \\
&+ R^3 \int_0^T \left\{ \frac{x^{2-\alpha}}{(2-\alpha)^3} \theta^3 x z^2 \right\}_{x=0} dt \\
&- R \int_0^T \left\{ \frac{\theta z_x^2}{(2-\alpha)} \right\}_{x=1} dt + \frac{R}{(2-\alpha)} \int_0^T \left\{ \beta^2 \theta x^{1-\alpha} z^2 \right\}_{x=0} dt \\
&- \frac{2R^2}{(2-\alpha)^2} \int_0^T \left\{ x^{2-\alpha} \beta \theta^2 z^2 \right\}_{x=0} dt + \frac{R^3}{(2-\alpha)^3} \int_0^T \left\{ x^{3-\alpha} \theta^3 z^2 \right\}_{x=0} dt. \\
&- \frac{R}{(2-\alpha)} \int_0^T \left\{ \beta \theta z^2 \right\}_{x=0} + \frac{R^2}{(2-\alpha)^2} \int_0^T \left\{ \theta^2 x z^2 \right\}_{x=0} dt.
\end{aligned} \tag{4.37}$$

Because of the  $\beta$  sign condition,  $\beta < 0$ , every term involving  $\beta$  from (4.37) is nonnegative. By [8] (Lemma 3.1), we have that for all  $v \in H_a^1(0, 1)$ , then  $xv^2(x) \rightarrow 0$  as  $x \rightarrow 0^+$ . Thus, the remaining terms vanish, and we have

$$\text{Boundary} \geq -\frac{R}{(2-\alpha)} \int_0^T \left\{ \theta z_x^2 \right\}_{x=1} dt. \tag{4.38}$$

From the form of (4.26) and (4.37), we choose the weight function  $\theta(t)$ . We will prove

the theorem using a more general function  $\theta(t)$  than that in [8]. For  $k > 0$ , consider the function

$$\theta(t) = \left( \frac{1}{t(t-T)} \right)^k. \quad (4.39)$$

Note that the choice for  $\theta(t)$  satisfies the limit conditions prescribed in Theorem 13 for  $t \rightarrow 0^-, t \rightarrow T^+$ . Then the time derivatives are

$$\theta_t = \frac{k(2t-T)}{(t(T-t))^{k+1}}, \quad (4.40)$$

and

$$\theta_{tt} = k \left( \frac{(4k+2)t^2 - 2t(2kT+T) + (k+1)T^2}{(t(T-t))^{k+2}} \right). \quad (4.41)$$

From (4.26), the space-time terms satisfy

$$\begin{aligned} \text{Space-time} = & -\frac{1}{2}R \iint_{Q_T} \theta_{tt} p z^2 dx dt - 2R^2 \iint_{Q_T} \theta \theta_t x^\alpha p_x^2 z^2 dx dt \\ & - R^3 \iint_{Q_T} \theta^3 x^{2\alpha-1} (2xp_{xx} + \alpha p_x) p_x^2 z^2 dx dt \\ & - R \iint_{Q_T} \theta x^{2\alpha-1} (2xp_{xx} + \alpha p_x) z_x^2 dx dt - R \iint_{Q_T} \theta (x^\alpha p_x)_{xx} z z_x dx dt. \end{aligned} \quad (4.42)$$

By the choice of  $p(x)$ , this simplifies to

$$\begin{aligned} \text{Space-time} = & -\frac{R}{(2-\alpha)^2} \iint_{Q_T} \theta_{tt} z^2 dx dt + \frac{R}{2(2-\alpha)^2} \iint_{Q_T} \theta_{tt} x^{2-\alpha} z^2 dx dt \\ & - \frac{2R^2}{(2-\alpha)^2} \iint_{Q_T} \theta \theta_t x^{2-\alpha} z^2 dx dt \\ & + \frac{R^3}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt + R \iint_{Q_T} \theta x^\alpha z_x^2 dx dt. \end{aligned} \quad (4.43)$$

Let us estimate these terms in (4.43). The leading term in  $z^2$  contains  $\theta^3$ . The leading term in  $z_x^2$  contains  $\theta$ . Thus, we want to estimate the first three terms in (4.43) with respect to the last two. Now, by the form of  $\theta$  and  $\theta_{tt}$  from (4.40), an estimate of the form  $|\theta_{tt}| \leq c\theta^3$  is possible if and only if  $k \geq 1$ . In this case,  $|\theta_{tt}| \leq cT^{4k-2}\theta^3$ . Thus, the second term from (4.43) gives

$$\left| \frac{R}{2(2-\alpha)^2} \iint_{Q_T} \theta_{tt} x^{2-\alpha} z^2 dx dt \right| \leq \frac{Rc(k)T^{4k-2}}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt. \quad (4.44)$$

