# A CLINICAL INVESTIGATION OF THE UNDERSTANDING OF EXPONENTS BY REMEDIAL ALGEBRA STUDENTS AT A FOUR YEAR COLLEGE

By

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ABSTRACT

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The purpose of the study was to gain information about the manner in which successful and unsuccessful college students in beginning remedial algebra and intermediate remedial algebra view and apply the algebraic processes relevant to the concept and related principles of exponent.

The theoretical aspect of the investigation took the form of a synthesis of the theories of Brunner, Skemp, Gagné, and Krutetskii, combined with the cognitive modes suggested by Erlwanger to develop a model for use in the determination of each student's "understanding" in terms of Skemp's "relational" and "instrumental", and Bruner's "iconic" and "symbolic" modes of knowledge representation. A "thinking-aloud" interviewing technique was used in an effort to rate each of fourteen remedial algebra student's thought processes as they solved a pre-determined set of problems pertaining to exponents. The students were designated as using one or more of the following levels of understanding: Instrumental-Iconic, Instrumental-

Symbolic, Relational-Iconic, or Relational-Symbolic.

It was found that both successful and unsuccessful students operate primarily at the Instrumental-Symbolic level, with the successful intermediate algebra students occasionally using a Relational-Symbolic mode, and only rarely a Relational-Iconic mode. Neither successful nor unsuccessful students used numerical imagery to any extent even when suggested by the interviewer. The major source of difficulty appeared to be in the transition from a purely arithmetical problem to an algebraic one involving a variable exponent and a variable base. It was determined that the beginning algebra students, both successful and unsuccessful, had an "understanding" only of positive integer exponents with integer bases. They had virtually no understanding of the exponential properties. The intermediate algebra students, additionally, had a limited instrumental understanding of negative integer, zero, and rational exponents as well as the exponential properties.

Suggestions for future research are indicated. Also, an extensive bibliography of learning theory as well as imagery research is provided.

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TABLE OF CONTENTS

Chapter Page

# CHAPTER I

## THE PROBLEM AND ITS BACKGROUND

### Introduction

Why do some students, after completing one, two, or more years of high school mathematics, including at least one algebra course, enroll in college, only to find that due to a low score on an entrance examination he/she has been assigned to a "remedial" algebra course? Once in the course, why do they continue to make errors on elementary material such as stating $(x^2)^3 = x^5$; $x^2 \cdot x^3 = x^6$; and $x^2 + x^3$, at various times to equal either $x^5$ or $x^6$? It would be tempting to classify such errors as "careless" due to lack of attention, or lack of concentration and effort on assigned problems. However, as the literature on this topic indicates, this answer is far too simplistic.

In addition to the above concerns, why do some remedial algebra students, when confronted with the dual tasks of $x^2 \cdot x^3$, and $(x^2)^3$, adopt the strategy of falling back to a "numerical imagery" mode and substitute a number such as 2 for x, and use this answer to aid in the determination of the correct result for the initial problem? Why do other students never adopt this procedure and depend entirely on their memory of the appropriate "law of exponents"? Do some students lack practice in the use of such imagery?

How does the understanding of the concept of exponent differ among students that use numerical imagery and those that do not? Do some students lack an understanding of the use of variable? Do some students lack an understanding of the concept of exponent?

This study was undertaken with the assumption that difficulty with the concept of exponent is pervasive among students in remedial algebra courses at the college level. Due to the spiral nature of mathematical learning, this study was undertaken with the further assumption that learner difficulties which occur in the more advanced topics of radicals and logarithms are due to a lack of understanding of the concept of exponent. Thus, the remedial algebra students' understanding of the concept of exponent is an area that is indeed worthy of investigation. Furthermore, this study was undertaken with the belief that learner difficulties with exponents for the most part, are not due to carelessness or lack of effort, but are attributable to more deep-rooted cognitive confusion, that can be amended by appropriate teaching and learning strategies.

It should be noted that this study was intended to investigate the thought processes of students with respect to the conceptual difficulties manifested in the learning of remedial algebra. In this regard, the topic of exponents was chosen due to the fact that inherent in this topic are the ideas of variable, equality, definitions, and generalizations. However, exponent is merely an exemplar of many equally appropriate topics that could have been chosen (for example, factoring).

Robert Davis (1967) has stated:

> The entire traditional ninth-grade algebra course is based on a sequence of tricks to get children to write down on paper what appear to be correct answers, although the student more often than not does not know what it all means, if anything (p. 16).

Evidence that college and university faculties must deal with students that lack an understanding of fundamental algebraic concepts that were previously assumed to be assimilated in a ninth-grade algebra course is strikingly presented by Keimig (1983) in a report on academic standards. She notes:

> The demographic depression and the prevailing mood of decline, diminished resources, and threatened retrenchment are new, at least to this generation of faculty. So are the kinds of students new to many institutions that have altered their admissions practices and curricula, as most institutions have done (p. 1).

Citing a report by Roueche (1981), Keimig (1983) elaborates on the characteristics of students entering college today:

> The average high school graduate today has a "B" average over four years of high school, yet reads at the eight-grade level, a loss of two grade levels in the last 10 years (p. 51).

Akst (1981), explaining that arithmetic and algebra have traditionally been considered primary or secondary school topics, states:

> Yet in 1980 there were more than 600,000 students enrolled in college remedial or developmental math courses covering precisely this content. To aid these students, twice as numerous as they had been ten years, earlier, virtually all campuses have established basic math programs... (p. 1).

Killian's (1980) research (cited in Keimig) indicates that fifty percent of freshmen are concrete thinkers with respect to Piaget's developmental stages (p. 46). Clearly, the changing nature of college enrollments will challenge the college mathematics professionals.

One of the underlying assumptions pertinent to today's higher educational system is that there is great diversity of intelligence within the student population. This intelligence factor is generally seen as coinciding with an achievement factor. That is, when great differences in performance are noted between individual students, it is expected that there is a like difference in intelligence. Fifty years ago only the most "capable" students were admitted to college in the United States. Today's college students appear to be more variable in terms of measurable attributes of background and ability. However, while admissions policies have become increasingly liberal, traditional expectations and attitudes toward student achievement persist. With regard to the study of mathematics, mathematics faculty frequently and implicitly divide the students into two groups: those who can "understand" theory, and those who must limit themselves to step-by-step procedures that depend primarily on rote-learning processes. It is difficult to determine if traditional assumptions are in fact valid or merely a self-fulfilling prophecy (Hackworth, 1981, pp. 51-53).

Research does exist to indicate factors other than intelligence play a role in achievement. It has been

conjectured that "cognitive entry skills" are accountable for approximately 50 percent of the student's variance from the norm with respect to achievement using conventional instruction techniques (Bloom, 1976) (reported in Hackworth, 1981, pp. 49-50). In addition, Bloom's research indicates that approximately 20 percent of variance in end of course achievement is based on "affective entry skills". Thus, according to Bloom, nearly 70 percent of end of course achievement can be attributed to factors which, at least partly, are deficient for the remedial mathematics students. However, there is some hope for both students and faculty. Bloom (1976) asserts that 95 percent of students can achieve mastery (90 percent) of the instructional objectives when provided with appropriate instruction (Hackworth, 1981 p. 51). In addition, Carroll (1963), cited in Hackworth (1981), suggests that based on his research, that if speed of learning is disregarded, there is a relatively small variation in achievement level within the population (p. 50).

One of the key difficulties with most remedial instructional programs is that they are based on knowledqe of the students "understanding" of assumed prerequisite concepts. However, several researchers have pointed out with dismay that instruments for measuring understanding are not available (see for example, Underhill, 1976). Thus, teachers are faced with with the twin delemmas of being asked not only to teach for understanding, but also to determine the mathematical understanding that remedial students bring to the learning situation. Additionally, there is little agreement as to the definition of mathematical "understanding".

David, Jockusch, and McKnight (1978) propose the following as a definition of understanding:

> Comparing input data with many existing things you already know; looking for apparent inconsistencies or contradictions; making careful note of the areas which can be used in the future to guide future selection of solution methods; trying to identify and retrieve an appropriate "assimilation paradigm" or schema, and to synthesize a new one if no appropriate old one can be found in memory; making a careful critical appraisal of how well the present situation matches the retrieved schema that has been selected; and trying to develop appropriate "meta-language" in order to be able to analyze the mathematical situation effectively (pp. 283-284).

Skemp (1976) has described two levels of understanding: "instrumental" understanding, and "relational" understanding. Skemp described "instrumental understanding" as "rules without reason" and "relational understanding" as "knowing what to do and why". Skemp states:

> In instrumental understanding I would until recently not have regarded as understanding at all. It is what I have in the past described as "rules without reasons", without realising that for many pupils and their teachers the possession of such a rule, and ability to use it, was what they meant by "understanding" (p. 20).

Skemp (p. 21) cautions that instrumental understanding "usually involves a multiplicity of rules" as students encounter new problems in mathematics. Eventually the massive number of such rules becomes unmanageable. In contrast, relational understanding has more general applications:

> To understand something means to assimilate it into an appropriate schema (Skemp, 1971, p. 46).

He further notes:

> An appropriate schema is one which takes into account the long term learning task, and not just the immediate one (Skemp, 1971, p. 51).

This then brings us to the heart of the problem for educators who are involved in remedial mathematics instruction at the college level: When a particular concept is to be presented by an instructor to the student in a learning situation, how does the instructor know if the "appropriate" schema is available? Secondly, if the determination is made that the schema is not available, what instructional techniques are appropriate? Due to the wide diversity of student backgrounds in a beginning algebra class (remedial), are there any commonalities with respect to relational understanding, in terms of prerequisites, that can be assumed?

## Purpose of Study

This study focused on the preceding questions for the concept, and the related principles, of exponent. It was the purpose of this study to investigate the following questions with respect to both successful and unsuccessful (as indicated by test scores) remedial students in both beginning algebra and intermediate algebra at the college level:

1. Do remedial algebra students have a relational, instrumental, or no understanding of the prerequisites conjectured as necessary (as advocated by Gagné) for success

in dealing with the concept of exponent.

2. Do remedial algebra students have a relational, instrumental, or no understanding of the concept of exponent?
   a. How does the understanding of positive, negative, (both integral and fractional) and zero exponents differ in the same student? Between students?
   b. How does the understanding of explicit number exponents and literal exponents differ in the same student? Between students?
3. Do remedial algebra students have the ability to generalize (as defined by Krutetskii) the various properties of exponents?
   a. Can the source of "false generalizations" be determined?
   b. Have students that appear to have generalized the properties of exponents (relational understanding), merely generalized the symbolic notation (instrumental understanding)?
4. What types of imagery (Bruner's enactive, iconic, and symbolic) do students use when working with the concept of exponent?
   a. Does the imagery used differ, and in what respect, for students at the relational and instrumental levels of understanding?
   b. Can a student who is operating at the instrumental level be "pushed" by way of hints and guided questioning to use numerical imagery as an aid

to relational understanding?

5. Do successful students (as determined by a letter grade on a test) differ from unsuccessful students with respect to the four questions above?

Despite its inherent limitations, a clinical interviewing methodology has been deemed by many researchers as the appropriate procedure for investigating internalized thought processes (see, for example, Suydam and Dessart, 1980; Lester, 1980; Fennema and Behr, 1980; Kantowski, 1977; Confrey and Lanier, 1980).

Seven students were selected from each of a remedial beginning algebra class and a remedial intermediate algebra class. One-half of these students were determined by their instructor as obtaining the highest scores, and one-half as obtaining the lowest scores, on a unit examination pertaining to exponents. A semi-structured interview procedure, using the "thinking-aloud" technique was followed as students solved problems relating to the concept and principles of exponent. The interviews were recorded on audio-tapes, and significant body actions of the interviewees were recorded on paper by the interviewer. The tapes were later qualitatively analyzed for any commonalities of behavior with respect to the research questions previously mentioned.

## Theoretical Background

A theoretical framework for the investigation of the preceding research questions was developed by combining critical features from each of the theories of Skemp, Bruner, Krutetskii, and Gagnè.

## Skemp's Theory of Understanding

Skemp (1979A) has proposed three types of "understanding". Two types: "instrumental" understanding, and "relational" understanding were deemed as pertinent to this study. Skemp gives the following definitions:

> "Instrumental understanding" is the ability to apply an appropriate remembered rule to the solution of a problem without knowing why the rule works.
> "Relational understanding" is the ability to deduce specific rules or procedures from more general mathematical relationships (p. 45).

Skemp's general definition of "understanding" is given in one of his earlier writings: "To understand something means to assimilate it into an appropriate schema" (Skemp, 1971, p. 45).

## Bruner's Theory of Knowledge Representation

Bruner has conjectured that human beings process and represent information by way of three parallel systems — one through manipulation and action, one through perceptual organization and imagery, and one through symbolic apparatus. Bruner has called these modes of representation, respectively "enactive", "iconic", and "symbolic" (Bruner, 1970, p. 291). He notes, "Their appearance in the life of the child is in that order, each depending upon the previous one for its development, yet all of them remaining more or less intact throughout life" (p. 291).

Bruner (1966B) has specified that learning of

mathematics follows precisely this order.

> We would suggest that learning mathematics reflects a good deal about intellectual development. It begins with instrumental activity, a kind of definition of things by doing them. Such operations become represented and summarized in the form of particular images. Finally, and with the help of a symbolic notation that remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with. But while, once abstraction is achieved, the learner becomes free in a certain measure of the surface appearance of things, he nonetheless continues to rely upon the stock of imagery he has built en route to abstract mastery. It is this stock of imagery that permits him to work at the level of heuristic, through convenient and nonrigorous means of exploring problems and relating them to problems already mastered (p. 68).

In a different statement relating to the extreme importance of adequate imagery background he states:

> We reached the tentative conclusion that it was probably necessary for a child, learning mathematics, to have not only a firm sense of the abstraction underlying what he was working on, but also a good stock of visual images for embodying them. For without the latter it is difficult to track correspondences and to check what one is doing symbolically (Bruner, 1966B, p. 66).

## Krutetskii's Theory of Mathematical Generalization

Krutetskii (1976) asserted:

> ...there has been no fixed definition of mathematical ability that would satisfy everyone. Perhaps the only thing about which all investigators agree is that one should distinguish between ordinary, "school" ability for mastering mathematical information, reproducing it, and using it independently and creative mathematical ability, related to the independent creation of the original product that has a social value (p. 21).

Krutetskii (1976) investigated the mathematical "ability" of "capable", "average", and "relatively incapable" students (p. 1976). Placed among the relatively incapable students were those students "...who could not work problems that went beyond the limits of the standard they had mastered", and students whose "...mathematical habits were formed with difficulty, required a large number of exercises, and were shaky — disintegrating easily in the absence of practice" (p. 177).

Krutetskii has subdivided mathematical ability into the categories: "generalization", "reversibility", "flexibility", and "curtailment" (Krutetskii, 1976, pp. 195-198).

Of particular interest to this study is Krutetskii's definition of the ability to generalize as the

> ...ability to see something general and known to him in what is particular and concrete (subsuming a particular case under a known general concept), and (2) the ability to see something general and still unknown to him in what is isolated and particular (to deduce the general from particular cases, to form a concept) (Krutetskii, 1976, p. 237).

Rachlin (1981) notes that Krutetskii's definition of generalization ability as the ability to generalize algebraic operations and mental processes, to a large extent is equivalent to the ability that is traditionally, in America, called "transfer".

Kruetskii (1976) reports that mathematically capable students readily found the generality behind particular, and externally different, details (frequently, "on the spot").

Incapable students on the other hand had to be extensively tutored on material which covered all the various cases and combination of irrevelant features, to attain even a very elementary degree of generalization (pp. 240-241).

Gagne's Theory of Hierarchical Prerequisites

Gagne's approach to the teaching of mathematical concepts comes primarily from a combination of the neobehaviorist psychological position and the task analysis model that historically has developed from the industrial and military training models. Gagné suggests that instruction should always begin with a task analysis of the instructional objectives. The primary question to Gagné is, "What should the learner be able to do when instruction is completed?" This "capability" must then be stated specifically and behaviorally (Crosswhite, et al, 1973, p. 7).

The capability can be conceived of as a terminal behavior which eventually is placed at the top of an often complex pyramid. This complex pyramid is developed by asking at each level, "What would the learner have to know in order to do this?" Gagné would continue this determination of prerequisite knowledge until the fundamental units of learning are reached — For Gagné this would be classical or conditioned responses (Crosswhite, et al, 1973, p. 7).

Gagné (1970) has stated:

> Thus it becomes possible to "work backward" from any given objective of learning to determine what the prerequisite learnings must be—if necessary, all the way back to chains and simple discriminations. When such an analysis is made, the result

> is a kind of map of what must be learned. Within this map, alternate "routes" are available for learning, some of which may be best for one learner, some for another. But the map itself must represent all of the essential landmarks; it cannot afford to omit some essential intervening capabilities (p. 242).

He further notes:

> Representations of learning hierarchies are limited to the description of and interrelations of intellectual skills. Thus, a hierarchy does not represent external conditions of learning, as they have been described in previous chapters. Accordingly, the learning hierarchy does not picture the procedures of instruction. The intention is not to depict how an individual may come to learn a particular intellectual skill — what kind of instruction to give, how much guidance of learning to introduce, what sequence of communications to follow, and so on. What is shown is only the internal conditions for learning, the prerequisite capabilities that will provide the positive transfer to a new learning event. Identifying these capabilities and assuring their availability are matters of critical importance for instruction (p. 242).

In this study the concern was not with Gagne's behavioristic approach to learning, but with his admonition that an analysis of prerequisite knowledge and skills is vital to the design of any instructional unit. Gagné has indicated the omission of any essential step in the hierarchy of prerequisite knowledge may lead to long-term "blocking" of future understanding (Gagné, 1970, p. 243). In particular, this investigation was interested in the question of whether a student's understanding of the concept and principles of exponents was "blocked" by the lack of understanding of prerequisite concepts.

A Theoretical Synthesis — Skemp, Bruner, Krutetskii, and Gagné

That portion of Skemp's theory which pertains to "instrumental" understanding and "rleational" understanding was combined with Bruner's modes of internal representation of information, "iconic" and "symbolic" to form a two-by-two matrix for the investigation of the understanding of exponents. The understanding of the concept of exponent by each student interviewed was qualitatively determined to belong to one cell of the following matrix, which was developed from suggestions by Erlwanger (1975C) and Alexander (1977):

| | | UNDERSTANDING | |
|---|---|---|---|
| | | Instrumental | Relational |
| MODE | Iconic | | |
| | Symbolic | | |

Problems and questions presented to the interviewees were designed to elicit information that enabled a diagnosis to be made with respect to a pre-determined set of conjectured prerequisite concepts as advocated by Gagné. In addition, problem sets and interviewer questions were arranged in an order that anticipated the determination of generalizability (as defined by Krutetskii) on various exponential concepts and principles.

## Definition of Terms

<u>Canceling</u> The dividing of a common factor from numerator and denominator of a fraction, based on the identity and multiplicative properties. For example:

$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{2}{2} \cdot \frac{2}{3} = 1 \cdot \frac{2}{3} = \frac{2}{3}.$$

<u>Explicit Number</u> A non-literal number such as 0, 2, ½, and $\sqrt{7}$.

<u>Exponential Definitions</u>

a. Positive Integer Exponent

If n is a positive integer (called the "exponent"), and a any real number, (called the "base"), then $a^n$ is a short notation for the number of factors of a. For example, $2^3$ means $2 \cdot 2 \cdot 2$.

b. Negative Integer Exponent

$a^{-n}$, where a is a non-zero real number and n is a positive real number, means the reciprocal of $a^n$. For example, $2^{-3}$ is equivalent to $1/2^3$.

c. Zero Exponent

$a^0$ is defined to be 1, where a is any non-zero real number. For example, $2^0$ is equivalent to 1.

d. Rational Exponent

$a^{n/m}$, where a, m, and n are real numbers, is the mth root of $a^n$ or the nth power of the mth root of a. For example, $8^{2/3}$ is the cube root of 8 squared; $\sqrt[3]{8^2} = 4$ , or equivalently the square of the cube root of 8; $(\sqrt[3]{8})^2 = 4$.

<u>Exponential Properties</u> For any of the variations of exponents

and bases previously mentioned, these are generalizations of manipulative techniques summarized in the following:

a. $a^m \cdot a^n = a^{m+n}$

As an example, $2^2 \cdot 2^3$ would be $(2 \cdot 2) \cdot (2 \cdot 2 \cdot 2)$ and by arithmetical properties equivalent to $2^{2+3}$ or $2^5$.

b. $(a^m)^n = a^{mn}$

As an example, $(2^2)^3$ by arithmetical properties would be $(2^2)\ (2^2)\ (2^2)$ and thus $2^{2+2+2} = 2^{3(2)} = 2^6$.

c. $a^m/a^n = a^{m-n}$

As an example, $2^4/2^3$ would be $(2 \cdot 2 \cdot 2 \cdot 2)/(2 \cdot 2 \cdot 2)$ and by arithmetical properties three of the four factors could thus be "canceled", $\frac{\cancel{2}^1 \cdot \cancel{2}^1 \cdot \cancel{2}^1 \cdot 2}{\cancel{2}_1 \cdot \cancel{2}_1 \cdot \cancel{2}_1}$, leaving one factor of 2, that is $2^{4-3} = 2$.

d. $(a \cdot b)^n = a^n/b^n$

As an example, $(2 \cdot 3)^2$ means $(2 \cdot 3) \cdot (2 \cdot 3)$ and by arithmetical properties is equivalent to $(2 \cdot 2) \cdot (3 \cdot 3)$ and thus $2^2 \cdot 3^2$.

e. $(a/b)^n = a^n/b^n$

As an example, $6^2/3^2$ is equivalent to $\frac{6 \cdot 6}{3 \cdot 3}$ and by "canceling" the factors of 3, $\frac{\cancel{6}^2 \cdot \cancel{6}^2}{\cancel{3}_1 \cdot \cancel{3}_1}$ equivalent to $2 \cdot 2$ or $2^2$, that is, $(6/3)^2$.

<u>False Generalization</u> This term is used in this present study to represent two possible kinds of erros. First, a student may change a problem to make it "fit" a known generalization, and secondly, the student may change a generalization to make it fit the problem. The second sense may, in fact, occur

when a student is not aware of the generalization, but forms an incorrect generalization on his/her own.

<u>Literal Number</u> In this study the term is equivalent to the term "variable".

<u>Numerical Imagery</u> In this study the term is used for the situation in which a student working with an expression containing variables, uses the substitution of explicit numbers for the variable in order to gain more understanding of the variable expression.

<u>Remedial Algebra</u> Algebra that is taught at the college level, but is a normal part of a standard high school algebra sequence. In this study, there are two such courses. Beginning Algebra, Math 111, is basically equivalent to the first year of high school algebra and Intermediate Algebra, Math 121, which is basically equivalent to the second year of high school algebra. They are remedial in the sense that a student having taken them in high school and understanding enough to successfully reach the minimum score on the ACT examination, is not required to take them at Ferris State College. Additionally, no credit toward a mathematics major is granted for these courses.

<u>Variable</u> In this study, the assumption was made that students would visualize "variable" as a placeholder for a number, or simply a letter which stands for a number. Variable is used in the present study in two senses. First, as a "generalized number" in such expressions as $2a + 3a = 5a$. That is, the expression is true when any number is substituted for a.

Secondly, as a "specific number", as in the solution to $x + 4 = 7$, where only the number 3 can be substituted for x to make an equality.

## Overview

In Chapter II, Review of the Literature, the concept of understanding as used in this study, is placed in an American historical perspective of understanding. Connectionism, as defined through the theories of major writers, such as Thorndike, is tied in historically with the concept of "meaningful learning", as developed from the theories of Brownell, Piaget, Bruner, and Skemp. In addition, what researchers had to say about "algebraic thinking" is reviewed. In this regard, studies are reviewed which have emphasized exponents, remedial algebra for college students, algebraic errors, and algebraic concepts. Also, to place Bruner's concept of imagery into the major present-day psychological theories of imagery, the works of the major researchers in the field of imagery are reviewed.

In Chapter III, Research Procedures, the method of selection of the interviewees is detailed. Also, the predetermined questions and problems for the interviews are indicated along with the theoretical basis for their selection. In addition, the interview techniques and procedures are elaborated.

A model for analysis of the interview data is presented in Chapter IV, Analysis Of Interviews. The theories of Skemp and Bruner are integrated to form a matrix, as suggested by

Erlwanger (1975C) and Alexander (1977), for the purpose of classifying the understanding of exponents exhibited by the individual during the interviews. Excerpts from the audio-taped interviews are presented verbatum so that the reader may see the model in action as it is applied to the research questions previously indicated.

This study was undertaken with the idea that while "understanding" is a somewhat elusive concept with respect to determination, its importance, however, cannot be overlooked, due to the massive number of "remedial" students entering the higher educational system in America.

It has been stated by Brown (1942):

> In general, I've come to the conclusion (there are exceptions, of course) that the ease and accuracy with which any educational outcome is measured is in direct proportion to its unimportance. That is, the easy items to measure accurately are the ones which make least difference whether they are measured or not (p. 354).

In a similar vein Craig (1966) notes:

> Finally, there is admittedly insufficient evidence of how well the concepts derived in psychological theory or the laboratory apply in the classroom. The risks of overgeneralization are great; but they may be reduced somewhat if extensions of psychological concepts of teaching are proposed, not as fact, but as hypotheses that merit further tryout in classrooms. This is what we have advised. With this proviso, we suggest that any remaining risk in the use of learning theories is preferable to the alternative hazards of dependence on hunches, uncritical imitation, or habit (p. 82).

Perhaps Bayles (1960) best articulated the hope for teaching the remedial mathematics students enrolling in

American colleges:

> In reality, preference for understanding-level over memory-level teaching means belief in the proposition that the only way to make teaching genuinely practical is to make it basically theoretical (p. 194).

It is with this attitude that this study was undertaken.

# CHAPTER II

## REVIEW OF THE LITERATURE

A thorough search of the literature has revealed no in-depth studies using interviews to examine college students' concepts of exponent. Several studies have classified the various types of errors that students make in algebraic study. Some researchers have studied and classified the factors which contribute to algebraic success. In a similar vein other studies have been carried out to classify concepts which are fundamental to algebraic learning.

Since this present study is about understanding, the review includes a historical perspective of "meaningful learning" therory. Additionally, because of the importance of imagery to this study, a review of imagery literature is included.

Elementary algebra is traditionally a course which is taught at either the eighth or ninth grade level of school, consequently, this review will cite not only literature pertinent to college students, but also will include pre-college studies. Since to some researchers algebra is "generalized arithmetic", it follows that there is not a fine line that distinguishes between studies which relate to arithmetic. This review then, does include literature which pertains particularly to arithmetic, but is deemed general enough to be pertinent to this investigation of algebra learning.

## Learning Theory

Heidbreder in 1924 (cited in Bourne, 1966, pp. 24-25) anticipated the two major lines of development which grew to characterize the theories of concept development. One line views the human organism as a passive recipient of environmental information. A "composite photograph" is gradually built up from examples in which the common features stand out and the irrelevant features are washed out. The learner is viewed as an organism which contributes nothing except memory of previous examples. Internal activity which operates on an incoming event is seen as non-existent as far as contributing to concept formation. That is, the learner is passive. This line of reasoning has been used in various forms to define the arguments of psychologists who adhere to the "associationistic" and "behavioristic" therories of learning.

A second theoretical line conjectures that humans are not passive learners, but actually are active participants in the concept formation process. The learner always develops some hypothesis about the unknown concept. Over a sequence of examples and nonexamples, the learner, after possibly forming and rejecting several hypotheses, will settle on a correct one and consequently the concept is formed. In this therory the learner is always active, registering information modifying any hypothesis which is incompatible with incoming data (Bourne, 1966, p. 25). In this way concepts are "constructed" by the learner.

In this section the major features of researchers in both the stimulus-response associational theory and the constructivist hypothesis-testing theories will be detailed. Additionally, the Gestaltist-field theory, which is in some sense a middle ground between the associationistic and the constructivist theories, will be reviewed.

## Associationism

The primary psychological figure in America in the early part of the 20th century was Edward L. Thorndike. Although Thorndike's work built upon the works of his teacher, William James, who in turn was influenced by Alexander Bain, who in turn was influenced by associationist theorists all the way back to Aristotle, it is Thorndike's name that has become predominant (Sandiford, 1942, p. 107). Thorndike is perhaps remembered best for his statement of the "law of effect", which in modern-day language is thought of as "reinforcement" (Resnick and Ford, 1981, pp. 12-13). In Thorndike's theory, learning is seen as a problem of establishing proper connections. This blended well with the doctrine of "social utility" prevalent in his day. The social usefulness could be used as a measure to decide which connections should be encouraged in the learner. With his tradition of laboratory experimentalism, he was thoroughly committed to the task of transforming laboratory findings into a theory of classroom instruction (McDonald, 1964, p. 8).

Thorndike formulated his "law of effect" based on animal experimentation. The experiment most frequently associated with Thorndike in this regard consisted of placing a cat in a wooden box that could be opened by tripping a latch. Eventually the trapped cat in frustration would accidentally trip the latch and escape. The cat would then be placed in the box again and the process repeated. Each time the experiment was repeated, less time would be required for the cat to escape. In Thorndike's view, the cat was not "figuring out" how to escape from the box, rather the reward of escape served to strengthen the bonds between the experimental situation and the response that permitted escape (Resnick and Ford, 1981, p. 12). Thus,

> When a modifiable connection between a situation and a response is made, and is accompanied or followed by a satisfying state of affairs, that connection's strength is increased: When made and accompanied, or followed by, an annoying state of affairs, its strength is decreased (Thorndike, 1913, p. 4).

Although he experimented mostly with animals, Thorndike believed his principles applied equally well to humans. He, along with other psychologists, called "connectionists" or "associationists" believed all human behavior could be analyzed in terms of two simple constructs. When broken down to irreducible units, behavior was found to consist of stimuli (external events) and responses (the subject's reaction to the stimuli). If a certain response given in reaction to a particular stimuli brought a reward of some nature, then a bond, or association (or "connection") was formed between the stimulus and response. The more frequently a particular stimulus - response pair was rewarded, the stronger the connection. Thus, the law of effect suggested that practice followed by reward was the principle manner in which human learning took place (Resnick and Ford, 1981, p. 13).

Thorndike started a tradition of bringing laboratory results to teachers and educators. He had a particular interest in the teaching of arithmetic and addressed this topic in his 1922 book: The Psychology of Arithmetic. What teachers needed to do was to find and make explicit the particular domain of bonds that constituted arithmetic. After this, "drill and practice" involved presenting bonds in a carefully fashioned manner so that important bonds were practiced frequently, and lesser bonds, less often. It should be noted that bonds had an effect on each other, and that any particular bond should be formed with the consideration for every other bond that has been or will be formed. The rewards that served to strengthen the particular bonds were obtained when arithmetic problems were made interesting, practical, and fun (Resnick and Ford, 1981, p. 15). This appears to be a part of Thorndike's theory of learning mathematics that was ignored by many of his disciples.

Thorndike's (cited in Alexander, 1977, p. 26) perception of the learning of algebra, in terms of topics covered, was indicated when he surveyed high school teachers as to which topics should be taught. His summary included:

> 1. Involved manipulation of polynomial expressions is not a justifiable way of using the high-school student's time.
> 2. Since the application of equations in other high-school subjects is chiefly in the proportion form, mastery of that form and other easy equation forms should be secured.

3. It would be profitable to extend the field of application of the construction of formulas as well as their evaluation.

4. There is need for the careful development of the art of criticism as applied to graphs.

5. The presentation of laws of means of mathematical graphs should be encouraged.

6. The function concept should be used when advantageous, but with economy (Thorndike, 1923, pp. 82-83).

That algebra for Thorndike was mainly rote, is indicated by his statement that learning algebra requires learning "...about one hundred and fifty rules" (p. 228).

Thorndike, however, also seemed to indicate that more than rote learning was necessary. He felt the formulations of "habits" and "principles" was an integral part of algebra. He notes (p. 239):

> If we leave to habit everything that can be done as well by habit, we gain an added dignity for the matters that really are matters of principle... It is not because he does not value rules and principles in algebra that the psychologist often prefers to use habits instead: it is rather because he does value principles and does not wish them to be misused and cheapened.

Thorndike's meaning of the words "habit" and "principle" can be gleaned from his statement with respect to literal numbers:

> The principle <u>We can represent numbers by letter</u>, develops into at least ten distinct habits of thought, namely:
>
> 1. A letter may mean a particular number of things, like men, boys, or eggs.
>
> 2. A letter may mean a particular number of units, like cents, quarts, feet.
>
> 3. A letter may mean any one of a number of numbers, like the number of dollars in the cost of any number of suits of clothes of a certain sort, or the number of square feet in any rectangle.
>
> 4. A letter may mean any number, as in $(p + q)(p - q) = p^2 - q^2$.
>
> 5. If you call a certain number p. you may

call 3 times that number q or r or s or any letter except p that you please, but it is commonly useful to call 3 times that number 3p.

6. If you call a certain number p, you may call 3 more than that number any letter except p that you please, but it is commonly useful to call it p + 3.

7,8,9. The same principle of consistency and utility with p - 3, p/3 and 3/p.

10. If we call a certain number (say, the profit Mr. A. made in January, 1922) p we don't call it anything else and don't call p something different so long as we are thinking about the problem to answer which we called that number p (Thorndike, 1923, pp. 228-229).

Although, specifically referring to arithmetic, Thorndike seems to indicate that his idea of mathematical learning is more than the simple stimulus - response episode that many of his followers advocated: He notes:

> ...I hasten to add that the psychologists of today do not wish to make the learning of arithmetic a mere matter of requiring thousands of disconnected habits, nor to decrease by one jot the pupil's genuine comprehension of its general truths. They wish him to reason not less than he has in the past, but more. They find, however, that you do not secure reasoning in a pupil by demanding it, and that his learning of a general truth without the proper development of organized habits back of it is likely to be, not a rational learning of that general truth, but only a mechanical memorizing of a verbal statement of it. (Thorndike, 1922, p. 174-175).

Thorndike's view of algebra learning can be summarized by the following statements:

> If the 'set' or attitude of the mind toward the first hundred or so operations with literal numbers is permitted to become that of learning a queer game, where you pretend to add, subtract, multiply and divide letters, there is certainty that these bonds themselves will be weak, and probability that all later practice will be much less effective than it should be (Thorndike, 1922, p. 329).

and

> Algebra to most learners...is in large measure forming more or less particular bonds or connections,

> such as $a \times ab = a^2b$, $a(a + b) = a^2 + ab$, a means 1a, $-a \times -b = +ab$, learning to operate several of these together as needed, organizing them further into more inclusive habits and insight, summing up what one has learned to do in rules, and thus gradually attaining a sense of what it is right to do with literal numbers and why (p. 246).

In summary, Thorndike believed the successful algebra student is one who, upon the presentation of a problem, perceives the structure of the problem, chooses the appropriate sequence of connections, and from memory is able to produce the necessary series of connections to solve the problem.

Thorndike's rules for generating specific sequences of drill and practice were largely intuitive. Which bonds were the easiest? How much practice was enough? How should practice be organized? Perhaps Thorndike's greatest legacy in mathematics education was the step he took in the direction of bringing psychological theory to bear on instructional problems. He focused attention on the content of learning mathematics. Some of the questions he posed are still impacting both research and education even today (Resnick and Ford, 1981, p. 16).

Simultaneously with Thorndike's development of connectionism was Watson's promotion of "behaviorism". By "behavior" Watson referred merely to muscular activity. Watson believed humans learned the same as Pavlov's dogs, in terms of classical conditioning. Watson formulated the theory that learning can be explained without Thorndike's law of effect. Learning is dependent upon "frequency" and "recency". Watson's principle of frequency states the more frequently one has responded to a particular stimulus, the more likely that same response will be given should the stimulus occur again. His principle of frequency said the same held true with respect to elapsed time since the response to the stimulus.

Watson and his disciple, Guthrie, and their followers, became known as "contiquity" theorists due to their belief that the stimulus - response bonds are strengthened by the response occurring in the presence of the stimulus (Sandigard, 1942, p. 107; Harrison, 1967, p. 28). In contrast, in "operant", or "instrumental" conditioning, the response must be made before either a positive or negative reinforcer. The "effect" of the stimulus response bond is the strengthening agent. The followers of this theory are known as "reinforcement" theorists (Klausmeier and Goodwin, 1971, p. 25).

Connectionism, as advocated by Thorndike, and Watson's behaviorism, fell into disfavor in America during the later 1920's and 1930's. This was for the most part based upon the growing mood in America of "education for democracy". Education became the hope of all classes, an instrument of social reform. John Dewey with his psychology of pragmatism that interrelated "aim", "interest", and "intelligent action" was chosen by educators as the prime spokesman against the anti-equalitarian ideas of Thorndike. Dewey attached the reflex arc of the connectionists, arguing that the stimulus and response were not sharply distinguishable, but must be viewed as organically related. For Dewey, "mediated experiences" were the main psychological event. What the learner is involved in when a stimulus is given will determine how the stimulus is perceived, and consequently the response (McDonald, 1964,

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p. 11). Dewey (1933) noted, "to grasp the meaning of a thing, an event, or a situation, is to see it in relation to other things; to note how it operates or functions, what consequences follow it, what uses it can be put to" (p. 137). In his attack on connectionism, and his view that learners' interests and aims are paramount in education, Dewey's philosophy was easily reconciled with the growing Gestaltist view of learning (Harrison, 1967, p. 30).

Gestalt Psychology of Learning

In the late 1920's and early 1930's the domination of the American psychological scene by behaviorism and connectionism was challenged by the importation of Gestalt psychology from Germany. A Cornell psychologist R.M. Ogden translated Koffka's German writings into English. Kohler's works were also translated. Although the Gestalt theory had been developing in Germany since first announced by Max Wertheimer in 1912, it was Kohler and Koffka's writings, plus their extended visits to America, that brought the new theory to the attention of the American psychologists. (Hilgard, 1964, p. 54).

