CHARACTERIZATIONS OF THE BLOCH SPACE AND RELATED SPACES

By

Karel Mattheus Rudolf Stroethoff

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1987

ABSTRACT

CHARACTERIZATIONS OF THE BLOCH SPACE AND RELATED SPACES

By

Karel Mattheus Rudolf Stroethoff

In the first chapter we give local and global Dirichlet-type characterizations for both the Bloch space and the little Bloch space, generalizing some of the characterizations for these spaces given in [2].

In the second chapter we characterize the Bloch space and the little Bloch space in terms of the pseudo-hyperbolic metric on the unit disk; it is shown that the Bloch space coincides with the class of analytic functions on the disk that are uniformly continuous with respect to the pseudo-hyperbolic metric.

In chapter three we further develop some of the results obtained by Baernstein in [3], where he proved that an analytic function on the disk belongs to the space *BMOA* if and only if the Möbius transforms of the function form a bounded family in the Nevanlinna class. We give a description of the space *VMOA* in terms of the Nevanlinna characteristic. A description of *VMOA* cannot be obtained by simply replacing Baernstein's boundedness condition by the corresponding vanishing condition (as is usually the case). We then formulate and prove analogous characterizations for the Bloch space and the little Bloch space in terms of an area version of the Nevanlinna characteristic.

In the fourth chapter we give a different proof of Baernstein's value distribution characterization for *BMOA* [3], Theorem 3, and we formulate and prove the corresponding description of the space *VMOA*. Defining an area version of the counting function used in the value characterizations for *BMOA* and *VMOA*, we obtain analogous results for the Bloch space and the little Bloch space.

Karel Mattheus Rudolf Stroethoff

In chapter five we give estimates for the growth of analytic functions in weighted Dirichlet space, which then are used to give necessary and sufficient conditions on the growth of an analytic function on the disk for inclusion in the Bloch space or the little Bloch space.

Chapter six briefly discusses cyclic vectors in the little Bloch space. We generalize a theorem of Anderson, Clunie and Pommerenke [1], Theorem 3.8.

In the seventh chapter we consider Hankel operators with integrable symbol. The Hankel operators that we study are defined by projecting onto the orthogonal complement of the Bergman space. We first prove that these Hankel operators transform in a unitarily equivalent way if the symbol is replaced by one of its Möbius transforms. We then restrict our attention to Hankel operators with conjugate analytic symbol, and show Sheldon Axler's results [2], Theorems 6 and 7, hold if the operator norm of the Hankel operator is obtained by putting a weighted L^p-norm on both its domain and its range.

REFERENCES:

[1] J. M. Anderson, J.G. Clunie and Ch. Pommerenke, "On Bloch functions and normal functions," Journal für die Reine und Angewandte Mathematik 270 (1974), 12-37.

[2] S. Axler, "The Bergman Space, The Bloch Space, and Commutators of Multiplication Operators," Duke Mathematical Journal 53 (1986), 315-332.

[3] A. Baernstein, "Analytic Functions of Bounded Mean Oscillation," pages 3-36 in *Aspects of Contemporary Analysis*, edited by D.A. Brannan and J.G. Clunie, Academic Press, London, 1980.

ACKNOWLEDGMENT

I would like to thank my advisor, Sheldon Axler. He has been an excellent teacher and he has provided me with very helpful guidance and advice. I thank him for the many valuable conversations; I particularly appreciate his interest, enthusiasm and encouragement.

TABLE OF CONTENTS

page

Chapter 0 1

Chapter 1 24

Chapter 2 43

Chapter 3 51

Chapter 4 63

Chapter 5 79

Chapter 6 103

Chapter 7 112

Bibliography 139

Chapter 0

In this chapter we give some background information, establish most of the notation for the chapters that follow, and list the major results of this thesis. Since we will be dealing with Bloch functions on the unit disk, we start with a theorem of the man whose name is attached to these functions.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk in the complex plane. The basic idea of a Bloch function on $\mathbb{D}$ goes back to the following theorem of André Bloch [9].

Bloch's Theorem *: There exists a finite positive constant b such that if g is an analytic function on $\mathbb{D}$, normalized so that $g(0) = 0$ and $g'(0) = 1$, then there is a disk Δ contained in $\mathbb{D}$ on which g is one-to-one and such that $g(\Delta)$ contains a disk of radius b.*

For $w \in \mathbb{C}$ and $0 < r < \infty$ we will use the notation $\Delta(w, r) = \{z \in \mathbb{C} : |z - w| < r\}$ for the open disk of radius r centered at w. For an analytic function f on $\mathbb{D}$ and a point z in $\mathbb{D}$ let $d_f(z)$ be the supremum of all $r > 0$ for which there exists an open connected neighborhood Ω of z in $\mathbb{D}$ such that f is one-to-one on Ω and $f(\Omega) = \Delta(f(z), r)$, unless $f'(z) = 0$ (and thus there are no such $r > 0$), in which case we let $d_f(z) = 0$. If $d_f(z) > 0$, then necessarily $d_f(z) < \infty$, and it is easy to show that the supremum in the definition of quantity $d_f(z)$ is actually attained, i.e., there exists an open connected neighborhood Ω of z in $\mathbb{D}$ such that f is one-to-one on Ω and $f(\Omega) = \Delta(f(z), d_f(z))$. A disk $\Delta(f(z), r)$ that is the image under f of an open connected neighborhood of z on which f is one-to-one, is called a schlicht disk of f around $f(z)$. Thus the number $d_f(z)$ is the radius of the largest schlicht disk of f around $f(z)$. The first systematic study of this quantity was done by W. Seidel and J.L. Walsh in [34]. As an easy consequence of Schwarz's Lemma we have:

$$d_f(0) \le |f'(0)|. \tag{0.1}$$

Another easy property is that for $\gamma \neq 0$:

$$d_{\gamma f}(z) = |\gamma|\, d_f(z). \tag{0.2}$$

For $\lambda \in \mathbb{D}$ let the Möbius function $\varphi_\lambda : \mathbb{D} \to \mathbb{D}$ be defined by

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \quad z \in \mathbb{D}. \tag{0.3}$$

Then it is easily verified that for an analytic function f on $\mathbb{D}$ and for every $\lambda \in \mathbb{D}$:

$$d_{f \circ \varphi_\lambda}(z) = d_f(\varphi_\lambda(z)). \tag{0.4}$$

Using this quantity, Bloch's Theorem can now be restated as follows:

There exists a finite positive constant b such that if g is an analytic function on $\mathbb{D}$, normalized so that $g(0) = 0$ and $g'(0) = 1$, then there exists a point $w \in \mathbb{D}$ for which $d_g(w) \ge b$.

If f is an analytic function on $\mathbb{D}$ and $f'(0) \neq 0$, then we can apply this version of Bloch's Theorem to the function $g = (f - f(0))/f'(0)$. Using the properties (0.1) and (0.2) it follows that there exists a point $w \in \mathbb{D}$ (depending on f) such that

$$d_f(0) \le |f'(0)| \le \frac{1}{b} d_f(w).$$

Observe that the above is trivially satisfied if $f'(0) = 0$ (with any $w \in \mathbb{D}$), so that the initial restriction that $f'(0) \neq 0$ can be removed. Now take $\lambda \in \mathbb{D}$. It is elementary to verify that $\varphi_\lambda'(0) = |\lambda|^2 - 1$, so that the above inequality and (0.4) give that for every λ in $\mathbb{D}$ there exists a point $w_\lambda \in \mathbb{D}$ for which

$$d_f(\lambda) \le (1 - |\lambda|^2)|f'(\lambda)| \le \frac{1}{b} d_f(w_\lambda). \tag{0.5}$$

Now, for an analytic function f on $\mathbb{D}$ we set

$$\|f\|_{\mathfrak{B}} = \sup_{z \in \mathbb{D}} (1 - |z|^2)|f'(z)|.$$

The Bloch space $\mathfrak{B}$ is the set of all analytic functions f on $\mathbb{D}$ for which $\|f\|_{\mathfrak{B}} < \infty$. Even though $\|\,.\,\|_{\mathfrak{B}}$ is not a norm, we will refer to $\|f\|_{\mathfrak{B}}$ as the Bloch norm of function f. The quantity $|f(0)| + \|f\|_{\mathfrak{B}}$ defines a norm on the linear space $\mathfrak{B}$, and we will see later that $\mathfrak{B}$ equipped with this norm is a Banach space.

Two quantities A_f and B_f, both depending on an analytic function f on $\mathbb{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on $\mathbb{D}$ we have:

$$\frac{1}{C} B_f \le A_f \le C\, B_f.$$

If the quantities A_f and B_f are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$. It follows from (0.5) that for an analytic function f on $\mathbb{D}$ we have the equivalence

$$\|f\|_{\mathfrak{B}} \approx \sup_{z \in \mathbb{D}} d_f(z) . \tag{0.6}$$

For a region $\Omega \subset \mathbb{C}$ let $H^\infty(\Omega)$ denote the algebra of all bounded analytic functions on Ω. We will simply write H^∞ for $H^\infty(\mathbb{D})$. It is clear that the image of a bounded analytic function cannot contain arbitrarily large schlicht disks, so that the equivalence (0.6) immediately gives us the inclusion $H^\infty \subset \mathfrak{B}$.

In the argument leading from Bloch's Theorem to the equivalence (0.6) the Möbius functions on the disk played an important role. For an analytic function f on $\mathbb{D}$ and a point $\lambda \in \mathbb{D}$, we will call the function $f \circ \varphi_\lambda - f(\lambda)$ a Möbius transform of function f. It follows from (0.4) and equivalence (0.6) that $\mathfrak{B}$ is invariant under Möbius transforms, i.e., if $f \in \mathfrak{B}$ and $\lambda \in \mathbb{D}$, then $f \circ \varphi_\lambda - f(\lambda) \in \mathfrak{B}$. This is also easy to see from the definition of the Bloch norm. Let f be an analytic function on $\mathbb{D}$, and let $\lambda \in \mathbb{D}$. We have already observed that $\varphi_\lambda'(0) = |\lambda|^2 - 1$, so that by the chain rule we have

$$(f \circ \varphi_\lambda)'(0) = f'(\lambda)\, \varphi_\lambda'(0) = (|\lambda|^2 - 1) f'(\lambda).$$

It follows that

$$\|f\|_{\mathfrak{B}} = \sup_{\lambda \in \mathbb{D}} |(f \circ \varphi_\lambda)'(0)| ,$$

hence for every $\lambda \in \mathbb{D}$:

$$\|f\|_{\mathfrak{B}} = \|f \circ \varphi_\lambda\|_{\mathfrak{B}}. \tag{0.7}$$

In [31] Rubel and Timoney showed that the Bloch space $\mathfrak{B}$ is maximal among all Möbius-invariant Banach spaces of analytic functions on $\mathbb{D}$ which have a decent linear functional.

Contained in the Bloch space is the little Bloch space $\mathfrak{B}_0$, which is by definition the set of all analytic functions f on $\mathbb{D}$ for which

$$(1-|z|^2)f'(z) \to 0 \text{ as } |z| \to 1^-.$$

It follows immediately from (0.5) that if f is in $\mathfrak{B}_0$ then $d_f(z) \to 0$ as $|z| \to 1^-$. That the converse is also true follows from the following result of Pommerenke ([27], Theorem 1):

If f is analytic on $\mathbb{D}$ and $d_f(z) \leq 1$ for all $z \in \mathbb{D}$, then for all $z \in \mathbb{D}$:

$$(1-|z|^2)\,|f'(z)| \leq \frac{2}{\sqrt{3}}\sqrt{d_f(z)}\,(3 - d_f(z)). \tag{0.8}$$

We can actually obtain a simpler proof as a result of the following theorem:

***Theorem 0.1** : Let f be an analytic function on $\mathbb{D}$. Then for every $z \in \mathbb{D}$ we have:*

$$(1-|z|^2)\,|f'(z)| \leq 4\sqrt{d_f(z)\ \|f\|_{\mathfrak{B}}}. \tag{0.9}$$

This theorem has the following corollary:

***Corollary 0.2** : Let f be an analytic function on $\mathbb{D}$. Then there exists a point $w \in \mathbb{D}$ for which*

$$d_f(w) \geq \frac{1}{20} \|f\|_{\mathfrak{B}} . \tag{0.10}$$

If for an analytic function g on the unit disk $g'(0) = 1$, then $\| g \|_{\mathfrak{B}} \geq 1$, and we see that Bloch's Theorem is a consequence of Corollary 0.2 (and conversely, it is easy to show that Corollary 0.2 is a consequence of Bloch's Theorem). A proof of Theorem 0.1 can be based on the following lemma which Edmund Landau used to give a proof of Bloch's Theorem (see [20], Satz 2).

***Lemma 0.3** : Let $0 < R < \infty$. Let g be analytic on the disk $\Delta(0, R)$, such that $g(0) = 0$ and $a = |g'(0)| > 0$. Suppose that $|g(z)| \leq M$ for all $|z| < R$. Then:*

$$d_g(0) \geq \frac{R^2 a^2}{6M} . \tag{0.11}$$

The following proof is derived from Landau's proof.

Proof:

Without loss of generality we can assume that $R = 1$ and $M = 1$ (otherwise consider the function h on $\Delta(0,1) = \mathbb{D}$ defined by $h(z) = g(Rz)/M$ for $z \in \mathbb{D}$).

Suppose that g is analytic on $\mathbb{D}$, such that $g(0) = 0$, $a = |g'(0)| > 0$, and $|g(z)| \leq 1$ for all $z \in \mathbb{D}$. We must show that $d_g(0) \geq a^2/6$. Let

$$g(z) = \sum_{n=1}^{\infty} a_n z^n \ , z \in \mathbb{D},$$

be the Taylor series expansion of function g. Then it is easy to show that $|a_n| \leq 1$ for all n in $\mathbb{N}$. In particular $a = |a_1| \leq 1$. So if we put $\rho = a/4$, then $0 < \rho \leq ¼$. Take a point w in $\Delta(0, a^2/6)$, and consider the function g_w defined on $\mathbb{D}$ by $g_w(z) = a_1 z - w$ for $z \in \mathbb{D}$. For $|z| = \rho$ we have

$$|g(z) - w - g_w(z)| \leq \sum_{n=2}^{\infty} |a_n||z|^n \leq \sum_{n=2}^{\infty} \rho^n = \frac{\rho^2}{1-\rho} \leq \frac{a^2}{12} \ .$$

Since $|w| < a^2/6$ we also have that for $|z| = \rho$, $|g_w(z)| \geq |a_1 z| - |w| > a^2/4 - a^2/6 = a^2/12$. Thus for all $|z| = \rho$ we have $|g(z) - w - g_w(z)| < |g_w(z)|$. By Rouché's Theorem the number of zeros of $g - w$ in $\Delta(0,\rho)$ is equal to the number of zeros of g_w in $\Delta(0,\rho)$, which is easily seen to be one. This shows that $\Delta(0, a^2/6)$ is a schlicht disk around $g(0) = 0$, so that $d_g(0) \geq a^2/6$, as was to be shown.□

Proof of Theorem 0.1:

Let f be an analytic function on $\mathbb{D}$. We must show that (0.9) holds. In view of (0.4) and (0.7) it suffices to show

$$|f'(0)| \leq 4\sqrt{d_f(0)\ \|f\|_{\mathcal{B}}} \ .$$

For $|z| \leq ½$

$$|f'(z)| \leq \frac{1}{(1-|z|^2)} \|f\|_{\mathcal{B}} \leq \frac{4}{3} \|f\|_{\mathcal{B}} \ ,$$

so that for $|z| \leq ½$:

$$|f(z) - f(0)| \leq \int_0^1 |z|\,|f'(tz)|\,dt$$

$$\leq \frac{1}{2}\,\frac{4}{3}\,\|f\|_{\mathfrak{B}} = \frac{2}{3}\,\|f\|_{\mathfrak{B}}\,.$$

Apply Lemma 0.3 with $g = f - f(0)$, $R = ½$, $M = (2/3)\,\|f\|_{\mathfrak{B}}$, and $a = |f'(0)|$. It follows from (0.11) that

$$|f'(0)|^2 \leq \frac{6M}{R^2}\,d_f(0) = 16\,d_f(0)\,\|f\|_{\mathfrak{B}}\,,$$

and the proof is complete.□

Proof of Corollary 0.2:

Let f be an analytic function on $\mathbb{D}$. If $\|f\|_{\mathfrak{B}} = 0$ then there is nothing to show, so assume that $\|f\|_{\mathfrak{B}} > 0$. Let $0 < \gamma < 1$. Choose a $w \in \mathbb{D}$ for which

$$(1 - |w|^2)\,|f'(w)| \geq \gamma\,\|f\|_{\mathfrak{B}}\,.$$

Then it follows from (0.9) that

$$d_f(w) \geq \left(\frac{\gamma}{4}\right)^2 \|f\|_{\mathfrak{B}}\,,$$

from which (1.10) follows by taking γ sufficiently large.□

We now turn from the geometric aspects of Bloch functions to the functional analytic aspects of the linear space $\mathfrak{B}$. In [13] the Bloch space is identified as the dual space of a Banach space whose norm is defined by an area integral. This implies that the Bloch space is a Banach space (which can also be proved directly from the definition). We will now introduce the Bergman spaces on the unit disk. Let A denote the usual Lebesgue area measure on the complex plane $\mathbb{C}$. For an analytic function f on $\mathbb{D}$ and $0 < p < \infty$ we define

$$\|f\|_{L_a^p} = \left(\int_{\mathbb{D}} |f|^p \, dA/\pi \right)^{1/p}.$$

The Bergman space L_a^p is defined to be the set of all analytic functions f on $\mathbb{D}$ for which $\|f\|_{L_a^p} < \infty$. The subscript a stands for "analytic." Clearly each Bergman space L_a^p is a linear space. For $1 \leq p < \infty$, $\|\,.\,\|_{L_a^p}$ is a norm on L_a^p, and equipped with this norm L_a^p becomes a Banach space. For $0 < p < 1$, $\|\,.\,\|_{L_a^p}$ is no longer a norm, but $\|f - g\|^p_{L_a^p}$ defines a translation invariant complete metric on L_a^p, so that L_a^p is a Fréchet space.

If $1 < p < \infty$, let $p' = p/(p-1)$ denote the conjugate index. The dual space of L_a^p can be identitied as $L_a^{p'}$: defining the pairing

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z)\, \overline{g(z)}\, dA(z)/\pi\,, \tag{0.12}$$

for $f \in L_a^p$, $g \in L_a^{p'}$, every bounded linear functional on L_a^p is of the form

$$f \mapsto \langle f, g \rangle, (f \in L_a^p) \tag{0.13}$$

for some unique $g \in L_a^{p'}$. Moreover, the norm of the linear functional in (0.13) is equivalent to the norm $\| g \|_{L_a^{p'}}$ (see [7] for a proof).

Very much in the same way the dual of the Bergman space L_a^1 can be identified as the Bloch space $\mathfrak{B}$. In [2] and in [13] the Bloch space is shown to be the dual of the space $\mathfrak{J}$ which is defined to be $\mathfrak{J} = \{f : f$ is analytic on $\mathbb{D}$ and $f' \in L_a^1\}$. The pairing used in both papers involves the derivative of the function in $\mathfrak{J}$. This is not parallel to the pairing in (0.12); it seems more natural to pair a Bloch function with a function in L_a^1. This was done by Sheldon Axler in [7]. We will outline his results. There is however a problem with the pairing as defined in (0.12): there exist $f \in L_a^1$ and $g \in \mathfrak{B}$ such that the product $f \bar{g}$ is not integrable over the disk $\mathbb{D}$. To overcome this problem define the pairing by

$$\langle f, g \rangle = \lim_{t \to 1^-} \int_{t\mathbb{D}} f(z)\, \overline{g(z)}\, dA(z)/\pi . \tag{0.14}$$

If $g \in \mathfrak{B}$, then (0.14) is defined for every $f \in L_a^1$ and the map

$$f \mapsto \langle f, g \rangle, (f \in L_a^1) \tag{0.15}$$

is a bounded linear functional on L_a^1 with norm equivalent to $\| g \|_{\mathfrak{B}} + | g(0) |$, and every bounded linear functional on L_a^1 is of the form (0.15) for some unique $g \in \mathfrak{B}$.

Finally, the dual space of the little Bloch space $\mathfrak{B}_0$ can be identified with L_a^1 : every bounded linear functional on $\mathfrak{B}_0$ is of the form

$$f \mapsto \langle f, g \rangle, (f \in \mathfrak{B}_0) \tag{0.16}$$

for some unique $g \in L_a^1$, and the norm of the linear functional in (0.16) is equivalent to the norm $\| g \|_{L_a^1}$.

For $\lambda \in \mathbb{D}$ recall the definition of the Möbius function φ_λ defined in (0.3):

$$\varphi_\lambda(z) = \frac{\lambda - z}{1 - \bar{\lambda} z}, \quad z \in \mathbb{D}.$$

The function φ_λ is easily seen to be it own inverse under composition:

$$(\varphi_\lambda \circ \varphi_\lambda)(z) = z \text{ for all } z \in \mathbb{D}.$$

The following identity can be obtained by straight forward computation:

$$\frac{1 - \bar{u}\,\varphi_\lambda(z)}{1 - \bar{u}\lambda} = \frac{1 - \overline{\varphi_\lambda(u)}\,z}{1 - \bar{\lambda} z}, \quad (u, \lambda, z \in \mathbb{D}). \tag{0.17}$$

The special case that $u = \lambda$ yields

$$(1 - \bar{\lambda}\,\varphi_\lambda(z))(1 - \bar{\lambda} z) = 1 - |\lambda|^2, \quad (\lambda, z \in \mathbb{D}). \tag{0.18}$$

If we substitute $u = \varphi_\lambda(z)$ in (0.17) and make use (0.18) we obtain the identity:

$$1 - |\varphi_\lambda(z)|^2 = \frac{(1 - |\lambda|^2)(1 - |z|^2)}{|1 - \bar{\lambda} z|^2}, \quad (\lambda, z \in \mathbb{D}). \tag{0.19}$$

A slightly different form in which we will frequently apply identity (0.19) is:

$$\frac{1-|\varphi_\lambda(z)|^2}{1-|z|^2} = |\varphi_\lambda{}'(z)| \ , (\lambda, z \in \mathbb{D}) . \tag{0.20}$$

For points λ, z in the disk $\mathbb{D}$ the pseudo-hyperbolic distance $\mathrm{d}(\lambda, z)$ between λ and z is defined by

$$\mathrm{d}(\lambda, z) = |\,\varphi_\lambda(z)\,| .$$

Then it can be shown that d is a metric on $\mathbb{D}$ (see, for example, [14], page 4). For each point $\lambda \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(\lambda,r)$ with pseudo-hyperbolic center λ and pseudo-hyperbolic radius r is defined by

$$D(\lambda,r) = \{\, z \in \mathbb{D} : \mathrm{d}(\lambda, z) < r \,\}.$$

The pseudo-hyperbolic disk $D(\lambda,r)$ is also a euclidean disk: its euclidean center and euclidean radius are:

$$w = \frac{1-r^2}{1-r^2|\lambda|^2}\,\lambda ,$$

and

$$s = \frac{1-|\lambda|^2}{1-r^2|\lambda|^2}\,r ,$$

respectively.

For a Lebesgue measurable set $K \subset \mathbb{C}$, let $|K|$ denote the measure of K with respect to the normalized Lebesgue area measure A/π. It follows immediately that:

$$|D(\lambda,r)| = \frac{(1-|\lambda|^2)^2}{(1-r^2|\lambda|^2)^2}\, r^2 . \tag{0.21}$$

For $\lambda \in \mathbb{D}$, the substitution $z = \varphi_\lambda(w)$ results in the Jacobian change in measure given by $dA(z)/\pi = |\varphi_\lambda'(w)|^2 \, dA(w)/\pi$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on $\mathbb{D}$ we have the following change-of-variable formulas:

$$\int_{D(\lambda,r)} h(z)\, dA(z)/\pi = \int_{D(0,r)} (h \circ \varphi_\lambda)(w)\, \frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda} w|^4}\, dA(w)/\pi , \tag{0.22a}$$

and

$$\int_{D(0,r)} (h \circ \varphi_\lambda)(z)\, dA(z)/\pi = \int_{D(\lambda,r)} h(w)\, \frac{(1-|\lambda|^2)^2}{|1-\bar{\lambda} w|^4}\, dA(w)/\pi . \tag{0.22b}$$

Many of the properties of the Bloch space and the little Bloch space are analogous to their counterparts in the classical Hardy space setting. Recall the definition of the Hardy spaces: for an analytic function f on $\mathbb{D}$ and $0 < p < \infty$ define

$$\|f\|_{H^p} = \left(\sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p} .$$

The Hardy space H^p is defined to be the set of all analytic functions f on the disk $\mathbb{D}$ for which $\| f \|_{H^p} < \infty$. For $1 \le p < \infty$, $\| \cdot \|_{H^p}$ is a norm on H^p, and equipped with this norm H^p is a Banach space; for $0 < p < 1$, H^p is a Fréchet space (see e.g. [12], Corollaries 1 and 2 on page 37).

Let $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$ denote the unit circle, and let μ_0 denote the normalized Lebesgue measure on $\partial\mathbb{D}$. If $f \in H^p$, for $0 < p < \infty$, then the radial limit

$$f^*(\zeta) = \lim_{r \to 1^-} f(r\zeta)$$

exists for $[\mu_0]$-a.e. $\zeta \in \partial\mathbb{D}$, and the function $f^* \in L^p(\partial\mathbb{D},\mu_0)$ ([12], Theorem 2.2).

If $1 < p < \infty$, let $p' = p/(p-1)$ denote the conjugate index. The dual space of H^p can be identitied as $H^{p'}$: the pairing is

$$\langle f, g \rangle = \int_{\partial\mathbb{D}} f^*(\zeta)\, \overline{g^*(\zeta)}\, d\mu_0(\zeta)\ , \tag{0.23}$$

for $f \in H^p$ and $g \in H^{p'}$ ([12], Theorem 7.3).

Before we give Charles Fefferman's identification of the dual space of H^1 we need to introduce more notation. A connected subset $I \subset \partial\mathbb{D}$ for which $\mu_0(I) > 0$ will be called an arc in $\partial\mathbb{D}$. For a function $g \in L^1(\partial\mathbb{D},\mu_0)$ and an arc I in $\partial\mathbb{D}$ let g_I denote the average of g over I:

$$g_I = \frac{1}{\mu_0(I)} \int_I g\, d\mu_0\ .$$

For a function $g \in L^1(\partial\mathbb{D},\mu_0)$ let

$$\| g \|_{BMO} = \sup \left\{ \frac{1}{\mu_0(I)} \int_I |g - g_I|\, d\mu_0 : I \text{ an arc in } \partial\mathbb{D} \right\}.$$

A function $g \in L^1(\partial\mathbb{D},\mu_0)$ for which $\| g \|_{BMO} < \infty$ is said to be of bounded mean oscillation. The set of all functions in $L^1(\partial\mathbb{D},\mu_0)$ that are of bounded mean oscillation is denoted by BMO. The class BMO was first introduced by John and Nirenberg in [18] (in the context of functions defined on cubes in $\mathbb{R}^n$).

