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Elias Manuel Lopez

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Major professor

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Elias Manuel Lopez

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ABSTRACT

LICCI GORENSTEIN IDEALS OF DEVIATION TWO

By

Elias Manuel Lopez

A new family of licci Gorenstein ideals of odd height and deviation 2, for height at least 7, is introduced. A characterization, up to deformations and specializations, of licci Gorenstein ideals of height 5 and deviation 2 is given in terms of a minimal set of generators. Also, a family of licci Gorenstein ideals of any even height larger or equal to 6 and deviation 2 defining a rigid algebra which are not hypersurface sections is constructed.

A Maria Julia y

Maria Alejandra

Por todo lo que son

Por todo lo que serán

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INTRODUCTION

Let (R, m) be a Gorenstein local ring, and let I be a proper ideal of R. The deviation of I, d(I), is the difference between the minimal number of generators of I and the grade of I. In some sense, it is a measure of the failure of I being a complete intersection. Some authors refer to it as "complete intersection defect".

For perfect Gorenstein ideals of height 3 and deviation 2 we have the result of Buchsbaum and Eisenbud [B–E, 2.1]. These ideals are generated by the 4×4 pfaffians of a 5×5 alternating matrix with entries in m. For perfect Gorenstein ideals of height 4 and deviation 2, which are generically complete intersections, we have the result of Vasconcelos and Villarreal [V–V, 1.1] (based on Herzog and Miller [H–M, 1.7]). These ideals are hypersurface sections, namely they can be written as (K, y), K a Gorenstein ideal of height 3 and deviation 2 and y a regular element on R/K.

In this dissertation, we give a structure theorem for licci Gorenstein ideals of height 5 and deviation 2. Moreover, we construct a class of rigid algebras defined by licci Gorenstein ideals of even height and deviation 2 which are not a hypersurface section for every even height larger or equal to 6. By licci we understand for an ideal to be in the linkage class of a complete intersection.

Deviation 2 is the smallest possible deviation for a perfect Gorenstein ideal not being a complete intersection [Ku]. The first non trivial examples of licci Gorenstein ideals of odd height and deviation 2, were constructed by Huneke and Ulrich [H–U–1]. If R is local Gorenstein, $n \geq 2$, X a $2n\times2n$ generic alternating matrix, Y a $2n \times 1$ generic matrix, then in $R[X, Y]_{(X,Y)}$ the ideal $H_n = (I_1(XY), Pf(X))$ is licci, Gorenstein, has deviation 2 and height $2n-1$. Moreover it is not a hypersurface section.

Our main tool is linkage of ideals. Linkage was first introduced by Peskine and Szpiro [P–S], and it has been found to be very useful as a classifying tool, as a method for proving results about special varieties [K–M, H–U–2], and for studying the divisor class group of rigid algebras [H–U–1]. Our main interest is for ideals I and J which are linked and the linking sequence is part of a minimal set of generators of I. In this case we write $I \to J$.

For the ideals of height 5 we are able to describe them, up to deformations and specializations in terms of a minimal set of generators. To accomplish this, we use induction on the number of steps needed to link such ideals to a complete intersection.

We are able to show that every licci, Gorenstein ideal I of height 5 and deviation 2 in a local Gorenstein ring is up to deformations and specializations, either H_3 or (H_2, x, y), x and y regular modulo H_2 (theorem 2.22). To obtain this result, we first characterize, up to deformations and specializations, all perfect almost complete intersections of height 4 and type 2 in R (proposition 2.16). Then we compute all possible ideals J and K such that $H_3 \to J \to K$ and $(H_2, x,y) \to J \to K$ (lemmas 2.17 to 2.21). It follows that K is either a hypersurface section or has a common specialization with H_3. Then, we prove our main result. As an application, we find that if $R = k[[x_1, \cdots, x_r]]$, then I is exactly of the desired form.

We also find a family K_n of licci Gorenstein ideals of height $2n-1$ and deviation 2, and show that K_4 is not obtained from any H_n by deformations, specializations and taking hypersurface sections.

This new family can be described as follows: with the same notation, let X_{ij} be the pfaffian of the matrix obtained from X after deleting the i–th and j–th rows and columns and let $[\ell_1 \cdots \ell_{2n}]^t = XY$. Then the ideal

$$K_n = (\ell_1, \cdots, \ell_{2n-3}, X_{2n-1\ 2n}, X_{2n-2\ 2n}, X_{2n-2\ 2n-1}, Pf(X))$$

is licci, Gorenstein, has height 2n–1, and for $n \geq 3$, it has deviation 2. Also there is a tight double link [K–M–2, 2.1] between H_n and K_n.

These ideas are discussed in Chapter II.

In even grade, the only known result for Gorenstein ideals of deviation 2 is due to Herzog and Miller, [H–M, 1.7] and Vasconcelos [V–V, 1.1]. They show that, under some mild hypothesis, a perfect Gorenstein ideal of height 4 and deviation 2 is a hypersurface section (this implies such ideals are licci). Kustin once asked whether a Gorenstein ideal of even height and deviation 2 is a hypersurface section. We answer this question negatively. We found a family E_n, $n \geq 4$, of licci rigid Gorenstein ideals of even height 2n–2 and deviation 2 which is not a hypersurface section for any height (theorem 3.7). We obtain E_n from (H_{n-1}, x_1, x_2, x_3), x_1, x_2, x_3 regular module H_{n-1}, by performing what Kustin and Miller call a semi–generic tight double link [K–M, 3.1]. The fact that E_n is rigid follows from [K–M, 3.2 and K] and the fact that E_n is not a hypersurface section follows from the fact that E_n is contained in the square of the maximal ideal.

We also find a family F_n of ideals which are obtained from E_n by specialization. These ideals F_n are easily described. Let k be a field, X a $2n-1 \times 2n-1$ generic alternating matrix, Y a $(2n-1)x1$ generic matrix, $n \geq 3$. In $k[X, Y]_{(X,Y)}$ consider the product $[\ell, \cdots, \ell_{2n-1}]^t = XY$, and let $X_{i_1 \cdots i_r}$ be the pfaffian of the matrix obtained from X by deleting the rows and columns $i_1, \cdots, i_r$. Then $F_n = (\ell_1, \cdots, \ell_{2n-4}, X_{2n-3\ 2n-2\ 2n-1}, X_{2n-3}, X_{2n-2}, X_{2n-1})$ is licci, Gorenstein, has deviation 2 and height 2n–2 (theorem 3.3). F_3 is a hypersurface section but F_n is not if $n \geq 4$ (corollary 3.10). We finish by showing that $k[X,Y]_{(X,Y)}/F_n$ is (R_2) but not (R_3) (proposition 3.11).

We want to close with a question that has been around for some while, but we want to formulate it again. If I is a perfect Gorenstein ideal of R, and I has deviation 2, is I licci? If the answer is yes, we would have to describe all Gorenstein ideals of height 5 and deviation 2.

CHAPTER I

BASIC RESULTS AND NOTATIONS

We will use as a general reference H. Matsumura's book Commutative Algebra [Mat]. Unless otherwise stated, (R, m) will always denote a local Gorenstein ring with identity.

We begin by establishing the notations needed for the manipulations of pfaffians.

Let X be a n×n generic matrix of indeterminates (that is $x_{ij} = -x_{ji}$, $x_{ii} = 0$). In $R[X] = R[\{x_{ij}/1 \le i,j \le n\}]$, the determinant of X is a perfect square. If n is odd, this determinant is zero. For n even, the pfaffian of X, denoted by Pf(X) is a uniquely defined square root of the determinant of X such that if X is specialized to s = diagonal $\{S,\cdots,S\}$, $S = \binom{0\ \ 1}{-1\ \ 0}$, then pf(s) = 1. [Jac, 6.4].

If $r < n$, we denote by $Pf_{i_1,\cdots,i_r}(X)$ the pfaffian of the alternating matrix obtained from X by deleting the rows and columns $i_1,\cdots,i_r$.

Let (i) denote the multi index $i_1,\cdots,i_r$. Define $\sigma(i)$ to be zero if (i) has a repeated index, and otherwise to be the sign of the permutation that rearranges $i_1,\cdots,i_r$ in ascending order (see [K]).

Let

$$|i| = \Sigma_{s=1}^{r} i_s$$

and let

(1) $$X_{(i)} = (-1)^{|i|+1}\sigma(i)\mathrm{Pf}_{(i)}(X)$$

If $r = n$, let $X_{(i)} = (-1)^{|i|+1}\sigma(i)$, and for $r > n$, let $X_{(i)} = 0$. The pfaffian of X can be expressed in terms of lower order pfaffians as follows:

(2) $$\mathrm{Pf}(X) = \Sigma_{j=1}^{n} x_{ij}X_{ij}$$

while

$$\Sigma_{i=1}^{n} x_{ik}X_{ij} = 0 \quad j \neq k$$

(see [Ar.E], page 142).

Let Y be a $n\times 1$ matrix of indeterminates over R[X], and write $[\ell_1,\cdots,\ell_n]^t = XY$.

Consider the product

$$Y^tX^t \begin{bmatrix} X_{1\,j} \\ \vdots \\ X_{j-1\,j} \\ 0 \\ X_{j+i\,j} \\ \vdots \\ X_{n\quad j} \end{bmatrix}.$$

Then, on one hand this product is

$$[\sum_{i=1}^{n} \ell_i X_{ij}]$$

while on the other, by (2) it is $[y_j \, Pf(X)]$. Then

$$(3) \qquad \Sigma_{i=1}^{n} X_{ij} \ell_i = y_j \, Pf(X).$$

Also, because X is alternating, we obtain

$$(4) \qquad \ell_1 y_1 + \cdots + \ell_n y_n = 0.$$

Let $a_1, \cdots, a_r$ be elements of R. We denote by $[a_1 \cdots a_r]$ the $1 \times r$ matrix whose entries are the a_i's and by $(a_1, \cdots, a_r)$ the ideal they generate.

If I is a proper ideal of R, $\mu(I)$ denotes the minimal number of generators of I, grade I its grade and ht(I) its height. $d(I) = \mu(I) - \text{grade } I$ is the deviation of I. If $d(I) = 0$, I is called a complete intersection and, in this case, I is generated by a regular sequence. If $d(I) \leq 1$, then I is called an almost complete intersection.

The ideal I is a hypersurface section if $I = (K, y)$ where grade $K = (\text{grade } I) - 1$ and y is regular on R/K.

Let (R, m) be a Cohen–Macaulay ring with canonical module K_R (for definitions, see [H–K]). We define the type of R, r(R), as $\mu(K_R)$.

If $R = P/I$, where P is a local Gorenstein ring and I is a perfect ideal and if

$$0 \to P^{g_n} \to \cdots \to P^{g_1} \to P$$

is the minimal resolution of R, then $r(R) = g_n$. We now list some basic facts of linkage of ideals.

<u>Definition 1.1</u>. Let (R, m) be a local Cohen–Macaulay ring, and let I and J be two ideals of R.

a) We say that I and J are <u>linked</u>, and write $I \sim J$, if there is a regular sequence $\underline{\alpha} = \alpha_1, \cdots, \alpha_g$ in $I \cap J$ such that $I = (\alpha):J$ and $J = (\underline{\alpha}):I$.

b) I and J are geometrically linked, if I and J are linked and have no associated primes in common.

Definition 1.1 (a) is what Peskine and Szpiro [P–S] call algebraic linkage. We drop the word algebraic in this work. Notice that I and J are not allowed to be the unit ideal. Moreover I and J are unmixed ideals of height g.

Definition 1.2. An ideal I of R is generically a complete intersection if I is unmixed and I_p is a complete intersection in R_p, for all $p \in \mathrm{Ass}(R/I)$.

Remark 1.3. With the notations of 1.1 (b), if I and J are geometrically linked, then they are generically complete intersections.

The proof of the following proposition can be found in Peskine and Szpiro [P–S].

Proposition 1.4. Let I be an unmixed ideal of height g of the Gorenstein local ring R, let $\underline{\alpha}$ be a regular sequence inside I with $(\underline{\alpha}) \neq I$, and set $J = (\underline{\alpha}):I$. Then

a) $I = (\underline{\alpha}):J$ (i.e. I and J are linked)

b) R/I is Cohen–Macaulay if and only if R/J is Cohen Macaulay

c) Let R/I be Cohen–Macaulay, then $K_{R/I} \cong J/(\underline{\alpha})$ and $K_{R/J} \cong I/(\underline{\alpha})$.

d) In addition to the assumptions of 1.4 (c), assume that proj. $\dim_R(R/I)$ is finite. Let F. be the minimal free resolution of R/I and K. the Koszul complex of $(\underline{\alpha})$. Let U: K. → F. be a morphism of complexes lifting the embedding $(\underline{\alpha}) \to I$. Then, the dual of the mapping cone of U, C(U*), is a resolution of R/J.

Definition 1.5. [H–U–2, 2.6] Let I and J be two ideals in R. We say that J is minimally linked to I, and write I → J, if I and J are linked with respect to the regular sequence $\underline{\alpha}$, and the elements of $\underline{\alpha}$ form part of a minimal generating set of I.

In general, minimal linkage is preferable to arbitrary linkage, partly because of the following observation, which follows directly from 1.4.c.

Remark 1.6. [H–U–2, 2.7] In adition to the assumptions of 1.4.c, let $k = R/m$.

a) $r(R/J) = \mu(I/(\underline{\alpha})) = \mu(I) - \dim_k((\underline{\alpha}+mI)/mI) \geq d(I)$. In particular, if R/J is Gorenstein, I is an almost complete intersection.

b) $r(R/J) = d(I)$ if and only if $I \to J$. In particular, if $I \to J$ and I is an almost complete intersection, then R/J is Gorenstein.

Definition 1.7. Let I be an unmixed ideal of R.

a) The linkage class (even linkage class) of I is the set of all R–ideals J which can be obtained from I by a finite number of links (even number of links).

b) [H–U–2, 2.9] We say that I is licci if I is in the linkage class of a complete intersection.

Recall that if X is a $n \times m$ matrix of indeterminates over R, R[X] denotes the polynomial ring $R[\{x_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq m\}]$, and (m, X) is the ideal of R[X] generated by m and the entries of X.

Definition 1.8. Let R be a local Gorenstein ring. Let I be an ideal of R of grade g, and let $\underline{f} = f_1, \cdots, f_n$ be a generating set of I, let X be a $g \times n$ generic matrix over R, and consider the R[X] regular sequence $\underline{\alpha}$ with

$$[\underline{\alpha}]^t = X[\underline{f}]^t$$

a) [H–U–1, 2.3], $L_1(f) = (\underline{\alpha})R[X] : IR[X]$ is called a first generic link of I.

It can be shown that this definition is essentially independent of the generating set $\underline{f}$ of I [H–U–1, 2.4].

b) [H–U–2, 2.12(a)] with the notation as in (a), set $L_0(I) = I$, $L_1(I) = L_1(f)$, $L_i(I) = L_1(L_{i-1}(I))$, $i > 0$. $L_i(I)$ is an i–th generic link of I.

Definition 1.9. Let I, X, g and $\underline{f}$ as in 1.8 or let I = R, g an arbitrary integer. Denote by R(X) the local ring $R[X]_{mR[X]}$

a) [H–U–2, 2.10(b)] If $[\underline{\alpha}]^t = X[\underline{f}]^t$ then $L^1(\underline{f}) = (\underline{\alpha})R(X):IR(X)$ is called a first universal link of I.

It can be shown that this definition is essentially independent of $\underline{f}$ and g [H–U–2, 2.11(b)].

b) [H–U–2, 2.12(b)] With the notations of (a), set $L^0(I) = I$, $L^1(I) = L^1(\underline{f})$, $L^i(I) = L^i(L^{i-1}(I))$, $i > 0$. $L^1(I)$ is called an i–th universal link of I.

Universal links have the following property.

Lemma 1.10. [H–U–3, 2.4] Set the notations as in 1.9, and assume R/m is infinite. Then I is licci if and only if for some $n > 0$, $L^n(I)$ is the unit ideal.

We will use the idea of deformation.

Definition 1.11. a) [H–U–2, 2.1.a] Let (R, I) and (S, J) be pairs of local Gorenstein rings R and S, and ideals $I \subset R$ and $J \subset S$. We say that (R, I) is isomorphic to (S, J) if there is an isomorphism $\varphi: R \to S$ with $\varphi(I) = J$.

b) [H–U–2, 2.2.a] With the notation of (a), we say that (S, J) is a deformation of (R, J) if there is a sequence $\underline{a}$, regular on S and on S/J such that $(S/(\underline{a})), \underline{a}+J/(\underline{a}))$ is isomorphic to (R, I). Equivalently, one says that (R, I) is a specialization of (S, J).

The following propositions give a connection between deformation and linkage.

Proposition 1.12. [H–U–1, 1.12] Let I be a Cohen–Macaulay ideal of grade g in R, $\underline{\alpha}$ a regular sequence inside I with $(\underline{\alpha}) \neq I$ and set $J = (\underline{\alpha}):I$. Let $\underline{\beta}$ be a sequence in R regular on $R/(\underline{\alpha})$, set $\overline{R} = R/(\underline{\beta})$, and set $\overline{I}$, $\underline{a}$, $\overline{J}$ the images of I, $(\underline{\alpha})$, J in $\overline{R}$. Then $\overline{J} = (\underline{a}): \overline{I}$, $\overline{I} = (a): \overline{J}$ and $\underline{\beta}$ is a regular sequence on R/I and R/J.

Proposition 1.13. [H–U–2, 2.16] Let (R, I) and (S, J) be pairs of local

Gorenstein rings R, S and Cohen–Macaulay ideals I of R, J of S, such that (S, J) is a deformation of (R, I). Let $I = I_0 \sim \cdots \sim I_n$ be a sequence of links in R.

Then, there is a sequence of links $J = J_0 \sim J_1 \sim \cdots \sim J_n$ in S such that (S, J_i) is a deformation of (R, I_i) for all $0 \leq i \leq n$.

Proposition 1.14. [H–U–2, 2.17] Let I be a Cohen–Macaulay ideal of R, and let $I = I_0 \sim \cdots \sim I_n$ be a sequence of links in $R_0 = R$. For $1 \leq i \leq n$. Consider the i–th generic link $L_i(I) \subsetneq R_i$, where R_i is a polynomial ring over R_{i-1}. Let $T = R_n$. Then, there is a $q \in \mathrm{Spec}(T)$ such that $m \subset q$ and $(T_q, L_i(I)T_q)$ is a deformation of (R, I_i), $0 \leq i \leq n$.

Remark 1.15. Let I be a Cohen–Macaulay ideal of R, let $(\overline{R}, \overline{I})$ be a specialization of (R, I), let T be as in 1.14, let $q \in \mathrm{Spec}(T)$ such that $m \subsetneq q$, let $S = T_q$ and let $\overline{S} = \overline{R} \otimes_R S$.