Koffka's 1924 book, Growth in Mind (cited in Hilgard and Bower, 1966), detailed the criticism of trial-and-error learning as conceived by Thorndike (and upon behaviorism). Koffka's view was supported by Kohler in his 1925, book Mentality of Apes. Kohler showed how apes could by-pass the laborious processes of stamping out incorrect responses and stamping in correct ones, as indicated in Thorndike's theories developed from experiments with cats (Hilgard and Bower, 1966, p. 231). A typical problem as described by Kohler, was the "detour" in which a barrier was placed between an ape subject and the goal object. The animal had to detour around the barrier to obtain the goal object, such as a banana. Other settings required the animal to join two sticks to form a rake in order to obtain food. The behaviorial descriptions of the animals solving problems indicated that the apes observed the goal object for some period of time, then very rapidly solved the problem. According to the Gestalt point of view, the animals were able to reorganize their perception of the world, thus achieving "insight" into the problem, and consequently perceive most aspects of the problem in its entirety or "whole", a "gestalt" (Ellis, 1972, pp. 185-186).

Gestalt principles were particularly attractive to the educational progressives, who under the leadership of people such as Dewey, had already taken the position against rote learning and in favor of insight as an educational goal. The principles made it unnecessary to rely on the conditioning stimulus-response process to explain all human learning (Woodring, 1971, p. 91). There were voices raised in protest against the acceptance of insight as a competitor to associative learning. Guthrie (1935) (cited in Hilgard, 1964) felt that sudden learning would have to fall into the category of luck, and hence lay outside the influence of science. Others thought of insight as an extreme case of transfer of training. Even others saw insight in the descriptive sense: learning could be <u>with</u> insight but not <u>by</u> insight (Hilgard, 1964, p. 61). Hilgard notes that some advocates of associationism taught practical skills by such methods as carefully constructed motion pictures, accompanying narratives, and practice in actual manipulations. The emphasis was not on habits, but upon inherent organization, meaningfulness of parts, and perceptual patterns. This is exactly the emphasis that one alerted to insight would propose (Hilgard, 1964, p. 62). Hilgard maintains that criticism of association theory was not based on the importance assigned to past experience, but rather on the notion that past experience guarantees the solution to a problem, no matter how the problem is presented. In contrast, the insight point of view is that, with sufficient past experience, more difficult problems can sometimes be solved by a learner due to the structural display of the problem, but other learners will solve the problem because they are better able to generalize and not be misled by the structural features (Hilgard, 1964, pp. 61-62).

Gestalt psychology had its greatest success in the field of perception. It demonstrated the role of organization and background upon phenomenally perceived processes in its assault on the associationist hypothesis that a percept is made up of sensation-like elements bound together. When the gestalt psychologists turned to learning theory they used the same arguments against the associationist's reflex arc of learning. Koffka's starting point for learning was the assumption that the laws of organization in perception are applicable to learning. Since the difficulty of a problem depends upon the initial structuring of the field, as is open to observation of the learner, the correct solution is largely a matter of perception (Hilgard and Bower, 1966, pp. 232-233). The guiding principle of gestalt learning theory is that psychological organization tends to move in one general direction, toward the "good" gestalt. A "good gestalt has such properties as regularity, simplicity and stability. Psychological organization tends to follow laws of "similiarity", "proximity", "closure", and "good continuation". As an example of the law of "closure", in a problematic situation the whole is seen as incomplete and a tension is set up toward completion. The strain for the learner to complete acts as an aid to learning, and the achievement of closure is satisfying. The gestalt preference is for conceiving psychological processes as a product of the present field, and the role of past experience is explained in a theory of "memory traces" (Harrison, 1967, p. 32). The essential features are:

> A trace is assumed from prior experience and represents the past in the present, a present process is assumed which can select and communicate with the trace, and there is a resulting process of recall or recognition. Against the theory that memory leads to decay and fuzziness, the Gestaltists theorize that it leads to changes in the direction of increased clarity. In skill learning the trace, as part of the field process, exerts an influence in the direction of making it similar to the process which initially produced the trace. While the skill is being learned, the trace is somewhat unstable. The trace and process interact and greater stability is gradually reached. Since the trace system obeys dynamic laws, it also undergoes stabilizing changes over periods of no practice, greater improvement of skill learning will occur with distributed practice as proposed to mass practice (Hilgard and Bower, 1966, p. 239).

Wertheimer (1959), summed very succintly the Gestaltist position on problem solving:

(S$_1$ is the situation in which thought process start, S$_2$ is when after a number of the process ends, the problem is solved).

When one grasps a problem situation, its structural features and requirements set up certain strains, stresses, tensions in the thinker. What happens in real thinking is that these strains and stresses are followed up, yield vectors in the direction of improvement of the situation, and change it accordingly. S$_2$ is a state of affairs that is held together by inner forces as a good structure in which there is harmony in the mutual requirements, and in which the parts are determined by the structure of the whole, as the whole is by the parts (p. 239).

The process does not involve merely the given parts and their transformations. It works in conjunction with material that is structurally relevant but is selected from past experience, from previous knowledge and orientation.

In all this, those movements and steps are strongly preferred which change the state of affairs in S$_1$ along a structurally consistent line into S$_2$ (pp. 239-240).

He added advice for teachers:

In the light of my teaching experiences it would seem best -- especially at the start to show, to "teach" as little as possible. It would seem desirable to avoid as much as possible the giving of ready-made steps. The child should be confronted with tasks with which he tries to deal himself. Let the child face problems, let him receive cooperative help as he needs it, but let him not simply copy or repeat given procedures. I would avoid as much as possible anything that might introduce a mechanical state of mind, an attitude of drill (p. 276).

Wertheimer pointed out the inadequacies of the associationistic theory of learning in comparison to the Gestalt conception of learning.

Association may mean chaining items together in an and-sum of connections that has by nature no structure, as in the usual theory of learning rote syllables. Or at the opposite extreme, it

> may mean realizing of structural belonging in which items require each other as parts in a context--including the enduring effects of that realization.
>
> Repetition may mean that the same piecemeal, blind connection occurs over and over again; or it may mean the change from an un-understood and sheer additive pairing to the realization of a structure in which the meaning of the items becomes that of parts in a characteristic whole.
>
> Trail and error may mean a heedless succession of blind proceeding with random order of directions; or, again, it may mean that some sensible hypothesis is structurally tested. In the latter case, the very failure may elucidate the situation and suggest another hypothesis which fits the given structure better.
>
> Learning on the basis of success may mean that an action is singled out because of the success that follows the action only factually, but is not understood; or it may mean that, in learning, a subject grasps why just this kind of action leads to just this effect for intrinsic structural reasons. It is the latter form of "learning by success" that enables the subject to vary his action in a structurally sensible way when the situation is no longer the same (pp. 250-251).

Wertheimer's statements show the progression of Gestalt psychology into more of a "field" theory of psychology. Field therories of learning give unusual prominence to the organized "whole"; whose properties and structure both explain the localized occurrence that it embraces and at the same time permits increased control over it (Hartmann 1942, p. 166). Phrased another way:

> Our motor performances have an identifiable beginning and end; our perceptions are clearly in terms of spread-out wholes or 'figures', such as entire persons, objects, and events; and our 'inner life' is almost always structured, being rarely in an unstructured or nonstructured phase (p. 171).

Hartmann (1942) notes that the "field" in theory is derived from the scientific use of the word, and quotes Kantor to indicate that it "consists of a definite frame of reference marking the limits of interactions of phenomena during the occurrence of events" (p. 172).

There are many versions of field theory, Gestalt theory being only one. Lewin's "field theory" was more explicit in his advise to educators. Lewin notes, "A teacher will never succeed in giving proper guidance to a child if he doesn't learn to understand the psychological world in which that individual child lives" (Lewin, 1942, p. 217). Lewin indicated psychologists and educators must not pick out isolated elements within a situation, but must characterize the situation as a whole. According to Lewin the "atmosphere" surrounding a learning situation is just as important as the bearing physical entities have on the field of gravity and field of electricity (Lewin, 1942, p. 218).

Lewin specified four types of learning: 1) as a change in "cognitive structure", 2) as a change in motivation, 3) as a change in ideology, and 4) as a voluntary control of body musculature (p. 220). With respect to learning as a change in cognitive structure, Lewin notes:

> It is correct that a change in cognitive structure may occur on the occasion of repeated experience. However, it is important to realize that it is not the repetition itself, but the change in cognitive structure which is essential for learning (p. 229).

He uses the scenario (later adopted by Skemp) of a person attempting to find his/her way about a new city without a map. Even after many trips the person is still basically lost. In contrast, once given a map, the person is able to view the situation as a whole, and consequently make the appropriate changes to his/her cognitive structure (p. 229). To Lewin, learning results from the two entities, the structure of the cognitive field, and from certain need or motivations. Entities pertaining to motivation are those such as basic needs, goal structures, level of aspiration, and group belongingness (pp. 238-239).

One other significant theory in the continuium of historical development of learning theory from pure stimulus-response theory to the cognitive learning theories prevelant in the 1970's and 1980's, is Tolman's "sign-gestalt" theory. Initially Tolman's theory was called "purposive behaviorism; and in addition to "sign-gestalt", has been called a "sign-significance" theory, and an "expectancy" theory (Hilgard and Bower, 1966, p. 191). Tolman's theory was a genuine "behaviorism" and as such rejected introspection and conscious experience. It was concerned only with objective behavior, but did, however, emphasize the cognitive nature of learning. Tolman's system was "molar" rather than the "molecular" behaviorism of Watson. An act of behavior was considered to have properties of its own, which could be described and identified regardless of whatever muscular, glandular, or neural processes underlie it. The molecular facts of physics and physiology, which are the basis of behavior, have identifying properties of their own, but these are not the properties of behavior as molar. Thus, Tolman's theory was independent of physiology (pp. 191-192). Although, the complete act of behavior is initiated by environmental stimuli and physiological states, certain processes intervene. These "intervening variables" include such processes as cognitions and purposes. For Tolman, the problem of psychological analysis at the molar level was to infer these processes which intervene between the initiation of action in the world of physics and physiology and the resulting observable consequences. Tolman believed that in learning an individual formed a cognitive "map" by learning, not movements, but meanings (pp. 193-195). Hilgard and Bower (1966) have noted the importance of Tolman's contribution to the American psychological movement:

> It was Tolman's contribution then to show that a sophisticated behaviorism can be cognizant of all the richness and variety of psychological events, and need not be constrained by an effort to build an engineer's model of the learning machine.
>
> With the diversification of behaviorism under the influence of Tolman and others, the old brittleness of Watsonian behaviorism has largely disappeared, and what virtues there are in the behavioristic position have now become part of the underlying assumptions of most American psychologists--without most of them thinking of themselves as behaviorists at all (p. 219).

The contemporary status of Tolman's sign-gestalt theory, as well as gestalt and field theories is rather a matter of opinion. Woodring (1971) feels that Gestalt psychology has already passed its peak and has now been absorbed into what is psychology. However, Kohler (1958) (cited in Hilgard, 1964) complained behaviorism was much in ascendance in American, and that problems such as perceptual contours were being reduced to atomistic terms. He took the position that the major teachings of Gestalt psychology had been neglected (p. 58). Hilgard feels, however, that the upsurge of interest in problem-solving and in cognition generally, have hidden beneath them the basic notion of "insight" as advocated by the Gestaltists (Hilgard, 1964, p. 581; Harrison, 1967, p. 35).

Behaviorism, which had suffered in the late 1920's and the early 1930's at the hands of the gestaltists, and at the hands of their own theorists such as Tolman, was given a new lease on life by Skinner's (1938) publication of The Behavior of Organisms. Skinner rejected intervening variables, and broke with conventional stimulus-response psychology by making a distinction between "respondent" and "operant" behavior. Respondent behavior is a direct response to a stimulus, as in classical conditioning. An example of respondent behavior would be the flow of saliva in response to food in the mouth. On the other hand, operant behavior simply happens, apparently spontaneously, rather than in response to a specific stimulus. As an example of operant behavior, a baby alone in a crib may twist and "coo" spontaneously, in response to nothing in particular (Hilgard, Atkinson, and Atkinson, 1979, p. 198). One aspect of Skinner's theory may be compared to Thorndike's law of effect: when the occurrence of an operant is followed by the presentation of a reinforcing stimulus, the strength is increased. What is important is that the response of the organism produces the reinforcing agent. This type of conditioning became know as "instrumental" conditioning to distinguish it from classical conditioning (Hilgard and Bower, 1966, p. 110).

Skinner is more interested in individual behavior as opposed to group behavior. He believes that given a strong independent variable, a reliable dependent variable, and adequate experimental controls, that behavior is lawful enough to permit accurate prediction of what an individual organism will do under a given set of conditions (Hill, 1964, p. 37). Skinner and his followers have classified reinforcement schedules, interpreted avoidance learning, analyzed verbal behavior, and attempted to interpret social phenomena in terms of learning principles (pp. 39-40).

In 1954 Skinner embarked upon a series of investigations which led to inventions designed to increase the efficiency of teaching in the areas of arithmetic, reading, spelling, and other school subjects. Skinner's inventions, and others modeled after his, came to be known as "teaching machines" (Hilgard and Bower, 1966, p. 132). Skinner (1971) has stated, "Fortunately for us all, the human organism is reinforced by many things. Success is one of them" (p. 39). Skinner held that since the teacher could not be with every student on all tasks, the machines, with "programs" that provided immediate feedback with respect to success, could have a significant impact by acting as reinforcers. This method of "programmed instruction" eventually became a major commercial and educational enterprise in the 1960's (Hilgard and Bower, 1966). The interest has been rekindled in the 1980's due to the advent of the micro-computer and the consequent proliferation of educational "software".

According to Hill (1964) in addition to Skinner, two other major approaches to stimulus-response theories of academic psychology have continued into modern times. The names of Guthrie and Hull are generally associated with these traditions (p. 28). Guthrie (1942), citing animal studies, reaffirmed his conviction that association by contiguity is the basic law of learning (pp. 46-47). Later, Guthrie did expand his theory to put more emphasis on the modification of stimulus reception by changes in receptor orientation. His expanded concept included scanning, a systematic variation in receptor orientation service to discover a stimulus. That is, what is being noticed becomes a signal for what is being done (Guthrie, 1959) (cited in Hilgard and Bower, 1966, p. 92). The importance of Guthrie's expanded theory was in the trend toward cognitive interest (Hill, 1964, p. 41). Hull, on the other hand postulated that a number of intervening variables serves to link the independent and dependent variables. The most basic of these intervening variables is "excitatory potential", which refers to the strength of the tendency to give a certain response to a particular stimulus. The excitatory potential for any given response depends upon a number of other intervening variables such as interest, habit, strength, drive, and incentive motivation. The importance of Hull's modification is primarily due to the cognitive slant given to his stimulus-response theory by the distinction between learning and performance (Hill, 1964, pp. 34-35).

W.F. Hill (1964, p. 27) has indicated that most learning theories within academic psychology have a stimulus-response orientation. He notes that this is true whether one considers conditioning, verbal learning or thinking. Hill notes that although in many cases the paradigm may be greatly modified and elaborated the stimulus and response theory is basic to most theoretical work in learning. Nowhere is this more evident than in the theories of Robert Gagne'. Gagne's approach to learning comes essentially from a combination of the neobehaviorist psychological position in combination with the task analysis model that dominates the fields of industrial and military training. Crosswhite, et al., 1973, p. 8). Gagne' (1970) has postulated that there are eight different types of "learning":

> Type 1: Signal Learning. The individual learns to make a general , diffuse response to a signal. This is the classical conditioned response of Pavlov (1927).
>
> Type 2: Stimulus-Response Learning. The learner acquires a precise response to a discriminated stimulus. What is learned is a connection (Thorndike, 1898) or a discriminated operant (Skinner, 1938), sometimes called an instrumental response (Kimble, 1961).
>
> Type 3: Chaining. What is acquired is a chain of two or more stimulus-response connections. The conditions for such learning have been described by Skinner (1938) and others, notably Gilbert (1962).
>
> Type 4: Verbal Association. Verbal association is the learning of chains that are verbal. Basically, the conditions resemble those for other (motor) chains. However, the presence of language in the human being makes this a special type because internal links may be selected from the individual's previously learned repertoire of language (see Underwood 1964b).
>
> Type 5: Discrimination Learning. The individual learns to make different identifying responses to as many different stimuli, which may resemble each other in physical appearance to great or lesser degree. Although the learning of each stimulus-response connection is a simple type 2 occurrence, the connections tend to interfere with each other's retention (Postman, 1961).
>
> Type 6: Concept Learning. The learner acquires a capability of making a common response to a class of stimuli that may differ from each other widely in physical appearance. He is able to make a response that identifies an entire class of concepts or events (see Kendler, 1964). Other concepts are acquired by definition, and consequently have the formal characteristics of rules.
>
> Type 7: Rule Learning. In simplest terms, a rule is a chain of two or more concepts. It functions to control behavior in the manner suggested by a verbalized rule of the form, "If A, then B". where A and B are previously learned concepts. However, it must be carefully distinguished from

> the mere verbal sequence, "If A, then B", which, ofcourse, may also be learned as type 4.
>
> Type 8: Problem Solving. Problem solving is a kind of learning that requires the internal events usually called thinking. Two or more previously acquired rules are somehow combined to produce a new capability that can be shown to depend on a "higher-order" rule (pp. 63-64).

Gagne' rejects the notion that learning is the same for all eight types; their differences are said to be more important than their similarities. In Gagne's system the eight types are hierarchically related in that each type requires the next lower type as a prerequisite--except possibly for Type 1 and Type 2 (Hilgard and Bower, 1966, p. 570). For Gagne', or the programmed-instruction position which is patterned after his theories, the objectives of instruction are "capabilities". These are behavioral products that can be specified operationally. He insists that objectives must be stated clearly in behavioral terms (Crosswhite, et al, 1973, p. 8). Additionally, Gagne' has extended the concept of hierarchical prerequisites to school subjects such as mathematics. The terminal behavior (capability) is placed at the top of what will eventually be a complex pyramid. Subsequent to analyzing the task Gagne' would ask, "What would you need to know in order to do that?" Once these prerequisites are determined, the question would be asked again, and continued until the pyramid is completed (Crosswhite, et al, p. 7). Gagne' has stated, "If one wants to insure that a student can learn some specific new activity, the very best guarantee is to be sure he has previously learned the prerequisite capabilities. When this in fact has been accomplished, it seems to me quite likely that he will learn the new skill without repetition" (Gagne', 1973, p. 111).

Although conditioning is at the base of his eight learning types, the sufficiency of conditioning is rejected:

> Despite the widespread occurrence of conditioned responses in our lives, they remain unrepresentative of most of the events we mean by the word "learning". Voluntary acts can be conditioned only with difficulty, if at all. If a child wants to learn to ride a bicycle, he will get no help in this activity by arranging the pairing of a conditioned and unconditioned stimulus, because voluntary control of his actions is not acquired in this way. The same is true, needless to say, for most other kinds of things he must learn, beginning with reading, writing, and arithmetic. There can be little doubt that Watson's idea that most forms of human learning could be accounted for as chains of conditioned responses is wildly incorrect; and this has been pretty generally conceded for many years (Gagne', 1970, pp. 12-13).

The message Gagne' transmits is, that faced with the problem of improving training, one should look for much less help from well known learning principles than from the implications of the techniques of task analysis, and component task sequencing. Gagne' has been especially active in applying the principles to general education. In particular student achievement in mathematics has been analyzed in terms of hierarchies of knowledge and component task achievement in the course of the acquisition of knowledge (Glaser, 1964, p. 173).

## Meaningful Learning Theory of Mathematics

P.D. Woodring (1971) notes that by 1940 many of the authors of educational psychology text books had become convinced that they must present a variety of psychological theories to teachers -- behavioristic, association theories, gestalt, and psychoanalytic interpretation (p. 92). McDonald (1964) notes that such theories as "functionalism", as expounded by Judd, absorbed the major ideas of other systems--heredity and the nervous system are there, social influences are prominent, measurement and individual differences receive their due, the psychological aspects of school subjects is included, and the cognitive processes are emphasized (p. 23).

Against this historical background of general learning theory, those psychologists and educators with a particular interest in mathematics learning were rebelling against the rote memorizations and drill and practice routines of the stimulus-response advocates. William Brownell (1935) in criticizing connectionist theory described the "meaning" theory of arithmetic:

> This theory makes meaning, the fact that children see sense in what they learn, the central issue in their arithmetic instruction. Drill is recommended when ideas and processes, already understood, are to be practiced to increase proficiency, to be fixed for retention, or to be rehabilitated after disuse. The "meaning" theory conceives of arithmetic as a closely knit system of understandable ideas, principles and processes. (p. 19).

Brownell (1973), identified four major faults of connectionist learning theory: 1) The magnitude of the task of attempting to memorize the required multitudes of bonds, 2) the use of product measures given to evaluate process measures, 3) the invalidity of the analysis of arithmetic performance by adults, and 4) the probability of negative effect on later mathematics learning. In a similar vein, he identified four instructional weaknesses that he attributed to connectionist learning theory. First, the teacher's attention is directed away from the processes by which children learn, while they are overly concerned about the product of learning. According to Brownell, since to connectionists all learning is the formation of connections, the process of learning is making of connections. He notes:

> ...identification of the learning process with the formation of connections, however valid for ultimate psychological and neurological theory, is not useful to teachers. Teaching is the guidance of learning. We can guide learning most effectively when we know what the learners assigned to us really do in the face of their learning tasks. In a word, we as teachers can be helpful in guidance to the degree to which we know our pupil's processes. I do not mean that the product of those processes is no concern of ours; but I do mean that processes are of at least equal importance with products. The teacher who knows the product which is to be finally achieved, but who also knows how to discover, evaluate, and direct the processes of her pupils as they approach this goal - that teacher is probably a good teacher (pp. 63-64).

A second instructional weakness of connectionist theory, according to Brownell, is over-rapid instruction. He indicates the connectionistic view of learning leads us to give the pupil at the outset the form of response which we want him to ultimately have, with the consequent result of memorization and superficial, empty verbalization. Brownell indicates that a more valid picture of learning should be plotted in terms of process and would look something like a series of steps, each successive one somewhat higher on the maturity scale than the preceeding one. Each stage serves its purpose for a time, then is abandoned, but not forgotten. The older procedure is overlaid by another, but the old pattern remains for use if for any reason the new procedure does not function smoothly (p. 64). Brownell indicates that educators must take the time to use whatever sensory aids that are available. He states:

> When the goal of understanding is accepted, the function of temporary aids is seen in its proper perspective. Such aids contribute meanings when meanings are needed; and the more meanings, the deeper the understanding, and the greater the chances of successful transfer to new and unfamiliar situations (p. 66).

He continues:

> Learning is progressive in character. The abstractions of mathematics are not to be attained all at once, by some coordinated effort of mind and will. Instead, we must start with the child wherever he is, at the foot of the ladder, or at some point higher up. Well chosen sensory aids reveal the nature of the final abstractions in a way which makes sense to the child. If he can work out the new relationships in a concrete way and can himself test their validity in an objective setting, he has faith and confidence at the start; and he is the readier to learn with understanding the more abstract representations of mathematics. Sensory aids, like many so-called crutches, are then not only admissable under the conception of learning which I am outlining: they are obligatory (p. 66).

The third instructional weakness of connectionist learning theory indicated by Brownell, is faulty practice. Brownell feels the connectionist exhortation of "make the proper connection under satisfying conditions; exercise the connection until it is firmly established", often leads to superficial learning. He feels that repetitive practice often "freezes" the learner at an undesirably low level of maturity. Brownell advocates that in the early stages of mathematics learning, activities should be instituted which will enable the learner to explore the new material. He advocates "varied" practice to aid the student in discovering the right combination of ideas. Repetitive practice should only be used to promote efficiency once the learning has occured (pp. 67-68).

The fourth instructional weakness attributed to the connectionistic school by Brownell was the use of inappropriate remedial measures. Brownell indicates that remediation for the connectionist would be merely to show the pupil the correct connection and have the student practice until this proper connection is formed. He believes that errors in mathematics are the result not of imperfectly learned symbols, but of imcomplete understandings, of inappropriate thought processes, and of faulty procedures. In Brownell's opinion mathematical errors comes from failure to traverse the stages and levels of thinking in an orderly fashion. Called upon to perform at a higher level than any he has yet attained, and without guidance to reach the higher level, the child will choose to either refuse to learn, try to learn by blindly following rules, or to fool the teacher by being proficient at a lower level procedure. Brownell notes:

> The child whose attitude toward mathematics has been ruined needs to have that attitude corrected. The working of masses of unenlightened and unenlightening examples and problems will not reach the source of difficulty. If the undesirable attitude arose because of inability to understand and of a consequent series of failures, the child's attitude will improve only when he understands and when he has had ample experiences of a successful kind. (p. 70).

The appropriate remedial instruction according to Brownell is a guided questioning approach.

> He should go as far as he can on his own; when he can go no further, he should be questioned and guided through questions to locate his difficulty and to analyze its nature. Through continued questioning he should be led to suggest possible next steps and then to evaluate these steps himself. But at all

> stages the student should be required to make sure of his own knowledge (to the extent that he has any) and he should be allowed to identify his deficiencies himself and to feel that he is making progress by his own efforts. Remedial instruction of this kind is worthy of the name, and the results justify the time and energy that must be expended to secure them (p. 71).

Brownell, citing Katona's research, indicates that problem solving which is based upon understanding is superior to "problem solving" (quotes by Brownell) based upon memorization. He notes also that understanding is a matter of degree, and that varying degrees of understanding react differently with problem solving. He further believed that the degree of understanding engendered is a function of instruction given, and that the form of instruction which enables the learner best to organize his previous experience of learning is favored to other kinds (Brownell, 1942, pp. 436-437).

Brownell (1948), writing with educational research in mind, indicated measures to be used in addition to speed and accuracy of problem solving. He suggests that more basic are changes in the process level, time of retention, and the ability to transfer the learned ideas to new situations. Brownell (1946) also identified the processes of understanding - analysis, synthesis, discrimination, generalization, and comparison. He notes:

> ...Understanding of a principle means that one can identify its appropriateness and usefulness in situations where it has not been seen or used previously. Understanding of a process implies that one knows when and how to use it effectively (p. 41).

Brownell (1942), sounding somewhat like Piaget, states:

> ...one needs only to point out that all learning starts with some inadequacy of adjustment, some disturbance of equilibrium -- and so, with a "problem" -- and that in the process of achieving adjustment and returning to a state of equilibrium one, "solves" the problem (p. 415).

That Brownell was in fact an admirer of Piaget's work, and in a sense an "American voice" for Piaget, is indicated by the following statement from an aritcle by Brownell (1942) on problem solving:

> Much of this chapter, an amount of space which may appear to be disproportionate, is given to Piaget's account of the development of ability in problem solving. The writer had as an alternative the possibility of citing numerous other valuable researches, but these would have had to be dismissed with a word. The decision in favor of Piaget's research was made in full recognition of the loss of prestige which it has suffered during the past few years. Some of Piaget's investigation have been repeated in this country and elsewhere, but with different results; and some of his interpretation have been challenged. Nevertheless, with all their limitations, Piaget's studies seem to provide the most illuminating single description of the way in which children attain power in problem solving (p. 428).

However, Brownell does indicate several criticisms of Piaget's theory. They include: the prejudicial character of the problem tasks, the definition of reasoning as a highly formalistic type of thinking, the impression that at definite ages children reach definite levels of thinking, and the fact that Piaget makes adult reasoning unlike children's problem solving (pp. 430-431). Perhaps as Piaget's theories were further explained and developed in America over the next twenty years, Brownell would have found himself primarily in agreement with Piaget.

## Piaget's Theory of Cognitive Development

Piaget is often referred to as a child psychologist, but he has characterized himself as a "genetic epistemologist". The central question guiding Piaget's research was not "what are children like?", but "How does the relationship between KNOWER and the KNOWN change with the passage of time?" Piaget's methodology consists of first observing children's reaction to their surroundings. Then, based on these observations, he forms hypotheses about the sort of biological and mental structures that underlie their reactions. Next, he recomposes the hypothesis in the form of questions or problems that he then poses to children in order to reveal their thinking processes and so test the hypotheses. Piaget was extremely productive, and over the years has produced large numbers of problem situations that subsequently have been used by researchers world wide (Thomas, 1979, pp. 289-292). In this regard Skemp (1979) reports that Flavell ntending to write a one chapter summary of Piaget's theory completed the task seven years later, after having completed a book of about 500 pages. Piaget himself published five books during the years 1923 to 1932. The books were referred to by Piaget as his "adolescent" works. These early books gave his preliminary and tentative conclusions about children's intellectual development. In the United States the books were initally well received, and during the 1920's and 1930's Piaget's work was highly regarded. However, his views, as expressed in his early books, came under extreme criticism and America's interest in his research faded. But with the translation and publication of several of Piaget's later books in the 1950's, interest was revived (Ginsburg and

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Opper, 1969, pp. 1-7).

Smock (1976) feels that Piaget was greatly influenced by advances in theoretical physics during the 1920's and 1930's. The fundamental aspects of relativity that are inherent in Piaget's theory are that a) conceptual judgements are always relative to the postion of the observer, and that b) analysis of knowledge acquisition requires a description of its operational basis; that is, the mental operations of the individual is associated with the construction and maintenance of consistent patterns (structure) of his constantly changing relations with his physical and social environments. Thus, Piaget is unique among psychologists in that his emphasis on a "constructivist" theory of knowledge is indissoluble from his interpretation of "operationism".. According to Piaget, "reality" is constructed by the child, not imminent in mind or stimulus (Smock, 1976, p. 10).

Smock (1976) notes that Piaget's view of a child as one who is seen facing phenomenal disorder from which he must construct a coherent view of reality is a very different child than the one confronted by a stable reality as viewed by the typical "naive realist" psychologist of America (p. 10).

Thomas (1979) has indicated that Piaget's theory of knowledge and its acquisition differs from the "common sense", or popular beliefs, in four fundamental aspects. First common sense holds that knowledge is a body of information of beliefs a person has acquired through instruction, or through direct experience with the world. Second, common sense holds that a person's knowledge is a fairly faithful and accurate representation of what the person has been taught or witnessed. Third, common sense indicates that an individual's vast storehouse of knowledge is increased if he/she adds to it from his/her daily experiences, item upon item. Fourth, common sense holds that whenever an item of knowledge is recalled from the storehouse of memory, the item can be recovered in essentially the same condition as it was when first acquired (p. 292).

Piaget in opposition to his "common sense" approach does not agree that knowledge is a body of acquired information, but instead he conceives of knowledge as a process. To know something means to act on that thing, with the action being either physical or mental or both. Young children have only physical knowledge of an item, but as they grow older, they gain more experience with such direct, physical knowing, and mature internally so that they are increasingly freed from direct physical behavior in order to know something. They become capable of producing mental images and symbols that represent objects and relationships, and consequently, the older child's knowledge increasingly becomes mental activity. He thinks about things by carrying out interiorized actions on symbolic objects. Thus to Piaget, knowledge is a process or repertoire of actions, rather than an inventory of stored information (Thomas, 1979, pp. 292-293).

Piaget diagrees with the "common sense" idea of the way objects or events are recorded in a child's mind (perception). To Piaget the child does not take in a picture of objective reality, but the picture he obtains is biased (act

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by both the child's past experiences and his current stage of internal maturation. Thus, the way two children know (action) the same object will not be identical (Thomas, 1979, p. 293).

Piaget does agree with the "common sense" idea that the result of a person's past actions can be stored as memories to be retrieved when wanted. He agrees that the quantity of memories increase with age and experience, but he does not believe that remembering is simply a matter of summoning images of past events from memory and placing them in consciousness whereby they can be viewed in their passive, original condition. Remembering is a reenacting of the original process of knowing, but it is not simply a repetition of the original knowing. The child's mind has been altered by subsequent experiences and internal maturation, thus the interiorized recitation or rehearsal of the stored event is, as it were, "now performed on somewhat altered mental state" (Thomas, 1979, pp. 293-294).

To summarize, Piaget's system views knowledge as a process of acting, either physically and/or mentally, on objects, images, and symbols. The objects are from the world of direct experience, while the images and symbols can be derived not only from the "real world, but from memory as well. As Piaget (cited in Thomas, 1979, p. 294) has stated, "All knowledge is continually in a course of development and of passing from a state of lesser knowledge to one which is more complete and effective".

The preceding paragraphs have indicated an overall

direction of intellectual development that Piaget's theory is designed to explain. The subsequent paragraphs in this section will detail the more technical aspects of the mechanisms that bring about these kinds of developments.

Piaget has used the term schemes of an action to refer to the general structure of this action which conserves itself during repetitions, consolidates itself by exercise, and applies itself to situations with varying surroundings. Piaget (1968) states:

> A "scheme" is that part of an action or operation which is repeatable and generalized in another action or operation; it is that part which is essentially characteristic of action or operation (p. 15).

According to Mick and Brazier (1979), Piaget uses the word "schema" to refer to simplified images (p. 52). Piaget's "schemes" parallels Bartlett's (1932) notion of a "schema" as "...an active organization of past reactions, or of past experiences, which must always be supposed to be operating in any well-adapted organic responses" (p. 201). Thomas (1979) indicates that the only difference between Piaget's "schemes" and "schemata" is the English translation. His later writings were translated as "scheme", while the initial translation of his earlier writings from the original French contained the word "schema". Consequently, many reviewers of Piaget's work use the word "scheme" (cf. Thomas 1979) and others the word "schema" (cf. Harrison, 1967) in a similar fashion.

According to Piaget, the purpose of all thought and behavior is to enable the individual to adapt to his/her environment in ever more satisfactory ways. A child's development must be conceptualized in terms of schemes. The child acquires ever greater quantities of schemes that become interlinked in ever more sophisticated patterns. Crucial to the understanding of the process of evolution of these schemes is Piaget's notion of assimilation and accomodation. Piaget used the word "assimilation" to refer to that process of taking in or understanding events of the environment by matching the perceived features of those events to the individual's schemes. Piaget (1958) has states:

> To assimilate an object to a schema means conferring to that object one of several meanings and it is that attribution of meaning which thus requires, even when it takes place by observation, a system of more or less complex inferences. In brief, one could say that assimilation is an association accompanied by inferences (p. 59).

However, when the perceived structures of the environment fail to fit the child's available schema, even with some perceptual shaping of that structure, one of two consequences will result. First the event is ignored or passed by, and thus not assimilated. The second possible outcome of a poor match between the perceived environment and the individuals available schemes is not of total rejection, but of dissatisfaction, along with continued attempts to achieve a match. In this case schemes themselves may be altered in form or multiplied to match the perceived environmental realities. Piaget used the word "accomodation" to identify this process of altering existing schemes in order to permit the assimilation of otherwise incomprehensible events (Thomas, 1979, pp. 294-299). The importance of the roles played by assimilation and accomodation in Piaget's theory can be noted in the following statement:

> Thus it may be seen that intellectual activity begins with confusion of experience and of awareness of the self, by virtue of the chaotic undifferentiation of accommodation and assimilation. In other words, knowledge of the external world begins with an immediate utilization of things, whereas knowledge of self is stopped by this purely practical and utilitarian contract. Hence there is simply interaction between the most superficial zone of external reality and the wholly corporal periphery of the self. On the contrary, gradually as the differentiation and coordination of assimilation and accommodation occur, experimental and accomodative activity penetrates to the interior of things, while assimilatory activity becomes enriched and organized.....Intelligence thus begins neither with knowledge of the self nor of things as such but with knowledge of their interaction, and it is by orienting itself simulaneously toward the two poles of that interaction that intelligence organized the world by organizing itself (Piaget, 1954, pp. 354-355).

Piaget (1954) has used the term <u>equilibrium</u> to specity the ideal relationship between assimilation and accomodation:

> In their initial directions, assimilation and accomodation are obviously opposed to one another, since assimilation is conservative and tends to subordinate the environment to the organism to the successive constraints of the environment... Assimilation and accomodation are therefore the two poles of an interaction between the organism and the environment, which is the condition for all biological and intellectual operation, and such an interaction presupposes from the point of departure an equilibrium between the two tendencies of opposite poles (pp. 352-353).

Piaget views knowledge as "invariance under transformation: (Smock, 1976, p. 11). Piaget distinguishes between two aspects of an act of knowing: the <u>figurative</u> and the <u>operative</u>. The figurative aspect refers to the actions by which the individual produces a "copy" of reality, and in this case cognition is concentrated on the state of reality rather than

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transformations. Figurative knowledge is acquired through the activity of accommodation to the properties of the object. Piaget makes three sub-divisions among the figurative components of knowing. First is perception, which is a system that refers to the senses. The copy of a child's environment based on perception is often inexact, as in the case of visual illusions. A second component of cognition is imitation. It is by imitation that the child reproduces actions of persons or things. Although the child is involved in action, imitation produces only a copy of reality and is thus considered figurative. A third aspect of figurative knowledge is mental imagery. Mental imagery is used to "picture" a copy of absent object or events (Ginsburg and Opper, 1969, pp. 152-153).

Operative knowledge is the dynamic aspect of cognition and refers to actions that are used to change reality. Operations are the "action schemata" that construct "logical" transformations, or coordinated actions, on "states" of reality. Such logical systems of transformations operate either upon representation of events, or upon the cognitive system's own logical operations (Smock, 1976, pp. 16-17). Piaget makes a distinction between concrete operations and formal operations. Operations are ways of manipulating objects in relation to each other. If the objects are physically present, and the individual actually moves the objects, or by observing the objects determines how they might be moved, then these operations are said to be concrete operations. However, when concrete operations are transposed into propositions about the relationships that exist, or might exist among objects, and then these propositions are mentally manipulated, these intellectual actions are called formal operations.