Define $BMOA = \{ f \in H^1 : f^* \in BMO \}$, and for $f \in BMOA$ set

$$\| f \|_{BMOA} = \| f^* \|_{BMO} .$$

Equipped with the norm $\| f \|_{BMOA} + | f(0) |$, $BMOA$ is a Banach space. For $0 < p < \infty$ it can be shown that for every analytic function f on $\mathbb{D}$:

$$\| f \|_{BMOA} \approx \sup_{\lambda \in \mathbb{D}} \| f \circ \varphi_\lambda - f(\lambda) \|_{H^p} . \tag{0.24}$$

Charles Fefferman proved that the dual space of H^1 can be identified with the space $BMOA$. There is however a problem with the pairing as defined in (0.23): there exist functions $f \in H^1$ and $g \in BMOA$ such that $f^* g^{*-}$ is not integrable over the circle $\partial\mathbb{D}$. Fefferman showed that if $g \in BMOA$, then the map

$$\psi(f) = \int_{\partial\mathbb{D}} f^*(\zeta) \, \overline{g^*(\zeta)} \, d\mu_0(\zeta) , \; f \in H^\infty \tag{0.25}$$

extends to a bounded linear functional on H^1 with norm equivalent to $\| g \|_{BMOA} + | g(0) |$, and every bounded linear functional on H^1 is of the form (0.25) for a unique $g \in BMOA$ (for a proof see [8]).

By using Taylor series it is easy to see that $| f'(0) | \leq \| f \|_{H^2}$ for every analytic function f on $\mathbb{D}$. It follows that for an analytic function f on $\mathbb{D}$ and a point $\lambda \in \mathbb{D}$:

$$(1 - |\lambda|^2)\,|f'(\lambda)| \le \|f \circ \varphi_\lambda - f(\lambda)\|_{H^2} . \tag{0.26}$$

Thus we have the inclusion $BMOA \subset \mathfrak{B}$.

Paley's integral inequalities (see Chapter 5) and a change-of-variable give us that for every analytic function f on $\mathbb{D}$:

$$\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2)\, dA(z)/\pi \le \|f \circ \varphi_\lambda - f(\lambda)\|^2_{H^2} \le$$

$$\le 2 \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2)\, dA(z)/\pi . \tag{0.27}$$

It follows from (0.24) and (0.27) that for every analytic function f on $\mathbb{D}$:

$$\|f\|_{BMOA} \approx \sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2)\, dA(z)/\pi .$$

In [32] Donald Sarason introduced the space VMO of functions of vanishing mean oscillation defined by

$$VMO = \{g \in L^1(\partial\mathbb{D},\mu_0) : \frac{1}{\mu_0(I)} \int_I |g - g_I|\, d\mu_0 \to 0 \text{ as } \mu_0(I) \to 0 \} .$$

Define $VMOA = \{ f \in H^1 : f^* \in VMO \}$. Since clearly VMO is contained in BMO , we have that $VMOA$ is contained in $BMOA$. It can be shown that analogous to equivalence (0.24), if $0 < p < \infty$, then for every analytic function f on $\mathbb{D}$:

$$f \in VMOA \Longleftrightarrow \left[\| f \circ \varphi_\lambda - f(\lambda) \|_{H^p} \to 0 \text{ as } |\lambda| \to 1^- \right]. \tag{0.28}$$

From (0.26) and (0.28) we get the inclusion $VMOA \subset \mathfrak{B}_0$. From (0.27) and (0.28) we see that for every analytic function f on $\mathbb{D}$:

$$f \in VMOA \Longleftrightarrow \left[\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- \right].$$

For an analytic function f on $\mathbb{D}$ and $t \in (0, 1)$ the dilate f_t is defined by $f_t(z) = f(tz)$ for $z \in \mathbb{D}$. It can be shown that an analytic function f on $\mathbb{D}$ belongs to $VMOA$ if and only if $\| f - f_t \|_{BMOA} \to 0$ as $t \to 1^-$. Since each dilate of an analytic function is continuous on a neighborhood of $\mathbb{D}$ it is easy to see that the space $VMOA$ is the closure in $BMOA$ of the set of all polynomials.

The dual space of $VMOA$ can be identified with H^1: if $g \in H^1$, then the map

$$\psi(f) = \int_{\partial\mathbb{D}} f^*(\zeta) \, \overline{g^*(\zeta)} \, d\mu_0(\zeta), \quad f \text{ a polynomial},$$

extends to a bounded linear functional on $VMOA$ with norm equivalent to $\| g \|_{H^1}$, and every bounded linear functional on $VMOA$ can be obtained in this way.

We now list the major results in this thesis.

In the following two theorems we give local and global Dirichlet-type characterizations for the Bloch space and the little Bloch space, generalizing some of the characterizations for these spaces given in [6]:

Theorem 1.7 : *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *Then for an analytic function* f *on* $\mathbb{D}$ *the following quantities are equivalent:*

(A) $\|f\|_{\mathfrak{B}}$;

(B) $$\sup_{\lambda \in \mathbb{D}} \left(\frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \right)^{1/p} + \sum_{k=1}^{n-1} |f^{(k)}(0)| \;;$$

(C) $$\sup_{\lambda \in \mathbb{D}} \left(\int_{D(\lambda,r)} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} \, dA(z)/\pi \right)^{1/p} + \sum_{k=1}^{n-1} |f^{(k)}(0)| \;;$$

(D) $$\sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \right)^{1/p} + \sum_{k=1}^{n-1} |f^{(k)}(0)| .$$

Theorem 1.9 : *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *Then for an analytic function* f *on* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) $$\frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \to 0 \text{ as } |\lambda| \to 1^- ;$$

(c) $$\int_{D(\lambda,r)} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- ;$$

(d) $$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- .$$

In [8] Baernstein proved that an analytic function on $\mathbb{D}$ belongs to the space *BMOA* if and only if the Möbius transforms of the function form a bounded family in the Nevanlinna class. The following theorem gives a description of the space $\dot{V}MOA$ in terms of the Nevanlinna chacteristic T :

Theorem 3.3 : *For an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in VMOA$;

(b) *for every $\rho > 0$ we have that* $T\left(\dfrac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) \to 0$ *as* $|\lambda| \to 1^-$.

The following two theorems give analogous characterizations for the Bloch space and the little Bloch space, in terms of T_a, an area version of the Nevanlinna characteristic:

Theorem 3.6 : *For an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(A) $f \in \mathfrak{B}$;

(B) $\sup_{\lambda \in \mathbb{D}} T_a(f \circ \varphi_\lambda - f(\lambda)) < \infty$.

Theorem 3.7 : *For an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) *For every $\rho > 0$ we have that* $T_a\left(\dfrac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) \to 0$ *as* $|\lambda| \to 1^-$.

In [8] Baernstein gave a value distribution characterization for the space *BMOA*. The following theorem describes the space *VMOA* in terms of the counting function N (for which the definition is given in chapter 4):

Theorem 4.3 : *For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in VMOA$;

(b) *for every $\delta > 0$ we have:*

$$\sup \{N(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\} \to 0 \text{ as } |\lambda| \to 1^- .$$

Defining N_a, an area version of the counting function N (see chapter 4), we have analogous results for the Bloch space and the little Bloch space:

Theorem 4.4 : *For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(A) $f \in \mathfrak{B}$;

(B) $\sup \{N_a(w, \lambda, f) : w \in \mathbb{C}, \lambda \in \mathbb{D} \text{ and } |f(\lambda) - w| \geq 1\} < \infty .$

Theorem 4.5 : *For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) *for every $\delta > 0$ we have:*

$$\sup \{N_a(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\} \to 0 \text{ as } |\lambda| \to 1^- .$$

For an analytic function f on $\mathbb{D}$ and $0 < r < 1$ let $\overline{M}(r,f) = \sum_{n=0}^{\infty} |a_n| r^n$,

where the numbers a_n are the Taylor coefficients of f at 0. Using this crude estimate on the growth of an analytic function we have the following results:

Theorem 5.3 : *Let* $0 < r < 1$. *For an analytic function* f *on* $\mathbb{D}$ *the following quantities are equivalent:*

(A) $\|f\|_{\mathfrak{B}}$;

(B) $\sup_{\lambda \in \mathbb{D}} \overline{M}(r, f \circ \varphi_\lambda - f(\lambda))$.

Theorem 5.4 : *Let* $0 < r < 1$. *For an analytic function* f *on* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) $\overline{M}(r, f \circ \varphi_\lambda - f(\lambda)) \to 0$ *as* $|\lambda| \to 1^-$.

The following result on cyclic vectors generalizes a theorem of Anderson, Clunie and Pommerenke ([2], Theorem 3.8). It is similar to a result of Brown and Shields for the Dirichlet space ([10], Theorem 1).

Corollary 6.4 : *Let* $f, g \in \mathfrak{B}_0$, *such that* $|f(z)| \geq |g(z)|$ $(z \in \mathbb{D})$, *and suppose that g is bounded and* g^2 *is cyclic for* $\mathfrak{B}_0$. *Then* f *is cyclic for* $\mathfrak{B}_0$.

Similar to Proposition 11 of [10] we have the following result:

Corollary 6.6 : *If* $f, g \in \mathfrak{B}_0 \cap H^\infty$, *and if* fg *is cyclic for* $\mathfrak{B}_0$, *then both* f *and* g *are cyclic for* $\mathfrak{B}_0$.

The Hankel operators H_f are defined by projecting onto the orthogonal complement of the Bergman space (see chapter 7). These Hankel operators transform in a unitarily equivalent way if the symbol is repaced by one of its Möbius transforms:

Theorem 7.1 : *Let $f \in L^1(\mathbb{D}, dA/\pi)$. For each $\lambda \in \mathbb{D}$ the Hankel operators H_f and $H_{f \circ \varphi_\lambda}$ are u nitarily equiva lent.*
More precisely, there exist unitary operators $U_1 : L_a^2 \to L_a^2$ and $U_2 : (L_a^2)^\perp \to (L_a^2)^\perp$ such that $U_1(H^\infty) \subset H^\infty$ and

$$U_2 \circ H_{f \circ \varphi_\lambda} = H_f \circ U_1 .$$

Corollary 7.2 : *Let $f \in L^1(\mathbb{D}, dA/\pi)$, and $0 < p \leq \infty$. If $H_f \in \mathcal{C}^p$, then for each $\lambda \in \mathbb{D}$*

$$H_{f \circ \varphi_\lambda} \in \mathcal{C}^p .$$

Sheldon Axler's results [6], Theorems 6 and 7 hold if the operator norm of the Hankel operator is obtained by putting a weighted L^p-norm $\| \, . \, \|_{p,\alpha}$ on both the domain and the range of the Hankel operator:

Theorem 7.3 : *Let $1 < p < \infty$ and $-1 < \alpha < p - 1$. Then for $f \in L_a^1$ the Bloch norm $\| f \|_{\mathfrak{B}}$ and the operator norm $\| H_{\bar{f}} \|_{p,\alpha}$ are equivalent.*
In particular, $H_{\bar{f}}$ is bounded as an operator on H^∞ with the weighted L^p-norm $\| \, . \, \|_{p,\alpha}$ on both the domain and the range of $H_{\bar{f}}$ if and only if $f \in \mathfrak{B}$.

Theorem 7.4 : *Let* $1 < p < \infty$ *and* $-1 < \alpha < p - 1$. *Then for* $f \in L_a^1$ *the Hankel operator* $H_{\bar{f}}$ *is compact as an operator on* H^∞ *with the weighted* L^p*-norm* $\| \cdot \|_{p,\alpha}$ *on both the domain and the range of* $H_{\bar{f}}$ *if and only if* $f \in \mathfrak{B}_0$.

Chapter 1

In this chapter we will give several Dirichlet-type characterizations for the Bloch space and the little Bloch space. Our point of departure is the following theorem which is taken from [6], where it is proved for $1 \le p < \infty$.

***Theorem 1.1** : Let $0 < p < \infty$ and let $0 < r < 1$. Then for an analytic function f on the unit disk $\mathbb{D}$ the following quantities are equivalent :*

(A) $\|f\|_{\mathfrak{B}}$;

(B) $\sup\limits_{\lambda \in \mathbb{D}} \|f \circ \varphi_\lambda - f(\lambda)\|_{L_a^p}$;

(C) $\sup\limits_{\lambda \in \mathbb{D}} \left(\dfrac{1}{|D(\lambda,r)|} \displaystyle\int_{D(\lambda,r)} |f(z) - f_{D(\lambda,r)}|^p \, dA(z)/\pi \right)^{1/p}$;

(D) $\sup\limits_{\lambda \in \mathbb{D}} \left(\dfrac{1}{|D(\lambda,r)|} \displaystyle\int_{D(\lambda,r)} |f(z) - f(\lambda)|^p \, dA(z)/\pi \right)^{1/p}$;

(E) $\sup\limits_{\lambda \in \mathbb{D}} \text{distance} \left(\bar{f}\,|_{D(\lambda,r)}, H^\infty(D(\lambda,r)) \right)$;

(F) $\sup\limits_{\lambda \in \mathbb{D}} \left(\text{area } f(D(\lambda,r)) \right)^{1/2}$;

(G) $\sup\limits_{\lambda \in \mathbb{D}} \left(\displaystyle\int_{D(\lambda,r)} |f'(z)|^2 \, dA(z)/\pi \right)^{1/2}$.

Whereas quantities *(B)*, *(C)*, and *(D)* in Theorem 1.1 are expressed for general p in $(0,\infty)$, quantity *(G)* is given only for the special case that $p = 2$. The question arises whether quantity *(G)* in Theorem 1.1 can be replaced by a more general quantity depending upon p and specializing to the above *(G)* in case $p = 2$.

Quantities *(C)* and *(D)* are local as opposed to quantity *(B)*, which is global, this leads to another question: is there a global version of quantity *(G)*?

These questions will be answered in Theorem 1.7, where we will also give equivalent quantities involving higher derivatives of the function.

The equivalences of Theorem 1.1 carry over to the little Bloch space. Several descriptions of this space are given in the following theorem which is taken from [6], where it is proved for $1 \le p < \infty$.

Theorem 1.2 : *Let* $0 < p < \infty$ *and let* $0 < r < 1$. *Then for an analytic function* f *on the unit disk* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) $\| f \circ \varphi_\lambda - f(\lambda) \|_{L^p_a} \to 0$ *as* $|\lambda| \to 1^-$;

(c) $\dfrac{1}{|D(\lambda,r)|} \displaystyle\int_{D(\lambda,r)} |f(z) - f_{D(\lambda,r)}|^p \, dA(z)/\pi \to 0$ *as* $|\lambda| \to 1^-$;

(d) $\dfrac{1}{|D(\lambda,r)|} \displaystyle\int_{D(\lambda,r)} |f(z) - f(\lambda)|^p \, dA(z)/\pi \to 0$ *as* $|\lambda| \to 1^-$;

(e) *distance* $\left(\bar{f}|_{D(\lambda,r)}, H^\infty(D(\lambda,r)) \right) \to 0$ *as* $|\lambda| \to 1^-$;

(f) *area* $f(D(\lambda,r)) \to 0$ *as* $|\lambda| \to 1^-$;

(g) $\displaystyle\int_{D(\lambda,r)} |f'(z)|^2 \, dA(z)/\pi \to 0$ *as* $|\lambda| \to 1^-$;

(h) $\| f - f_t \|_{\mathfrak{B}} \to 0$ *as* $t \to 1^-$.

Although the definition of the Bloch space only involves the first derivative of the function, the following lemma gives characterizations involving higher derivatives.

Lemma 1.3 *: Let $n \in \mathbb{N}$. Then for an analytic function f on $\mathbb{D}$ the following quantities are equivalent:*

(A) $\| f \|_{\mathcal{B}}$;

(B) $$\sup_{z \in \mathbb{D}} (1 - |z|^2)^n \, |f^{(n)}(z)| + \sum_{k=1}^{n-1} |f^{(k)}(0)| \, .$$

Proof :

For $n = 1$ the equivalence of the two quantities is precisely the definition of the Bloch norm. By induction it suffices to show that for a fixed $n \in \mathbb{N}$, for every analytic function f on $\mathbb{D}$ the quantities

$$(B_n) \quad \sup_{z \in \mathbb{D}} (1 - |z|^2)^n \, |f^{(n)}(z)| + \sum_{k=1}^{n-1} |f^{(k)}(0)|$$

and

$$(B_{n+1}) \quad \sup_{z \in \mathbb{D}} (1 - |z|^2)^{n+1} \, |f^{(n+1)}(z)| + \sum_{k=1}^{n} |f^{(k)}(0)|$$

are equivalent.

Let g be an analytic function on $\mathbb{D}$, and let $w \in \mathbb{D}$. Then:

$$\begin{aligned}
|g(w) - g(0)| &\leq \int_0^1 |w|\, |g'(tw)|\, dt \\
&= \int_0^1 \frac{|w|}{(1-|tw|)^{n+1}} (1-|tw|)^{n+1} |g'(tw)|\, dt \\
&\leq \int_0^1 \frac{|w|}{(1-t|w|)^{n+1}}\, dt \; . \sup_{z \in \mathbb{D}} (1-|z|)^{n+1} |g'(z)| \\
&\leq \frac{1}{n(1-|w|)^n} \sup_{z \in \mathbb{D}} (1-|z|)^{n+1} |g'(z)| \; .
\end{aligned}$$

Thus

$$(1-|w|)^n\, |g(w)| \leq \frac{1}{n} \sup_{z \in \mathbb{D}} (1-|z|)^{n+1}\, |g'(z)| + |g(0)| \; .$$

Put $g = f^{(n)}$, multiply by $(1+|w|)^n$ (which is less than 2^n), and take the supremum over all $w \in \mathbb{D}$, to get

$$\sup_{w \in \mathbb{D}} (1-|w|^2)^n\, |f^{(n)}(w)| \leq \frac{2^n}{n} \sup_{z \in \mathbb{D}} (1-|z|^2)^{n+1}\, |f^{(n+1)}(z)| + |f^{(n)}(0)| \; .$$

Hence quantity (B_n) is less than or equal $2^n/n$ times quantity (B_{n+1}).

For the converse, fix $z \in \mathbb{D}$ and put $r = (1-|z|)/2$. Again let g be an analytic function on $\mathbb{D}$. By the Cauchy Integral Formula

$$g'(z) = \frac{1}{2\pi i} \int_{|w - z| = r} \frac{g(w)}{(w - z)^2}\, dw \ ,$$

so that

$$|g'(z)| \leq \frac{1}{r}\, sup\ \{|g(w)| : |w - z| = r\} . \tag{1.1}$$

If $|w - z| = r$, then $|w| \leq |z| + r = (1 + |z|)/2$. By the analyticity of g,

$$sup\ \{|g(w)| : |w - z| = r\} \leq sup\ \{|g(w)| : |w| \leq (1 + |z|)/2\} .$$

Multiply both sides of inequality (1.1) by $(1 - |z|)^{n+1} = 2^{n+1} r^{n+1}$ to get

$$(1 - |z|)^{n+1} |g'(z)| \leq 2^{n+1} sup\ \{r^n\ |g(w)| : |w| = (1 + |z|)/2\} .$$

For $|w| = (1 + |z|)/2$ we have $1 - |w| = (1 - |z|)/2 = r$, so it follows that

$$(1 - |z|)^{n+1} |g'(z)| \leq 2^{n+1} sup\ \{ (1 - |w|)^n\ |g(w)| : |w| = (1 + |z|)/2\} . \tag{1.2}$$

Put $g = f^{(n)}$, multiply by $(1 + |z|)^{n+1}$ (which is less than 2^{n+1}), and take the supremum over all $z \in \mathbb{D}$, getting

$$\sup_{z \in \mathbb{D}} (1 - |z|^2)^{n+1} |f^{(n+1)}(z)| \leq 2^{2n+2} \sup_{w \in \mathbb{D}} (1 - |w|^2)^n\ |f^{(n)}(w)| .$$

It follows that quantity (B_{n+1}) is less than or equal 2^{2n+2} times quantity (B_n). This completes the induction and the lemma is proved. □

The equivalences of Lemma 1.3 carry over to the little Bloch space, as is shown in the following lemma.

Lemma 1.4 *: Let $n \in \mathbb{N}$. Then for an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) $(1 - |z|^2)^n f^{(n)}(z) \to 0$ *as* $|z| \to 1^-$.

Proof:

For $n = 1$ the equivalence of the two statements is precisely the definition of the little Bloch space. By induction it suffices to show that for a fixed $n \in \mathbb{N}$, for every analytic function f on $\mathbb{D}$ the statements

(b_n) $\quad (1 - |z|^2)^n f^{(n)}(z) \to 0$ as $|z| \to 1^-$

and

(b_{n+1}) $\quad (1 - |z|^2)^{n+1} f^{(n+1)}(z) \to 0$ as $|z| \to 1^-$

are equivalent.

Let $n \in \mathbb{N}$ be fixed. That statement (b_n) implies statement (b_{n+1}) follows easily from (1.2) (applied to $g = f^{(n)}$).

For the converse suppose that f is an analytic function on $\mathbb{D}$ satisfying condition (b_{n+1}). Let $0 < r < 1$. Let g be an analytic function on $\mathbb{D}$, and let $w \in \mathbb{D}$. Then as in the proof of Lemma 1.3 :

$$|g(w) - g(rw)| \leq \int_r^1 |w|\ |g'(tw)|\ dt$$

$$\leq \frac{1}{n(1-|w|)^n}\ sup\ \{\ (1-|z|)^{n+1}\ |g'(z)| : r|w| \leq |z| < 1\ \}\ .$$

Therefore we have

$$(1-|w|)^n\ |g\,(w)| \leq \frac{1}{n}\ sup\ \{\ (1-|z|)^{n+1}\ |g'(z)| : r|w| \leq |z| < 1\ \} + (1-|w|)^n\ |g(rw)|$$

In the above inequality put $g = f^{(n)}$. For given $\varepsilon > 0$, choose $\rho \in (0, 1)$ such that $(1-|z|^2)^{n+1}\ |f^{(n+1)}(z)| < \varepsilon$ whenever $\rho < |z| < 1$. For $\rho < r < 1$ it follows from the above inequality that $(1-|w|)^n\ |f^{(n)}(w)| \leq \varepsilon/n + (1-|w|)^n\ |f^{(n)}(rw)|$ whenever we have $\rho/r < |w| < 1$. Hence $(1-|w|^2)^n f^{(n)}(w) \to 0$ as $|z| \to 1^-$, i.e., f satisfies (b_n) . This completes the induction, and the lemma is proved. □

For the statement and proof of the following lemma we need more notation. For a point $\lambda \in \mathbb{D}$ and $0 < r < 1$, let

$$D^2(\lambda,r) = \cup\{\ D(w,r) : D(w,r) \cap D(\lambda,r) \neq \emptyset\ \}\ .$$

Since φ_λ is injective, $D(w,r) \cap D(\lambda,r) \neq \varnothing$ if and only if $D(\varphi_\lambda(w),r) \cap D(0,r) \neq \varnothing$. It follows that $D^2(\lambda,r) = \varphi_\lambda(D^2(0,r))$. It is easily seen that $D^2(0,r) = D(0,s)$, where $s = s(r) = ((3 + r^2)r)/(1 + 3r^2)$. Thus we have $D^2(\lambda,r) = D(\lambda,s)$. Note also that $D^2(\lambda,r/3) \subset D(\lambda,r)$.

***Lemma 1.5** : Let $0 < r < 1$, and let q be a real number. Then there exists a constant C (depending on r and q) such that for every pair of non-negative measurable functions u and v on $\mathbb{D}$ satisfying*

$$u(\lambda) \le \frac{1}{|D(\lambda,r)|^q} \int_{D(\lambda,r)} v(z)\ dA(z)/\pi\ ,\ (\lambda \in \mathbb{D}) \tag{1.3}$$

we have:

$$\int_{D(\lambda,r)} u(z)\ dA(z)/\pi \ \le \frac{C}{|D(\lambda,r)|^{q-1}} \int_{D^2(\lambda,r)} v(z)\ dA(z)/\pi\ ,\ (\lambda \in \mathbb{D})\,. \tag{1.4}$$

Proof:

Fix $0 < r < 1$, and let q be a real number. Let u and v be a pair of non-negative measurable functions on $\mathbb{D}$ satisfying (1.3). Using characteristic functions (1.3) can be rewritten as:

$$u(\lambda) \le \frac{1}{|D(\lambda,r)|^q} \int_{\mathbb{D}} \chi_{D(\lambda,r)}(z)\ v(z)\ dA(z)/\pi\ ,\ (\lambda \in \mathbb{D})\,. \tag{1.5}$$

Take $w \in \mathbb{D}$. Integrating both sides of (1.5) over $D(w,r)$ and applying Fubini's Theorem, we get:

$$\int_{D(w,r)} u(\lambda)\, dA(\lambda)/\pi \leq \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \chi_{D(w,r)}(\lambda)\, \chi_{D(\lambda,r)}(z) \frac{1}{|D(\lambda,r)|^q}\, dA(\lambda)/\pi \right) v(z)\, dA(z)/\pi . \tag{1.6}$$

Using $\chi_{D(w,r)}(\lambda)\, \chi_{D(\lambda,r)}(z) = \chi_{D(w,r) \cap D(z,r)}(\lambda)$ and $|D(\lambda,r)| \approx |D(w,r)|$ whenever $\lambda \in D(w,r)$, we see that there exists a constant C (depending on r and q) for which the inner integral at the right hand side of (1.6) is smaller than

$$\frac{C}{|D(w,r)|^q} |D(w,r) \cap D(z,r)| .$$

Clearly, for $z \in \mathbb{D}$, $|D(w,r) \cap D(z,r)| \leq |D(w,r)|\ \chi_{D^2(w,r)}(z)$. Hence for each $z \in \mathbb{D}$, the inner integral at the right hand side of (1.6) is smaller than

$$\frac{C}{|D(w,r)|^{q-1}} \chi_{D^2(w,r)}(z) ,$$

and (1.4) follows. □

The following lemma is due to Luecking ([24], Lemma 2.1). We will give a different proof. As in Luecking's proof we will use induction on n, but at a "different level." Lemma 1.5 will play a crucial role in our proof: it will be the tool to go from a pointwise estimate to one involving integrals.

Lemma 1.6 : *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *There exists a constant* C *(depending on n, p, and r) such that for every analytic function* f *on* $\mathbb{D}$*:*

$$|f^{(n)}(\lambda)|^p \leq C \frac{1}{|D(\lambda,r)|^{1+np/2}} \int_{D(\lambda,r)} |f|^p \, dA/\pi \ , \ (\lambda \in \mathbb{D}) . \tag{1.7}$$

Proof:

Let $0 < p < \infty$ and $0 < r < 1$ be fixed. By induction on $n \in \mathbb{N}$ we will show that (1.7) holds for every analytic function f on $\mathbb{D}$ and for each $\lambda \in \mathbb{D}$.

First assume that $n = 1$. By a normal families argument, there exists a constant $C_{p,r}$ such that for every analytic function g on $\mathbb{D}$:

$$|g'(0)|^p \leq C_{p,r} \int_{D(0,r)} |g|^p \, dA/\pi \ . \tag{1.8}$$

Take $\lambda \in \mathbb{D}$. Applying (1.8) to $g = f \circ \varphi_\lambda$ and using change-of-variable formula (0.22b), we get:

$$(1 - |\lambda|^2)^p |f'(\lambda)|^p \leq C_{p,r} \int_{D(\lambda,r)} |f(w)|^p \frac{(1 - |\lambda|^2)^2}{|1 - \bar{\lambda} w|^4} \, dA(w)/\pi \ . \tag{1.9}$$

Since

$$\frac{(1 - |\lambda|^2)^2}{|1 - \bar{\lambda} w|^4} \leq \frac{16}{(1 - |\lambda|^2)^2} \ ,$$

it follows from (1.9) that

$$|f'(\lambda)|^p \le 16\, C_{p,r} \frac{1}{(1-|\lambda|^2)^{2+p}} \int_{D(\lambda,r)} |f|^p \; dA/\pi \, .$$

Hence

$$|f'(\lambda)|^p \le C_{1,p,r} \frac{1}{|D(\lambda,r)|^{1+p/2}} \int_{D(\lambda,r)} |f|^p \; dA/\pi \, , \tag{1.10}$$

and (1.7) is proved for $n = 1$.