For the third term from (4.43), we must estimate  $|\theta\theta_t|$ . By definition from (4.39) and (4.40), we have

$$\theta\theta_t = k(2t - T) \left( \frac{1}{t(T-t)} \right)^{2k+1}, \quad (4.45)$$

so that once again  $|\theta\theta_t| \leq c\theta^3$  if and only if  $k \geq 1$ . If this is the case, then we can estimate the third term as

$$\left| \frac{2R^2}{(2-\alpha)^2} \iint_{Q_T} \theta\theta_t x^{2-\alpha} z^2 dx dt \right| \leq \frac{R^2c(k)T^{2k-1}}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt. \quad (4.46)$$

Finally, we consider the first term from (4.43). By Young's Inequality, we can estimate the second derivative,

$$\theta_{tt} \leq c\theta^m = c \left( \theta x^{p_1} z^2 \right)^{r_1} \left( \theta^3 x^{2-\alpha} z^2 \right)^{r_2}, \quad (4.47)$$

where  $0 < r_2 \leq r_1$  and

$$\begin{aligned} r_1 + r_2 &= 1 \\ r_1 + 3r_2 &= m \end{aligned} \quad (4.48)$$

$$(2-\alpha)r_2 + p_1r_1 = 0.$$

The first two equations restrict  $m$  to  $1 < m < 3$ . The inequality  $\theta_{tt} \leq c\theta^m$  can be only be



satisfied if  $k(1 - m) + 2 \leq 0$ . If this is the case, then

$$\begin{aligned}
& \left| \frac{R}{(2 - \alpha)^2} \iint_{Q_T} \theta_{tt} z^2 dx dt \right| \\
& \leq \frac{Rc(k)T^{-2+2k(m-1)}}{(2 - \alpha)^2} \iint_{Q_T} \left( \theta x^{p_1} z^2 \right)^{r_1} \left( \theta^3 x^{2-\alpha} z^2 \right)^{r_2} dx dt \\
& \leq \epsilon \frac{Rc(k)T^{-2+2k(m-1)}}{(2 - \alpha)^2} \iint_{Q_T} \theta x^{p_1} z^2 dx dt \\
& + \left( \frac{1}{\epsilon} \right)^{\frac{r_1}{r_2}} \frac{Rc(k)T^{-2+2k(m-1)}}{(2 - \alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt
\end{aligned} \tag{4.49}$$

Since  $\alpha < 2$ , and  $r_2 \leq r_1$ , then

$$x^{p_1} = x^{(\alpha-2)\frac{r_2}{r_1}} \leq x^{\alpha-2}.$$

If  $\alpha \neq 1$ , then Hardy's inequality ([8] Lemma 2.1) gives

$$\iint_{Q_T} \theta x^{p_1} z^2 dx dt \leq \iint_{Q_T} \theta x^{\alpha-2} z^2 dx dt \leq \frac{C}{(\alpha - 1)^2} \iint_{Q_T} \theta x^\alpha z_x^2 dx dt. \tag{4.50}$$

Combining (4.44), (4.46), (4.49), and (4.50) gives

$$\begin{aligned}
\text{Space-time} & \geq \frac{R^3}{(2 - \alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt + R \iint_{Q_T} \theta x^\alpha z_x^2 dx dt \\
& - \frac{Rc(k)T^{4k-2}}{(2 - \alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt - \frac{R^2 c(k)T^{2k-1}}{(2 - \alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt \\
& - \epsilon \frac{Rc(k)T^{-2+2k(m-1)}}{(2 - \alpha)^2(1 - \alpha)^2} \iint_{Q_T} \theta x^\alpha z_x^2 dx dt \\
& - \left( \frac{1}{\epsilon} \right)^{\frac{r_1}{r_2}} \frac{Rc(k)T^{-2+2k(m-1)}}{(2 - \alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt.
\end{aligned} \tag{4.51}$$