It should be noted that just not any manipulation of an object is an operation. To be classified as operations, these actions must be internalizable, reversible, and coordinated into systems that have laws that apply to the total system, and not just the single manipulation (Thomas, 1979, p. 308). As Piaget (1972) has noted,

> operations...are actions, since they are carried out on objects before being performed on symbols. They are internalizable, since they can also be carried out in thought without losing their original character of actions. They are reversible... Finally, since operations do not exist in isolation, they are connected in the form of structured wholes (p. 8).

Piaget recognized that from day to day intellectual growth is continuous with no major jumps ahead from one day to the next. However, when he viewed the entire spectrum of growth years, he was able to distinguish what breaks indicated that the child had completed one phase of development and was now engaged in a further one. Consequently, Piaget identified a number of different series of developmental stages. For example, one series concerned understanding of physical causality, another the steps in imitation and play, another in the conception of moral principles and justice, and others in understanding number, space, and movement. However, underlying these specific series of stages is a basic set that provides for overall sensorimotor-intellectual development. Most interpreters of Piaget have viewed his writings as advocating four major periods of intellectual development: sensorimotor (birth to 2 years); preoperational (2 years to 7 years); concrete operational ( 7 years to 11 years); and formal operational (11 years and above) (Thomas, 1979, pp. 303-318; Ginsburg and Opper, 1969, p. 26). It should be noted that there is some measure of confusion among Piaget's reviewers as to the number of major periods, and corresponding ages that he advocates. Some interpreters see him as advocating three major periods, others four, and some five (cf. Mick and Brazier, 1979, p. 46 and Phillips, 1969, p. 11). This disagreement seems to hinge on the fact that sometimes Piaget will call a period of growth a "stage" and in another instance a "substage". Piaget (1958, pp. 4-15) indicates three stages (with various substages) while in another article he indicates four stages (1963, p. 3).

The close interplay between motor activity and perception in infants led Piaget to label the first two years as a sensorimotor period. During this period infants discover the relationships between sensations and motor behavior. They learn, for example, how far to reach when grasping, what happens when they push their food to the edge of the table, and that their hand is part of their body while the chair is not. Through many "experiments" infants develop a conceptualization of themselves as separate from the external world. A major discovery for them is the concept of "object permanence" - an awareness that an object continues to exist even when it is not present to the senses. For example, if a cloth is used to cover a toy within reach of an eight-month old, the infant will not attempt to search for the toy, and will act as if it failed to exist. On the other hand a ten-month old will continue to search for the hidden object. The older child appears to realize the object still exists although it is out of sight. It should be noted that even during this sensorimotor period, Piaget views the infant's activities, such as grasping and letting go, and other repetitive activities as indicating the acquiring knowledge is not accomplished by the environment's imposing reality on the child's mind. Rather, the repetitive activity is purposive, designed either to preserve or rediscover an act or skill. That is, it functions as practice (Hilgard, Atkinson, and Atkinson, 1979, pp. 70-71; Thomas, 1979, pp.304-305). It should be noted that Piaget has subdivided the sensorimotor level of development into six stages: reflexive, primary circular reactions, secondary circular reaction, coordination of secondary schemes, tertiary circular reaction, and beginning of thought (Ginsburg and Opper, 1969, pp. 29-66). Ginsburg and Opper have noted Piaget's notion of "curiosity" or "novelty" has impact even at this young age. According to Piaget's idea, the child actively seeks out new stimulation. Also, this motivations aspect is a relativistic concept. That which catches a child's curiosity is not the object per se, but rather the relation between the new object and the child's previous experience (p. 39).

The second period of intellectual development in Piaget's theory has been labeled the <u>preoperational thought period</u>.

The preoperational stage extends from the start of organized symbolic behavior, especially language, until about seven years of age. During this period the child will reconstruct his experiences from the sensorimotor level into representational thought. Piaget indicates that during this period when the child considers static states of a situation he will explain them in terms of the static perceptual configurations at the given moment, rather than in terms of the changes leading from one situation to another. When the child does consider transformations, he assimilates them to his own actions and not as reversible operations. The child lacks any notion of conservation at this level. For example, when having transferred a given quantity of liquid into a beaker which is more elongated than the initial container, the child believes the quantity has increased because the form of the container is different (Inhelder and Piaget, 1958, pp. 245-248). However, as the child reaches the age of five or six he enters into a stage Piaget calls "intuitive thought". It is a stage of transition between depending solely on perception, and depending on logical thinking. A child at the age of four or five, presented with a row of six red beads on a table, when asked to put an equivalent number of blue beads, will put a row of blue beads the same length as the red beads, without bothering to count. A child at the age of six or seven will line up the blue beads opposite the red ones, thus showing progress toward recognizing equivalent quantities. However, if the experimenter spreads out the r

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the red beads to form a longer line, the six or seven year old child thinks the number of blue and red beads are no longer equivalent. The child is unduly influenced by the perception of line length rather than the logic of quantity (Piaget, 1950, p. 132) (cited in Thomas, 1979, p. 311).

The next level of development in Piaget's scheme is called the concrete operational stage, and covers the approximate ages from seven to twelve years. Inhelder and Piaget (1958) state:

> In a general sense, by concrete operations we mean actions which are not only internalized, but are also integrated with other actions to form reversible systems. Secondly, as a result of their internalized and integrated nature, concrete operations are actions accompanied by an awareness on the part of the subject of the techniques and coordinations of his own behavior. These characteristics distinguish operations from simple goal-directed behavior, and they are precisely those characteristics not found at this first stage: the subject acts only with a view toward achieving the goal; he does not ask himself why he succeeds (p. 6).

They note also:

> From the standpoint of form, concrete operations consist of nothing more than a direct organization of immediately given data. The operations of classification, serial ordering, equalization, correspondence, etc., are means of inserting a set of class inclusions of relations into a particular content (for example: lengths, weights, etc.), means which it is presented to the subject ( p. 249).

By "internalizable" actions Piaget means that actions can be carried out in thought and not lose the "original character of actions" (Piaget, 1953, p. 8). By "reversible operations" he means "...the capacity to execute the same action in both directions, but being conscious that we are dealing with the same action" (Piaget, 1957, p. 44) (cited in Battro, 1973, p. 152). Or in another place, "reversibility" is d

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is defined as the permanent possibility of returning to the starting point of the operation in question" (Inhelder and Piaget, 1958, p. 272). Piaget has designated two distinct and complementary forms of reversibility: negation or inversion, in which one can return to the starting point by canceling an operation which has already been performed: and reciprocity, in which one can return to the starting point by logical compensating a difference. At the concrete operational level, the two forms of reversibility are employed independently of each other, but are not integrated into a structured whole (pp. 272-278).

The primary differences between the preoperational level and the concrete operational level of thinking relate not only to the greater command of the notions of conservation and reversibility, but also to the fact that the concrete operations child is capable of decentering his attention, of recognizing the way two or more dimensions of an event interact to produce a given result. This is illustrated by the previous example of the two beakers, one wide and one narrow. Where as the preoperational child believed the elongated beaker contained more water, the concrete operations child considers both dimensions simultaneously and recognizes their interaction (Thomas, 1979, p. 315).

Inhelder and Piaget (1958) have indicated that the most general property characterizing formal thought is a combinational system whereby the individual is able to determine all possible outcomes of an experiment. Operations are no longer restricted to dealing only with physical objects or reality. The individual at the formal thought stage can logically derive all possibilities in terms of hypothetical statements. Piaget has referred to this as "reflective thinking". These possibilities can be considered as a system of second order operations due to the fact that they describe relations between hypotheses rather than referring to objects directly (pp. 341-342). Piaget has indicated that the adolescent's thought structures come to approximate a mathematical lattice structure when the individual is using methods involving proportionality (Inhelder and Piaget, 1958, pp xxii-xiii). Piaget did not start with the idea that cognitive development conforms to the laws which govern mathematical structures. However, Piaget (1971) notes that many years of research into cognitive development led him to conclude that thought structures do come to approximate mathematical structures. He states, "yet mathematics constitutes a direct extention of logic itself, so much so that it is actually impossible to draw a firm line of demarcation between these two fields (p. 44).

The fourth developmental stage designated by Piaget is the <u>formal operational stage</u>. This period extends from approximately eleven years of age to about fifteen years of age. This stage is characterized by the development of formal, abstract thought operations, with which the child can reason with hypotheses rather than being restricted to reasoning only in terms of objects. Inhelder and Piaget (1958) have stated:

> Formal thinking is both thinking about thought

> (propositional logic is a second-order operational system which operates on propositions whose truth, in turn, depends on class, relations, and numerical operations) and a reversal of relations between what is real and what is possible (the empirically given comes to be inserted as a particular sector of the total set of possible combinations (pp. 342-343).

Inhelder and Piaget (1958) note in the concrete operational stage children can use both of the complementary forms of reversibility (inversion for classes and numbers and reciprocity for relations), but they never integrate them into the single total system found in formal logic. In comparison, the adolescent child:

> ...superimposes propositional logic on the logic of classes and relations. Thus, he gradually structures a formal mechanism (reaching an equilibrium point at about 14-15 years) which is based on both the lattice structure and the group of four transformations. This new integration allows him to bring inversion and reciprocity together in a single whole. As a result, he comes to control not only hypothetico-deductive reasoning and experimental proof based on the variation of a single factor with the others held constant (all other things being equal), but also a number of operational schemata which he will use repeatedly in experimental and logico-mathematical thinking (p. 335).

Piaget notes the attaining of the formal operations stage is not the end of intellectual growth. The framework of thought is complete, but the framework is not entirely filled in. The most obvious distinction between adult and adolescent thought is the greater lingering egocentrism displayed by adolescents. The adolescent, with the newly acquired skills of logical thought, is an idealist who expects the world to be logical. Such an individual fails to recognize or accept the reality that people do not operate solely on the basis of logic, and thus the adolescent becomes a reformer and critic of the older generation, envisioning a glorious future in which the younger generation will right today's wrongs. According to Piaget this idealism and egocentrism is tempered and becomes more realistic when the youth enters to occupational world or else enters a field of professional training. Consequently, over the years of youth and adulthood the framework of thought is filled in with more complex schemes or greater knowledge. This is, even though the adolescent is capable of all forms of adult logic, the adult in a sense does know more (Thomas, 1979, p. 317; Inhelder and Piaget, 1958, pp. 340-346).

Piaget (1976) has indicated that there are four factors which explain mental development; maturation, experience, social transmission, and equilibration. He states that each of these factors although vital to the sort of knowledge the child acquires in terms of mental schemes, and the time at which they are acquired, none is by itself sufficient to account for mental development. However, when taken together these factors regulate the four stages of cognitive development previously described (p. 74).

Piaget (1976) acknowledges that internal maturation plays a role throughout mental growth. Heredity not only provides the newborn child with the equipment to cope with problems, but also establishes a time schedule for new developmental possibilities throughout the child's growing years. Each act of maturation creates possibilities for new schemes to be created which would have been impossible previously. However, Piaget is explicit when noting that intelligence is not programmed. Thus, even though an individual is organically mature enough for a particular scheme to be created, the extent to which such potentialities are realized is determined by the types of experiences a child has with his environment. (pp. 72-74; Thomas, 1979, pp. 300-301).

Piaget (1976) has separated interaction with the environment into two varieties direct, unguided physical experience, and the guided transmission of knowledge, that is, education in the broad sense (social transmission) (pp. 72-74). In the case of physical experience the child manipulates, observes, listens to, and smells objects to note what occurs when they are acted upon. It is not the observation of the passive objects that develops intelligence, but the set of conclusions that child draws from those actions that bring about events and influence objects. Piaget (1976) has called this developing of conclusions, based on learning the result of the coordination, "logico-mathematical" experience. Piaget notes that physical experience is not a simple recording of phenomena, but "constitutes an active structuration", since it always involves an assimilation to logico-mathematical structures. He offers the example that comparing two weights presupposes the establishment of a relation, and therefore the "construction" of a logical form (p. 73).

Social transmission is the educative factor (broad sense) in that it refers to the transmission of knowledge to the individual from without. However, for a child to receive societal information, the child must have cognitive structures available which enable him to assimilate the information.

In the case of schooling the linguistic structure would be vital. Piaget (1976) indicates that even though social transmission is important, it by itself is insufficient. He states:

> Although necessary and essential, it also is insufficient by itself. Socialization is a structuration to which the individual contributes as much as he receives from it, whence the interdependence and isomorphism of "operation" and "cooperation". Even in the case of transmissions in which the subject appears most passive, such as school-teaching, social action is ineffective without an active assimilation by the child, which presupposes adequate operatory structures (pp. 73-74).

An example of the above could be cited when a parent (or school teacher) attempts to teach advanced mathmetics to a child that does not have the mental structures available to understand.

Equilibration is the factor which maintains a balance among the previous three factors. Piaget (1976) has defined "equilibrium" (later referred to as "equilibration") as a series of active compensations on the part of the individual in response to external disturbance, and an adjustment "that is both retroactive (loop systems or feedbacks) and anticipatory, constituting a permanent system of compensations" (p. 74). Equilibration is a process of "self-regulation" (p. 74). Piaget maintains that growth in knowledge is due to a conflict between events and structures, an imbalance between new experiences and what the individual knows. When new experiences are neither too novel nor too familar, they may be assimilated and may influence and change those existent structures. That is, these structures have accommodated to the new information. This cognitive conflict creates the transition from one stage to another (Sigel and Cocking, 1977, p. 21).

It should be noted that Piaget believes that affectivity and motivation cannot be isolated from the cognitive and intellectual evolutionary factors. He states (1959):

> It may even seem that affective dynamic factors provide the key to all mental development... There can be no affective states without the intervention of perceptions or comprehensions which constitute their cognitive structure... The two aspects, affective and cognitive, are at the same time inseparable and irreducible. (p. 75).

Piaget explains stage changes and motivation in terms of "cognitive conflict" and "logical necessity" (Sigel and Cocking, 1977, p. 22). Certain environmental demands and events require responses that have an inherent logic. The child thus learns to respond to these demands in logical fashion out of necessity. The situation of logical necessity frequently creates for the child a cognitive conflict. As an example, the child must learn that two objects cannot be in the exact same space at the same time. The conflicts that children have in their own activities as well as with other people, creat a "disequilibrium", or tension, and it is in the solution to these problems that propels the child from one competence level to another (p. 22).

Piaget also stresses the role of "self-regulations" in development. He notes (1976):

> ...it is impossible to interpret the development of affective life and of motivations without stressing the all-important role of self-regulations, whose importance, moreover, all the schools have emphasized, albeit under various names (p. 75).

The importance that Piaget attaches to these self-regulations (partial compensations) and the equilibration factor cannot be over-emphasized. Such importance is indicated by Piaget's (1976) statement:

> The regulations are directly dependent on the equilibration factor, and all later development (whether of thought, or moral reciprocity, or of cooperation) is a continuous process leading from the regulations to reversibility and to the extension of reversibility. Reversibility is a complete--that is totally balanced system of compensations in which each transformation is balanced by the possibility of an inverse or reciprocal.
>
> Thus equilibration by self-regulation constitutes the formative process of the structures we have described (pp. 75-76).

Educational Implications

Piaget's theory suggests a general proposition that should have important consequences for education. This proposition is that the young child differs in several ways from an adult: in methods of approaching reality, in the views of the world, and in the uses of language. Piaget's investigations concerning concepts of number and verbal communication have enabled him to contribute to a basic change in the way educators view learners. As a result of Piaget's work, educators have become convinced that the child is not just a minature and less wise adult, but a unique being with a mental structure which is distinctive and qualitatively different from that of adults. For instance, the child below seven years of age sincerely believes that water may gain or lose quantity when poured back and forth from different shaped containers (Ginsburg and Opper, 1969, p. 219).

It should be noted that Piaget considers himself a student of human knowledge and development, not of educational design. However, by implication, a surprising variety of educational recommendations and practices have been derived from his theory (Resnick and Ford, 1981, p. 186). Perhaps the single most important proposition that can be inferred from Piaget's work for use in the classroom, is that children learn best from concrete activities. For this reason a school should encourage the pupils' activity, and his manipulation and exploration of objects. Sigel and Cocking (1977) paraphrase Piaget: "Each time one teaches a child something he could have discovered for himself, the child is kept from inventing it and consequently from understanding it completely" (p. 20). Since Piagetian theory indicates that social interaction is essential to developing a multiperspective view which is essential for objectivity, the teacher should encourage group activities and projects (p. 20).

The teacher does, however, have an indispensable role in the education process. The teacher's role should be that of a mentor, stimulating initiation and research. He should determine the child's current stage of development in the various areas the curriculum is designed to promote, and then construct the initial devices and create the situations which present useful problems to the child. The teacher is needed to provide counter examples to encourage reflection and reconsideration of hastily drawn conclusions. The teacher should not be a go-between that transmits ready-made solutions and facts to the student (Piaget, 1974, pp. 15-16).

The teacher must be acutely aware of Piaget's notions of assimilation, accommodation, and equilibration. According to Piaget, these factors are primary in motivating students to

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to learn. Piaget (1959) states: "...to say that a subject is interested in a certain result or object thus _means_ that he assimilates it or anticipates an assimilation and to say that he needs it means that _he possesses schemas requiring its utilization_" (p. 86). Disequilibrium occurs when the student assimilates data from his immediate experience into existing mental structures. As cognitive structures change to accommodate the new data, equilibrium is restored. Cognitive conflict generates a need to establish equilibrium between the new information and existing schemas. This condition is the motivation for cognitive activities. Equilibration theory holds that what is learned depends on what the learner can take from the given situation by means of the cognitive structures available to him. The child will take interest in those situations that generates cognitive conflict. If the task demands are so unusual as to be unassimilable or so obvious as to require no mental work, the student will not be motivated (Smock, 1976, pp. 14-15). Thus, the teacher must choose materials and topics with extreme care. Lovell (1971) notes:

> The job of the teacher is to use his professional skill and provide learning situations for the child which demand thinking skills just ahead of those which are available to him. It is a question of keeping the carrot just ahead of the donkey's nose. When a child is almost ready for an idea, the learning situation provided by the teacher may well "precipitate" the child's understanding of that idea (p. 17).

Smock (1976) indicates that a large amount of confusion concerning Piagetian theory for instructional practice and educational research derives from a failure to consider the "figurative" and "operative" aspects of intellectual functioning. Figurations and associated acts are based on physical and social environment. On the other hand, operational knowledge is not based on abstractions from physical objects and specific events, but rather is derived by abstractions from coordinated actions relevant to those events. That is, operations are those action schemata that construct "logical" transformation of "states". Although, admitting that figurations and operations have implications for, but not necessarily causal effect on each other, Smock feels that relationships between figurative and operational structures has not been completely researched. He believes that the failure to consider both aspects has lead to many contradictory research findings with respect to Piagetian theory. In particular, he indicates that many studies have generated little or no "cognitive conflict" and consequent negative findings, due to the fact that any disparity belongs to the experimenter's reality and is external to the child's logical operational system (pp. 16-20). Supporting Piaget's view that intellectual development brings a gradual transformation of overt actions into mental operations, Harrison (1967) proposes that the teacher should assist the internalization and schematization process by having students perform actions with less and less direct support from external activities. For example, a student might be led to operate directly on physical objects, then on pictorial representations, and finally on cognitive anticipations of operations not actually performed until the original external operations take place internally independent of the environment (pp. 73-74).

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Specifically referring to the educational difficulties in the learning and teaching of mathematics, Piaget has indicated that the problem is not one of aptitude, but rather from affective blocking or inadequate preparation. He feels the failure of formal education can be traced to the fact that it begins with language and illustrations rather than real practical action. Piaget emphasizes that verbalization does not guarantee understanding, nor does understanding depend on verbalization. When the teacher tries to impart knowledge in a verbal manner, the result is frequently learning of a very superficial nature (Ginsburg and Opper, 1969, pp. 221-222). Preparation for mathematics learning should begin in the home with concrete manipulations that foster awareness of basic logical, numerical, and mensurational relationships. This type of activity should be systematically developed and expanded throughout the primary grades until it takes the form of elementary physical and mechanical experiments by the time secondary education begins (Piaget, 1951, pp. 95-98) (cited in Harrison, 1967, p. 71). Duckworth (1964) quotes Piaget as taking exception to those who would <u>teach</u> the "structure" of mathematics to children. Piaget stated:

> Teaching means creating situations where structures can be discovered; it does not mean transmitting structures which may be assimilated at nothing other than a verbal level (p. 498).

Finally, Resnick and Ford (1981) suggest that in terms of contributions to mathematics education, Piaget's greatest contribution is perhaps the clinical interview technique.

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They state:

> It provides a means by which teachers can understand what children understand. This is a crucial step in an educational strategy that seeks to match instruction to children's development. Teachers can cultivate their own skills of observing and questioning, as well as of setting interesting problems. As they become better at this, they begin to note details of children's thinking that had not been apparent before and find themselves able to follow children's lines of reasoning more clearly. Under these conditions, mistakes are not seen as poor thinking, but as information about each child's current understanding. On this basis, tasks and questions can be posed that represent the best match in terms of intellectual "stretching" ... (p. 192).

Comparison With Stimulus-Response and Gestalt Theories

Piaget indicates that the fundamental relation involved in all developement and all learning is not the relation of association. In the stimulus response schema, the relation between stimulus and response is understood to be association. Piaget believes the fundamental relation is assimilation rather than association. He feels that a stimulus is important only when it can be assimilated to a cognitive structure. Piaget indicates that the activity on the part of the learner is underplayed in the stimulus-response schema. Existing between the stimulus and response is an organism with its cognitive structures. The response exists in the cognitive structure before the stimulus is enacted. Piaget cites a study by Berlyne, who spent a year working in Geneva, in which an attempt was made to translate Piaget's theories on the development of operations into Hullian stimulus-response language. Berlyne found it necessary to introduce the concepts of "transformation responses" and "internal reinforcements".

Piaget noted that these are nothing more than his "operations" and "equilibrations". Piaget, referring to his theory, states:

> So you see that it is indeed a stimulus-response theory, if you will, but first you add operations then you add equilibration. That's all we want! (Piaget, 1976, pp. 78-79).

Piaget (1976) indicates less disagreement with Gestalt theory. He notes that he had been aware of the works of Wertheimer and Kohler when he began his own research, that he would have become a gestaltist (p. 121). The two theories parallel each other in that cognitive activities and the reality on which they act are structured totalities. Both consider intelligence and perception as part of the system of equilibrations. The primary difference between the two theories is that the gestalt is more static than Piaget's dynamic and modifiable schema. In addition, Piaget's schema is always the result of differentiation, generalization, and integration of earlier schemata, while the gestalt is an outgrowth of a certain level of neural maturation, given a particular perceptual field, and not of past environmental interactions (p. 121; Harrison, 1967, p. 67).

Bruner's Theory of Mathematics Learning

The noteworthy flaws in Piaget's extensive theory of cognitive development, as far as many American psychologists were concerned in the 1950's, was Piaget's lack of definite statements as to how or why a child passes from one stage of operations to another, and Piaget's belief that logical structures are independent of language (Kagan, 1966, pp. 98-112). In the 1950's American experimental psychologists, ur

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under the leadership of Jerome Bruner, were beginning extensive research into the analysis of the variables affecting and processes underlying concept learning. These psychologists not only found points of disagreement with Piaget, but also pushed the associationistic view of learning into disfavor in the United States (Bourne, et al, 1979, p. 169).

Bruner and his colleagues conducted experiments with adults in which they examined the strategies people employed in the complex process of sorting and classifying that constitutes concept development. Those pioneering efforts of Bruner and his associates resulted in the publication of A study of Thinking (Bruner, Goodnow, and Austin, 1956). A major finding from these concept attainment studies was the realization that researchers can describe and evaluate strategies in a systematic way in terms of their objectives. It was this laboratory experimentation with adults that led Bruner to begin to examine the cognitive processes of children. He became particularly interested in how children mentally represented the concepts and ideas they were learning.

That Bruner is essentially Piagetian, is well illustrated by the fact that Bruner's 1966 publication Studies in Cognitive Growth is dedicated to Piaget (Bruner, 1966A, p. xv). He notes, "many points of disagreement are nevertheless minor by comparison with the points of fundamental agreement we share with Professor Jean Piaget" (p. xv). Bruner, like Piaget, was a constructivist, that is, reality is constructed by the individual. Anglin (1973) gives a succinct description of Bruner's theory, which shows the many parallels to Piaget's

theory:

> Though obviously different in basic ways, several modes of acquiring knowledge - including the perception of an event, the attainment of a concept, the solution of a problem, the discovery of a scientific theory, and the mastery of a skill - can be described at a formal level in ways that are strikingly similar. In short, each can be viewed as a kind of problem whose solution is actively, though not necessarily consciously, constructed. Construction usually involves a recursive process, in which the first step is an inferential leap from sense data to a tentative hypothesis achieved by relating incoming information to an internally stored model of the world based upon past experience. The second step is essentially a confirmation check in which the tentative hypotehsis is tested against further sense data. In the face of a match the hypothesis is maintained, in the face of a mismatch the hypothesis is altered in a way that acknowledges the discrepant evidence. That the sense data might be called cues, clues, instances, or experimental results; that the hypothesis might be called a category, a rule, a principle, or a theory; that the internal model might be called a generic coding system, a focus, a system of representation, a cognitive structure, a schema, or a paradigm; that the recursive process might be called inference and confirmation check, strategy, analysis by synthesis, induction and deduction, conjecture and refutation, or the hypothetico-deductive method all these should not be allowed to obscure the underlying formal similarity of diverse kinds of mental activities (p. xxii).

For both Bruner and Piaget, cognitive development involves qualitative rather than quantitative changes in the cognitive structures which are available at different age levels. Both believe in the importance of action in infancy. Both view growth in childhood as moving the individual from a state of domination by immediacy and appearance to a state whereby the individual is able to realize connectedness over time as well as invariance in face of surface change (Anglin, 1973, xviii).

However, there are some major differences between the theories of intellectual development by Bruner and Piaget. Perhaps, the most critical difference concerns the significance attributed to cultural agents in the course of development. Piaget's theory, while not being a wholly hereditary account, basically gives little weight to the environment and views the course of development as relatively fixed. Piaget seemed little concerned with "the American question" of accelerating intellectual development. (Anglin, 1973, p. xx). In contrast Bruner, who believes that a theory of development should go hand in hand with a theory of instruction, argued that mental growth to a very considerable extent is dependent upon mastering techniques that are embodied in the culture and thus passed down to the child by cultural agents (Bruner, 1966B. p. 21). In line with this belief, Bruner and his colleagues consistently attempted to design pedagogical means for accelerating intellectual achievements. Another real difference between Bruner and Piaget concerns the question of whether the changes that occur in a child's problem solving skills at around six or seven years of age are mediated by language. Bruner believes that symbolic, especially linguistic transformations play an important role in guiding thought. Piaget argues that the linguistic correlates of success in Bruner's studies were symptoms of concrete operation, but in no way could be causally linked to the increased competence. It should be pointed out that Piaget's theory is more of a "descriptive" nature, while Bruner's theory is more fundamentally psychological, and in comparison to Piaget's theory, of a "prescriptive" nature. Also, in contrast to Piaget, Bruner had a persistent concern with the relevance of his theories to practical concerns, particularly education (Anglin, 1973, pp. xviii-xxi).

Piaget's theory of equilibration, as the mechanism responsible for the child advancing from one stage to another, was criticized by Bruner for its lack of "specificity" and for its "circularity" of prediction about growth. However, Bruner did admit that "cognitive conflict" notion of growth is valid, but noted, "The rub is that there are many cognitive conflicts of this kind that do not lead a child to grow" (Bruner, 1966A, p. 4). In fact Bruner proposed a three "stage" model of intellectual development, whereby growth was given an "impulsion" by disequilibrium.

Bruner (1966B) stated:

> What comes out of this picture, rough though I have sketched it, is a view of human beings who have developed three parallel systems of processing information and for representing it - one through manipulation and action, one through perceptual organization and imagery, and one through symbolic apparatus (p. 28).

Bruner (1966A) calls the mode based on manipulation and action enactive (pp. 16-21). This is a mode for representing past events through an appropriate motor response. This mode is thought to be the only way infants can remember things during what Piaget has called the sensorimotor stage. Adults would use this mode for example, when riding a bicycle for the first time in years. Resnick and Ford (1981) suggest that children who add numbers by perhaps tapping their fingers are using the enactive mode of representation (p. 112).

Bruner (1966A) called the mode of representation based on perceptual organization and imagery the iconic mode (p. 21). Images stand for perceptual events in the same manner that a picture stands for an object pictured. The mode of representation based on symbolic apparatus, Bruner (1966A) called the symbolic mode. Bruner hypothesized such properties as "categoriality", "hierarchy", "predication", "causation", and "modification" for the symbolic mode, and noted that any symbolic activity is not logical without these properties. According to Bruner, the three modes of representation occur in the life of the child in the order of enactive, iconic, and symbolic. Each mode depends on the preceeding one for its development, yet all remain more or less intact throughout life. At the enactive level, actions cannot be transformed. At the iconic level, which is based on images, they can be transformed but lack a generalization quality. It is the attainment of the symbolic level and the successful internalization of language, that enables the individual to represent and transform environment regularities with power and strategy. At the symbolic level, the child can use actions and images arbitrarily, and perhaps will use all modes of representation simultaneously. Bruner indicated that without special training in the symbolic representation of experience, a child will grow into adulthood depending in large measure on the iconic and enactive modes of representing and organizing the world (pp. 21-48) (Harrison, 1967, p. 82).

Bruner does not seem to distinguish between the words "thinking" and "understanding" (Herscovics, 1979, p. 98).

However, he has postulated two modes of thinking: "intuitive" thinking and "analytic" thinking. According to Bruner (1960) thinking does not advance in well-defined steps, but seems to involve an implicit perception of the total problem with only very little awareness of the processes used. Typically intuitive thinking rests on familiarity with the domain of knowledge involved and with its structure. On the otherhand, analytic thinking proceeds characteristically one step at a time. A specific plan of attack as well as full awareness of the operations employed is normal (pp. 55-68).

Bruner (1966B) indicated that good teaching involves instruction which facilitates student comprehension of a structure of a discipline, and in instruction which develops learner understanding of process. He favored a "discovery" format of teaching. "Optimal structure" for Bruner referred to a set of propositions for which a larger body of knowledge could be generated. Since ideal structure depends upon its power for simplifying information, generating new propositions, and increasing the manipulative ability of a body of knowledge, structure must always be related to present ability and knowledge of the learner (pp. 44-48).

He states:

> ...a theory of instruction seeks to take account of the fact a curriculum reflects not only the nature of knowledge itself but also the nature of the knower and of the knowledge getting process... To instruct someone in these disciplines is not a matter of getting him to commit results to mind. Rather, it is to teach him to participate in the process that makes possible the establishment of knowledge. We teach a subject not to produce little living libraries on the subject, but rather to get a student to think mathematically for himself

> to consider matters as a historian does, to take part in the process of knowledge getting. Knowing is a process, not a product (p. 72).

Reporting on his collaboration with Dienes, Bruner (1966B) had specific ideas with respect to mathematics learning:

> We reached the tentative conclusion that it was probably necessary for a child, learning mathematics to have not only a firm sense of the abstraction underlying what he was working on, but also a good stock of visual images for embodying them. For without the latter it is difficult to track correspondences and to check what one is doing symbolically (p. 66).

Again, with respect to mathematics learning, he made the following observations about children using a wooden model in attempting to square quadratic expressions:

> The children always began by constructing an embodiment of some concept, building a concrete model for purposes of operational definition. The fruit of the construction was an image that "stood for" the concept. From there on, the task was to provide means of representation that were free of particular manipulations and specific images. Only symbolic operations provide the means of representing an idea in this way. ...what struck us about the children as we observed them is that they not only understood the abstractions they had learned but also had a store of concrete images that served to exemplify the abstractions. When they searched for a way to deal with new problems, the task was usually carried out not simply by abstract means but also by "matching up" images (p. 65).

And in another statement:

> We would suggest that learning mathematics reflects a good deal about intellectual development. It begins with instrumental activity, a kind of definition of things by doing them. Such operations become represented and summarized in the form of particular images. Finally and with the help of symbolic notation the remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with. But while, once abstraction is achieved, the learner becomes free in a certain measure of the surface appearance of things he nonetheless continues to rely upon the stock of imagery he has built en route to abstract mastery. It is this stock of imagery that permits him to work at the level

> of heuristic, through convenient and nonrigorous means of exploring problems and relating them to problems already mastered (p. 68).

Bruner (1966B) disagreed with Piaget's stage development of "readiness". He believed that complex notions could be presented in an understandable form at any age. He states:

> Any idea or problem or body of knowledge can be presented in a form simple enough so that any particular learner can understand it in a recognizable form (p. 44).

This often misunderstood statement, is not advocating, for example, symbolic, mathematics to very young children, but indicates that, if a teacher is clever enough, any idea can be presented in a somewhat recognizable form in one of the three modes: enactive, iconic, or symbolic, to any child. One consequence of the previous statement by Bruner is that he advocates a "spiral" curriculum (Bruner, 1973A, pp. 423-425). What he is not advocating is the premature presentation of symbols:

> What is most important for teaching basic concepts is that the child be helped to pass progressively from concrete thinking to the utilization of more conceptually adequate modes of thought. But it is futile to attempt this by presenting formal explanations based on a logic that is distant from the child's manner of thinking and sterile in its implications for him. Much teaching in mathematics is of this sort. The child learns not to understand mathematical order but rather to apply certain devices of recipes without understanding their significance and connectedness (p. 416).

He further states: "For it is in the development of symbolic representation that one finds, perhaps, the greatest thicket of psychological problems (1973B, p. 327)." And, finally, with respect to the teaching of algebra:

> We provide training in addition, then we move to abstract symbols like a + a + a and see whether 3a emerges as the answer. Then we test further to see whether the child has grasped the idea of repeated addition--(or) multiplication. We devise techniques of instruction along the way to aid the child in building a generic code to use for all sorts of quantities. If we fail to do this, we say that the child has learned in rote fashion or that--we have given the child 'mechanical' rather then insightful' ways of solving the problem. The distinction is not between mechanical and insightful, really, but whether or not the child has grasped and can use the generic code we have set out to teach him (1973C, p. 222).

Thus, on the basis of Bruner's theory, a researcher investigating a student's understanding of algebra should note the actions that are performed, the imagery used, as well as any symbolic system used.

Skemp's Theory of Mathematical Understanding

Richard Skemp (1979B) of Great Britain has developed a comprehensive model of intelligence which he offers as a replacement for the current models that are based on "I.Q.". Skemp (1979A) indicates that such I.Q. models can only rank order individuals with respect to intelligence. They do not deal with what intelligence is for, how it works, or how learners can be aided to make the best use of whatever "intelligence" they possess (p. 44). Skemp's model purports to do all of these.

The starting point for Skemp is his observation that most human behavior is "goal directed"; together with the conjecture that cumulatively, success in achieving goals is a major factor of survival. For goal directed activity operating on the physical environment, there exists a "director system", which receives information about the present stage of the "operand" (what is being acted on), compares this with a selected "goal state", and with the aid of a "plan" which it constructs from available "schemas", takes the operand from its present state to its goal state and keeps it there. This director system, Skemp called "delta-one" (Skemp, 1979A, p. 44). Skemp has also designated a "delta-two" director system. However, its operands are not in the "outside" environment, but in delta-one. That is, they are "mental objects". The purpose of delta-two is to optimize the functioning of delta-one. The job of delta-two is goal-directed mental activity of many kinds which includes "learning". Learning includes the construction and testing by delta-two within delta-one of the schemas which delta-one must have to function properly.

Skemp (1979B) makes a distinction between the environment in which physical actions and activities take place and a "mental realm" (p. 21). Individulas build "mental models" of the physical environment. For a director system to function properly, the mental models must match "actuality". Important, also, is the accuracy with which the goal state as represented in an individual's mental model "matches the consequences of achieving that state", and the accuracy and completeness of the paths by which the goal state may be reached (p. 23). Skemp notes that mental models differ in significant ways from physical ones. One of the chief characteristics of these mental models "is that their elements nearly always represent not just one actual object or event, but what is common to a number of these." He calls a mental entity of this type a "concept". By representing what is common to a variety of actual states, concepts enable an individual to act successfully in a wide range of actual systems by means of the same director system. Skemp calls models made of a number of interconnected concepts a "conceptual structure". The process by which certain qualities of objects and events are internalized as concepts, he call "abstraction". Skemp emphasizes that the development of conceptual structures and the ability to use the process of abstraction is indicative of the director system's openness to change - to "learn". A distinctive quality of intelligence can be inferred by an organism's ability to construct new director systems, and to improve the ones it has (pp. 24-25). He states:

> Intelligent learning is now conceptualized as a process involving two director systems. These are:
> (a) a director system delta-one, whose state can be changed in this way. Such a director system is described as teachable.
> (b) a director system delta-two, who operand is the director system described in (a), and whose function is to take it towards states which make possible optimal functioning (pp. 89-90).

Skemp notes that when an individuals director system does not have the ability to change as in "intelligent learning" and change toward a goal-state gives rise to frustration and the inability to change away from an anti-goal state gives rise to anxiety (p. 16).

Skemp (1979B) calls the formation of a concept based on incoming sensory-data "perceptual learning", and such a concept a "primary concept". Concepts which the individual derives from other concepts are called "secondary concepts" (pp. 118-120). For Skemp a "schema" is "a structure of connected concepts" (p. 190). He terms the ability to makes one's mental processes and schemas the objects of conscious view "reflective intelligence" (p. 175). Skemp writes of "assimilation" - "This emphasis in our experience of actuality towards what is like our existing schemas", "expansion" of schemas, and "reconstruction" of schemas. It is interesting to note that Skemp in this 1979 formulation of his theory does not use the word "accomodation", but earlier (Skemp, 1972, p. 190), did use this word.