Now assume that (1.7) holds for $n \ge 1$. Then

$$|f^{(n)}(\lambda)|^p \le C_{n,p,r/3} \frac{1}{|D(\lambda,r/3)|^{1+np/2}} \int_{D(\lambda,r/3)} |f|^p \, dA/\pi \, . \tag{1.11}$$

Apply Lemma 1.5 with $u = f^{(n)}$, $v = C_{n,p,r/3} f$, $q = 1 + np/2$, and r replaced by $r/3$. Using $D^2(\lambda,r/3) \subset D(\lambda,r)$, we get

$$\int_{D(\lambda,r/3)} |f^{(n)}|^p \; dA/\pi \le C'_{n,p,r} \frac{1}{|D(\lambda,r/3)|^{np/2}} \int_{D(\lambda,r)} |f|^p \; dA/\pi \, . \tag{1.12}$$

Inequality (1.10) applied to $f^{(n)}$ gives us

$$|f^{(n+1)}(\lambda)|^p \le C_{1,p,r/3} \frac{1}{|D(\lambda,r/3)|^{1+p/2}} \int_{D(\lambda,r/3)} |f^{(n)}|^p \, dA/\pi \, . \tag{1.13}$$

Combining (1.12) and (1.13) yields

$$
\begin{aligned}
|f^{(n+1)}(\lambda)|^p &\leq C_{1,p,r/3}\, C'_{n,p,r/3} \frac{1}{|D(\lambda,r/3)|^{1+np/2+p/2}} \int_{D(\lambda,r)} |f|^p \, dA/\pi \\
&\leq C_{n+1,p,r} \frac{1}{|D(\lambda,r)|^{1+(n+1)p/2}} \int_{D(\lambda,r)} |f|^p \, dA/\pi ,
\end{aligned}
$$

which is (1.7) for $n+1$. This completes the induction, and the lemma is proved. □

Theorem 1.7 : *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *Then for an analytic function* f *on* $\mathbb{D}$ *the following quantities are equivalent:*

(A) $\quad \| f \|_{\mathfrak{B}}$;

(B) $$\sup_{\lambda \in \mathbb{D}} \left(\frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \right)^{1/p} + \sum_{k=1}^{n-1} |f^{(k)}(0)| ;$$

(C) $$\sup_{\lambda \in \mathbb{D}} \left(\int_{D(\lambda,r)} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} \, dA(z)/\pi \right)^{1/p} + \sum_{k=1}^{n-1} |f^{(k)}(0)| ;$$

(D) $$\sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \right)^{1/p} + \sum_{k=1}^{n-1} |f^{(k)}(0)| .$$

Remark 1.8 :

(1) Of special interest are the cases where $np = 2$. For $n = 1$ and $p = 2$ quantity *(B)* specializes to quantity *(G)* of Theorem 1.1, and quantity *(D)* gives a global version of quantity *(G)* of Theorem 1.1:

$$\|f\|_{\mathfrak{B}} \approx \sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \right)^{1/2},$$

which should be compared with the known equivalence

$$\|f\|_{BMOA} \approx \sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi \right)^{1/2}.$$

(2) For $n = 2$ and $p = 1$ both quantities (B) and (C) specialize to the local Besov-type equivalence

$$\|f\|_{\mathfrak{B}} \approx \sup_{\lambda \in \mathbb{D}} \int_{D(\lambda,r)} |f''(z)| \, dA(z)/\pi + |f'(0)| .$$

The Besov space $\{f \in H(\mathbb{D}) : f'' \in L_a^1\}$ is minimal among all Möbius-invariant Banach spaces of analytic functions on $\mathbb{D}$ (see [3] or [5]). The above equivalence says that the Bloch space $\mathfrak{B}$ is the set of analytic functions on $\mathbb{D}$ whose restrictions to pseudo-hyperbolic disks (of a fixed pseudo-hyperbolic radius) are uniformly in the Besov space.

For $n = 2$ and $p = 1$ quantity (D) specializes to the global Besov-type equivalence

$$\|f\|_{\mathfrak{B}} \approx \sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f''(z)| (1 - |\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi + |f'(0)| .$$

(3) In the case that $n = 1$ quantities *(C)* and *(D)* are of interest because the quantity

$$\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} \, dA(z)/\pi$$

is invariant under Möbius transformations of f.

Proof of Theorem 1.7:

Take $0 < p < \infty$, $0 < r < 1$, and $n \in \mathbb{N}$. Let f an analytic function on $\mathbb{D}$. We will use the same letter C to denote a constant independent of the function f even though the constant changes from one appearance to the next. Apply Lemma 1.5 with $u = |f^{(n)}|^p$, $v = C\,|f|^p$, $q = 1 + np/2$, where C is the constant of (1.7). We get:

$$\int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \le C \, \frac{1}{|D(\lambda,r)|^{np/2}} \int_{D^2(\lambda,r)} |f|^p \, dA/\pi \, .$$

Replace f by $f - f(\lambda)$, and write $D(\lambda,s) = D^2(\lambda,r)$. It follows from the above inequality that

$$\frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \le C \, \frac{1}{|D(\lambda,s)|} \int_{D(\lambda,s)} |f - f(\lambda)|^p \, dA/\pi \, . \quad (1.14)$$

Combining (1.14) and Theorem 1.1 we get

$$\sup_{\lambda \in \mathbb{D}} \left(\frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \right)^{1/p} \leq C \quad \|f\|_{\mathfrak{B}} .$$

By Lemma 1.3 we also have

$$\sum_{k=1}^{n-1} |f^{(k)}(0)| \leq C \quad \|f\|_{\mathfrak{B}} ,$$

thus it follows that quantity (B) is less than or equal C times quantity (A).

For the converse apply Lemma 1.6 to $f^{(n)}$, to get

$$|f^{(n+1)}(\lambda)|^p \leq C_{1,p,r} \frac{1}{|D(\lambda,r)|^{1+p/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi .$$

Multiply both sides of this inequality by $(1-|\lambda|^2)^{(n+1)p}$. Since $|D(\lambda,r)| \approx (1-|\lambda|^2)^2$ we get

$$\left((1-|\lambda|^2)^{n+1} |f^{(n+1)}(\lambda)| \right)^p \leq C \frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi . \qquad (1.15)$$

By subharmonicity of the function $|f^{(n)}|^p$ we have

$$|f^{(n)}(0)|^p \leq \frac{1}{r^2} \int_{D(0,r)} |f^{(n)}|^p \, dA/\pi .$$

It follows that

$$\sup_{\lambda \in \mathbb{D}} \, (1 - |\lambda|^2)^{n+1} \, |f^{(n+1)}(\lambda)| \; + \; |f^{(n)}(0)| \; \leq$$

$$\leq \; C \, \sup_{\lambda \in \mathbb{D}} \left(\frac{1}{|D(\lambda,r)|^{1 - np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \; dA/\pi \right)^{1/p} ,$$

and by using Lemma 1.3 it follows that quantity *(A)* is less than or equal C times quantity *(B)*. This completes the proof that quantities *(A)* and *(B)* are equivalent.

That quantities *(B)* and *(C)* are equivalent is an immediate consequence of the fact that $(1 - |z|^2)^2 \approx |D(\lambda,r)|$, whenever $z \in D(\lambda,r)$.

For $z \in D(\lambda,r)$ we have $(1 - |\varphi_\lambda(z)|^2)^2 > (1 - r^2)^2$, thus

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p \, (1 - |z|^2)^{np - 2} (1 - |\varphi_\lambda(z)|^2)^2 \; dA(z)/\pi \; \geq$$

$$\geq \; (1 - r^2)^2 \int_{D(\lambda,r)} |f^{(n)}(z)|^p \, (1 - |z|^2)^{np - 2} \; dA(z)/\pi \, , \qquad (1.16)$$

and it follows that quantity *(D)* is greater than or equal a constant times quantity *(C)*. To complete the proof we will show that quantity *(D)* is less than or equal a constant times quantity *(A)*. Again we make use of Lemma 1.3.

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p \, (1 - |z|^2)^{np - 2} \, (1 - |\varphi_\lambda(\xi)|^2)^2 \; dA(z)/\pi \; \leq$$

$$\leq \left(\sup_{z \in \mathbb{D}} \, (1 - |z|^2)^n \, |f^{(n)}(z)| \right)^p \int_{\mathbb{D}} \left(\frac{1 - |\varphi_\lambda(z)|^2}{1 - |z|^2} \right)^2 dA(z)/\pi \, .$$

Now, the integral at the right of the above inequality is 1 [by (0.20) the integrand is equal to $|\varphi_\lambda'(z)|^2$, so that by changing to the variable $w = \varphi_\lambda(z)$ the integral is transformed into one with integrand identically equal to 1]. Thus quantity *(D)* is less than or equal to quantity *(B)* of Lemma 1.3, and the proof is complete. □

The equivalences of Theorem 1.7 carry over to the little Bloch space, as is shown in the following theorem.

Theorem 1.9 : *Let* $0 < p < \infty$, $0 < r < 1$, *and* $n \in \mathbb{N}$. *Then for an analytic function* f *on* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in \mathcal{B}_0$;

(b) $$\frac{1}{|D(\lambda,r)|^{1-np/2}} \int_{D(\lambda,r)} |f^{(n)}|^p \, dA/\pi \to 0 \text{ as } |\lambda| \to 1^- ;$$

(c) $$\int_{D(\lambda,r)} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- ;$$

(d) $$\int_{\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- .$$

Proof:

Take $0 < p < \infty$, $0 < r < 1$, and $n \in \mathbb{N}$. Let f be an analytic function on $\mathbb{D}$. With the help of Theorem 1.2, it follows immediately from inequality (1.14) that *(a)* implies *(b)*. Conversely inequality (1.15) together with Lemma 1.4 give that *(b)* implies *(a)*.

The equivalence of statements *(b)* and *(c)* follows immediately from the fact that for $z \in D(\lambda,r)$ we have $(1-|z|^2)^2 \approx |D(\lambda,r)|$.

That *(d)* implies *(c)* is a consequence of inequality (1.16). To complete the proof we will show that *(a)* implies *(d)*. Suppose that *(a)* holds, i.e., $f \in \mathcal{B}_0$. By Lemma 1.4 we have that $(1-|z|^2)^n |f^{(n)}(z)| \to 0$ as $|z| \to 1^-$. Then it is easy to see that

$$\int_{\mathbb{D}\setminus\delta\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \to 0 \text{ as } \delta \to 1^- . \quad (1.17)$$

By Lemma 1.3 there is a constant C such that $(1-|z|^2)^n |f^{(n)}(z)| \leq C$ for every $z \in \mathbb{D}$. It follows from the chain of inequalities

$$\int_{\delta\mathbb{D}} |f^{(n)}(z)|^p (1-|z|^2)^{np-2} (1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \leq$$

$$\leq C^p \int_{D(0,\delta)} \left(\frac{1-|\varphi_\lambda(z)|^2}{1-|z|^2} \right)^2 dA(z)/\pi$$

$$= C^p \int_{D(0,\delta)} |\varphi_\lambda'(z)|^2 \, dA(z)/\pi \qquad \text{[by identity (0.20)]}$$

$$= C^p \int_{D(\lambda,\delta)} 1 \, dA/\pi = C^p \, |D(\lambda,\delta)| ,$$

that for every $\delta \in (0, 1)$ we have

$$\int_{\delta\mathbb{D}} |f^{(n)}(z)|^p \, (1-|z|^2)^{np-2}\,(1-|\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \;\to\; 0 \text{ as } |\lambda|\to 1^- . \quad (1.18)$$

Combining (1.17) and (1.18) yields that *(d)* holds. □

Chapter 2

In this chapter we will give characterizations of the Bloch space and the little Bloch space in terms of the pseudo-hyperbolic metric. It will be shown that the Bloch space consists of those analytic functions on the disk that are uniformly continuous with respect to the pseudo-hyperbolic metric. A similar description will be given for the little Bloch space. We will also consider the real harmonic Bloch space on the unit disk. First we will show that for an analytic function on the disk the Bloch norm and the supremum of the oscillations of the function over pseudo-hyperbolic disks of a fixed radius are equivalent quantities.

Theorem 2.1 *: Let* $0 < r < 1$ *. For* f *analytic on* $\mathbb{D}$ *the following quantities are equivalent:*

(A) $\|f\|_{\mathcal{B}}$;

(B) $\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda, r)} |f(z) - f(\lambda)|$.

Proof:

Fix $0 < r < 1$, and let f be analytic on $\mathbb{D}$. It follows from the identity

$$f'(0) = \frac{2}{r^4} \int_{D(0,r)} \bar{z} \, f(z) \, dA(z)/\pi$$

that

$$|f'(0)| \le \frac{2}{r^4} \, r \int_{D(0,r)} |f(z)| \, dA(z)/\pi$$

$$\le \frac{2}{r} \sup_{z \in D(0,r)} |f(z)| .$$

Replacing f by $f \circ \varphi_\lambda - f(\lambda)$, we get the inequality

$$(1 - |\lambda|^2)\,|f'(\lambda)| \leq \frac{2}{r} \sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)|, \tag{2.1}$$

and it follows that

$$\|f\|_{\mathfrak{B}} \leq \frac{2}{r} \sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)|.$$

On the other hand, as in the proof of Lemma 1.3, for $|w| < r$ we have

$$\begin{aligned} |f(w) - f(0)| &\leq \int_0^1 \frac{|w|}{1 - t^2 |w|^2}\, dt \,.\, \|f\|_{\mathfrak{B}} \\ &\leq \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) \|f\|_{\mathfrak{B}} \,. \end{aligned}$$

Replacing f by $f \circ \varphi_\lambda - f(\lambda)$ yields

$$|f(\varphi_\lambda(w) - f(\lambda)| \leq \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) \|f\|_{\mathfrak{B}} \,,$$

whenever $|w| < r$. Hence

$$\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)| \leq \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) \|f\|_{\mathfrak{B}} \,, \tag{2.2}$$

and the theorem is proved.□

A function f on $\mathbb{D}$ is uniformly continuous with respect to the pseudo-hyperbolic metric if

$$\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda, r)} |f(z) - f(\lambda)| \to 0 \text{ as } r \to 0^+. \tag{2.3}$$

Let UC denote the class of <u>all</u> functions $f: \mathbb{D} \to \mathbb{C}$ which are uniformly continuous with respect to the pseudo-hyperbolic metric. Let $H(\mathbb{D})$ denote the set of all analytic functions on $\mathbb{D}$.

Corollary 2.2 : $\mathfrak{B} = UC \cap H(\mathbb{D})$

Proof:

If $f \in UC \cap H(\mathbb{D})$, then f satisfies (2.3). In particular, for some $r \in (0, 1)$ we have

$$\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda, r)} |f(z) - f(\lambda)| \leq 1,$$

so that by Theorem 2.1 $f \in \mathfrak{B}$.

For the converse suppose that $f \in \mathfrak{B}$. Taking the limit $r \to 0^+$ in (2.2) yields (2.3), hence $f \in UC \cap H(\mathbb{D})$, and the corollary is proved.□

Remark 2.3 : *For an <u>arbitrary</u> function $f: \mathbb{D} \to \mathbb{C}$ to be uniformly continuous with respect to the pseudo-hyperbolic metric f must satisfy the little-o condition* (2.3). *However, for an <u>analytic</u> function $f: \mathbb{D} \to \mathbb{C}$ to be uniformly continuous with respect to the pseudo-hyperbolic metric it is sufficient (and of course necessary) that f satisfies the big-O condition*

$$\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)| < \infty,$$

for some $r \in (0, 1)$.

As usual, the equivalences of the previous theorem carry over to the little Bloch space. This is expressed in the following theorem.

***Theorem 2.4** : Let* $0 < r < 1$ *. For an analytic function* f *on* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) $\sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)| \to 0$ *as* $|\lambda| \to 1^-$.

Proof:

That *(b)* implies *(a)* follows immediately from (2.1).

For the converse, suppose that $f \in \mathfrak{B}_0$. From the proof of Theorem 2.1 we see that for $t \in (0, 1)$ and $\lambda \in \mathbb{D}$

$$\sup_{z \in D(\lambda,r)} |f(z) - f_t(z) - (f(\lambda) - f_t(\lambda))| \le \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) \|f - f_t\|_{\mathfrak{B}} . \tag{2.4}$$

Using the triangle inequality it follows from (2.4) that for $t \in (0, 1)$ and $\lambda \in \mathbb{D}$

$$\sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)| \le \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) \|f - f_t\|_{\mathfrak{B}} + \sup_{z \in D(\lambda,r)} |f_t(z) - f_t(\lambda)| . \tag{2.5}$$

Let $t \in (0, 1)$. The dilate f_t is analytic in a neighborhood of the disk, so clearly

$$\sup_{z \in D(\lambda,r)} |f_t(z) - f_t(\lambda)| \to 0 \text{ as } |\lambda| \to 1^-,$$

and it follows from inequality (2.5) that

$$\limsup_{|\lambda| \to 1^-} \sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)| \le \frac{1}{2} \log\left(\frac{1+r}{1-r}\right) \|f - f_t\|_{\mathfrak{B}}.$$

Since $f \in \mathfrak{B}_0$, we have $\|f - f_t\|_{\mathfrak{B}} \to 0$ as $t \to 1^-$, hence the above inequality yields

$$\limsup_{|\lambda| \to 1^-} \sup_{z \in D(\lambda,r)} |f(z) - f(\lambda)| = 0,$$

which implies that *(b)* holds.□

Let $h(\mathbb{D})$ denote the set of all real harmonic functions on $\mathbb{D}$. Define the real harmonic Bloch space B to be the class of all real harmonic functions u on $\mathbb{D}$ for which

$$\|u\|_B = \sup_{z \in \mathbb{D}} (1 - |z|^2)\,|(\nabla u)(z)| < \infty,$$

where ∇u denotes the gradient of u. If f is analytic on $\mathbb{D}$, and $u = Re\, f$, then it follows from the Cauchy-Riemann equations that $|f'| = |\nabla u|$, and consequently $\|f\|_{\mathfrak{B}} = \|u\|_B$. It follows immediately that $B = Re\, \mathfrak{B}$.

So if $u \in B$, then $f \in \mathfrak{B}$, so that $f \in UC$, and hence $u \in UC$. Thus we have the inclusion $B \subset UC \cap h(\mathbb{D})$. We claim that the converse is also true, i.e., in analogy to

Corollary 2.2 we have the following result:

Theorem 2.5 *:* $B = UC \cap h(\mathbb{D})$.

Proof:

We make use of the fact that the conjugate function operation is a bounded operator in the $L^1(\mathbb{D}, dA/\pi)$ norm (for a proof see [7]): there is a constant C such that for every real harmonic function u on $\mathbb{D}$

$$\int_{\mathbb{D}} |\tilde{u}| \, dA/\pi \leq C \int_{\mathbb{D}} |u| \, dA/\pi .$$

Let $0 < r < 1$. Dilating the above inequality gives that for every real harmonic function u on $\mathbb{D}$

$$\int_{D(0,r)} |\tilde{u}| \, dA/\pi \leq C \int_{D(0,r)} |u| \, dA/\pi . \tag{2.6}$$

Suppose that $u \in UC \cap h(\mathbb{D})$. Let f be analytic on $\mathbb{D}$ such that $u = Re\, f$. Since u is uniformly continuous with respect to the pseudo-hyperbolic metric we can pick $0 < r < 1$ such that

$$\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda, r)} |u(z) - u(\lambda)| \leq 1 . \tag{2.7}$$

Let $\lambda \in \mathbb{D}$ be fixed. Using the change-of-variable formula (0.22a) and formula (0.21) for the normalized area of a pseudo-hyperbolic disk, we have:

$$\frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f - f(\lambda)|\ dA/\pi \le \frac{(1-|\lambda|^2r^2)}{r^2(1-|\lambda|r)^4}\int_{D(0,r)} |f\circ\varphi_\lambda - f(\lambda)|\ dA/\pi$$

$$\le \frac{1}{r^2(1-r)^4}\int_{D(0,r)} |f\circ\varphi_\lambda - f(\lambda)|\ dA/\pi\ . \qquad (2.8)$$

Write $f = u + i\,\tilde{u}$. It is easily seen that $(u\circ\varphi_\lambda - u(\lambda))^{\sim} = \tilde{u}\circ\varphi_\lambda - \tilde{u}(\lambda)$, so by (2.6) we have

$$\int_{D(0,r)} |\tilde{u}\circ\varphi_\lambda - \tilde{u}(\lambda)|\ dA/\pi \le C\int_{D(0,r)} |u\circ\varphi_\lambda - u(\lambda)|\ dA/\pi\ .$$

Since $f\circ\varphi_\lambda - f(\lambda) = u\circ\varphi_\lambda - u(\lambda) + i\,(\tilde{u}\circ\varphi_\lambda - \tilde{u}(\lambda))$, the above inequality and the triangle inequality give us that

$$\int_{D(0,r)} |f\circ\varphi_\lambda - f(\lambda)|\ dA/\pi \le C\int_{D(0,r)} |u\circ\varphi_\lambda - u(\lambda)|\ dA/\pi\ . \qquad (2.9)$$

From (2.7) we see that the integral at the right of (2.9) is bounded by r^2. Combining this with (2.8) yields

$$\frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f\circ\varphi_\lambda - f(\lambda)|\ dA/\pi \le \frac{C+1}{(1-r)^4}\ .$$

By Theorem 1.1 *(D)*, we have $f \in \mathfrak{B}$, so that $u = Re\, f \in B$, as was to be shown.□

Remark 2.6 : *For an arbitrary function* $u : \mathbb{D} \to \mathbb{R}$ *to be uniformly continuous with respect to the pseudo-hyperbolic metric* u *must satisfy the little-o condition* (2.3). *However, for an real harmonic function* $u : \mathbb{D} \to \mathbb{R}$ *to be uniformly continuous with respect to the pseudo-hyperbolic metric it is sufficient (and of course necessary) that* u *satisfies the big-O condition*

$$\sup_{\lambda \in \mathbb{D}} \sup_{z \in D(\lambda, r)} |u(z) - u(\lambda)| < \infty ,$$

for some $r \in (0, 1)$.

Just as BMO is closed under the conjugate function operation, so is B, the class of real harmonic functions on the disk that are uniformly continuous with respect to the pseudo-hyperbolic metric.

Corollary 2.7 : *If* $u \in UC \cap h(\mathbb{D})$, *then* $\tilde{u} \in UC \cap h(\mathbb{D})$.

Proof:

Suppose that $u \in UC \cap h(\mathbb{D})$. By Theorem 2.5, $u \in B$. Thus $u = Re f$, with $f \in \mathfrak{B}$. Then $-if \in \mathfrak{B}$, so that $\tilde{u} = Re(-if) \in B$, and by Corollary 2.2 we are done.□

Chapter 3

In this chapter we describe some spaces of analytic functions on the unit disk in terms of Nevanlinna characteristics. Our starting point is Baernstein's characterization for the space $BMOA$; he proved that an analytic function on the unit disk belongs to the space $BMOA$ if and only if the Möbius transforms of the function form a bounded family in the Nevanlinna class. We give a similar description of the space $VMOA$. This description cannot be obtained by simply repacing Baernstein's boundedness condition by the corresponding vanishing condition (as is usually the case). We then formulate and prove analogous characterizations for the Bloch space and the little Bloch space in terms of an area version of the Nevanlinna characteristic.

For f analytic on $\mathbb{D}$ the Nevanlinna characteristic $T(f)$ is defined by

$$T(f) = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta .$$

The Nevanlinna class is the set $N = \{f \in H(\mathbb{D}) : T(f) < \infty\}$.
Let $0 < p < \infty$, then it follows from the inequality $p \log^+ x \le x^p$ that

$$p \, \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \, d\theta \le \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta , \tag{3.1}$$

hence

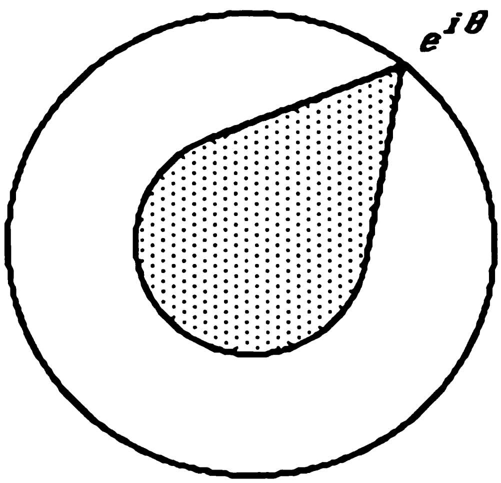
$$T(f) \le \frac{1}{p} \|f\|_{H^p}^p , \text{ for } 0 < p < \infty . \tag{3.2}$$

It follows from (3.2) that for $f \in BMOA$

$$\sup_{\lambda \in \mathbb{D}} T(f \circ \varphi_\lambda - f(\lambda)) < \infty ,$$

i.e., the family $\{f \circ \varphi_\lambda - f(\lambda) : \lambda \in \mathbb{D}\}$ is bounded in the Nevanlinna class N. In [8] Baernstein proved that the converse is also true. Before stating his result we need to introduce more notation:

Fix $0 < \alpha < \pi/2$. For $e^{i\theta} \in \partial\mathbb{D}$ let $\Gamma_\alpha(e^{i\theta})$ denote the Stolz region based at $e^{i\theta}$, i.e., $\Gamma_\alpha(e^{i\theta})$ is the interior of the convex hull of the circle $|z| = \sin \alpha$ and the point $e^{i\theta}$:



The non-tangential maximal function $\mathcal{N}_\alpha(f)$ of a complex function f defined on $\mathbb{D}$ is defined by

$$(\mathcal{N}_\alpha(f))(e^{i\theta}) = \sup \{|f(z)| : z \in \Gamma_\alpha(e^{i\theta})\} .$$

Note that $(\mathcal{N}_\alpha(f))(e^{i\theta}) \geq |f^*(e^{i\theta})|$ if f has a non-tangential limit $f^*(e^{i\theta})$ at $e^{i\theta}$.

In [8] Baernstein proved the following "John-Nirenberg type" of theorem:

Theorem 3.1 : *There exists an absolute constant K such that for each* $0 < \alpha < \pi/2$ *and f analytic on* $\mathbb{D}$ *the following statements are equivalent :*

(A) $\{f \circ \varphi_\lambda - f(\lambda) : \lambda \in \mathbb{D}\}$ *is bounded in the Nevanlinna class N ;*

(B) There exists a constant $\beta = \beta(\alpha, f)$ *for which*

$$\mu_0(\{e^{i\theta} :\ \mathcal{N}_\alpha(f \circ \varphi_\lambda - f(\lambda))(e^{i\theta}) > t\ \}) < K\, e^{-\beta t}, \tag{3.3}$$

for all $\lambda \in \mathbb{D}$ *, and for all* $0 < t < \infty$.

As Baernstein indicated ([8], Corollary 5.2), Theorem 3.1 has as an immediate consequence:

Theorem 3.2 : *For an analytic function f on* $\mathbb{D}$ *the following statements are equivalent:*

(A) $f \in BMOA$;

(B) $\sup\limits_{\lambda \in \mathbb{D}} T(f \circ \varphi_\lambda - f(\lambda)) < \infty$.

What about the space *VMOA* ? One may be tempted to replace the above big-O condition *(B)* in Theorem 3.2 by the corresponding little-o condition, and ask whether

$$f \in VMOA \iff T(f \circ \varphi_\lambda - f(\lambda)) \to 0 \text{ as } |\lambda| \to 1^- ? \tag{3.4}$$

The answer is negative: the condition at the right of (3.4) is certainly necessary for f to be in *VMOA* (this follows from (3.2)), but not sufficient. That the condition is not sufficient follows from the observation that it is trivially satisfied when $\|f\|_\infty \leq 1/2$ (because this implies that $T(f \circ \varphi_\lambda - f(\lambda)) = 0$ for all $\lambda \in \mathbb{D}$), but not every analytic function f on $\mathbb{D}$ for which $\|f\|_\infty \leq 1/2$ is contained in $VMOA$.