To make the Space-time terms nonnegative, we must choose  $\epsilon = \epsilon(T)$  appropriately, and then  $R = R(\epsilon, T, \alpha)$  from there. The term containing  $\epsilon$  must be dominated then by the second term, so  $\epsilon$  needs to be on the same order as  $T^{2-2k(m-1)}(2-\alpha)^2(1-\alpha)^2$ . In terms of  $T$ , this means that  $\left(\frac{1}{\epsilon}\right)^{\frac{r_1}{r_2}}$  is of order  $\left(T^{-2+2k(m-1)}\right)^{\frac{r_1}{r_2}}$ . From here, we can choose  $R$ . By comparing the first and last terms of (4.51),

$$R^3 \gtrsim RT^{-2+2k(m-1)} \left(T^{-2+2k(m-1)}\right)^{\frac{r_1}{r_2}} \left(\frac{1}{(1-\alpha)^2(2-\alpha)^2}\right)^{\frac{r_1}{r_2}},$$

so that

$$R \gtrsim T^{\left(1+\frac{r_1}{r_2}\right)(k(m-1)-1)} \left(\frac{1}{|(1-\alpha)(2-\alpha)|}\right)^{\frac{r_1}{r_2}}.$$

Also, by comparing the first, third and fourth terms of (4.51),

$$R \gtrsim T^{2k-1}.$$

From these, choose  $R$  sufficiently large, so that

$$R \geq \max \left\{ T^{\left(1+\frac{r_1}{r_2}\right)(k(m-1)-1)}, T^{2k-1} \right\}. \quad (4.52)$$

Then, for the chosen  $\epsilon$  small enough and  $R$  large enough as in (4.52), from (4.51), the first two terms will dominate, giving

$$\text{Space-time} \geq \frac{R^3}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt + R \iint_{Q_T} \theta x^\alpha z_x^2 dx dt \geq 0. \quad (4.53)$$

Thus, with  $R$  large enough as in (4.52), we have from (4.25), (4.38), and (4.53),

$$\|f e^{-R\sigma}\|^2 \geq \langle P_1 z, P_2 z \rangle = \text{Space-time} + \text{Boundary} \quad (4.54)$$

$$\begin{aligned} &\geq \frac{R^3}{(2-\alpha)^2} \iint_{Q_T} \theta^3 x^{2-\alpha} z^2 dx dt \\ &\quad + R \iint_{Q_T} \theta x^\alpha z_x^2 dx dt - \frac{R}{(2-\alpha)} \int_0^T \left\{ \theta z_x^2 \right\}_{x=1} dt. \end{aligned} \quad (4.55)$$

Finally, we recall that  $w = e^{R\sigma} z$ . We use this to obtain the bounds in the original variable.

So,

$$\begin{aligned} R^3 \theta^3 x^{2-\alpha} w^2 + R \theta x^\alpha w_x^2 &= R^3 \theta^3 x^{2-\alpha} e^{2R\sigma} z^2 + R \theta x^\alpha (R \sigma_x e^{R\sigma} z + e^{R\sigma} z_x)^2 \\ &\leq c \left( R^3 \theta^3 x^{2-\alpha} e^{2R\sigma} z^2 + R \theta x^\alpha e^{2R\sigma} z_x^2 \right). \end{aligned} \quad (4.56)$$

To match (4.55), multiply by  $e^{-2R\sigma}$  to get

$$R^3 \theta^3 x^{2-\alpha} w^2 e^{-2R\sigma} + R \theta x^\alpha w_x^2 e^{-2R\sigma} \leq c \left( R^3 \theta^3 x^{2-\alpha} z^2 + R \theta x^\alpha z_x^2 \right). \quad (4.57)$$

From (4.55) and (4.57), the Carleman Estimate follows.

$$\begin{aligned} &\iint_{Q_T} \left( R \theta x^\alpha w_x^2 + R^3 \theta^3 x^{2-\alpha} w^2 \right) e^{-2R\sigma} dx dt \\ &\leq C \iint_{Q_T} e^{-2R\sigma} f^2 dx dt + C \int_0^T \left\{ R \theta e^{-2R\sigma} w_x^2 \right\}_{x=1} dt. \end{aligned} \quad (4.58)$$

□

## APPENDIX

# Appendix

## Minimum Energy: Approximate Control

In this appendix, we now consider one brief application regarding the energy for the non-degenerate parabolic system. Let  $\Omega$  be a domain, let  $\omega \subset \Omega$ . Consider the interior control problem

$$\begin{cases} u_t - \Delta u = g\chi_\omega, & (x, t) \in \Omega \times (0, T), \\ u = 0, & (x, t) \in \partial\Omega \times (0, T), \\ u(x, 0) = 0, & x \in \Omega. \end{cases} \quad (\text{A.1})$$

Here,  $g$  is a control function satisfying the *approximate controllability* condition. That is, instead of driving the solution to an exact final state at a given end time as we have observed before, the control function should only have the restriction that it drives the solution close to a specified final state.