In S consider the sequence of localizations of generic links

$$IS \sim L_1(I)S \sim \cdots \sim L_n(I)S.$$

Then in $\overline{S}$ we have the sequence of links $\overline{I}\ \overline{S} \sim L_1(I)\overline{T} \sim \cdots \sim L_n(I)\overline{S}$, where $L_i(I)\overline{S} = L_i(\overline{I})\overline{S}$ and $(\overline{S}, L_i(I)\overline{S})$ is a specialization of $(S, L_i(I))$.

Proposition 1.16. [U–1] Let I be a licci ideal of R. Then, there is a deformation (S, J) of (R, I) such that J is generically a complete intersection.

Definition 1.17. Let R be a formal power series ring over a field k, and I an ideal of R. We say that the pair (R, I) defines a rigid algebra A = R/I if the first upper cotangent module $T^1(A/k, A)$ vanishes (for definitions see [L–M]).

In this case (R, I) has the following property: let (S, J) be a deformation of (R, I), with S also a formal power series ring over k. Then, there is a set of indeterminates $\underline{Z}$ such that $(R[|\underline{Z}|], IR[|\underline{Z}|]$ is isomorphic to (S, J).

For more details on deformation theory, see [Ar. M].

Let (R, I) and (S, J) be pairs of local Gorenstein rings R and S and ideals I of R and J of S. We write (R, I) ≈ (S, J) if there is a pair (T, K), T a local Gorenstein ring, with char R = char S = char T, such that (T, K) is a deformation of (R, I) and of (S, J).

Definition 1.18. [He, 2.2] The equivalence relation generated by ≈ for the pair (R, I) is called the Herzog class of (R, I). Note that ≈ need not be an equivalence relation.

Definition 1.19. [He, 1.5] Let R be a formal power series ring over a field and I a Cohen–Macaulay ideal of R such that R/I is reduced. I is called strongly nonobstructed if $I/I^2 \otimes_{R/I} K_{R/I}$ is Cohen–Macaulay.

Herzog [He] is able to show that if $R = k[[x_1,\cdots,x_r]]$ and if I is strongly nonobstructed, Cohen–Macaulay and reduced, then (R, I) has a deformation (S,J), $S = k[[T_1,\cdots,T_n]]$ and S/J rigid such that the following holds: let (P, K) be any pair in the Herzog class of (R, I), $P = k[[z_1,\cdots,z_m]]$, K reduced, then there is a finite set of indeterminates $\underline{w}$ such that (P, K) is a specialization of $(S[|\underline{w}|], JS[|\underline{w}|])$. In particular, all rigid algebras in the Herzog class of (R, I) are isomorphic, after adjoining power series variables. We use these ideas together with the following propositions.

Proposition 1.20. [B] Let $R = k[[x_1,\cdots,x_r]]$ and I a licci reduced ideal of R. Then I is strongly nonobstructed.

Proposition 1.21 [U–1, Cor 2.2]. Let k be a field, $R = k[[x_1,\ldots,x_n]]$ and let I be a licci ideal of R. Then, there is a pair (R', I') in the same Herzog class of (R, I) such that R'/I' is reduced.

Finally, we will use the following result.

Proposition 1.22 [H–U–3]. Let (R, m) be a local Gorenstein ring with infinite residue class field and let I be a licci Gorenstein of R. Then there is a sequence of minimal links $I = I_0 \to I_1 \to \cdots \to I_{n-1} \to I_n$ where I_n is a complete intersection and I_{2i} is Gorenstein for $0 \leq 2i \leq n$.

CHAPTER II

LICCI GORENSTEIN IDEALS OF DEVIATION TWO AND ODD HEIGHT

Let (R, m) be a local Gorenstein ring. In this chapter we will study licci Gorenstein ideals of R of odd height and deviation 2.

For $n \geq 2$, let X be a $2n \times 2n$ generic alternating matrix over R and Y a $2n \times 1$ generic matrix over R[X]. Denote by $[\ell_1 \cdots \ell_{2n}]^t = XY$. Set $S = R[X, Y]$. Consider the ideal $H_n = (\ell_1, \cdots \ell_{2n}, Pf(X))$. Using the ideas of Huneke and Ulrich, one can show that in $S_{(m, X, Y)}$, H_n is licci, Gorenstein, has deviation 2 and height 2n–1. [H–U–1, prop. 5.8 to 5.12], [K, introduction].

In this chapter we will produce a new family of licci Gorenstein ideals of odd height and deviation 2, and we will establish a structure theorem for licci Gorenstein ideals of height 5 and deviation 2.

We begin with the following sequence of lemmas.

<u>Lemma 2.1</u> [J–P, th 2.3]. With the previous notations, in R[X, Y]

$$ht(Pf_{2r}(X)) = \begin{bmatrix} 2n-2r+2 \\ 2 \end{bmatrix} \quad 0 \leq 2r \leq 2n$$

where $Pf_{2r}(X)$ is the ideal generated by the $2r \times 2r$ pfaffians of X.

<u>Proposition 2.2</u> [H, 1.1]. Let P be a Cohen Macaulay domain, and let M be a P–module having a finite resolution

$$0 \longrightarrow P^m \xrightarrow{A} P^n \longrightarrow M \longrightarrow 0$$

Denote by by S(M) the symmetric algebra of M and by $I_t(A)$ the ideal generated by the t×t minors of A. Then, the following are equivalent

i) S(M) is a domain

ii) grade $(I_t(A)) \geq m+2-t$ for $1 \leq t \leq m$.

Recall that if M has a presentation of the form

$$P^m \xrightarrow{(a_{ij})} P^n \longrightarrow M \longrightarrow 0$$

then S(M) is isomorphic to $P[T_1,\cdots,T_n]/J$ where J is the ideal generated by the m linear forms $\Sigma_{j=1}^n a_{ij}T_j$.

With this notation, Huneke shows that if ii) holds, then ht J = m.

In addition to the previous notations, lets assume R is a domain. Then we can now establish the following proposition.

Proposition 2.3. The ideal $(\ell_1,\cdots,\ell_{2n-2})$ is a prime ideal in R[X, Y] of height 2n–2, $n \geq 3$.

Proof. Let A be the matrix obtained from X by deleting the last 2 rows. Then $[\ell_1,\cdots,\ell_{2n-2}]^t = AY$.

First, because R is a Cohen–Macaulay domain, R[X] is a Cohen–Macaulay domain. Consider now the following complex:

$$0 \longrightarrow (R[X])^{2n-2} \xrightarrow{A} (R[X])^{2n} \longrightarrow M \longrightarrow 0.$$

Because A is a matrix with maximum rank over R(X), the map from $(R[X])^{2n-2}$ to $(R[X])^{2n}$ is injective. Moreover

$$S(M) = \frac{R[X]\,[Y]}{(\Sigma_{j=1}^{2n} x_{ij}y_j \mid 1 \le i \le 2n-2)} \cong \frac{R[X, Y]}{(\ell_1, \ldots, \ell_{2n-2})}.$$

Thus, by proposition 2.2, it is enough to check that grade $(I_t(A)) \geq (2n-2)+2-t = 2n-t$, $1 \leq t \leq 2n-2$.

Because R[X] is Cohen–Macaulay, grade equals height. Let T be the matrix formed with the first 2n–2 columns of A. Then T is alternating and $ht(I_t(T)) \leq ht(I_t(A))$. We will show that for $t \leq 2n-4$, $ht(I_t(T)) \geq 2n-t$.

Buchsbaum and Eisenbud show [B–E, 2.6] that for T

$$I_{2r}(T) \subseteq I_{2r-1}(T) \subseteq Pf_{2r}((T)) \subseteq \sqrt{I_{2r}(T)}\,.$$

Then, by proposition 2.1 and the definition of radical of an ideal

$$ht(I_{2r}(T)) = ht(I_{2r-1}(T)) = ht(Pf_{2r}(T)) = \begin{bmatrix} 2n-2r \\ 2 \end{bmatrix}.$$

Hence, for $2r \leq 2n-4$

$$ht(I_{2r}(T)) = \frac{(2n-2r)(2n-2r-1)}{2} \geq 2n-2r.$$

Also, for $2r-1 \leq 2n-5$

$$ht(I_{2r-1}(T)) = ht(I_{2r}(T)) = (n-r)(2n-2r-1) \geq (2n-2r-1)+2 = 2n-(2r-1).$$

Thus, we have shown that if $t \leq 2n-4$, then

$$ht(I_t(A)) \geq ht(I_t(T)) \geq 2n-t.$$

Hence, we are left with the cases t = 2n–3 and t = 2n–2. For the case t = 2n–3, consider the (2n–2)×2n matrix

$$D = \begin{bmatrix} 0 & \cdot\,\cdot & x_{12} & x_{13} & x_{14} & 0 & \cdots & 0 \\ -x_{12} & \ddots & & & & & & \vdots \\ -x_{13} & & \ddots & & & & & \vdots \\ -x_{14} & & & \ddots & & & & \vdots \\ 0 & & & & \ddots & & & 0 \\ \vdots & & & & & \ddots & & x_{14} \\ 0 & 0 & -x_{14} & -x_{13} & -x_{12} & 0 & x_{12} & x_{13} \end{bmatrix}.$$

We obtain D by specializing A. Here $x_{ii+j} \longrightarrow x_{11+j}$, $1 \le i \le 3$, and $x_{ij} \longrightarrow 0$ otherwise. Therefore

$$ht(I_{2n-3}(D)) \le ht(I_{2n-3}(A)).$$

Now, $I_{2n-3}(D)$ contains

$$\det \begin{bmatrix} x_{14} & \cdots & 0 \\ & \ddots & 0 \\ * & & x_{14} \end{bmatrix} = x_{14}^{2n-3}$$

$$\det \begin{bmatrix} x_{13} & x_{14} & 0 & \cdots & 0 \\ & \ddots & & & 0 \\ * & & & \ddots & x_{14} \\ & & & & x_{13} \end{bmatrix} \equiv x_{13}^{2n-3} \bmod(x_{14})$$

and

$$\det\begin{bmatrix} x_{12} & x_{13} & x_{14} & 0 \cdots 0 & 0 \\ & \ddots & & & \\ & & \ddots & & x_{14} \\ * & & & \ddots & x_{13} \\ & & & & x_{12} \end{bmatrix} \equiv x_{12}^{2n-3} \bmod (x_{13}, x_{14}).$$

Hence $ht(I_{2n-3}(D)) \geq 3$ and therefore, $ht(I_{2n-3}(A)) \geq 3$. For $t = 2n-2$, we specialize $x_{14} \longrightarrow 0$ and let $\overline{D}$ be the matrix obtained from D in this way. Then $I_{2n-2}(\overline{D})$ contains

$$\det\begin{bmatrix} x_{13} & 0 \cdots\cdots & 0 \\ & & \\ * & & \\ & & 0 \\ & & x_{13} \end{bmatrix} = x_{13}^{2n-2}$$

and

$$\det\begin{bmatrix} x_{12} & x_{13} & 0 & \cdots & 0 \\ & & & & \\ * & & & & 0 \\ & & & & x_{13} \\ & & & & x_{12} \end{bmatrix} \equiv x_{12}^{2n-2} \bmod (x_{13})$$

Then $ht(I_{2n-2}(\overline{D})) \geq 2$, and therefore $ht(I_{2n-2}(A)) \geq 2$. It follows then that $ht(I_t(A)) \geq 2n-t$. ▮

Recall that $S = R[X, Y]$, where R is a local Gorenstein ring domain, $n \geq 2$, X is a $2n\times 2n$ generic alternating matrix over R, Y a $2n\times 1$ generic matrix over

R[X]. We have the following corollaries.

Corollary 2.4. $(\ell_1,\cdots,\ell_{2n-1})$ is a complete intersection in $S_{(m, X, Y)}$.

Proof. We first treat the case where R = D is a Cohen–Macaulay domain. By 2.3 $(\ell_1,\cdots,\ell_{2n-2})$ is a prime ideal in D[X, Y] of height 2n–2. Denote by "——" images in $D[X, Y]/(x_{i2n-1}, x_{j2n}, i \neq 2n, j \neq 2n-1)$. If $\ell_{2n-1} \in (\ell_1,\cdots,\ell_{2n-2})$, then $\overline{\ell}_{2n-1} = x_{2n-1\,2n}\, y_{2n} \in (\overline{\ell}_1,\cdots,\overline{\ell}_{2n-2}) = (\Sigma_{j=1}^{2n-2} x_{ij}y_j,\ 1 \leq i \leq 2n-2)$, which is false. Then $(\ell_1,\cdots,\ell_{2n-1})$ is a complete intersection in D[X, Y].

We now prove how the claim follows for an arbitrary local Cohen–Macaulay ring R. Let V be a prime ring of R localized at the contraction of m, let $p \in \operatorname{Spec} S_{(m,X,Y)}$ with $(\ell_1,\cdots,\ell_{2n-1}) \subset p$, and let $(\pi) = p \cap V$. We have to show that $\dim S_p \geq 2n-1$.

First consider the case where π is not regular on S_p. Then since R is Cohen–Macaulay, $\dim S_p/\pi S_p = \dim S_p$. On the other hand, $D = V/(\pi)$ is a field and $S_p/\pi S_p$ is faithfully flat over $D[X,Y]_{p\cap D[X,Y]}$. Then by what we have shown before, $2n-1 \leq \dim D[X,Y]_{p\cap D[X,Y]} \leq \dim S_p/\pi S_p = \dim S_p$.

Now consider the case where π is regular on S_p. In this case take D = V. Now D is a domain and S_p is faithfully flat over $D[X,Y]_{p\cap D[X,Y]}$. Again as above, $\dim S_p \geq 2n-1$.

Corollary 2.5. The ideal $(\ell_1,\cdots,\ell_{2n-2}, \mathrm{Pf}(X))$ is a complete intersection in $S_{(m,X,Y)}$.

Proof. It runs parallel to the proof of 2.4, after noticing that $\mathrm{Pf}(X) \notin (\ell_1,\cdots,\ell_{2n-2})$. Grade, with the notations as 2.4, D[X, Y] by assigning $\deg x_{ij} = 0$, $\deg y_j = 1$. Then Pf(X) has degree 0, but each ℓ_i has degree 1. Then we are done.

Corollary 2.4 is the key of the following lemma. In it we prove all our claims about H_n.

Lemma 2.6. H_n is a licci Gorenstein ideal of height 2n–1 and deviation 2.

Also, in S, a homogeneous resolution of H_n starts and ends like

$$0 \to S(-(4n-3)) \to \cdots \to S^{2n}(-2) \oplus S(-n) \to S.$$

Proof. By induction on n, consider first the case n = 2. H_2 can be viewed as the 4×4 pfaffians of the following 5×5 generic matrix

$$\begin{bmatrix} 0 & y_1 & y_2 & y_3 & y_4 \\ -y_1 & 0 & x_{34} & -x_{24} & x_{23} \\ -y_2 & -x_{34} & 0 & x_{14} & -x_{13} \\ -y_3 & x_{24} & -x_{14} & 0 & x_{12} \\ -y_4 & -x_{23} & x_{13} & -x_{12} & 0 \end{bmatrix}.$$

Then H_2 is Gorenstein of height 3 and deviation 2 [B–E, 2.1]. Also, [B–E, proof of 2.1] the projective dimension of S/H_2 is three, and hence it is perfect. Therefore, H_2 is licci [W]. It also follows [B–E, proof of 2.1] that if deg x_{ij} = deg y_j = 1, then a homogeneous resolution of S/H_2 starts and ends like

$$0 \to S(-5) \to \cdots \to S^5(-2) \to S$$

and notice that 5 = 4(2)–3.

Assume now that the result is true for n–1. Let X' be a 2(n–1)×2(n–1) generic alternating matrix over R, Y' a 2(n–1)×1 generic matrix over R[X']. Then in S' = R[X', Y'], $H_{n-1} = (I_1(X'Y'), Pf(X'))$ is licci, Gorenstein has deviation 2 and height 2(n–1)–2 = 2n–3. Grade S' by assigning deg x_{ij} = deg y_j = 1. A homogeneous resolution of H_{n-1} starts and ends like

$$0 \to S'(4(n-1)-3) \to \cdots \to S'^{2n-2}(-2) \oplus S'(-(n-1)) \to S'.$$

Write $[\ell'_1,\cdots,\ell'_{2n-2}]^t = X'Y'$. If X(resp Y) is a $2n\times 2n$ generic matrix over R (resp $2n\times 1$ generic matrix over R[X]), then we can view S' as a subring of S = R[X, Y], X'(resp. Y') obtained from X(resp. Y) by deleting the last 2 rows and columns (resp. last 2 columns). Then $H_{(n-1)}$ can be viewed as a licci ideal of S, with same height, deviation and resolution

$$0 \to S(-(4(n-1)-3)) \to \cdots \to S^{2n-2}(-2) \oplus S(-(n-1)) \to S.$$

Then $Pf(X') = X_{2n-1\ 2n}$. Write $[\ell_1,\cdots,\ell_{2n}]^t = XY$. Consider in S, the ideal $(H_{(n-1)}, y_{2n-1}, y_{2n}) = (\ell_1,\cdots,\ell_{2n-2}, X_{2n-1\ 2n}, y_{2n-1}, y_{2n})$. A homogeneous resolution of this ideal is

$$0 \to S(-(4n-5)) \to \cdots \to S^2(-1) \oplus S^{2n-2}(-2) \oplus S(-(n-1)) \to S.$$

Consider the ideal $(\ell_1,\cdots,\ell_{2n-2}, y_{2n})$, which, by (2.3), is a complete intersection. The Koszul complex of $\ell_1,\cdots,\ell_{2n-2}, y_{2n}$ is

$$0 \to S(-(4n-3)) \to \cdots \to S^{2n-2}(-2) \oplus S(-1) \to S$$

and then a resolution of $L_n = (\ell_1,\cdots,\ell_{2n-2}, y_{2n}) : (H_{(n-1)}, y_{2n-1}, y_{2n})$ starts and ends like

$$0 \longrightarrow S(-(3n-2)) \oplus S(-(4n-4)) \longrightarrow \cdots \longrightarrow S(-1) \oplus S^{2n-1}(-2) \longrightarrow S.$$

Then L_n is generated by $(\ell_1,\cdots,\ell_{2n-2}, y_{2n-1})$ and an element of degree 2. Consider ℓ_{2n-1}. Because

$$\ell_{2n-1}\, y_{2n-1} = -\Sigma_{i=1}^{2n-2}\, \ell_i y_i - \ell_{2n} y_{2n},$$

then $\ell_{2n-1} y_{2n-1} \in (\ell_1, \cdots, \ell_{2n-2}, y_{2n})\; \ell_{2n-1} X_{2n-1,2n}$ in $(\ell_1, \cdots, \ell_{2n-2}, y_{2n})$. Also, because $Pf(X)y_{2n} = \sum_{i=1}^{2n-1} X_{i2n}\, \ell_i$ then

$$X_{2n-1\; 2n}\, \ell_{2n-1} = Pf(X)y_{2n} - \sum_{i=1}^{2n-2} X_{i2n}\, \ell \in (\ell_1 - \ell_{2n-2}, y_{2n}).$$

Hence $\ell_{2n-1} \in L_n$. A similar argument to the one used in 2.4, shows that $\ell_{2n-1} \notin (\ell_1, \cdots, \ell_{2n-2}, y_{2n})$. Then $(\ell_1, \cdots, \ell_{2n-1}, y_{2n}) \subseteq L_n$ and $L_n/(\ell_1, \cdots, \ell_{2n-2}, y_{2n})$ is minimally generated by one element of degree 2. Then $(\ell_1, \cdots, \ell_{2n-1}, y_{2n}) = L_n$. Hence in $S_{(m,X,Y)}$, L_n and $(H_{n-1}, y_{2n-1}, y_{2n})$ are linked, $L_n \longrightarrow (H_{n-1}, y_{2n-1}, y_{2n})$ and $(H_{(n-1)}, y_{2n-n}, y_{2n}) \longrightarrow L_n$. Then $r(S/L_n) = 2$, $d(L_n) = 1$, by (1.6).