Let's return to $BMOA$ and rewrite the condition in Theorem 3.2.
Let $\rho > 0$. If $f \in BMOA$, then also $f/\rho \in BMOA$, so that

$$\sup_{\lambda \in \mathbb{D}} T\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) < \infty .$$

It follows that for f analytic on $\mathbb{D}$:

$$f \in BMOA \iff \left[\forall \rho > 0 : \sup_{\lambda \in \mathbb{D}} T\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) < \infty\right]. \tag{3.5}$$

Having replaced the big-O condition in Theorem 3.2 by a collection of big-O conditions in (3.5), going to the corresponding little o-conditions yields the following:

Theorem 3.3 : *For an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in VMOA$;

(b) *for every $\rho > 0$ we have that* $T\left(\dfrac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) \to 0$ *as* $|\lambda| \to 1^-$.

Before the proof we need to relate the Nevanlinna characteristic and the H^2 - norm of an analytic function. We'll do this not just for the H^2 - norm, but for any H^p - norm:

Lemma 3.4 : *Let* $0 < p < \infty$. *For an analytic function* f *on* $\mathbb{D}$:

$$\|f\|_{H^p}^p = p^2 \int_0^\infty \rho^{p-1} T\left(\frac{f}{\rho}\right) \, d\rho \, . \tag{3.6}$$

Proof:

Let $0 < p < \infty$. Integration by parts yields the formula:

$$\int_0^1 t^{p-1} \log \frac{1}{t} \, dt = \frac{1}{p^2} \, .$$

Thus for $0 \le x < \infty$ we have:

$$\int_0^\infty \rho^{p-1} \log^+ \frac{x}{\rho} \, d\rho = \int_0^x \rho^{p-1} \log \frac{x}{\rho} \, d\rho = x^p \int_0^1 t^{p-1} \log \frac{1}{t} \, dt = \frac{1}{p^2} x^p \, .$$

For an analytic function f on $\mathbb{D}$ and $0 < r < 1$ an application of Fubini's Theorem gives:

$$\int_0^\infty \rho^{p-1} \left(\frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{f(re^{i\theta})}{\rho} \right| \, d\theta \right) d\rho = \frac{1}{p^2} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \, . \tag{3.7}$$

Taking the limit as $r \to 1^-$, and using the Monotone Convergence Theorem we get (3.6).□

Now we are ready for the proof of Theorem 3.3.

Proof of Theorem 3.3 :

Let f be an analytic function on $\mathbb{D}$. We have already seen that condition *(b)* in Theorem 3.3 is necessary.

To prove the sufficiency, suppose that f satisfies condition *(b)*. Our first step is in showing that $f \in BMOA$. Choose an $r \in (0, 1)$ such that $T(f \circ \varphi_\lambda - f(\lambda)) < 1$ whenever $r < |\lambda| < 1$. Note that $g \in N \Longleftrightarrow g \circ \varphi_\lambda \in N$ (This follows easily from the fact that each function in the Nevanlinna class N is the quotient of two H^∞-functions). Pick w such that $r < |w| < 1$. Then $T(f \circ \varphi_w - f(w)) < 1$, so that $f \circ \varphi_w \in N$, and therefore $f \in N$. Thus $log^+|f|$ has a harmonic majorant, call it h. Then for $\lambda \in \mathbb{D}$, $h \circ \varphi_\lambda$ is a harmonic majorant of $f \circ \varphi_\lambda$, whence

$$T(f \circ \varphi_\lambda) \leq (h \circ \varphi_\lambda)(0) = h(\lambda).$$

Using the inequality $log^+(x + y) \leq log^+x + log^+y + log\, 2$, it follows that for $|\lambda| \leq r$:

$$T(f \circ \varphi_\lambda - f(\lambda)) \leq h(\lambda) + log^+|f(\lambda)| + log\, 2\,.$$

Hence the family $\{f \circ \varphi_\lambda - f(\lambda) : \lambda \in \mathbb{D}\}$ is bounded in N, and by Theorem 3.2 we have $f \in BMOA$.

Since $f \in BMOA$ we can apply Theorem 3.1. Let β be such that (3.3) holds. Then for $\lambda \in \mathbb{D}$ and $t > 0$:

$$\mu_0(\{e^{i\theta} : |f^*(\varphi_\lambda(e^{i\theta})) - f(\lambda)| > t\}) < K e^{-\beta t}.$$

Using the above inequality as well as the distribution function for the log^+, it follows that for every $\rho > 0$:

$$T\left(\frac{f\circ\varphi_\lambda - f(\lambda)}{\rho}\right) \le \frac{1}{2\pi}\int_0^{2\pi} log^+ \frac{|f^*(\varphi_\lambda(e^{i\theta})) - f(\lambda)|}{\rho}\, d\mu_0(e^{i\theta})$$

$$= \int_\rho^\infty \frac{1}{t}\,\mu_0(\{e^{i\theta} : |f^*(\varphi_\lambda(e^{i\theta})) - f(\lambda)| > t\})\, dt$$

$$\le \int_\rho^\infty \frac{1}{t}\, K e^{-\beta t}\, dt \le \frac{K}{\rho\beta}\, e^{-\beta\rho}\ .$$

Now let $\varepsilon > 0$ be given. Choose R > 0 such that $K\, e^{-\beta R} < (\varepsilon^2\beta^2)/8$. Then integrating the above inequality we get

$$\int_R^\infty \rho\, T\left(\frac{f\circ\varphi_\lambda - f(\lambda)}{\rho}\right) d\rho < \frac{\varepsilon^2}{8}\ . \tag{3.8}$$

By the Lebesgue Dominated Convergence Theorem:

$$\int_0^R \rho\, T\left(\frac{f\circ\varphi_\lambda - f(\lambda)}{\rho}\right) d\rho \to 0 \text{ as } |\lambda| \to 1^-\ .$$

Choose $\delta \in (0\,,1)$ such that

$$\int_0^R \rho \, T\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) d\rho < \frac{\varepsilon^2}{8} \tag{3.9}$$

whenever $1 - \delta < |\lambda| < 1$. Using the formula of Lemma 3.4, it follows from (3.8) and (3.9) that

$$\| f \circ \varphi_\lambda - f(\lambda) \|_{H^2}^2 = 4 \int_0^\infty \rho T\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) d\rho < \varepsilon^2 ,$$

hence $\| f \circ \varphi_\lambda - f(\lambda) \|_{H^2} < \varepsilon$, whenever $1 - \delta < |\lambda| < 1$. Therefore $f \in VMOA$, and the theorem is proved.□

The classical Nevanlinna characteristic T is defined in terms of log^+, which only measures the values of the function that are of modulus bigger than 1. Instead we could define

$$T'(f) = \sup_{0 \le r < 1} \frac{1}{2\pi} \int_0^{2\pi} log\,(1 + |f(re^{i\theta})|)\, d\theta ,$$

for an analytic function f on $\mathbb{D}$, and we obtain a characteristic equivalent to T.

In fact, for $0 \le x < \infty$, $log^+x \le log\,(1 + x) \le log^+x + log\,2$, so that for every analytic function f on $\mathbb{D}$,

$$T(f) \le T'(f) \le T(f) + log\,2 .$$

The characteristic T' also measures values of the function that are of modulus less than 1.

Theorem 3.5 : *For an analytic function* f *on* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in VMOA$;

(b) $T'(f \circ \varphi_\lambda - f(\lambda)) \to 0$ *as* $|\lambda| \to 1^-$.

Proof:

For $x \geq \rho > 0$,

$$\log^+ \frac{x}{\rho} \leq \frac{\log(1 + 1/\rho)}{\log(1+\rho)} \log(1+x),$$

so that

$$T\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) \leq \frac{\log(1+1/\rho)}{\log(1+\rho)} T'(f \circ \varphi_\lambda - f(\lambda)).$$

So if $T'(f \circ \varphi_\lambda - f(\lambda)) \to 0$ as $|\lambda| \to 1^-$, then $f \in VMOA$.
The inequality $\log(1+x) \leq x$ implies that

$$T'(f \circ \varphi_\lambda - f(\lambda)) \leq \| f \circ \varphi_\lambda - f(\lambda) \|_{H^1} \leq \| f \circ \varphi_\lambda - f(\lambda) \|_{H^2},$$

so that $T'(f \circ \varphi_\lambda - f(\lambda)) \to 0$ as $|\lambda| \to 1^-$, when $f \in VMOA$.□

For f analytic on $\mathbb{D}$ the area version of the Nevanlinna characteristic, $T_a(f)$, is defined by

$$T_a(f) = \int_{\mathbb{D}} \log^+ |f| \, dA/\pi .$$

The area-Nevanlinna class is the set $N_a = \{f \in H(\mathbb{D}) : T_a(f) < \infty\}$.

Let $0 < p < \infty$. Integrating both sides of inequality (3.1) gives, in analogy to (3.2):

$$T_a(f) \le \frac{1}{p} \|f\|^p_{L^p_a}, \text{ for } 0 < p < \infty. \tag{3.10}$$

So the area-Nevanlinna class contains all Bergman spaces. Analogous to Baernstein's characterization for the space *BMOA* given in Theorem 3.2 we have following result for the Bloch space:

Theorem 3.6 : *For an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(A) $f \in \mathfrak{B}$;

(B) $\sup_{\lambda \in \mathbb{D}} T_a(f \circ \varphi_\lambda - f(\lambda)) < \infty$.

Proof:

That *(A)* implies *(B)* follows from (3.10) and the Garcia-norm characterization for the Bloch space [Theorem 1.1 *(B)*].

For the converse, let f be an analytic function on $\mathbb{D}$ and suppose that

$$M = \sup_{\lambda \in \mathbb{D}} T_a(f \circ \varphi_\lambda - f(\lambda)) < \infty.$$

Fix $0 < r < 1$, and let $z, \lambda \in \mathbb{D}$ with $d(z, \lambda) < r$. Put $u = \varphi_\lambda(z)$, then $|u| < r$. Using that the function $log^+|f \circ \varphi_\lambda - f(\lambda)|$ is subharmonic on $\mathbb{D}$ we have

$$log^+|f(z) - f(\lambda)| = log^+|(f \circ \varphi_\lambda)(u) - f(\lambda)|$$

$$\leq \frac{1}{(1-r)^2} \int_{\Delta(u,1-r)} log^+|(f \circ \varphi_\lambda)(w) - f(\lambda)| \, dA(w)/\pi$$

$$\leq \frac{1}{(1-r)^2} T_a(f \circ \varphi_\lambda - f(\lambda)) \leq \frac{M}{(1-r)^2} .$$

Since $x \leq exp(log^+x)$ for all $x \geq 0$ it follows that $|f(z) - f(\lambda)| \leq exp(M/(1-r)^2)$, and it follows from Theorem 2.1 that $f \in \mathfrak{B}$, as was to be shown.□

A description of the little Bloch space in terms of the area-Nevanlinna characteristic is contained in the following theorem which is analogous to the description of the space $VMOA$ given in Theorem 3.3.

Theorem 3.7 : *For an analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) *For every $\rho > 0$ we have that* $T_a\left(\dfrac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) \to 0$ *as* $|\lambda| \to 1^-$.

Proof:

That *(b)* is implied by *(a)* follows easily from (3.10) and the Garcia-norm characterization for the little Bloch space [Theorem 1.2 *(b)*].

For the converse, suppose that f is an analytic function on $\mathbb{D}$ for which *(b)* holds. Fix $0 < r < 1$. Let $z, \lambda \in \mathbb{D}$ such that $d(z, \lambda) < r$. Then, as in the proof of Theorem 3.6:

$$\frac{|f(z) - f(\lambda)|}{\rho} \leq exp\left[\frac{1}{(1-r)^2} T_a\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right)\right]. \tag{3.11}$$

Given $\varepsilon > 0$, choose $0 < \rho < \varepsilon/2$. Since *(b)* holds we can choose a $\delta \in (0\,,1)$ for which

$$T_a\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\rho}\right) < (1-r)^2 \log 2\,, \tag{3.12}$$

whenever $0 < 1 - |\lambda| < \delta$. Combining (3.11) and (3.12) we get that for $0 < 1 - |\lambda| < \delta$ $|f(z) - f(\lambda)| \leq 2\rho < \varepsilon$. We conclude that

$$\sup_{z \in D(\lambda, r)} |f(z) - f(\lambda)| \to 0 \text{ as } |\lambda| \to 1^-,$$

so that by Theorem 2.4, $f \in \mathfrak{B}_0$, and we are done.□

Chapter 4

In this chapter we will give a different proof of Baernstein's value distribution characterization for *BMOA* [8], Theorem 3, and then formulate and prove the corresponding description for the space *VMOA* . Defining an area version of the counting function used in the value distribution characterizations for *BMOA* and *VMOA*, we obtain analogous results for the Bloch space and the little Bloch space.

The Green's function for the unit disk is given by

$$g(z,\lambda) = \log \frac{1}{|\varphi_\lambda(z)|}, \quad \text{for } z, \lambda \in \mathbb{D} .$$

For a nonconstant analytic function f on $\mathbb{D}$ let $\{z_n(f)\}$ denote the zeros of f in $\mathbb{D}$, listed in increasing moduli and repeated according to multiplicities. Following Baernstein we define $N(w, \lambda, f)$, the "counting function for value w started at λ", by

$$N(w, \lambda, f) = \sum_n g(z_n(f - w), \lambda) .$$

Note that $g(z, 0) = \log(1/|z|)$, so that

$$N(w, 0, f) = \sum_n \log \frac{1}{|z_n(f - w)|} ,$$

the usual counting function. It is clear from the definition of the counting function that

$$N(w, \lambda, f) = \infty \text{ if } f(\lambda) = w \; ; \tag{4.1a}$$

$$N(w, \lambda, f) = 0 \text{ if } f \text{ omits the value } w \; . \tag{4.1b}$$

The following properties of the counting function, which are easily verified, are useful: For $w \in \mathbb{C}$, $\alpha \in \mathbb{C}\backslash\{0\}$, $\lambda \in \mathbb{D}$ and f analytic on $\mathbb{D}$ we have:

$$N(w, \lambda, f) = N(w + \alpha, \lambda, f + \alpha) \tag{4.2a}$$

$$N(w, \lambda, f) = N(\alpha w, \lambda, \alpha f) \tag{4.2b}$$

$$N(w, \lambda, f) = N(0, 0, f \circ \varphi_\lambda - w) \, . \tag{4.2c}$$

The following theorem is due to Baernstein ([8], Theorem 3). We will give a simpler proof of his theorem.

Theorem 4.1: *For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(A) $f \in BMOA$;

(B) $\sup \{N(w, \lambda, f) : w \in \mathbb{C}, \lambda \in \mathbb{D} \text{ and } |f(\lambda) - w| \geq 1\} < \infty$.

Just as in Baernstein's proof we will need to relate the Nevanlinna characteristic of an analytic function with its counting function. This is done in the following classical result.

Cartan's Formula : *For a nonconstant analytic function f on* $\mathbb{D}$:

$$T(f) = \frac{1}{2\pi} \int_0^{2\pi} N(e^{i\theta}, 0, f) \, d\theta + \log^+ |f(0)| \, . \tag{4.3}$$

A proof of Cartan's Formula can be found in [17], pages 214-215, for the case that f is analytic on a neighborhood of $\mathbb{D}$. The general case follows easily by looking at the dilates f_t of f. Using the Monotone Convergence Theorem we see that $T(f_t)$ increases to $T(f)$ and for each θ in $(0, 2\pi)$ we have that $N(e^{i\theta}, 0, f_t)$ increases to $N(e^{i\theta}, 0, f)$ as we take the limit $t \to 1^-$. For these dilates f_t we know that (4.3) holds, so that another application of the Monotone Convergence Theorem gives that (4.3) holds for f.

Proof of Theorem 4.1:

Let f be a nonconstant analytic function on $\mathbb{D}$. By Jensen's Formula we have:

$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\, d\theta = \sum_{n\,:\,|z_n(f)|<r} \log\frac{r}{|z_n(f)|} + \log|f(0)|.$$

Thus

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+|f(re^{i\theta})|\, d\theta \geq \sum_{n\,:\,|z_n(f)|<r} \log\frac{r}{|z_n(f)|} + \log|f(0)|,$$

which, after taking the limit $r \to 1^-$, gives us the inequality

$$T(f) \geq N(0, 0, f) + \log|f(0)| . \tag{4.4}$$

Replacing f by $f \circ \varphi_\lambda - w$, and making use of (4.2c) the above inequality yields

$$N(w, \lambda, f) \leq T(f \circ \varphi_\lambda - w) - \log|f(\lambda) - w| .$$

Using the inequality $\log^+(x+y) \le \log^+ x + \log^+ y + \log 2$, we get

$$N(w, \lambda, f) \le T(f \circ \varphi_\lambda - f(\lambda)) + \log^+|f(\lambda) - w| - \log|f(\lambda) - w| + \log 2 .$$

So if $|f(\lambda) - w| \ge 1$, then we have

$$N(w, \lambda, f) \le T(f \circ \varphi_\lambda - f(\lambda)) + \log 2 .$$

The above inequality and Theorem 3.2 show that *(A)* implies *(B)*.

To prove the converse suppose that

$$M = \sup\{N(w, \lambda, f) : w \in \mathbb{C}, \lambda \in \mathbb{D} \text{ and } |f(\lambda) - w| \ge 1\} < \infty .$$

By Cartan's Formula

$$T(f \circ \varphi_\lambda - f(\lambda)) = \frac{1}{2\pi} \int_0^{2\pi} N(e^{i\theta}, 0, f \circ \varphi_\lambda - f(\lambda))\, d\theta .$$

Now, using (4.2a) and (4.2c), for every $0 \le \theta \le 2\pi$ we have $N(e^{i\theta}, 0, f \circ \varphi_\lambda - f(\lambda))$ $= N(e^{i\theta} + f(\lambda), \lambda, f) \le M$, so it follows that

$$T(f \circ \varphi_\lambda - f(\lambda)) \le M, \text{ for all } \lambda \in \mathbb{D},$$

and hence, by Theorem 3.2, $f \in BMOA$. □

Before going to *VMOA* let's rewrite the condition in Theorem 4.1 for inclusion in *BMOA* . Suppose that $f \in BMOA$, and let $\delta > 0$. Since $f/\delta \in BMOA$, it satisfies condition *(B)* of Theorem 4.1. By (4.2b), $N(w, \lambda, f) = N(w/\delta, \lambda, f/\delta)$. Therefore we must have that for an analytic function f on $\mathbb{D}$:

$$f \in BMOA \iff$$
$$\left[\forall \delta > 0 : \sup \{N(w, \lambda, f) : w \in \mathbb{C}, \lambda \in \mathbb{D} \text{ and } |f(\lambda) - w| \geq \delta\} < \infty \right]. \quad (4.5)$$

We will show that the little-o condition corresponding to the big-O condition in (4.5) will give a necessary and sufficient condition for inclusion in the space *VMOA* . This will be made precise in Theorem 4.3.

In the proof of Theorem 4.3 we will need to relate the counting function N of an analytic function to the H^2-norm of the function. As is shown in the following lemma, this can be done for not just for the H^2-norm but for any H^p-norm of an analytic function.

Lemma 4.2 : *Let* $0 < p < \infty$ *. For an analytic function* f *on* $\mathbb{D}$ *with* $f(0) = 0$, *we have:*

$$\|f\|_{H^p}^{p} = \frac{p^2}{2\pi} \int_{\mathbb{C}} |w|^{p-2} N(w, 0, f) \, dA(w) .$$

Proof:

Fix $0 < p < \infty$, and let f be an analytic function on $\mathbb{D}$ with $f(0) = 0$. By Cartan's Formula and (4.2b), for every $\rho > 0$:

$$T\left(\frac{f}{\rho}\right) = \frac{1}{2\pi} \int_0^{2\pi} N(e^{i\theta}, 0, \frac{f}{\rho})\ d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} N(\rho e^{i\theta}, 0, f)\ d\theta\ .$$

Multiply by ρ^{p-1} and integrate with respect to ρ over the interval $(0, \infty)$. By the formula (3.6) of Lemma 3.4 we get

$$\|f\|_{H^p}^p = p^2 \int_0^{\infty} \rho^{p-1} \left(\frac{1}{2\pi} \int_0^{2\pi} N(\rho e^{i\theta}, 0, f)\ d\theta \right) d\rho$$

$$= \frac{p^2}{2\pi} \int_{\mathbb{C}} |w|^{p-2} N(w, 0, f)\ dA(w)\ ,$$

and the lemma is proved. □

Theorem 4.3 *: For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in VMOA$;

(b) *for every $\delta > 0$ we have:*

$$\sup\{N(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\} \to 0 \text{ as } |\lambda| \to 1^-.$$

Proof:

Let f be a nonconstant analytic function on $\mathbb{D}$. Let $\delta > 0$. Making use of Cartan's Formula and the equations (4.2) we see

$$T\left(\frac{f\circ\varphi_\lambda - f(\lambda)}{\delta}\right) = \frac{1}{2\pi}\int_0^{2\pi} N\left(e^{i\theta}, 0, \frac{f\circ\varphi_\lambda - f(\lambda)}{\delta}\right)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} N(\delta e^{i\theta} + f(\lambda), \lambda, f)\, d\theta$$

$$\leq \sup\{N(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\},$$

so that, by Theorem 3.3, *(b)* implies *(a)*.

To prove the other implication we make use of Lemma 4.2. In this lemma take $p = 2$, and replace f by $f\circ\varphi_\lambda - f(\lambda)$, we get the formula

$$\|f\circ\varphi_\lambda - f(\lambda)\|^2_{H^2} = \frac{2}{\pi}\int_{\mathbb{C}} N(w + f(\lambda), \lambda, f)\, dA(w). \tag{4.6}$$

We will also need Lehto's Theorem [21], which states that for a function g, analytic on a neighborhood of $\mathbb{D}$, the function $w \mapsto N(w, 0, g)$ is subharmonic on $\mathbb{C}\setminus\{g(0)\}$. Let g be an analytic function on $\mathbb{D}$ for which $g(0) = 0$. Let $0 < r < 1$. Applying Lehto's Theorem to the dilate g_r of g we get that for $\delta > 0$ and for $|u| \geq \delta$

$$N(u, 0, g_r) \leq \frac{1}{\pi\delta^2}\int_{|u-v|<\delta} N(v, 0, g_r)\, dA(v). \tag{4.7}$$

Taking the limit where $r \to 1^-$, we get

$$N(u, 0, g) \leq \frac{1}{\pi\delta^2}\int_{|u-v|<\delta} N(v, 0, g)\, dA(v).$$

Apply the inequality to $g = f \circ \varphi_\lambda - f(\lambda)$. Using equations (4.2) we get

$$N(u + f(\lambda), \lambda, f) \leq \frac{1}{\pi \delta^2} \int_{|u - v| < \delta} N(v + f(\lambda), \lambda, f)\ dA(v)\ .$$

Replacing $u + f(\lambda)$ by w yields the formula

$$N(w, \lambda, f) \leq \frac{1}{\pi \delta^2} \int_{|w - z| < \delta} N(z, \lambda, f)\ dA(z)\ . \tag{4.8}$$

Combining (4.6) and (4.8) gives us that for $|f(\lambda) - w| \geq \delta$

$$N(w, \lambda, f) \leq \frac{1}{2\delta^2} \|f \circ \varphi_\lambda - f(\lambda)\|^2_{H^2} ,$$

so that

$$sup\ \{N(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\} \leq \frac{1}{2\delta^2} \|f \circ \varphi_\lambda - f(\lambda)\|^2_{H^2} ,$$

from which it follows that *(a)* implies *(b)* .□

Now we will turn to the Bloch space and the little Bloch space. Defining an area version of the counting function used in the value distribution characterizations for *BMOA* and *VMOA*, we obtain analogous results for the Bloch space and the little Bloch space.

Define an area version N_a of the counting function N as follows: given an analytic function f on $\mathbb{D}$ we first define $N_a(0, 0, f)$ by

$$N_a(0,0,f) = \int_0^1 2r\, N(0,0,f_r)\, dr\,,$$

and, mimicking (4.2c), for $w \in \mathbb{C}$ and $\lambda \in \mathbb{D}$ define $N_a(w,\lambda,f)$ by

$$N_a(w,\lambda,f) = N_a(0,0,f\circ\varphi_\lambda - w)\,.$$

Observe that $N_a(w,\lambda,f) = 0$ if f omits the value w, but that (4.1a) is not necessarily true for counting function N_a. It follows immediately from the definition that properties (4.2) do hold for counting function N_a: for $w \in \mathbb{C}$, $\alpha \in \mathbb{C}\backslash\{0\}$, $\lambda \in \mathbb{D}$ and f analytic on $\mathbb{D}$ we have:

$$N_a(w,\lambda,f) = N_a(w+\alpha,\lambda,f+\alpha) \tag{4.9a}$$

$$N_a(w,\lambda,f) = N_a(\alpha w,\lambda,\alpha f) \tag{4.9b}$$

$$N_a(w,\lambda,f) = N_a(0,0,f\circ\varphi_\lambda - w)\,. \tag{4.9c}$$

Theorem 4.4 : *For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(A) $f \in \mathfrak{B}$;

(B) $\sup\,\{N_a(w,\lambda,f) : w \in \mathbb{C}, \lambda \in \mathbb{D}$ *and* $|f(\lambda) - w| \geq 1\} < \infty$.

Proof:

Let f be a nonconstant analytic function on $\mathbb{D}$, and let $0 < r < 1$. By inequality (4.4) we have:

$$N(0,0,f_r) \le T(f_r) - \log|f(0)|.$$

Multiply the above inequality by $2r$ and integrate with respect to r over the interval (0, 1) to get:

$$N_a(0,0,f) \le T_a(f) - \log|f(0)|. \tag{4.10}$$

Just as in the proof of Theorem 4.1 it follows that if $|f(\lambda) - w| \ge 1$, then we have

$$N_a(w,\lambda,f) \le T_a(f \circ \varphi_\lambda - f(\lambda)) + \log 2.$$

Theorem 3.6 and the above inequality show that *(A)* implies *(B)*.

Note that integrating Cartan's Formula gives us the formula

$$T_a(f) = \frac{1}{2\pi}\int_0^{2\pi} N_a(e^{i\theta},0,f)\, d\theta + \log^+|f(0)|. \tag{4.11}$$

To prove the converse we use this formula and proceed as in the proof of Theorem 4.1.□

The value distribution characterization for the Bloch space carries over to the little Bloch space in the same way as going from *BMOA* to *VMOA*.

Theorem 4.5 : *For a nonconstant analytic function f on $\mathbb{D}$ the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) *for every $\delta > 0$ we have:*

$$sup\ \{N_a(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\} \to 0 \text{ as } |\lambda| \to 1^-.$$

Proof:

Let f be a nonconstant analytic function on $\mathbb{D}$. Let $\delta > 0$. Making use of (4.11) and the equations (4.9), as in the proof of Theorem 4.3, we have for every $\delta > 0$

$$T_a\left(\frac{f \circ \varphi_\lambda - f(\lambda)}{\delta}\right) \leq sup\ \{N_a(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta\},$$

so that, by Theorem 3.7, *(b)* implies *(a)*.