Let  $u_1 \in L^2(\Omega)$  be a final state for (A.1). For  $u_0 \in L^2(\Omega)$ ,  $g \in L^2(\omega \times (0, T))$ , there is a unique solution  $u \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega))$ . Given  $\epsilon > 0$ , we say then that the approximate controllability condition is satisfied if

$$\|u(x, T) - u_1\|_{L^2} \leq \epsilon. \quad (\text{A.2})$$

Then, we know (see [19] for example) that if  $u_0 \in L^2(\Omega), u_1 \in L^2(\Omega)$ , that (A.1) is approximately controllable in the sense of (A.2). Furthermore,  $\tilde{g}$  is a control function for (A.1), where  $\tilde{g}$  is given by the solution to the adjoint equation

$$\begin{cases} \tilde{g}_t + \Delta \tilde{g} = 0, & (x, t) \in \Omega \times (0, T), \\ \tilde{g} = 0, & (x, t) \in \partial\Omega \times (0, T), \\ \tilde{g}(x, T) = g_0 & x \in \Omega, \end{cases} \quad (\text{A.3})$$

where  $g_0$  is the (unique) minimizer of the functional

$$J_\epsilon(u_1, g) = \frac{1}{2} \int_0^T \int_\omega |g|^2 dx dt + \epsilon \left( \int_\Omega g^2 dx \right)^{\frac{1}{2}} - \int_\Omega u_1 g dx, \quad (\text{A.4})$$

in the sense that  $J_\epsilon(u_1, g_0) = \min_{g \in L^2(\Omega)} J_\epsilon(u_1, g)$ . We will observe that in fact, this  $\tilde{g}$  also has the minimum energy.

**Lemma 3.** *Let  $g_0$  be the minimizer of (A.4). Let  $\tilde{g}(x, t)$  be the solution to (A.3) with initial data  $g_0$ . Then  $\tilde{g}$  also minimizes the energy. That is, for any control function  $g$  for (A.1), we have*

$$\int_0^T \int_\omega |g|^2 dx dt \geq \int_0^T \int_\omega |\tilde{g}|^2 dx dt. \quad (\text{A.5})$$

*Proof.* Let  $g(x, t)$  be any control function for (A.1). Then, decompose  $g(x, t)$  as the sum of functions

$$g(x, t) = \tilde{g}(x, t) + h(x, t), \quad (\text{A.6})$$

where  $\tilde{g}(x, t)$  solves (A.3) with initial data  $g_0$ , where  $g_0$  is the minimizer of (A.4). Then, upon multiplication of  $\tilde{g}$  and integration, we get

$$\int_0^T \int_{\omega} (\tilde{g} + h) \tilde{g} dx dt = \int_0^T \int_{\omega} |\tilde{g}|^2 dx dt + \int_0^T \int_{\omega} \tilde{g} h dx dt. \quad (\text{A.7})$$

On the other hand, using the definition of the decomposition and (A.1), we have the relation

$$\begin{aligned} & \int_0^T \int_{\omega} (\tilde{g} + h) \tilde{g} dx dt \\ &= \int_0^T \int_{\Omega} (u_t - \Delta u) \tilde{g} dx dt \\ &= \int_0^T \int_{\Omega} u (\tilde{g}_t + \tilde{g}) dx dt + \int_{\Omega} u(T) \tilde{g}(T) dx \\ &= \int_{\Omega} u(T) \tilde{g}(T) dx, \end{aligned} \quad (\text{A.8})$$

since  $\tilde{g}$  solves the adjoint equation. Now, by using the definition for the approximate controllability condition, (A.7), and (A.8), we obtain