Consider now the complete intersection $(\ell_1, \cdots, \ell_{2n-1})$ (see 2.4). Its homogeneous Koszul complex is

$$0 \longrightarrow S(-(4n-2)) \longrightarrow \cdots \longrightarrow S^{2n-1}(-2) \longrightarrow S.$$

Then a resolution of $(\ell_1, \cdots, \ell_{2n-1})\colon L_n$ starts and ends like

$$0 \longrightarrow S(-(4n-3)) \longrightarrow \cdots \longrightarrow S^{2n}(-2) \oplus S(-n) \longrightarrow S.$$

Hence $(\ell_1, \cdots, \ell_{2n-1})\colon L_n$ is generated by $(\ell_1, \cdots, \ell_{2n-1})$, one element of degree 2 and one element of degree n. Consider ℓ_{2n} and Pf(X). (3) and (4) in chapter (I) show that $(\ell_1, \cdots, \ell_{2n}, Pf(X)) = H_n \subsetneq (\ell_1, \cdots, \ell_{2n-1})\colon L_n$. If $\ell_{2n} \in (\ell_1, \cdots, \ell_{2n-1})$, then $\ell_{2n} \equiv 0$ (mod(x_{ij}, $i \leq 2n-1$, $j \leq 2n-1$, y_{2n}) which is not the case. As in 2.5, $Pf(X) \notin (\ell_1,\cdots,\ell_{2n})$. Then $H_n \subseteq (\ell_1,\cdots,\ell_{2n-1}): L_n$ and $(\ell_1,\cdots,\ell_{2n-1}): L_n/(\ell_1,\cdots,\ell_{2n-1})$ is minimally generated by one element of degree 2 and one element of degree n. Then $H_n = (\ell_1,\cdots,\ell_{2n-1}): L_n$. Hence in $S_{(m,X,Y)}$, $L_n \to H_n$ and $H_n \to L_n$. Then $d(H_n) = 2$, $r(S/H_n) = 1$, ht $H_n = 2n-1$. The last twist of a homogeneous resolution of H_n is 4n–3, and H_n double linked to $(H_{n-1}, y_{2n-1}, y_{2n})$, then H_n is licci. Now the result follows. ∎

In some unpublished notes, Andrew Kustin obtains H_n from (H_{n-1}, y, z), y, z regular modulo H_{n-1}, by what Kustin and Miller call semi generic tight double link [K–M, 3.1]. In those notes, he proves, by induction, that H_n is licci, Gorenstein, has height 2n–1 and deviation 2. But moreover, he shows that in k[[X, Y]], k a field, H_n defines a rigid algebra (see also [K]). We will use this fact several times.

Corollary 2.5 gives the following lemma.

Lemma 2.7. The ideal $J_n = (\ell_1,\cdots,\ell_{2n-2}, X_{2n-1\ 2n}, Pf(X))$ is a licci ideal of type 2, deviation 1 and height 2n–1 in $S_{(m,X,Y)}$.

Proof. We will show $H_n \to J_n$ via the regular sequence

$$\underline{\beta} = \ell_1,\cdots,\ell_{2n-2}, Pf(X).$$

Grade S by assigning deg x_{ij} = deg y_j = 1. A homogeneous free resolution of H_n (see 2.6) is

$$0 \to S(-(4n-3)) \to \cdots \to S^{2n}(-2) \oplus S(-n) \to S$$

while the Koszul complex of $\underline{\beta}$ is

$$0 \to S(-(5n-4)) \to \cdots \to S^{2n-2}(-2) \oplus S(-n) \to S.$$

Then a resolution of (β): H_n starts and ends like

$$0 \to S^2(-(5n-6)) \to \cdots \to S^{2n-2}(-2) \oplus S(-(n-1)) \oplus S(-n) \to S.$$

Therefore, (β): H_n is generated by (β) and an element of degree n−1. Consider $X_{2n-1\ 2n}$, which has the appropriate degree. By (3) in Chapter (I), we see that

$$X_{2n-1\ 2n}\, \ell_{2n-1} = \mathrm{Pf}(X)y_{2n} - \Sigma_{i=1}^{2n-2} X_{i2n}\, \ell_i \in (\beta)$$

and

$$X_{2n-1\ 2n}\, \ell_{2n} = \Sigma_{i=1}^{2n-2} X_{i2n-1}\, \ell_i - \mathrm{Pf}(X)y_{2n-1} \in (\beta).$$

It follows then that $J_n \subseteq (\beta)$: H_n.

Suppose $X_{2n-1\ 2n} \in (\beta)$. Then, there would be homogeneous elements $a_1, \cdots, a_{2n-2}$, $b \in S$ such that

$$X_{2n-1\ 2n} = a_1 \ell_1 + \cdots + a_{2n-2}\ell_{2n-2} + b\mathrm{Pf}(X).$$

Because $n = \deg \mathrm{Pf}(X) > \deg X_{2n-1\ 2n} = n-1$, $b = 0$.

If we assign now $\deg x_{ij} = 0$, $\deg y_j = 1$, then $X_{2n-1\ 2n}$ would have degree 0, but the ℓ_i's would have degree 1. This is impossible. Therefore $X_{2n-1\ 2n} \notin (\beta)$.

Hence $J_n/(\beta)$ is minimally generated by $X_{2n-1\ 2n}$. Also (β):$H_n/(\beta)$ is generated by an element of degree n−1. It follows then $J_n = (\beta)$: H_n.

Then, in $S_{(m,X,Y)}$, $J_n = (\beta)$: H_n, and by 1.4(a), J_n and H_n are linked. Moreover $H_n \to J_n$ and hence J_n has type 2, by 1.6(b). Thus, J_n is not a complete intersection but by 1.6(a), $d(J_n) \leq 1$. This implies $d(J_n) = 1$. J_n is also licci because so is H_n. Now we are done. ▮

Lemma 2.8. The ideal $(\mathcal{I}) = (\ell_1,\cdots,\ell_{2n-3}, X_{2n-1\ 2n}, \mathrm{Pf}(X))$, $n \geq 2$ is a complete intersection in $S_{(m,X,Y)}$.

Proof. We use induction on n. For n = 2

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} \\ -x_{12} & 0 & x_{23} & x_{24} \\ -x_{13} & -x_{23} & 0 & x_{34} \\ -x_{14} & -x_{24} & -x_{34} & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

and $(\mathcal{I})$ is $(x_{12}y_2 + x_{13}y_3 + x_{14}y_4, x_{12}, x_{12}x_{34} + x_{14}x_{23} - x_{13}x_{24})$, which has height 3.

Consider now the case n = 3, and let p be a prime ideal containing $(\mathcal{I})$ such that ht $p \leq 5$. If p contains x_{12}, x_{13} and x_{23}, then p also contains

$$\Sigma_{j=4}^{6} x_{ij}y_j, \ 1 \leq i \leq 3$$

and hence ht $p \geq 6$. We may then assume w.l.o.g, that $x_{12} \notin p$.

Then, there is a matrix A invertible over S_p such that

$$A^tXA = \left[\begin{array}{cc|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & & 0 \\ \hline & 0 & X' \end{array}\right]$$

and the entries of X' are generic over $R[x_{12}^{-1}, \{x_{ij}, i \leq 2\}]$. If Y is replaced by $A^{-1}Y$, then in S_p, $(\mathcal{I})$ becomes $(y_1, y_2, \ell_1', x_{12}', \mathrm{Pf}(X'))$, which has height 3+2 = 5.

Assume the result is true for n–1, $n \geq 4$. Let p be a prime ideal of $S_{(m,X,Y)}$ such that $(\gamma) \subset p$ and ht $p \leq 2n-1$. Then p does not contain the ideal I generated by the entries of the $(2n-4)\times(2n-4)$ matrix obtained from X, by deleting the last 4 rows and columns. In efect, for n = 4, if $I \subset p$, then p contains

$$(\Sigma_{j=5}^{8} x_{ij}y_j \mid 1 \leq i \leq 4),$$

which is a complete intersection [Ho], and the x_{ij}, $1 \leq i,j \leq 4$ are regular modulo this ideal. Then $ht(p) \geq 6+4$, which is impossible. For $n \geq 5$, ht $(p) \leq 2n-1 < (n-2)(2n-5) = $ ht I.

We may assume then, w.l.o.g., that $x_{12} \not\in p$. Then, there is a matrix A, invertible over S_p, such that

$$A^tXA = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{0} & X' \end{array}\right]$$

and the entries of X' are regular over $R[\{x_{ij} \mid i \leq 2\}, x_{12}^{-1}]$. If $Y' = A^{-1}Y$, and Y' is obtained from Y" by deleting the first 2 rows, let $[\ell_1' \cdots \ell_{2n-2}']^t = X'Y'$. Then (γ) becomes

$$(y_1, y_2, \ell_1', \ldots, \ell_{2n-5}', Pf(X'))$$

which, by induction hypothesis, has height $(2n-3) + 2 = 2n-1$. Then ht $p = 2n-1 = ht(\gamma)$ and γ is a regular sequence in $S_{(m,X,Y)}$. ∎

<u>Theorem 2.9</u>. In $S_{(m,X,Y)}$, for $n \geq 3$, the ideal

$$K_n = (\ell_1, \ldots, \ell_{2n-3}, x_{2n-2\ 2n}, x_{2n-2\ 2n-1}, x_{2n-1\ 2n}, Pf(X))$$

is a licci Gorenstein ideal of height 2n–1 and deviation 2.

Proof. We first show that $J_n \longrightarrow K_n$, where J_n is the ideal of 2.7, via the regular sequence (γ) of 2.8.

Grade S as usual. A resolution of J_n is (proof of 2.8)

$$0 \longrightarrow S^2(-(5n-6)) \longrightarrow \cdots \longrightarrow S^{2n-2}(-2) \oplus S(-(n-1)) \oplus S(-n) \longrightarrow S,$$

while the Koszul complex of γ is

$$0 \longrightarrow S(-(6n-7)) \longrightarrow \cdots \longrightarrow S^{2n-3}(-2) \oplus S(-(n-1)) \oplus S(-n) \longrightarrow S.$$

Then a resolution for $(\gamma)\colon J_n$ starts as

$$S^{2n-3}(-2) \oplus S^3(-(n-1)) \oplus S(-n) \longrightarrow S.$$

Then $(\gamma)\colon J_n$ is generated by γ and 2 elements of degree n–1.

Consider $X_{2n-2\;2n}$ and $X_{2n-2\;2n-1}$, which have degree n–1. By (3) in chapter (I) we have

$$\ell_{2n-2}X_{2n-2\;2n} = fy_{2n} - \Sigma_{i=1}^{2n-3} X_{i2n}\ell_i - X_{2n-1\;2n}\ell_{2n-1} \in (\gamma)$$

and

$$\ell_{2n-2}X_{2n-2\;2n-1} = fy_{2n-1} - \Sigma_{i=1}^{2n-3} X_{i2n-1}\ell_i + X_{2n-1\;2n}\ell_{2n-1} \in (\gamma).$$

Therefore $K_n \subseteq (\gamma)\colon J_n$. If $X_{2n-2\;2n} \in (\gamma)$, then for homogeneous elements $a_1, \cdots, a_{2n-3}$, b, c in S, we have

$$X_{2n-2\ 2n} = a_1\ell_1+\cdots+a_{2n-3}\ell_{2n-3} + bX_{2n-1\ 2n} + cPf(X),$$

and, by degree reasons, $b \in R$, $c = 0$. Then

(1) $$X_{2n-2\ 2n}-bX_{2n-1\ 2n} = a_1\ell_1+\cdots+a_{2n-3}\ell_{2n-3}.$$

If $\deg x_{ij} = 0$ and $\deg y_j = 1$, then (1) holds if and only if $a_1 = \cdots = a_{2n-3} = 0$. Then

(2) $$X_{2n-2\ 2n} = b\, X_{2n-1\ 2n}.$$

If we specialize, by making $x_{i\ 2n-2} = 0$, the left hand side of (2) is nonzero, but the right hand side is. Similarly, $X_{2n-2\ 2n-1} \in (\mathfrak{I}, X_{2n-2\ 2n})$ if and only if

(3) $$X_{2n-2\ 2n-1} = bX_{2n-2\ 2n} + c\, X_{2n-1\ 2n},\ b, c \in R.$$

If we specialize by assigning $x_{i2n-2} = x_{i2n-1} = 0$ the right hand side of (3) is zero, but the left hand side is different from zero.

If we assign $\deg x_{ij} = 1$, $\deg y_j = 0$, then, say, ℓ_1 is in

$$(\ell_1,\cdots,\ell_{2n-3}, X_{2n-1\ 2n}, X_{2n-2\ 2n}, X_{2n-2\ 2n-1}, Pf(X))$$

if and only if $\ell_1 \in (\ell_2,\cdots,\ell_{2n-3})$, which is contrary to 2.3.

If $\deg x_{ij} = 0$, $\deg y_j = 1$, then $Pf(X)$ is in

$$(\ell_1,\cdots,\ell_{2n-3}, X_{2n-2\ 2n}, X_{2n-1\ 2n}, X_{2n-2\ 2n-1})$$

if and only if

$$\mathrm{Pf}(X) \in (X_{2n-2\ 2n}, X_{2n-1\ 2n}, X_{2n-2\ 2n-1}).$$

Specialize by assigning $x_{ij} \longrightarrow 0$ $i, j \leq n$ and $x_{ij} \longrightarrow x_{ij}$ otherwise. Then X becomes

$$\overline{X} = \left[\begin{array}{c|c} 0_{n\times n} & A \\ \hline -A^t & B \end{array}\right]$$

where A is a generic n×n matrix. Then $\det(\overline{X}) = (\det A)^2 \neq 0$ and hence, $\mathrm{Pf}(\overline{X}) \neq 0$.

However, for $i, j \in \{2n-2, 2n-1, 2n\}$ (note $n \geq 3$), if we delete the i–th rows and j–th columns, we get

$$\left[\begin{array}{c|c} 0_{n\times n} & * \\ \hline * & D \end{array}\right]$$

where D is a$(n-2)\times(n-2)$ matrix. So we get that this matrix has determinant 0. In particular

$$\overline{X}_{2n-1\ 2n} = \overline{X}_{2n-2\ 2n} = \overline{X}_{2n-2\ 2n-1} = 0.$$

Then K_n is minimally generated by $(\mathfrak{X}, X_{2n-2\ 2n}, X_{2n-2\ 2n-1})$ and it is contained in $(\mathfrak{X})\colon J_n$. Also $(\mathfrak{X})\colon J_n/(\mathfrak{X})$ is generated by 2 elements of degree n–1. Then $K_n = (\mathfrak{X})\colon J_n$.

Then, in $S_{(m,X,Y)}$, $K_n \longrightarrow J_n$ and $J_n \longrightarrow K_n$. By 1.6(b), then

$$d(K_n) = r(S_{(m,X,Y)}/J_n) = 2$$

and

$$r(S_{(m,X,Y)}/K_n) = d(J_n) = 1$$

as claimed. Also, since J_n is licci, so is K_n. ∎

Theorem 2.9 presents a family of Gorenstein ideals of deviation 2 and any odd height larger or equal to 5. K_3 and H_3 have height 5, and if deg x_{ij} = deg y_j = 1, they are generated by 6 elements of degree 2 and one element of degree 3 (later we will see they have a common specialization).

We turn our attention to the case n = 4.

In this case set k = Q, S = k[X, Y], X an 8×8 generic alternating matrix over k, Y a 8 × 1 generic matrix over k[X].

Then, the minimal free resolution of H_4 [K, 5.5 and 6.1] starts as

$$\cdots S^{99} \longrightarrow S^{37} \longrightarrow S^{9} \longrightarrow S$$

and in particular the second Betti number of H_4 is 37.

Several applications of the mapping cone produce a (not necessarily minimal) resolution of K_4 that starts as

$$\cdots \longrightarrow S^{d} \longrightarrow S^{9} \longrightarrow S$$

and $d \geq 44$. That means then that H_4 and K_4 are in different Herzog classes.

The minimal resolution for H_3 is

$$0 \longrightarrow S \longrightarrow S^7 \longrightarrow S^{22} \longrightarrow S^{22} \longrightarrow S^7 \longrightarrow S.$$

If x, y are regular on S/H_3, their Koszul complex is

$$0 \longrightarrow S \longrightarrow S^2 \longrightarrow S$$

and hence a resolution for (H_3, x, y) starts as

$$\cdots S^{37} \longrightarrow S^9 \longrightarrow S$$

and hence K_4 is not in the same Herzog class as (H_3, x, y). The minimal resolution for H_2 is

$$0 \longrightarrow S \longrightarrow S^5 \longrightarrow S^5 \longrightarrow S.$$

If x, y, z, w are regular on S/H_2, their Koszul complex is

$$0 \longrightarrow S \longrightarrow S^4 \longrightarrow S^6 \longrightarrow S^4 \longrightarrow S$$

and the resolution for (H_2, x, y, z, w) starts as

$$\cdots \longrightarrow S^{31} \longrightarrow S^9 \longrightarrow S.$$

We see then that (H_2, x, y, z, w) and K_4 are in different Herzog classes.

So, it is still left to determine if for some $n > 4$, $(K_4, t_1,\cdots,t_r)$ and $(H_n,x_1,\cdots,x_e)$ are in the same Herzog class, where $t_1,\cdots,t_r$ form a regular sequence modulo K_4 and $x_1,\cdots,x_e$ form a regular sequence modulo H_n. The answer is in the following sequence of lemmas.