To prove the other implication we need an area-version of Lemma 4.2. If $0 < p < \infty$, the function f is analytic on $\mathbb{D}$, and $f(0) = 0$, then applying Lemma 4.2 to the dilates f_r of f and subsequently integrating with respect to r over the interval (0, 1) yields the formula

$$\|f\|^p_{L^p_a} = \frac{p^2}{2\pi} \int_{\mathbb{C}} |w|^{p-2} N_a(w, 0, f)\ dA(w)\ .$$

In the above formula take $p = 2$ and for $\lambda \in \mathbb{D}$ replace f by $f \circ \varphi_\lambda - f(\lambda)$;analogous to (4.6) we get:

$$\| f \circ \varphi_\lambda - f(\lambda) \|^2_{L^2_a} = \frac{2}{\pi} \int_{\mathbb{C}} N_a(w + f(\lambda), \lambda, f) \; dA(w) . \qquad (4.12)$$

Integrating (4.7) with respect to r over the interval (0, 1) gives that for an analytic function g on $\mathbb{D}$ for which $g(0) = 0$ and for $|u| \geq \delta > 0$ we have

$$N_a(u, 0, g) \leq \frac{1}{\pi \delta^2} \int_{|u - v| < \delta} N_a(v, 0, g) \; dA(v) .$$

As in the proof of Theorem 4.3 it follows that whenever $|f(\lambda) - w| \geq \delta$ we must have

$$N_a(w, \lambda, f) \leq \frac{1}{\pi \delta^2} \int_{|w - z| < \delta} N_a(z, \lambda, f) \; dA(z) . \qquad (4.13)$$

Combining (4.12) and (4.13) we get

$$sup \{ N_a(w, \lambda, f) : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta \} \leq \frac{1}{2\delta^2} \| f \circ \varphi_\lambda - f(\lambda) \|^2_{L^2_a} ,$$

from which it follows that *(a)* implies *(b)*.□

For a nonconstant analytic function f on $\mathbb{D}$ it is easy to compute $N_a(0, 0, f)$. Let $\{z_n\}$ denote the zeros of f in $\mathbb{D}$, as usual, listed in increasing moduli and repeated according to multiplicities. Then for every $0 < r < 1$:

$$N(0, 0, f_r) = \sum_n \chi(|z_n| < r) \log \frac{r}{|z_n|} ,$$

thus we have

$$N_a(0,0,f) = \sum_n \int_{|z_n|}^{1} 2r \log \frac{r}{|z_n|}\, dr = \sum_n \frac{1}{2}\left(\log \frac{1}{|z_n|^2} - (1-|z_n|^2)\right).$$

Using power series it is elementary to show that

$$\frac{1}{2}(1-|z|^2)^2 \le \log \frac{1}{|z|^2} - (1-|z|^2), \quad z \in \mathbb{D},$$

so that we have the inequality

$$\sum_n (1-|z_n|^2)^2 \le 4\, N_a(0,0,f).$$

If $f \in N_a$, i.e., $T_a(f) < \infty$, and if $f(0) \neq 0$, then it follows from (4.9) that $N_a(0,0,f) < \infty$, so that by the above inequality

$$\sum_n (1-|z_n|^2)^2 < \infty. \tag{4.14}$$

The condition $f(0) \neq 0$ is no restriction: if $f(0) = 0$, then write $f(z) = z^m g(z)$ $(z \in \mathbb{D})$ for an $m \in \mathbb{N}$ and an analytic function g on $\mathbb{D}$ for which $g(0) \neq 0$. It is easy to see that then also $g \in N_a$, so that the zeros of g satisfy (4.14). It is then clear that also the zeros of f satisfy (4.14). Thus we have given a proof that the zeros $\{z_n\}$ of a function f in the area-Nevanlinna class N_a must satisfy (4.14). In [16], Andrei Heilper showed that conversely, all sequences $\{z_n\}$ in $\mathbb{D}$ satisfying (4.14) can be obtained as zerosets of functions in the area-Nevanlinna class.

For a nonconstant analytic function f on $\mathbb{D}$ and $0 < r < 1$ let $n(f, r)$ denote the number of zeros of f in $D(0, r)$, counted according to multiplicities. Then

$$n(f) = \lim_{r \to 1^-} n(f, r)$$

denotes the number of times (counting multiplicities) that f assumes the value 0. In [29] Pommerenke showed that a Bloch function f which safisfies the valence condition

$$\sup_{u \in \mathbb{C}} \int_{|w - u| < 1} n(f - w)\, dA(w) < \infty,$$

must belong to $BMOA$. If f is univalent (or finitely-valent), then it is trivial that the above condition is satisfied, thus univalent (or finitely-valent) Bloch functions belong to $BMOA$. We will give a necessary and sufficient condition on a Bloch function for inclusion in the space $BMOA$.

It is elementary to show that

$$N(0, 0, f_r) = \int_0^r \frac{n(f, t)}{t}\, dt .$$

Thus it follows that

$$N_a(0,0,f) = \int_0^1 \left(2r \int_0^r \frac{n(f,t)}{t}\, dt \right) dr$$

$$= \int_0^1 \left(\frac{1}{t} - t \right) n(f,t)\, dt\ .$$

Hence

$$N_a(0,0,f) + \int_0^1 t\, n(f,t)\, dt = N(0,0,f)\, .$$

Take $\lambda \in \mathbb{D}$ and $w \in \mathbb{C}$. Replacing f by $f \circ \varphi_\lambda - w$ yields

$$N_a(w,\lambda,f) + \int_0^1 t\, n(f \circ \varphi_\lambda - w, t)\, dt = N(w,\lambda,f)\, , \tag{4.15}$$

which (in view of Theorems 4.1 and 4.4) implies that for a Bloch function f to belong to *BMOA* it is necessary and sufficient that

$$sup\ \{ \int_0^1 t\, n(f \circ \varphi_\lambda - w, t)\, dt : \lambda \in \mathbb{D},\ w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq 1 \} < \infty\, .$$

Note that $n(f \circ \varphi_\lambda - w, t)$ is the number of zeros of $f - w$ in the pseudo-hyperbolic disk $D(\lambda, t)$, counted according to multiplicities. Thus the above condition is trivially satisfied if f is univalent (or finitely-valent).

Using Theorems 4.2 and 4.5 we see from (4.15) that a little Bloch function f must belong to $VMOA$ if and only if

$$\forall \delta > 0 : \left[\sup \left\{ \int_0^1 t\, n(f \circ \varphi_\lambda - w, t)\, dt : w \in \mathbb{C} \text{ and } |f(\lambda) - w| \geq \delta \right\} \to 0 \text{ as } |\lambda| \to 1^- \right].$$

Chapter 5

In this chapter we give estimates for the growth of analytic functions in weighted Dirichlet spaces, which then are used to give necessary and sufficient conditions on the growth of an analytic function on the disk for inclusion in the Bloch space or the little Bloch space. For the Bloch space and the little Bloch space we establish certain weighted Dirichlet-type conditions, and we investigate the question of whether analogous results are true for the spaces *BMOA* and *VMOA*.

We start with a lemma that gives estimates for the weighted Bergman norms of an analytic function and its derivative.

Lemma 5.1 : *Let* $-1 < \alpha < \infty$. *For an analytic function* f *on* $\mathbb{D}$ *we have*

$$\frac{1}{\alpha+1}\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha+2}\, dA(z)/\pi \le \int_{\mathbb{D}} |f(z) - f(0)|^2 (1-|z|^2)^{\alpha}\, dA(z)/\pi \le$$

$$\le \frac{\alpha+3}{\alpha+1}\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha+2}\, dA(z)/\pi \; . \qquad (5.1)$$

Proof:

Let $-1 < \alpha < \infty$. For an analytic function f on $\mathbb{D}$ with Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \; , \; z \in \mathbb{D},$$

it is easily seen that

$$\int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\alpha} \, dA(z)/\pi = \sum_{n=0}^{\infty} |a_n|^2 \beta(n, \alpha), \tag{5.2}$$

where

$$\beta(n, \alpha) = \int_{\mathbb{D}} |z|^{2n} (1-|z|^2)^{\alpha} \, dA(z)/\pi = \int_0^1 x^n (1-x)^{\alpha} \, dx .$$

Then we have that

$$\beta(n, \alpha) = \frac{n! \, \Gamma(\alpha+1)}{\Gamma(n+\alpha+2)} . \tag{5.3}$$

For the derivative of f we have

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha+2} \, dA(z)/\pi = \sum_{n=1}^{\infty} |a_n|^2 n^2 \beta(n-1, \alpha+2) .$$

Using (5.3) and the properties of the Gamma-function it is easy to verify that

$$n^2 \beta(n-1, \alpha+2) = \frac{n(\alpha+1)}{n+\alpha+2} \beta(n, \alpha) .$$

Thus we have

$$\frac{1}{\alpha+1} n^2 \beta(n-1, \alpha+2) \leq \beta(n, \alpha) \leq \frac{\alpha+3}{\alpha+1} n^2 \beta(n-1, \alpha+2) ,$$

and (5.1) follows immediately. □

In the above proof, for each $n \in \mathbb{N}$, $(\alpha + 1)\,\beta(n, \alpha) = n\,\beta(n-1, \alpha+1)$ increases to 1 as α decreases to -1. If we take $f \in H^2$ then we have

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} |a_n|^2,$$

so that for each $m \in \mathbb{N}$,

$$\left| \|f\|_{H^2}^2 - (\alpha+1) \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\alpha}\, dA(z)/\pi \right|$$

$$= \sum_{n=1}^{\infty} |a_n|^2 (1 - (\alpha+1)\beta(n, \alpha))$$

$$\leq \sum_{n=1}^{m} |a_n|^2 (1 - (\alpha+1)\beta(n, \alpha)) \; + \; 2 \sum_{n=m+1}^{\infty} |a_n|^2,$$

which implies that

$$(\alpha+1) \int_{\mathbb{D}} |f(z)|^2 (1-|z|^2)^{\alpha}\, dA(z)/\pi \;\rightarrow\; \|f\|_{H^2}^2 \text{ as } \alpha \rightarrow -1^{+}.$$

Taking the limit in (5.1) where $\alpha \rightarrow -1^{+}$, we thus obtain Paley's integral inequalities (see [14], Lemma 3.2), which we will use later in this chapter:

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)\, dA(z)/\pi \le \|f - f(0)\|^2_{H^2} \le 2\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)\, dA(z)/\pi . \quad (5.4)$$

For an analytic function f on $\mathbb{D}$ with Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \quad z \in \mathbb{D},$$

set

$$\overline{M}(r,f) = \sum_{n=0}^{\infty} |a_n| r^n , \text{for } 0 \le r < 1 .$$

The quantity $\overline{M}(r,f)$ is a very crude estimate on the growth of the modulus of the function f. In the following lemma we give an estimate on $\overline{M}(r,f)$ in terms of a weighted Dirichlet norm of the function f.

Lemma 5.2 : *Let* $0 < \alpha < \infty$. *For an analytic function* f *on* $\mathbb{D}$ *for which* $f(0) = 0$, *we have for all* $r \in [0, 1)$ *the inequality:*

$$(1-r^2)^{\alpha/2}\overline{M}(r,f) \le \sqrt{\frac{\alpha+1}{\alpha}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha}\, dA(z)/\pi \right)^{1/2} . \quad (5.5)$$

Proof:

Let $0 < \alpha < \infty$, and $r \in [0, 1)$. For the derivative f' of f we have

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \, dA(z)/\pi = \sum_{n=1}^{\infty} |a_n|^2 n^2 \beta(n-1,\alpha) .$$

Using the Cauchy-Schwarz inequality we have

$$\overline{M}(r,f) \le \left(\sum_{n=1}^{\infty} \frac{r^{2n}}{n^2 \beta(n-1,\alpha)} \right)^{1/2} \left(\sum_{n=1}^{\infty} |a_n|^2 n^2 \beta(n-1,\alpha) \right)^{1/2}$$

$$\le \left(\sum_{n=1}^{\infty} \frac{r^{2n}}{n^2 \beta(n-1,\alpha)} \right)^{1/2} \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^\alpha \, dA(z)/\pi \right)^{1/2} . \tag{5.6}$$

We need to estimate the infinite sum in (5.6). It follows from (5.3) that

$$n^2 \beta(n-1,\alpha) = \frac{n\alpha}{n+\alpha} \frac{n!\,\Gamma(\alpha)}{\Gamma(n+\alpha)} ,$$

therefore

$$\sum_{n=1}^{\infty} \frac{r^{2n}}{n^2 \beta(n-1,\alpha)} \le \sum_{n=1}^{\infty} \frac{n+\alpha}{n\alpha} \frac{\Gamma(n+\alpha)}{n!\,\Gamma(\alpha)} r^{2n}$$

$$\le \frac{\alpha+1}{\alpha} (1-r^2)^{-\alpha} ,$$

which together with (5.6) gives the desired inequality. □

We will use Lemma 5.2 to obtain characterizations for the Bloch space and the little Bloch space in terms of quantity $\overline{M}$. The lemma can also be used to prove a result due to V.S. Zakharyan [36]:

Let $0 < \alpha < \infty$ *. If* f *is an analytic function on* $\mathbb{D}$ *for which*

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^{\alpha} \, dA(z)/\pi < \infty , \tag{5.7}$$

Then

$$(1-r^2)^{\alpha/2} \overline{M}(r,f) \to 0 \ \ as \ \ r \to 1^{-}. \tag{5.8}$$

Proof:

Fix $0 < \alpha < \infty$. Let f be an analytic function on $\mathbb{D}$ with Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} a_n z^n , \ \ z \in \mathbb{D}.$$

For $N \in \mathbb{N}$ apply Lemma 5.2 to the function g_N defined by

$$g_N(z) = f(z) - \sum_{n=0}^{N} a_n z^n, \ z \in \mathbb{D} .$$

We get

$$(1-r^2)^{\alpha/2} \overline{M}(r,f) \le$$
$$\le (1-r^2)^{\alpha/2} \sum_{n=0}^{N} |a_n| r^n + \sqrt{\frac{\alpha+1}{\alpha}} \left(\int_{\mathbb{D}} |g_N'(z)|^2 (1-|z|^2)^{\alpha} \, dA(z)/\pi \right)^{1/2} . \tag{5.9}$$

It follows from (5.9) that

$$\limsup_{r \to 1^-} (1-r^2)^{\alpha/2} \overline{M}(r,f) \le \sqrt{\frac{\alpha+1}{\alpha}} \left(\int_{\mathbb{D}} |g_N'(z)|^2 (1-|z|^2)^\alpha \, dA(z)/\pi \right)^{1/2}. \quad (5.10)$$

In (5.10) let $N \to \infty$. Since f satisfies (5.7) the integral at the right of (5.10) tends to 0 and (5.8) follows. □

Theorem 5.3 : *Let* $0 < r < 1$. *For an analytic function* f *on* $\mathbb{D}$ *the following quantities are equivalent:*

(A) $\|f\|_{\mathfrak{B}}$;

(B) $\sup_{\lambda \in \mathbb{D}} \overline{M}(r, f \circ \varphi_\lambda - f(\lambda))$.

Proof:

Fix $0 < r < 1$. Let f be analytic on $\mathbb{D}$. By Lemma 5.2

$$(1-r^2)\, \overline{M}(r, f - f(0)) \le \sqrt{2} \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2)^2 \, dA(z)/\pi \right)^{1/2}.$$

Combining the above inequality with Lemma 5.1 we get

$$\overline{M}(r, f - f(0)) \le \frac{\sqrt{2}}{(1-r^2)} \|f - f(0)\|_{L_a^2}.$$

Let $\lambda \in \mathbb{D}$. Applying the above inequality to $f \circ \varphi_\lambda - f(\lambda)$ we get

$$\overline{M}(r, f \circ \varphi_\lambda - f(\lambda)) \leq \frac{\sqrt{2}}{(1 - r^2)} \| f \circ \varphi_\lambda - f(\lambda) \|_{L_a^2}, \tag{5.11}$$

and with the help of Theorem 1.1 it follows that quantity *(B)* is less than or equal to a constant times the Bloch norm of f.

To show the converse, note that $|f'(0)|\, r \leq \overline{M}(r, f)$, so that

$$(1 - |\lambda|^2)|f'(\lambda)| \leq \frac{1}{r} \overline{M}(r, f \circ \varphi_\lambda - f(\lambda)) , \tag{5.12}$$

which implies that the Bloch norm of f is less than or equal quantity *(B)*. □

As usual the equivalences of the previous theorem carry over to the little Bloch space, and we have:

Theorem 5.4 : *Let* $0 < r < 1$. *For an analytic function* f *on* $\mathbb{D}$ *the following statements are equivalent:*

(a) $f \in \mathfrak{B}_0$;

(b) $\overline{M}(r, f \circ \varphi_\lambda - f(\lambda)) \to 0$ *as* $|\lambda| \to 1^-$.

Proof:

Fix $0 < r < 1$. Let f be analytic on $\mathbb{D}$. It follows immediately from (5.12) that *(b)* implies *(a)*. The converse follows from (5.11) and Theorem 1.2. □

We now wish to investigate the spaces *BMOA* and *VMOA*. In view of Theorem 1.7, comparison of the two equivalences

$$\|f\|_{\mathcal{B}} \approx \sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2)^2 \, dA(z)/\pi \right)^{1/2},$$

and

$$\|f\|_{BMOA} \approx \sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi \right)^{1/2},$$

leads to the following question:

Question *: Let $0 < p < \infty$ and let f be an analytic function on $\mathbb{D}$. Are the following true?*

(i) $f \in BMOA \iff \sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi < \infty$?

(ii) $f \in VMOA$

$$\iff \left[\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- \right] ?$$

We do not know an answer for the above question. The classical results of Littlewood and Paley ([22], Theorems 5 and 6, page 54) and a change of variables give the following implications for an analytic function f on $\mathbb{D}$:

(I) For $0 < p \leq 2$:

$$\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi < \infty \implies f \in BMOA;$$

$$\left[\int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{p-2} (1 - |\varphi_\lambda(z)|^2) \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- \right] \implies f \in VMOA.$$

(II) For $2 \le p < \infty$:

$$f \in BMOA \implies \sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)\, dA(z)/\pi < \infty;$$

$$f \in VMOA \implies \left[\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)\, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- \right].$$

As mentioned above, these implications follow from Littlewood and Paley's theorems, but we can also prove them directly, using Theorem 1.7:

Proof:

(I) Let $0 < p \le 2$. If for an analytic function f on $\mathbb{D}$

$$\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)\, dA(z)/\pi < \infty,$$

then it follows from Theorem 1.7 that $f \in \mathfrak{B}$. Since we have

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_\lambda(z)|^2)\, dA(z)/\pi$$

$$\le \|f\|_{\mathfrak{B}}^{2-p} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)\, dA(z)/\pi ,$$

both implications in *(I)* follow.

(II) Let $2 \le p < \infty$. Now make use of the inequality

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2) \, dA(z)/\pi$$

$$\le \|f\|_{\mathfrak{B}}^{p-2} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_\lambda(z)|^2) \, dA(z)/\pi \quad ,$$

and since $BMOA \subset \mathfrak{B}$, the implications in *(II)* follow immediately. □

What we can prove is the following theorem.

Theorem 5.5 : *Let* $0 < p < \infty$, $0 < \sigma < 1$. *Then for an analytic function* f *on* $\mathbb{D}$ *we have the following two implications:*

(i) $$\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)^\sigma \, dA(z)/\pi < \infty$$

implies that $f \in BMOA$ *;*

(ii) $$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)^\sigma \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^-$$

implies that $f \in VMOA$.

The proof of Theorem 5.5 makes use of the following weighted Garcia-norm equivalences for the Bloch space and the little Bloch space.

Lemma 5.6 : *Let* $-1 < \alpha < \infty$ *and* $0 < p < \infty$. *Then for an analytic function* f *on* $\mathbb{D}$ *we have:*

(i) $$\|f\|_{\mathfrak{B}} \approx \sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^p \, (1-|z|^2)^\alpha \, dA(z)/\pi \right)^{1/p};$$

(ii) $$f \in \mathfrak{B}_0 \Leftrightarrow \left[\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^p \, (1-|z|^2)^\alpha \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- \right].$$

Proof:

Take $-1 < \alpha < \infty$ and $0 < p < \infty$. Let f be an analytic function on $\mathbb{D}$. Choose a number $s \in (1, \infty)$ small enough such that $s\alpha > -1$. Let s' denote the conjugate index of s, i.e., $s' = s/(s-1)$. An application of Hölder's inequality gives that

$$\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^p \, (1-|z|^2)^\alpha \, dA(z)/\pi$$

$$\leq \left(\int_{\mathbb{D}} (1-|z|^2)^{s\alpha} \, dA(z)/\pi \right)^{1/s} \left(\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(z)|^{ps'} \, dA(z)/\pi \right)^{1/s'}. \quad (5.13)$$

Now, since $s\alpha > -1$, we have that the integral at the left of (5.13) is finite, in fact it is equal to $1/(s\alpha + 1)$. It follows that

$$\left(\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(z)|^p \, (1-|z|^2)^\alpha \, dA(z)/\pi \right)^{1/p} \leq \left(\frac{1}{s\alpha + 1}\right)^{1/ps} \|f \circ \varphi_\lambda - f(\lambda)\|_{L_a^{ps'}}$$

(5.14)

To obtain an inequality in the other direction choose a number $q \in (1,\infty)$ large enough so that $q > \alpha + 1$. By Hölder's inequality

$$\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^{p/q}\, dA(z)/\pi$$

$$= \int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^{p/q} (1 - |z|^2)^{\alpha/q} (1 - |z|^2)^{-\alpha/q}\, dA(z)/\pi$$

$$\leq \left(\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^{p} (1 - |z|^2)^{\alpha}\, dA(z)/\pi \right)^{1/q} \left(\int_{\mathbb{D}} (1 - |z|^2)^{-\alpha q'/q}\, dA(z)/\pi \right)^{1/q'}.$$

Because $q - 1 > \alpha$, we have $-\alpha q'/q = -\alpha/(q - 1) > -1$, and thus the integral at the right of the last inequality is finite, in fact it is equal to $(q - 1)/(q - 1 - \alpha)$. It follows that

$$\| f \circ \varphi_\lambda - f(\lambda) \|_{L_a^{p/q}} \leq \left(\frac{q - 1}{q - 1 - \alpha}\right)^{(q-1)/p} \left(\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^{p} (1 - |z|^2)^{\sigma}\, dA(z)/\pi \right)^{1/p}. \tag{5.15}$$

By Theorem 1.1 equivalence *(i)* follows from (5.14) and (5.15), and statement *(ii)* is obtained by using Theorem 1.2. □

Proof of Theorem 5.5:

Let $0 < \sigma < 1$. First we will prove that both statements hold for integers $p > 2$. Let n be an integer, $n > 2$. Then $q = n/(n - 1) \leq 2$. Let g be an analytic function on $\mathbb{D}$ for which $g(0) = 0$. An application of Hölder's inequality and (5.4) gives

$$\| g \|_{H^q} \le \| g \|_{H^2} \le \left(2 \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2)\, dA(z)/\pi \right)^{1/2} . \tag{5.16}$$

Let f be analytic on $\mathbb{D}$, and assume for the moment that $f(0) = 0$. Apply (5.16) to the function $g = f^{n-1}$. This yields

$$\| f \|_{H^n}^{2(n-1)} \le 2\,(n-1)^2 \int_{\mathbb{D}} |f(z)|^{2(n-2)}\, |f'(z)|^2\, (1 - |z|^2)\, dA(z)/\pi \; .$$

Writing $\beta = n + \sigma - 2$, and using Hölder's inequality with index $n/2$, which has conjugate index $n/(n-2)$, we get

$$\| f \|_{H^n}^{2(n-1)} \le 2\,(n-1)^2 \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^{2\beta/n}\, |f(z)|^{2(n-2)} (1 - |z|^2)^{1 - 2\beta/n}\, dA(z)/\pi$$

$$\le 2\,(n-1)^2 \left(\int_{\mathbb{D}} |f'(z)|^n\, (1 - |z|^2)^{\beta}\, dA(z)/\pi \right)^{2/n} \times$$

$$\times \left(\int_{\mathbb{D}} |f(z)|^{2n}\, (1 - |z|^2)^{\frac{n - 2\beta}{n-2}}\, dA(z)/\pi \right)^{1 - 2/n} . \tag{5.17}$$

Now let $\lambda \in \mathbb{D}$, and in (5.17) replace f by $f \circ \varphi_\lambda - f(\lambda)$. Then (5.17) becomes

$$\| f \circ \varphi_\lambda - f(\lambda) \|_{H^n}^{2(n-1)} \le$$

$$\leq 2\,(n-1)^2 \left(\int_{\mathbb{D}} |f'(\varphi_\lambda(z))|^n\, |\varphi_\lambda'(z)|^n\, (1-|z|^2)^{\beta}\, dA(z)/\pi \right)^{2/n} \times$$

$$\times \left(\int_{\mathbb{D}} |f(\varphi_\lambda(z)) - f(\lambda)|^{2n}\, (1-|z|^2)^{\frac{n-2\beta}{n-2}}\, dA(z)/\pi \right)^{1-2/n} . \quad (5.18)$$

Making use of identity (0.20) and the change-of-variable formula (0.22a) we see that the first integral at the right hand side of (5.18) is equal to

$$\int_{\mathbb{D}} |f'(\varphi_\lambda(z))|^n \left(\frac{1-|\varphi_\lambda(z)|^2}{1-|z|^2}\right)^{n-2} (1-|z|^2)^{\beta}\, |\varphi_\lambda'(z)|^2\, dA(z)/\pi =$$

$$= \int_{\mathbb{D}} |f'(\varphi_\lambda(z))|^n\, (1-|\varphi_\lambda(z)|^2)^{n-2}\, (1-|z|^2)^{\sigma}\, |\varphi_\lambda'(z)|^2\, dA(z)/\pi$$

$$= \int_{\mathbb{D}} |f'(w)|^n\, (1-|w|^2)^{n-2}\, (1-|\varphi_\lambda(w)|^2)^{\sigma}\, dA(w)/\pi \quad .$$

Since $\sigma < 1$ we have that the exponent of $(1 - |z|^2)$ in the second integral at the right hand side of (5.18) is bigger than -1. By Lemma 5.6 there exists a constant C such that the second integral at the right of (5.18) is less than or equal $C\, \|f\|_{\mathcal{B}}$. It follows that

$$\|f \circ \varphi_\lambda - f(\lambda)\|_{H^n}^{2(n-1)} \leq$$

$$\leq 2\,(n-1)^2\, C^{1-2/n}\, \|f\|_{\mathcal{B}}^{1-2/n} \left(\int_{\mathbb{D}} |f'(z)|^n\, (1-|z|^2)^{n-2}\, (1-|\varphi_\lambda(z)|^2)^{\sigma}\, dA(z)/\pi \right)^{2/n} .$$

(5.19)

An important observation to make is that the conditions in statements *(i)* and *(ii)* imply that $f \in \mathfrak{B}$ (by Theorem 1.7), so both statements *(i)* and *(ii)* follow at once from inequality (5.19).

The general case is easily reduced to the previous case, again making use of the Bloch norm of f. Let $0 < p < \infty$. Choose an integer $n > 2$ such that $n \geq p$. Then we have

$$\int_{\mathbb{D}} |f'(z)|^n \, (1 - |z|^2)^{n-2} \, (1 - |\varphi_\lambda(z)|^2)^\sigma \, dA(z)/\pi$$

$$\leq \|f\|_{\mathfrak{B}}^{n-p} \int_{\mathbb{D}} |f'(z)|^p \, (1 - |z|^2)^{p-2} \, (1 - |\varphi_\lambda(z)|^2)^\sigma \, dA(z)/\pi \ .$$

This completes the proof of this theorem. □

The following assertion, which is implicit in the results of V.V. Peller ([26], Theorem 2' on page 454), is an immediate consequence of the above theorem.