It is not the purpose of this review to completely explore Skemp's model of intelligence in detail (Skemp's explanation required 324 pages), but to give a synopsis that will aid in the description of his theories with respect to the teaching and learning of mathematics. His model of intelligence is primarily Piagetian, though he denies it emphatically. That Skemp's "assimilation", "expansion", and "reconstruction" equals Piaget's "assimilation" and "accommodation" is obvious. However, Skemp feels that Piaget fails to distinguish between "psychological" and "physiological" assimilation. Skemp indicates that his use of "schema" as a mental structure existing in its own right contrasts with Piaget's "schema" as being linked to actions. Importantly, Skemp believes that "Piaget's view that any act or mental state, can in itself be described as intelligent" is inferior to Skemp's notion that intelligence cannot be inferred from behavior in itself, but only from "changes in behavior" (Skemp, 1979B, pp. 212-221).

That Skemp's theoretical differences with Piaget are not major has been seen by the following reviewers. Bolton (1977) states, "...the main features of Skemp's schematic learning are essentially Piagetian" (p. 137). Fehr (1966) commenting on Skemp's early work on concepts, comments, "This is quite in agreement with Piaget's philosophy, developed through his study of children's thinking, namely, that to 'know a concept is to act on it'" (p. 224). Harrison (1967) writes, "Skemp's assertion that he has added to Piaget's description in his discussion of reflective intelligence seems somewhat more illusory than real in the light of an examination of Piaget's whole theory" (p. 125).

The primary feature of Skemp's theory is the fact that he has in particular addressed the teaching and learning of mathematics for not only young children, but also those at the secondary level. In this regard, he has extended the work of Piaget and Bruner, whose research in mathematics learning dealt mainly with young children that had not attained the formal level of thought. Skemp has written extensively about the teaching and learning of mathematics, particularly algebra, as it pertains to students that are on the verge of the formal stage, or have in fact reached it.

Skemp views algebra as a "generalized arithmetic". This is indicated by his statement, "...the number schema, combined with the idea of a variable, together lead straight into algebra" (Skemp, 1971, p. 237). In another place he wrote:

(cited in Harrison, 1967, p. 106).

> Just as the number concepts and arithmetical processes are abstractions from and generalization of experiences with material objects, so are the algebraic concepts generalizations of experiences with numbers and arithmetical processes.

Skemp (1971) as previously noted, indicates that concepts were either "primary", which are based on sensory experiences, or "secondary", which are derived from other concepts (pp. 118-119). He believes that all mathematical concepts are secondary concepts. Skemp states that, "a concept is a (learnt) realization of some regularity in actuality" (Skemp, 1979B, p. 29). He also ordered concepts: "If concept A is an example of concept B, then we shall say that B is of a higher order then A" (Skemp, 1971, p. 25). He defines relationships between concepts and speaks of a "conceptual hierachy" for interrelated concepts (p. 25). Of extreme importance to the field of mathematics, and for this study as well, is Skemp's assertion that:

> In general, concepts of a higher order than those which a person already has cannot be communicated to him by a definition, but only by collecting together, for him to experience, suitable examples (p. 26).

Definitions are useful for adding precision to the boundaries of a known concept, or of indicating its relation to other concepts. They can also be used to communicate new concepts of a <u>lower</u> order (p. 26).

Skemp, like Bruner, gives a great amount of credit to language in the intellectual developmental process. He defines a "language" to be a "set of symbols, with their associated concepts, such that for certain relations between the symbols there correspond certain relations between the concepts" (Skemp, 1979B, p. 159). A symbol is a primary concept, and reflective thought is closely connected with associating the involved concepts with a symbol. He suggests that verbal symbols concentrate our attention on one part of a schema at a time, and is analytic; whereas visual symbols is better at showing how parts relate to each other, and is mainly synthetic (p. 158). If concepts are sufficiently well formed to be activated by encountering of thinking of the associated symbol, then symbols aid in helping to show structure, making routine thinking automatic, understanding information, and retrieving information. Symbols are essential for reflective activity; they help acquire the raw data, and then enable the individual to manipulate the ideas, first in one way and then in the other (p. 160).

Particularly in mathematics, symbols make a major contribution. It is essential in mathematics that elementary processes become automatic, thus freeing the individual's attention to concentrate on new ideas. In mathematics, this is done by detaching the symbols from their concepts, and manipulating them accoring to well-formed habits without attention to their meaning. This automatic performance of routine tasks must be clearly distinguished from the mechanical manipulation of meaningless symbols, which is not mathematics. A machine does not know what it is doing. A mathematician, working automatically, can at any time he wishes pause and re-attach their meanings to the symbols; and he must be able to pass easily from one form of activity to the other, according to the requirements of the task (Skemp, 1971, p. 89). Skemp emphasizes that without the ability of the individual to invest symbols with the appropriate meanings, they are useless (p. 89). It follows that in using symbols in the teaching and learning of mathematics, it is imperative that both the student and teacher have the same concepts associated with the symbols (Skemp, 1979, p. 153).

Skemp (1979B, p. 158) notes that reflective activity which involves making an idea conscious seems to involve associating the idea with a symbol. Concepts are elusive objects and it may be that symbols are the most abstract kind of concept of which an individual can be aware. It is largely by the use of symbols that humans achieve voluntary control over their thoughts (Skemp, 1971, p. 83). A far reaching type of reflective activity is that which leads to what Skemp call "mathematical generalization". An example is offered that deals with exponential notation. After defining the notation by examples such as:

$$a^2 = a \cdot a$$
$$a^3 = a \cdot a \cdot a$$
$$a^4 = a \cdot a \cdot a \cdot a, \text{ etc.}$$

it can be seen rather easily that

$$a^2 \cdot a^3 = a \cdot a \cdot a \cdot a \cdot a$$

Having developed these schema the individual can "formulate" it symbolically as $a^m \cdot a^n = a^{m+n}$, where a is any real number and m and n are natural numbers. If the learner reflects on the "form" of the method, while ignoring the content and original restructions, and considers such "illegal" operations as $2^0 \cdot 2^3 = 2^{0+3} = 2^3$ (as being valid operations), and $2^{-3} \cdot 2^3 = 2^{-3+3} = 2^0$ and determines that logically $2^0 = 1$ and $2^{-3} = 1/2^3$, then consequently obtains an extension of the concept of exponent and the method involved, and writes $a^m a^n = a^{m+n}$ for all <u>integers</u> m and n; Skemp would say the individual has used mathematical generalization (pp. 59-61; Mick and Brazier, 1979, pp. 60-61).

Perhaps Skemp's major contribution to the teaching and learning of mathematics is his conjectures pertaining to "understanding". He states:

> To understand a concept, a group of concepts, or symbols, is to connect it with an appropriate schema. To understand an experience is to realize it within an appropriate schema (Skemp, 1979B, p. 148),

and,

> We may mis-understand, meaning we think we understand something when we do not...because we connect it wrongly to an appropriate schema...or we may connect it to an inappropriate schema (p. 148).,

and also,

> In the complete absence of an appropriate schema, not only can there not be any understanding at all, but there may be complete unawareness of its absence (p. 147).

Skemp (1979A) indicates that an "appropriate schema" is "appropriate to the subject matter" and "appropriate to the goal to be achieved" (p. 45). He notes, without details, that there could possibly be other considerations.

Skemp does not elaborate on what an "appropriate" schema is, but does indicate the "qualities" of a schema. In short he notes "the difference between a schema and a set of isolated concepts is, that in the schema the concepts are connected..." (Skemp, 1979B, p. 187). Skemp describes "connections" as either "associative" or "conceptual", but cautions that a connection which is conceptual for one person may be associative for another. The higher the proportion of conceptual links to associative links the higher the quality of the schema. A critical feature for connected concepts is that the "activation" of one should lead to the activation of the other (pp. 187-189). Skemp uses the analogy of a "cognitive map" (like Tolman) to indicate that a schema is "a structure of connected concepts" (p. 190). Among the features important for the effective use of a schema are, relevance to the task at hand, extent of its domain, the quality of organization which makes it possible to use concepts for a lower or higher order, the quality of connections (associative or conceptual), the degree to which it can function when confronted with irrelevant information, the degree to which it can assimilate new facts and assimilatory power relative to other schemas (pp. 190-191). Apparently for Skemp, the learning of a new concept in algebra would require a well-conceptualized bank of prerequisite concepts.

Skemp (1979B) has put forth two types of understanding: "instrumental" and "relational":

> Instrumental understanding, in a mathematical situation, consists of recognizing a task as one of a particular class for which one already knows a rule. To find the area of a rectangle, multiply the length by the breadth. For a triangle, calculate half times the base times the perpendicular height.... The kind of learning which makes this possible, which I call instrumental learning, is the memorizing of such rules.
>
> Relational understanding, in contrast, consists primarily in relating a task to an appropriate schema... If he has an appropriate parent schema, someone who does not know the rule can still calculate the area of a given trapezium by dividing it into

a triangle and a parallelogram... (p. 259).

In an earlier article Skemp (1976) indicates that the terms instrumental understanding and relational understanding were suggested to him by Steig Mellin-Olsen of Bergen University. Mellin-Olsen and Skemp justify their terminology by their belief that students and teachers say they understand something if they can use a rule or procedure to obtain the correct answer, thus using mathematics in the same way an individual learns to use an instrument. In relational understanding the individual is aware of the underlying mathematical relationships - which includes instrumental understanding (p. 20).

Skemp (1976) notes that it is not a self-evident truth that mathematics teachers should strive for relational understanding. Instrumental understanding has several (apparent) advantages: instrumental mathematics is usually easier to "understand" ("minus times minus equal plus"), the rewards are more immediate (success breeds confidence and effort), often less content coverage is required, and the right answer can usually be obtained faster (even theoretical mathematicians use instrumental thinking). On the other hand relational understanding is more adaptable to new tasks (one does not need a new rule for each new situation), it is easier to remember (but harder to learn), it often can become a goal in itself, and relational schemas are organic in quality--once an individual sees relational understanding as a goal, he will seek out new material to understand (pp. 23-24).

Skemp believes the long-term educational advantages of relational understanding are obvious, but offers several reasons why instrumental mathematics is taught in many classrooms: (1) relational understanding takes too long when only one technique or problems needs to be learned, (2) the topic is too difficult, but the students need to "know" it for an examination, (3) a skill is needed in another area, such as science, before all the prerequisite schemas are available for relational understanding, and (4) a new teacher may choose to teach instrumentally because all of the other faculty teach this way. Situational factors which contribute to the adopting of an instrumental technique by a given teacher include; good examination scores are needed to impress supervisors, over-burdened syllabi, the difficulty of assessing relational understanding, and the psychological difficulty of adapting to relational goals after many years of teaching instrumentally (p. 24).

Skemp cautions that aside from which type of understanding is better, there are some very serious mathematical mismatches that can occur. Students whose goal is to understand instrumentally, may be taught by a teacher that wants them to understand relationally, or the other way about. Also, the text may aim at either instrumental or relational understanding (perhaps changing according to topic), and thus create more chances for mis-matches. Skemp believes that much of the "modern mathematics" used texts whose aim may have been relational understanding, but were taught by teachers with an instrumental philosophy. Consequently, students would have been better off to have a "traditional" text, for at least they would have become proficient at many mathematical techniques that would have been of use in other subjects (p. 22).

Backhouse (1978) as well as Byers and Herscovics (1977) have disagreed with Skemp's distinguishing between two types of understanding. Byers and Herscovics believe there should instead be four categories. Byers and Herscovics would add "intuitive" and "formal" understanding. Intuitive understanding (like Bruner) "is the ability to solve a problem without prior analysis of the problem", and formal understanding "is the ability to connect mathematical symbolism with relevant mathematical ideas and to combine these ideas into chains of logical reasoning" (Byers and Herscovics, pp. 24-27). Herscovics suggests that the introduction of a formal approach to the teaching of algebra without the formal notation being given meaning is what leads to instrumental understanding. The significance of research into "understanding" is expressed by Herscovics (1976):

> Because it has been so difficult to define understanding, it has often been expressed in the form of vague generalities. This inability to define understanding clearly has made it difficult to counter the criticism of those evaluators using exclusively as their criteria the so-called products of learning, namely, the right answers (p. 104).

Skemp (1979A) did amend his theory to add formal understanding, which he calls "logical" understanding. He also conjectures two "modes of mental activity", "intuitive" and "reflective" (p. 45). His logical understanding, like Byers and Herscovics, is the ability to connect mathematical symbolism and notation with relevant mathematical ideas and to combine these into chains of logical reasoning (p. 45). Skemp compares this with Bruner's "analytic apparatus of one's craft", and notes that the objective is not the development of new schemas or concepts, but to ascertain that the existing schemas which have been constructed and the solutions which have been devised are accurate (p. 47). Skemp's logical understanding may be thought of as the techniques which are involved in formal proof.

It should be pointed out that there is not general agreement as to Skemp's logical understanding. Choat (1981) has argued strenously that relational understanding, "...provides for the formation of concepts in two categories - physical experience and logico-mathematical experience" (p. 19). He states:

> Logic is implicit throughout relational understanding and does not apply only to inductions to regulate form so agreement cannot be reached to accept a separate category of logical understanding as defined by Skemp (p. 19).

In fact Choat indicates that at about the "junior school" (England) age, children are deemed to no longer need play and physcial experience in learning, and that subsequent teaching is based on the medium of language, and a consequent formal approach, which leads inevitably to instrumental understanding (p. 21).

This summary of Skemp's theory will end with two statements by Skemp (1971) which emphasize in no uncertain terms his stance against what he considers the prevalence of the teaching of mathematics based on meaningless rules.

> Instruction of this kind may fairly be described as a series of insults to the intelligence; for they purport to be based on reason, but (usually) are not. To the extent what is being communicated is not intelligible, the receiver is trying to accomodate his schemas to assimilate meaninglessness. To do this would be equivalent to destruction of

> these schemas; the mental equivalent of bodily injury (p. 117).

## Imagery Research

That a review of the research on "imagery" was essential for any study which relates to the uses of Bruner's "iconic" mode or Piaget's "concrete" stage is succinctly illustrated by Clements (1981) in an article in which he discusses the somewhat confused state of imagery research. He notes that the theoretical formulation of imagery might seem irrelevant to mathematic educators, but that the recent use of visual aids and the growing realization and research indicating that imagery can be of fundamental importance in the teaching and learning of mathematics, is likely to generate in the mechanisms of how images are evoked. He notes:

> A second, and equally important, reason why mathematics educators should concern themselves with theoretical formulations of imagery is that unless they do they will not be able to take full advantage of the substantial body of research on imagery which has been carried out by psychologists, following the remarkable return to favour of imagery research in the 1970's. Mathematics educators will continue to give the impression of naivete' in matters related to research concerning imagery and spatial ability unless they become better acquainted with the relevant psychological literatures (p. 7).

Psychologists, in addition, to studying the "what" and "how" of mental events have generally been interested also in the "why" of mental events. Due to the fact that mental activities occur completely "inside the head" no other person can be aware of the actual occurrence of a mental event. Since scientists can only study observable forms of behavior, the study of mental imagery has a long and controversial history (Holt, 1970, p. 254).

The following is a short synopsis of the early history of mental imagery in America as reported in Kosslyn and Jolicoeur (1980, pp. 139-144).

Self-reporting methods are the most ancient techniques for studying imagery. The belief that people differed in their ability to recall sensory experiences was recorded by Fechner in 1860. In 1883, Galton developed this idea more thoroughly by way of a "breakfast table" questionnaire, whereby people were asked to imagine their breakfast table that particular morning with respect to color, brightness, and detail. Slightly over 10 percent of his subjects appeared not to have images. The nonimagers tended to be successful scholars and scientists, while the higher imagers were women and children. Kosslyn and Jolicoeur note that McKellar (1965) found that virtually all members of Mensa (who score very high on IQ tests) report some imagery in contrast to Galton's findings. Thus, as the authors note, either Galton's methods were flawed or times change (p. 141). Betts (1909) refined questionnaires by Galton by having people assign a numerical value to vividness with respect to sensory modes. He found that auditory and visual imagery were slightly more vivid than other modes.

In the early 1900's Wilhelm Wundt, often identified as the founder of "modern psychology", and his followers believed that all thought processes were accompanied by images. When they could not substantiate their beliefs by research, a hotly contested psychological debate ensued. Somewhat as a result, the behavioristic philosophy of John Watson gained wide acceptance. Watson rejected the ideas of mental images and studied behavior in its own rights, disregarding any attempts to draw conclusions about internal events from observation. Consequently, imagery fell out of favor during the reign of the behavioristic brand of psychology in the United States. It was not until the 1960's that the limitations of behaviorism were perceived and "cognition" came to the forefront, that imagery research once again flourished.

Contemporary research in imagery is in the following areas (Kosslyn, 1980, pp. 2-3):

1. Attempts to determine the effects of the use of imagery on a person's ability to perform various tasks.

2. Attempts to demonstrate how imagery is involved with modality-specific perception - e.g. auditory image vs. visual image.

3. Attempts to determine situations in which people use imagery spontaneously.

4. Attempts to ascertain the structure of imagery.

During the 1970's several researchers have put forth theoretical models of the imagery construct. The models fall primarily into either of two classes; "picture-in-the-mind" theories, and "propositional" theories (Clements, 1981, p. 3).

One of the most influential picture-in-the-mind theories was developed by Paivio. Paivio's "dual coding" theory suggests that a particular stimuli may bring into play both an imagery system and a linguistic system. To Paivio, memory is dual--both pictorial and verbal. Images are thought to be concrete and parallel while verbal representations are abstract and sequential. In a finding that has definite implications for mathematics, Paivio indicates that concrete words are remembered better than abstract words, and that pictures are remembered better than either. Paivio's model has been criticized for failing to address the questions of generation, inspection, and transformation of images (Pinker & Kosslyn, 1983, p. 48-49).

Another picture-in-the-mind theory was developed by Kosslyn. In this model the "picture" is a two-dimensional rapidly decaying icon of how the object really looked at some particular time. The images have no long term memory role. Long-term information is stored as abstractions, and images are "generated" out of these representations. In Kosslyn's view the same storage format does not apply for the storing and generating of mental images and verbal information. Images have two components, a pictorial "surface representation", and a "deep representation" contained in long term memory. Long term memory stores not just quantitative facts but also lists of facts in "symbolic" format which allows an individual to generate images in novel combinations. Kosslyn has used the analogy of a cathode ray tube (CRT) to describe his model, and has used a computer simulation program to produce a "picture" of a "car" on the CRT (Kosslyn, 1980, p. 154; Clements 1981, p. 4-6).

One researcher who feels he has supplied empirical evidence for the picture-in-the-mind model was Shepard (1978) (cited by Clements, 1981, p. 4, and Yuille and Marschark, 1983, p. 143). Shepard designed experiments which purported to show that mental images could be "rotated" in a manner analogous to the way the physical object would be rotated. His conclusions are based on the "time lag" for the subject to mentally rotate an object.

The preceding theories are models that use the image as an essential element of cognition. That these models are not universally accepted is well illustrated by Pylyshyn. In a debate that has some of the same features as "cognition" vs. "stimulus-response", Pylyshyn (1973, 1980) (cited in Yuille and Marschark, 1983, p. 147) argued that images do not exist as "pictures-in-the-mind", but that knowledge is stored as a set of propositions. He feels that there exists basic similarities between cognitive operations and computational procedures. That is, cognitive processes can be modeled by formal operations operating on symbolic structures, much like computer data structures. The computer is the basic tool for evaluation by these "computation" theorists. These theorists reject the notion that it is important to postulate imagery and believe that both verbal information and mental images can be generated from a set of propositions which represent a persons knowledge and are stored in memory. Pylyshyn has said that Kosslyn's "picture" of a car on a CRT is immaterial, that what really counts is the computation and symbolic procedures that were used to develop the "image", and is therefore redundant (Clements, 1981, p. 6).

Anderson (1978) believes the problem of defining imagery may be insoluble. He claims there is no way to determine which group is correct, barring strong physiological data. He feels both models can be made to fit the data. He advocates as a solution a "hybrid" or dual-code model such as Piaget's. That this debate between the two groups is viewed as exceedingly serious for the future of psychological study is indicated by Yuille (1983).

> However, there is a stronger, more important conclusion being offered here: that a continuation of the current type of theorizing and research related to imagery will improve the likelihood of success of the computational approach, and that approach will be disasterous, not only for imagery models, but for psychology as a whole. The current crisis in imagery research stems from the weak theoretical concepts which characterize the field. For the most part, definitions of imagery seem to be based upon intuitive notions of mental processes, and the interpretation of research results appear to originate from the same intuitive source. This is not an argument that intuition is not a useful heuristic, but rather that it is not a sufficient basis for an empirical science. What is required from aspiring theorists is a solid definition of imagery, and its relationship to other cognitive processes (pp. 280-281).

There are several other theories which vary somewhat from the "computational" and "picture-in-the-mind" tenets. Clements (1981) cites an example of one such theory: the theoretic account of imagery by the French psychologist M. Denis. He insists imagery is an active constructive process. He distinguishes between life-time figural schemata developed throughout an individual's life and stored in long-term memory, and the mobilization of figural schema in a particular situation, which gives rise to mental images. Denis conjectures that "representational units" are stored in memory. These units are hierarchically organized components organized according to their probability of contributing to image formation. A key feature in an "activational process" which can be applied selectively to the task at hand. The word "pie" could evoke the image of a person eating an apple pie or a circular pie diagram for the solution of fraction problems. Denis feels the contention that imagery may result in faster and better recall is not necessarily true. He believes that on some tasks the extensive use of visual imagery may be a hindrance (p. 6).

Clements (1981) notes that several other imagery theories are implicit in what is known as the "representational-development hypothesis" (p. 6). The three main features of this hypothesis are: (1) internal representations used changes with age, (2) the representational forms of adults are stronger than representational forms of children, and (3) the earlier preferred forms are not erased but are supplemented, and overshadowed, by later representational forms. From this comes the claim that older people think more "abstractly", while young children rely more on images.

Of particular interest to this study is the fact that the leading proponents (implicitly) of the representational-development theory are Piaget and Bruner. They did not denote a specific theory of imagery, but it is embedded in their total theories.

<u>Piaget and Inhelder</u> (1971) (cited in Clements, 1981) denote a difference between "reproductive" images which are analogous to known objects or events and "anticipatory" images (at about seven or eight years of age) in which objects or events not previously perceived are represented. Also, Piaget and Inhelder refer to images as "static", "kinetic" or "transformational". Inhelder and Piaget indicate that images are

not derived from perception, but are "an interiorization of imitation". To Piaget it is evident that children are incapable of forming mental images until they have proceeded through a stage whereby objects can only be represented by actions performed on the objects (sensorimotor period). Only after about 18 months of age does imagery representation come, and children can think about things that are not directly perceived. Kosslyn (1980, p. 460) notes that it is extremely difficult to consider Piaget's theory of imagery in isolation since it is embedded in the framework of his total theory. Kosslyn takes issue with Piaget and Inhelder in their insistence that images are "interiorized imitations". Kosslyn states, "It never is clear exactly what is imitated, or how such imitation occurs" (p. 410). Kosslyn describes Piaget's theory as "describing" a phenomenon rather than explaining it. In a rather strongly worded criticism he notes:

> Piaget and Inhelder's account is more on the level of intentionality..., and hence is open to multiple interpretations on the level of the function of the brain. They do not specify how interiorized imitation operates, nor have they specified the format or content of the image. This level of discourse will never process adequacy, and hence seems of limited value" (p. 411).

Kosslyn also writes:

> Given the long-standing popularity of the idea, there is surprisingly little evidence that young children really do utilize mental imagery more than do adults (p. 415).

However, that Piaget has strong support for his imagery theory, comes from Yuille and Marschark (1983, p. 150). They note that Piaget's model of cognition contains both an amodal and imagery code. Piaget's memory consists of mental structures and schemata. Images are not elements as such in these structures, but are tools that the system can use in problem solving. When Piaget indicates that young children can imitate aspects of the environment, or anticipate consequences of their actions by using images, he implies that such actions could not take place without the aid of images. Yuille and Marschark state:

> The Piagetian approach may be a possible resolution to the computational-imagery debate. However, before such an alternative becomes viable, the meaning of the concept "scheme" and the nature of the mental structure must be elaborated. Although this constitutes a major task, it might be a more worthwhile expenditure of effort than the continuation of the current debate (p. 150).

Bruner (1970) built somewhat upon Piaget's notions of the child's interactive relationship with the environment to devise a model of how such interactive episodes are represented in the mind. Bruner felt that the most important thing about "memory" was not storage of past experience, but the retrieval of relevant information. The information needs to have been coded, and have some method of being processed in order to be adequate. This system of coding and processing, Bruner called "representation". He called the three modes of representation (discussed elsewhere) "enactive", "iconic", and "symbolic". He states:

> Iconic representation summaries events by the selective organization of percepts and of images, by the spatial, temporal, and qualitative structures of the perceptual field and their transformed images. Image "stands for" perceptual events in the close but conventionally selective way that a picture stands for the object pictured (p. 291).

That Bruner did not have a complete theory of imagery is evident in his statement that:

> We know little about the conditions necessary for the growth of imagery and iconic representation, or to what extent parental or environmental intervention effects it during the earliest years. In ordinary adult learning a certain amount of motoric skill and practice seems to be necessary pre-conditions for the development of a simultaneous image to represent the sequence of acts involved.

That Bruner (1973) held imagery to be a key ingredient of his model is indicated by his observations when reporting on younger children learning to use wooden models to square binomial expressions:

> The children always began by constructing an embodiment of some concept, building a concrete form of operational definition. The fruit of the consruction was an image...(p. 432).

And also,

> ...they had not only understood the abstractions they had learned but also had a store of concrete images that served to exemplify the abstractions (p. 433).

Bruner (1973) has offered the advice that mathematics should begin by "instrumental activity" which become "summarized in the form of a particular images", and finally using symbolic notation "that remains invariant across transformations in imagery" the abstract properties are grasped (p. 436). He indicates that imagery is not the desired end result:

> Translation of experience into symbolic form... opens up realms of intellectual possibility that are orders of magnitude beyond the most powerful image-forming system (Bruner, 1970, p. 295).

But he does indicate that once abstraction has occured. It is, "...this stock of imagery that permits him to work at the level of heuristic..." (Bruner, 1973A, p. 436).

Kosslyn (1980) attacks Bruner et al for their lack of specificity. He states:

> Simply positing that equilibration (resulting in reduction in conflict) is accomplished is not enough; we need to know how this operation is supposed to occur. Without such specification, the notion becomes very difficult to disprove. As the theory now stands, Bruner, Olver, and Greenfield probably can account for any finding or its converse with nearly equal ease (p. 410).

Clements (1981) comments on the fact that "spatial ability" and "mathematical ability" do not necessarily go hand-in-hand. He discusses a study that he and a colleague did with first-year engineering student in Papua, New Guinea. In this study they concluded that there was a tendency for students who preferred to process mathematical information by verbal-logical means to out-perform more visual students on both spatial and mathematical tests. Clements notes that after reviewing the literature pertaining to spatial ability, visual imagery, and mathematical learning, that in some circumstances imagery use can have a detrimental effect on abstraction of concepts. He acknowledges that many studies, for example, Kent and Hedger (1980), have proven the worth of the visual-imagery mode in mathematical problem solving. Clements details his belief that teachers of mathematics generally end up forcing the verbal mode of learning on students (p. 3).

Wirszup (1974) reported on the significant amount of research the Russians have done with respect to the learning and teaching of geometry. As part of their program, they have researched the relationship between spatial notions and imagination, the relationship between geometry and other branches of mathematics and using visual principles in intuition.

Wirszup indicates that the Russians agree with Piaget that geometry in the western world is started much too late -- the teaching of geometry should start early in elementary school (p. 3).

Krutetskii (1976) studied the extent to which students rely on visual images in problem-solving. He examined the role of the "verbal-logical" and "visual-pictorial" components of mental activity with respect to mathematics learning. He had two significant conclusions. One, the ability to visualize abstract mathematical concepts and the ability for spatial geometric concepts are not necessary components in mathematics ability. Secondly, the ability to visualize abstract mathematical relationships and the ability for spatial geometric concepts showed a very high correspondence (p. 315).

In the United States such investigators as Jencks and Peck (1972) have followed up the Soviet ideas and have attempted to develop more "concrete" procedures in the teaching of arithmetic. They note:

> Fundamental to the idea of mental imagery is the use of something -- frequently concrete objects -- from which the learners can find answers for himself (p. 643).

They advocate the use of such objects as graph paper, yardsticks, diagrams, floor tiles and such. They state their investigations (like Bruner) show that a mistake is made, "when symbolic rules procede mental imagery necessary to give an arithmetic process a common sense foundation" (p. 644).

Clements (1981) reports research that deals also with older students. Threadgil-Sowder and Juilf's (1980) research indicated that low mathematical ability students showed significant improvement using manipulative materials, but that high ability students found a symbolic treatment more beneficial. Horwitz (1981) in a study of college students concluded that visual properties of a problem affects the solvability by low performance students but not by high performance students (Clements, 1982, p. 35).

This review of the literature on imagery is drawn to a conclusion by a quotation from Clements (1982):

> ...it is obvious that, considering the large expenditure of time and money on the research efforts which have been described, we know precious little about how and when use of imagery is likely to facilitate mathematics learning. This state of affairs should not be regarded as a signal for a lessening of research activity aimed at increasing our understanding of the role of visual imagery in mathematics learning (p. 36).

He notes that one only has to review the literature, for example, on concept images in geometry, "...to be convinced that the different images which one associates with certain mathematical tasks can substantially affect both performance on, and understanding of, those tasks" (p. 36).

## Related Mathematical Studies

Most research studies dealing with the acquisition and/or formation of mathematical concepts, cognitive processes, and "remedial" students have been done at the elementary school age level. There have been some few studies that dealt specifically with algebra learning and teaching at the intermediate school and high school levels. Fewer studies still have been conducted with respect to algebra teaching and learning with college-age students. Since many remedial courses in algebra at the college level cover essentially the same material as beginning and intermediate algebra courses at the intermediate and high school level, this review included studies which pertain to algebra learning and teaching regardless of the age level of the students. Additionally, studies which did not directly relate to algebra, but were deemed to have significant implications for algebra learning were also reviewed. It should be noted that the number of studies which used an interviewing technique in an effort to determine the cognitive processes used by college students in the learning of algebra, is rather insignificant in comparison to the number of "statistically oriented" studies.

## Exponents

A thorough data base search shows virtually no significant research has been done with exponent concepts as a dominating interest. Nearly all research into algebra, and errors in algebra, list exponents in a "minor" role. Typically, exponents are noted under "variable errors" or "generalization errors". This is pointed out dramatically in a study by Alexander (1977). In a thorough review of the literature dealing with both algebraic concepts and error diagnosis at the high school level, he reviewed 17 studies dating back to 1910. Of these studies 7 reported that exponents were among the many factors causing difficulty in algebra. Alexander (p. 10) cites a study by Fossler (1924, p. 15), which lists $2x^{\frac{1}{2}} \cdot 2x^{\frac{1}{2}} = 5x$ as a misconception students had with exponents. Farha

(1935, p. 21) (cited in Alexander, p. 18) reports, "94 percent of all students showed some confusion in dealing with exponents".

Another study, based on the National Assessment of Education Progress (NAEP) mathematics assessment testing, during the 1977-1978 school year, reported results that have implication for the study of exponents. A National Council of Mathematics Teachers booklet (edited by Corbitt, 1981) compiled, and commented on the results of this assessment. The following results are pertinent to this proposed study:

Given the problem: "$a^4/a^{20}$ = ", 30% of the students that had completed one year of algebra, and 25% of those students that had completed two years of algebra, gave the incorrect response of $1/a^5$ (p. 65).

Given the problem: " $\sqrt{b^{36}}$ = ", 35% of the students with one year of algebra, and 47% of the students with two years of algebra gave the incorrect response of $b^6$ (p. 65). The authors of the report write:

> Students generally could simplify expressions involving positive exponents, but had difficulty with negative exponents and radicals (p. 63).

and

> ...both older groups agreed that "there is always a rule to follow in solving mathematics problems". The students may be concentrating on mastering rules to the extent of ignoring concomitant understanding... (p. 146).

The above would indicate that Skemp's warning of a "multiplicity of meaningless" rules in algebra should be viewed with grave concern.

Shlomo Vinner (1977) researched the concept of exponent as related to the "definitional approach" -- some types of numbers are defined by means of a "lower" type. His sample was one hundred ninety-five college freshmen, all of whom had attended a calculus course, and fifty-six students, at the upper level of undergraduate mathematics studies. He attempted to elicit information concerning the three formula $a^n = a.a.a...a; a^{-m} = 1/a^m \ (a \neq o); \ a^{n/m} = \sqrt[m]{a^n}$, where a is a real number, with m and n whole numbers. The students were to determine if these formulae represented theorems, laws, facts about numbers, a definition, or axioms. Only about one-fourth of the freshmen, and about one-half of the upper-classmen, correctly identified all the defining formulae. Two conclusions of Vinner's are pertinent to this study. First, he advocates that the definitional approach should not be used until the graduate level. He states, "To teach the definitional approach before the student is at the suitable intellectual stage is just useless (although he might pass the exam)" (pp. 24-25). Secondly, he describes a "naive Platonic" attitude that immature (mathematically) students have toward mathematics. These students view the existence of mathematical objects as analogous to the existence of concrete objects (p. 19). These students believe that, somehow, all arithmetic operations are discovered. The notion that mathematicians define operations, contradicts the naive Platonic philosophy. Vinner's research does seem to have implications for both teachers and textbook authors in the defining of symbols and other entities in devising instructional materials for a remedial algebra class.

Another study, by S. Rachlin (1981), although not limited explicitly to exponents, did deal with them as part of several other problems. This research stands out due to the fact that college level students' knowledge of basic algebra was studied by clinical means. Rachlin used a Soviet "ascertaining" experiement as advocated by Krutetskii. Using both interview procedures, and paper and pencil tests, he developed a case study for each of four students that were very successful, grade-wise, in a remedial basic algebra class at the college level. All four students had been very successful in two years of high school algebra. Rachlin investigated Krutetskii's "generalization", "reversibility" and "transfer" in terms of Skemp's "relational" and "instrumental" understanding. He concluded that success in a basic algebra course could be used to imply generalization ability (defined by Krutetskii) but could not be used to imply reversibility, or flexibility. Also, he concluded that the students differed greatly in their relational understanding (Skemp's) of particular topics, even though their test scores were nearly identical in the basic algebra class.

Of particular significance to this study, are the following statements by Rachlin:

> Rules such as whether to add or multiply exponents in a particular situation, although finely tuned for a test, were applied uncertainly a few weeks later (p. 261).

For example, while working with multiplication of polynomials with variable exponents, all of the subjects indicated some confusion over whether to add or multiply exponents. This confusion arose again when multiplying radicals with different indices (p. 248).

Recall Skemp's admonition presented in Chapter I, "An appropriate schema is one which takes into account the long term learning task" in light of the preceding paragraph.

Rachlin states:

> The common behavior observed in the subjects' solution to the Generalization Test was their apparent ease in generalizing the first rule: $(a^m)^n = a^{mn}$, to all its variants (p. 243).

However, during the application of the second rule: $(A+B)^2 = A^2 + 2AB + B^2$, some confusion did arise as the variants became more complicated:

4) $(a^3 + b)^2 = a^5 + 2a^3b + b^2$

5) $(2x^{4n} + ty)^2 = 4x^{6n} + 4x^{4ny} + y^2$ (p. 86).

Rachlin calls these "false generalizations". Perhaps only the operational symbols in the first rule have been generalized, and not the concept. This study will attempt to distinguish between concept generalization (relational understanding) and symbolic generalization (instrumental understanding).

Also of interest to the present study is Rachlin's comment:

> ...only three subjects initiated the use of arithmetic variants of the algebraic tasks as a heuristic for solving the tasks. The fourth subject used arithmetic variants only if she was directed to by the interviewer. Two subjects experienced difficulty in transferring processes from arithmetic to algebra (p. 247).

This study will attempt to determine if, when, and under what conditions, remedial algebra students use numerical imagery as an aid for symbolic tasks.

Concept Learning

Davis, Jockusch, and McKnight (1978) over a several year period investigated the "cognitive" or "information handling" processes that are involved when students in grades seven, eight or nine begin the study of algebra or "advanced" arithmetic. They formulated a rather thorough definition of "understanding": (noted earlier)

> What is "understanding"? We presume it is (at least in large part) a composition of many of the items discussed early in these notes: comparing input data with many existing things you already know; looking for patterns, contrasts, comparisons; looking for apparent inconsistencies or contraditions; making careful note of the cues which can be used in the future to guide future selections of solution methods; trying to identify and retrieve an appropriate "assimilation paradigm" or schema, and to synthesize a new one if no appropriate old one can be found in memory; making a careful critical appraisal of how well the present situation matches the retrieved schema that has been selected: and trying to develop appropriate "metalanguage" in order to be able to analyze the mathematical situation effectively (p. 283).

This perhaps is Skemp's "appropriate schema". They point out that many students do not realize that mathematics can be understood, but imagine that all people deal with mathematics by reliance on memorized rote procedures. There are other students who do recognize that there is such a thing as understanding, know when they do understand, and recognize the value of understanding in terms of retaining knowledge as well as the fact that it is the kind of knowledge that can be combined with other knowledge to allow greater power in thinking about novel problems (p. 283).

Davis, et al, state that they have become convinced that the "purpose" or "intent" cannot be avoided -- students accomplish exactly what they set out to accomplish. If they merely desire an answer to get the problem "right", or to please the teacher, this is what they will get. On the other hand, if a student is really trying to "learn mathematics" by the definition of understanding (above), this is what they will do (p. 276).