Lemma 2.10. Let (R, m) be a local Gorenstein ring, and let I be a hypersurface section in R. Then, any specialization of I is a hypersurface section.

Proof. Write I = (K, y), y regular on R/K. If $\underline{\alpha}$ is a regular sequence on R and on R/I, then the specialization via $\underline{\alpha}$ is $(R/(\underline{\alpha}), (I,(\underline{\alpha}))/(\underline{\alpha}))$. Now

$$\frac{(I,\ \underline{\alpha})}{(\underline{\alpha})} = \frac{(K,y,\underline{\alpha})}{(\underline{\alpha})}$$

and y, $\underline{\alpha}$ are regular on R/K. Because R is local, $\underline{\alpha}$, y are regular on R/K and hence

$$\frac{(K,y,\underline{\alpha})}{(\underline{\alpha})} = \frac{(K,\underline{\alpha},y)}{(\underline{\alpha})}$$

with y regular on $R/(K, \underline{\alpha})$. ∎

Lemma 2.11. Let (R, m) and I as in 2.10. Assume R/m is infinite. If I is licci, so is K.

Proof. Write I = (K, y), y regular on R/K. By 1.10, there is an n such that $L^n(I) = L^n(K, y) = (1)$. Because $y \in m$, it is enough to show $L^n(K, y) = (L^n(K),y)$ since $(L^n(K), y) = (1)$ if and only if $L^n(K) = (1)$, and by induction on n, we only need to consider the case n = 1.

Let g = grade I, $\underline{f} = f_1,\cdots,f_r$ generators of K. Then $(f_1,\cdots,f_r, y) = I$. Let X be a g×(r+1) generic matrix over R. Then in $R(X) = R[X]_{mR[X]}$, x_{gr+1} is invertible, and after row and column operations X becomes

$$\left[\begin{array}{c|c} X' & 0 \\ \hline 0 & 1 \end{array}\right]$$

with X' generic over $R[\{x_{ir+1}, x_{gj}, 1 \le i \le g, 1 \le j \le r+1\}, x_{rg+1}^{-1}]$. Then y is an element of the regular sequence $\underline{\alpha}$ given by

$$[\underline{\alpha}]^t = \left[\begin{array}{c|c} X' & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} \underline{f} \\ y \end{array}\right]$$

and then $L^1(K, y) = (L^1(K), y)$. ∎

Lemma 2.12. Let k be a field, $S = k[[T_1, \cdots, T_r]]$ and let I be an ideal of S such that S/I is rigid. If I is a hypersurface section, then $I \not\subseteq m_S^2$.

Proof. Write I = (K, y), y regular on S/K. Let x be an indeterminate over S. Then (S[[x]], (K, x)) is a deformation of (S, (K,y)) via the regular sequence x–y. Because (S, I) is rigid, for some of indeterminate z, (S[[x]], (K, x)) is isomorphic to (S[[z]], (K, y) S[[z]]). If $I = (K, y) \subseteq (m_{S[[z]]})^2$, then $(K,x) \subseteq (m_{s[[x]]})^2$ and hence $x \in (m_{s[[x]]})^2$, which is impossible.

The following proposition is important by itself. It gives a criterion for a licci ideal to be a hypersurface section.

Proposition 2.13. Let k be an infinite field and S a formal power series ring over k. Let I be a licci ideal of S, and let (T, J), $T = k[[\underline{x}]]$, be in the same Herzog class as (S, I). Then I is a hypersurface section if and only if J is.

Proof. Assume I is a hypersurface section, I = (K, y), y regular on S/K. Because I is licci, then by lemma 2.11, K is licci. By proposition 1.21, there is a deformation $(\tilde{S}, \tilde{K})$ of (S, K) which is reduced. Also, by 1.13, $\tilde{K}$ is licci and hence strongly nonobstructed by 1.20.

By the discussion following definition 1.19, there is a pair $(\approx{S}, \approx{K})$, $\approx{S}$ a power series ring over k, such that $(\approx{S}, \approx{K})$ is rigid, reduced, and for some indeterminate Y, $(\approx{S}[[Y]], (\approx{K}, Y))$ is a rigid deformation of (S, I). Let (T, J) be a pair in the same Herzog class of (S, I). Then J is licci [U–2], and hence (T, J) has a deformation

$(\tilde{T}, \tilde{J})$ with $\tilde{T}/\tilde{J}$ reduced. By the same discussion following definition 1.19, $(\tilde{T}, \tilde{J})$ is a specialization of

$(\tilde{\tilde{S}}[[Y, \underline{w}]], (\tilde{\tilde{K}}, Y)\,\tilde{\tilde{S}}[[Y, \underline{w}]])$ for a finite set of indeterminates $\underline{w}$. Then $\tilde{J}$, and hence J, is a hypersurface section, by 2.10. ∎

Another criterion for an ideal I being a hypersurface section is given by Herzog and Miller [H–M, 1.3]. They show that if k is an algebraic closed field and I is a Gorenstein ideal of deviation 2 of $R = k[[x_1,...,x_e]]$ such that I is generically a complete intersection and I/I^2 is a R/I – Cohen–Macaulay module, then $I = (J, t_1,...,t_r)$, $t_1,...,t_r$ regular on R/J if and only if $\mu(I) - \mu(H_1(I)) \geq r$, where $H_1(I)$ is the first Koszul homology module of I.

We come back now to our question prior to lemma 2.10, and assume the answer is yes. Then repeated applications of proposition 2.13 would imply that H_n is in the same Herzog class as $(K_4,t_1,\cdots,t_r)$ or K_4 is in the same Herzog class as $(H_n, x_1,\cdots,x_e)$ for n = 2, 3 or 4. But we saw this is impossible.

We summarize our findings in the next result.

Theorem 2.14. $(S, (K_4, t_1,\cdots,t_r))$ $(t_1,\cdots,t_r$ a regular sequence on $S/K_4)$ is not in the same Herzog class as $(T, (H_n, x_1,\cdots,x_e))$, $(x_1,\cdots,x_e$ regular sequence on $T/H_n)$ for any $n \geq 2$.

We now want to study the case n = 3 more closely. In this case ht K_3 = ht $H_3 = 5$.

We begin with the following lemma due to Brodman.

Lemma 2.15. [Br, 3. Satz] Let R be a local noetherian ring with infinite residue class field, and let I be an ideal of R of grade $g > 0$ which is generically a complete intersection. Then there are $x_1,\cdots,x_g$ in I such that

i) $x_1,\cdots,x_g$ form a regular sequence.

ii) $(x_1,\cdots,x_g)_p = I_p$ for all minimal primes p of R/I.

iii) $\{x_1, \cdots, x_g\}$ is part of a minimal set of generators of I.

We now recall our notation: (R, m) is always a local Gorenstein ring. We will assume also that 2 is a unit in R, and that R/m is infinite.

<u>Proposition 2.16</u>. Let I be a perfect almost complete intersection ideal of R of height 4 and type 2. Further assume (R, I) has a deformation which is generically a complete intersection. Then there is a pair (S, J) in the same Herzog class as (R, I) such that J is licci and either:

i) $J = I_1(AB)$, A a 5×5 alternating matrix, B a 5×1 matrix, a_{ij}, $b_j \in m_S$, or

ii) J is a hypersurface section.

<u>Proof</u>. We can assume, after deforming, that I is generically a complete intersection.

Let K be a minimal geometric link of I, which exists by 2.15. Then K is perfect, Gorenstein, generically a complete intersection (1.3) and $d(K) \leq 2$.

If $d(K) = 0$, K is a complete intersection, say, $K = (a_1, a_2, a_3, a_4)$. If x_1, x_2, x_3, x_4 are indeterminates over R, then

$$(R', K') = (R[x_1,x_2,x_3\ x_4]_{(x_1,x_2,x_3,x_4)}, (x_1,x_2,x_3,x_4))$$

is a deformation of (R, K) via the regular sequence $\beta = \{x_i - a_i \mid 1 \leq i \leq 4\}$. This sequence is regular on R'/K' if and only if

$$(x_1, x_2, x_3, x_4, x_1-a_1, x_2-a_2, x_3-a_3, x_4-a_4)$$

is a complete intersection in R'. But this later ideal is

$$(x_1, x_2, x_3, x_4, a_1, a_2, a_3, a_4)$$

which is a complete intersection in R'. Notice also that $R'/(\underline{\beta}) \cong R$ and $K'+(\underline{\beta})/(\underline{\beta}) \cong K$.

Let $K' \underset{(\underline{\gamma})}{\sim} J'$ where (R', J') is a deformation of (R, J) (1.13). Since

$$r(R'/J') = 2 = \mu(K'/(\underline{\gamma})),$$

two elements which are part of a minimal set of generators of K', say x_1 and x_2 are in $(\underline{\gamma}) \subsetneq J'$. Then J' is a hypersurface section.

Now assume K is not a complete intersection. Because K is Gorenstein and perfect, then, [Ku], $d(K) \neq 1$.

If $d(K) = 2$, then $K \longrightarrow I$ (1.6(b)). Because K is generically a complete intersection, then K is a hypersurface section [V–V, 1.1]. Namely, K = (L, t), L a Gorenstein ideal of height 3 and deviation 2 and t regular on R/L. It follows then I is licci.

Thus, there is a 5×5 alternating matrix $D = [d_{ij}]$ with entries in m such that L is generated by the 4×4 pfaffians of D, [B–E, 2.1], i.e. $L = (D_i, 1 \leq i \leq 5)$ where D_i are the pfaffians (with signs) of D.

We now compute all possible minimal links of K. Let $(\underline{\alpha}) = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the regular sequence defining $K \longrightarrow I$. Then, there is a 4×6 matrix T with entries in R such that

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = T \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ t \end{bmatrix} .$$

Because the link is minimal, a maximal minor of T is invertible. Thus, if "——" denotes the reduction modulo m, $\overline{T}$ has rank 4 over $\overline{R}$. Therefore, the matrix formed with the first 5 columns of $\overline{T}$ has rank at least 3 over $\overline{R}$. Then, after performing row operations, one can assume that

$$T = \left[\begin{array}{c|ccc} & s_1 & q_1 & r_1 \\ I_3 & s_2 & q_2 & r_2 \\ & s_3 & q_3 & r_3 \\ \hline 0_{1\times 3} & s_4 & q_4 & r_4 \end{array}\right]$$

and because the link is minimal, one of the element of the last row is a unit.

Let

$$M = \begin{bmatrix} 1 & 0 & 0 & -s_1 & q_1 \\ 0 & 1 & 0 & -s_2 & q_2 \\ 0 & 0 & 1 & -s_3 & q_3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

then M is invertible over R, M^tDM is alternating and for $1 \le i \le 3$

$$D_i' = (M^tDM)_i = D_i - s_iD_4 + q_iD_5.$$

Hence, we can perform column operations in the first 5 columns of T and after doing so, T becomes

$$\left[\begin{array}{ccc|ccc} & I_3 & & 0 & 0 & r_1 \\ & & & 0 & 0 & r_2 \\ & & & 0 & 0 & r_3 \\ \hline 0 & 0 & 0 & s_4 & q_4 & r_4 \end{array}\right]$$

we now deform (R, K) to (S, P), where $S = R[Z, w]_{(m,Z,w)}$, where $Z = [z_{ij}]$ is a 5×5 generic matrix over R, w an indeterminate over R[Z] and $P = (Z_1, Z_2, Z_3, Z_4, Z_5, w)$. We deform here using the regular sequence $z_{ij}-d_{ij}$, $w-t$, $1 \le i, j \le 5$.

Let now $J = (\underline{\alpha}):P$. Then (S, J) is a deformation of (R, I) (1.13).

If r_4 is a unit (or if one of the r_i's is a unit) then $w+S_4Z_4+q_4Z_5 \in (\underline{\alpha}) \subsetneq J$ (or $w+Z_i \in (\underline{\alpha}) \subsetneq J$) and hence J is a hypersurface section.

If none of the r_i's is a unit, then we can assume s_4 is a unit (the case q_4 a unit is symmetric). We deform then r_1, r_2, r_3 and r_4 to undeterminates over S. By (1.13) the link also deforms. Now specialize r_1, r_2, r_3 to zero. Then, after column operations T becomes

$$\left[\begin{array}{c|c} I_3 & 0_{3\times 3} \\ \hline 0_{1\times 3} & 1 \ 0 \ r \end{array}\right]$$

and $\underline{\alpha} = Z_1, Z_2, Z_3, Z_4 + rw$, which is a regular sequence, because r and w are regular modulo (Z_1, Z_2, Z_3). Then (by 1.12) the linkage also specializes. Our ambient ring is now $S = R[Z, w, r]$. Grade S by assigning degree 1 to all indeterminates.

The homogeneous minimal resolution of P is [B–E]

$$0 \longrightarrow S(-6) \longrightarrow \cdots \longrightarrow S^5(-2) \oplus S(-1) \longrightarrow S$$

while the Koszul complex of $\underline{\alpha}$ is

$$0 \longrightarrow S(-8) \longrightarrow \cdots \longrightarrow S^4(-2) \longrightarrow S$$

and hence a homogeneous resolution of $(\underline{\alpha})$:P starts as $\cdots \longrightarrow S^5(-2) \longrightarrow (S)$ and hence $(\underline{\alpha}):P/(\underline{\alpha})$ is generated by an element of degree 2.

Consider z_{45}. Because $z_{45}Z_4$ and $z_{45}Z_5$ are in (Z_1, Z_2, Z_3), then

$$z_{45}rw = z_{45}(Z_4+rw) - z_{45}Z_4 \in (\underline{\alpha})$$

and

$$z_{45}rZ_5 = r(z_{45}Z_5) \in (\underline{\alpha})$$

and it follows that $(\underline{\alpha}, z_{45}r) \subsetneq (\underline{\alpha}):P$.

If for homogeneous elements a_1, a_2, a_3, a_4 in R[Z, w, r]

$$z_{45}r = a_1Z_1 + a_2Z_2 + a_3Z_3 + a_4(Z_4+rw)$$

then by degree reasons, $a_i \in R$, $1 \leq i \leq 4$. But this is impossible.

Then $J = (\underline{\alpha}):P = (Z_1, Z_2, Z_3, Z_4 + rw, z_{45}r)$, $(R[Z, r,w]_{(m,Z,r,w)}, J)$ is a deformation of (R, I). Moreover $J = (I_1(AB))$, where

$$A = \begin{bmatrix} 0 & -z_{14} & z_{34} & 0 & z_{13} \\ z_{14} & 0 & z_{24} & 0 & z_{12} \\ -z_{34} & -z_{24} & 0 & 0 & z_{23} \\ 0 & 0 & 0 & 0 & -r \\ -z_{13} & -z_{12} & -z_{23} & r & 0 \end{bmatrix} \quad B = \begin{bmatrix} -z_{25} \\ z_{35} \\ z_{15} \\ w \\ z_{45} \end{bmatrix}$$

as claimed. ∎

Notice that if J is a hypersurface section, almost complete intersection of height 4 and type 2, then J = (P, t) where P is an almost complete intersection of height 3 and type 2, which are described in [B–E].

In the following sequence of lemmas, all ideals J and K with $H_3 \to J \to K$ or $(H_2, y, z) \to J \to K$, y, z regular modulo H_2, are computed, up to deformations and specializations. J turns out to be an almost complete intersection of type 2, while K is a perfect Gorenstein ideal of deviation at most 2. We will observe that K is either a hypersurface section, or it is obtained from H_3 by deformations and specializations. Figure 1 summarizes our findings. Arrows mean minimal links and the numbers associated to the arrows indicate the number of the lemma in which the link is computed, h.s.s. means hypersurface section, while s.h.h.s (d.h.s.s.) means single hypersurface section (double hypersurface secton). The other notations are standard.

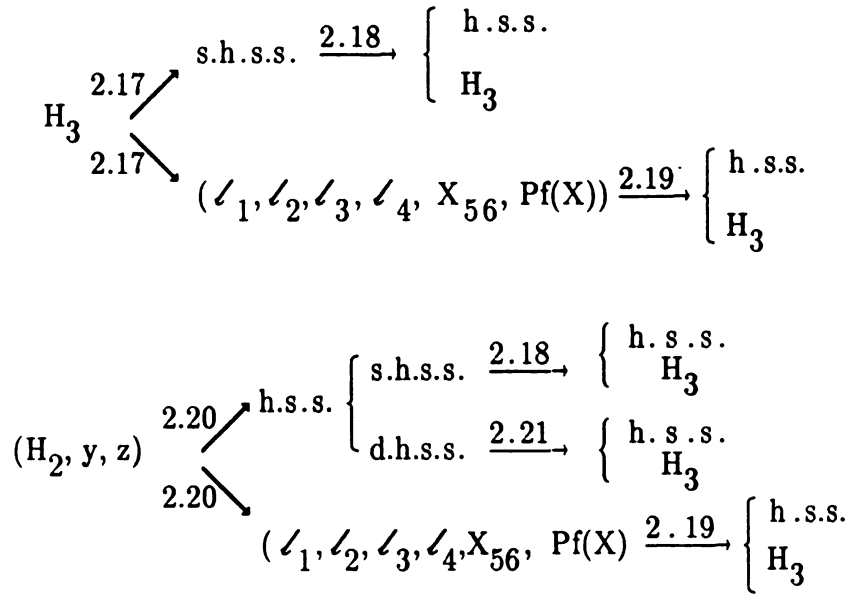

Figure 1

<u>Lemma 2.17</u>. Let X be a 6×6 generic alternating matrix over R, Y a 6×1 generic matrix over R[X]. In $S = R[X, Y]_{(m,X,Y)}$ consider the ideal $H = H_3$ generated by $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6$ and Pf(X), where $[\ell_1\ \ell_2\ \ell_3\ \ell_4\ \ell_5\ \ell_6]^t = XY$.

Let J be an ideal of S such that $H \longrightarrow J$. Then there is a pair (S', J') in the same Herzog class as (S, J) such that either

i) J' is a hypersurface section or

ii) There is a 6×6 alternating matrix A and a 6×1 matrix B, $a_{ij}, b_j \in m_{S'}$ such that if $[t_1 \cdots t_6]^t = AB$, then $J' = (t_1, t_2, t_3, t_4, A_{56}, Pf(A))$.