Corollary 5.7 : *Let* $1 < p < \infty$. *If* f *is an analytic function on* $\mathbb{D}$ *for which*

$$\int_{\mathbb{D}} |f'(z)|^p \, (1 - |z|^2)^{p-2} \, dA(z)/\pi \ < \infty \ , \tag{5.20}$$

then $f \in$ *VMOA.*

Hong Oh Kim proved that a Blaschke product satisfying (5.20) must be a finite Blaschke product ([19], Theorem 1.1 on page 176). A simple proof is provided by the previous corollary, since it is easy to see that *VMOA* cannot contain Blaschke products with infinitely many zeros in $\mathbb{D}$. (In fact if b is a Blaschke product and $\lambda \in \mathbb{D}$ is a zero for b, then $\| b \circ \varphi_\lambda - b(\lambda) \|_{H^2} = \| b \circ \varphi_\lambda \|_{H^2} = 1$. Thus b is not contained in *VMOA* if it has infinitely many zeros in $\mathbb{D}$.)

In [19] Hong Oh Kim also proved the following result:

If $f \in \mathfrak{B}$ and

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^\alpha \, dA(z)/\pi < \infty , \tag{5.21}$$

for $0 < \alpha + 1 < p < \infty$, then $f \in H^q$ for all $0 < q < \infty$.

We can give a simple proof of this result using the same idea as in the proof of Theorem 5.5.

Proof:

Take $0 < \alpha + 1 < p < \infty$, and let $f \in \mathfrak{B}$, and suppose that (5.21) holds. Let $n \geq 2$ be an integer such that $n > p$. Put $\beta = \alpha + n - p$. As in the proof of Theorem 5.5 we have

$$\|f\|_{H^n}^{2(n-1)} \leq 2(n-1)^2 \left(\int_{\mathbb{D}} |f'(z)|^n (1-|z|^2)^\beta dA(z)/\pi \right)^{2/n} \times$$

$$\times \left(\int_{\mathbb{D}} |f(z)|^{2n} (1-|z|^2)^{\frac{n-2\beta}{n-2}} \, dA(z)/\pi \right)^{1-2/n} . \tag{5.22}$$

Now estimate the first integral at the right of (5.22) as follows

$$\int_{\mathbb{D}} |f'(z)|^n (1-|z|^2)^{\beta} \, dA(z)/\pi \le \|f\|_{\mathfrak{B}}^{n-p} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{\alpha} \, dA(z)/\pi < \infty .$$

The exponent $(n - 2\beta)/(n - 2) = (2p - 2\alpha - n)/(n - 2)$ in the second integral at the right of (5.22) is easily seen to be greater than -1, so that also this integral is finite (as a consequence of Lemma 5.6). Thus $f \in H^n$, for arbitrary integers $n > p$. Hence $f \in H^q$ for all $0 < q < \infty$. □

After these digressions, Theorem 5.5 and our question preceeding it should be compared with the following theorem.

Theorem 5.8 : *Let* $0 < p < \infty$, $1 < \eta < \infty$. *Then for an analytic function f on* $\mathbb{D}$ *we have:*

(i) $$\|f\|_{\mathfrak{B}} \approx \sup_{\lambda \in \mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)^{\eta} \, dA(z)/\pi \right)^{1/p} ;$$

(ii) $$f \in \mathfrak{B}_0 \Leftrightarrow \left[\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)^{\eta} \, dA(z)/\pi \to 0 \text{ as } |\lambda| \to 1^- \right].$$

Proof:

Take $0 < p < \infty$, and let $1 < \eta < \infty$. Using the definition of the Bloch norm, identity (0.20) and the change-of-variable formula (0.22a), we have that for an analytic function f on $\mathbb{D}$:

$$\int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} (1-|\varphi_\lambda(z)|^2)^{\eta} \, dA(z)/\pi \le$$

$$\le \|f\|_{\mathcal{B}}^{p} \int_{\mathbb{D}} \frac{(1-|\varphi_\lambda(z)|^2)^{\eta}}{(1-|z|^2)^2}\, dA(z)/\pi$$

$$= \|f\|_{\mathcal{B}}^{p} \int_{\mathbb{D}} (1-|\varphi_\lambda(z)|^2)^{\eta-2}\, |\varphi_\lambda'(z)|^2\, dA(z)/\pi$$

$$= \|f\|_{\mathcal{B}}^{p} \int_{\mathbb{D}} (1-|z|^2)^{\eta-2}\, dA(z)/\pi \;=\; \|f\|_{\mathcal{B}}^{p}\, \frac{1}{\eta-1}\,.$$

Hence

$$\int_{\mathbb{D}} |f'(z)|^p\, (1-|z|^2)^{p-2}\, (1-|\varphi_\lambda(z)|^2)^{\eta}\, dA(z)/\pi \;\le\; \frac{1}{\eta-1}\, \|f\|_{\mathcal{B}}^{p}\,. \tag{5.23}$$

Let g be an analytic function on $\mathbb{D}$, then by the subharmonicity of $|g|^p$ we have

$$|g(0)|^p \le 4 \int_{D(0,\frac{1}{2})} |g(z)|^p\, dA(z)/\pi\,.$$

If $z \in D(0,\frac{1}{2})$, then $(1-|z|^2)^{\eta+p-2} \ge \delta = min\,\{1, (3/4)^{\eta+p-2}\}$, and therefore

$$|g(0)|^p \le \frac{4}{\delta} \int_{D(0,\frac{1}{2})} |g(z)|^p\, (1-|z|^2)^{\eta+p-2}\, dA(z)/\pi$$

$$\le \frac{4}{\delta} \int_{\mathbb{D}} |g(z)|^p\, (1-|z|^2)^{\eta+p-2}\, dA(z)/\pi\,. \tag{5.24}$$

Let $\lambda \in \mathbb{D}$. Applying inequality (5.24) to the derivative of the Möbius transform $f \circ \varphi_\lambda - f(\lambda)$ of f, we get

$(1 - |\lambda|^2$

Using ide

at the rig

It follo

$(1 - |$

The eq

also g

necess

$|f'(z)$

$$(1-|\lambda|^2)^p\,|f'(\lambda)|^p \le \frac{4}{\delta}\int_{\mathbb{D}} |f'(\varphi_\lambda(z))|^p\,|\varphi_\lambda'(z)|^p\,(1-|z|^2)^{\eta+p-2}\,dA(z)/\pi\ . \quad (5.25)$$

Using identity (0.20) and the change-of-variable formula (0.22a) we see that the integral at the right of (5.25) is equal to

$$\int_{\mathbb{D}} |f'(\varphi_\lambda(z))|^p \left(\frac{1-|\varphi_\lambda(z)|^2}{1-|z|^2}\right)^{p-2} (1-|z|^2)^{\eta+p-2}\,|\varphi_\lambda'(z)|^2\,dA(z)/\pi$$

$$= \int_{\mathbb{D}} |f'(\varphi_\lambda(z))|^p\,(1-|\varphi_\lambda(z)|^2)^{p-2}\,(1-|z|^2)^{\eta}\,|\varphi_\lambda'(z)|^2\,dA(z)/\pi$$

$$= \int_{\mathbb{D}} |f'(w)|^p\,(1-|w|^2)^{p-2}\,(1-|\varphi_\lambda(w)|^2)^{\eta}\,dA(w)/\pi\ .$$

It follows from (5.25) that

$$(1-|\lambda|^2)^p\,|f'(\lambda)|^p \le \frac{4}{\delta}\int_{\mathbb{D}} |f'(z)|^p\,(1-|z|^2)^{p-2}\,(1-|\varphi_\lambda(z)|^2)^{\eta}\,dA(z)/\pi\ . \quad (5.26)$$

The equivalence in *(i)* follows immediately from (5.23) and (5.26). Inequality (5.26) also gives that the condition in *(ii)* is sufficient for f to be in $\mathcal{B}_0$. To prove the necessity, suppose that $f \in \mathcal{B}_0$. Given $\varepsilon > 0$, choose an $r \in (0, 1)$ such that $|f'(z)|\,(1-|z|^2) < \varepsilon$ whenever $r \le |z| < 1$. Then we have

$$\int_{\mathbb{D}\setminus r\mathbb{D}} |f'(z)|^p\,(1-|z|^2)^{p-2}\,(1-|\varphi_\lambda(z)|^2)^{\eta}\,dA(z)/\pi\ \le$$

$$\leq \varepsilon^p \int_{\mathbb{D}\setminus r\mathbb{D}} \frac{(1-|\varphi_\lambda(z)|^2)^\eta}{(1-|z|^2)^2}\, dA(z)/\pi$$

$$\leq \varepsilon^p \int_{\mathbb{D}} (1-|\varphi_\lambda(z)|^2)^{\eta-2}\, |\varphi_\lambda'(z)|^2\, dA(z)/\pi$$

$$= \varepsilon^p \int_{\mathbb{D}} (1-|z|^2)^{\eta-2}\, dA(z)/\pi \;=\; \frac{\varepsilon^p}{\eta-1}\,.$$

On the other hand

$$\int_{r\mathbb{D}} |f'(z)|^p\, (1-|z|^2)^{p-2}\, (1-|\varphi_\lambda(z)|^2)^\eta\, dA(z)/\pi \;\leq$$

$$\leq \|f\|_{\mathcal{B}}^p \int_{r\mathbb{D}} \frac{(1-|\varphi_\lambda(z)|^2)^\eta}{(1-|z|^2)^2}\, dA(z)/\pi$$

$$\leq \|f\|_{\mathcal{B}}^p \int_{r\mathbb{D}} (1-|\varphi_\lambda(z)|^2)^{\eta-2}\, |\varphi_\lambda'(z)|^2\, dA(z)/\pi$$

$$= \|f\|_{\mathcal{B}}^p \int_{D(\lambda,r)} (1-|z|^2)^{\eta-2}\, dA(z)/\pi\;.$$

It follows that

$$\int_{\mathbb{D}} |f'(z)|^p\, (1-|z|^2)^{p-2}\, (1-|\varphi_\lambda(z)|^2)^\eta\, dA(z)/\pi \;\leq$$

$$\leq \frac{\varepsilon^p}{\eta-1} + \|f\|_{\mathcal{B}}^p \int_{D(\lambda,r)} (1-|z|^2)^{\eta-2}\, dA(z)/\pi\;,$$

and we get the necessity of the condition in *(ii)*, since for every fixed $r \in (0, 1)$ we have

$$\int_{D(\lambda,r)} (1-|z|^2)^{\eta-2}\, dA(z)/\pi \;\to\; 0 \text{ as } |\lambda| \to 1^-.$$

This completes the proof of this theorem. □

It would be interesting to have characterizations of $BMOA$ and $VMOA$ involving the pseudo-hyperbolic disks $D(\lambda,r)$. Recall that an analytic function f on $\mathbb{D}$ belongs to $\mathfrak{B}$ if and only if

$$\sup_{\lambda\in\mathbb{D}} \int_{D(\lambda,r)} |f'|^2\, dA/\pi < \infty$$

for some $r \in (0, 1)$, and that f belongs to $\mathfrak{B}_0$ if and only if

$$\int_{D(\lambda,r)} |f'|^2\, dA/\pi \;\to\; 0 \text{ as } |\lambda| \to 1^-$$

for some $r \in (0, 1)$. The following theorem should be compared with these results.

Theorem 5.9 : *For an analytic function f on $\mathbb{D}$ we have:*

(i) $$f \in BMOA \iff \sup_{\lambda\in\mathbb{D}} \int_0^1 \left(\int_{D(\lambda,r)} |f'|^2\, dA/\pi \right) dr < \infty;$$

(ii) $$f \in VMOA \iff \left[\int_0^1 \left(\int_{D(\lambda,r)} |f'|^2\, dA/\pi \right) dr \to 0 \text{ as } |\lambda| \to 1^- \right].$$

Proof:

Let f be an analytic function on $\mathbb{D}$. Using characteristic functions we have

$$\int_{D(\lambda,r)} |f'|^2 \, dA/\pi = \int_{\mathbb{D}} |f'(z)|^2 \chi_{D(\lambda,r)}(z) \, dA(z)/\pi \ ,$$

thus

$$\int_0^1 \left(\int_{D(\lambda,r)} |f'|^2 \, dA/\pi \right) dr = \int_{\mathbb{D}} |f'(z)|^2 \left(\int_0^1 \chi_{D(\lambda,r)}(z) \, dr \right) dA(z)/\pi$$

$$= \int_{\mathbb{D}} |f'(z)|^2 \, (1 - |\varphi_\lambda(z)|) \, dA(z)/\pi \ ,$$

and both *(i)* and *(ii)* follow at once. □

The previous theorem can be used to give yet another proof of Pommerenke's result ([29], Satz 1) which states that for an analytic function f on $\mathbb{D}$ which is one-to-one, containment in $\mathfrak{B}$, or in $\mathfrak{B}_0$, implies that the function already belongs to *BMOA* , or *VMOA* , respectively.

Proof:

Suppose that $f \in \mathfrak{B}$ is one-to-one. Let $\lambda \in \mathbb{D}$. Since for every $z \in D(\lambda,r)$

$$|f(z) - f(\lambda)| \le \|f\|_{\mathfrak{B}} \, \log \frac{1}{1-r} \ ,$$

we have the inclusion

$$\{f(z) - f(\lambda) : z \in D(\lambda,r)\} \subset D(0, \|f\|_{\mathfrak{B}} \log \frac{1}{1-r}),$$

and it follows that

$$\int_{D(\lambda,r)} |f'|^2 \, dA/\pi = |\{f(z) - f(\lambda) : z \in D(\lambda,r)\}| \leq \|f\|_{\mathfrak{B}}^2 \left(\log \frac{1}{1-r}\right)^2 . \quad (5.27)$$

Thus

$$\int_0^1 \left(\int_{D(\lambda,r)} |f'|^2 \, dA/\pi \right) dr \leq \|f\|_{\mathfrak{B}}^2 \int_0^1 \left(\log \frac{1}{1-r}\right)^2 dr < \infty ,$$

and by *(i)* of Theorem 5.9 we have that $f \in BMOA$.

If $f \in \mathfrak{B}_0$, then by Theorem 1.2 we have for each $r \in (0, 1)$

$$\int_{D(\lambda,r)} |f'|^2 \, dA/\pi \to 0 \text{ as } |\lambda| \to 1^- ,$$

so that by (5.27) and the Lebesgue Dominated Convergence Theorem we have that

$$\int_0^1 \left(\int_{D(\lambda,r)} |f'|^2 \, dA/\pi \right) dr \to 0 \text{ as } |\lambda| \to 1^- ,$$

thus, by Theorem 5.9, $f \in VMOA$. □

Chapter 6

In this chapter we briefly discuss cyclic vectors in the little Bloch space. We generalize a theorem of Anderson, Clunie and Pommerenke and obtain a result very similar to one of Brown and Shields in the context of Dirichlet spaces.

First some notation and a definition. In order to be able to compare our result for the little Bloch space with a result of Brown and Shields for the Dirichlet space we will give a general definition for a cyclic vector.

Let $\mathfrak{S}$ be a Banach space of analytic functions on $\mathbb{D}$ which contains the polynomials as a dense subset, which is invariant under multiplication by the function z, and for which all the point evaluations are bounded linear functionals on $\mathfrak{S}$. For a function $f \in \mathfrak{S}$ let $[f]_{\mathfrak{S}}$ denote the closure of the set $\{ pf : p$ is a polynomial $\}$ in the Banach space $\mathfrak{S}$.

Definition : *A function $f \in \mathfrak{S}$ is called a cyclic vector in $\mathfrak{S}$ or cyclic for $\mathfrak{S}$, if $[f]_{\mathfrak{S}} = \mathfrak{S}$.*

The little Bloch space $\mathfrak{B}_0$ furnishes an example of such a Banach space $\mathfrak{S}$. That $\mathfrak{B}_0$ is invariant under multiplication by the function z is easy to see, and that the polynomials form a dense subset of $\mathfrak{B}_0$ is proved in [2],Theorem 2.1. In general, it is easy to show that a cyclic vector in $\mathfrak{S}$ has no zeros in $\mathbb{D}$ (see, for example, [35], Proposition 4). In the case of the little Bloch space Anderson, Clunie and Pommerenke proved the following result [2], Theorem 3.8:

For $f \in \mathfrak{B}_0$ the condition $\inf\{|f(z)| : z \in \mathbb{D}\} > 0$ implies that f is cyclic for $\mathfrak{B}_0$.

In Corollary 6.4 we will extend their result and prove:

If $f \in \mathfrak{B}_0$, $g \in \mathfrak{B}_0 \cap H^\infty$, $|f(z)| \geq |g(z)|$ *in* $\mathbb{D}$, *and if* g^2 *is cyclic for* $\mathfrak{B}_0$, *then* f *is cyclic for* $\mathfrak{B}_0$.

We are actually able to prove a result not just for cyclic vectors in $\mathfrak{B}_0$ but one that gives an inclusion relation for the sets $[f]_{\mathfrak{B}_0}$ introduced above. This will be the content of Theorem 6.3.

The Dirichlet space $\mathcal{D} = \{f \in H(\mathbb{D}) : f' \in L_a^2\}$ is another example of a Banach space $\mathcal{E}$ of analytic functions on $\mathbb{D}$. Our Corollary 6.4 should be compared with the following result of Brown and Shields [10], Theorem 1:

If $f \in \mathcal{D}$, $g \in \mathcal{D} \cap H^\infty$, $|f(z)| \geq |g(z)|$ *in* $\mathbb{D}$, *and if* g^2 *is cyclic for* $\mathcal{D}$, *then* f *is cyclic for* $\mathcal{D}$.

In [10], Proposition 11, Brown and Shields proved also that:

If $f, g \in \mathcal{D} \cap H^\infty$, *and if* fg *is cyclic for* $\mathcal{D}$, *then both* f *and* g *are cyclic for* $\mathcal{D}$.

This is also true for bounded functions in the little Bloch space. In Theorem 6.5 we will give an inclusion relation for the sets $[f]_{\mathfrak{B}_0}$ introduced above. As a corollary we get:

If $f, g \in \mathfrak{B}_0 \cap H^\infty$, *and if* fg *is cyclic for* $\mathfrak{B}_0$, *then both* f *and* g *are cyclic for* $\mathfrak{B}_0$.

In the proofs of Theorems 6.2 and 6.5 we will need to use the following two lemmas. Recall that for an analytic function g on $\mathbb{D}$, and for $0 < t < 1$ the dilate g_t of g is defined by the equation $g_t(z) = g(tz)$ $(z \in \mathbb{D})$.

***Lemma 6.1** : Let $g \in \mathfrak{B}_0$. Then:*

$$\sup_{z \in \mathbb{D}} (1 - |z|)\, |g_t{}'(z)|\, \log\left(\frac{1 - t|z|}{1 - |z|}\right) \to 0 \text{ as } t \to 1^- . \tag{6.1}$$

Proof:

Since $g_t{}'(z) = t\, g'(tz)$, and $(1 - t|z|)\, |g'(tz)| \le \|g\|_{\mathfrak{B}}$ we have the inequality

$$(1 - |z|)\, |g_t{}'(z)|\, \log\left(\frac{1 - t|z|}{1 - |z|}\right) \le \|g\|_{\mathfrak{B}}\ \frac{1 - |z|}{1 - t|z|} \log\left(\frac{1 - t|z|}{1 - |z|}\right) . \tag{6.2}$$

Take $0 < r < 1$. It is elementary to show that

$$\max_{|z| \le r} \frac{1 - |z|}{1 - t|z|} \log\left(\frac{1 - t|z|}{1 - |z|}\right) = \frac{1 - r}{1 - tr} \log\left(\frac{1 - tr}{1 - r}\right) . \tag{6.3}$$

It follows from (6.2) and (6.3) that

$$\max_{|z| \le r} (1 - |z|)\, |g_t{}'(z)|\, \log\left(\frac{1 - t|z|}{1 - |z|}\right) \to 0 \text{ as } t \to 1^- . \tag{6.4}$$

Now let $\varepsilon > 0$ be given. Since $g \in \mathfrak{B}_0$, we can choose an $r \in (0, 1)$ such that $(1 - |w|)\, |g'(w)| < \varepsilon$ whenever $r^2 < |w| < 1$. Then for $r < |z| < 1$ and $r < t < 1$ we have $(1 - t|z|)\, |g_t{}'(z)| = t(1 - |tz|)\, |g'(tz)| < t\varepsilon < \varepsilon$. Because $x \log(1/x) < x$ for $0 < x < 1$, we have that

$$(1 - |z|) \log\left(\frac{1 - t|z|}{1 - |z|}\right) \le (1 - t|z|) ,$$

so that for $r < t < 1$

$$\sup_{r<|z|<1} (1-|z|)\,|g_t'(z)|\,\log\left(\frac{1-t|z|}{1-|z|}\right) < \varepsilon \,. \tag{6.5}$$

Our claim (6.1) follows readily from (6.4) and (6.5). □

Lemma 6.2 : *Let $f \in \mathcal{B}_0$ be nonvanishing in $\mathbb{D}$. Suppose that for $h \in \mathcal{B}_0$:*

$$\|(\frac{f}{f_t} - 1)\,h_t\|_{\mathcal{B}} \to 0 \ \ as\ \ t \to 1^-. \tag{6.6}$$

Then $[h]_{\mathcal{B}_0} \subset [f]_{\mathcal{B}_0}$.

Proof:

Let $f \in \mathcal{B}_0$ be nonvanishing in $\mathbb{D}$, and suppose that $h \in \mathcal{B}_0$ satisfies (6.6). We will have to show that $h \in [f]_{\mathcal{B}_0}$. Let $\varepsilon > 0$ be given. Since $h \in \mathcal{B}_0$, $\|h_t - h\|_{\mathcal{B}} \to 0$ as $t \to 1^-$. With (6.6) it follows that we can choose a $t \in (0, 1)$ for which

$$\|(\frac{f}{f_t} - 1)\,h_t\|_{\mathcal{B}} < \varepsilon \ \text{ and } \ \|h_t - h\|_{\mathcal{B}} < \varepsilon \,. \tag{6.7}$$

The function h_t/f_t is analytic in a neighborhood of $\overline{\mathbb{D}}$, hence we can find a sequence of polynomials (p_n) (of course depending on the t that we picked) such that the functions $p_n - h_t/f_t$ and their derivatives converge to 0 uniformly on $\mathbb{D}$. We claim that in fact

$$\|f(p_n - \frac{h_t}{f_t})\|_{\mathcal{B}} \to 0 \ \text{ as } \ n \to \infty\,. \tag{6.8}$$

To prove this claim, write $g_n = p_n - h_t/f_t$. Then by the choice of the sequence of polynomials $g_n \to 0$ and $g_n' \to 0$ uniformly on $\mathbb{D}$ as $n \to \infty$. Using the product rule for differentiation we see that

$$(1-|z|^2)\left|\frac{d}{dz}(fg_n)\right| \le (1-|z|^2)|f'(z)||g_n(z)| + (1-|z|^2)|f(z)||g_n'(z)| \ . \quad (6.9)$$

Again using the inequality $x \log(1/x) < 1$ for $0 < x < 1$, it follows from

$$|f(z) - f(0)| \le \log\left(\frac{1}{1-|z|}\right) \|f\|_{\mathfrak{B}} ,$$

that

$$(1-|z|^2)|f(z) - f(0)| \le 2\|f\|_{\mathfrak{B}} ,$$

which combined with (6.9) gives that

$$\|fg_n\|_{\mathfrak{B}} \le \|f\|_{\mathfrak{B}} \|g_n\|_\infty + (2\|f\|_{\mathfrak{B}} + |f(0)|)\|g_n'\|_\infty \ . \quad (6.10)$$

Now, since both $\|g_n\|_\infty$ and $\|g_n'\|_\infty$ tend to 0 as $n \to \infty$, our claim (6.8) follows immediately from (6.10).

We are now ready to finish the proof. By (6.8) there is a polynomial p such that

$$\|f(p - \frac{h_t}{f_t})\|_{\mathfrak{B}} < \varepsilon \ . \quad (6.11)$$

By the triangle inequality

$$\|fp - h\|_{\mathcal{B}} \le \|f(p - \frac{h_t}{f_t})\|_{\mathcal{B}} + \|(\frac{f}{f_t} - 1)h_t\|_{\mathcal{B}} + \|h_t - h\|_{\mathcal{B}} ,$$

so that (6.7) and (6.11) imply that $\|fp - h\|_{\mathcal{B}} < 3\varepsilon$. We conclude that $h \in [f]_{\mathcal{B}_0}$ which implies that $[h]_{\mathcal{B}_0} \subset [f]_{\mathcal{B}_0}$. □

Theorem 6.3 : *Let $f, g \in \mathcal{B}_0$, such that $|f(z)| \ge |g(z)|$ $(z \in \mathbb{D})$, and suppose that f is nonvanishing and that g is bounded. Then $[g^2]_{\mathcal{B}_0} \subset [f]_{\mathcal{B}_0}$.*

Proof:

Let $f, g \in \mathcal{B}_0$, such that $|f(z)| \ge |g(z)|$ $(z \in \mathbb{D})$, and suppose that f is nonvanishing and that g is bounded. It is easy to see that $g^2 \in \mathcal{B}_0$. So by Lemma 6.2 it suffices to show that (6.6) holds with $h = g^2$. As in the proof of Theorem 2.1 note that

$$|f(z) - f_t(z)| \le \|f\|_{\mathcal{B}} \, log\left(\frac{1 - t|z|}{1 - |z|}\right) . \tag{6.12}$$

It is elementary to check that

$$\frac{d}{dz}\left((\frac{f}{f_t} - 1)g_t^2\right) = 2\frac{f(z) - f_t(z)}{f_t(z)} g_t(z) g_t'(z) + \frac{f'(z) - f_t'(z)}{f_t(z)} g_t^2(z) - \frac{(f(z) - f_t(z))f_t'(z)}{f_t^2(z)} g_t^2(z) .$$

Using that $|f_t(z)| \ge |g_t(z)|$ $(z \in \mathbb{D})$ it follows that

$$\left|\frac{d}{dz}\left((\frac{f}{f_t} - 1)\, g_t^{\,2}\right)\right| \leq 2\,|f(z) - f_t(z)|\,|g_t'(z)| + |f'(z) - f_t'(z)|\,\|g\|_\infty +$$

$$+ |f(z) - f_t(z)|\,|f_t'(z)| \;. \quad (6.13)$$

Using the definition of the Bloch norm it follows from (6.12) and (6.13) that

$$\|(\frac{f}{f_t} - 1)\, g_t^{\,2}\|_{\mathfrak{B}} \leq 4\,\|f\|_{\mathfrak{B}} \sup_{z\in\mathbb{D}} (1-|z|)\,|g_t'(z)|\,\log\left(\frac{1-t|z|}{1-|z|}\right) +$$

$$+ \|g\|_\infty\,\|f - f_t\|_{\mathfrak{B}} + 2\,\|f\|_{\mathfrak{B}} \sup_{z\in\mathbb{D}} (1-|z|)\,|f_t'(z)|\,\log\left(\frac{1-t|z|}{1-|z|}\right) . \quad (6.14)$$

Now, by Lemma 6.1 the first and the third term at the right of inequality (6.14) tend to 0 as we take the limit where $t \to 1^-$. Since $f \in \mathfrak{B}_0$ also $\|f - f_t\|_{\mathfrak{B}} \to 0$, and our claim that

$$\|(\frac{f}{f_t} - 1)\, g_t^{\,2}\|_{\mathfrak{B}} \to 0 \text{ as } t \to 1^-$$

follows immediately. □

The following corollary is an immediate consequence of Theorem 6.3 and the definition of a cyclic vector.