$$\begin{aligned} & \frac{1}{2} \int_0^T \int_{\omega} |g|^2 dx dt \\ &= \frac{1}{2} \int_0^T \int_{\omega} |\tilde{g} + h|^2 dx dt \\ &= \int_0^T \int_{\omega} \frac{1}{2} \tilde{g}^2 + \tilde{g} h + \frac{1}{2} h^2 dx dt \\ &\geq \frac{1}{2} \int_0^T \int_{\omega} \tilde{g}^2 dx dt + \int_0^T \int_{\omega} \tilde{g} h dx dt \\ &= \int_{\Omega} u(T) \tilde{g}(T) dx - \frac{1}{2} \int_0^T \int_{\omega} \tilde{g}^2 dx dt, \text{ by (A.8)} \\ &= \int_{\Omega} u_1 \tilde{g}(T) dx + \int_{\Omega} (u(T) - u_1) \tilde{g}(T) dx - \frac{1}{2} \int_0^T \int_{\omega} \tilde{g}^2 dx dt \\ &\geq \int_{\Omega} u_1 \tilde{g}(T) dx - \epsilon \|\tilde{g}(t)\|_{L^2(\Omega)} - \frac{1}{2} \int_0^T \int_{\omega} \tilde{g}^2 dx dt \\ &= -J_{\epsilon}(u_1) \end{aligned} \quad (\text{A.9})$$

$$= \frac{1}{2} \int_0^T \int_{\omega} |\tilde{g}(x, t)|^2 dx dt.$$

Thus, as claimed,

$$\int_0^T \int_{\omega} |g(x, t)|^2 dx dt \geq \int_0^T \int_{\omega} |\tilde{g}(x, t)|^2 dx dt. \quad (\text{A.10})$$

□



## BIBLIOGRAPHY

## BIBLIOGRAPHY

- [1] F. ALABAU-BOUSSOUIRA, P. CANNARSA, AND G. FRAGNELLI *Carleman estimates for degenerate parabolic operators with applications to null controllability*, J. Evol. Equ., 6 (2006) no. 2 pp. 161-204.
- [2] A. AMIROV AND M. YAMAMOTO, *Unique continuation and an inverse problem for hyperbolic equations across a general hypersurface*, J. Phys.: Conf. Ser., 12 (2005) pp. 1-12.
- [3] A. BENSOUSSAN, G. DA PRATO, M. C. DELFOUR, AND S. K. MITTER, *Representation and Control of Infinite Dimensional Systems*, Birkhäuser Boston 2007.
- [4] C. BARDOS, G. LEBEAU AND J. RAUCH, *Sharp sufficient conditions for the observation, control and stabilization of waves from the boundary*, SIAM J. Control Optim., 30 (1992) no. 5 pp. 1024-1065.
- [5] L. BAUDOUIN, M. BUHAN AND S. ERVEDOZA, *Global Carleman estimates for waves and applications*, Comm. Partial Differential Equations, 38 (2013) no. 5 pp. 823-859.
- [6] L. BAUDOUIN, E. CRÉPEAU AND J. VALEIN, *Global Carleman estimate on a network for the wave equation and application to an inverse problem*, Math. Control Relat. Fields, 1 (2011) no. 3 pp. 307-330.
- [7] P. CANNARSA AND G. FLORIDIA, *Approximate multiplicative controllability for degenerate parabolic problems with Robin boundary conditions*, Commun. Appl. Ind. Math., 2 (2011) no. 2 DOI: 10.1685/journal.caim.376.
- [8] P. CANNARSA, P. MARTINEZ, AND J. VANCOSTENOBLE, *Carleman Estimates for a class of degenerate parabolic operators*, SIAM J. Control Optim., 47 (2008) no. 1 pp. 1-19.
- [9] P. CANNARSA, P. MARTINEZ, AND J. VANCOSTENOBLE, *Global Carleman Estimates for Degenerate Parabolic Operators with Applications*, American Mathematical Soc., 2015.
- [10] P. CANNARSA, P. MARTINEZ, AND J. VANCOSTENOBLE, *Null controllability of degenerate heat equations*, Adv. Differential Equations 10 (2005) no. 2 pp. 153-190.