<u>Proof</u>. Because H is Gorenstein and has deviation 2, $r(R/J) = 2$ and $d(J) = 1$. Consider the regular sequence $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ defining $H \longrightarrow J$. It is given by the product

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{bmatrix} = T \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \\ \zeta_5 \\ \zeta_6 \\ Pf(X) \end{bmatrix}$$

where T is a 5×7 matrix with entries in S. Because $H \longrightarrow J$, one maximal minor of T is invertible. Thus, if "——" denotes the reduction modulo the maximal ideal of S, then $\overline{T}$ has rank 5 over $\overline{S}$, and therefore the matrix formed with the first 6 columns of $\overline{T}$ has rank at least 4 over $\overline{S}$. Hence, after row operations, T becomes

$$\left[\begin{array}{cccc|ccc} & & & & d_1 & q_1 & s_1 \\ & I_4 & & & d_2 & q_2 & s_2 \\ & & & & d_3 & q_3 & s_3 \\ & & & & d_4 & q_4 & s_4 \\ \hline 0 & 0 & 0 & 0 & d_5 & q_5 & s_5 \end{array}\right]$$

and one of the elements of the last row is a unit.

As in 2.16, we can perform column operations in the first 6 columns of T.

After doing so, T becomes

$$\left[\begin{array}{c|ccc} & & & s_1 \\ I_4 & 0_{4\times 2} & & s_2 \\ & & & s_3 \\ \hline & & & s_4 \\ 0 & d_5 & q_5 & s_5 \end{array}\right].$$

If neither d_5 nor q_5 are units we can deform them to indeterminates and s_5 is a unit. We get the regular sequence

$$\gamma' = \{\ell_i + s_i \,\mathrm{Pf}(x),\, d_5\ell_5 + q_5\ell_6 + s_5 \,\mathrm{Pf}(X) \mid 1 \leq i \leq 4\}.$$

By specializing d_5 and q_5 to zero, this sequence becomes $\gamma = \{\ell_1, \ell_2, \ell_3, \ell_4, \mathrm{Pf}(X)\}$ which is regular, by corallary 2.5.

Then d_5, q_5 are regular on $S/(\gamma')$, and hence by 1.12 (γ'): H specializes to (γ):H and by the proof of 2.7, this link is $(\ell_1, \ell_2, \ell_3, \ell_4, X_{56}, \mathrm{Pf}(X))$.

If d_5 is a unit (the case q_5 a unit is symmetric), then by column operations in the first 6 columns of T, this matrix becomes

$$\left[\begin{array}{c|ccc} & & & s_1 \\ I_4 & 0_{4\times 2} & & s_2 \\ & & & s_3 \\ \hline & & & s_4 \\ 0_{1\times 4} & 1 & 0 & s_5 \end{array}\right]$$

and $\gamma = \{\ell_i + s_i \mathrm{Pf}(X)\ i \leq i \leq 5\}$.

If none of these s_i's is a unit, we can deform them to variables, and then specialize to zero. γ becomes $\{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$, which is a regular sequence, by

2.4, and $\mathfrak{I}$:H is a hypersurface section, by the proof of 2.6.

Otherwise, w.l.o.g., s_5 is a unit.

After row operations and column operations in the first 6 columns of T, we can assume that T is

$$\left[\begin{array}{c|c} I_5 & \begin{matrix} 0_{4\times 2} \\ 0 \ 1 \end{matrix} \end{array}\right]$$

and $\mathfrak{I} = \{\ell_1, \ell_2, \ell_3, \ell_4, \ell_5 + \mathrm{Pf}(X)\}$.

By (3) in chapter I we see that

$$\ell_5(X_{56}+y_6) = (\mathrm{Pf}(X)+\ell_5)y_6 - \Sigma_{i=1}^{4} X_{i6}\ell_i$$

and by (3) and (4) in chapter I,

$$\ell_6(X_{56}+y_6) = -(\mathrm{Pf}(X)+\ell_5)y_5 + \Sigma_{i=1}^{4}(X_{i5}-y_i)\ell_i$$

and hence $y_6+X_{56} \in (\mathfrak{I})$:H and then the link is a hypersurface section. ∎

Lemma 2.18. Let X be a 5×5 generic alternating matrix over R, Y a 5×1 generic matrix over R[X], v an indeterminate over R[X, Y]. In S = $R[X, Y, v]_{(m,X,Y,v)}$, consider the ideal J = $(\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, v)$, where $[\ell_1 \ldots \ell_5]^t = XY$, and let J ⟶ K. Then K is a licci, Gorenstein ideal of height 5 and deviation at most 2, and there is a pair (S', K') in the same Herzog class as (S, K) such that either

i) K' is a hypersurface section or

ii) K' = H_3.

Proof. Observe first that $(S, (\ell_1,\ell_2,\ell_3,\ell_4,\ell_5))$ is in the same Herzog class as the pair obtained in 2.16 (i). Actually, such a pair is a specialization of $(S, (\ell_1,\cdots,\ell_5))$.

Then $(\ell_1,\cdots,\ell_5)$ is licci, (2.16 and 1.13), is an almost complete intersection, has height 4 and type 2.

Because v is regular on $S/(\ell_1,\cdots,\ell_5)$, then J is licci of type 2, deviation 1 and height 5.

Hence if $J \longrightarrow K$, then K is licci, Gorenstein of height 5 and $d(K) \leq 2$.

Let $\gamma = \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5$ be the regular sequence defining the link $J \longrightarrow K$. Then, for some 5×6 matrix T with entries in S we have

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{bmatrix} = T \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ v \end{bmatrix}.$$

Because $J \longrightarrow K$, a maximal minor of T is invertible, and if "——" denotes the reduction modulo the maximal ideal of S, then $\overline{T}$ has rank 5 on $\overline{S}$, and therefore the matrix formed with the first 5 columns of $\overline{T}$ has rank at least 4 over $\overline{S}$.

Then, after row and column operations, in the first 5 columns of T, T becomes

$$\left[\begin{array}{c|cc} I_4 & 0_{4\times 1} & \begin{matrix} r_1 \\ r_2 \\ r_3 \\ r_4 \end{matrix} \\ \hline 0_{1\times 4} & d_5 & r_5 \end{array}\right]$$

and one of the elements of the last row is a unit.

If one of the r_i's is a unit, then $v+\ell_i$ or $v+d_5\ell_5$ is in $(\mathcal{J}) \subsetneq K$, and hence K is a hypersurface section.

Otherwise, none of the r_i's is a unit. Then we can deform them to variables and then $\mathcal{J}$ becomes

$$\gamma' = (\ell_i + r_i v \mid 1 \le i \le 5).$$

Consider, then the following generic matrices over R

$$X' = \left[\begin{array}{ccccc|c} & & & & & r_1 \\ & & & & & r_2 \\ & & X & & & r_3 \\ & & & & & r_4 \\ & & & & & r_5 \\ \hline -r_1 & -r_2 & -r_3 & -r_4 & -r_5 & 0 \end{array}\right] \qquad Y' = \left[\begin{array}{c} Y \\ \hline v \end{array}\right]$$

Then, if $[\ell'_1, \cdots, \ell'_6]^t = X'Y'$, $\ell'_i = \ell_i + r_i v$, $1 \le i \le 5$. Moreover,

$$(R[X', Y']_{(m,X',Y')}, (\ell'_1, \cdots, \ell'_5))$$

is $(S, \mathcal{J}')$ and

$$(R[X', Y']_{(m,X',Y')}, (\ell'_1, \cdots, \ell'_5, v))$$

is a deformation of (S, J), modulo the regular sequence $r_1 \cdots, r_5$.

Moreover $(\ell'_1, \cdots, \ell'_5) : (\ell'_1, \cdots, \ell'_5, v) = H_3$ (proof of 2.6) and we are done. ∎

<u>Lemma 2.19</u>. Set the notations as in 2.17, in S consider the ideal

$$J = (\ell_1, \ell_2, \ell_3, \ell_4, X_{56}, Pf(X)),$$

and let J ⟶ K. Then K is licci of height 5 and deviation at most 2 and there is a pair (S', K') in the same Herzog class as (S, K) such that either

i) K' is a hypersurface section or

ii) $K' = H_3$.

Proof. That K is licci, Gorenstein of height 5 and $d(K) \leq 2$ is clear (1.6 and 2.7). Let $\underline{\gamma}$ be the regular sequence defining J ⟶ K. Then there is a 5×6 matrix T with entries in S such that

$$[\underline{\gamma}] = T \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ X_{56} \\ Pf(X) \end{bmatrix}.$$

By standard arguments (see the proof of 2.16 to 2.18) T becomes

$$\left[\begin{array}{c|ccc} I_3 & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} t_1 \\ t_2 \\ t_3 \end{matrix} & \begin{matrix} s_1 \\ s_2 \\ s_3 \end{matrix} \\ \hline 0_{2\times 3} & \begin{matrix} r_4 \\ r_5 \end{matrix} & \begin{matrix} t_4 \\ t_5 \end{matrix} & \begin{matrix} s_4 \\ s_5 \end{matrix} \end{array}\right]$$

and at least one element of the fourth row and one element of the fifth row, in different columns, are units.

Assume first r_4 is a unit (the case r_5 a unit is symmetric). Then, after row operations T becomes.

$$\left[\begin{array}{c|cc} & t_1 & s_1 \\ I_4 & t_2 & s_2 \\ & t_3 & s_3 \\ & t_4 & s_4 \\ \hline 0_{1\times 4} & t_5 & s_5 \end{array}\right]$$

and one element of the last row is a unit. If t_5 is a unit, we may assume then that

$$\mathcal{I} = \{\ell_i + s_i \operatorname{Pf}(X),\ X_{56} + s_5 \operatorname{Pf}(X) \mid 1 \le i \le 4\},$$

and because

$$(y_6 + \Sigma_{i=1}^4 X_{i6}s_i + \ell_5 s_5) \operatorname{Pf}(X) = \Sigma_{i=1}^4 (\ell_i + s_i \operatorname{Pf}(X))X_{i6} + \ell_5(X_{56} + s_5 \operatorname{Pf}(X))$$

is in $(\mathcal{I})$, $y_6 + \Sigma_{i=1}^4 X_{i6}s_i + \ell_5 s_5 \in (\gamma){:}J$ and the link is a hypersurface section.

If t_5 is not a unit, then s_5 is a unit and we get

$$\mathcal{I} = \{\ell_i + t_i X_{56},\ \operatorname{Pf}(X) + t_5 X_{56} \mid 1 \le i \le 4\}.$$

Since t_5 is not a unit, we may deform it to a variable and specialize to 0. Then

$$\mathcal{I} = \{\ell_i + t_i X_{56},\ \operatorname{Pf}(X) \mid 1 \le i \le 4\}.$$

If none of the t_i's is a unit, again we may deform them to variables and specialize to 0. Then $\mathcal{I} = \{\ell_1, \ell_2, \ell_3, \ell_4, \operatorname{Pf}(X)\}$ and $\mathcal{I}{:}J = H_3$ by 2.7.

If, say t_4 is a unit, we may assume then $\mathcal{I} = \ell_1, \ell_2, \ell_3, \ell_4 + X_{56}, \operatorname{Pf}(X)$. Notice our ground ring is still S. Assuming $\deg x_{ij} = \deg y_j = 1$. A resolution for J, which was obtained in the proof of 2.7, is

$$0 \longrightarrow S^2(-9) \longrightarrow \cdots \longrightarrow S^5(-2) \oplus S(-3) \longrightarrow S$$

while the Koszul complex of $\mathfrak{I}$ is

$$0 \longrightarrow S(-11) \longrightarrow \cdots \longrightarrow S^4(-2) \oplus S(-3) \longrightarrow S.$$

Then a resolution of $\mathfrak{I}$: J starts as

$$S^6(-2) \oplus S(-3) \longrightarrow S.$$

Consider $\ell_5 - X_{46}$ and $\ell_6 + X_{45}$, which have degree 2. Notice that, because of (3) in chapter I

$$\ell_4(\ell_5 - X_{46}) = -y_6 \mathrm{Pf}(X) + \Sigma_{i=1}^{3} X_{i6}\ell_i + (X_{56} + \ell_4)\ell_5 \in (\mathfrak{I})$$

and

$$\ell_4(\ell_6 + X_{45}) = y_5 \mathrm{Pf}(X) - \Sigma_{i=1}^{3} X_{i5}\ell_i + (X_{56} + \ell_4)\ell_6 \in (\mathfrak{I})$$

then

$$K = (\ell_1, \ell_2, \ell_3, \ell_4 + X_{56}, \ell_5 - X_{46}, \ell_6 + X_{45}, \mathrm{Pf}(X)) \subseteq (\mathfrak{I}):J.$$

By setting $y_i = 0$, these elements minimally generate K if and only if X_{46}, X_{56}, X_{45}, Pf(X) minimally generate some ideal, which was the case of the proof of 2.9. Then $K = (\mathfrak{I}):J$ and $d(K) = 2$.

Specialize (S,K) via the regular sequence

$$* = \{y_3+x_{12}, y_2-x_{13}, y_1+x_{23}, y_5, x_{25}, x_{26}, x_{34}, x_{45}, x_{46}\ x_{56}\}.$$

We obtain a pair (S', K'), where

$$K' = (x_{14}y_4+x_{16}y_6, x_{24}y_4, x_{36}y_6, 2(x_{14}x_{23}-x_{13}x_{24}), 2(x_{15}x_{23}+x_{12}x_{35}),$$

$$2(x_{16}x_{23}+x_{12}x_{36}), x_{16}x_{24}x_{35}-x_{15}x_{24}x_{36}),$$

and then, because 2 is a unit, we get

$$K' = (x_{14}y_4+x_{16}y_6, x_{24}y_4, x_{36}y_6, x_{14}x_{23}-x_{13}x_{24}, x_{15}x_{23}+x_{12}x_{35},$$

$$x_{16}x_{23}+x_{12}x_{36}, x_{16}x_{24}x_{35}-x_{15}x_{24}x_{36}).$$

That is $K' \doteq (I_1(AB), Pf(A))$ where

$$A = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ -x_{12} & 0 & x_{23} & x_{24} & 0 & 0 \\ -x_{13} & -x_{23} & 0 & 0 & x_{35} & x_{36} \\ -x_{14} & -x_{24} & 0 & 0 & 0 & 0 \\ -x_{15} & 0 & -x_{35} & 0 & 0 & 0 \\ -x_{16} & 0 & -x_{36} & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -x_{23} \\ x_{13} \\ -x_{12} \\ y_4 \\ 0 \\ y_6 \end{bmatrix}.$$

Here we use the fact that 2 is a unit. We now have to show that (S', K') is a specialization of (S, K). It is enough to show that ht K' = 5.

Let $p \supset K'$, be a prime ideal in S'. If x_{23} is in p, K' contains also $x_{14}y_4+x_{16}y_6$, $x_{36}y_6$, $x_{12}x_{35}$ and $x_{13}x_{24}$ and then, ht $p \geq 5$. If $x_{23} \notin p$, then in S_p, K' becomes

$$(x_{14}y_4+x_{16}y_6,\ x_{24}y_4,\ x_{36}y_6,\ x_{14}-\frac{x_{13}x_{24}}{x_{23}},\ x_{15}+\frac{x_{12}x_{35}}{x_{23}},$$

$$x_{16}+\frac{x_{12}x_{36}}{x_{23}},\ x_{16}x_{24}\,x_{35}-x_{15}x_{24}x_{36})$$

which contains 3 linear forms, $x_{14}-\frac{x_{13}x_{24}}{x_{23}}$, $x_{15}+\frac{x_{16}x_{35}}{x_{23}}$, $x_{16}+\frac{x_{12}x_{36}}{x_{23}}$, and $x_{24}y_4$, $x_{36}y_6$. Then ht K' $\geq$ 5, and then, because K' is a specialization of an ideal of height 5, then ht K' = 5. Also notice that K' is a specialization of H_3.

Finally, assume that neither r_4 and r_5 are units. Then row and column operations in the first 4 columns, T becomes

$$\left[\begin{array}{c|ccc} & 0 & 0 & 0\\ I_3 & 0 & 0 & 0\\ & 0 & 0 & 0\\ \hline & r'_4 & 1 & 0\\ 0 & & & \\ & r'_5 & 0 & 1 \end{array}\right]$$

where r'_4, r'_5 are still not units. Then

$$\mathcal{I} = \{\ell_1, \ell_2, \ell_3,\ X_{56}+r'_4\ell_4,\ \mathrm{Pf}(X) + r'_5\ell_5\}.$$

Now we deform r'_4 and r'_5 to indeterminates and specialize them to zero. $\mathcal{I}$ then becomes $\{\ell_1, \ell_2, \ell_3, X_{56}, \mathrm{Pf}(X)\}$, which is a regular sequence by 2.8. Then, the linkage specializes (1.12). Moreover $(\mathcal{I})$:J = $(\ell_1, \ell_2, \ell_3, X_{45}, X_{46}, X_{56}, \mathrm{Pf}(X))$ (2.9). If we apply the specialization * to (S, K), we obtain the pair (S', K') obtained before. Now we are done. ▮

We now turn our attention to the following case: let (R, m) as usual, and let X' a 4×4 generic alternating matrix over R, Y' a 4×1 generic matrix over R[X'] and y, z indeterminates over R[X', Y']. Set S = R[X', Y', y, z]. We are interested in all ideals J and K of $S_{(m,X',Y',y,z)}$ such that $(H_2, y_2) \longrightarrow J \longrightarrow K$. Because H_2 is a deviation 2 Gorenstein ideal of height 3, we may assume it is generated by the 4×4 pfaffians of a 5×5 generic alternating matrix X[B–E, 2.1]. We write ℓ_i for $(-1)^{i+1} X_i$. Write then S = R[X, y, z].

<u>Lemma 2.20</u>. With the previous notation, let $I = (H_2, y, z) \longrightarrow J$. Then there is a pair (S', J') in the same Herzog class as $(S_{(m,X,y,z)}, J)$ such that either

i) J' is a hypersurface section or

ii) there is a 6×6 alternating matrix A and a 6×1 matrix B $a_{ij}, b_j \in m_{S'}$, such that if $[t_1,\cdots,t_6]^t = AB$, then $J = (t_1,\cdots,t_4, A_{56}, Pf(A))$.

<u>Proof</u>. Let $\underline{\gamma}$ be the regular sequence defining $I \longrightarrow J$. Then there is a 5×7 matrix with coefficients in $S_{(m,X,y,z)}$ such that

$$[\underline{\gamma}] = T \begin{bmatrix} \ell_1 \\ \ell_2 \\ \ell_3 \\ \ell_4 \\ \ell_5 \\ y \\ z \end{bmatrix}$$

Because $I \longrightarrow J$, a maximal minor of T is invertible. By using standard arguments, after row operations and column operations in the first 5 columns of T, T becomes

$$\left[\begin{array}{c|cccc} & & & s_1 & d_1 \\ I_3 & 0_{3\times 2} & & s_2 & d_2 \\ & & & s_3 & d_3 \\ \hline 0_{2\times 3} & t_4 & r_4 & s_4 & d_4 \\ & t_5 & r_5 & s_5 & d_5 \end{array}\right]$$

and at least one element of the fourth row and one element of the fifth row, in different columns, are units.