In an observation that has relevance to this study dealing with exponents, Davis, et al, note that all teachers they observed emphasized the fact that algebraic statements are in fact statements about numbers, and urged students to use numerical substitutions to check relationships involving variables. In their words, "Students use this strategy all too infrequently..." (p. 127). Also relevant to this present study is their confirmation of Vinner's caution about the "definitional approach" to algebra. They point out that many students are confused by the definition of $b^0 = 1$ based on logical arguments, such as $b^0 \cdot b^{0+2} = b^2$. Many students believe that mathematics was created in one "mega-creation" of the entire system, and are astounded that we are trying to decide what $2^0$ "ought" to be (p. 118-119). Davis, et al, note that the, "process of recognizing the general and separating it from the specific, by imagining variation where no actual variation has been presented...(p. 95) is fundamental to mathematical thinking. This agrees with Skemp's "mathematical generalization".

Another study which is deemed pertinent to the one at hand, is a study by Harrison (1967). Harrison used Skemp's early work with "reflective intelligence" tests in combination with a battery of aptitude tests in an attempt to predict performance scores. The sample was six classes of grade eight students. The significant finding was that Skemp's notion of "reflecting thinking" in relationship to mathematics learning was validated.

Erlwanger's (1974) case studies of six elementary school children that were being taught mathematics in an "Individualized Instruction" curriculum has implications not only for elementary teaching, but all levels of educational endeavor. All six students were of average or above intelligence (I.Q.). Four of the students were considered by their teachers to be superior in mathematics ability. The case studies indicated that each child appeared to have "a stable, cohesive system of interrelated ideas, beliefs, emotions, views, and so on about mathematics and mathematics learning". The children developed a view of mathematics as a set of rules for putting symbolic patterns on paper. They also held strange (in terms of "adult" thinking) views of the purpose of mathematics and the relation between rules and answers. The children were dependent upon a formal type of thinking about rules in which explanations involved patterns of symbols (Erlwanger, 1975A, p. 157).

The case studies of "Benny" and "Mat" in particular raise grave doubts about using written test results as the sole indicator of an individual's mathematical knowledge, particularly where the tests are totally of a symbolic nature. Mat, for example, when asked to add "3/4 + 1/4"; used the algorithm a/b + c/d = a+c/b+d to get 4/8, and then used the rule for reducing to get 1/2. When the interviewer had him obtain 4/4 using wooden blocks, Mat said he would choose the 1/2, "...that's what method they've taught me to do in my booklet..." (p. 251). Benny had a unique method of operating on decimals and fractions: 5/10 = 1.5, because the one stands for 10; the decimal; then there's 5...shows how many ones". Benny also thought 4/11 = 1.5, and also that 11.4 = 1.5. His consistent pattern was: ab/c + a . (b + c), or ab/c = b/ac = a . (b + c) (Erlwanger, 1975B, pp. 5-6). If either Mat or Benny got a wrong answer, when they checked the answer sheet, they merely looked for symbolic patterns to generalize. This procedure has implications for Davis' "generalization by instance", and Skemp's "mathematical generalization", as well as Rachlin's false generalizations. Teachers must distinguish between "concept" generalization and "symbolic" generalization.

Benny's case study has serious implications for a rule-governed content-area such as algebra. When Benny was asked if he had ever discovered other ways of doing mathematics than he had been taught, he responded, "No! I stick with the rules." When the interviewer asked Benny if this was because it was easier, he said "Yes. When you hardly know how to do it." (p. 17).

Erlwanger (1975A) postulated "conceptual" thinking and "procedural" thinking. He has described these as "algorithm" thought and "non-algorithmic" thought. Conceptual thinking describes the situation when a child is thinking about "mathematical quantities" or "mathematical relations". In procedural thinking, the child is primarily interested with the application of some rule (often self-developed on the basis of an incorrect generalization). Both modes of thinking could possibly involve algorithmic thinking. However, in conceptual thinking the emphasis would be on the development of an appropriate algorithm, while in procedural thinking the emphasis would be on trying to follow an algorithm step-by-step (pp. 192-193). Elsewhere he has categorized these as "formal" and "intuitive" thinking (1974, p. 31). It is interesting to note that Erlwanger's modes of thinking parallel somewhat Skemp's "relational" and "instrumental" understanding.

Erlwanger's work is of major importance because, like Piaget, he shows educators that an interview technique is a necessary, and viable technique, for ascertaining the mathematical knowledge and ideas an individual student possesses. In conclusion of the presentation of Erlwanger's studies a final statement is offered as to the importance and implications of his work for college remedial algebra teachers and learners. He (1974, p. 285) writes, "The case studies suggests that unless a child's "wrong" ideas, beliefs, and views are detected and corrected, they may develop and become more complex."

Another group of studies conducted by Krutetskii over a number of years also has major implications for this present study.

Krutetskii (1976) rejected the product measurement of mathematics "ability level" as being "the study of the level of knowledge and skills that is attained" (p. 13), and indicates that the attempt to find purely statistical methods for describing the qualitative distinctions among abilities will fail. He indicates that one must study and analyze the "process" of attaining results in problem-solving to discover the "psychological essence" of the results (p. 13). For the Soviets, testing followed by factor analysis without incorporating the psychological analysis of process, "has not proved its value" (p. 36).

The Russians somewhat rejected Piaget's stage theory and believed that cognitive development and schooling are closely linked. They believed that the curriculum can alter the development stages of Piaget (Carpenter, 1980, p. 128).

Krutetskii (1976) has postulated three stages in mathematical problem solving: (1) gathering information, (2) processing information, and (3) retaining information. He relates the stages to one or more of the following abilities (defined by Krutetskii):

1. The ability to "generalize" both to place a "particular case under a general rule", and to go from "the particular to what is unknown and general (p. 335).
2. The ability of "flexibility": the ability to switch easily "to a new method of operation, from one mental operation to another" (p. 227).
3. The ability to reverse ("reversibility") from a direct train of thought -- a "reconstruction of its direction " (p. 287).
4. The ability to curtail ("curtailment") the mental process (unconsciously) in solving problems of a familiar type. However, when necessary the student can return to full detailed reasoning (p. 264).

Krutetskii's (1976) studies are deemed particularly important because of the breaking down the concept of "mathematical ability" into components which can be "ascertained" qualitatively during problem solving. Krutetskii's studies are known for the remarkable ability, as related to the above factors, of some of the students -- (for example, Sonya, p 251). However, of concern to this study is Krutetskii's rather pessimistic view of "incapable students". As opposed to capable students that generalized "correctly and immediately", incapable students "cannot generalize according to essential features, even with help from the experimenter and after a number of intermediate, single-type practice examples" (p. 155). On the curtailment aspect, incapable pupils were always marked by superfluous comprehensiveness, detail, and unnecessary activity...At the same time their reasoning was not distinguished by accuracy..." (p. 267).

On flexibility, it was, "...as if the solution that had been found cut off any possibility of their switching to a new method of operations" (p. 278). Discussing an incapable student Krutetskii notes that after a lengthly time the experimenter taught her to use a faster, easier method, "but afterward she could not reproduce the habitual method right away. And only after a half-hour did she recall the old one, but...forgot the new one" (p. 278). Likewise for reversibility, "...incapable pupils saw the second problem given them as a reverse problem only in elementary cases..." (p. 289).

Confrey and Lanier (1980) used Krutetskii's conceptualization of abilities in an experiment with general mathematics students at the intermediate school level with approximately the same results as Krutetskii for "incapable students". However, they offered some suggestions for educators. They suggest: teachers should be aware of the psychological processes of teaching and learning mathematics (as well as logical), consideration of how abilities (both student's and teacher's) influence the process of teaching mathematics, careful attention to the assumptions that are made with respect to ability present in the students, and consider developing teaching units designed to teach specific abilities (as indicated by Krutetskii) (pp. 551-554).

Serious implication for this present study is indicated by Confrey and Lanier's report that the students viewed mathematics as a collection of symbols to be manipulated according to certain rules. The students had little, if any, concrete or mental representation to fall back on. The reason for doing mathematics was to get answers, which are either right or wrong if the teacher or answer key so indicates, understanding is not involved. (p. 554-555).

Wagner (1981) reporting on a study that was designed to investigate students' "conservation of equation and function under transfer of variable", documented two common misconceptions about variables. One misconception is that changing a variable symbol changes its referent. A second misconception is that the linear ordering of the alphabet corresponds to the linear ordering of the counting numbers (p. 116).

Wagner used the equation(s) $7w + 22 = 109$ and $7n + 22 = 109$ and asked the students "which would be larger, w or n"? In a sample of size 29, she found that less than one-half of the students did not conserve equation. These were not all young children in her study. The fact that one-third of the students who had studied algebra did not recognize the w and n would be the same, would seem to have very significant implication for the teaching of the concept of variable (pp. 109-116).

Matz (1979) reported similar data. She found variable errors such as, "concluding that 4x = 46 given the x = 6, or that xy = -8 given that x = -3 and y = -5" (p. 133).

The first error is a place-value error while the second is an implicit addition inferred error. Matz feels that the critical step in the transition from arithmetic to algebra is the concept of a "symbolic value". For students who initially fail to realize that a letter represents a number, "operating with a letter seems totally underconstrained" (p. 131). She notes that letters are not a very intuitive notation for symbolic values since they do not obviously refer to a number. She believed the usual procedure of using boxes for missing numbers in arithmetic does not naturally generate to the concept of a symbolic value. Boxes are not manipulated in sentences, nor are they divided by two. Boxes have an inherent connotation of being filled that letters do not. Students when faced with a letter will not be able to realize that variables can be "instantiated" by replacing them with a number (p. 131).

Other researchers have found student difficulty caused by the multiple-meanings of a variable. Adda (1982) notes that $x + 3$, $2x + 3 = 0$, and $ax^2 + bx + c = 0$, where both "parameters" and "unknowns" are referred to by letters has great potential as a source of confusion (p. 210-211). Kuchemann (1978) has found similary difficulties with the concept of variable. Buxton's (1982) research has lead her to believe that mathematical symbols not only have a cognitive aspect, but an emotional side as well. She has compared the "panic" that mathematical symbols create for a student pushing them about instrumentally to the fear and embarrassment that an illiterate adult faces when they see printed words (p. 215-220).

Behr, Erlwanger, and Nichols (1980) designed an experiment to determine how students view the equals sign. It turned out that students see the equal sign as a "do something" symbol -- an operator symbol such as in "2 + 3 = 5". In an interview setting it was found that students would try to turn statements such as "3 = 3" or "2 + 3 = 3 + 2" into an addition or subtraction problem; they were not able to view it as a relational symbol (pp. 13-15). Matz (1979) noticed a similar situation with students in algebra with respect to the difference between "tautologies" and "constraint equtations". As an example she notes the syntatic similarity between the two semantically different statements: $4(x + 3) = 4x + 12$ and $3x + 3 = 2x + 7$.

That these difficulties with variables and the equal sign are not limited to young children was shown dramatically by Rosnick & Clement (1980) who asked first-year engineering students to write an equation to symbolize the statement "There are six times as many students as professors at this university" (p. 4).

In a group of 150 only 63% were able to answer correctly. They conducted clinical interviews and tutoring sessions with students that had incorrect answers. The interviews gave evidence that students changed their behavior and gave answers to similar problems, but Rosnick & Clement believe that "conceptual understanding of equation and variable remained, for the most part, unchanged" (p. 23). They feel it is essential that students be able to view variables as standing for "number", and their studies show the "shortcomings of an educational system that focuses primarily on manipulative skills" (p. 23).

Matz (1979) sums up her extensive investigations of algebra errors by indicating that most errors are not the result of carelessness, but rather are systematic and rule-based. The two most common errors are (1) use of a known rule (as is) to a new inappropriate situation, and (2) incorrectly adapting a known rule so that it can be used to solve a new problem (p. 95). VanLehn (1982) also found that younger students were very systematic in their development of "buggy" algorithms in subtraction.

Another study which has some implications for the teaching and learning of algebra at the college level was performed by Gage (1976). Gage used a self-designed concept attaiment test in conjuction with selected concepts from beginning algebra to determine the effect of positive and negative instances of concept on its acquisition. Her determination was that when a positive instance was followed by a negative instance, the concept is more easily acquired. Most algebra texts when dealing with a topic such as exponents, will usually show each "law" as being separate, with no problems presented where none of the "laws" apply.

Yonis (1968) conducted an investigation which he claimed shows that the use of "enactive" and "iconic" modes of teaching were superior in teaching elementary algebraic principles to "low achieving" college students. He made provisions for the use of concrete referents for symbols by the use of diagrams and a calculator. His results showed that this was better than the "traditional" lecture method. However, the study seems to place an undue amount of emphasis on the calculator as the "concrete" referent.

A study which relates significantly to the learning of algebra at the college level was conducted by Parete (1978). Parete using tasks that were designed to determine Piagetian stages, determined that the 231 college students tested could be divided as follows: 21% early-concrete, 30% late-concrete, 10% transitional, 29% early-formal, and 10% in the formal stages.

Of particular significance to any study pertaining to remedial mathematics is research into the recently named malady of Mathematics Anxiety. It is defined literally as "uneasiness or apprehension regarding mathematics" (Widmer and Chavez, 1982, p. 272).

Fennema and Behr (1980) argue that confidence and anxiety are similar in relation to mathematical learning. They also believe that there are sex-related differences in the confidence/anxiety dimension that helps explain the fewer number of females than males entering mathematics related fields of study. They also indicate that their research shows that high-anxiety students appear to perform better in a highly structured learning situation. (pp. 334-335).

Shodahl and Diers (1984) list several possible reasons for math anxiety: unintelligible texts, lack of concrete learning experiences in the Piagetian sense, lack of teacher encouragement to develop an overall picture with a consequent trust in intuition, and teaching which leaves out an explanation of process. They have developed a "Math Without Fear" course which they indicate reduces anxiety. The course is taught jointly by a mathematician and a psychologist. The course involves guest speakers to acquaint the student with the importance of mathematics, journal-keeping, rapport-building, relaxation techniques, and the dispelling of the mystique and myths which surrounds mathematics (pp 32-35).

In a similar vein, Sequin (1984) has found success in reducing math anxiety by a combination exercise and relaxation program held prior to mathematics classes (pp. 33-35).

Although, not explicity referring to math anxiety, but more generally to test anxiety, Russo (1984) indicates that anxiety is due to a preoccupation with failure, negative comparisons, thoughts and feelings of inadequacy, and self-blame and criticism. He suggests relaxation and self-monitoring in an attempt to see the task at hand in a more positive light (p. 164).

However, Greenwood (1984) insists that sex-related differences in mathematical employment is generally a societal problem and has little to do with math anxiety. In fact, he argues that the principle cause of math anxiety lies in teaching methodologies based on the "explain-practice-memorize" paradigm with a consequent lack of the understanding of process. He believes that until we, "apply the problem-solving process to the teaching and learning of arithmetic and basic mathematics concepts and skills, we will continue to produce young adults who suffer from math anxiety" (p. 663).

This review of the literature is brought to a close by a thought-provoking statement that was made by Doyle (1983) after researching the literature for those factors which contribute to success in the academic setting:

> The central point is that the type of tasks which cognitive psychology suggests will have the greatest long-term consequences for improving the quality of academic work are precisely those which are the most difficult to install in classrooms (p. 196).

## Summary: Review of Literature

The review of the literature explored the historical movement of "meaningful learning" by tracing its development through the connectionistic and behavioristic eras, through the period of opposition to the above theories by Brownell and the Gestaltists, and finally into the modern day era of Piagetian cognitive theory.

Modern-day imagery theory was described in an effort to place Bruner's, Skemp's, and Piaget's developmental concept of imagery into a more theoretical framework.

Finally, in this review a compilation of studies detailing the learning of algebraic concepts was discussed. The work of Krutetskii in terms of his descriptions of various abilities was given prominence. Several studies were cited showing the apparent inadequacy of "rule-based" understanding to provide a suitable background for student success in algebra. The work of Erlwanger and Matz were given special emphasis in this regard.

The review of the literature served as a theoretical background for the research procedures and questions detailed in Chapter III; RESEARCH PROCEDURES.

## CHAPTER III

## RESEARCH PROCEDURES

Since this study was basically a qualitative investigation of the thought processes of students, a clinical interview method of gathering data was deemed to be the most appropriate procedure to follow.

That the clinical interview is in fact a viable methodology for research in the cognitive domain was illustrated well by Piaget and Krutetskii (reported in CHAPTER II). The procedure has been well documented by several American researchers as well (Erlwanger, 1974; Easley, 1977; Ginsburg, 1981; and Confrey and Lanier, 1980).

### The Interviews

The interviews were held in a quiet comfortable air-conditioned office with side-by-side desks for the interviewer and the interviewee. The desk of the interviewer was large enough, and situated so that a tape-recorder could be placed close enough for easy recording, but yet not be the dominant article on the desk.

The interviewer started the session by an explanation on the order of the following, which was designed to put the student at ease as much as possible:

> What I am attempting to do is determine how students think about algebra — how they look at

> it; what goes through their minds as they work a certain problem. I will be talking with several students, so in order to be able to determine if there are common ways that students think, or common errors that students make, I would like to tape-record our talk. I will give you several problems to work out. While you are working these problems, I want you to tell me what's going through your mind — that is, think out loud so that I can better determine how you are working the problem. I want you to do your best, but keep in mind this is not a test. I am not after right or wrong answers. I'm interested in what you are thinking as you solve the problems.

The tape-recorder was then turned on.

The problems presented had been typed on sheets of paper with ample work space provided for each problem. No more than five problems were presented on one piece of paper; that is, when a student and interviewer were through with a set of five problems, then an additional set of five was presented. It was felt that presenting 49 problems initially would somewhat awe and perhaps discourage the students. In order not to fatigue either the student or interviewer, no session lasted more than one hour. If the problem sets had not been completed, an additional appointment was made. The range of times for the interviews were from one hour to one hour and 45 minutes.

To facilitate the analysis of the taped interviews, the students were asked to state the number of the problem before starting to work the problem. The student was encouraged to not only "think-aloud", but write steps on the paper (in order to coordinate the written work and oral statements).

Although interviews were built around the problem sets, the interviewer was free to challenge answers, offer hints of encouragement, challenge and contradict, or to present related problems in an effort to "draw out" the thought processes of the students. The interviewer recorded any significant non-verbal or significant body actions during the interview.

Due to the somewhat ambiguous privacy laws with respect to student records and information, at the conclusion of the interview each student was asked to sign a "consent form" (Appendix C), which allowed the obtaining of academic information and the publishing of the results. This form was based on a similar form by Rachlin (1981). Additionally, the students were asked to fill out a form detailing their mathematics background.

## Selection of the Sample

Eight students were selected from a section of beginning algebra (Math 111), and eight students from a section of intermediate algebra (Math 121) at Ferris State College, Big Rpaids, Michigan during the summer of 1984 for the interviews. The selections were from those students determined by their instructors to have scored very low (four students from beginning algebra, four from intermediate algebra), or very high (four students from beginning algebra, and four from intermediate algebra) on a unit test pertaining to exponents that is a normal part of each course. Two of these students failed to show up for the interview appointment, even after repeated appointments, thus fourteen students were actually interviewed.

The beginning algebra course, Math 111, is a course comparable to a ninth-grade algebra course. It is a terminal course for many of the "job oriented" programs at Ferris State College, as well as a prerequisite for the mathematically unprepared student whose program (e.g. business and health programs) calls for an intermediate algebra course. Students are usually placed in Math 111 by; not having taken, or by not having successfully completed a second-year algebra course; and/or having received less than a "C" grade in a first-year high school algebra course; and/or scoring between seventeen and nineteen on the mathematics portion of the ACT examination.

Intermediate algebra, Math 121, is a terminal course for many four-year programs in the business, health, and technical programs at Ferris State College, as well as a prerequisite for any unprepared student whose program requires further mathematics courses, such as trigonometry or college algebra. Students are placed in this course by having having taken no second-year algebra course in high school, or having received a grade of "C" or less in such a course, and/or having scored between twenty and twenty-two on the ACT test. In any event, the guidelines are very flexible, and are ignored frequently, based on the wishes of the student.

The courses are taught in the traditional lecture method.

The classes meet four days a week, with class size varying from twenty-five to thirty-five. The ages of the students in this sample varied from 18 years to approximately 55 years. The textbooks used in Math 111 (Lial and Miller, 1980) and Math 121 (Wooton and Drooyan, 1980) both develop properties of exponents from the use of numerical examples involving positive exponents, then extend the properties to zero, negative and rational exponents by use of logical arguments. It is worth noting that their problem sets consist of positive examples of the particular property involved. That is the bases are usually the same or the exponents the same. Both texts define "variable" to be a letter which represents a number (Lial and Miller, p. 10; Wooton and Drooyan, p. 2). In a personal conversation the instructor of the students indicated that due to the shortness of time (10 weeks) that the course was taught as a "tool" type course, or program prerequisite.

Approximately thirty-five percent of the students enrolled in Math 111 either withdraw from class or receive a grade of less than "C" in the class. Very rarely does a student that received less than a "C" in Math 111 manage to sucessfully ("D" or better) complete a Math 121 class (Totten, 1983).

## Selection of the Questions

The hierarchy of prerequisite knowledge for the understanding of the concept of exponent was derived from an analysis procedure based on Gagne's learning hierarchy model

(Gagné, 1970). The hierarchy chart (Appendix A) was then developed and presented to two professors in the Mathematics Department at Ferris State College for validation.

In an attempt to put the students "at ease" and to verify that portion of the hierarchy of prerequisite knowledge leading up to the concept of exponent, the students were asked the following questions: (Simplify)

I. 1. $2 \cdot 3 + 4 - 12$

2. $15 - 3 - (2 + 1)$

3. $\frac{2 \cdot 4 \cdot 6}{8}$

4. $\frac{2 \cdot 5 + 6}{8}$

5. $a + a + a$

6. $2a + 3a$

7. $2a + 3a - b$

8. $\frac{4xyz}{2x}$

9. $\frac{a + b}{a}$

10. $3 + 4(x + 5)$

Due to the difficulties with respect to "variable" and the "equal sign", that were pointed out by Wagner (1981), Matz (1979), and Erlwanger, et al (1980), the following questions were developed to probe the student's understanding of variable and equation:

II. 1. What does it mean to you when we say $2x + 3x = 5x$?

2. What does it mean to you when we say solve for x in $x + 4 = 7$?

3. Which of the indicated equations would be the correct answer for the following:

   Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors. (Rosnick and Clement, 1983, p. 4)

   a. $P = 6S$

   b. $S = 6P$

4. Is the following true or false?

   $4x = 46$ given that $x = 6$ (Matz, 1979, p. 134)

5. Without solving, could you compare the solutions for W and N in the following:

   $7W + 22 = 109$; $7N + 22 = 109$ (Wagner, 1981, p. 109)

In the spring of 1984, a diagnostic examination (Appendix B) was given to selected Math 111 and Math 121 classes. The test was given to Math 111 classes after the completion of the unit on exponents, and to the Math 121 classes prior to covering the unit on exponents and radicals. The most commonly missed problems were: (Simplify)
$x^0$; $2^0$; $8^{2/3}$; $4^3/2^2$; $2^3 + 2^2$; $x^3 + x^2$; $(xy)^2$; and $(x^3)^2$.

In addition, for each class a record of the most frequently missed problems on the exponential unit examinations were kept. In Math 111 twenty-seven students out of one hundred fifty-nine missed $3^4 \cdot 3^2$, and sixty-six students missed $4^2 + 4^3$. In Math 121, out of thirty-two students, twenty-three missed $2^5 \cdot 5^5$, twenty-eight missed $14^4/7^5$, and thirty-two missed $x^{-3}/(x^{-1} + x^2)$.

The development of the next set of questions was based on the examination results cited above, the National Assessment of Education Progress results (reported in CHAPTER I), reports of problem difficulties in Alexander's (1977) and Rachlin's (1981) research, as well as the theories of Bruner, Skemp, and Krutetskii. The following problems were presented to all students. Again, no more than five problems were presented at one time.

III. Find the value and/or simplify the following:

1. $2^3$
2. $(-2)^3$
3. $-2 \cdot 3^2$
4. $2^2 \cdot 3^2$
5. $2^2 \cdot 2^3$
6. $2^2 + 2^3$
7. $3^2 + 2^3$
8. $(2^2)^3$
9. $4^4/2^2$
10. $(2 \cdot 3)^2$
11. $(2 + 3)^0$
12. $(2 + 3)^2$
13. $4^4/4^2$
14. $5^{-2}$
15. $(2 + 3)^{-1}$
16. $14^2/7^2$
17. $2^{-2} \cdot 2^3$
18. $2^3/4^{-1}$
19. $(2^{-1})^{-2}$

20. $(xy)^3$

21. $(x^2)^3$

22. $x^a \cdot x^b$

23. $m^n/m^2$

24. $a^{-b}$

25. $x^2 + x^3$

26. $x^a y^b$

27. $(x^6)^{1/2}$

In addition the Math 121 students were given the following:

28. $\sqrt{4} \cdot \sqrt{9}$

29. $\sqrt{14} \;/\; \sqrt[3]{7}$

30. $(x + 3)^2$

31. $(\sqrt{2} + \sqrt{3})^2$

32. $\sqrt{6} - \sqrt{3}$

33. $(x^{3n} + 2)^2$

34. $a^{1/n} \cdot a^{1/m}$

(Problems 32 and 33 from Rachlin, 1981)

The problems were grouped on the basis of the following system:

The problems with like bases and multiplication or division were used to determine if students use exponential properties, or use the operations of arithmetic to arrive at the result.

The problems with addition or different bases were designed to check for false generalizations and/ or instrumental understanding or relational understanding.

The problems with variable bases and/or variable exponents were designed to see if students had generalized the pertinent exponential property, if any type of imagery was used, and for relational or instrumental understanding.

Those problems with zero or negative exponents were for the purpose of determining if students have "mathematically generalized" the system of positive exponents, and if so, whether this was understood instrumentally or relationally.

The problems that include radicals in addition to meeting the design criteria as indicated above, were to see if students viewed the exponential notation and radical notation as parallel notation for the same quantities.

## A Model For Analysis of the Protocols

Erlwanger (1975C, pp. 13-16) theorized that students in arithmetic operate congitively in one of three systems:

"Basic System" - the student operating in this cognitive mode show a spontaneity and natural insight about quantities and any relations between the quantities. The behavioral characteristics of students operating in the Basic System show little body movement, and an appearance of deep thought independent of the written work or diagrams on paper. The student shows a great deal of confidence in any judgement made or answer obtained.

"Perceptual Manipulation System" - the student is continuously involved with an attempt to convert all written or oral quantities given into a diagram. The student is not concerned with quantitative judgements or about any relations between quantities, but only about procedures and actions for transforming quantities. The behavior associated with this mode shows frequent eye movements between the given problem and his construction. The student has rapid hand and body movement in dealing with his real or imagined construction. Any oral description given by the student is accompanied by actions indicated above, along with frequent touching of his constructions.

"Notational-System Manipulation System" - the student operating in this system is concerned totally with arrangement and order of written terms and their manipulation. He is concerned with the recognition of the type of problem by the position of terms and symbols. From the relative position of the terms and symbols the student hopes to select the appropriate algorithm and apply it. The student is only concerned with manipulating symbols and not about any qualitative judgements about the quantities involved. The behavior associated with a student operating in this system shows hand, eye, and body movements all related to and determined by the written form of the problem...Also, there is frequent hand and eye movements between the written terms and symbols. This student also shows inflexibility in oral descriptions of the problem, and also shows a rigidity when pointing or touching terms and symbols that are being used. The student shows a lack of confidence in answers obtained.

Alexander (1977) believed since Erlwanger's model was designed for elementary children that there was one significant obstacle in attempting to apply the model to adolescent thinking. He conjectured that some students at the formal level of thought could apply algorithms and perform manipulation of symbols with "spontaneity and confidence" (p. 66). Consequently, Alexander used Bruner's notions of enactive, iconic, and symbolic representations of Erlwanger's Basic, Perceptual Manipulation and Notational-System Manipulation modes of congitive activity (pp. 84-85). Alexander believed that Erlwanger's one dimensional model was not adequate in that an individual at the formal operational level could operate on symbols in either a "mechanical" or "insigtful" manner. Thus, he developed a two by three matrix whereby the three schemas; enactive, iconic, and symbolic were paired with the two levels of thought; mechanical and insightful. That perhaps Alexander's model could be somewhat ambiguous in terms of the word "insightful" is suggested by Bruner (from whom Alexander borrowed the word):

> The distinction is not between mechanical and insightful really, but whether or not the child has grasped and can use the generic code we have set to teach him (Bruner, 1973, p. 223).

The key is the words "generic code". This would appear to be Skemp's "conceptual structure" or "appropriate schema" (p. 46).

However, the designations of enactive, iconic, and symbolic seem to be extremely relevant to a study of the mathematical thinking of college students. Bruner (1966), when discussing the roles of the enactive, iconic, and symbolic representations, noted:

> When the learner has a well-developed symbolic system, it may be possible to bypass the first two stages. But one does so with the risk that the learner may not possess the imagery to fall back on when his symbolic transformations fail to achieve a goal in problem solving (1966, p. 49).

It was one of the purposes of this study to see if in fact college students do have imagery "to fall back on", or are they operating symbolically without any real understanding of the situation. This study (as did Alexander) eliminated the "enactive" mode of representation, based upon the fact that physical materials are not available in the algebra classes at Ferris State College, nor were they available during the interview. Therefore, the model used to analyze the interviews in this study combined Skemp's levels of understanding (instrumental and relational) with Bruner's iconic and symbolic modes of representation of knowledge to form a two-by-two matrix as previously indicated in Chapter I (p. 15).

The following characteristics of student behavior with respect to cell assignment of the matrix was suggested by Alexander (1977, pp. 85-88).

A. <u>The Instrumental - Iconic Mode</u>

> It is concerned with some diagramatic or imagined representation of the problem. It is not concerned with the relation between the problem and the representation or any general principles and relationships which may be abstracted from the imagery employed.

The associated behavior is marked by:

1. Eye and hand movements related to imagined or real transformations on the representation of the problem. These movements are done in either a random order or in a rigid pattern indicative of a habitual rather than thoughtful response and there is little or no eye or hand movement to the original problem.

2. Little oral description of the actions or transformations or oral description which lacks spontaneity.

3. Lack of confidence in any answer obtained.

B. <u>Relational - Iconic Mode</u>

It is concerned with some diagramatic or imagined representation of the problem, the relation between the problem and the representation, and general principles and relationships which may be abstracted from the imagery employed.

The associated behavior is marked by:

1. Eye and hand movements related to imagined real transformations on the representation of the problem with, at least initially, eye or hand movement relating the image to the original problem.

2. The actions or transformations are not random, but follow a flexible pattern related to the nature of the problem.

3. The actions or transformation are accompanied by spontaneous oral descriptions.

4. Confidence in any answer obtained.

C. <u>Instrumental - Symbolic Mode</u>

It is concerned with symbolic aspects of the problem : the recognition of the type of problem from symbolic cues, the selection of an appropriate algorithm, the manipulation of terms, expressions, and symbols, but not with general principles and relationships inherent in logical thought.

The associated behavior is marked by:

1. Eye and hand movements determined by the written form of the problem and performed either in a random order or in a rigid pattern dictated by the algorithm selected.

2. Little oral description of the actions, or oral description which lacks spontaneity.

3. Lack of confidence in any answer obtained.

D. Relational - Symbolic Mode

It is concerned with the symbolic aspects of the problem: the comprehension of the problem, and pertinent relationships which exist among the involved quantities, from symbolic cues, the selection of appropriate principles and algorithms for the solution of the problem.

The associated behavior is marked by:

1. Little movement of eye or hand.

2. Spontaneous oral descriptions or relations observed or algorithms selected.

3. Confidence in any answer obtained.

Each student will be assigned to at least one cell for each problem in the interview sequence based upon his/her written work of the problem, his/her spoken words, as well as body actions.

The use of the model in assigning students' mathematical thinking to various cells as they work problems will be seen in operation in Chapter IV, Analysis of the Interviews. One short example is offered here in order that the reader may see the model in action.

Sue, a Math 121 (intermediate) student, has been given problem #17: $2^{-2} \cdot 2^{3}$.

Sue: (long pause) umm...(then writes $2^{1}$)

Interviewer: Now, tell me what you did there.

Sue: Well, what I'm trying to do, I've got the same bases, so what I just did is added the exponents.

It is clear that Sue is in the Instrumental-Symbolic mode from her hesitancy, and the fact that she fell back on a rule. However, she does switch to the Relational-Iconic mode when pushed by the interviewer.

Interviewer: (As Sue stares at the paper apparently in deep thought) You're trying to decide if that is really right?

Sue: Right.

Interviewer: And how do you decide?

Sue: Well, you'd have to go one-half times one-half ...times 8. One-fourth of 8 is 2.

## Summary

Using the models of Erlwanger (1975C) and Alexander (1977) as guides, a model was developed that related Skemp's instrumental and relation understanding to Bruner's iconic and symbolic mode of knowledge representation for use with an interviewing procedure to analyze the thought processes of 14 remedial algebra students at the college level as they worked designated problems. A transcription to paper was made from the audio-tapes of all material deemed relevant to the study and will be detailed in the next chapter.

# CHAPTER IV

# ANALYSIS OF THE INTERVIEWS

## The Use of the Problem Sets

This study was intended to investigate the understanding of college remedial algebra students with respect to the concept and principles of exponent, variable, and equation. Additionally, the intention was to investigate the students' prerequisite knowledge for the concept of exponent, as well as the role played by imagery in the understanding of all the above. The problem sets (listed in Chapter III, pp. 138-142) were broken into eight categories for the determination of the students understanding with respect to the above concerns.

1. The following problems were used for the purpose of the determination of prerequisite knowledge.

   a. Order of operations, use of parentheses, distributive property.

      I. 1. $2 \cdot 3 + 4 - 12$

      2. $15 - 3 - (2 + 1)$

      10. $3 + 4(x + 5)$

   b. Reducing fractions

      I. 3. $\frac{2 \cdot 4 \cdot 6}{8}$

      4. $\frac{2 \cdot 5 + 6}{8}$

8. $\frac{4xyz}{2x}$

9. $\frac{a + b}{a}$

2. The following problems were used for the investigation of the students' understanding of variable, and/or equation.

I. 5. a + a + a

6. 2a + 3a

7. 2a + 3a - b

II. 1. What does it mean to you when we say 2x + 3x = 5x?

2. What does it mean to you when we say solve for x in x + 4 = 7?

3. Which of the indicated equations would be the correct answer for the following:

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors. (Rosnick, Clement, 1983, p. 4)

a. P = 6S

b. S = 6P

4. Is the following true or false? 4x = 46 given that x = 6 (Matz, 1979, p. 134)

5. Without solving, could you compare the solutions for W and N in the following:

   $7W + 22 = 109$; $7N + 22 = 109$

   (Wagner 1981, p. 109)

3. The following problems were used for the investigation of the students' understanding of the definition of "exponent". (All of the remaining problems are from problem set III.)

   1. $2^3$
   2. $(-2)^3$
   11. $(2 + 3)^0$
   14. $5^{-2}$
   24. $a^{-b}$
   27. $(x^6)^{\frac{1}{2}}$

4. The following problems were used to investigate the students' understanding of the exponential properties $a^m \cdot a^n = a^{m+n}$ and $a^m/a^n = a^{m-n}$.

   5. $2^2 \cdot 2^3$
   13. $4^4/4^2$
   17. $2^{-2} \cdot 2^3$
   22. $x^a \cdot x^b$
   23. $m^n/m^2$
   34. $a^{1/n} \cdot a^{1/m}$

5. The following problems were used to investigate the students' understanding of the exponential properties $(ab)^n = a^n b^n$ and $(a/b)^n = a^n/b^n$.

4. $2^2 \cdot 3^2$

10. $(2 \cdot 3)^2$

16. $14^2/7^2$

20. $(xy)^3$

28. $\sqrt{4} \cdot \sqrt{9}$

6. The following problems were for the investigation of the students' understanding of the exponential property $(a^m)^n = a^{mn}$.

8. $(2^2)^3$

19. $(2^{-1})^{-2}$

21. $(x^2)^3$

7. The following problems were used to determine if the students used "false generalizations" of the various exponential properties. Also, these were used to see if the students' could use the concept of exponent with the required arithmetic operation to complete the problem.

3. $-2 \cdot 3^2$

6. $2^2 + 2^3$

7. $3^2 + 2^3$

9. $4^4/2^2$

11. $(2 + 3)^0$

12. $(2 + 3)^2$

15. $(2 + 3)^{-1}$

18. $2^3/4^{-1}$

25. $x^2 + x^3$

26. $x^a y^b$

29. $\sqrt{14} / \sqrt[3]{7}$

32. $\sqrt{6} - \sqrt{3}$

8. The following problems were used to investigate the students' ability to square a binominal as well as their generalization of the exponential properties.

30. $(x + 3)^2$

31. $(\sqrt{2} + \sqrt{3})^2$

33. $(x^{3n} + 2)^2$

It should be noted that the categories were not intended to be unique. For example, problem III. 9. $4^4/2^2$ was listed under "false generalizations", but it could easily be solved by a relational understanding of the property $a^m/a^n = a^{m-n}$. That is, since $2^2 = 4$, the problem could well be seen as $4^4/4^1$ (or $2^8/2^2$). Another good example is problem III. 27. $(x^6)^{\frac{1}{2}}$, which is listed under both the category for definition and the property $(a^m)^n = a^{mn}$.

The problems were deliberately kept on the "simple" side to allow for an examination of the students' thought processes as they relate to exponents without involving any other concepts or "rules". Only category 7 violates this somewhat, by requiring that the students be acquainted with the algorithm for the squaring of a binomial. In this regard, as indicated in Chapter III, the Math 111 (beginning algebra) students were not asked to do problem set III. (28-34). Also, the Math 111 students were not asked any problems involving radicals since the Math 111 courses do not cover problems of this type. It was anticipated that the Math 121 (intermediate algebra) students would at least recall having seen this algorithm, and thus, would not be "scared off" by the form of the problem, and could concentrate on the questions by the interviewer.