Corollary 6.4 : *Let $f, g \in \mathfrak{B}_0$, such that $|f(z)| \geq |g(z)|$ $(z \in \mathbb{D})$, and suppose that g is bounded and g^2 is cyclic for $\mathfrak{B}_0$. Then f is cyclic for $\mathfrak{B}_0$.*

Theorem 6.5 : *Let $f, g \in \mathfrak{B}_0 \cap H^\infty$, and suppose that f is nonvanishing. Then $[f g]_{\mathfrak{B}_0} \subset [f]_{\mathfrak{B}_0}$.*

Proof:

Take $f, g \in \mathfrak{B}_0 \cap H^\infty$, and suppose that f is nonvanishing. It is easy to see that then their product $h = fg$ is in $\mathfrak{B}_0$. By Lemma 6.2 it suffices to show that the function h satisfies (6.6). It follows from

$$\left| \frac{d}{dz}\left((\frac{f}{f_t} - 1) h_t \right) \right| = \left| \frac{d}{dz}((f - f_t)\, g_t) \right|$$

$$\leq |f'(z) - f_t'(z)|\; \| g \|_\infty + |f(z) - f_t(z)|\; |g_t'(z)|,$$

and inequality (6.12) that

$$\| (\frac{f}{f_t} - 1)\, h_t \|_{\mathfrak{B}} \leq \| f - f_t \|_{\mathfrak{B}}\; \| g \|_\infty +$$

$$+\ 2 \| f \|_{\mathfrak{B}} \sup_{z \in \mathbb{D}} (1 - |z|)\, |g_t'(z)|\, \log\left(\frac{1 - t|z|}{1 - |z|} \right). \qquad (6.15)$$

Both terms at the right of inequality (6.15) tend to 0 as $t \to 1^-$ (that the second term tends to zero follows from Lemma 6.1). Thus

$$\| (\frac{f}{f_t} - 1)\, h_t \|_{\mathfrak{B}} \to 0 \text{ as } t \to 1^-,$$

and by Lemma 6.2 we are done. □

The following corollary is an immediate consequence of Theorem 6.5 and the definition of a cyclic vector.

Corollary 6.6 : *If $f, g \in \mathfrak{B}_0 \cap H^\infty$, and if $f\,g$ is cyclic for $\mathfrak{B}_0$, then both f and g are cyclic for $\mathfrak{B}_0$.*

Chapter 7

In this chapter we consider Hankel operators with integrable symbol. The Hankel operators that we study are defined by projecting onto the orthogonal complement of the Bergman space. We first prove that these Hankel operators transform in a unitarily equivalent way if the symbol is replaced by one of its Möbius transforms. We then restrict our attention to Hankel operators with conjugate analytic symbol, and show that Sheldon Axler's results [6], Theorems 6 and 7, hold if the operator norm of the Hankel operator is obtained by putting a weighted L^p-norm on both its domain and its range.

Recall that for $0 < p < \infty$ the Bergman space L_a^p is defined as the space of analytic functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{L_a^p} = \left(\int_{\mathbb{D}} |f(z)|^p \, dA(z)/\pi \right)^{1/p} < \infty .$$

For $1 \leq p < \infty$ the Bergman space L_a^p is a Banach space. The Bergman space L_a^2 is a Hilbert space; it is a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA/\pi)$ with inner product given by

$$\langle f, g \rangle = \int_{\mathbb{D}} f(z) \overline{g(z)} \, dA(z)/\pi ,$$

for $f, g \in L^2(\mathbb{D}, dA/\pi)$. Point evaluation is a bounded linear functional on the Hilbert space L_a^2, thus for every $\lambda \in \mathbb{D}$ there exists a unique function $k_\lambda \in L_a^2$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle \text{ , for all } f \in L_a^2 .$$

These functions k_λ ($\lambda \in \mathbb{D}$) are called the reproducing kernels for L_a^2. It is easy to verify that for every $\lambda \in \mathbb{D}$ the reproducing kernel k_λ is given by the formula

$$k_\lambda(z) = \frac{1}{(1 - \bar{\lambda} z)^2} \text{ , for } z \in \mathbb{D} .$$

Because of the reproducing property of k_λ we have $\langle k_\lambda , k_\lambda \rangle = k_\lambda(\lambda)$. Using the above formula for k_λ it follows at once that

$$\| k_\lambda \|^2_{L_a^2} = \frac{1}{(1 - |\lambda|^2)^2} .$$

Let P denote the orthogonal projection of $L^2(\mathbb{D}, dA/\pi)$ onto L_a^2. In view of the above formula for the reproducing kernels it is easy to see that for $g \in L^2(\mathbb{D}, dA/\pi)$ we have the following formula for its projection $P(g)$:

$$(P(g))(z) = \int_{\mathbb{D}} \frac{g(w)}{(1 - \bar{w} z)^2} \, dA(w)/\pi \text{ , for } z \in \mathbb{D} . \tag{7.1}$$

The map $I - P$ is the orthogonal projection of $L^2(\mathbb{D}, dA/\pi)$ onto $(L_a^2)^\perp$ [the orthogonal complement of L_a^2 in $L^2(\mathbb{D}, dA/\pi)$]. For a function $f \in L^\infty(\mathbb{D}, dA/\pi)$, the Hankel operator $H_f : L_a^2 \to (L_a^2)^\perp$ is defined by $(H_f)(g) = (I - P)(fg)$, $g \in L_a^2$.
Observe that the integral in (7.1) makes sense even when $g \in L^1(\mathbb{D}, dA/\pi)$, so we can extend the definition of P to $L^1(\mathbb{D}, dA/\pi)$. We want to consider Hankel operators H_f

for which the symbol f is in $L^1(\mathbb{D},dA/\pi)$. To do this restrict the domain of H_f to H^∞ and define H_f by

$$(H_f)(g) = (I - P)(fg),\ g \in H^\infty.$$

Using (7.1) for the product fg and for $g = P(g)$ we get the following formula for the Hankel operator H_f ; for $f \in L^1(\mathbb{D},dA/\pi)$ and $g \in H^\infty$ we have:

$$(H_f g)(z) = \int_{\mathbb{D}} \frac{f(z) - f(w)}{(1 - \overline{w} z)^2}\, g(w)\ dA(w)/\pi \ , \text{ for } z \in \mathbb{D} . \tag{7.2}$$

In [6] Sheldon Axler showed that for $f \in L_a^2$ the Hankel operator $H_{\overline{f}}$, densely defined on H^∞ with the L_a^2-norm, is bounded if and only if $f \in \mathfrak{B}$. It follows that for every f in L_a^2 the Hankel operator of each Möbius transform of $\overline{f}$ has norm equivalent to the norm of $H_{\overline{f}}$. In the next theorem we will show that Hankel operators transform in a unitarily equivalent way if the symbol is replaced by one of its Möbius transforms. This implies that the Hankel operator of each Möbius transform of a given function has the same norm as the Hankel operator of the given function (as densely defined operators on H^∞ with the L_a^2-norm).

Theorem 7.1 : *Let $f \in L^1(\mathbb{D},dA/\pi)$. For each $\lambda \in \mathbb{D}$ the Hankel operators H_f and $H_{f \circ \varphi_\lambda}$ are unitarily equivalent.*

More precisely, there exist unitary operators $U_1 : L_a^2 \to L_a^2$ and $U_2 : (L_a^2)^\perp \to (L_a^2)^\perp$ such that $U_1(H^\infty) \subset H^\infty$ and

$$U_2 \circ H_{f \circ \varphi_\lambda} = H_f \circ U_1 .$$

Proof:

Take $f \in L^1(\mathbb{D}, dA/\pi)$ and $g \in H^\infty$. Let $\lambda \in \mathbb{D}$. By (7.2) we have for $z \in \mathbb{D}$

$$(H_{f \circ \varphi_\lambda}(g))(z) = \int_{\mathbb{D}} \frac{f(\varphi_\lambda(z)) - f(\varphi_\lambda(w))}{(1 - \bar{w} z)^2}\, g(w)\; dA(w)/\pi \;. \tag{7.3}$$

In (7.3) make the substitution $u = \varphi_\lambda(w)$. Making use of identity (0.17) we have

$$\frac{1}{(1 - \overline{\varphi_\lambda(u)}\, z)^2} \frac{(1 - |\lambda|^2)^2}{|1 - \bar{\lambda} u|^4} = \frac{(1 - \bar{u}\lambda)^2}{(1 - \bar{\lambda} z)^2 (1 - \bar{u}\varphi_\lambda(z))^2} \frac{(1 - |\lambda|^2)^2}{|1 - \bar{\lambda} u|^4}$$

$$= \frac{(1 - |\lambda|^2)^2}{(1 - \bar{\lambda} z)^2} \frac{1}{(1 - \bar{u}\varphi_\lambda(z))^2 (1 - \bar{\lambda} u)^2} \;,$$

so that change-of-variable formula (0.22a) transforms (7.3) into

$$(H_{f \circ \varphi_\lambda}(g))(z) = \frac{(1 - |\lambda|^2)^2}{(1 - \bar{\lambda} z)^2} \int_{\mathbb{D}} \frac{f(\varphi_\lambda(z)) - f(u)}{(1 - \bar{u}\varphi_\lambda(z))^2} \frac{1}{(1 - \bar{\lambda} u)^2}\, g(\varphi_\lambda(u))\; dA(u)/\pi$$

$$= (1 - |\lambda|^2) k_\lambda(z) \int_{\mathbb{D}} \frac{f(\varphi_\lambda(z)) - f(u)}{(1 - \bar{u}\varphi_\lambda(z))^2} (1 - |\lambda|^2) k_\lambda(u)\; (g \circ \varphi_\lambda)(u)\; dA(u)/\pi$$

$$= (1 - |\lambda|^2) k_\lambda(z)\;\; H_f((1 - |\lambda|^2) k_\lambda\; (g \circ \varphi_\lambda))(\varphi_\lambda(z)) \,.$$

Thus we have

$$H_{f \circ \varphi_\lambda}(g) = (1 - |\lambda|^2) k_\lambda\;\; H_f((1 - |\lambda|^2) k_\lambda\; (g \circ \varphi_\lambda)) \circ \varphi_\lambda \;. \tag{7.4}$$

Define the operator $U : L^2(\mathbb{D},dA/\pi) \to L^2(\mathbb{D},dA/\pi)$ by

$$U(g) = (1 - |\lambda|^2)\, k_\lambda\, (g \circ \varphi_\lambda), \text{ for } g \in L^2(\mathbb{D},dA/\pi).$$

Since $(1 - |\lambda|^2)\, k_\lambda = - \varphi_\lambda'$, we have for $g \in L^2(\mathbb{D},dA/\pi)$

$$\| U(g) \|^2_{L^2(\mathbb{D},\, dA/\pi)} = \int_{\mathbb{D}} |(g \circ \varphi_\lambda)(z)|^2 |\varphi_\lambda'(z)|^2\, dA(z)/\pi = \| g \|^2_{L^2(\mathbb{D},\, dA/\pi)},$$

so that U is well-defined. For $g, h \in L^2(\mathbb{D},dA/\pi)$ we have

$$\langle U(g), h \rangle = \int_{\mathbb{D}} (1 - |\lambda|^2)\, k_\lambda(z)\, g(\varphi_\lambda(z))\, \overline{h(z)}\, dA(z)/\pi .$$

In the above integral make the substitution $u = \varphi_\lambda(z)$. We get

$$\langle U(g), h \rangle = \int_{\mathbb{D}} (1 - |\lambda|^2)\, k_\lambda(\varphi_\lambda(u))\, g(u)\, \overline{h(\varphi_\lambda(u))}\, |\varphi_\lambda'(u)|^2\, dA(u)/\pi .$$

Now using the identity (0.18) it is easy to verify that

$$k_\lambda(\varphi_\lambda(u))\, |\varphi_\lambda'(u)|^2 = \overline{k_\lambda(u)} ,$$

so that we have

$$\langle U(g), h \rangle = \int_{\mathbb{D}} g(u)\, \overline{(1-|\lambda|^2)\, k_\lambda(u)\, h(\varphi_\lambda(u))}\ dA(u)/\pi = \langle g, U(h) \rangle .$$

Hence U is a self-adjoint operator on $L^2(\mathbb{D}, dA/\pi)$.

Take $g \in L^2(\mathbb{D}, dA/\pi)$ and put $h = U(g)$. Differentiating the identity $\varphi_\lambda(\varphi_\lambda(z)) = z$ we see that for each $z \in \mathbb{D}$

$$(1-|\lambda|^2)^2\, k_\lambda(z)\, k_\lambda(\varphi_\lambda(z)) = 1 ,$$

so that

$$U(h)(z) = (1-|\lambda|^2)^2\, k_\lambda(z)\ k_\lambda(\varphi_\lambda(z))\ g(z) = g(z) ,$$

and thus $U \circ U = I$. Hence U is a unitary operator on $L^2(\mathbb{D}, dA/\pi)$.

Observe that $U(L_a^2) \subset L_a^2$, $U(H^\infty) \subset H^\infty$, and $U((L_a^2)^\perp) \subset (L_a^2)^\perp$. The first two of these inclusions are obvious from the definition of U. The last inclusion follows from the first since the operator U is self-adjoint. Let $U_1 : L_a^2 \to L_a^2$ and $U_2 : (L_a^2)^\perp \to (L_a^2)^\perp$ be the restrictions of U to L_a^2 and $(L_a^2)^\perp$ respectively. Then both U_1 and U_2 are unitary operators and $U_1(H^\infty) \subset H^\infty$. We claim that

$$U_2 \circ H_{f \circ \varphi_\lambda} = H_f \circ U_1 .$$

Let $g \in H^\infty$, then it follows from (7.4) that

$$H_{f \circ \varphi_\lambda}(g) = (1-|\lambda|^2)\, k_\lambda\ (H_f \circ U_1)(g) \circ \varphi_\lambda ,$$

so that

$$
\begin{aligned}
(U_2 \circ H_{f \circ \varphi_\lambda})(g) &= (1-|\lambda|^2)\, k_\lambda \,(H_{f \circ \varphi_\lambda}(g) \circ \varphi_\lambda) \\
&= (1-|\lambda|^2)^2\, k_\lambda \,(k_\lambda \circ \varphi_\lambda)\,(H_f \circ U_1)(g) \\
&= (H_f \circ U_1)(g) \ ,
\end{aligned}
$$

and our claim is verified. This completes the proof of Theorem 7.1. □

In order to state a corollary of the above theorem we need to introduce more notation. For a linear operator $S : L_a^2 \to (L_a^2)^{\perp}$, densely defined on H^∞, let $\| S \|$ denote the operator norm of S obtained by putting the L^2-norm on both the domain and the range of S, i.e.,

$$
\| S \| = sup \ \{ \| S\,(g) \|_{L^2(\mathbb{D}, dA/\pi)} : g \in H^\infty \text{ and } \| g \|_{L_a^2} \leq 1 \} \ .
$$

Let $\mathcal{L}(L_a^2, (L_a^2)^{\perp})$ denote the set of all bounded linear operator $T : L_a^2 \to (L_a^2)^{\perp}$, densely defined on H^∞. For $T \in \mathcal{L}(L_a^2, (L_a^2)^{\perp})$, define its singular numbers $s_n(T)$ by

$$
s_n(T) = inf \ \{ \| T - F \| : F \in \mathcal{L}\,(L_a^2, (L_a^2)^{\perp}) \text{ has rank at most } n \} \ ,
$$

for $n \in \mathbb{N}_0$. Note that $s_0(T) = \| T \|$. For $0 < p < \infty$ the Schatten-von Neumann class $\mathcal{C}^p$ is defined to be the set of all bounded linear operators $T : L_a^2 \to (L_a^2)^{\perp}$, densely defined on H^∞, for which

$$
\| T \|_{\mathcal{C}^p} = \left(\sum_{n=0}^{\infty} s_n(T)^p \right)^{1/p} < \infty \ .
$$

Let $\mathcal{C}^\infty$ denote the set of all bounded linear operators $T : L_a^2 \to (L_a^2)^\perp$, densely defined on H^∞, which are compact. Then clearly $\mathcal{C}^p \subset \mathcal{C}^\infty$ for $0 < p < \infty$. Take f in $L^1(\mathbb{D}, dA/\pi)$ and suppose that $\lambda \in \mathbb{D}$. Let U_1 and U_2 be the unitary operators of Theorem 7.1. If for an $n \in \mathbb{N}_0$ operator $F \in \mathcal{L}(L_a^2, (L_a^2)^\perp)$ has rank at most n, then also $U_2 \circ F \circ U_1^{-1}$ has rank at most n. Since U_1 and U_2 are unitary operators it follows that

$$\| H_{f \circ \varphi_\lambda} - F \| = \| H_f - U_2 \circ F \circ U_1^{-1} \| ,$$

which implies that for each $n \in \mathbb{N}_0$

$$s_n(H_{f \circ \varphi_\lambda}) = s_n(H_f) .$$

Thus we get the following corollary.

Corollary 7.2 *: Let $f \in L^1(\mathbb{D}, dA/\pi)$, and $0 < p \leq \infty$. If $H_f \in \mathcal{C}^p$, then for each $\lambda \in \mathbb{D}$*

$$H_{f \circ \varphi_\lambda} \in \mathcal{C}^p .$$

Before we proceed note that equation (7.4) can be used to obtain a formula for $H_f(k_\lambda)$. Since $(1 - |\lambda|^2)^2 k_\lambda (k_\lambda \circ \varphi_\lambda) = 1$, it follows from (7.4) that

$$H_{f \circ \varphi_\lambda}(k_\lambda) = k_\lambda \; H_f(1) \circ \varphi_\lambda = k_\lambda \; (f \circ \varphi_\lambda - P(f) \circ \varphi_\lambda) .$$

Replacing f by $f \circ \varphi_\lambda$ we get the formula

$$H_f(k_\lambda) = (f - P(f \circ \varphi_\lambda) \circ \varphi_\lambda)\, k_\lambda \,. \tag{7.5}$$

Let $1 < p < \infty$ and $-1 < \alpha < p - 1$. For a Lebesgue measurable function g on $\mathbb{D}$ let the weighted L^p-norm of g be defined by

$$\| g \|_{p,\alpha} = \left(\int_{\mathbb{D}} |g(z)|^p \, (1 - |z|^2)^\alpha \, dA(z)/\pi \right)^{1/p} .$$

For $f \in L_a^1$ think of $H_{\bar f}$ as an operator from H^∞ to the class of all functions on $\mathbb{D}$. The operator norm $\| H_{\bar f} \|_{p,\alpha}$ of $H_{\bar f}$ is obtained by putting the weighted L^p-norm $\| \,.\, \|_{p,\alpha}$ on both the domain and the range of $H_{\bar f}$, i.e.,

$$\| H_{\bar f} \|_{p,\alpha} = sup \; \{ \| H_{\bar f}\, g \|_{p,\alpha} : g \in H^\infty \text{ and } \| g \|_{p,\alpha} \le 1 \} .$$

Thus $\| H_{\bar f} \|_{2,0}$ coincides with our notation $\| H_{\bar f} \|$ used before Corollary 7.2. In [6] Sheldon Axler showed that the operator norm $\| H_{\bar f} \|$ and the Bloch norm $\| f \|_{\mathfrak{B}}$ are equivalent. In the following theorem we extend this result to the operator norms $\| H_{\bar f} \|_{p,\alpha}$.

Theorem 7.3 : *Let $1 < p < \infty$ and $-1 < \alpha < p - 1$. Then for $f \in L_a^1$ the Bloch norm $\| f \|_{\mathfrak{B}}$ and the operator norm $\| H_{\bar f} \|_{p,\alpha}$ are equivalent.*
In particular, $H_{\bar f}$ is bounded as an operator on H^∞ with the weighted L^p-norm $\| \,.\, \|_{p,\alpha}$ on both the domain and the range of $H_{\bar f}$ if and only if $f \in \mathfrak{B}$.

In [6] Sheldon Axler also showed that the Hankel operator $H_{\bar{f}}$ is compact if and only if f is in the little Bloch space $\mathfrak{B}_0$. That this result remains true when we view $H_{\bar{f}}$ as an operator on H^∞ with the weighted L^p-norm $\| \,.\, \|_{p,\alpha}$ on both the domain and the range of $H_{\bar{f}}$, is the content of the following theorem.

Theorem 7.4 : *Let $1 < p < \infty$ and $-1 < \alpha < p - 1$. Then for $f \in L_a^1$ the Hankel operator $H_{\bar{f}}$ is compact as an operator on H^∞ with the weighted L^p-norm $\| \,.\, \|_{p,\alpha}$ on both the domain and the range of $H_{\bar{f}}$ if and only if $f \in \mathfrak{B}_0$.*

For the proofs of Theorems 7.3 and 7.4 we need a series of lemmas. The first of these lemmas gives estimates on some integrals, and will be used in the next lemmas.

Lemma 7.5 : *Let $0 < \beta < \infty$. Then there exists a finite positive constant C (depending on β) such that for every $t \in (0, 1)$ we have*

$$(a) \quad \int_{-\pi}^{\pi} \frac{1}{|1 - t e^{i\theta}|^{2\beta}} \, d\theta \le C \; ((1-t)^{1-2\beta} + 1) \;, \text{ if } 0 < \beta < \tfrac{1}{2} \;; \qquad (7.6a)$$

$$(b) \quad \int_{-\pi}^{\pi} \frac{1}{|1 - t e^{i\theta}|^{2\beta}} \, d\theta \le C \; (1 + \log \frac{1}{1-t}) \;, \text{ if } \beta = \tfrac{1}{2} \;; \qquad (7.6b)$$

$$(c) \quad \int_{-\pi}^{\pi} \frac{1}{|1 - t e^{i\theta}|^{2\beta}} \, d\theta \le C \; \frac{1}{(1-t)^{2\beta-1}} \;, \text{ if } \tfrac{1}{2} < \beta < \infty. \qquad (7.6c)$$

Proof:

Take $0 < \beta < \infty$. It is elementary to show that

$$|1-te^{i\theta}|^2 = (1-t)^2 + 2t\,(1-\cos\theta)$$

$$= (1-t)^2 + 4t\,\sin^2\frac{\theta}{2}$$

$$\geq (1-t)^2 + 4t\,\frac{\theta^2}{\pi^2}\,.$$

Thus for $\frac{1}{2} \leq t < 1$ we have

$$|1-te^{i\theta}|^2 \geq (1-t)^2 + 2\,\frac{\theta^2}{\pi^2}\,,$$

and it follows that

$$\int_{-\pi}^{\pi} \frac{1}{|1-te^{i\theta}|^{2\beta}}\,d\theta \leq \int_{-\pi}^{\pi} \frac{1}{\left((1-t)^2 + 2\,\frac{\theta^2}{\pi^2}\right)^{\beta}}\,d\theta\,. \tag{7.7}$$

The substitution $\theta\sqrt{2} = \pi(1-t)\,x$ in the integral at the right of (7.7) yields

$$\int_{-\pi}^{\pi} \frac{1}{|1-te^{i\theta}|^{2\beta}}\,d\theta \leq \frac{\pi}{\sqrt{2}}\,\frac{1}{(1-t)^{2\beta-1}} \int_{-\sqrt{2}/(1-t)}^{\sqrt{2}/(1-t)} \frac{1}{(1+x^2)^{\beta}}\,dx\,. \tag{7.8}$$

Now we have to distinguish three cases.

Case (a): $0 < \beta < \frac{1}{2}$. We estimate the integral at the right of inequality (7.8).

$$\int_0^{\sqrt{2}/(1-t)} \frac{1}{(1+x^2)^\beta}\, dx \le \int_0^1 \frac{dx}{(1+x^2)^\beta} + \int_1^{\sqrt{2}/(1-t)} \frac{dx}{x^{2\beta}}$$

$$= \int_0^1 \frac{dx}{(1+x^2)^\beta} + \left(\frac{\sqrt{2}}{1-t}\right)^{1-2\beta} \cdot \frac{1}{1-2\beta}$$

$$\le K\left(1+\left(\frac{1}{1-t}\right)^{1-2\beta}\right).$$

Thus we have

$$\int_{-\sqrt{2}/(1-t)}^{\sqrt{2}/(1-t)} \frac{1}{(1+x^2)^\beta}\, dx \le 2K\left(1+\left(\frac{1}{1-t}\right)^{1-2\beta}\right),$$

and with the help of (7.8) inequality (7.6a) follows immediately.

Case (b): $\beta = ½$. In this case

$$\int_1^{\sqrt{2}/(1-t)} \frac{dx}{x} = \log \sqrt{2} + \log \frac{1}{1-t},$$

so that the same estimates as in the previous case show that

$$\int_{-\sqrt{2}/(1-t)}^{\sqrt{2}/(1-t)} \frac{1}{(1+x^2)^\beta}\, dx \le 2K\ (1 + \log \frac{1}{1-t}),$$

which combined with (7.8) gives inequality (7.6b).

Case (c): $\frac{1}{2} < \beta < \infty$. It follows from (7.8) that

$$\int_{-\pi}^{\pi} \frac{1}{|1 - t e^{i\theta}|^{2\beta}} \, d\theta \le \frac{\pi}{\sqrt{2}} \frac{1}{(1-t)^{2\beta - 1}} \int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{\beta}} \, dx \ .$$

Since $\beta > \frac{1}{2}$ the improper integral in the above inequality is finite, and (7.6c) follows. This completes the proof of this lemma. □

The next lemma will play a crucial role in the proof of Theorem 7.3, where it will be used twice.

Lemma 7.6 : *Let* $0 < \alpha < 1$. *Then there exists a finite positive constant* C (*depending on* α) *such that for every analytic function* f *on* $\mathbb{D}$ *and for all* $z \in \mathbb{D}$:

$$\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar{w} z|^2} \frac{1}{(1 - |w|^2)^{\alpha}} \, dA(w) \ \le \ \frac{C}{(1 - |z|^2)^{\alpha}} \ \|f\|_{\mathfrak{B}} \ . \qquad (7.9)$$

Proof:

Take $0 < \alpha < 1$. Let f be an analytic function on $\mathbb{D}$. Fix a point $z \in \mathbb{D}$. In the integral at the left of (7.9) make the change of variables $\lambda = \varphi_z(w)$. We get

$$\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar{w} z|^2} \frac{1}{(1 - |w|^2)^{\alpha}} \, dA(w) \ =$$

$$= \frac{1}{(1-|z|^2)^\alpha} \int_{\mathbb{D}} \frac{|f(\varphi_z(\lambda)) - f(z)|}{|1-\bar{\lambda} z|^{2(1-\alpha)}} \frac{1}{(1-|\lambda|^2)^\alpha} \, dA(\lambda) \, .$$

Since $\|f \circ \varphi_z\|_{\mathfrak{B}} = \|f\|_{\mathfrak{B}}$ it suffices to show that there exists a finite positive constant C such that for every analytic function f on $\mathbb{D}$ and for all $z \in \mathbb{D}$:

$$\int_{\mathbb{D}} \frac{|f(\lambda) - f(0)|}{|1-\bar{\lambda} z|^{2(1-\alpha)}} \frac{1}{(1-|\lambda|^2)^\alpha} \, dA(\lambda) \leq C \;\; \|f\|_{\mathfrak{B}} \, . \tag{7.10}$$

Fix $f \in \mathfrak{B}$, and let $z \in \mathbb{D}$. Using that for every $\lambda \in \mathbb{D}$,

$$|f(\lambda) - f(0)| \leq \|f\|_{\mathfrak{B}} \; log \, \frac{1}{1-|\lambda|} \, ,$$

the integral at the left of (7.10) is less than or equal to

$$\|f\|_{\mathfrak{B}} \int_{\mathbb{D}} log \, \Big(\frac{1}{1-|\lambda|}\Big) \frac{1}{|1-\bar{\lambda} z|^{2(1-\alpha)}} \frac{1}{(1-|\lambda|^2)^\alpha} \, dA(\lambda) \, ,$$

so it suffices to show that

$$\sup_{z \in \mathbb{D}} \int_{\mathbb{D}} log \, \Big(\frac{1}{1-|\lambda|}\Big) \frac{1}{|1-\bar{\lambda} z|^{2(1-\alpha)}} \frac{1}{(1-|\lambda|^2)^\alpha} \, dA(\lambda) < \infty \, . \tag{7.11}$$

It is easy to see that the integrals in (7.11) depend only on the modulus $\rho = |z|$ of z in the disk $\mathbb{D}$, so we have to show that there exists a finite positive constant C such that for all $\rho \in [0, 1)$ we have

$$\int_0^1 r \log\left(\frac{1}{1-r}\right) \frac{1}{(1-r^2)^\alpha} \left(\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2(1-\alpha)}}\, d\theta\right) dr \le C\,. \qquad (7.12)$$

Distinguish the following three cases.