If s_5 or d_5 are units (and likewise for any other s_i or d_i) $y+t_5\ell_4+r_5\ell_5$ or $z+t_5\ell_4+r_5\ell_5$ are in $(\mathfrak{I})$ and hence $\mathfrak{I}$:I is a hypersurface section.

Then, we may assume that T is:

$$\left[\begin{array}{c|cc} & s_1 & d_1 \\ & s_2 & d_2 \\ I_5 & s_3 & d_3 \\ & s_4 & d_4 \\ & s_5 & d_5 \end{array}\right]$$

and none of the s_i's, d_i's are units $1 \leq i \leq 5$. Therefore we may deform all of them to indeterminates and specialize all of them, but s_4, d_5, to 0. Then

$$\mathfrak{I} = \{\ell_1, \ell_2, \ell_3, \ell_4+sy,\ \ell_5 + dz\}$$

and $ht(\mathfrak{I}) = 5$.

If we assign $\deg x_{ij} = \deg y = \deg z = \deg s = \deg d = 1$, then, a resolution for I is

$$0 \longrightarrow S(-7) \longrightarrow \cdots \longrightarrow S^5(-2) \oplus S^2(-1) \longrightarrow S,$$

while the Koszul complex of $\mathfrak{I}$ is

$$0 \longrightarrow S(-10) \longrightarrow \cdots \longrightarrow S^5(-2) \longrightarrow S.$$

Then, a resolution of $(\mathfrak{I})$: I starts as

$$S^5(-2) \oplus S(-3) \longrightarrow S.$$

Consider x_{45} sd, which has degree 3. Because

$$x_{45}\, sdy = x_{45}d(\ell_4+sy) - x_{45}d\ell_4 \in (\underline{\gamma})$$

and

$$x_{45}\, sdz = x_{45}\, s(\ell_5+dz) - x_{45}s\ell_5 \in (\underline{\gamma})$$

it follows that $x_{45}sd \in (\underline{\gamma})$: I. If deg $x_{ij} = 1$, deg y = deg $z = 2$, deg d = deg $s = 0$, then $(\underline{\gamma})$ is generated by homogeneous elements of degree 2, but x_{45} sd has degree 1. Then $x_{45}\, sd \notin (\underline{\gamma})$. Thus $J' = (\underline{\gamma}):I = (\ell_1,\ell_2,\ell_3,\ell_4+sy,\ \ell_5+dz,\ x_{45}\, sd)$ which can be viewed as $(t_1, t_2, t_3, t_4, A_{56}, \mathrm{Pf}(A))$ where, if

$$A = \begin{bmatrix} 0 & x_{45} & x_{24} & -x_{25} & 0 & 0 \\ -x_{45} & 0 & x_{34} & -x_{35} & 0 & 0 \\ -x_{24} & -x_{34} & 0 & -x_{23} & -d & 0 \\ x_{25} & x_{35} & x_{23} & 0 & 0 & -s \\ 0 & 0 & d & 0 & 0 & 0 \\ 0 & 0 & 0 & s & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -x_{13} \\ x_{12} \\ x_{15} \\ x_{14} \\ z \\ y \end{bmatrix}$$

then the t_i's are as in (ii). ∎

In lemma 2.20 (i), J is a hypersurface section, namely $J = (N, t)$, t regular on S/N and N is a type 2 almost complete intersection of height 4, which is licci because J is licci (2.11). By 2.16, after deforming and specializing if needed, N is either of the form $I_1(AB)$, A a 5×5 alternating matrix, B a 5×1 matrix, or N is a hypersurface section. In the first case, if $J \longrightarrow K$, then K is described in 2.18. We now study the second case.

Lemma 2.21. Let J be a licci, almost complete intersection ideal of R of height 5 and type 2 which is a double hypersurface section. If J $\longrightarrow$ K, K is licci, Gorenstein, has height 5 and deviation at most 2, and K is a hypersurface section.

Proof. Write J = (N, y, z), y, z regular on R/N, N a height 3, type 2 almost complete intersection. Let Y, Z be indeterminates over R. Then (R', J') = ($R[Y, Z]_{(m,Y,Z)}$, (m, Y, Z)) is a deformation of (R, J). Let J' ~ K' such that (R',K') is a deformation of (R,K) (such K' exists by 1.13). Hence J' $\longrightarrow$ K' and we will show that K' is a hypersurface section.

Let γ be the regular sequence defining J' $\longrightarrow$ K'. Then, for some matrix T with coefficients in R', we have

$$[\gamma]^t = T \begin{bmatrix} Y \\ Z \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

where N = (a_1, a_2, a_3, a_4). By standard arguments, we may assume that T is

$$\left[\begin{array}{c|c} 1 & \\ 0 & \\ 0 & T' \\ 0 & \\ 0 & \end{array}\right].$$

Hence γ contains $Y+t_{12}Z+\Sigma_{i=1}^{4} t_{1\,i+2}\, a_i$ and hence K' is a hypersurface section.
∎

We are now ready to prove the main result of this chapter.

Theorem 2.22. Let (R, m) be a local Gorenstein ring in which 2 is a unit, and assume the residue class field of R is infinite. Let I be a licci, Gorenstein ideal of R of height 5 and deviation 2. Then, there is a pair (R', I') in the same Herzog class as (R, I) for which either

i) I is a double hypersurface section or

ii) there is a 6×6 alternating matrix A and a 6×1 matrix B, such that $I' = (I_1(AB), Pf(A))$, $a_{ij}, b_j \in m_{s'}$.

Proof. By the conditions on R and because I is licci and Gorenstein, there is a sequence of links

$$I = I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n$$

where I_n is a complete intersection and I_{2i} is Gorenstein $0 \leq 2i \leq n$ (1.22).

Therefore I_2 is Gorenstein of height 5 and deviation at most 2 (1.6). If I_2 has deviation 0, then it is a complete intersection. If not, because I_2 is perfect, then by [Ku], $d(I_2) = 2$, and I_2 is linked in two less steps to a complete intersection. Therefore, by induction hypothesis, we may assume that either I_2 is a complete intersection or after deformations and specializations, I_2 is as (i) or (ii).

Assume first $d(I_2) = 2$. Then, because $d(I_2) = r(R/I_1) = 2$, $I_2 \longrightarrow I_1$, and because $d(I_1) = r(R/I_0) = 1$, $I_1 \longrightarrow I_0$.

If $L_1(I_2)$ is a first generic link of I_2 in some polynomial ring $R[\underline{Z}]$, and if $L_2(I_2)$ is any second generic link of I_2 in some polynomial ring R[Z, W], then for some p in Spec(R[Z, W]), $m \subset p$, $(R[Z, W]_p, L_i(I_2)R[Z, W]_p)$ is a deformation of (R, I_{2-i}), $0 \leq i \leq 2$ (proposition 1.14).

Therefore we can replace R by $R[Z, W]_p$, I_{2-i} by $L_i(I_2)R[Z, W]_p$. Change notations and call this ring R, and the ideals I_2, I_1 and I_0. Then R is Gorenstein local, 2 is a unit, $I_2 \longrightarrow I_1 \longrightarrow I_0$ and these links deform and specialize (proposition 1.13 and remark 1.15). Therefore, and by induction hypothesis, we may assume I_2 is actually equal to the ideal described in i) or ii) and we can also assume all matrices are generic.

Assume I_2 is as (i). Then I_2 is a double hypersurface section. Then lemmas 2.20, 2.16, 2.18, 2.21 and 2.19 describe I.

If I_2 is as (ii), then lemmas 2.17, 2.16, 2.18, 2.21 and 2.19 describe I.

So, we now assume I_2 is a complete intersection, and $I \longrightarrow I_1 \longrightarrow I_2$. For I_1 we know $d(I_1) = 1$ and because $r(R/I_1) = 1$, $I_1 \longrightarrow I$. Hence $I_2 \sim I_1 \longrightarrow I$.

Let $(\underline{\alpha})$ be the linking sequence from I_2 to I_1. Then since $r(R/I_1) = 2 = \mu(I_2/(\underline{\alpha}))$, three elements of I_2 which are part of a minimal set of generators of I_2 are in $(\underline{\alpha})$.

Deform (R, I_2) to

$$(R[T_1, T_2, T_3, T_4, T_5]_{(T_1,T_2,T_3,T_4,T_5)}, (T_1, T_2, T_3, T_4, T_5)).$$

Then $(\underline{\alpha})$ contains 3 of the T_i's and hence I_1 is a double hypersurface section. Lemma 2.21 describes I. Now we are done. ∎

Corollary 2.23. Let (R, I) be as in 2.22. Then the minimal resolution of R/I is either

i) $$0 \longrightarrow R \longrightarrow R^7 \longrightarrow R^{22} \longrightarrow R^{22} \longrightarrow R^7 \longrightarrow R$$

or

ii) $$0 \longrightarrow R \longrightarrow R^7 \longrightarrow R^{13} \longrightarrow R^{13} \longrightarrow R^7 \longrightarrow R.$$

Proof. By 2.22, (R, I) is in the same Herzog class of $(R[X, Y]_{(m,X,Y)}, H_3)$ for a 6×6 generic alternating matrix X and a 6×1 generic matrix Y, or in the same Herzog class as $(R[Z, t, w]_{(m,Z,t,w)}, (H_2, t, w))$ for some 5×5 generic alternating matrix Z and indeterminates t and w. A resolution for H_3 is (i) and a resolution for (H_2, t, w) is (ii) [K]. Now we use the fact that the Betti numbers are invariants of the Herzog class.

Corollary 2.24. Let J be a licci almost complete intersection ideal of R of height 5 and type 2. Then there is a pair (R', J') in the same Herzog class as (R, J) such that either:

i) J is a hypersurface section or

ii) There is a 6×6 alternating matrix A and a 6×1 matrix B, a_{ij}, $b_j \in m_{R'}$, such that if $[\ell_1,\ldots,\ell_6]^t = AB$, $J' = (\ell_1, \ell_2, \ell_3, \ell_4, A_{56}, Pf(X))$.

Proof. Let J ⟶ I. Then J is licci, Gorenstein, has height 5 and deviation at most 2.

If d(I) = 0, I is a complete intersection and then, by the proof of 2.22, J is a hypersurface section, and it is described in 2.16.

Otherwise d(I) = 2 [Ku]. Then, there is a pair (R', I') in the same Herzog class of (R, I) such that I' is as (i) or (ii) of theorem 2.22. If I' is as i, lemmas 2.20 and 2.16 describe J. If I' is as (ii), lemmas 2.17 and 2.16 describe J. ∎

Corollary 2.25. Let $R = k[[x_1,\cdots,x_r]]$, where k is an infinite field with char $k \neq 2$ and let I be a licci Gorenstein ideal of R of height 5 and deviation 2. Then, either

a) I is a double hypersurface section or

b) $I = (I_1(AB), Pf(A))$, where A is a 6×6 alternating matrix, B a 6×1 matrix, a_{ij}, b_j not units.

Proof. By (2.22), (R, I) is in the same Herzog class as $(R[[X, Y]]), H_3)$ or $(R[[Z,t,w]], (H_2,t,w))$, where X is a 6×6 generic alternating matrix, Y a 6×1 generic matrix. But since $R[[X,Y]]/H_3$ and $R[[Z,t,w]]/(H_2,t,w)$ are rigid [H–U–1, K], it follows from the discussion following (1.19) that (R, I) is a specialization of $(R[[X,Y,w]], H_3 R[[X,Y,w]])$ or $(R[[Z,t,w,v]], (H_2,t,w) R[[Z,t,w,v]])$. ∎

CHAPTER III

THE EVEN GRADE CASE

In this chapter, we construct a family of licci, Gorenstein ideals of deviation 2 and even grade that are not hypersurface sections in any even grade larger or equal to 6.

The only positive result known on even grade is the result of Herzog and Miller [H–M, th. 1.7]. They show that if R is a local Gorenstein ring in which 2 is a unit, and if I is a perfect, Gorenstein, generically a complete intersection ideal of R of height 4 and deviation 2 such that I/I^2 is Cohen–Macaulay, then I is a hypersurface section.

Vasconcelos and Villarreal show that the condition of I/I^2 being Cohen–Macaulay follows from the other assumptions [V–V, th. 1.1].

We now construct a family of licci Gorenstein ideals of even height and deviation 2.

Let k be a field, n an integer, larger or equal to three, X a $(2n-1)\times(2n-1)$ generic alternating matrix, Y a $(2n-1)\times 1$ generic matrix over k[X]. Let R = k[X,Y], $S = R_{(X,Y)}$ and in this ring consider the ideal

$$I_n = (\ell_1,\cdots,\ell_{2n-2}, x_{2n-1}, y_{2n-1}) = (H_{n-1}, y_{2n-1}),$$

where $[\ell_1,\cdots,\ell_{2n-1}]^t = XY$.

Notice I_n is licci, Gorenstein, has even height 2n–2 and deviation 2.

Lemma 3.1. With the above notations,

$$G_n = (\ell_1,\ldots,\ell_{2n-4}, X_{2n-3\ 2n-2\ 2n-1}, X_{2n-1}, y_{2n-1})$$

is a licci ideal of deviation 1, height 2n–2 and type 2 in S.

<u>Proof</u>. This is essentially lemma 2.7.

<u>Lemma 3.2</u>. $\underline{\alpha}_n = (\ell_1,\ldots,\ell_{2n-4}, X_{2n-3\ 2n-2\ 2n-1}, X_{2n-1})$ is a complete intersection in S.

<u>Proof</u>. By induction on n. For n = 3,

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

and

$$(\underline{\alpha}_3) = (x_{12}, x_{13}y_3 + x_{14}y_4 + x_{15}y_5, x_{23}y_3 + x_{24}y_4 + x_{25}y_5, x_{13}x_{24}-x_{14}x_{23}).$$

If $T = S/(x_{12})$, then T is a Cohen–Macaulay domain. A direct application of (2.2) shows that $(x_{13}y_3 + x_{14}y_4 + x_{15}y_5, x_{23}y_3 + x_{24}y_4 + x_{25}y_5)$ is a prime ideal of height 2 (it actually defines $Sym(I_2(Z))$ where Z is a 2×3 generic matrix) (see also [Ho]) and $x_{13}x_{24}-x_{14}x_{23}$ is not in this ideal. Hence $(\underline{\alpha})$ is a complete intersection.

For $n \geq 4$, let $p \in Spec(S)$ be such that ht $p \leq 2n-2$ and $(\underline{\alpha}_n) \subsetneq p$. Then, as in 2.8, p does not contain the ideal generated by the entries of the matrix formed with the first 2n–4 rows and 2n–4 columns of X. Then, we can assume, w.l.o.g., that x_{12} is invertible in S_p. Hence, there is a matrix A, invertible over S_p, such that

$$A^T X A = \left[\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & X' \end{array}\right]$$

and the entries of X' are generic over $k[x_{12}^{-1}, \{x_{ij}, i \le 2\}]$. If $Y'' = A^{-1}Y$, then (α_n) becomes (y_1, y_2, α_{n-1}), which by induction hypothesis, has height $(2n-4)+2 = 2n-2$. ∎.

We can establish now the following proposition.

Proposition 3.3. With the previous notations,

$$F_n = (\ell_1,\dots,\ell_{2n-4}, X_{2n-1}, X_{2n-2}, X_{2n-3}, X_{2n-3\ 2n-2\ 2n-1})$$

is a licci Gorenstein ideal of height 2n–2 and deviation 2.

Proof. Grade R by assigning $\deg x_{ij} = \deg y_j = 1$. A resolution for G_n starts and ends like (2.7)

$$0 \to R^2(-(5n-10)) \to \cdots \to R(-1) \oplus R^{2n-4}(-2) \oplus R(-(n-2)) \oplus R(-(n-1)) \to R$$

while the Koszul complex for $(\underline{\alpha}_n)$ is

$$0 \to R(-(6n-11)) \to \cdots \to R^{2n-4}(-2) \oplus R(-(n-2)) \oplus R(-(n-1)) \to R.$$

Therefore a resolution for $(\underline{\alpha}_n)$: G_n starts as

$$\cdots \to R^{2n-4}(-2) \oplus R(-(n-2)) \oplus R^3(-(n-1)) \to R.$$

Consider X_{2n-2} and X_{2n-3}, which have degree (n–1). Let X' (resp. Y') be the matrix obtained from X (resp Y) by deleting the (2n–2)th row and column (resp.

(2n–2)th row).

Let $[\ell_1, \ldots \ell_{2n-3}\ \ell_{2n-1}]^t = X'Y'$, and notice $X_{2n-2} = -Pf(X')$. Then

$$X_{2n-2}y_{2n-1} = -y_{2n-1}\, Pf(X')$$

$$= \Sigma_{i=1}^{2n-3} X_{i2n-2\ 2n-1}\, \ell_i$$

$$= \Sigma_{i=1}^{2n-4} X_{i2n-2\ 2n-1}\, \ell_i + X_{2n-3\ 2n-2\ 2n-1}\, \ell_{2n-3}$$

$$- (\Sigma_{i=1}^{2n-3} X_{i2n-2\ 2n-1}\, x_{i2n-2})\, y_{2n-2}$$

$$= \Sigma_{i=1}^{2n-4} X_{i2n-2\ 2n-1}\, \ell_i + X_{2n-3\ 2n-2\ 2n-1}\, \ell_{2n-3} - X_{2n-1}\, y_{2n-2}$$

$$\in (\underline{\alpha}_n).$$

Similarly, $X_{2n-3}\, y_{2n-1} \in (\underline{\alpha}_n)$, and then $F_n \subsetneq (\underline{\alpha}): G_n$. If $X_{2n-3} \in (\underline{\alpha})$, then for homogeneous elements $a_1, \cdots, a_{2n-4}$, b, c in R,

$$X_{2n-3} = \Sigma_{i=1}^{2n-4} a_i \ell_i + bX_{2n-3\ 2n-2\ 2n-1} + c\, X_{2n-1}.$$

Then deg b = 1. If we set deg $x_{ij} = 0$, deg $y_i = 2$, then $a_1 = \cdots = a_{2n-4} = 0$. Therefore

$$X_{2n-3} = b\, X_{2n-3\ 2n-2\ 2n-1} + c\, X_{2n-1}.$$

Specialize, by assigning $x_{i2n-3} = 0$. Then X_{2n-1} becomes zero but

$$X_{2n-3} = b\, X_{2n-3\ 2n-2\ 2n-1} \neq 0,$$

but this is impossible, because X_{2n-3} is an irreducible element of R and neither b nor $X_{2n-3\ 2n-2\ 2n-1}$ are units in R.