## The Interviews

This study was designed to investigate the following questions (as indicated in Chapter I):

1. Do remedial algebra students have a relational, instrumental, or no understanding of the prerequisites conjectured as necessary (as advocated by Gagné) for success in dealing with the concept of exponent.
2. Do remedial algebra students have a relational, instrumental, or no understanding of the concept of exponent?
   a. How does the understanding of positive, negative, (both integral and fractional) and zero exponents differ in the same student? Between students?
   b. How does the understanding of explicit number exponents and literal exponents differ in the same student? Between students?
3. Do remedial algebra students have the ability to generalize (as defined by Krutetskii) the various properties of exponents?
   a. Can the source of "false generalizations" be determined?

   b. Have students that appear to have generalized the properties of exponents (relational understanding), merely generalized the symbolic notation (instrumental understanding)?

4. What types of imagery (Bruner's enactive, iconic, and symbolic) do students use when working with the concept of exponent?
   a. Does the imagery used differ, and in what respect, for students at the relational and instrumental levels of understanding?
   b. Can a student who is operating at the instrumental level be "pushed" by way of hints and guided questioning to use numerical imagery as an aid to relational understanding?
5. Do successful students (as determined by a letter grade on a test) differ from unsuccessful students with respect to the four questions above?

In an attempt to gain insight into the research questions, the interview audio-tapes were analyzed in terms of Skemp's relational and instrumental understanding paired with Bruner's iconic and symbolic modes of representation (as discussed in Chapter III). The student interviews will be detailed one at a time, by tracing through the various categories. In the interest of space and redundancy, excerpts from each category will not be presented for every student. Sufficient excerpts are chosen to present the reader with enough background to see the type of thinking that the individual uses. Excerpts were selected that best seemed to indicate the type of thinking the students used in solving the problem sets. Those chosen indicate not only Relational-Symbolic, Instrumental-Symbolic, Relational-Iconic, and Instrumental-Iconic, but also show the wavering of the thought process and the consequent transfer from one mode of understanding to another.[1]

Each of the interview excerpts are numbered for the purpose of referencing. The problem under discussion will be indicated after the excerpt number, or else will be indicated in parentheses when the excerpt references another problem during the discussion. Items of information that are not part of the dialogue will also be presented to the reader by use of parentheses. All excerpts are indented, and each individual's dialogue is typed single spaced. In the following fictitious example, the excerpt is number 1000, the problem is $x^y \cdot x^4$, and the reader is told there is a long pause, and that problem number 6 is $2^4 \cdot 2^3$. Jed is the student being interviewed.

1000: $x^y \cdot x^4$

Jed: x to the 4y power, no wait,...that's not right. When you multiply, you (long pause)...yeah, you multiply exponents.

Interviewer: Is this problem similar to problem number 6? $(2^4 \cdot 2^6)$

Math 111 High Examination Scores

The first group of student interviews are from that group of Math 111 (beginning algebra) students that were

1. More information regarding transcripts and protocols may be obtained by contacting the author.

designated by their instructor as having ranked at the top of the class on a unit test dealing with exponents.

Art had no high school algebra, but has taken Math 111 two times previously and has failed. This time Art is doing well by memorizing rules. He eventually received a B+ for a final grade. Art's instrumental understanding is indicated in the following excerpt. Art has successfully performed the operations on problem I. 1.

1. $2 \cdot 3 + 4 - 12$

Interviewer: It looks as if you grouped the two and three together, and the four and twelve together, could you have grouped the three and four together?

Art: I work according to rules, which is multiplication first. You never deviate from the rules unless you are told to do so.

However, the fact that Art does have some appreciation for "rules" as conventions is indicated by his next comment.

Interviewer: Why do you suppose we have that rule?

Art: It seems probably for uniformity.

That Art possesses some relational understanding of arithmetic is shown in the following. Art has multiplied out the numerator and divided to correctly get 6.

2. $\frac{2 \cdot 4 \cdot 6}{8}$

Interviewer: Could you have taken a shortut there - reduced or canceled?

Art: Uh...I assume I could factor out an eight and a four, and two.

Interviewer: Would you show me?

Art: Factor out $2 \cdot 2 \cdot 2 \cdot 2 \cdot 3/2 \cdot 2 \cdot 2$ and cancel out the twos (which he then does correctly).

Art reverts to the Instrumental-Symbolic mode on the next problem.

3. $\frac{2 \cdot 5 + 6}{8}$

Art: 2 times 5 is 10 + 6 is 16 over 8,...again I suppose I could...

Interviewer: What answer do you get?

Art: 2.

Interviewer: The other way — could you reduce this problem?

Art: No, not unless I factored it out...you've got the factor of ten which is 2·5 and the factor of six, which is two times 3 (writes $\frac{2 \cdot 2 \cdot 5 \cdot 3}{2 \cdot 2 \cdot 2}$), which is fifteen over two, which doesn't cancel out right.

Interviewer: In problem number 3. ($\frac{2 \cdot 4 \cdot 6}{8}$) you were able to cancel the two into the eight, could I do that here?

Art: I would just become more confused. It's not to complex to grasp, it's just not logical.

Art's last statement should not be taken necessarily as relational understanding. The "logical" means the violation of his rule to perform the operations in the numerator then the denominator fist, then divide. Art also has difficulty with the concept of variable.

4. a + a + a

Art: a to the third power...no, that's not right, three a, because a to the third power would be a times a times a.

Interviewer: So, a plus a plus a is three a?

Art: Yes.

Interviewer: How do you see that, what if...

Art: There are three sets of a

Interviewer: Suppose I asked you to explain to me why it's three a's, could you use a number or something?

Art: In more complex equations I could.

Interviewer: Suppose a were five, how would that work out?

Art: Then again, I would have to use more complex equations.

Interviewer: Suppose a were five there, what would you get?

Art: You mean right here was five a? If a were five a that would be seven a.

Art does eventually see that idea of a = 5, and gets 15.

Art's Instrumental-Symbolic understanding of algebra fractions is indicated in the next sequence. Art has successfully completed problem I. 8. $\left(\frac{4xyz}{2x}\right)$ by correctly dividing out the factors. His statement in excerpt 3. that it was "not logical" to reduce on problem 4. was merely because it conflicted with his other answer or rule, is brought out in this sequence.

5. $\frac{a + b}{a}$

Art: a + b would simply be a + b over a. We could get rid of the a's and it would be b over 1 or simply b.

Interviewer: Suppose, I doubt that you have problem 8. or problem 9. right, let's say problem 9. $\left(\frac{a + b}{a}\right)$.

Art: Well, you could turn it around into a multiplication problem.

Interviewer: O.k.

Art: Simply stated would be...huh...b...uh... times, uh let me see (long pause, sigh)...uh, o.k... o.k, that would be b times a equals ab not a + b.

Art has switched into the Relational-Symbolic mode due to the fact that even though struggling he was able to use the relationship between multiplication and division to contradict an answer that he had obtained while in the Instrumental-Symbolic mode of understanding.

6. $\frac{a + b}{b}$

Interviewer: So what does that tell you?

Art: Which just proves my theory.

Interviewer: Which tells you, you made a mistake on that or...?

Art: (Shakes head yes, agrees with the answer b) If I disagree with the b.

Interviewer: Could you justify for me it is b?

Art: Seems logically that I could.

Problem number 4. $\frac{2 \cdot 5 + 6}{8}$, is pointed out to Art. Eventhough Art briefly was in the Relational-Symbolic mode, he now reverts to Instrumental-Symbolic. Apparently his work with variables has now caused difficulty with the method that was "not logical" before.

7. $\frac{2 \cdot 5 + 6}{8}$

Interviewer: Could you do some canceling here?

Art: Well let's see, we had 2·2·3·5 and we had 2·2 (crosses out the twos)

It is then pointed out to Art that he had decided before that the answer was 2, and he then changed his mind and decided 2 was correct. Art's lack of confidence in his answer is one symptom of the Instrumental mode of understanding.

Art's understanding of variable seems to waver between instrumental and relational. In problem I. 10. 3 + 4 (x + 5), Art correctly obtains $4x + 23$. When the interviewer pushes him to show that the answer is not $27x$, the best Art can do is "you could prove it if you integrate it into a more complex set of conditions". Even when pushed, Art never considers a counter-example by substituting a number, but the following sequence shows that Art does at times move out of his rule-based mode of understanding into a Relational-Iconic mode (even though his computation is bad).

8. What does it mean to you when we say $2x + 3x = 5x$?

Art: It means we have 2 times the variable x plus 3 times the variable x and combine the 2 and 3 because we have the plus sign, which means 5x, knowing that we cannot combine x. When we have plus, we are simply left with the variable x.

Interviewer: Suppose I looked at that left side and I said x is 100. What would the simplification of the left side be?

Art: 5x is 500, or you could say 200 plus 300.

Art's idea of exponent is for the most part an instrumental one. Even though he does have rules, he is not quite sure of these.

9. $2^2 \cdot 3^2$.

Art: O.k, that would be simply 2 times 2 equals 4, and 3 times 3 equals 9. And then we multiply 9 times 4, which equals...uh 36.

Interviewer: Is there another way I could have done that? Suppose I could combine the exponents somehow or the base?

Art: No.

Interviewer: Suppose I had 2 times 3 is 6, squared is 36. Is that an accident?

Art: I don't know, I've never really checked into it. I suppose we could check into it right now.

Interviewer: O.k. Let's do.

Art then uses an example of $2^3 \cdot 3^2$, gets 6 for the base and can't decide what to do with the exponents. He does not want to change the problem even though the interviewer points out that the bases are different.

That the interview has challenged Art's idea of there being one way to do all types of problems is brought out by the next excerpt.

10. $2^2 + 2^3$

Art: Now, I'm starting to think about this. Could we go back?

Interviewer: Sure.

Art goes back to problem 4. $2^2 \cdot 3^2$, and gets $6^4$, which he calls 24. He is corrected by the interviewer and together they decide $6^4$ is a "large number", and thus incorrect because it doesn't equal thirty-six.

Interviewer: This one (problem 4) gives us 36. Looks like we multiplied the 2 and the 3 and kept the exponent.

Art: It's starting to boggle my mind.

Interviewer: Sometimes they do.

Art: Sometimes I go back to my book, and often times look at the rules. You know Knute Rockne said, "Practice, practice, practice."

Art did decide "any number to the zero power equals one" on problem 11. $(2 + 3)^0$, but gave $(-5)(-5) = 25$ for $5^{-2}$. Although, with help from the interviewer, he did finally remember that $5^{-2}$ was 1/25.

However, that Art has not generalized the properties of exponents to include exponents other than positive whole numbers is indicated by the following excerpt. When confronted with the problem $(2^{-1})^{-2}$, Art decides that it is $1/2 \cdot 1/2 = 1/4$. The interviewer and Art look back at problem 8. $(2^2)^3$, which with help from the interviewer, Art had finally decided was equivalent to $2^6$.

11. $(2^{-1})^{-2}$

Interviewer: Let's transfer this over to problem 19. We've decided that $(2^2)^3$ is $2^6$. Could I use the same property or do the negative exponents lead to something different? How do you view the negative exponent? For example could that $(2^{-1})^{-2}$ be $2^2$?

Art: That's what I was trying to show right here, I had 1/2 times 1/2; but that's not...

Interviewer: Which is 1/4, but if I multiplied the exponents, I'd get 2 to the second,...

Art: I'm confused about this rule. I'll tell you I'm really confused about it.

That Art is working continuously in the Instrumental-Symbolic mode is verified by the next excerpt.

12. $(xy)^3$

Art: O.k., that would be just xy to the third. (writes $(xy)^3$) Some people I'm sure would view that as x to the third y to the third.

Interviewer: Are those the same? x to the third y to the third is that the same as...

Art: No.

Interviewer: What if I'm one of those who think it's x to the third y to the third, and I ask you to show me I'm wrong.

Art: I'd say go back to the text and prove it.

Interviewer: The text would prove it?

Art: The text would give the rule.

That Art sometimes does remember the correct rule, is demonstrated by the fact that on problem 23. $(m^n/m^2)$, he immediately wrote $m^{n-2}$.

The fact that the Instrumental-Iconic mode is often as misleading for a student as the Instrumental-Symbolic mode is indicated by Art's response to problem 27. $(x^6)^{\frac{1}{2}}$. Art drew the following diagram: xxx/xxx and immediately gave the correct answer of $x^3$. It was only upon further questioning that it could be determined that Art had no understanding of rational exponents.

Art's philosophy of learning algebra is summed up by a comment made while working on problem III. 9. $4^4/2^2$. Art works the problem in a Relational-Iconic fashion by writing $\frac{2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2\cdot2}{2\cdot2}$ and canceling out the twos to get 64.

Interviewer: Can't divide the 2 into the four or subtract exponents or anything like that?

Art: Then again I'm just in an elemental class right now and that's the way I keep from being confused.

Interviewer: Some people do subtract exponents on this type of problem.

Art: If you could show me a trick, I'd be happy to use it.

Art works reasonably well as long as he can use numerical imagery and properties of arithmetic, but as problems become symbolic, Art shifts to an instrumental mode based strictly on symbolic manipulation. He then is open to any "trick" that reminds him of a property that he vaguely remembers.

Dea is another Math 111 student that did well on the unit examination on exponents that was taken approximately two weeks prior to the interview. She eventually received an "A" for the class. Dea is much more confident of her answers than Art. This is apparently due to the fact that she has more of a relational understanding than Art. Dea makes an observation about how she views numbers. Dea has correctly evaluated 15 - 3 - (2 + 1) by first combining the 2 and the 1 in parentheses.

13. 15 - 3 - (2 +1)

Dea: I don't know why I do it that way. I guess it's...I always group. See I see numbers in groups. Maybe that's what helps me in algebra. Instead of seeing the 2 + 1, I see a 3, now this one here, (starts doing problem 3. $\frac{2\cdot 4\cdot 6}{8}$), I would do this as 24 times 2. That would be easier for me than 8 times 6, which is 24 times 2 is 48, divided by 8, is 6.

Interviewer: O.k. Any shortcuts on that maybe? Any canceling perhaps?

Dea: Yeah, you could cancel this out. 2 times 4 would be 8 and cancel the 8 out and get your 6.

Interviewer: O.k.

Dea: But, I don't see that as a canceling problem. I would not recognize that as a short step. I would just go ahead and figure that out.

Dea then works problem I. 4. $\frac{2\cdot 5 + 6}{8}$ properly and gets 2. When asked if she could "cancel" here, she indicates that this is addition and therefore cannot be reduced.

However, Dea's relational understanding does not hold up when the problem involves variables.

14. $\frac{a + b}{a}$

Dea: a goes into a, 1 time plus b.

Interviewer: You can always cancel?

Dea: You can just...Yes, that's the way I do it. You know if you have the same variables, or letters or whatever, you just divide. I see it as division, but I just cancel when I see it.

Interviewer: Could I ask you to go back to problem 4. $(\frac{2 \cdot 5 + 6}{8})$?

Dea: 2 times 5 is 10 + 6 is 16 divided by 8 is 2.

Interviewer: What I am asking is if you could cancel there.

Dea: (sighs) (long pause) Let's see...yeah, you could...No, I don't see where you can cancel before you perform the operation.

Interviewer: O.k.

Dea: I'm thinking 2 goes into this (2·5) five times and 2 goes into 6 three times. So, let's see that would be 5 plus 6, which is 11 fourths. I guess it has to be multiplication on top before you can.

Interviewer: So what about number 9. $(\frac{a + b}{a})$?

Dea: Hum...(pause - no response)

Interviewer: How do you check these out? I see I've created a little puzzlement in your mind.

Dea: I know it.

Interviewer: You look at that and you're asking yourself can I cancel or can't I? Is there any way, in your own mind, to check this out?

Dea: The way I see it anything over itself is 1. That's the way I see it is 1 + b. I don't know how I could check it out.

The interviewer once more has Dea compare problem 3. $\frac{2 \cdot 4 \cdot 6}{8}$ and problem 4. $\frac{2 \cdot 5 + 6}{8}$ and problem 8. $\frac{4xyz}{2x}$ and problem 9. $\frac{a + b}{a}$, and then pushes agian to see if Dea can "check it out". Finally, on her own, Dea uses numerical imagery.

15. $\frac{a + b}{a}$

Dea: Um, let me think here. Yeah, if you had the variables. If you just punched in a = 2 and b = 3.

Interviewer: O.k. Try that.

Dea: You'd have 2 + 3 on top, which is 5 divided by 2, which equals five-halfs.

Dea then correctly decides that the "a" cannot be canceled. However, Dea's numerical imagery technique is short-lived. On the very next problem she not only does not use numerical imagery, but makes an error by confusing what Matz (1979, p. 137) called "tautologies" and "constraint" equations.

16. 3 + 4 (x + 5)

Dea: You've got 4x plus 20 plus 3 would be 4x + 23.

Interviewer: You can't combine the 4x and 23?

Dea: No, they are unlike terms. Now if you were to add the 3 and 4 and go 7 times...that would be 7x + 35. (long pause) See they don't work out the same. There again you should do multiplication before addition.

Interviewer: Some people mistakenly add the 4x and the 23 and get 27x. Suppose you had to verify which answer is correct. How could you check it out?

Dea: Uh...Yea...let's see could I put a number in there...(mutters to herself)...couldn't do that. I don't really know how to go about checking that particular problem. I think what you'd have to do is put 0 on the other side and solve for x. Then just check it out and see if it equals 0.

Dea then solves 4x + 23 + 0, get x = 23/4 and tries to put this value of x in $3 + 4(x + 5) = 0$, becomes confused and drops it. (That this method would actually work out is an "accident" as far as Dea's thinking is concerned.)

Dea also had difficulty with the "students and professors" problem II. 3.

17.

Dea: (reads problem again) It would be 6S = P.

Interviewer: Could you give me your logic on that?

Dea: When I read that, I see 6 times the students. So if you have 6 times students equals professors.

Interviewer: What if there were 8 professors, how many students?

Dea: Uh..., 8 times 6 is 48.

Interviewer: 48 students?

Dea: There are 6 times as many students as professors. (Does not see the inconsistency with her answer.)

Dea's notions of exponent are in general of a relational type, but she has not generalized the properties in all cases.

18. $2^2 \cdot 3^2$

Dea: You've got 4 times 9 which equals 36.

Interviewer: Could you have done that another way?

Dea: Um...No not that I see.

Interviewer: How about the 2 times 3 first...

Dea: Yeah, you would combine them that way.

Interviewer: Is that just an accident?

Dea: I don't know. I've never done it that way. I was told to do the exponents first. So, I don't even visualize that problem. That might be an accident.

However, on problem 10. $(2 \cdot 3)^2$ which is the reverse of pr

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problem 4. $2^2 \cdot 3^2$. Dea has no difficulty using the exponential property and immediately wrote $2^2 \cdot 3^2$. That Dea has a relational understanding of this property, $(ab)^n = a^n b^n$, was also indicated on this problem.

19. $(2 \cdot 3)^2$

(Dea has correctly done this computation in the two ways indicated above and is being questioned by the interviewer.)

Interviewer: How do you know those are the same? $((2^2 \cdot 3^2)$ and $(2 \cdot 3)^2)$.

Dea: This is saying $((2 \cdot 3)^2)$ 2 times 3 times 2 times 3, and just rearrange it. (writes $(2 \cdot 3)(2 \cdot 3) = 2 \cdot 2 \cdot 3 \cdot 3$)

Dea does have only an Instrumental-Symbolic understanding of zero and negative exponent.

20. $(2 + 3)^0$

Dea: 5 to the 0 equals 1

Interviewer: Why?

Dea: Because its a rule. Anything to the 0 power equals 1.

Interviewer: Why not 0 rather than 1?

Dea: Oh, I don't know why I do that. It's a rule that stuck in my head. The way I remember it is, since it's in the problem, it has to equal something to keep its place.

Interviewer: Well, in that case, could we have called $5^0$ five? And let that hold its place?

Dea: Well, you probably could, but I don't think it would work out. I don't know why they come up with that 1. Well, yeah, I do. One times anything is one - generally in multiplication. It doesn't in addition.

21. $5^{-2}$

Dea: Let's see...I know how to get the answer, but I don't know how to explain it to you.

Interviewer: O.k. What's the answer.

Dea: It's 1 over, bring the 5 down and make a positive 2. One over 5 squared. I don't know how or why, but I know that's what is supposed to be done.

Dea is able to correctly work out other problems dealing with negative exponents by using the definition. On problem 17. $2^{-2} \cdot 2^{3}$, when pushed by the interviewer, she does use the appropriate exponential property. However, on problem 19. $(2^{-1})^{-2}$, she comes up with two different answers using the definition and never does see the involved property.

That Dea wavers between a Relational-Symbolic and Instrumental-Symbolic mode of understanding with respect to exponents, is indicated in the following excerpts:

22. $(x^{2})^{3}$

Dea: That another exponent times x squared times 3 equal $x^{6}$.

Interviewer: Could you verify that?

Dea: Yes. You could take $x \cdot x$ three times (writes $x \cdot x \cdot x \cdot x \cdot x \cdot x$).

Dea is obviously in the Relational-Symbolic mode in the excerpt above, but when only variables are involved, the situation changes.

23. $x^{a}\ x^{b}$

Dea: Uh...let me see now...unknown exponents would be x times x. I don't think that can be changed in any way. Those are all unknowns... (long pause). We could put x squared ab. (writes $x^{2ab}$). I would think. Yeah,...No, Yeah.

Interviewer: Could you verify that?

(Dea now resorts to numerical imagery in her attempt to verify her answer and transfers into a Relational-Iconic mode.)

Dea: Well yeah, you could put a number in. Take $x = 2$, $a = 2$, $b = 3$. I don't think they'll get too big that way. (writes $2^2 \cdot 2^3$). Which would be...$2^5$. So the bases stay the same. That goes right back to adding exponents. That would be $x^{a+b}$.

However, on problem 26. $x^a y^b$, Dea answers $(xy)^{a+b}$, and proving Skemp's admonition that a "multiplicity of rules" can get out of hand, has difficulty on problem 27. $(x^6)^{\frac{1}{2}}$. Dea answers this as $(x^6)^{-2} = x^{-12} = 1/x^{12}$, because "the one-half power is the same as minus 2".

In general, Dea seems to operate according to rules with some relational understanding with the non-variable type problems. However, she fluctuates between relational and instrumental when variables are involved. In comparison to Art, Dea does frequently resort to numerical imagery.

Dea and Art are representative of the type of understanding exemplified by the "better" Math 111 (beginning) algebra students during the interviews. Dea remembered the properties better than Art, although neither appeared to have generalized all properties. Both used many false generalizations. Dea did appeal to numerical imagery to aid in the understanding of symbolic relationships. However, she was not consistent in this regard. The interview for the other student in this category will not be detailed here due to the fact that for the most part, it would merely be repetitious of the types of thinking shown by Art and Dea.

Math 111 Low Examination Scores

The four Math 111 students that scored the lowest on the exponent unit examination in general exhibited a lower level of understanding of the concepts involved in the problem sets than Art and Dea. Their understanding was primarily at the instrumental level on arithmetic problems, while the Instrumental-Symbolic mode was the mainstay for the variable type problem. However, one student, Tim, did not differ significantly from Art and Dea in his thinking. (Although he did eventually receive an F for the class). The other three students were very consistent in their thinking. Again, due to the repetitive nature of their interviews, only one will be detailed in this section.

Max is a student that has returned to school after several years. This is the second time Max has enrolled in Math 111, after having received an F previously. He received a D- for his final grade this time. Max had no high school algebra. He needs credit in Math 111 and Math 121 to meet the requirements of the specific program in which he is enrolled. Max does reasonably well on the arithmetic type prerequisite category problems. He correctly does problem 1. $2 \cdot 3 + 4 - 12$, problem 2. $15 - 3 - (2 + 1)$, and problem 3. $\frac{2 \cdot 4 \cdot 6}{8}$. However, on problem 3., he is asked by the interviewer if he could have "reduced" before doing the multiplication. Max writes: $\frac{\not{2}^{1} \cdot \not{4}^{2} \cdot \not{6}^{3}}{\not{8}_{4}}$ to get a different answer of 3/2. He is not able to see what went wrong, but does believe 6 is correct.

Max would not attempt to reduce in the next problem

$\frac{(2\cdot5 + 6)}{8}$ because "the 5 was odd and could not be reduced by 2". That Max does not really understand the situation relationally is brought out when he gets to problem 9. $\frac{a + b}{a}$

24. $\frac{a + b}{a}$

Max: That's b. The a's cancel out.

Interviewer: The a's cancel?

Max: Yes, they are common factors.

In an attempt to get Max to use numerical imagery to detect his error, the interviewer pushes.

Interviewer: Suppose I doubted that the answer is b. Is there any way that you could justify that it really is b?

Max: Well, you can't divide the a into the b, but the two a's cancel. I know exactly what you want.

Interviewer: Some people cancel the a into the a, get 1, and give an answer of 1 + b.

Max: Well, you would have a 1. It goes in there one time. I see what you are talking about now.

Interviewer: So you think it's 1 + b?

Max: Probably.

The interviewer asks Max to look back at problem 8. $\frac{4xyz}{2x}$, which Max had done correctly, using the reducing idea.

Interviewer: Up in number 8., you got 1's when you canceled out the x's.

Max: Yes.

Interviewer: When you multiplied by the one, the product just stayed the same?

Max: Yes.

Interviewer: Down here since the 1 is added (emphasizes this word), you have to leave the one?

That Max is thinking in the Instrumental-Symbolic mode is indicated by the fact that he recognizes the difference (somewhat) between addition and multiplication with the arithmetic problems, but even with the strong hint by the interviewer, he finally concludes the answer is "b".

On problem I. 10. 3 + 4(x + 5) Max exhibits a common tendency of the Instrumental-Symbolic mode of thinking, to dwell on the physical arrangement of the symbols. Max successfully completes problem 1. 2·3 + 4 - 12 by indicating that "you always do multiplication first". Even when the interviewer suggests adding the 3 and the 4 first Max will not change.

25. 3 + 4(x + 5)

Max: That would be 7x plus...uh 35.

Interviewer: How did you get the 7?

Max: I add the 3 and the 4 then times the x and times the 5.

Interviewer: Let's go back and look at problem 1. (2·3+4-12). Now you didn't get a 7 here because you said you prefer to...

Max: Do multiplication first. (long pause) Now why didn't I do multiplication first?

Interviewer: Yes, is there a times sign between the 4 and parentheses?

Max: I would think so. Just times what's inside.

Interviewer: Why don't you try it this second way, and see if you get the same answer.

Max: You get 3 + 4x + 20, which is 4x + 23.

Interviewer: Which answer would you go with?

Max then makes the correct choice. He has somewhat of a relational understanding of variable. However, the various uses of variable along with the idea of equation does give him difficulty. Max correctly answers problem 5. a + a + a. However, when pushed to explain why it was not $a^3$, he never considers substituting a number. On problem 6. 2a + 3a and problem 7. 2a + 3a - b, he also does well. Max uses the only "concrete imagery" of the interviews. When challenged as to whether the a's and b's could be added in problem 7., he responds, "You can't add oil and water". The following excerpt starts with II. 1. It is here that Max shows some relational understanding of variable.

26. What does it mean to you when we say 2x + 3x = 5x?

Max: That means you combine like terms. 3x plus 2x equals 5x.

Interviewer: How do you view the x there?

Max: x is an unknown.

Interviewer: If x were 2, what would this statement mean?

(Max substitutes in both sides and gets 2·2 + 3·2 = 5·2).

Interviewer: Are those equal?

(Max adds in the following manner 2·2 and answers "yes").

$$\begin{array}{r} 2\cdot 2 \\ \underline{3\cdot 2} \\ 5\cdot 2 \end{array}$$

Interviewer: Suppose x is 10, what would I get on the left — without adding them up?

Max: 20 plus...

Interviewer: No, No, don't add them up.

Max: Oh, you get 50, just 5 times 10.

Even though he is doing poorly in the class, Max has just exhibited more of an understanding of variable than some of the "better" students. However, he does have difficulty in comparing the use of variable in the problem just discussed and problem 2. What does it mean to you when we say solve for x in $x + 4 = 7$?

27.

Max: It means you get the difference between plus 4 and 7. (He gets 3).

Interviewer: Do you view the x in problem 2 as being different than the x in problem 1? Are these problems two different types?

Max: No, it's the same thing, x is still the unknown except in problem 1, I've got 2 times the unknown.

Until this stage of the interview, Max seemed to be on par with Art and Dea. It was with the start of the questions on exponents that his Instrumental-Symbolic mode of thinking became completely dominant. Max had a rule for every problem and would tend to exhibit superficial confidence by sticking with his results in the face of contrary evidence.

28. $-2 \cdot 3^2$

Max: I've got to multiply both sides of this. If I multiply 2 times 3, I have to multiply two times the power. (Max gets $-6^4$).

Interviewer: Could you possibly do that another way?

(Max writes $-6 \cdot 6$)

Interviewer: Are there other possibilities?

Max: I could get this. (Writes $-2 \cdot 9 = -18$)

Interviewer: Now, we've got three different answers here. Which do we go with?

Max: I think this one ($-6^4$), because we had to multiply both sides of the problem by 2.

Obviously, Max is confusing the distributive property with exponential properties. Max also has difficulty with exponential properties.

29. $2^2 \cdot 3^2$

Max: I'll get 4 for the power and 6 for the base. You add the exponents. (Writes $6^4$).

Interviewer: Try problem number 5. ($2^2 \cdot 2^3$) (Max writes $4^5$).

Interviewer: Now how did you get that?

Max: I'm multiplying these numbers (2 and 2) and adding the exponents. If I multiplied them, it wouldn't come out right.

Interviewer: I don't understand what do you mean?

Max: Multiply the exponent (writes $2 \cdot 3 = 6$). That's not right.

Interviewer: Suppose you didn't know any of these properties. Suppose you didn't know that property of adding exponents that you are using. Could you do it another way?

Max: I'd have to go 2 times 2 and 2 times 2 times 2.

Interviewer: And what would that be equal to? (Max multiplies and gets 32).

Interviewer: I'll tell you, $4^5$ is equal to 1024. That's not equal to 32. Which answer would you go with?

Max: I'd put my faith in this one. (points to $4^5$). I've been taught to add the exponents.

Max has shown that he does not have relational understanding of positive exponents.

Max continues throughout the interview to "solve" all problems dealing with exponents by the "application" of some rules. He usually is somewhat right, but misses the relevant features of the problem that should enable him to distinguish the appropriate property. Without belaboring the point by giving the transcriptions verbatum, it should be indicated that Max made the following "errors" with respect to positive exponents:

$4^4/2^2 = 2^2$; $4^4/4^2 = 1^2$; $14^2/7^2 = 2$; $(x^2)^3 = x^5$; $m^n/m^2 = n/2$; and $x^a y^b = xy^{ab}$. He did get $(2^2)^3 = 2^6$, but changed to $2^5$ when challenged.

Max does have some instrumental understanding of zero and negative exponents. He attempts to think of negative exponents in the Instrumental-Iconic mode, but the imagery used is not really appropriate.

30. $(2 + 3)^0$

Max: One

Interviewer: Why is that?

Max: Zero exponent.

Interviewer: Always?

Max: That's what I understand.

Interviewer: Why do we have zero exponents? When will we ever use those?

Max: I believe it's an absolute. I'm not very sure. But you've got to have zero exponents in order to have negative exponents.

Interviewer: Why do we have negative exponents?

Max: If you take six apples away from four apples, you're in the hole.

Max is able to write $5^{-2} = 1/25$ and $a^{-b} = 1/a^b$. However, Max again resorts to false generalizations in the application of properties. He makes the following errors: $(2^{-1})^{-2} = 1/2^3$; $2^{-1} \cdot 2^3 = 4^1$; and $2^3/4^{-1} = -2$.

In summary, Max appears to use some imagery while using an instrumental level of thinking. He does have some relational understanding, but his tendency to place total authority in algorithms robs him of the benefits of this understanding.

It was stated previously that Max's interview was typical of the students, that did poorly on the exponential unit examination. It should be noted that this is true with respect to the tendency to rely on rules to solve problems. Max was atypical in the sense that he had more of an understanding (both relational and instrumental) of variable than the others. Two students responded "$a^3$" for the answer to a + a + a. When asked what "a represented", they both answered "1". The other person was correct on a + a + a = 3a, but on the next problem 2a + 3a gave $5a^2$ as the answer. Apparently, the concept of variable, as it relates to exponents, is a troublesome one for some students.

As reported earlier in this section, the interviews for the other three students in the "lower" category of the Math 111 students will not be extensively detailed. However, the following excerpts are enlightening as to the type of "understanding" that some students have with respect to the combination of variables and exponents. Len, and Nan are the students in the following excerpts:

31. a + a + a

Len: a to the third. (writes $a^3$).

Interviewer: What procedure did you use to get that?

Len: a + a + a...exponents, I'm just adding the a's.

Interviewer: Is there any way to check this out? Suppose you were not sure whether it was a to the third or something else? (long pause) Is "a" a number, or a letter...what does it represent?

Len: One,...right? (laughs) (long pause)

Interviewer: Let me ask you this; suppose a were 2, what would you have?

Len: If a stood for 2, then 6.

Interviewer: How did you get that?

Len: Well a, a, a. If a were 1, you'd get 3, so if it's 2, you have two, four, six.

It is interesting to note that Len would have the correct answer on a paper and pencil test. His confusing of the notation as he passed back and forth from explicit numbers to variables was such that the errors "self-corrected".

32. a + a + a

(Nan has just correctly responded "3a")

Interviewer: How do you view the "a" there? What is it?

Nan: Everybody hates those letters. How can you add a "z" to an "a"?

Interviewer: What does a represent?

Nan: I guess a could represent 1, couldn't it?

Interviewer: Does that check out over here? (points to a + a + a = 3a)

Nan: 1 plus 1 plus 1 = 3. There would be no a.

Interviewer: What does the 3a represent? Is that 3 + a, or 3 times a or...?

Nan: Just generally means there are three a's... Would be 3 + a.

(After several conversational exchanges, the interviewer has Nan substitute in the number 2 in a + a + a = 3a and get 6 = 6.)

Nan: I never thought of it that way. Nobody ever taught it that way.

If Nan's memory is true, apparently she has never been out of the Instrumental-Symbolic mode since starting algebra.

The total lack of confidence in her own understanding, as well as her view of algebra is expressed by her statements while working on problem III. 10. $2^2 \cdot 3^2$. Nan indicated the numbers were "rigged", and it was luck that $2^2 \cdot 3^2 = (2 \cdot 3)^2$. The interviewer then gave her another problem $5^2 \cdot 3^2$ and together they checked out the fact that $(5 \cdot 3)^2$ and $5^2 \cdot 3^2$ were equal by using $25 \cdot 9$ and $15^2$ to get 225. Nan had said that these two would not be the same, and indicated her surprise by, "They are not supposed to do that are they?...There is supposed to be only one given way to get an answer."

<u>Math 121 High Examinations Scores</u>

Interview excerpts will be presented for only two students in this section. Judy's excerpts are, in general, representative of her relational understanding, while Alan's show his inclination toward an instrumental mode of understanding. These excerpts express quite well the range of student responses of those students that did well on the exponential unit examination.

Judy has a good overall grade point average for all courses. She completed one year of high school algebra, as well as Math 111. She received an A grade in both of these courses, and eventually received an A in Math 121.

Judy will not be pushed into "canceling" out the 2 in problem 4. $\frac{2 \cdot 5 + 6}{8}$, and sticks with her result of 2.

33.

Interviewer: I thought when you had times, you could cancel.

Judy: It has to be all times.

Interviewer: I wonder why that is. What have I really done wrong if I cancel the 2?

Judy: I don't know.

(Judy is "pushed" by the interviewer and finally exhibits her appreciation for the structure of arithmetic.)

Judy: (writes $\frac{2(5 + 3)}{8}$)...and now divide out the 2.

That Judy can consistently use the distributive property is shown in problem 6. 2a + 3a, when she once more justifies her answer by writing (2 + 3)a = 5a.

Judy's responses in the next excerpt show that she does use the Relational-Iconic mode of thinking of "variable".

34. What does it mean to you when we say 2x + 3x = 5x?

Judy: I'm not sure what you want.

Interviewer: How do you view the x?

Judy: As a variable. There is no value for x yet.

Interviewer: Well, let's suppose x is 15, could you tell me the value of the left side without substituting 15 on the left side?

Judy: 2 times 15 plus 3 times 15...

Interviewer: No, that's not what I want you to do.

Judy: x is 15? (Interviewer agrees). Then x is 15 on the right and you get 75.

Judy then does problem 2. What does it mean to you when we say solve for x in x + 4 = 7?, by giving the result of 3. When asked the difference in problem 1 and problem 2, she states, "In problem two x has only one value, but in problem one it can have many values".

Judy comes as near as any student to expressing relational understanding of the "students and professors" problem.

35.

Judy: (long pause, while looking at the problem) b. (S = 6P).

Interviewer: What makes you choose b?

Judy: Because there are six times as many students as professors. So for every professor there are six students. (writes $S = 6 \cdot 6$, $36 = 36$)

Judy's lack of confidence betrays her wavering into the Instrumental-Symbolic mode in this next exchange.

Interviewer: But it seems if there are six times as many students, you would multiply the number of students by 6.

Judy: (long pause) Well, one student equals six professors if you want to equal them out, right? (laughs) I'm terrible at story problems.

Interviewer: So, which are you going to go with? Are you going to stick with your answer or did I change your mind?

Judy: I'm going to switch to a. (P = 6S)

Judy uses a Relational-Iconic mode when dealing with exponential properties.

36. $2^2 \cdot 3^2$

(Judy writes $4 \cdot 9 = 36$)

Judy: I could have gone $2 \cdot 3$ is 6.

Interviewer: O.k. Do that.

Judy: $2 \cdot 3$ to the second power is also 36.

Interviewer: Now you've got me confused. On problem 3.

$(-2\cdot3^2)$, you told me you had to do the exponentaiation first.

Judy: They are not to the same power. (points to problem 3.)

Interviewer: How do you know that works?

Judy: That's what we learned in Math 121.