Case (a): $\frac{1}{2} < \alpha < 1$. Applying Lemma 7.5 with $\beta = 1-\alpha$, so that $0 < \beta < \frac{1}{2}$, we have

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho\, e^{i\theta}|^{2(1-\alpha)}}\, d\theta \le C\,((1-r\rho)^{2\alpha-1} + 1) \le 2C\,,$$

and (7.12) follows immediately.

Case (b): $\alpha = \frac{1}{2}$. Then Lemma 7.5 gives us that

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2(1-\alpha)}}\, d\theta \le C\,\left(1 + \log \frac{1}{1-r\rho}\right) \le C\,\left(1 + \log \frac{1}{1-r}\right),$$

from which (7.12) follows easily.

Case (c): $0 < \alpha < \frac{1}{2}$. Applying Lemma 7.5 with $\beta = 1-\alpha$, so that $\frac{1}{2} < \beta < 1$, we have

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho\, e^{i\theta}|^{2(1-\alpha)}}\, d\theta \le C\, \frac{1}{(1-r\rho)^{1-2\alpha}} \le C\, \frac{1}{(1-r)^{1-2\alpha}}\,,$$

which implies that

$$\int_0^1 r \log\left(\frac{1}{1-r}\right) \frac{1}{(1-r^2)^\alpha} \left(\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2(1-\alpha)}} \, d\theta \right) dr \le$$

$$\le C \int_0^1 \log\left(\frac{1}{1-r}\right) \frac{1}{(1-r)^{1-\alpha}} \, dr = C/\alpha^2,$$

and (7.12) follows immediately. This completes the proof of this lemma. □

We will need estimates on the weighted L^p-norms of the reproducing kernels k_λ. These are obtained in the following lemma.

Lemma 7.7 *: Let* $1 < p < \infty$ *and* $-1 < \alpha < p - 1$. *Then there exists a finite positive constant C such that for every* $\lambda \in \mathbb{D}$ *:*

$$\frac{1}{C} \frac{1}{(1-|\lambda|^2)^{2p-\alpha-2}} \le \| k_\lambda \|_{p,\alpha}^p \le C \frac{1}{(1-|\lambda|^2)^{2p-\alpha-2}}.$$

Proof:

Take $1 < p < \infty$ and let $-1 < \alpha < p - 1$. In the formula

$$\| k_\lambda \|_{p,\alpha}^p = \int_{\mathbb{D}} |k_\lambda(z)|^p (1-|z|^2)^\alpha \, dA(z)/\pi$$

make the substitution $z = \varphi_\lambda(w)$. This yields the formula

$$\| k_\lambda \|_{p,\alpha}^{p} = \frac{1}{(1-|\lambda|^2)^{2p-\alpha-2}} \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\bar{\lambda} w|^{2(\alpha-p+2)}} \, dA(w)/\pi ,$$

so it suffices to show the following two statements:

$$\sup_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\bar{\lambda} w|^{2(\alpha-p+2)}} \, dA(w)/\pi < \infty ; \tag{7.13a}$$

$$\inf_{\lambda \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\bar{\lambda} w|^{2(\alpha-p+2)}} \, dA(w)/\pi > 0 . \tag{7.13b}$$

As in the proof of Lemma 7.6 we have for $\rho = |\lambda|$

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^\alpha}{|1-\bar{\lambda} w|^{2(\alpha-p+2)}} \, dA(w)/\pi =$$

$$= \int_0^1 r\,(1-r^2)^\alpha \left(\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2(\alpha-p+2)}} \, d\theta \right) dr \tag{7.14}$$

Put $\beta = \alpha - p + 2$. Then it follows from $\alpha < p-1$ that $\beta < 1$. It is however not ruled out that β is negative. We will first show that statement (7.13a) holds. Distinguish the following four cases.

Case 1: $\frac{1}{2} < \beta < 1$. Then by (7.6c)

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2\beta}} \, d\theta \le C \frac{1}{(1-r\rho)^{2\beta-1}} \le C \frac{1}{(1-r)^{2\beta-1}} ,$$

so that the integral in (7.14) is less than or equal to

$$C \int_0^1 r\,(1-r^2)^\alpha \frac{1}{(1-r)^{2\beta-1}}\,dr \le C\,2^{2\beta-1} \int_0^1 r\,(1-r^2)^{\alpha-2\beta+1}\,dr$$

$$\le C\,2^{2\beta-1} \frac{1}{2(\alpha-2\beta+2)} \le \frac{2C}{p-1},$$

since $\alpha - 2\beta + 2 = 2(p-1) - \alpha > p-1$, and $2\beta - 1 < 1$. This proves (7.13a) for this case.

Case 2: $\beta = ½$. Then by (7.6b) we have

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2\beta}}\,d\theta \le C\,(1 + log \frac{1}{1-r\rho}) \le C\,(1 + log \frac{1}{1-r}),$$

and it follows that the integral in (7.14) is less than or equal

$$C \int_0^1 r\,(1-r^2)^\alpha\,(1 + log \frac{1}{1-r})\,dr < \infty,$$

since $\alpha > -1$, and (7.13a) follows.

Case 3: $0 < \beta < ½$. Then by (7.6a)

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2\beta}}\,d\theta \le C\,((1-r\rho)^{1-2\beta} + 1) \le C\,((1-r)^{1-2\beta} + 1),$$

so that the integral in (7.14) is less than or equal

$$C \int_0^1 r\,(1-r^2)^{\alpha}\,((1-r)^{1-2\beta}+1)\,dr \le C\,\left(\frac{1}{2(\alpha-2\beta+2)}+\frac{1}{2(\alpha+1)}\right)$$

$$\le \frac{C\,(p+\alpha)}{2(p-1)(\alpha+1)}\,,$$

since $\alpha - 2\beta + 2 > p - 1$, and it follows that (7.13a) holds.

Case 4: $\beta \le 0$. Then the trivial estimate

$$\int_{-\pi}^{\pi} \frac{1}{|1-r\rho e^{i\theta}|^{2\beta}}\,d\theta \le \frac{2\pi}{2^{2\beta}}\,,$$

shows that the integral in (7.14) is bounded uniformly in $\rho \in [0, 1)$. This completes the proof of statement (7.13a). To show that statement (7.13b) holds we need to consider only two cases.

Case 1: $0 < \beta < 1$. Then the trivial inequality

$$\frac{1}{|1-\bar{\lambda} w|^{2\beta}} \ge \frac{1}{2^{2\beta}}\,,$$

implies that statement (7.13b) is true.

Case 2: $\beta \le 0$. Now using the inequalities

$$\frac{1}{|1-\bar{\lambda} w|^{2\beta}} \geq \frac{1}{(1-|w|)^{2\beta}} \geq \frac{2^{2\beta}}{(1-|w|^2)^{2\beta}} ,$$

we get

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha}}{|1-\bar{\lambda} w|^{2\beta}} \, dA(w)/\pi \;\geq\; 2^{2\beta} \int_{\mathbb{D}} (1-|w|^2)^{\alpha-2\beta} \, dA(w)/\pi \; .$$

Since $\alpha < p - 1$ and $p > 1$ we have that $\alpha - 2\beta + 1 = 2p - \alpha - 3 > p - 2 > -1$, so that the last integral converges to a positive number. Thus statement (7.13b) is proved and the proof of this lemma is complete. □

Proof of Theorem 7.3:

Let $1 < p < \infty$ and $-1 < \alpha < p - 1$. Take $f \in L_a^1$. Fix $g \in H^{\infty}$, and let $z \in \mathbb{D}$. Then it follows from (7.2) that

$$|(H_{\bar{f}}\, g)(z)| \leq \int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1-\bar{w} z|^2} \, |g(w)| \, dA(w)/\pi \; .$$

It is easily verified that the inequalities $1 < p < \infty$ and $-1 < \alpha < p - 1$ imply that we must have $(0, p-1) \cap (\alpha, \alpha+1) \neq \varnothing$. Choose $\gamma > 0$ such that $p\gamma \in (0, p-1) \cap (\alpha, \alpha+1)$; then clearly $\gamma < 1$. Writing p' for the conjugate index of p, i.e., $p' = p/(p-1)$, it follows immediately from $0 < p\gamma < p - 1$ that $0 < p'\gamma < 1$.

Applying Hölder's inequality we get

$$|(H_{\bar f}\, g)(z)| \le \int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar w z|^2} |g(w)| \frac{1}{(1-|w|^2)^{\gamma}} (1-|w|^2)^{\gamma}\, dA(w)/\pi$$

$$\le \left(\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar w z|^2} \frac{1}{(1-|w|^2)^{p'\gamma}}\, dA(w)/\pi \right)^{1/p'} \times$$

$$\times \left(\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar w z|^2} |g(w)|^p (1-|w|^2)^{p\gamma}\, dA(w)/\pi \right)^{1/p} . \qquad (7.15)$$

By Lemma 7.6 there is a finite positive constant C_1 such that

$$\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar w z|^2} \frac{1}{(1-|w|^2)^{p'\gamma}}\, dA(w)/\pi \le \frac{C_1}{(1-|z|^2)^{p'\gamma}} \|f\|_{\mathfrak{B}} .$$

Using this estimate in (7.15) and taking p-th powers it follows that

$$|(H_{\bar f}\, g)(z)|^p \le \frac{C_1^{p-1}}{(1-|z|^2)^{p\gamma}} \|f\|_{\mathfrak{B}}^{p-1} \int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar w z|^2} |g(w)|^p (1-|w|^2)^{p\gamma}\, dA(w)/\pi .$$

Integrating the above inequality and applying Fubini's Theorem we get

$$\|H_{\bar f}\, g\|_{p,\alpha}^p = \int_{\mathbb{D}} |(H_{\bar f}\, g)(z)|^p (1-|z|^2)^{\alpha}\, dA(z)/\pi \le C_1^{p-1} \|f\|_{\mathfrak{B}}^{p-1} \times$$

$$\times \int_{\mathbb{D}} |g(w)|^p (1-|w|^2)^{p\gamma} \left(\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \bar w z|^2} \frac{1}{(1-|z|^2)^{p\gamma-\alpha}}\, dA(z)/\pi \right) dA(w)/\pi . \qquad (7.16)$$

By the choice of γ we have that $0 < p\gamma - \alpha < 1$, so that we can apply Lemma 7.6 once more: there exists a finite positive constant C_2 such that

$$\int_{\mathbb{D}} \frac{|f(w) - f(z)|}{|1 - \overline{w} z|^2} \frac{1}{(1 - |z|^2)^{p\gamma - \alpha}} \, dA(z)/\pi \leq \frac{C_2}{(1 - |w|^2)^{p\gamma - \alpha}} \|f\|_{\mathfrak{B}} .$$

Therefore it follows from (7.16) that

$$\|H_{\bar{f}}\, g\|_{p,\alpha}^{p} \leq C_1^{p-1} C_2 \|f\|_{\mathfrak{B}}^{p} \int_{\mathbb{D}} |g(w)|^p (1 - |w|^2)^{\alpha} \, dA(w)/\pi ,$$

which implies that there exists a finite positive constant C for which

$$\|H_{\bar{f}}\|_{p,\alpha} \leq C \|f\|_{\mathfrak{B}} .$$

For the converse, fix $0 < r < 1$, and for $\lambda \in \mathbb{D}$ consider the reproducing kernels k_λ. Since f is analytic we have $P(\bar{f} \circ \varphi_\lambda) = \bar{f}(\lambda)$, and (7.5) gives us $H_{\bar{f}} k_\lambda = (\bar{f} - \bar{f}(\lambda))\, k_\lambda$, so that

$$\int_{D(\lambda,r)} |(H_{\bar{f}}\, k_\lambda)(z)|^p (1 - |z|^2)^{\alpha} \, dA(z)/\pi = \int_{D(\lambda,r)} |f(z) - f(\lambda)|^p \, |k_\lambda(z)|^p (1 - |z|^2)^{\alpha} \, dA(z)/\pi . \tag{7.17}$$

Take $z \in D(\lambda,r)$, then z can be written as $z = \varphi_\lambda(u)$ where $|u| < r$. Using the identity (0.18) it is easy to verify that

$$k_\lambda(z) = \frac{(1-\bar{\lambda}u)^2}{(1-|\lambda|^2)^2} .$$

We also have

$$1-|z|^2 = \frac{(1-|\lambda|^2)(1-|u|^2)}{|1-\bar{\lambda}u|^2} ,$$

so that

$$|k_\lambda(z)|^p \, (1-|z|^2)^\alpha = \frac{1}{(1-|\lambda|^2)^{2p-\alpha}} \, (1-|u|^2)^\alpha \, |1-\bar{\lambda}u|^{2p-2\alpha}$$

$$\geq \frac{1}{(1-|\lambda|^2)^{2p-\alpha}} \, (1-|u|)^{2p-\alpha} \, (1+|u|)^\alpha .$$

Since $2p - \alpha > \alpha + 2$ we have that $(1-|u|)^{2p-\alpha} \geq (1-|u|)^{\alpha+2}$ and it follows from the above inequality that there is a number $\delta > 0$ independent of $\lambda \in \mathbb{D}$ [in fact, we can take $\delta = min\,\{(1-r), (1-r)^{\alpha+1}\}$], such that for all $z \in D(\lambda,r)$

$$|k_\lambda(z)|^p \, (1-|z|^2)^\alpha \geq \frac{\delta}{(1-|\lambda|^2)^{2p-\alpha}} . \tag{7.18}$$

Combining (7.17) and (7.18) we have

$$\int_{D(\lambda,r)} |(H_{\bar{f}}\, k_\lambda)(z)|^p \, (1-|z|^2)^\alpha \, dA(z)/\pi \;\geq\; \frac{\delta}{(1-|\lambda|^2)^{2p-\alpha}} \int_{D(\lambda,r)} |f(z)-f(\lambda)|^p \, dA(z)/\pi ,$$

which, together with formula (0.21) for the normalized Lebesgue area of a pseudo-hyperbolic disk, gives us that

$$\frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f(z)-f(\lambda)|^p\, dA(z)/\pi \le$$

$$\le \frac{1}{r^2\delta}\,(1-|\lambda|^2)^{2p-\alpha-2}\int_{D(\lambda,r)} |(H_{\bar f}k_\lambda)(z)|^p\,(1-|z|^2)^\alpha\, dA(z)/\pi$$

$$\le \frac{1}{r^2\delta}\,(1-|\lambda|^2)^{2p-\alpha-2}\,\|H_{\bar f}k_\lambda\|^p_{p,\alpha} \tag{7.19}$$

Now, making use of Lemma 7.7 and the definition of the operator norm $\|H_{\bar f}\|_{p,\alpha}$ we have

$$\|H_{\bar f}k_\lambda\|^p_{p,\alpha} \le \|H_{\bar f}\|^p_{p,\alpha}\,\frac{C}{(1-|\lambda|^2)^{2p-\alpha-2}},$$

and it follows from (7.19) that

$$\frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f(z)-f(\lambda)|^p\, dA(z)/\pi \le \frac{C}{r^2\delta}\,\|H_{\bar f}\|^p_{p,\alpha},$$

and by Theorem 1.1 there exists a finite positive constant C' such that

$$\|f\|_{\mathfrak{B}} \le C'\,\|H_{\bar f}\|_{p,\alpha},$$

completing the proof of this theorem. □

Proof of Theorem 7.4 :

Let $1 < p < \infty$ and $-1 < \alpha < p-1$. Take $f \in L_a^1$.

By Theorem 7.3 there is a finite positive constant C such that

$$\| H_{\bar{f}} - H_{\bar{f}_t} \|_{p,\alpha} \leq C \, \| f - f_t \|_{\mathfrak{B}} \, .$$

If $f \in \mathfrak{B}_0$, then it follows from the above inequality that $H_{\bar{f}_t} \to H_{\bar{f}}$ in operator norm. Since each of the operators $H_{\bar{f}_t}$ is compact, it follows that $H_{\bar{f}}$ is compact.

For the converse, suppose that $H_{\bar{f}}$ is compact. For each $\lambda \in \mathbb{D}$ let n_λ be function

$$n_\lambda = \frac{k_\lambda}{\| k_\lambda \|_{p,\alpha}} \, .$$

Put $\omega_\alpha(z) = (1 - |z|^2)^\alpha$, for $z \in \mathbb{D}$. For any $s \in [1, \infty)$ let $L^{s,\alpha}$ denote the measure space $L^s(\mathbb{D}, \omega_\alpha \, dA/\pi)$. If p' denotes the conjugate index of p , i.e., $p' = p/(p - 1)$, then the dual of the space $L^{p,\alpha}$ can be identified with $L^{p',\alpha}$; the pairing is given by

$$(f, g) = \int_{\mathbb{D}} f(z) \, \overline{g(z)} \, (1 - |z|^2)^\alpha \, dA(z)/\pi \, ,$$

for $f \in L^{p,\alpha}$, $g \in L^{p',\alpha}$.

We claim that $n_\lambda \to 0$ weakly in $L^{p,\alpha}$ as $|\lambda| \to 1^-$.

That the set $\{g\omega_\alpha^{-1/p'} : g \in L^\infty(\mathbb{D}, dA/\pi)\}$ is dense in $L^{p',\alpha}$ follows easily from the fact that $L^\infty(\mathbb{D}, dA/\pi)$ is dense in $L^{p'}(\mathbb{D}, dA/\pi)$. Since $\{n_\lambda : \lambda \in \mathbb{D}\}$ is norm-bounded in $L^{p,\alpha}$ it suffices to show that $(n_\lambda, g\omega_\alpha^{-1/p'}) \to 0$ as $|\lambda| \to 1^-$, for all $g \in L^\infty(\mathbb{D}, dA/\pi)$. Fix a $g \in L^\infty(\mathbb{D}, dA/\pi)$. Noting that $\omega_\alpha^{-1/p'} \omega_\alpha = \omega_{\alpha/p}$ we have the estimate

$$|(k_\lambda, g\, \omega_\alpha^{-1/p'})| \leq \| g \|_\infty \int_{\mathbb{D}} |k_\lambda| \omega_{\alpha/p} \, dA/\pi = \| g \|_\infty \, \| k_\lambda \|_{1,\alpha/p} \, . \quad (7.20)$$

We would like to estimate the norm $\| k_\lambda \|_{1,\alpha/p}$, but Lemma 7.7 does not apply to this norm. The idea is to use Lemma 7.7 with an index slightly bigger than 1, but not too big, so that the necessary estimates work out. It is easy to see that we can choose a number q such that $1 < q < p$ and $-1 < q\alpha/p < q - 1$. Now, by Lemma 7.7 there exists a finite positive constant C' such that for every $\lambda \in \mathbb{D}$

$$\| k_\lambda \|_{q,q\alpha/p} \leq C' \frac{1}{(1-|\lambda|^2)^{2-\alpha/p-2/q}} .$$

By Hölder's inequality $\| k_\lambda \|_{1,\alpha/p} \leq \| k_\lambda \|_{q,q\alpha/p}$, so that we have

$$|(k_\lambda, g\,\omega_\alpha^{-1/p'})| \leq C' \| g \|_\infty \frac{1}{(1-|\lambda|^2)^{2-\alpha/p-2/q}} . \tag{7.21}$$

By Lemma 7.7 there is a finite positive constant C such that for every $\lambda \in \mathbb{D}$

$$\frac{1}{\| k_\lambda \|_{p,\alpha}} \leq C \, (1-|\lambda|^2)^{2-\alpha/p-2/p} . \tag{7.22}$$

Combining (7.21) and (7.22) we get

$$|(n_\lambda, g\,\omega_\alpha^{-1/p'})| \leq C C' \| g \|_\infty (1-|\lambda|^2)^{\frac{2}{q}-\frac{2}{p}},$$

which implies that $(n_\lambda, g\omega_\alpha^{-1/p'}) \to 0$ as $|\lambda| \to 1^-$, and the claim is proved.

Now, since $H_{\bar{f}}$ is a compact operator and $n_\lambda \to 0$ weakly in $L^{p,\alpha}$ as $|\lambda| \to 1^-$, we must have $\| H_{\bar{f}}\, n_\lambda \|_{p,\alpha} \to 0$ as $|\lambda| \to 1^-$. It follows from (7.18) and Lemma 7.7 that

$$\frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f(z)-f(\lambda)|^p \, dA(z)/\pi \le \frac{C}{r^2\delta} \, \| H_{\bar f}\, n_\lambda \|_{p,\alpha}^p ,$$

therefore we have

$$\frac{1}{|D(\lambda,r)|}\int_{D(\lambda,r)} |f(z)-f(\lambda)|^p \, dA(z)/\pi \;\to 0 \text{ as } |\lambda| \to 1^- ,$$

and by Theorem 1.2 it follows that $f \in \mathfrak{B}_0$. □

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] J.M. Anderson, "Bloch functions: The basic theory," pages 1-17 in *Operators and Function Theory,* edited by S.C. Power, Reidel Publishing Company, Dordrecht, 1985.

[2] J.M. Anderson, J. Clunie and Ch. Pommerenke, "On Bloch functions and normal functions," Journal für die Reine und Angewandte Mathematik 270 (1974), 12-37.

[3] J. Arazy and S.D. Fisher, "Some aspects of the minimal, Möbius-invariant space of analytic functions on the unit disc," pages 24-44 in *Interpolation Spaces and Allied Topics in Analysis*, edited by M. Cwikel and J. Peetre, Lecture Notes in Mathematics, Vol. 1070, Springer-Verlag, Berlin-New York, 1984.

[4] J. Arazy and S.D. Fisher, "The uniqueness of the Dirichlet space among Möbius-invariant Hilbert spaces," Illinois Journal of Mathematics 29 (1985), 449-462.

[5] J. Arazy, S.D. Fisher and J. Peetre, "Möbius invariant function spaces," Journal für die Reine und Angewandte Mathematik 363 (1985), 110-145.

[6] S. Axler, "The Bergman Space, The Bloch Space, and Commutators of Multiplication Operators," Duke Mathematical Journal 53 (1986), 315-332.

[7] S. Axler, "Bergman spaces and their operators," to appear.

[8] A. Baernstein, "Analytic Functions of Bounded Mean Oscillation," pages 3-36 in *Aspects of Contemporary Complex Analysis*, edited by D.A. Brannan and J.G. Clunie, Academic Press, London, 1980.

[9] A. Bloch, "Les théorèmes de M. Valiron sur les fonctions entières et la théorie de l'uniformisation," Annales de la Faculté des Sciences de l'Université de Toulouse, 3^{e} Série, 17 (1925), 1-22.

[10] L. Brown and A. L. Shields, "Cyclic vectors in the Dirichlet space," Transactions of the American Mathematical Society 285 (1984), 269-304.

[11] J.A. Cima, "The basic properties of Bloch functions," International Journal of Mathematics and Mathematical Sciences 2 (1979), 369-413.

[12] P.L. Duren, *Theory of H^p spaces*, Academic Press, New York, 1970.

[13] P.L. Duren, B.W. Romberg and A.L. Shields, "Linear functionals on H^p spaces with $0 < p < 1$," Journal für die Reine und Angewandte Mathematik 238 (1969), 32-60.

[14] J.B. Garnett, *Bounded Analytic Functions*, Academic Press, New York, 1981.

[15] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators,* Translations of Mathematical Monograph, Vol. 18, American Mathematical Society, Providence, Rhode Island, 1969.

[16] A. Heilper, "The zeros of functions in Nevanlinna's area class," Israel Journal of Mathematics, Vol. 34 (1979), 1-11.

[17] E. Hille, *Analytic function theory*, Vol.II, Ginn and Company, New York, 1962.

[18] F. John and L. Nirenberg, "On functions of bounded mean oscillation," Communications on Pure and Applied Mathematics 14 (1961), 415-426.

[19] H.O. Kim, "Derivatives of Blaschke Products," Pacific Journal of Mathematics 114 (1984), 175-190.

[20] E. Landau, "Über die Blochsche Konstante und zwei verwandte Weltkonstanten," Mathematische Zeitschrift 30 (1929), 608-634.

[21] O. Lehto, "A majorant principle in the theory of functions," Mathematica Scandinavica 1 (1953), 1-17.

[22] J.E. Littlewood and R.E.A.C. Paley, "Theorems on Fourier Series and Power Series (II)," Proceedings of the London Mathematical Society 42 (1936), 52-89.

[23] D.H. Luecking, "A technique for characterizing Carleson measures on Bergman spaces," Proceedings of the American Mathematical Society 87 (1983), 656-660.

[24] D.H. Luecking, "Forward and reverse Carleson inequalities for functions in Bergman spaces and their derivatives," American Journal of Mathematics 107 (1985), 85-111.

[25] D.H. Luecking, "Representation and duality in weighted spaces of analytic functions," Indiana University Mathematics Journal 34 (1985), 319-336, and Erratum in same journal, Vol. 35 (1986), 927-928.

[26] V.V. Peller, "Hankel operators of class $\mathcal{C}^p$ and their applications (rational approximation, Gaussian processes, the problem of majorizing operators)," Mathematics of the USSR-Sbornik 41 (1982), 443-479.

[27] Ch. Pommerenke, "On Bloch functions," Journal of the London Mathematical Society (2), 2 (1970), 689-695.

[28] Ch. Pommerenke, *Univalent functions*, Vandenhoek & Ruprecht, Göttingen, 1975.

[29] Ch. Pommerenke, "Schlichte Funktionen und analytische Funktionen von beschränkter mittlerer Oszillation," Commentarii Mathematici Helvetici 52 (1977), 591-602.

[30] Ch. Pommerenke, "On univalent functions, Bloch functions and VMOA," Mathematische Annalen 236 (1978), 199-208.

[31] L. A. Rubel and R. M. Timoney, "An extremal property of the Bloch space," Proceedings of the American Mathematical Society 75 (1979), 45-49.

[32] D. Sarason, "Functions of vanishing mean oscillation," Transactions of the American Mathematical Society, 207 (1975), 391-405.

[33] D. Sarason, *Function Theory on the Unit Circle*, Lecture notes of Virginia Polytechnic Institute and State University, Blacksburg, Virginia, 1978.

[34] W. Seidel and J.L. Walsh, "On the derivatives of functions analytic in the unit circle and their radii of univalence and p-valence," Transactions of the American Mathematical Society, 52 (1942), 128-216.

[35] A. L. Shields, "Cyclic vectors in Banach spaces of analytic functions," pages 315-349 in *Operators and Function Theory*, edited by S.C. Power, Reidel Publishing Company, Dordrecht, 1985.

[36] V.S. Zakharyan, "On the growth of functions with finite Dirichlet integral, " (in Russian) page 98 in *Complex Analysis and its Applications to Partial Differential Equations*, edited by Helmuth Malonek and Volker Pluschke, Kongreß- und Tagungsberichte der Martin-Luther-Universität, Halle-Wittenberg, Halle, 1984.