Similarly, $X_{2n-2} \in (\underline{\alpha}, X_{2n-3})$ only if, for homogeneous elements

$$X_{2n-2} = a\, X_{2n-3\ 2n-2\ 2n-1} + b\, X_{2n-3} + c\, X_{2n-1}.$$

Then deg a = 1. If we specialize by assigning $x_{i\ 2n-2} = 0$, we obtain

$$X_{2n-2} = a\, X_{2n-3\ 2n-2\ 2n-1} \neq 0,$$

which is impossible. In the same way

$$X_{2n-1} \notin (\ell_1, \cdots, \ell_{2n-4}, X_{2n-3\ 2n-2\ 2n-1}, X_{2n-3}, X_{2n-2})$$

Finally, as in 2.9, $\ell_1 \in (\ell_2, \ldots, \ell_{2n-4}, X_{2n-3\ 2n-2\ 2n-1}, X_{2n-3}, X_{2n-2}, X_{2n-1})$ if and only if $\ell_1 \in (\ell_2, \ldots, \ell_{2n-4})$, which is contrary to 2.4.

Then $F_n \subsetneq (\underline{\alpha}_n):G_n$ and $(\underline{\alpha}_n):G_n/(\underline{\alpha}_n)$ is minimally generated by 2 elements of degree n–1. Then $F_n = (\underline{\alpha}_n): G_n$, and $\underline{\alpha}_n$ form part of a minimal generating set of F_n.

Hence in $S = k[X, Y]_{(X,Y)}$, F_n and G_n are linked $F_n \to G_n$ and $G_n \to F_n$. Then $r(S/G_n) = d(F_n) = 2$ and $r(S/F_n) = d(G_n) = 1$. ∎

Proposition 3.3 produces a family of licci, Gorenstein ideals of every height larger or equal to 4 and deviation 2.

When n = 3, F_3 has height 2(3)–2 = 4 and then by [H–M] and [V–V], it is a hypersurface section. We ask if F_n, $n \geq 4$, is a hypersurface section.

Before we answer this question we need some definitions and propositions.

Definition 3.4. [K–M–2, def. 3.1] Let I be a grade g Gorenstein ideal of the local Gorenstein ring P. Let $\underline{a}$ be a $1 \times n$ vector which generates I, t an integer with $0 \le t \le g-1$, Y a $(g-1) \times (n-t)$ matrix of indeterminates, and v an indeterminate. Define $\underline{b}$ by the product

$$[\underline{b}]^t = \left[\begin{array}{c|c} I_t & \\ \hline 0 & Y \end{array} \right] [\underline{a}]^t.$$

Let y be an element of IP[[Y]] with $\underline{b}$, y regular and $\mu(I/(\underline{b}, y)) \ge 2$. If w is any element of P[[Y]] such that $J = (\underline{b}, y): I$ is generated by $(\underline{b}, y, w)$ and $K = (\underline{b}, w+vy): J$, then $P[[Y, v]]/K$ is called a semi generic tight double link of P/I.

Kustin and Miller show that K is a grade g Gorenstein ideal, and if $P = k[[x_1, \cdots, x_n]]$ and if I is a grade g ideal of P such that P/I is rigid, then any semi generic tight double link of P/I is also rigid [K–M, prop. 3.2 and Cor. 3.7].

We now begin the construction of our example.

Set the notations as in the beginning of this chapter, let $n \ge 4$, and let X'(resp. Y') be the matrix obtained from X (resp. Y) by deleting the last 3 rows and columns. Let P" = k[[X', Y]], P' = k[[X, Y]].

In P" consider the ideal $L_n = (\ell'_1, \ldots, \ell'_{2n-4}, f', y_{2n-3}, y_{2n-2}, y_{2n-1})$ where $[\ell'_1, \ldots, \ell'_{2n-4}]^t = X' \, Y'$, $f' = -\mathrm{Pf}(X')$, (actually it is $X_{2n-3\; 2n-2\; 2n-1}$).

Then, L_n has height $2n-2$ and deviation 2, it is Gorenstein, licci and P''/L_n is rigid ([H–U–1], [K]). Let $\underline{b}$ be defined by

$$[b]^t = \left[\begin{array}{c|ccc} & x_{1\,2n-3} & x_{1\,2n-2} & x_{1\,2n-1} \\ & \vdots & & \vdots \\ I_{2n-3} & & & \\ & x_{2n-4\,2n-3} & x_{2n-4\,2n-2} & x_{2n-4\,2n-1} \\ & x_{2n-2\,2n-1} & x_{2n-3\,2n-2} & x_{2n-3\,2n-1} \end{array}\right] \begin{bmatrix} \ell'_1 \\ \vdots \\ \ell'_{2n-4} \\ f' \\ y_{2n-3} \\ y_{2n-2} \\ y_{2n-1} \end{bmatrix}.$$

That is

$$b_1 = \ell'_1 + x_{12n-3}\, y_{2n-3} + x_{12n-2} y_{2n-2} + x_{12n-1}\, y_{2n-1} = \ell_1$$

$\vdots$

$$b_{2n-4} = \ell'_{2n-4} + x_{2n-4\,2n-3} y_{2n-3} + x_{2n-4\,2n-2} y_{2n-2} + x_{2n-4\,2n-1} y_{2n-1} = \ell_{2n-4}$$

$$b_{2n-3} = f' + x_{2n-2\,2n-1}\, y_{2n-3} + x_{2n-3\,2n-2}\, y_{2n-2} + x_{2n-3\,2n-1}\, y_{2n-1}$$

Then, we notice that $[\ell_1,\dots,\ell_{2n-1}]^t = XY$ and that $y_{2n-1} \in L_n P'$ is such that ($\underline{b}$, y_{2n-1}) is a complete intersection contained in L_n P', by 2.5. Also $\mu(L_n/(\underline{b},y_{2n-1})$ = 2, being generated by the images of y_{2n-3} and y_{2n-2}.

Let $w = X_{2n-1} - x_{2n-2\,2n-1}\,\ell_{2n-2} + x_{2n-3\,2n-2}\,\ell_{2n-3}$.

Lemma 3.5. With the previous notations, $T_n = (\underline{b}, y_{2n-1}, w)$ is a licci, type 2, deviation 1 ideal of height 2n–2 in P'.

Proof. With $C = (\underline{b}, y_{2n-1})$: L_n and denote by "——" reduction modulo

$$(\underline{x}) = (x_{2n-3\,2n-1}, x_{2n-3\,2n-2}, x_{2n-2\,2n-1}),$$

which is a complete intersection in S'.

Then $(\overline{\underline{b},\ y_{2n-1}}) = (\ell_1,\ell_2,\dots,\ell_{2n-4}, f', y_{2n-1})$, which by corollary 2.5 is a regular sequence in $\overline{P}'$. Then $(\underline{x})$ is a regular sequence on $P'/(\underline{b}, y_{2n-1})$ and therefore $\overline{C} = (\overline{b, y_{2n-1}}) : \overline{L}_n = (\underline{\overline{b}}, y_{2n-1}) : L_n$, and $(\underline{\gamma})$ is also a regular sequence on S'/C (1.12).

In $S''' = k[X, Y]/(\underline{\gamma})$ assign $\deg x_{ij} = \deg y_j = 1$. Then a resolution of $L_n S'''$ in S''' is ([K]), or (2.6)

$$0 \longrightarrow S'''(-(4n-8)) \longrightarrow \ldots \longrightarrow S'''^{2n-4}(-2) \oplus S'''(-(n-2)) \oplus S'''^{3}(-1) \longrightarrow S'''$$

while the Koszul complex of $\underline{b}, y_{2n-1}$ is

$$0 \longrightarrow S'''(-(5n-9)) \longrightarrow \cdots \longrightarrow S'''^{2n-4}(-2) \oplus S'''(-(n-2)) \oplus S'''(-1) \longrightarrow S'''.$$

Then a resolution of $\overline{C}\, S'''$ in S''' starts as

$$S'''^{2n-4}(-2) \oplus S'''(-(n-2)) \oplus S'''(-(n-1)) \oplus S'''(-1) \longrightarrow S'''$$

Now $\overline{X_{2n-1}}$ has degree n–1, and it is in $\overline{C}\, S'''$ but not in $(\overline{\underline{b}, y_{2n-1}})$. Then

$$\overline{C}\, S''' = (\overline{b}, \overline{y}_{2n-1}, \overline{X_{2n-1}})\, S'''$$

and therefore

$$\overline{C} = (\overline{b}, \overline{y}_{2n-1}, \overline{X_{2n-1}})\, P' = \overline{T}_n.$$

Consider now $y_{2n-2}\, w$ and $y_{2n-3}\, w$.

First let X" (resp Y") be the matrix obtained from X(resp. Y) by deleting the (2n–1)th row and column (resp (2n–1)th row). Let $[\ell''_1, \ldots, \ell''_{2n-2}]^t = X''Y''$. Since $X_{2n-1} = Pf(X'')$, by (3) and (4) in chapter (I) we get

$$y_{2n-2}w = y_{2n-2}(X_{2n-1} - x_{2n-2\ 2n-1}\ell_{2n-2} + x_{2n-3\ 2n-2}\ \ell_{2n-3})$$

$$= y_{2n-2}\ X_{2n-1} - y_{2n-2}x_{2n-2\ 2n-1}\ell_{2n-2} + y_{2n-2}\ x_{2n-3\ 2n-2}\ \ell_{2n-3}$$

$$= \Sigma_{i=1}^{2n-4}(X_{2n-1})_{i2n-2}\ell_i'' + X_{2n-3\ 2n-2\ 2n-1}\ \ell_{2n-3}''$$

$$- y_{2n-2}x_{2n-2\ 2n-1}\ell_{2n-2} + y_{2n-2}x_{2n-3\ 2n-2}\ \ell_{2n-3}.$$

Now we add and subtract

$$\Sigma_{i=1}^{2n-3}((X_{2n-1})_{i\ 2n-2}x_{i\ 2n-1})y_{2n-1}.$$

Notice then that

$$(X_{2n-1})_{i\ 2n-2}x_{i\ 2n-1}y_{2n-1} + (X_{2n-1})_{i\ 2n-2}\ \ell_i'' = (X_{2n-1})_{i\ 2n-2}\ \ell_i.$$

Hence, we obtain

$$= \Sigma_{i=1}^{2n-4}(X_{2n-1})_{i2n-2}\ \ell_i + X_{2n-3\ 2n-2\ 2n-1}\ \ell_{2n-3}$$

$$- (\Sigma_{i=1}^{2n-3}\ (X_{2n-1})_{i\ 2n-2}x_{i\ 2n-1})y_{2n-1} + \left[\sum_{\substack{i=1 \\ i \neq 2n-2}}^{2n-1} \ell_i y_i\right] x_{2n-2\ 2n-1}$$

$$+ x_{2n-3\ 2n-2}\ y_{2n-2}\ell_{2n-3}$$

$$= \Sigma_{i=1}^{2n-4}[(X_{2n-1})_{i\ 2n-2} - y_i x_{2n-2\ 2n-1}]\ell_i$$

$$+\ell_{2n-3}[X_{2n-3\;2n-2\;2n-1}+y_{2n-3}x_{2n-2\;2n-1}$$

$$+x_{2n-3\;2n-2}y_{2n-2}+x_{2n-3\;2n-1}y_{2n-1}]$$

$$-[\Sigma_{i=1}^{2n-3}(X_{2n-1})_{i\;2n-2}\,x_{i\;2n-1}$$

$$+\ell_{2n-1}x_{2n-2\;2n-1}+x_{2n-3\;2n-1}\ell_{2n-3}]y_{2n-1}\in(\underline{b}, y_{2n-1})$$

Also

$$y_{2n-3}w = y_{2n-3}(X_{2n-1}-x_{2n-2\;2n-1}\ell_{2n-2}+x_{2n-3\;2n-2}\ell_{2n-3})$$

$$= y_{2n-3}\,X_{2n-1}-x_{2n-2\;2n-1}y_{2n-3}\,\ell_{2n-2}+x_{2n-3\;2n-2}y_{2n-3}\ell_{2n-3}$$

$$=\Sigma_{i=1}^{2n-4}(X_{2n-1})_{i\;2n-3}\ell'_{i}-(X_{2n-1})_{2n-3\;2n-2}\ell'_{2n-2}$$

$$-x_{2n-2\;2n-1}y_{2n-3}\ell_{2n-2}+x_{2n-3\;2n-2}y_{2n-3}\ell_{2n-3}$$

We now add and subtract

$$\Sigma_{i=1}^{2n-3}((X_{2n-1})_{i\;2n-2}x_{i\;2n-1})y_{2n-1}.$$

We obtain then

$$=\Sigma_{i=1}^{2n-4}(X_{2n-1})_{i\;2n-3}\ell_{i}-X_{2n-3\;2n-2\;2n-1}\ell_{2n-2}$$

$$-(\Sigma_{i=1}^{2n-3}(X_{2n-1})_{i\;2n-2}\,x_{i\;2n-1})\,y_{2n-1}-x_{2n-2\;2n-1}y_{2n-3}\ell_{2n-2}$$

$$-\left[\sum_{\substack{i=1\\ i\neq 2n-3}}^{2n-1} y_i \ell_i\right] x_{2n-3\ 2n-2}$$

$$= \Sigma_{i=1}^{2n-4} ((X_{2n-1})_{i2n-3} + y_i x_{2n-3\ 2n-2}) \ell_i$$

$$- \ell_{2n-2}(X_{2n-3\ 2n-2\ 2n-1} + x_{2n-2\ 2n-1} y_{2n-3} + x_{2n-3\ 2n-2} y_{2n-2}$$

$$+ x_{2n-3\ 2n-1} y_{2n-1})$$

$$- y_{2n-1}(\Sigma_{i=1}^{2n-2} (X_{2n-1})_{i\ 2n-3}\ x_{i\ 2n-1} + \ell_{2n-1} x_{2n-3\ 2n-2}$$

$$- x_{2n-3\ 2n-1} \ell_{2n-2}) \in (\underline{b}, y_{2n-1}).$$

On the other hand, we had seen that $\overline{T}n = \overline{C}$ ("——" denote reduction module $(\underline{\gamma}) = (x_{2n-2\ 2n-1}, x_{2n-3\ 2n-2}, x_{2n-3\ 2n-1})$ and that γ is a regular sequence on P'/C. Then also $T_n = C$. Moreover $L_n \longrightarrow T_n$ and hence T_n has type 2, deviation 1 and height 2n–2. ▮

Remark 3.6. Let R' = k[X, Y] and grade R' by assigning deg $x_{ij} = 1$, degy_j = n–3. Then L_n R' has a presentation

$$R'^{2n-3}(-(n-2)) \oplus R'^{3}(-(n-3)) \longrightarrow R',$$

and hence a homogeneous resolution of T_n R' starts and ends as

$$0 \to R'^{2}(-(n-2)(2n-3)) \to \cdots \to R'^{2n-3}(-(n-2)) \oplus R'(-(n-3)) \oplus R'(-(n-1)) \to R'.$$

Let now v be an indeterminate over P' and let P = P'[[v]]. In this ring, we consider, with the above notations, the complete intersection ($\underline{b}$, $w+vy_{2n-1}$)

$$\eta = X_{2n-2} - vy_{2n-2} - x_{2n-2\ 2n-1}\ell_{2n-1} + x_{2n-3\ 2n-1}\ell_{2n-3}$$

and

$$\zeta = X_{2n-3} + vy_{2n-3} - x_{2n-3\ 2n-1}\ell_{2n-2} - x_{2n-3\ 2n-2}\ell_{2n-1}$$

with the above notations, we have the following theorem.

Theorem 3.7. Let $n \geq 4$. In P, the ideal $E_n = (\underline{b}, w+vy_{2n-1}, \eta, \zeta)$ is a licci, Gorenstein ideal of height 2n–2 and deviation 2, P/E_n is rigid and E_n is not a hypersurface section.

Proof. We will show $T_n \longrightarrow E_n$ and $E_n \longrightarrow T_n$ and that E_n is a semi–generic tight double link of L_n. Then the conditions on the height, deviation and type will be satisfied. Because L_n is rigid [H–U–1][K], then P/E_n is also rigid, E_n being a semi generic tight double link of L_n. Also, E_n is contained in the square of the maximal ideal, and hence, by (2.12), E_n is not a hypersurface section.

Therefore we only have to compute those links. Set deg x_{ij} and deg y_j as in 3.6, and set deg $v = 2$. A resolution for $(\underline{b}, y_{2n-1}, w)\,R'$ was found in 3.6

$$0 \longrightarrow R'^{2}(-(2n-3)(n-2)) \longrightarrow \cdots \longrightarrow R'^{2n-3}(-(n-2)) \oplus R'(-(n-3)) \oplus R'(-(n-1)) \longrightarrow R'$$

while the Koszul complex of $(\underline{b}, y_{2n-1}v+w)\,R'$ is

$$0 \longrightarrow R'(-(2n-3)(n-2)-(n-1)) \longrightarrow \cdots \longrightarrow R'^{2n-3}(-(n-2)) \oplus R'(-(n-1)) \longrightarrow R'$$

and then a resolution for $(\underline{b}, w+y_{2n-1}v)\,R' : (b, y_{2n-1}, w)\,R'$ starts as

$$R'^{2n-3}(-(n-2)) \oplus R'^{3}(-(n-1)) \longrightarrow R'.$$

Then E_n is generated by $(\underline{b}, y_{2n-1}v+w)$ and 2 elements of degree n–1.

Consider

$$\eta = X_{2n-2} - vy_{2n-2} - x_{2n-2\ 2n-1}\ell_{2n-1} + x_{2n-3\ 2n-1}\ell_{2n-3}$$

and

$$\zeta = X_{2n-3} + vy_{2n-3} + x_{2n-3\ 2n-1}\ell_{2n-2} - x_{2n-3\ 2n-2}\ell_{2n-1}$$

Let X" (resp. Y") the matrix obtained from X (resp. Y) be deleting the (2n–2)th row and column (resp. (2n–2)th row). Then $X_{2n-2} = -\mathrm{Pf}(X")$. Let also

$$[\ell'_1,\ldots,\ell'_{2n-3}\ell'_{2n-1}]^t = X"Y"$$

then, by (1), (3) and (4) in chapter (I), we have

$$\eta y_{2n-1} = (X_{2n-2} - vy_{2n-2} - x_{2n-2\ 2n-1}\ell_{2n-1} + x_{2n-3\ 2n-1}\ell_{2n-3})y_{2n-1} =$$

$$X_{2n-2}y_{2n-1} - vy_{2n-2}y_{2n-1} - x_{2n-2\ 2n-1}\ell_{2n-1}y_{2n-1} + x_{2n-3\ 2n-1}\ell_{2n-3}y_{2n-1}$$

which by (3) in Chapter I becomes

$$= -\Sigma_{i=1}^{2n-4}(X_{2n-2})_{i\ 2n-1}\ell'_i - (X_{2n-2})_{2n-3\ 2n-1}\ \ell'_{2n-3} - vy_{2n-2}y_{2n-1}$$

$$- x_{2n-2\ 2n-1}\ \ell_{2n-1}y_{2n-1} + x_{2n-3\ 2n-1}\ell_{2n-3}\ y_{2n-1}$$

Now, we add and substract

$$(\Sigma_{i=1}^{2n-3}X_{i\ 2n-2\ 2n-1}x_{i\ 2n-2})y_{2n-2}.$$

We obtain then, by also setting the indices in increasing order

$$\Sigma_{i=1}^{2n-4} X_{i\;2n-2\;2n-1}\ell_i + X_{2n-3\;2n-2\;2n-1}\ell_{2n-3} - X_{2n-1}y_{2n-2}$$

$$-vy_{2n-2}y_{2n-1} - x_{2n-2\;2n-1}\ell_{2n-1}y_{2n-1} + x_{2n-3\;2n-1}\ell_{2n-3}y_{2n-1}.$$

And by (4) in Chapter I, we get

$$\Sigma_{i=1}^{2n-4} X_{i\;2n-2\;2n-1}\ell_i + X_{2n-3\;2n-2\;2n-1}\ell_{2n-3} - X_{2n-1}y_{2n-2}$$

$$-vy_{2n-2}y_{2n-1} + x_{2n-2\;2n-1}\sum_{i=1}^{2n-2}\ell_i y_i + x_{2n-3\;2n-1}\ell_{2n-3}y_{2n-1}$$

Set $A_i = X_{i\;2n-2\;2n-1} + x_{2n-2\;2n-1}y_i$, $1 \leq i \leq 2n-4$.