Interviewer: Suppose I doubted that $(2\cdot3)^2$ is the same as $2^2\cdot3^2$. Could you show me it has to be?

Judy: (writes $(2\cdot3)\,(2\cdot3)$)

Interviewer: Is that the same as the other?

Judy then writes $2\cdot3\cdot2\cdot3 = 2\cdot2\cdot3\cdot3 = 2^2\cdot3^2$.

On problem 9. $4^4/2^2$ Judy is one of the few students who can use an exponential property. However, she does not do this until pushed by the interviewer. Judy has written $4^4/2^2 = 256/4 = 64$ when the next excerpt begins.

37. $4^4/2^2$

Interviewer: Is there any way I could shorten that down? Divide the 2 into the 4 or subtract exponents or whatever?

Judy: The bases have to be the same.

Interviewer: Well, could you make the bases the same?

Judy: Probably, I'd have to figure out what the exponent would have to be up here if I make 4 into 2 times 2. (writes $2^8/2^2 = 2^6$)

Interviewer: Could you have made the bases 4?

Judy: Yeah. I could have done that too by putting a 4 in the denominator.

However, Judy has not generalized all the exponential properties. She has no difficulty in explaining relationally all the problems that pertain to $a^m\cdot a^n = a^{m+n}$ and $a^m/a^n = a^{m-n}$, but does have difficulty looking at the "reverse" of

$(ab)^n = a^n b^n$ and $(a/b)^n = a^n/b^n$.

38. $(2 \cdot 3)^2$

Judy: This would be 6 to the second power equals 36.

Interviewer: Suppose I don't work this the way you do, but go $2^2 \cdot 3^2$. Is that alright?

Judy: ...Um huh (yes)...that's 4 times 9 which is 36 also.

Interviewer: Was I just lucky there because of the 2 and 3's? Given a problem of multiplication to a power, could I always do that?

Judy: As long as it's multiplication.

Interviewer: Is there a property dealing with this?

Judy: Not that I'm aware of.

This could be taken as just momentary forgetfulness on Judy's part, but the next excerpt adds evidence of lack of knowledge of the equivalent division property.

39. $14^2/7^2$

Judy: (long pause, starts to cross out the 7 and 14 but changes her mind.)

Interviewer: You were going to divide the 7 into the 14, but decided it didn't work?

Judy: I don't know why I did that? (Judy then gets $14^2/7^2 = 196/49 = 4$)

Interviewer: Could we have done that another way? It may be accidental, but...

Judy: 2 to the second would have been 4.

Interviewer: Is that just luck?

Judy: I don't know what would have happened to the 2 down here. (writes $7^2 = 14$, but corrects it).

Interviewer: Is there any way we can check that out? If we had 14 to the 7th power, we would be all day.

Judy: Not that I know of.

With the help of the interviewer, Judy writes $14^2/7^2$ = $\frac{14 \cdot 14}{7 \cdot 7}$ = 2·2 = 4, but still does not recognize the generalization.

Judy has no difficulty with the exponential problems that contain either variable bases or exponents. When pushed, she can fall back on numerical imagery to justify her conclusions.

40. $x^a \cdot x^b$

Judy: That's $x^{a+b}$

Interviewer: Suppose someone doubted that and thought it was $x^{a+b}$. Is there any way you could check this... to help decide between...

Judy: Well, if you took 1 for x.

Interviewer: What about a and b?

Judy: Use a = 2 and b = 3.

(Judy then gets $1^2 \cdot 1^3 = 1^5 = 1$, and concludes that 1 was a poor choice for x, when the interviewer indicated that $1^6$ also equals 1.

Judy handled the problems well which involved zero and negative exponents. It was apparent that her thinking was primarily instrumental due to the fact that she could offer no reason why "any number to the zero power is one". However, she did recognize that negative exponents were used for "making fractions". Judy did exhibit a relational understanding of rational exponent on problem 27.

41. $(x^6)^{\frac{1}{2}}$

Judy: $x^3$

Interviewer: How did you get that?

Judy: ½ of 6 is 3. You times the outer exponent by the inner exponent.

Interviewer: Why do we have fractional exponents? (pause) Why do we have one-half powers?

Judy: We make square roots out of them. (Judy writes $\sqrt{x^6} = x^3$).

Interviewer: How do we know the square root is $x^3$? (Judy writes $x^3 \cdot x^3 = x^6$)

Judy was the only student to show at least some instrumental understanding of the problem $\sqrt{14}/\sqrt[3]{7}$.

42. $\sqrt{14}/\sqrt[3]{7}$

Judy: I can't do that unless I change it to fractions.

Interviewer: Do that.

Judy: (writes $14^{1/2}/7^{1/3}$)

Judy is stymied, but with the interviewers help is able to finally simplify a similar problem, $\sqrt{2}/\sqrt[3]{2}$, by switching to fractional exponents: $2^{1/2}/2^{1/3} = 2^{1/2-1/3} = 2^{3/6-2/6} = 2^{1/6} = \sqrt[6]{2}$. However, she still could not complete the initial problem. When she tries problem 27. $a^{1/m} \cdot a^{1/n}$, she easily added the exponents as fractions, but after the interviewer had her consider the problem as $\sqrt[m]{a} \cdot \sqrt[n]{a}$, she could go no further and did not appear to see the equivalency between the problems.

Judy had little trouble with $(x + 3)^2$ and $(\sqrt{2} + \sqrt{3})^2$, but needed encouragement on $(x^{3n} + 2)^2$. She initially wrote $x^{6n^2} + 4$, but immediately scribbled this out and then completed the problem as the product two binomials — $(x^{3n} + 2)(x^{3n} + 2)$. It was only with the interviewers suggestion that she saw the problem as the square of a binomial.

In general, Judy appears to operate at either a Relational-Iconic or Relational-Symbolic level, and only occasionally dropping into the Instrumental-Symbolic mode of thinking.

Alan had no high school algebra, but did receive an A in a beginning algebra course at the college level. He eventually received an A- in the Math 121 class.

Alan's interview will not be detailed to the extent of Judy's. However, there are several problems which point out their different levels of understanding. Some of these will be presented in order that the reader may obtain a more precise picture of the thought processes of these more "capable" Math 121 students.

That Alan is not as proficient (as Judy) at the instrumental level is shown in the following excerpt.

42. 15 - 3 - (2 + 1)

Alan: O.k. Again, I would just work from left to right, just going from 15 - 3, you'd get 12. Then I'd do what's in the parentheses first, and then just subtract that (answers 9).

Interviewer: Could you have removed the parentheses first before you did all the addition and so on?

Alan: Well...

Interviewer: There's four numbers there, could you somehow write the four numbers without the parentheses and still come up with a 9?

Alan: You couldn't take the parentheses away and do it.

Interviewer: Well, what if I did take the parentheses away? Would that change anything?

Alan: It would change the answer.

Interviewer: Well, but I want it equivalent. If I drop the parentheses, what about the signs in here?

Alan: You'd have to change the plus to a minus.

Interviewer: O.k. Would you change the 2 to a minus?

Alan: No.

Alan does have a well developed sense of numerical imagery which helps him make decisions.

43. a + a + a

Alan: Oh...(long pause) Here again, just add straight across, three a's = 3a.

Interviewer: What if I thought that were $a^3$? (pause) Well, what did you write down underneath there?

Alan: Just checking something out.

Interviewer: Oh, good, now that's what I want to know. Now what did you do on that?

Alan: What I did is try to figure out exactly what 3a would be. I put a value on a of 5, which I got 5 + 5 + 5 = 15. Then 3 times a would be 15, since 3 times 5 = 15.

Alan showed that he was extremely capable when operating in the Relational-Iconic mode. The prime example is the "students and professors" problem.

44.

Alan: This b is correct and the a is incorrect.

Interviewer: O.k. Why did you pick the second one? (S = 6P)

Alan: Well, because there's 6 times as many students as professors, so...

Interviewer: So, let me ask you on that, then, how come you multiplied the professors by 6 if the students...

Alan: Because there are less professors than students. There's 6 times less professors than students.

Interviewer: Suppose that I thought it was a, could you justify that it was really b? Other than what you're saying. Could you show me I'm wrong if I chose a?

Alan: Well, just by putting in some type of number.

Interviewer: O.k., do that.

Alan: Six times as many students, so I'd give a value of 1 to professors and 6 to students. $1 = 6 \cdot 6$, which is not right, 1 does not equal 36.

Interviewer: Then yours would be what?

Alan: Mine would be $6 = 6 \cdot 1$, which is $6 = 6$.

That Alan has not generalized the exponential properties, and seems to have difficulty distinguishing among them is indicated by the following excerpt.

45. $2^2 \cdot 2^3$

Alan: O.k., here you can't multiply the two bases because your exponents are different, so I'd go back to my simple form. (writes $4 \cdot 8 = 32$)

Interviewer: Now could we have manipulated with the exponents there somehow? Is there a property involved?

Alan: No, because they're different exponents.

Interviewer: O.k., but you've got the same base. Does that matter?

Alan: No. To be able to combine the two bases, you've got to have the same exponents.

That Alan has difficulty with false generalizations is shown in his solutions to the following two problems.

46. $(2 + 3)^2$

Alan: Um...here I would add what's in the middle, what's in the brackets, then square it, $5^2 = 25$.

Interviewer: Could I square first?

Alan: Well, you could have, $2^2 = 4$ and $3^2 = 9$, no, you couldn't.

Interviewer: Could you have told me that before you got 13?

Alan: Not without working it out, no.

Alan is in fact prophetic as indicated by his answer to problem 25. $x^2 + x^3$. Alan listed the answer as $x^5$.

Alan's thinking on zero, negative, and rational exponents paralleled Judy's in the sense that they both used the Instrumental-Symbolic mode. Judy was more efficient at this than Alan.

Alan obtained the correct answers to $(x + 3)^2$ and $(\sqrt{2} + \sqrt{3})^2$ by writing each factor and using the "FOIL" method. However, the fact that Alan has not really generalized the concept of exponent, nor the properties, is exemplified by his answer to $(x^{3n} + 2)^2$. Alan again uses the FOIL method, but gave an answer of $x^{2^{9n^2}} + 4x^{3n} + 4$. Apparently, the concept of exponent combined with variable stretched his exponent schema too far.

Math 121 Low Examination Scores

Todd's interview excerpts are very much representative of the students who scored poorly on the exponential unit examination. In general, they are very rule oriented without any relational understanding or imagery to fall back on in case the rule is not an exact fit.

Todd eventually received a D grade for the class. He did not take an algebra course in high school, but did take the Math 111 class and obtained a C for the final grade.

Todd seems to have parallel systems of thinking. When dealing with numbers from arithmetic, he does reasonably well, but his tendency to use the Instrumental-Symbolic mode of thinking whenever variables appear in a problem overcome any remembrance or understanding of arithmetic procedures. The following excerpts are used to illustrate this.

47. $2 \cdot 3 + 4 - 12$

Todd: That would be 2 times 3 is 6 plus 4 is 10, minus 12 is negative 2.

Interviewer: Could we have added the 3 and the 4 first?

Todd: No, that's the way I learned it.

Todd is not really off-base yet, but in the next excerpt he applies a different system for a similar problem.

48. $3 + 4(x + 5)$

Todd: Uh...3 plus 4 would be 7, umm...would give you 7x + 35.

Interviewer: Would it have been possible to multiply by the 4 first?

Todd: Yes, it would be 4x + 5...no, you couldn't do that, it would goof you up.

Interviewer: How about 4x + 20?

Todd: Oh yeah, I see. That would be 4x + 20 plus 3. That would be 7x + 23. No, you couldn't do that. There are no parentheses around it...I'd have to go with the 7x + 35.

49. $\frac{2 \cdot 5 + 6}{8}$

Todd: O.k. Um...I see now that I've worked problem 3. $\left(\frac{2 \cdot 4 \cdot 6}{8}\right)$ that I can cancel. (Todd "cancels" by 2 and gets 11/4 for the answer.)

Todd then, on his own, decides to perform the operations in the numerator and obtain 16/8 = 2.

Interviewer: Now obviously both of these, 11/4 and 2 aren't correct. In problem 3. $\left(\frac{2 \cdot 4 \cdot 6}{8}\right)$ you did it two ways and got 6 both ways. What went wrong here?

Todd: I'm not sure. (He then decides that 2 is the correct answer.)

50. $\frac{a + b}{a}$

Todd: That would just be $\frac{a + b}{a}$. This is an add problem. You can't cancel unless it's multiplication.

Interviewer: What about problem 4. $(\frac{2 \cdot 5 + 6}{8})$? Does that adding matter, or does the fact that I've got $2 \cdot 5$ enable me to cancel?

Todd: Yes. The $2 \cdot 5$ lets me cancel.

Todd does illustrate some relational thinking in a continuation of the above excerpt.

Interviewer: What if I really thought I could cancel and get $1 + b$? Is there any way I could check my answer?

Todd: Times it by a.

Interviewer: Would you do that for me?

Todd: (writes $a(1 + b) = a + ab$). It $(1 + b)$ doesn't work.

Todd has only an Instrumental-Symbolic notion of variable. He does state that, "x is just replacing a number" in problem II. 1. $2x + 3x = 5x$ (meaning?). However, he is not able to follow this up until the interviewer requests that he substitute 10 for x. When asked to compare the "tautology", $2x + 3x = 5x$ in problem 1. and the "constraint equation", $x + 4 = 7$, Todd dwells on the 2 in 2x, and finally concludes the two problems would be essentially the same if the second were $2x + 4 = 7$.

Todd does use numerical imagery in the "student and professor" problem. He uses 10 for the nubmer of professors to get 60 for the number of students. He chose b. (S = 6P) and sticks with it. However, after this, the interviewer can not get Todd to use numerical imagery for justification. Todd always uses a rule instead.

Todd is aware of the exponential properties, but only in an instrumental way. In the following excerpt, the interviewer tries several times to get Todd to use the meaning of exponent along with arithmetic operations. Todd keeps using as his authority the property $(a \cdot b)^n = a^n b^n$.

51. $2^2 \cdot 3^2$

Todd: That would be $4 \cdot 9$ is 37. (He corrects to 36).

Interviewer: Maybe it's an accident, but it looks as if I multiplied the 2 and the 3 to get 6 and squared that, I would get the 36 answer. Could I multiply first and then square?

Todd: (pause) Yes, they are both the same. So as far as that goes, you could multiply and square.

Interviewer: Do you mean if the powers are the same, I can...?

Todd: Yes. It's like if you had an x and a y. (writes $x^2y^2$) uh...(long pause), no...uh...

Interviewer: Why don't you do this. Write $(2 \cdot 3)^2$. (Todd writes this.) Is this the same as $2^2 \cdot 3^2$?

Todd: Yes, you'd square the 2 and then square the 3.

Interviewer: Yes, but that's what I'm wondering, can I really do that? Could you show me other than the fact that both are 36? Can you take the $(2 \cdot 3)^2$ and change it into $2^2 \cdot 3^2$?

Todd: 2 times 3 is 6, squared is 36.

The interviewer tries again, but Todd keeps using the property in order to verify the particular case.

Todd makes the following false generalizations: $4^4/2^2 = 2^{4-2} = 4$; $14^2/7^2 = 2^0 = 1$ (does not recognize that 1 is not a reasonable answer); $(xy)^3 = x^3y^3 = xy^6$; $\sqrt{6} - \sqrt{3} = \sqrt{3}$.

Todd does correctly identify $(2 + 3)^0$ by noting "anything to the 0 power is 1", but then he generalizes that $(2 + 3)^0 = 2^0 + 3^0$. He doesn't see the difference until questioned by the interviewer. He then recognized that the addition "makes it wrong".

Todd, however, has no relational understanding of negative exponents.

52. $5^{-2}$

Todd: That's an exponent?

Interviewer: Yes, that's a negative 2 power.

Todd: I'm lost.

The interviewer suggests "possibilities" of $-5^2$ or $1/5^2$, and Todd immediately remembers the definition and does reasonably well on the other problems that have negative exponents by using this definition. He does not use the exponential properties for the negative exponents.

Todd does have some instrumental understanding of rational exponents.

53. $(x^6)^{½}$

Todd: That would be...(very long pause) 6 times ½ is $x^3$.

Interviewer: Why do we need fractional exponents?

Todd: I couldn't tell you.

Interviewer: What does it mean to have a one-half power? Does that have any meaning at all?

Todd: It's a root — a square root.

Todd's instrumental understanding of "roots" along with his misuse of exponential properties cause him to retract his correct applications of the rules.

Interviewer: Is $x^3$ really the square root of $x^6$?

Todd: (pause) No, $x^3$, that would be the square root of 9.

Interviewer: What's the square root of 25?

Todd: 5.

Interviewer: How do you know?

Todd: 5 times 5.

Interviewer: Now, is $x^3$ the square root of $x^6$?

Todd: No, that's the square root of $x^9$.

Interviewer: Check it out for me. (Todd multiplies $x^3 \cdot x^3$ and gets $x^6$.)

Interviewer: So, is $x^3$ the square root of $x^6$?

Todd: No.

On the problems dealing with the form $(a + b)^2$, Todd gave $x^2 + 9$ as the answer to $(x + 3)^2$. He was reminded of problem 12. $(2 + 3)^2$ where the distributing of the exponent did not work out. He then changes his mind. He tries to use the FOIL method, but still gets $x^2 + 9$. On $(\sqrt{2} + \sqrt{3})^2$. he reverts again and gets $2 + 3 = 5$. However, on the problem $(x^{3n} + 2)^2$, he did exhibit an instrumental understanding of the property $(a^m)^n$ by getting $x^{6n} + 4$.

Other students in the same category as Todd (test scores) used very nearly the same thinking mode of viewing the problem sets — Instrumental-Symbolic. They differed on the specific problems missed, but did share the characteristic of generalizing the properties of exponent to situations, where in fact they did not apply. It appeared that anytime the problem situation became a little "fuzzy", these students lost confidence in their ability and consequently placed their trust in the most likely algorithm that they remembered.

## Observations Across Students

The problem sets were subdivided into eight categories (pp. 150-154) for the purpose of tracing the students' thought processes through the entire interview. The categories were designed to examine the students' understanding, both instrumental and relational (or lack of either) with respect to prerequisite arithmetic knowledge, variable, equality, the various properties of exponent, and additionally "false generalizations" of any of these. Another interest was the type and quantity of imagery used. The format followed here will be to list each category along with pertinent references from the interviews to lend support as to either the existence or non-existence of understanding in the designated category.

### Category 1: Prerequisite Knowledge

This category was intended to investigate the students' knowledge of arithmetic operations, arithmetic properties, and simple operational type problems involving variables. The following problems are in this category: (from problem set I)

1. $2 \cdot 3 + 4 - 12$

2. $15 - 3 - (2 + 1)$

3. $\frac{2 \cdot 4 \cdot 6}{8}$

4. $\frac{2 \cdot 5 + 6}{8}$

8. $\frac{4xyz}{2x}$

9. $\frac{a + b}{a}$

10. $3 + 4(x + 5)$

All students interviewed obtained the correct result for problem 1. The important feature of this problem is the order in which the operations are performed. Due to the fact that it is a "convention" that multiplication is performed before addition, those students who said, "It's a rule", were considered to have attained the appropriate prerequisite knowledge. It is only when this problem is viewed in conjunction with problem 10. that the individual student's consistency can be observed. All students interviewed correctly answered problem 1., but one student from the low scoring Math 111, and one from the low scoring Math 121 obtained an answer of $7x + 35$ on problem 10.

The remaining problems 3, 4, 8, and 9 were used to investigate the students' understanding of the "reducing" process involving fractions. As indicated earlier, the categories are overlaping in some respects. Problems 4 and 9 were for the purpose of determining the thinking of students when they generalized an algorithm inappropriately. All of the Math 111 students "canceled" out the a in $\frac{a + b}{a}$, but none initially "canceled" the 2 in $\frac{2 \cdot 5 + 6}{8}$. The fact that many of the students were in an instrumental mode was shown by the fact that the interviewer "talked" them into dividing by 2 in the second problem.

Interestingly, the situation was reversed with the Math 121 students. Only one student incorrectly divided by a, and he eventually reversed himself by the use of numerical imagery (Todd). However, five of the seven students incorrectly reduced by 2 in the second problem. Most, however, did obtain the correct answer by performing the indicated operation.

Perhaps the above indicates that the beginning algebra students are more attuned to arithmetic procedures and just "grind out" their results, whereas the Math 121 students are more concerned with finding an algorithm from their more ample supply. Apparently the Math 111 students saw the more obvious "a" in the numerator and denominator, while the Math 121 students seemed to notice the multiplication symbol and divided out the common "factors".

None of the students obtained an incorrect answer for the other two problems. It is important to note that many did not use a reducing procedure in $\frac{2 \cdot 4 \cdot 6}{8}$ until prodded by the interviewer. One could conjecture that many were led "down the garden path" to divide inappropriately on the other two problems. One of the characteristics of relational understanding is faith and confidence in an answer. Only Judy exhibited this understanding by correctly factoring $\frac{2(5 + 3)}{8}$, then dividing (excerpt 33).

It would appear that both the Math 121 and Math 111 groups in general have a rather limited instrumental type understanding of the fraction and rational expression problems.

<u>Category 2: Variable and Equality</u>

Category 2 problems were designed to investigate the students' understanding of the different uses of variables and the equals sign. The following problems were used for this purpose.

I. 5. $a + a + a$

6. $2a + 3a$

7. $2a + 3a - b$

II. 1. What does it mean to you when we say $2x + 3x = 5x$?

2. What does it mean to you when we say solve for $x$ in $x + 4 = 7$?

3. Which of the indicated equations would be the correct answer for the following:

Write an equation using the variables S and P to represent the following statement: "There are six times as many students as professors at this university". Use S for the number of students and P for the number of professors. (Rosnick, Clement, 1983, p. 4)

a. $P = 6S$

b. $S = 6P$

4. Is the following true or false? $4x = 46$ given that $x = 6$ (Matz, 1979, p. 134)

5. Without solving, could you compare the solutions for W and N in the following: $7W + 22 = 109$; $7N + 22 = 109$ (Wagner 1981, p. 109)

The three problems from problem set I. could have been placed in the category of prerequisite knowledge, but were placed in this category because of the interviewer's emphasis on numerical imagery in the questioning process. Three Math 111 students from the "lower" level did answer $a^3$ on the problem a + a + a, and two followed up with $5a^2$ on problem 6. None of the Math 121 students did this. The Math 121 students were also able to "justify" their answers by choosing a particular value for a, to a much greater extent than the Math 111 students. Most of the Math 111 students eventually did use numerical imagery, but most, only after being helped by the interviewer. However, only one student, Judy (Math 121), used the distributive property to justify her answer to 2a + 3a.

The problem taken from Matz's (1979) research; if x = 6 does 4x = 46?, and the problem originated by Wagner (1981); Compare W and N given that 7W + 22 = 109 and 7N + 22 = 109, were not missed by any student. One student did note, "I used to think W and N were equal", on Wagner's problem. Perhaps Matz and Wagner were working with less symbolically experienced students.

Problem II. 3., the "students and professors" problem was primarily an exercise in viewing the surface feature "six times as many students" for the Math 111 students in both levels. Just as reported by Rosnick and Clements (1983, p. 4), they could not see their answers were incorrect even after substituting a number, which created an inequality. Four of the Math 121 students also viewed the problem in this fashion.

They seemed somehow to view one professor and a group of six students as being "equal". Perhaps in this situation mental imagery acted as a detriment to understanding. Three of the Math 121 students operated in the Relational-Iconic mode on this problem and easily showed their understanding of the particular problem as well as the concept of equality (excerpts 35 and 44).

One ominous implication of this category was the number of students that did not see a difference in the use of the variable x in the tautology $2x + 3x = 5x$, problem II. 1. and the conditional equation $x + 4 = 7$ in problem II. 2. Several students at both the Math 111 and Math 121 level viewed both problems as a "solve for x" type, or "add like terms type", and only with pushing by the interviewer were able to substitute numbers in $2x + 3x = 5x$ and then recognize that it held for all numbers. One A student in Math 121 (no excerpt), after being asked to substitute 10 for x and getting an equality, noted "I never thought of it like that before."

Category 3: Definition of Exponent

The purpose of this category was to investigate the students' understanding of positive, negative, and zero integral exponents. Additionally, a positive rational exponent was used in an attempt to determine not only the understanding of this form of exponent, but the understanding of the relationship between exponents and radicals as well. The problems were all from problem set III, and are as follows:

1. $2^3$

2. $(-2)^3$

11. $(2 + 3)^0$

14. $5^{-2}$

24. $a^{-b}$

27. $(x^6)^{1/2}$

All Math 111 students evaluated $2^3$ correctly. One lower level student gave $1/2^3$ for the result to $(-2)^3$. Apparently her negative exponent schema was activated by the appearance of both a negative number and an exponent. One other student needed three attempts before arriving at the correct result. Most of the students wrote down the three factors of negative two and used a step-by-step procedure. However, all students except one were able to give the signs of $(-2)^{50}$ and $(-2)^{51}$ immediately upon being asked to do so by the interviewer. Again, it looks as if the Math 111 group will use arithmetic procedures in preference to other methods.

All of the Math 121 students handled the two problems dealing with positive integral powers with ease, and also were able to give the signs of $(-2)^{50}$ and $(-2)^{51}$. However, approximately half did use a step-by-step procedure to evaluate $(-2)^3$.

For the problem $(2 + 3)^0$, all students except one gave the correct result. Eventually the one student also determined the correct result. Most of the Math 111 students appeared fearful of both zero and negative exponents, and no student showed confidence in the answer given (for example Max, excerpt 30). A small minority did change their answer to two ($2^0 + 3^0 = 1 + 1$) when questioned by the interviewer. The understanding was strictly at the Instrumental-Symbolic level for this problem.

The Math 121 responses paralleled the Math 111 group. One student answered 2, and one other student wavered between 1 and 2 (excerpt 51). No student in either group presented a logical argument when the interviewer asked, "Why do we have 0 exponents?". One student did respond, "one times anything is one...", (Dea, excerpt 20), but could not follow up on this. Perhaps one Math 121 student summed up the general feeling toward any exponent other than positive integers. When asked, "Why do we have negative exponents," he responded, "Just to screw us".

On the problem $5^{-2}$ four of the Math 111 students designated the result as "-25". There were two students from each level that missed this problem. Two of the Math 121 students in the lower level missed this problem. All students were able to recall the correct result when aided by the interviewer. That most students learned from this problem is evidenced by the fact that all Math 121 students, and all but one Math 111 student, gave the correct response to $a^{-b}$. However, no student in either group demonstrated any relational understanding when quizzed about "Why do we have negative exponents?" (excerpt 16)

Problem 27., $(x^6)^{\frac{1}{2}}$, gave the Math 111 student a chance to generalize from whole number exponents to rational number exponents due to the fact that these exponents are not covered as a part of the Math 111 course. It is possible that some of these students did cover this topic in high school. Three students answered correctly, but only one actually "extended" the appropriate exponential property without aid from the interviewer. Incorrect responses were, $(x^6)^{-2}$, $x^{6\frac{1}{2}}$, $x^9$, and 3x. Of those that gave $x^3$, one indicated $1/2 \div x^6 = x^3$ and one used the diagram $x \cdot x \cdot x / x \cdot x \cdot x$ (Art, excerpt 12).

The Math 121 students had covered this topic, therefore the interviewer was primarily interested in checking if these students could see the relationships with the $\sqrt{x^6}$. All of the Math 121 students multiplied the exponents to get $x^3$ except one student who answered $1/x^3$. This student was the only one who did not recognize the one-half power as equivalent to the square root. Judy was the prime example of the Relational-Symbolic understanding as she immediately wrote the answer and then verified that it was the same as the square root by multiplying $x^3 \cdot x^3$ to get $x^6$ (excerpt 41).

The interviews indicated that most students in Math 111 had an instrumental understanding of zero, negative integer, and fractional exponents. None of the students exhibited a relational understanding of rational exponents.

Category 4: $a^m \cdot a^n = a^{m+n}$; $a^m/a^n = a^{m-n}$

These problems were designed to investigate if the students had generalized these properties to several variations of the variables. The "generalization" ability was in Krutetskii's (1976) first sense — "subsuming a particular case under a known general concept" (p. 237). The problems in this category are taken from problem set III.

5. $2^2 \cdot 2^3$

13. $4^4/4^2$

17. $2^{-2} \cdot 2^3$

22. $x^a \cdot x^b$

23. $m^n/m^2$

34. $a^{1/n} \cdot a^{1/m}$ (Math 121 students only)

With respect to $a^m \cdot a^n = a^{m+n}$ in the group of Math 111 students, six of the seven students gave the answer 32 for $2^2 \cdot 2^3$. However, all six used the arithmetic version and gave $4 \cdot 8 = 32$. When pressed by the interviewer, four of these students were able to determine it was $2^5$. Not all were confident when they concluded, "You add the exponents". That the students were perhaps wise to use the obvious arithmetic procedure rather than the property, is indicated by the $4^5$ response given by the one person to get the wrong answer (excerpt 29). The fact that none of the Math 111 students had generalized this property to variants other than positive integer exponents is indicated by the results from problems 17 and 22. On the problem $2^{-2} \cdot 2^3$, five people determined the correct procedure, but all worked with the fractional equivalency of $2^{-2}$. Some few, at the prodding of the interviewer, did apply the property, but had no confidence other than the fact the answer was the same. The incorrect responses were 1/4 and -32. On the problem $x^a \cdot x^b$ only one Math 111 student gave the correct response. He was able to use numerical imagery in his justification. The other answers given were $x^{ab}$ and $x^{2ab}$.

For the Math 121 students the interviews showed four students recognized the generalization on the problem $2^2 \cdot 2^3$. However, the fact that they had not generalized the property to other variations is indicated in the results to $2^{-2} \cdot 2^3$. Not a single student added the exponents initially. That these students are more capable when dealing with negative exponents is illustrated by the fact that six of the seven students changed $2^{-2}$ to 1/4 and completed the problem correctly. One student listed $4^{-6}$ as the result. That for many students in algebra working with "letters" is easier than working with "numbers" is manifested by the fact that four of the seven Math 121 students gave an answer of $x^{a+b}$ for $x^a \cdot x^b$. Two students were able to use numerical imagery to aid in the determination of their result (excerpt 40). The only incorrect response from the Math 121 group was $x^{ab}$. One student that eventually received an A in the class obtained an answer of $x^{a+b}$, but stated she obtained the answer by using the rule, "but $x^{a+b}$ doesn't make sense." The Math 121 students were also asked to do the problem $a^{1/m} \cdot a^{1/n}$. Again showing the symbolic generalization used, all students added the exponents. However, the results have implications for the prerequisite category. In addition to the correct answer $a^{(n+m)/mn}$, the result $a^{1/n+m}$ was indicated three times. Most, however, did eventually add correctly when the error was pointed out by the interviewer. None of the students could convert to radical form and show any relational understanding (excerpt 42).

For the division algorithm, success at the

Instrumental-Symbolic level was approximately the same. Three of the four low level Math 111 students answered the problem $4^4/4^2$ correctly using arithmetic manipulation. One student did attempt to use the property, but designated $1^2$ as the answer. One student in the high level Math 111 group did use the property correctly. However, one student also answered $1^2$. Most students were able to "recall" the algorithm with a "reminder" from the interviewer. Only two of the Math 111 students were able to give the correct answer for $m^n/m^2$. The most popular incorrect answer was n/2. (the "m's cancel")

On the problem $4^4/4^2$, all Math 121 students answered correctly with four students (high level) applying the algorithms immediately. All other students used the arithmetic method, but used the algorithm with a slight hint from the interviewer. When challenged on "subtracting the exponents", three of the students were able to write the factors in the numerator and denominator and divide out two of the fours, thus illustrating relational understanding of the property. On the problem with a literal base, $m^n/m^2$, three people used the property correctly. Of the three, one had scored low on the exponential test. Two other students answered "can't be done", and one student (low scorer) gave an answer of $1^{n-2}$.

Again, it appears that the Math 111 students prefer to use the arithmetic method, because both the multiplication and division schemas with respect to exponents are incomplete. The Math 121 students prefer the algorithms, but in fact show very little more relational understanding of these properties than the Math 111 students.

Category 5: $(ab)^n = a^n b^n$; $(a/b)^n = a^n/b^n$

The problems in this category were as follows from problem set III.

4. $2^2 \cdot 3^2$

10. $(2 \cdot 3)^2$

16. $14^2/7^2$

20. $(xy)^3$

28. $\sqrt{4} \cdot \sqrt{9}$ (Math 121 students only)

This category caused much more difficulty than the preceding category as far as relational understanding was concerned.

For the purpose of discussion $(2 \cdot 3)^2$ and $(xy)^3$ will be paired, and $2^2 \cdot 3^2$ and $14^2/7^2$ will be paired.

The Math 111 students took the obvious arithmetic path on $(2 \cdot 3)^2$ and got $6^2 = 36$. It was only when the interviewer wondered if it was an "accident" that $2^2 \cdot 3^2$ also gave 36, that the real interest in the problem began. Only one student immediately quoted a property. However, several students showed a relational understanding of the arithmetic properties of exponent to show that $(2 \cdot 3) \cdot (2 \cdot 3)$ is equal to $(2 \cdot 2) \cdot (3 \cdot 3)$. (Dea, excerpt 19). On the similar problem $(xy)^3$, two high level students and one low level student used the property to immediately get the result. One high level student wrote 3x3y, and one low level answered 3xy. The students were able to view this as three factors of xy much more readily than they could view three factors of $(2 \cdot 3)$. Perhaps the fixation with arithmetic operations made $2 \cdot 3$ too enticing to ignore.

On the "reverse" direction problems $2^2 \cdot 3^2$ and $14^2/7^2$, success was not nearly so good. On $2^2 \cdot 3^2$ all students wrote $4 \cdot 9 = 36$. When pushed as to whether the $2 \cdot 3$ could be done first, the lack of understanding of this property was shown. One student answered that it would by $6^4$ ("You add exponents when multiplying"), however, that conflicted with the known result of 36, thus he did change to $6^2$ without any justification. Another student, Dea (excerpt 18) can only answer that it "might be an accident". Nan (p. 182) thought the numbers were "rigged". The lack of "reversibility" was universal on this property for the Math 111 students. When challenged to use a property, a typical answer (excerpt 29) was $6^4$. That success was less on $14^2/7^2$ for the Math 111 students, was due only to the fact that it was arithmetically more difficult to compute. Four students used long division to complete the problem (after squaring numerator and denominator). One high level and one low level student did use a "property". Both gave an answer of 2 ("Divide 14 by 7, subtract the exponents"). Apparently the students ignored the zero exponent. Art did write $2^2/1^2$, but his lack of confidence, and oral responses indicated he was primarily guessing.

The Math 121 students, to a great extent, paralleled the Math 111 students in their lack of relational understanding of the properties in this category. Judy (excerpt 38) who, in fact, seemed to have the widest range of relational understanding was not aware of a property that dictated that $(2 \cdot 3)^2 = 2^2 \cdot 3^2$. However, in excerpt 36, Judy had concluded that there was a property $2^2 \cdot 3^2 = (2 \cdot 3)^2$. Likewise (excerpt 39) Judy claims ignorance of any "shorcut" on $14^2/7^2$, although she very quickly saw that $2^2$ was the same answer that she had obtained using arithmetic. Some students did try to use a "property". One student wrote $2^2$, one answered $2^0$, and another one decided $2^1$. All students, when aided by the instructor, were able to use the definition of exponent to write $\frac{14 \cdot 14}{7 \cdot 7}$ and consequently concluded that $2^2$ was correct. All of the Math 121 students gave "$2 \cdot 3 = 6$ as their answer for $\sqrt{4} \cdot \sqrt{9}$. All agreed that it was the same as $\sqrt{36}$. However, the numbers were so obvious, it could not be ascertained if the students were using a property or just focusing on the arithmetic equivalency.

In general, the students in both Math 111 and Math 121 exhibited little evidence of acquaintance with these properties from this category. Consequently, what understanding they possessed was either at the arithmetic level or the instrumental level.

<u>Category 6: $(a^m)^n = a^{mn}$</u>

The problems used to investigate the understanding of this exponential property are listed below. All are from problem set III.

8. $(2^2)^3$

19. $(2^{-1})^{-2}$

21. $(x^2)^3$

On $(2^2)^3$ all of the Math 111 students except one did "what is in parentheses first", wrote $4^3$, then multiplied to get 64. When pushed by the interviewer, all of these students lacked confidence as to whether the exponents could be added or multiplied. Only after calculating $2^5$ and $2^6$ or writing $(2\cdot2)\cdot(2\cdot2)\cdot(2\cdot2)$ were they convinced to multiply the exponents. One of the students discussed previously in the interview portion of this document, Nan, insisted that it was "just luck" that 64 was obtained when, at the interviewer's instigation, she used $2^6$. Interestingly, Dea (excerpt 18) had insisted that exponentiation must be done when working with $2^2\cdot3^2$, but insisted that the 2 should not be squared first on $(2^2)^3$. She did use the property and multiplied the exponents to achieve the correct result. On $(x^2)^3$ two low level students and one high level student gave $x^5$ and 6x as the answer respectively. All other students gave $x^6$ as the answer. However, only one (Dea) used the property. All others wrote something on the order of $(x^2)\cdot(x^2)\cdot(x^2)$ to arrive at the result. One student (high level) responded, when asked about the possibility of adding or multiplying exponents, "that's what I hate about algebra, when you multiply, you add." On $(2^{-1})^{-2}$ only two students (low level) applied the algorithm initially. Their lack of confidence in their answers indicated that they were not really sure if in fact there was such a property. Four students eventually arrived at the result by using the definition of exponents and the consequent complex fraction. That the Math 111 students do not see the exponential properties as applying for negative exponents is indicated by Dea (excerpts 21 and 22).

Without hesitation, she gave $x^6$ as the result to $(x^2)^3$, but even with suggestions by the interviewer was never able to successfully accommodate her schema to allow for the negative exponents.