- [11] P. CANNARSA, P. MARTINEZ, AND J. VANCOSTENOBLE, *Persistent regional controllability for a class of degenerate parabolic equations*, Comm. Pure Appl. Math. 3 (2004) no. 4 pp. 607-635.
- [12] C. CASTRO AND E. ZUAZUA, *Concentration and lack of observability of waves in highly heterogeneous media*, Arch. Ration. Mech. Anal. 164 (2002) pp. 39-72.
- [13] N. CÎNDEA, E. FERNÁNDEZ-CARA AND A. MÜNCH, *Numerical controllability of the wave equation through primal methods and Carleman estimates*, ESAIM Control Optim. Calc. Var., 19 (2013) pp. 1076-1108.
- [14] D. D. S. FERREIRA, *Sharp  $L^p$  Carleman estimates and unique continuation*, Duke Math. J., 129 (2005) no. 3 pp. 503-550.
- [15] T. DUYCKAERTS, X. ZHANG AND E. ZUAZUA, *On the optimality of the observability inequalities for parabolic and hyperbolic systems with potentials*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 25 (2008) no. 1 pp. 1-41.
- [16] L. C. EVANS, *Partial Differential Equations*, American Mathematical Society, Providence, 2010.
- [17] C. FABRE, *Exact boundary controllability of the wave equation as the limit of internal controllability*, SIAM J. Control Optim., 30 (1992) no. 5 pp. 1066-1086.
- [18] C. FABRE AND J.-P. PUEL, *Pointwise controllability as limit of internal controllability for the wave equation in one space dimension*, Port. Math., 51 (1994) no. 3 pp. 335-350.
- [19] E. FERNÁNDEZ-CARA AND E. ZUAZUA, *The cost of approximate controllability for heat equations: the linear case*, Adv. Differential Equations, 5 (2000) pp. 465-514.
- [20] G. FLORIDIA, *Approximate controllability for nonlinear degenerate parabolic problems with bilinear control*, J. Differential Equations, 257 (2014) no. 9 pp. 3382-3422.
- [21] G. FLORIDIA, *Well-Posedness for a class of nonlinear degenerate parabolic equations*, in M. de León, W. Feng, Z. Feng, J. Lopez-Gomez, X. Lu, J.M. Martell, J. Parcet, D. Peralta-Salas, and W. Ruan, Eds., *Dynamical Systems and Differential Equations, AIMS Proceedings 2015*, pp. 455-463, 2015.
- [22] A. V. FURSIKOV AND O. Y. IMANUVILOV, *Controllability of evolution equations*, Volume 34 of Lecture Notes Series. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

- [23] X. FU, *A weighted identity for partial differential operators of second order and its applications*, C. R. Acad. Sci. Paris, Ser. I 342 (2006) pp. 579-584.
- [24] X. FU, J. YONG AND X. ZHANG, *Exact controllability for multidimensional semilinear hyperbolic equations*, SIAM J. Control Optim., 46 (2007) pp. 1578-1614.
- [25] D. GILBARG AND N.S. TRUDINGER, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag Berlin Heidelberg, 2001.
- [26] L. F. HO, *Observabilité frontière de l'équation des ondes*, C.R. Math. Acad. Sci. Paris Sér. I Math. 304 (1987) pp. 367-370.
- [27] O. Y. IMANUVILOV, *On Carleman estimates for hyperbolic equations*, Asymptot. Anal., 32 (2002) no. 3-4 pp. 185-220.
- [28] O. Y. IMANUVILOV AND M. YAMAMOTO, *Global uniqueness and stability in determining coefficients of wave equations*, Comm. Partial Diff. Eqns., 26 (2001) no. 7-8 pp. 1409-1425.
- [29] O. Y. IMANUVILOV AND M. YAMAMOTO, *Carleman estimates for the non-stationary Lamé system and the application to an inverse problem*, ESAIM Control Optim. Calc. Var., 11 (2005) no. 1 pp. 1-56.
- [30] C. KENIG, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation* in J. García-Cuerva, Ed., *Harmonic Analysis and Partial Differential Equations: Proceedings of the International Conference held in El Escorial, Spain, June 9-13, 1987*, Springer Berlin Heidelberg, 1989.
- [31] H. KOCH AND D. TATARU, *Carleman estimates and unique continuation for second-order elliptic equations with nonsmooth coefficients*, Comm. Pure Appl. Math., 54 (2001) no. 3, pp. 339-360.
- [32] V. KOMORNIK, *Exact controllability and Stabilization, The Multiplier Method*. RAM: Research in Applied Mathematics. Masson, Paris, 1994.
- [33] V. KOMORNIK, *Rapid boundary stabilization of the wave equation*, SIAM J. Control Optim., 29 (1991) no. 1 pp. 197-208.
- [34] J. LAGNESE, *Note on boundary stabilization of the wave equation*, SIAM J. Control Optim., 26 (1988) no. 5 pp. 1250-1256.