Then, the last expression becomes

$$\Sigma_{i=1}^{2n-4} A_i\ell_i + X_{2n-3\;2n-2\;2n-1}\ell_{2n-3} - X_{2n-1}y_{2n-2} - vy_{2n-1}y_{2n-2}$$

$$+ x_{2n-2\;2n-1}\ell_{2n-3}y_{2n-3} + x_{2n-2\;2n-1}\ell_{2n-2}y_{2n-2}$$

$$+ x_{2n-3\;2n-1}\ell_{2n-3}y_{2n-1} =$$

$$\Sigma_{i=1}^{2n-4} A_i\ell_i + \ell_{2n-3}(X_{2n-3\;2n-2\;2n-1} + x_{2n-2\;2n-1}y_{2n-3}$$

$$+ x_{2n-3\;2n-1}y_{2n-1})$$

$$- y_{2n-2}(X_{2n-1} + vy_{2n-1} - x_{2n-2\;2n-1}\ell_{2n-2})$$

$$= \Sigma_{i=1}^{2n-4} A_i \ell_i +$$

$$\ell_{2n-3}(X_{2n-3\;2n-2\;2n-1} + x_{2n-2\;2n-1}y_{2n-3} + x_{2n-3\;2n-2}y_{2n-2}$$

$$+ x_{2n-3\;2n-1}y_{2n-1})$$

$$- y_{2n-2}(X_{2n-1} + vy_{2n-1} - x_{2n-2\;2n-1}\ell_{2n-2} + x_{2n-3\;2n-2}\ell_{2n-3})$$

$$\in (\underline{b}, y_{2n-1}v+w).$$

Let now $\tilde{X}$ (resp. $\tilde{Y}$) the matrix obtained from X (resp. Y) by deleting the (2n–3)th row and column [resp. (2n–3)th row)]. Then $X_{2n-3} = Pf(\tilde{X})$. Let

$$[\tilde{\ell}_1,\dots,\tilde{\ell}_{2n-4}\tilde{\ell}_{2n-2}\tilde{\ell}_{2n-1}] = \tilde{X}\,\tilde{Y}.$$

Therefore, by (1), (3) and (4) in Chapter I, we have

$$\zeta y_{2n-1} =$$

$$(X_{2n-3} + vy_{2n-3} - x_{2n-3\;2n-2}\ell_{2n-1} + x_{2n-3\;2n-1}\ell_{2n-2})y_{2n-1} =$$

$$X_{2n-3}y_{2n-1} + vy_{2n-3}\,y_{2n-1} - x_{2n-3\;2n-2}\ell_{2n-1}y_{2n-1}$$

$$+ x_{2n-3\;2n-1}\ell_{2n-2}y_{2n-1}.$$

By (3) in chapter (I) this expression becomes

$$\sum_{i=1}^{2n-4} (X_{2n-3})_{i\;2n-1}\tilde{\ell}_i + X_{2n-3\;2n-2\;2n-1}\tilde{\ell}_{2n-2} + vy_{2n-3}y_{2n-1}$$

$$- x_{2n-3\ 2n-2}\ell_{2n-1}y_{2n-1} + x_{2n-3\ 2n-1}\ell_{2n-2}y_{2n-1}.$$

Add and subtract

$$\left(\sum_{i=1}^{2n-2}(X_{2n-3})_{i\ 2n-1}x_{i\ 2n-3}\right)y_{2n-3}$$

we obtain then

$$\Sigma_{i=1}^{2n-4}(X_{2n-3})_{i\ 2n-1}\ell_i + X_{2n-3\ 2n-2\ 2n-1}\ell_{2n-2}+vy_{2n-3}y_{2n-1}$$

$$-\left(\Sigma_{i=1}^{2n-2}(X_{2n-3})_{i\ 2n-1}x_{i\ 2n-3}\right) y_{2n-3}-x_{2n-3\ 2n-2}\ell_{2n-1}y_{2n-1}$$

$$+x_{2n-3\ 2n-1}\ell_{2n-2}y_{2n-1}.$$

Which by (2) and (4) in chapter I becomes

$$\Sigma_{i=1}^{2n-4}(X_{2n-3})_{i\ 2n-1}\ell_i+ X_{2n-3\ 2n-2\ 2n-1}\ell_{2n-2}+vy_{2n-3}y_{2n-1}$$

$$+X_{2n-1}y_{2n-3}+\left(\Sigma_{i=1}^{2n-2}\ell_iy_i\right)x_{2n-3\ 2n-2} + x_{2n-3\ 2n-1}\ell_{2n-2}y_{2n-1}.$$

Set $B_i = (X_{2n-3})_{i\ 2n-1} + y_ix_{2n-3\ 2n-2}$, $1 \leq i \leq 2n-4$.
Then we obtain

$$\Sigma_{i=1}^{2n-4}B_i\ell_i+\ell_{2n-2}(X_{2n-3\ 2n-2\ 2n-1}+y_{2n-1}\ x_{2n-3\ 2n-2}$$

$$+ x_{2n-3\ 2n-2}y_{2n-2}) - y_{2n-3}(X_{2n-1} + vy_{2n-1}+ \ell_{2n-3}x_{2n-3\ 2n-2})$$

$$= \Sigma_{i=1}^{2n-4} B_i \ell_i +$$

$$\ell_{2n-2}(X_{2n-3\ 2n-2\ 2n-1} + y_{2n-1}x_{2n-3\ 2n-1} + x_{2n-3\ 2n-2}y_{2n-2}$$

$$+ x_{2n-2\ 2n-1}y_{2n-3})$$

$$- y_{2n-3}(X_{2n-1} + vy_{2n-1} + \ell_{2n-3}x_{2n-3\ 2n-2} - x_{2n-2\ 2n-1}\ \ell_{2n-2})$$

$$\in (\underline{b}, w + y_{2n-1}\ v).$$

Therefore, $(b, w + y_{2n-1}v, \eta, \zeta)\ R' \subseteq (\underline{b}, w + y_{2n-1}v)\ R':(\underline{b}, w, y_{2n-1})R'$. If $\eta \in (b, w + y_{2n-1}\ v)$, then the image of η is in the image of $(\underline{b}, w + y_{2n-1}v)$ in

$$\frac{R'}{(v,\ x_{2n-3\ \ 2n-2},\ x_{2n-3\ \ 2n-1},\ x_{2n-2\ \ 2n-1})}.$$

Let "——" denote images in this ring. Suppose that

$$\overline{\zeta} = \overline{X}_{2n-3} \in (\ell_1,\ldots,\ell_{2n-4}, X_{2n-3\ 2n-2\ 2n-1}, \overline{X}_{2n-1})$$

where $\overline{X}$ is the matrix such that $\overline{x}_{ij} = x_{ij}$ for

$$(i, j) \notin \{(2n-2, 2n-1), (2n-3, 2n-2), (2n-3, 2n-1)\}$$

and $\overline{x}_{ij} = 0$ if (i, j) is in this set.

As usual it would follow that $\overline{X}_{2n-3}$ is in $(X_{2n-3\ 2n-2\ 2n-1}, \overline{X}_{2n-1})$ (see proof of 2.9). But that says that $X_{2n-3} \equiv 0 \bmod(\{x_{1j} | 2 \le j \le 2n-4\} \cup \{x_{i\ 2n-3}\})$ and this is false. If $\eta \in (b, w+y_{2n-1}v, \zeta)$ then, as usual,

$$\overline{\eta} = \overline{X}_{2n-2} \in (X_{2n-3\ 2n-2\ 2n-1}\overline{X}_{2n-3}, \overline{X}_{2n-1})$$

and hence

$$\overline{X}_{2n-2} \equiv 0 \bmod(\{x_{1j} | 2 \le j \le 2n-4\} \cup \{X_{i\ 2n-2}\})$$

and this is false.

If $\ell_1 \in (\ell_2,\dots,\ell_{2n-4}, w+y_{2n-1}v, \eta, \zeta)$, then in $\overline{R'}$ $\ell_1 \in (\ell_2,\dots,\ell_{2n-4})$ and this is false.

Thus, we have seen $(\underline{b}, w+y_{2n-1}v, \eta, \zeta)\, R'$ is not a complete intersection. Moreover η and ζ are homogeneous of degree n–1 and part of a minimal system of generators of

$$\frac{(\underline{b},\ w+y_{2n-1}\,v)R''\ :(b,\ w,\ y_{2n-1}R')}{(\underline{b},w+y_{2n-1}v)R'}$$

On the other hand, we had seen that this module is generated by two homogeneous elements of degree n–1. Then

$$(\underline{b},\ w+y_{2n-1}v)R':(\underline{b},\ w,\ y_{2n-1})R' = (\underline{b},\ +\ y_{2n-1}v,\ \eta,\ \zeta)R'$$

Hence in particular, $(b,\ w+y_{2n-1}v)P\colon T_n = E_n$. Now we are done and all our claims follow. ∎

We are interested in the relation between E_n and F_n. Before studying it, we need a lemma.

Lemma 3.8. Let $\overline{X}$ be the matrix obtained in 3.7, let $n \geq 3$ and let Y as usual. In $k = [\overline{X}, Y]$ the ideal

$$\overline{F}_n = (\ell_1, \dots, \ell_{2n-4}, \overline{X}_{2n-3}, \overline{X}_{2n-2}, \overline{X}_{2n-1}, X_{2n-3\ 2n-2\ 2n-1})$$

has height 2n–2.

Proof. Induct on n. Notice first that $\overline{F}_3 = F_3$ and hence ht $\overline{F}_3 = 2(3)-2 = 4$.

For larger n notice that the ideal generated by the entries of the matrix formed with the first 2n–4 rows and columns of $\overline{X}$ has height (n–2) (2n–5) (2.1). Let p be in Spec($k[\overline{X}, Y]$) be such that ht $p \leq 2n-2$ and $\overline{F}_n \subsetneq p$. Then (see the proof of 2.8) one of those entries, w.l.o.g., x_{12}, is not in p.

Then x_{12} is invertible in $k[\overline{X}, Y]_p$. Hence there is a matrix A, invertible over $k[\overline{X}, Y]_p$ such that

$$A^T X A = \left[\begin{array}{cc|cc} 0 & 1 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & 0 & & \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{0}} & X' & X'' \\ \cline{3-4} & & (-X'')^t & 0_{3\times 3} \end{array}\right]$$

and the entries of X' and X" are generic over $k[\{x_{ij} \mid i \leq 2\}, x_{12}^{-1}]$. Replace Y by $A^{-1}Y$. Then in $k[\overline{X}, Y]_p$, $\overline{F}_n$ is $(y_1, y_2, \overline{F}_{n-1})$ which, by induction has height $2(n-1)-2+2 = 2n-2$. ∎

Theorem 3.9. (S, E_n) and (S, F_n) are in the same Herzog class.

Proof. In $k[\overline{X}, Y]$, both pairs are $(k[\overline{X}, Y], \overline{F}_n)$, and grade E_n = grade $\overline{F}_n$ = grade F_n, by 3.8. ▮

Now we can answer the question posed after 3.3.

Corollary 3.10. F_n is not a hypersurface section if $n \geq 4$.

Proof. Theorem 3.9, theorem 3.7 and proposition 2.13. ▮

We close this chapter with the following proposition about F_n.

Proposition 3.11. Let $S = k[X, Y]_{(X,Y)}$ and let $n \geq 3$. Then S/F_n is (R_2) but not (R_3). In particular, S/F_n is a normal domain, but it is not rigid.

Proof. For n = 3

$$X = \begin{bmatrix} 0 & x_{12} & x_{13} & x_{14} & x_{15} \\ -x_{12} & 0 & x_{23} & x_{24} & x_{25} \\ -x_{13} & -x_{23} & 0 & x_{34} & x_{35} \\ -x_{14} & -x_{24} & -x_{34} & 0 & x_{45} \\ -x_{15} & -x_{25} & -x_{35} & -x_{45} & 0 \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

and

$$F_3 = (x_{12}, x_{13}y_3 + x_{14}y_4 + x_{15}y_5, x_{23}y_3 + x_{24}y_4 + x_{25}y_5,$$

$$x_{14}x_{25} - x_{15}x_{24}, x_{13}x_{24} - x_{14}x_{23}, x_{13}x_{25} - x_{15}x_{23}).$$

Let p be a prime ideal of S of height at most 6 containing F_3. Because $x_{12} \in p$, $\{x_{ij} | i \leq 2, 3 \leq j \leq 5\} \not\subset p$, and therefore one of those indeterminates is not in p. Since each one of them appears in 3 generators of F_3, one can see then that F_3S_p has at least 4 elements, part of a regular system of parameters of S_p. Hence

(R/F_3) is (R_2).

If $q = (\{x_{12}\} \cup \{x_{ij} | i = 1,2,\ j = 1,2,3\})$. Then ht $q = 7$ and $F_3 \subset q$. However F_3S_q has at most 3 elements part of a regular system of parameters of S_q. Hence S/F_3 is not (R_3).

For $n \geq 4$, let $p \in \mathrm{Spec}(S)$ such that ht $p \leq 2n$. Let X' be the matrix formed with the first 2n–4 rows and 2n–4 columns of X. Then $I_1(X)' \not\subseteq p$. Otherwise, for $n = 4$, p would also contain $\Sigma_{j=5}^{7} x_{ij}y_j$, $1 \leq i \leq 3$ and then ht $p \geq 9$. For larger n $I_1(x') \not\subseteq p$, since $ht(I_1(X')) > (n-2)(2n-5)$, (see 2.1) and $(n-2)(2n-5) > 2n$ if $n > 4$.

Then, w.l.o.g., $x_{12} \notin p$. Hence, in S_p there is an invertible matrix A such that

$$A^TXT = \left[\begin{array}{cc|c} 0 & 1 & \\ -1 & 0 & 0 \\ \hline \multicolumn{2}{c|}{0} & X' \end{array}\right].$$

and the entries of X' are generic over $k[x_{12}^{-1}, \{x_{ij} | i \leq 2\}]$. If we replace Y by $A^{-1}Y$, then in S_p, we have $F_nS_p = (y_1, y_2, F_{n-1})$, $n \geq 4$, and then, by induction S/F_n is (R_2).

To show that R/F_n is not R_3, localize at the prime

$$p = (\{x_{ij}/2n-5 \leq i,\ 2n-5 \leq j)\}$$

and observe that $(R/F_n)_p$ is obtained from R/F_3 by a purely trascendental extension of the residual class field of R/F_3.

SUMMARY

We keep the notation corresponding to each result.

(1) The ideal $(\ell_1,\dots,\ell_{2n-2})$ is a prime ideal of height 2n–2 in R[X, Y] where R is a Cohen–Macaulay domain (proposition 2.3).

(2) The ideal $K_n = (\ell_1,\dots,\ell_{2n-3}, X_{2n-1\ 2n}, X_{2n-2\ 2n}, X_{2n-2\ 2n-1}, Pf(X))$ is a licci, Gorenstein ideal of height 2n–1 and deviation 2 (theorem 2.9).

(3) Let (S, I) be a pair where $S = k[[\underline{W}]]$ and I a licci ideal of S. Let $(\tilde{S}, \tilde{I})$ be in the same Herzog class as (S, I). Then I is a hypersurface section if and only if $\tilde{I}$ is. (proposition 2.13)

(4) (S, K_4) is not in the same Herzog class as (S, H_n) for any $n \geq 2$ (theorem 2.14).

(5) Let I be a perfect almost complete intersection ideal of height 4 and type 2 in R. Then, there is a pair (S, J) in the same Herzog class as (R, I) such that either:

i) J is a hypersurface section or

ii) There is a 5×5 alternating matrix A and a 5×1 matrix B, a_{ij}, $b_y \in m_s$, such that $J = I_1(AB)$ (proposition 2.16).

(6) If I is a licci Gorenstein ideal of R of height 5 and deviation 2, then there is a pair (R', I') in the same Herzog class as (R, I) such that either:

i) I' is a double hypersurface section or

ii) there is a 6×6 alternating matrix A and a 6×1 matrix B such that I' = $(I_1(AB), Pf(A))$, $a_{ij}, b_j \in m_{R'}$. (theorem 2.22).

(7) With the notations of (6) if J is a licci type 2 almost complete intersection of height 5. Then, there is a pair (R',J') in the same Herzog class as (R,J) such that either:

i) J' is a hypersurface section or

ii) $J' = (\ell_1,\ldots,\ell_4, A_{56}, Pf(A))$ (corollary 2.24).

(8) Let $R = k[[x_1,\ldots,x_n]]$, and let J be a licci Gorenstein ideal of R of height 5 and deviation 2. Then either I is a specialization of H_3 or a specialization of (H_2, x, y) (corollary 2.25).

(9) F_n is a licci Gorenstein ideal of deviation 2 and height 2n–2 in $k[X, Y]_{(X,Y)}$. (proposition 3.3).

(10) E_n is a licci Gorenstein ideal of height 2n–2 and deviation 2 is S, such that S/E_n is rigid and E_n is not a hypersurface section (theorem 3.7).

(11) (S, E_n) and (S, F_n) are in the same Herzog class, for $n \geq 4$ (theorem 3.9).

(12) F_n is not a hypersurface section, $n \geq 4$ (corollary 3.10).

(13) R/F_n is (R_2) but not (R_3) (proposition 3.11).

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