Only one Math 121 student used the arithmetic procedure on $(2^2)^3$, all others immediately multiplied the exponents and were able to use the definition of exponent to verify their answer. They were equally successful with $(x^2)^3$. With the negative exponents $(2^{-1})^{-2}$, two students used the definition and worked the problem in the fraction mode. All others used the property to immediately get their answer, and when pushed, illustrated a relational understanding by going back to the fractional mode to verify that the property really applies.

The Math 111 students generally showed little understanding, either instrumentally or relationally of the property $(a^m)^n = a^{mn}$. Their primary technique of solving the problems was to use the definition of exponent, then count the factors. When this technique failed to apply directly, as in the problem with negative exponents, they had much more difficulty.

The Math 121 student showed a relational understanding of this property. However, it should be pointed out that this problem dealing with negative exponents was the fourth problem pertaining to negative exponents presented during the interview. Perhaps any uneasiness these students felt with respect to negative exponent had dissipated by this time.

Category 7: False Generalizations

The problems in this category were developed to be used in conjunction with problems from other categories. It cannot be said that a student has formed a concept, or generalized a concept or principle if the student applies the principle to negative instances as well as positive instances. For example, if a student multiplies the exponents when given $(x^2)^3$, but also multiplies them when given $x^2 \cdot x^3$, then he is generalizing falsely. Skemp (1979B) has stated "...the possession of a concept can be evidenced in this way, namely by distinguishing between examples and non-examples..." (p. 120). Confrey and Lanier (1980) noted, "...generalization is not a unitary process, but requires differentiation among the relevant variables, the constants and the irrelevant variables" (p. 551). The following problems were designed to investigate "false generalization" (Rachlin, 1981, p. 49). The previous categories did offer opportunities for the students to falsely generalize exponential properties. Many such false generalizations were pointed out in the discussions of these categories. All problems are from problem set III.

3. $-2 \cdot 3^2$

6. $2^2 + 2^3$

7. $3^2 + 2^3$

9. $4^4/2^2$

11. $(2 + 3)^0$

12. $(2 + 3)^2$

15. $(2 + 3)^{-1}$

18. $2^3/4^{-1}$

25. $x^2 + x^3$

26. $x^a y^b$

29. $\sqrt{14} \,/\, \sqrt[3]{7}$ (Math 121 students only)

32. $\sqrt{6} - \sqrt{3}$ (Math 121 students only)

Of the seven Math 111 students, the number of students falsely generalizing on the non-variable problems were as follows: $-2 \cdot 3^2$, one answer of $-6^4$ (excerpt 28); $2^2 + 3^3$, none; $3^2 + 2^3$, none; $4^4/2^2$, three answers of $2^2$; $(2 + 3)^2$, one answer of $4 + 9 = 13$; $(2 + 3)^{-1}$, one answer of $-2 - 3 = -5$; and $2^3/4^{-1}$, none. Several students did get the wrong answer for $2^3/4^{-1}$ due to arithmetic errors — one student designated $4^{-1} = -4$. It is worth noting that the students took the obvious arithmetic procedure to arrive at the correct solution, but all showed a lack of confidence to the question, "Is there a property that would apply here?" (excerpts 10 and 20). Several students did then generalize falsely, but immediately retracted when the result conflicted with their previous answer. Three people (two low level and one high level) gave $x^5$ as the result for $x^2 + x^3$. This is in contrast to no false generalizations for $2^2 + 2^3$. Several students did show relational understanding by using such procedures as $x^2 + x^3 = x \cdot x + x \cdot x \cdot x$ to indicate that $x^5$ was incorrect. Two students used numerical imagery in their justification. For the problem $x^a y^b$, four Math 111 students (three high level, one low) falsely generalized by indicating the following: $xy^{ab}$, $xy^{a+b}$, $(xy)^{a+b}$.

In the Math 121 group, none of the students gave an incorrect response to $-2 \cdot 3^2$, $2^2 + 3^2$ or $3^2 + 2^3$. Only one student was somewhat tempted to apply a property when it was suggested by the interviewer. The problem $4^4/2^2$ did cause three people to give an answer of $2^2$ (excerpt 51). All three did change their mind after using the arithmetic operations. Two people showed relational understanding by converting to either a common base of 2 or 4, then using the appropriate algorithm (excerpt 35). None of the students answered $(2 + 3)^0$ or $(2 + 3)^2$ incorrectly, although two students did write $2^0 + 3^0$ and then change their mind. Two Math 121 students (one high level, one low level) had difficulty with $(2 + 3)^{-1}$, both said it was -5. When asked about 1/5 as a result, both decided that either answer was correct "depending on how you look at it". All students in the Math 121 group answered $2^3/4^{-1}$ correctly, although some did have difficulty with the fractions. None considered converting to a base of 2 and using the appropriate property. Very nearly the same results appeared for the Math 121 students as for the Math 111 students on $x^2 + x^3$. None of these students falsely generalized on $2^2 + 2^3$, but two students (one high level, one low level) gave $x^5$, two other students initially wrote the $x^5$, but reflected and changed. The two students that wrote $x^5$ corrected themselves with a slight "push" by the interviewer. None of the students gave an incorrect answer on $x^a y^b$, although one student had difficulty convincing himself not to use a property. Apparently, $x^2 + x^3$ caused more difficulty than $x^a y^b$ because "the bases are the same". On the problem $\sqrt{14}$ / $\sqrt[3]{7}$ the false generalizations were in the direction of "can't". Only two students were able to convert to exponential form (excerpt 42), but neither could complete the problem. Two students (low level) gave an answer of $\sqrt{3}$ for $\sqrt{6} - \sqrt{3}$.

It appears that both Math 111 and Math 121 students falsely generalize the exponential properties. This is evidenced by the fact that practically no students gave incorrect responses with the arithmetic type problems, but in every case with the variable type problems, one or more students would incorrectly apply a property. Additionally, the interviews showed many of the students lacked confidence with respect to their answer, and would frequently change their result and then change it back.

Category 8: $(a + b)^2 = a^2 + 2ab + b^2$

This category was designed not so much to see if the Math 121 students could square a binomial, but more for the purpose of determining if they correctly used the exponential properties when exponents were not the most obvious feature of the problem.

Two students (low level) of the seven were not able initially to give the correct response to $(x + 3)^2$, but instead gave $x^2 + 9$. When reminded by the interviewer of their answer to $(2 + 3)^2$, one relented and used the FOIL method to get the correct result. The other student tried, but still did not get the answer (excerpt 53). The same two students immediately made the similar false generalization on the next problem and answered $2 + 3 = 5$ for $(\sqrt{2} + \sqrt{3})^2$.

However, two other students gave answers of $8 + 2 \cdot \sqrt{6}$ (because $(\sqrt{3})^2 = 6$), and 11 (because $\sqrt{6} + \sqrt{6} = 6$). The curcial feature of $(x^{3n} + 2)^2$ was the power the students would obtain for the first term of the trinomial answer. Responses were $2x^{3n}$, $x^{6n^2}$, $x^{9n^2}$ (twice), $x^{2^{9n^2}}$, and the correct result 6n (2 times). Most students were able to correct their answer with some hints by the interviewer. One student, showing consistency left out the middle term on all three problems.

This category showed rather well that the students had not, for the most part, generalized the properties $a^m \cdot a^n = a^{m+n}$ or $(a^m)^n = a^{mn}$. They had shown by their work on previous problems that they were aware of the properties and could also combine "like terms", but in fact only two of the seven students could put it all together on a more "complex" example.

## SUMMARY - ANALYSIS OF INTERVIEWS

In this section excerpts from the tape-recorded dialogue between an interviewer and students were presented from both a beginning and an intermediate algebra class as they worked selected problems while "thinking aloud". The interviewer questioned, gave hints, contradicted, and posed other problems in an effort to determine if the students were using either the Instrumental-Symbolic, Instrumental-Iconic, Relational-Symbolic, or the Relational-Iconic, mode of understanding. Selected excerpts from the interviews were presented as representive of the thinking processes that were used by the students while working the problem sets.

The problem sets were subdivided into eight categories in order to analyze the totality of the student responses for the purpose of investigating the research questions posed in Chapter I. The categories, along with some overall evaluation of the student responses are listed below:

1. Prerequisite knowledge The students interviewed in both beginning and intermediate algebra appear to have a limited instrumental understanding of the conjectured prerequisite knowledge of numerical fractions and rational expressions.

2. Variable and Equality The students interviewed in both algebra sections appear to have some difficulty with variable. They, in general, believe a variable, "stands for a number". They also can use Instrumental-Symbolic procedures for combining like terms. However, their difficulty with the concepts of variable and equality is evidenced by the fact that very few realize that numerical substitution is one appropriate means of testing a conjecture about variables. Additionally, some students did not notice a contradiction when asked to substitute a number for a variable, doing so and getting an inequality, thus indicating a lack of relational understanding of the equality concept. It was also found that students did not recognize the use of a variable for a tautology as opposed to

the use of a variable for a constraint equation.

3. <u>Definition of Exponent - Various Forms</u> In general, both the beginning and intermediate algebra students had a relational understanding of positive integer exponents. For the zero exponent both groups appear to have an instrumental understanding. However, the beginning algebra students exhibited very little understanding of negative integer or fractional exponents, while the intermediate students interviewed were judged to be at the Instrumental-Symbolic level of understanding with respect to both the negative integer and fractional exponents. Three of the intermediate students showed at least some relational understanding of fractional exponents.

4. <u>Exponential Properties $a^m \cdot a^n = a^{m+n}$ and $a^m/a^n=a^{m-n}$</u> Both groups of students were judged to lack relational understanding of these properties. The beginning algebra students tended to use arithmetic procedures to avoid having to use the properties, while the intermediate group could use the algorithms better, but showed very little relational understanding. A few of the students did exhibit relational understanding by using the exponential definitions (when working with positive integer exponents) to justify their use

of these properties.

5. <u>Exponential Properties $(ab)^n = a^n b^n$ and $(a/b)^n = a^n/b^n$</u>

   The students interviewed were judged to have at most some instrumental understanding of this category. The beginning, algebra students avoided the use of these whenever possible, while the intermediate students had some instrumental understanding. Both groups appeared to have great difficulty with the reverse of the properties - i.e. $a^n b^n = (ab)^n$.

6. <u>Exponential Property $(a^m)^n = a^{mn}$</u>

   Interviews of the beginning algebra students seemed to indicate that they had very little understanding of any type with respect to this property. Some few did show instances of instrumental understanding. The intermediate students on the other hand were judged to possess a relational understanding of this property.

7. <u>False Generalizations</u>

   The beginning algebra students in particular showed evidence of false generalizations. Once they have committed a rule to memory, they use it, whether appropriate or not. The lower level students in the intermediate group tended to do likewise. The better intermediate students did use false generalizations, but appeared to be more cautious about applying an algorithm. Some students did use numerical imagery as a basis for their decisions.

8. <u>Squaring A Binomial $(a + b)^2 = a^2 + 2ab + b^2$</u>
   This category was used only for the intermediate algebra students. The primary purpose was to look for the generalization of the properties $a^n \cdot a^m = a^{m+n}$ and $(a^m)^n = a^{mn}$. The majority of the students were able to square the binomials presented, but most obtained a wrong answer for the exponent of the first term of the answer when working with literal exponents.

CHAPTER V

# SUMMARY AND DISCUSSION

## Procedure

During the summer of 1984, fourteen students from Ferris State College, Big Rapids, Michigan were slected to participate in this study of the investigation of college remedial algebra students' understanding of the concept and principles of exponent. Seven students were selected from a beginning remedial algebra class, and seven students were selected from an intermediate remedial algebra class. Approximately one-half of the students from each class scored the highest grades of all students on an exponential unit examination given previous to the beginning of this study. The other students scored the lowest grades relative to all students on the exponential unit examination.

Forty-nine questions were developed based on errors from exponent pre-tests, prerequisite knowledge for the concept of exponent, and the various definitions and properties of exponent.

The primary instrument for investigating the students' understanding of exponent was an interview with each student of approximately ninety minutes. The students were asked to "think aloud" as they worked through the set of problems. Although the problems were pre-determined, the interviewer was free to develop new problems during the interview, offer hints of encouragement, question solutions, question procedures, and contradict. The item of interest was the student's thought process, therefore the interviewer was free to move in any direction that was necessary to "draw out" the student's understanding.

The interviews were audio-recorded for later analysis. The tape of each interview was analyzed by use of a two-by-two matrix developed by combining Skemp's (1979B) theory of relational and instrumental understanding with two of Bruner's (1973) modes of knowledge representation — iconic and symbolic. The students were rated as they worked various problems as belonging to either of the four cells: Instrumental-Iconic, Instrumental-Symbolic, Relational-Iconic, and Relational-Symbolic. Rating techniques were adapted from Erlwanger (1975C) and Alexander (1977) in order to determine the mode of thinking.

Selected excerpts of the interviews for the various levels of student "capabilities" were given verbatum in order for the reader to judge the validity of the analysis model Additionally, the problem set was subdivided into eight categories for the purpose of tracing an individual's thought process through the entire interview with respect to each of eight items of interest; prerequisite knowledge, the exponential properties, various definitions of exponent, and false generalizations. The problem set results from all interviews were then tabulated, and the various answers listed. These were coordinated with the verbatum interview excerpts to specify as far as possible the various levels of understanding at which the groups as entities appeared to be functioning.

## RESULTS

This study was designed to be an investigation into the thought processes of how students deal with the concept and principles of exponents. Several researchers have detailed both the dangers of attempting to draw firm conclusions from an interview setting, and the benefits of the proper utilization of the procedure (for example, Erlwanger, 1974). The results here, then, are of a somewhat tentative nature. At best they can serve as guides to future studies.

This study began with a goal of investigating the following questions:

1. Do remedial algebra students have a relational, instrumental, or no understanding of the prerequisites conjectured as necessary (as advocated by Gagne') for success in dealing with the concept of exponent.
2. Do remedial algebra students have a relational, instrumental, or no understanding of the concept of exponent?
   a. How does the understanding of positive, negative, (both integral and fractional) and zero exponents differ in the same student? Between students?
   b. How does the understanding of explicit

number exponents and literal exponents differ in the same student? Between students?

3. Do remedial algebra students have the ability to generalize (as defined by Krutetskii) the various properties of exponents?
    a. Can the source of "false generalizations" be determined?
    b. Have students that appear to have generalized the properties of exponents (relational understanding), merely generalized the symbolic notation (instrumental understanding)?
4. What types of imagery (Bruner's enactive, iconic, and symbolic) do students use when working with the concept of exponent?
    a. Does the imagery used differ, and in what respect, for students at the relational and instrumental levels of understanding?
    b. Can a student who is operating at the instrumental level be "pushed" by way of hints and guided questioning to use numerical imagery as an aid to relational understanding?
5. Do successful students (as determined by a letter grade on a test) differ from unsuccessful students with respect to the four questions above?

The results of the investigation into these questions will now be addressed, one at a time in numerical order.

1. Do remedial algebra students have a relational, instrumental, or no understanding of the prerequisites conjectured as necessary (as advocated by Gagne') for success in dealing with the concept of exponent.

The prerequisites, as developed in terms of Gagne's chart of hierarchical prerequisite knowledge (Appendix A), primarily were those of arithmetic operations and knowledge of variable and equality.

As indicated by the interview excerpts and interview tabulation, the students in both the beginning and intermediate levels appear to have relational understanding of the operations involving integers. The interviews showed that both beginning and intermediate algebra students had a very limited instrumental understanding of fractions and rational expressions. A particularly troublesome area was the false generalization of "canceling" inappropriately. For example, the problems $\frac{a + b}{a}$ and $\frac{2 \cdot 5 + 6}{8}$ were missed frequently due to the "crossing out" of a common number which actually was not a common factor of both numerator and denominator. Also, many of the intermediate algebra students had difficulty adding the exponents in the problem $a^{1/m} \cdot a^{1/n}$ due to the lack of understanding of the process used in adding fractions which contain literal numbers.

2. Do remedial algebra students have a relational, instrumental, or no understanding of the concept of exponent?

a. How does the understanding of positive, negative, (both integral and fractional) and zero exponents differ in the same student? Between students?

b. How does the understanding of explicit number exponents and literal exponents differ in the same student? Between students?

The results of the interviews lend weight to the conjecture that both beginning and intermediate algebra students, both successful and unsuccessful, have a relational understanding of the concept of positive integer exponent. It is with the use of the various exponential properties that instrumental understanding replaces relational understanding. The beginning algebra students tend to avoid the exponential properties to the greatest extent possible. The intermediate algebra students are much more attuned to algorithms, but tend to use the exponential properties in inappropriate situations (false generalizations) (excerpt 46).

In particular, the use of a variable makes the erroneous application of an exponential property much more likely. The interview results indicate that those problems with variable exponents are more difficult for students at all levels than those with variable bases. When both the base and exponent were literal numbers, all levels of students interviewed were as likely to use an incorrect application of a property as the correct one. For example, the two problems $x^a \cdot x^b$ and $x^a y^b$, lead to many different answers. Of those students who obtained the correct results, many seemed to have instrumental understanding to such a low degree that their answer was little better than a guess. Only a very few of the students could use numerical imagery as a crutch for decision-making on these problems.

The understanding of the zero exponent for all levels of students was shown to be at the instrumental level (excerpt 30). All students did recognize "anything to the zero power is 1". No student was able to offer an adequate response to the question, "Why do we need a zero exponent?", or "Why do we have a zero exponent." Most of the students' difficulties appeared at the false generalization level on the problem $(2 + 3)^0$. A few students distributed the zero and obtained two for an answer.

After variable exponents, the major difficulty appeared to be with negative exponents. For both the beginning and intermediate algebra students, it was as if the exponential properties were not intended to be used with negative exponents. At both levels the students tended to use the definition to convert to fractions, and then work with positive exponents. This showed a relational understanding of the definition, but not even instrumental understanding for the exponential properties with respect to negative exponents (excerpt 17).

In all the variations of exponent, the intermediate algebra students exhibited both more relational and instrumental understanding than the beginning algebra students. The unsuccessful intermediate algebra students appeared to have less understanding, of both types, than the successful beginning algebra students. However, apparently due to their longer "exposure", they were more apt to recall a procedure when prodded by the interviewer.

The understanding of rational exponents was virtually non-existent at either the relational or instrumental level for the beginning algebra students (excerpt 12). Rational exponents had not been covered as a topic in their class, therefore, it is not an unexpected result. Several students did give the correct result for $(x^6)^{\frac{1}{2}}$, but the interviews showed that since they were expected to do something, some students added and some multiplied. None mentioned the idea of a square root. On the other hand, the intermediate algebra students did, in general, show a relational understanding on $(x^6)^{\frac{1}{2}}$ by using not only the correct property, but recognizing the equivalency with the notion of square root (excerpt 41). However, most manifested, at best, an Instrumental-Symbolic understanding of the problem $a^{1/m} \cdot a^{1/n}$.

3. Do remedial algebra students have the ability to generalize (as defined by Krutetskii) the various properties of exponents?
    a. Can the source of "false generalizations" be determined?
    b. Have students that appear to have generalized the properties of exponents (relational

understanding), merely generalized
the symbolic notation (instrumental
understanding)?

It cannot be stated with any degree of confidence that remedial algebra students have the ability to generalize relationally the properties of exponent. The interviews indicated only one student out of fourteen who was at least "close" to having generalized the properties (Judy). Several students would do quite well on one or two properties, or on all properties in one direction, but then fail on another property, or the same property in the reverse direction (excerpt 38).

The source(s) of false generalizations appear to come from a total reliance on instrumental understanding when using exponents (excerpt 23).

Those students operating strictly at the instrumental level have no other way except total memorization of symbols to serve as a check to false generalization, therefore, most beginning algebra students and a majority of the intermediate algebra students engaged in false generalization. Those students who generalize falsely do not appear to notice all the relevant features of a problem, and frequently home in on one surface feature that matches a feature of an algorithm available to them. For example, several students saw only the multiplication and exponent for the problem $x^a y^b$, and obtained $xy^{a+b}$, obviously ignoring the essential part of the algorithm pertaining to the base.

Although variable exponents and bases are primarily a source of false generalizations, several students did engage in such thinking with explicit numbers (excerpt 28). In most situations where errors were made on the arithmetic problems, the interviewee could get the answer arithmetically with appropriate suggestions from the interviewer. However, when asked about a property for the problem, it was as if the arithmetic qualities of the numbers vanished, and in a sense the problems were of a variable nature (excerpt 29). The numbers were symbols to be manipulated.

4. What types of imagery (Bruner's enactive, iconic, and symbolic) do students use when working with the concept of exponent?
   a. Does the imagery used differ, and in what respect, for students at the relational and instrumental levels of understanding?
   b. Can a student who is operating at the instrumental level be "pushed" by way of hints and guided questioning to use numerical imagery as an aid to relational understanding?

As expected (due to lack of concrete materials), no student used the enactive mode of imagery. Most students interviewed worked primarily in the symbolic mode. This included both successful and unsuccessful students from both beginning and intermediate algebra. None of the students used the iconic mode spontaneously. The primary difference in imagery usage between the relational thinkers and instrumental thinkers appeared to be the fact that the students with relational understanding recognized that they could use numerical imagery. Judy was a good example of this (excerpt 40). This was not always possible for the people with instrumental understanding. Another A student in the intermediate algebra section indicated that "she had never thought of it that way." when pushed to use numerical substitution.

This investigation idicated that a student could be pushed to use numerical imagery as an aid to understanding, at least in the short term. Some students did not use numerical imagery initially, but after being pressed by the interviewer, used it frequently thereafter in an apparent relational manner. Dea (excerpts 14, 15 and 16) is a good example. Initially, she was not inclined to use numerical imagery, but after finally using it to determine that she cannot divide out the a in $\frac{a + b}{a}$, she then readily uses it in the next excerpt. Only a long-term evaluation could determine if Dea, in fact, will continue to use numerical imagery as an aid to understanding.

5. Do successful students (as determined by a letter grade on a test) differ from unsuccessful students with respect to the four questions above?

    a. Prerequisite Understanding

    The beginning algebra students interviewed did not differ to any great extent in their understanding of prerequisite knowledge.

both successful and unsuccessful students had at most an instrumental understanding. In the intermediate algebra group, the unsuccessful student had only an instrumental understanding of the prerequisites. However, the successful student group contained students that understood instrumentally and students that understood relationally. Judy (excerpts 33-41) and Alan (excerpts 42-46) differed in their understanding, but both were very successful grade-wise in the class.

b. Concept of Exponent

This question was dealt with thoroughly in the main question number 2. Generally, the interview results could be capsulized to the findings that both successful and unsuccessful students had relational understanding of positive integer exponents, instrumental understanding of zero exponents and negative exponents, and limited instrumental understanding of variable exponents.

c. Generalization and False Generalization

Both the unsuccessful and successful beginning algebra students, according to the interview results, had not generalized the concept of exponent to all of the variants and properties. Both groups appeared equally likely to generalize falsely on variable related problems. The successful students knew the "rules" better on the arithmetic problems. (Max, excerpts 24-30, and Dea, excerpts 13-23).

The unsuccessful intermediate algebra students were as likely to falsely generalize as the beginning algebra students. However, they were more likely to falsely generalize than the successful intermediate algebra students. Todd is an example of such a student (excerpts 47-53). The successful intermediate algebra students also used false generalization, but particularly on variable problems.

d. Imagery

For the beginning algebra students interviewed, imagery was for the most part non-existent. It was only at the instigation of the interviewer that numerically imagery was initially used.

As indicated in the main question number 4, the successful and unsuccessful student at the intermediate level used very little imagery. Some of the successful students used it when pushed by the interviewer.

## DISCUSSION

This investigation would lead one to believe that relational understanding as defined by Skemp (1979B) has little bearing on the success or lack of success in either a beginning or intermediate algebra class. All of the successful students at the beginning level and at least half of the successful students at the intermediate level lacked relational understanding with respect to exponential concepts and principles.

Bruner (1966) conjectured that mathematics learning should proceed in the order enactive mode, iconic mode, and finally symbolic mode of representation. The omitting of the first two modes was cause for concern for the long-term mathematical education of the individual. The interview data showed that few students view algebra in any mode other than symbolic. The data also has implication for the findings of other researchers.

Krutetskii (1976) indicated that "incapable students":

> Cannot generalize mathematical material according to essential features even with help from the experimenter and after a number of intermediate, single-type practice exercises (p. 254).

This investigation shows that rarely had students, at any level generalized the concepts and principles of exponent.

Gagne' (1970) interated the importance of prerequisite knowledge in mathematics learning. Skemp (1979) also indicated that conceptual structures can only be built from available lower order concepts. The students in this study, as indicated by the interviews, had minimal relational understanding of the conjectured prerequisite concepts.

If one assumes any "integrity" for the course offerings and resultant grades attained by the students in this study, the dramatic implication can only be that these students have accomplished a goal that the above theorists indicate they should not achieve. Perhaps a plausible explanation is offered by Robert Davis (1967).

Davis, after observing a teacher write on the board such problems as $x^2 \cdot x^3 = x^5$, $p^{10} \cdot p^7 = p^{17}$, and so on for 45 minutes a day for two consecutive days, had the following to say:

> Whenever I have described this lesson to mathematicians, I have usually found it neither necessary nor prudent to add any further remarks. To professional mathematicians, the lesson speaks for itself. To mathematics educators, however, it may be appropriate to claim that what is at work here is clearly a teacher belief system of a particularly Thorndikean sort. These stimuli were being connected to the responses with such tenacity as to explain how it can happen that a college freshman can write $x^2 + x^3 = x^5$ (p. 19).

The successful students in this study had received excellent grades on an exponential examination approximately two weeks previous to the interviews. The teacher indicated that for examinations he gave problems from the text. One can conjecture that the students earnestly studied the text and sample board problems, took the examination, and promptly forgot the rules (the "bonds" were broken). As Skemp (1979B) would say there was a match among the students, the text, and the teacher.

This is not an indictment of the instructor. He indicated that given the short span of ten weeks, along with a required syllabus, and students that were "unprepared", that he treated the courses as "tool" courses. He tried to get the students to learn enough to function in their particular program, or to meet a prerequisite for the next course.

If, in fact, the interview findings are representative of remedial students' understanding of algebra and conceptualization of the learning process in algebra as they exist at various two-year colleges, four-year colleges, and universities; then the findings are perhaps an indictment of the higher-educational system. Perhaps educational institutions are writing off any hope for these students to "understand" in favor of a stimulus-response situation which will at least enable the student to "pass" the math requirements and remain a bona-fide tuition paying student. Is it reasonable to enroll students who have a history of virtually no success in mathematics, give them a quick ten-week course consisting of lectures at the chalk board, place them in groups from thirty-five up to one-hundred-fifty, and expect them to understand? To do so violates virtually all that we have learned from cognitive research in the preceding fifty years by such men as Piaget, Bruner and Skemp.

Surely, the purpose of a mathematics course at the college level can not be justified by the concerned academic department wanting the course(s) to be "on the student's record". This would appear to be one of the real possibilities based on the consistent lack of any relational understanding by the majority of interviewees in this study. Apparently, students are "successfully" completing these two courses, and thus, meeting the required prerequisites for their particular programs.

The above would indicate there are two possible programmatic scenarios for these students. First, perhaps the understanding of the algebraic course material is not really needed in the involved programs, at least in any major role. In this case, one must question the expenditure of resources for all parties involved for the "window-dressing" of particular courses. Secondly, perhaps the algebra topics covered in the two remedial courses are indeed required for the programs, but are "taught" again by other instructors as the topics are needed in the programs. That is, the algebra courses could be justified by the conjecture that even though the students have little relational understanding, it is easier to memorize the involved algebraic concepts and principles the second time because "the students have seen the material before."

One must question the educational priorities of a system, which enables a student to methodically move up the educational ladder by continually memorizing material and consequently passing examinations. If such a student had understood relationally at the elementary school level, perhaps he/she would have understood the middle school mathematics, and eventually the high school mathematics. That is, relational understanding at any educational level equips the student to have a fair chance for success of such understanding at the next level. In this event, the student would have been able to enroll in a "college algebra" course when coming to college (if needed). It seems that at each educational level the system can not adequately respond due to the "failure" of the educational system to provide relational understanding at the previous level.

In summary, the only real answer to the massive numbers of remedial students enrolling at the college level would appear to be a real dedication by the educational system at all levels to expend the time and financial resources necessary to aid students in obtaining a relational, as opposed to an instrumental, understanding of mathematics.

## LIMITATIONS OF THE STUDY

1. This study dealt with remedial algebra students at Ferris State College who were enrolled during a summer term. Are the students who attend college during the summer quarter and enroll in remedial algebra representative of all such students who enroll during the academic year? Perhaps only those who failed the class previously, or those "bright" enough to move ahead, enroll in summer. All of these unanswered questions limit the scope of this study.

2. The small sample size from each of the two levels of algebra students limits any conclusion only to very tentative

conclusions, and only with respect to the fourteen students actually interviewed.

3. The model used in this study was derived from models developed by Erlwanger and Alexander (previously cited) for the purposes of investigating elementary school children, and middle school-age children respectively. There were two difficulties noticed during the application of this model to college-age students. First, students at this age operating at an instrumental level of understanding are often supremely confident of their answers. They believe they truly "understand". Thus, they have all the characteristics the model predicts for students at the relational level. Many times during the interviews, it was only after several problems that the instrumental level of understanding of an individual could be perceived. In this sense, the study is limited by the author's ability to actually perceive the correct level of understanding.

4. Additionally, college-age students, particularly the "bright" ones, have the ability to "learn" from the prodding and hints from the interviewer, as well as previously worked problems. As an example of this, frequently students who had no inclination to use numerical imagery, used it consistently after a suggestion by the interviewer. Due to the fact that the forty-nine problems were for the most part closely related conceptually, this study is limited by the students' ability to learn during the interview.

## SUGGESTIONS FOR FUTURE RESEARCH

1. The interview data indicates that successful students in general lack any real understanding (relational) of the algebraic processes related to the concept and principles of exponent. The implications of the interview results are so ominous with respect to any future mathematical endeavors by these students, that the study should be replicated at other institutions, not only with exponential concepts, but other algebraic concepts, and sampling more mathematically advanced students as well.

2. Doyle (1983) has indicated that "low ability" students in mathematics do not do well in an unstructured setting. They do not develop the strategies and "higher order executive routines" that enable them to be successful (pp. 175-177). Confrey and Lanier (1980) have suggested that Krutetskii's abilities such as generalization reversibility, flexibility, and curtailment should be addressed directly in a teaching situation rather than embedded in lessons on other concepts (p. 553).

The suggestion here is for a researcher to attempt to design such a course that is relevant to college-age algebra students.

3. Nigel Ford (1981) after reviewing the literature on the assessment of learning in higher education, subscribed to the belief that the quickest way to change student learning is to change the assessment system (p. 372).

Many students in the interviewing sessions obviously believed that answers were the most important aspect of algebra. All one had to do was memorize how to do enough problems and algebra was easy. One student accused the interviewer of "messing her up for the next test", by having her think of some aspects of exponents that she had never considered.

Is it too late for relational understanding to become a personal goal of remedial algebra students? By by-passing the first two levels of Bruner's knowledge representation, have many of the students become mired in the Instrumental-Symbolic mode due to affective reasons?

It is suggested here that it would be worthwhile to teach a remedial algebra class modeled after the "mathematics laboratory" idea of Fitzgerald (1972) (cited in Fey, 1980, p. 414):

> Primarily, a mathematics laboratory is a state of mind. It is characterized by a questioning atmosphere and a continuous involvement with problem solving situations. Emphasis is placed upon discovery resulting from student experimentation. A teacher acts as a catalyst in the activity between students and knowledge.
>
> Secondarily, a math lab is a physical plant equipped with material objects...Since a student learns by doing, the lab is designed to give him the objects with which he can do and learn.

Not only teach for understanding, but <u>test</u> for understanding as well. Perhaps it would take two terms to "cover" the material usually covered in one term. However, if the experiment could shed light on the question of whether it is actually <u>possible</u> to teach remedial algebra college students from a relational standpoint, then it would contribute greatly to what is known about such students.

This study began initially by detailing the massive numbers of remedial algebra students who are presently enrolled in higher educational institutions in the United States. According to Akst (1981), there were 600,000 such students in 1980. Stanley Erlwanger (1974) observed at the conclusion of his case studies with Benny et al:

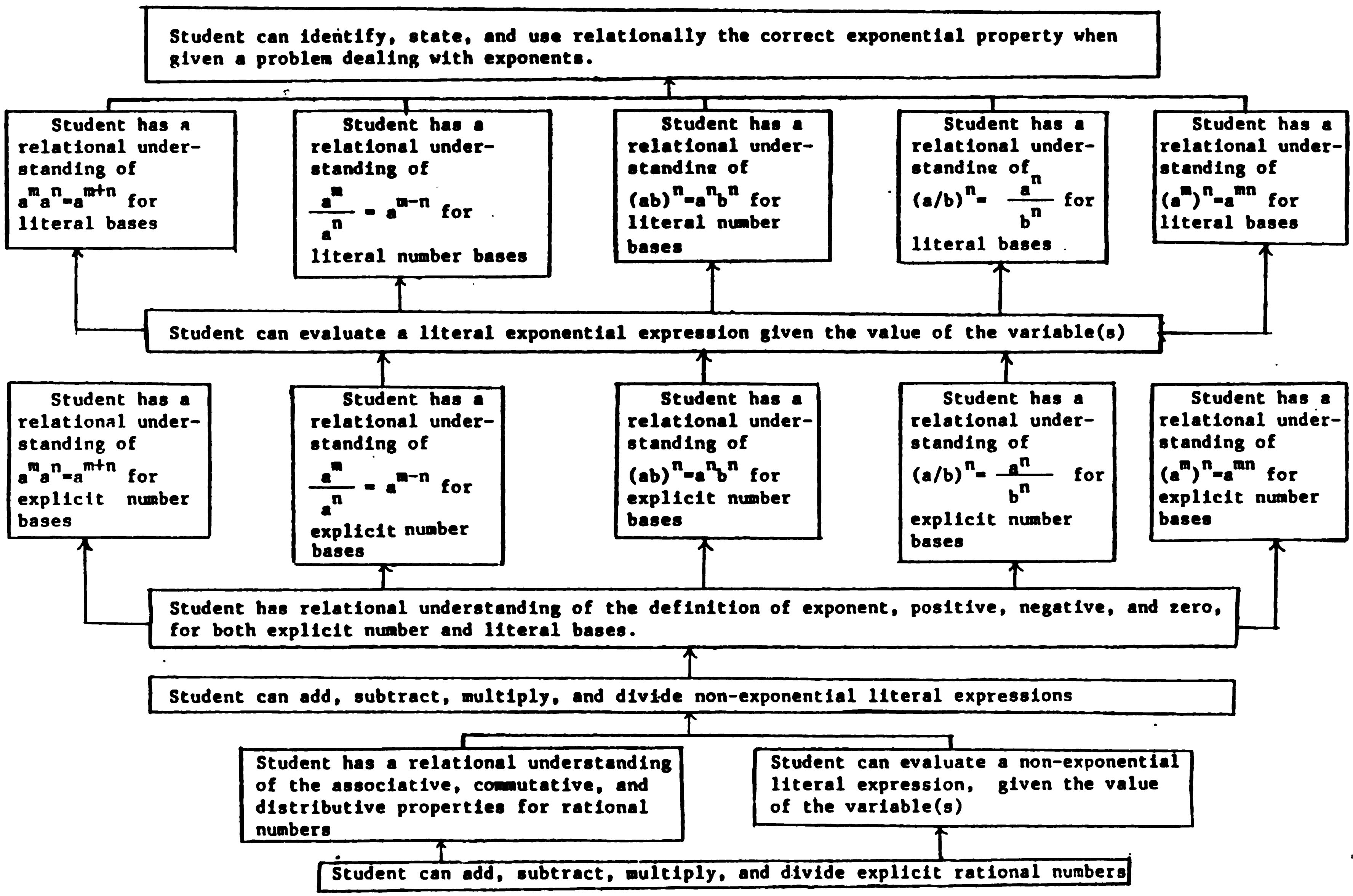
> The case studies suggest that unless a child's wrong ideas, beliefs and views are detected and corrected, they may develop and become more complex (p. 285).

This study is concluded with the observation that the findings herein indicate that Erlwanger was correct. Benny has grown up. Benny is now in college.

# APPENDICES

# APPENDIX A

Hierarchy of Prerequisite Knowledge for Exponents



# APPENDIX B

## Exponent Pre-Test

1. Write $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$ in exponential form
2. Simplify: $2^3 \cdot 2^2$
3. Simplify: $2 \cdot 3^2$
4. Simplify: $2^2+3^2$
5. Simplify: $2^3+2^2$
6. Simplify: $\frac{2^5}{2^2}$
7. Simplify: $(2^3)^2$
8. Simplify: $\frac{4^3}{2^2}$
9. Simplify: $2^0$
10. Simplify: $8^{2/3}$
11. Simplify: $(2 \cdot 3)^2$
12. Write: $a \cdot a \cdot a \cdot a \cdot a$ in exponential form
13. Simplify: $x^3 \cdot x^2$
14. Simplify: $xy^2$
15. Simplify: $x^2+y^2$
16. Simplify: $x^3+x^2$
17. Simplify: $\frac{x^5}{x^2}$
18. Simplify: $(x^3)^2$
19. Simplify: $\frac{y^3}{x^2}$
20. Simplify: $x^0$
21. Simplify: $(xy)^2$

# APPENDIX C

## CONSENT FORM

I confirm that my participation in Mr. Wilson's research into the learning of algebra is voluntary.

I recognize also, that my participation will in no way affect my grade in my mahtematics course.

I consent to allow Mr. Wilson to discuss my work in my algebra class with my instructor.

I realize that my responses may be published in a research report, but understand that this will be done in an anonymous fashion.

Name (Printed) ______________________________

Signature ____________________________________

Date________________________

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