- [35] I. LASIECKA, R. TRIGGIANI, AND P.-F. YAO, *Inverse observability estimates for second-order hyperbolic equations with variable coefficients*, J. Math. Anal. Appl. 235 (1999) pp. 13-57.
- [36] J.-L. LIONS, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Recherches en Mathématiques Appliquées, Tomes 1 et 2, Masson, Paris, 1988.
- [37] J.-L. LIONS, *Control of Distributed Singular Systems*, Bordas, Paris, 1985.
- [38] J.-L. LIONS, *Exact controllability, stabilization and perturbations for distributed systems*, SIAM Rev., 30 (1988) no. 1 pp. 1-68.
- [39] J.-L. LIONS AND E. MAGENES, *Non-Homogeneous Boundary Value Problems and Applications* 3 vols. Springer Berlin Heidelberg, 1972-1973.
- [40] Y. LIU, *Some sufficient conditions for the controllability of wave equations with variable coefficients*, Acta Appl. Math, 128 (2013) pp. 181-191.
- [41] F. MACIÀ AND E. ZUAZUA, *On the lack of observability for wave equations: a gaussian beam approach*, Asymptot. Anal., 32 (2002) no. 1 pp. 1-26.
- [42] P. MARTIN, L. ROSIER, AND P. ROUCHON, *Null controllability of the structurally damped wave equation with moving point control*, SIAM J. Control Optim., 51 (2013) no. 1 pp. 660-684
- [43] J. RALSTON, *Solutions of the wave equation with localized energy*, Comm. Pure Appl. Math, 22 (1969) no. 6, 807-823.
- [44] J. RALSTON, *Gaussian beams and the propagation of singularities* in Walter Littman, Ed., *Studies in Partial Differential Equations*, MAA Studies in Mathematics, 23 (1989) pp. 206-248.
- [45] J.-P. RAYMOND, *Optimal Control of Partial Differential Equations*, Online, retrieved from <https://www.math.univ-toulouse.fr/~raymond/book-ficus.pdf> .
- [46] V.G. ROMANOV, *Carleman estimates for second-order hyperbolic equations*, Sib. Math. J., 46 (2006) no. 1 pp. 131-151.
- [47] C. SOGGE, *Oscillatory integrals and unique continuation for second order elliptic differential equations*, J. Amer. Math. Soc., 2 (1989) no. 3 pp. 491-515.

- [48] C. SOGGE, *Uniqueness in Cauchy problems for hyperbolic differential operators*, Trans. Amer. Math. Soc., 333 (1992) no. 2 pp. 821-833.
- [49] D. TATARU, *Carleman estimates and unique continuation for solutions to boundary value problems*, J. Math. Pures Appl. (9), 78 (1996) no. 1 pp. 65-98.
- [50] P.-F. YAO, *On the inequalities for exact controllability of wave equations with variable coefficients*, SIAM J. Control. Optim., 37 (1999) pp. 1568-1599.
- [51] J. ZABCZYK, *Mathematical Control Theory: An Introduction*, Systems & Control: Foundations & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992.
- [52] X. ZHANG, *Explicit observability inequalities for the wave equation with potential and its application*, Proc. R. Soc. A, 456 (2000) pp. 1101-1115.
- [53] X. ZHANG, *Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities*, SIAM J. Control Optim., 39 (2000) no. 3 pp. 812-834.
- [54] E. ZUAZUA, *Controllability and Observability of Partial Differential Equations: Some Results and Open Problems*, in C.M. Dafermos and E. Feireisl, Eds., *Handbook of Differential Equations: Evolutionary Equations Vol. 3*, Elsevier/North-Holland, Amsterdam, (2007) pp. 527-621.
- [55] E. ZUAZUA, *Some results and open problems on the controllability of linear and semi-linear heat equations*, in F. Colombini and C. Zuily, Eds., *Carleman Estimates and Applications to Uniqueness and Control Theory*, Birkhäuser, Basel, (2001) pp. 191-211.
- [56] E. ZUAZUA, *Propagation, observation, and control of waves approximated by finite difference methods* SIAM Rev., 47 (2005) no. 2 pp. 197-243.
- [57] E. ZUAZUA, *Propagation of waves: numerical approximation and control*, Bol. Soc. Esp. Math Apl., 25 (2003) pp. 55-126.