This is to certify that the

dissertation entitled

Minimum Distance Estimation In
An Additive Effects Outliers Model

presented by

Sunil Kumar Dhar

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Statistics

H Koul
Major professor

Date April 8, 1988

MINIMUM DISTANCE ESTIMATION IN AN ADDITIVE EFFECTS OUTLIERS MODEL

by

Sunil Kumar Dhar

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Statistics and Probability

1988

ABSTRACT

MINIMUM DISTANCE ESTIMATION IN AN ADDITIVE EFFECTS OUTLIERS MODEL

BY

SUNIL KUMAR DHAR

Consider the additive effects outliers (A.O.) model where one observes $Y_{j,n} = X_j + v_{j,n}$, $0 \le j \le n$, with $X_j = \rho X_{j-1} + \epsilon_j$, $j = 0, \pm 1, \pm 2, \ldots$, $|\rho| < 1$. The sequence of r.v.s $\{X_j, j \le n\}$ is independent of $\{v_{j,n}, 0 \le j \le n\}$ and $v_{j,n}$, $0 \le j \le n$, are i.i.d. with d.f. $(1-\gamma_n)I[x \ge 0] + \gamma_n L_n(x)$, $x \in \mathbb{R}$, $0 \le \gamma_n \le 1$, where the d.f.s L_n, $n \ge 0$ are not necessarily known and ϵ_j's are i.i.d.. This thesis discusses the class of minimum distance estimators of ρ defined by Koul (1986, *Ann. Stat.* 14, 1194–1213) under the above A.O. model. These estimators are shown to be asymptotically normally distributed and their influence functions are also computed.

The second part of the thesis presents the asymptotic behavior of another class of minimum distance estimators defined by Heathcote and Welsh (1983, *J. Appl. Prob.* 20, 737–753) under the above model. This class of estimators is obtained by minimizing the negative of log modulus square of the empirical characteristic function (e.c.f.) of the residuals $Y_{j,n} - tY_{j-1,n}$, $1 \le j \le n$, as a function of t, for each value of the e.c.f.. Uniform consistency and uniform strong consistency of these estimators are proven, uniformity being taken over all possible estimators defined above. The weak convergence of these estimators to a Gaussian process is also established under the above A.O. model.

In both the problems it is observed that the asymptotic biases vanish if $\sqrt{n}\,\gamma_n = o(1)$ or if $\sqrt{n}\,\gamma_n = O(1)$ and $Z_n \to 0$ in probability, where Z_n is a r.v. with the d.f. L_n. The asymptotic biases are non–vanishing when $\sqrt{n}\,\gamma_n \to \gamma$ and $Z_n \to Z$ in probability, where Z is some r.v. and $0 < \gamma \leq 1$.

To my parents and my wife

ACKNOWLEDGEMENTS

I wish to express my heart felt thanks to Professor Hira Lal Koul for being my guide during my entire stay here. I sincerely thank him for suggesting the problem, inspiring and guiding me during the preparation of this dissertation.

My sincere thanks to Professor James Hannan for his constant help and encouragement. His constructive criticism was a great help in improving the results of my thesis.

I am thankful to Professors James Hannan, Habib Salehi and Joseph Gardiner for serving on my committee. To professor Habib Salehi I extend my special thanks for always taking interest in my well being.

I wish to express my gratitude to all my teachers for everything they taught me.

Finally, thanks to Ms. Cathy Sparks for typing the manuscript.

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CHAPTER 1.

0. **Introduction and Summary.**

Let F and L_n, $n \geq 0$, be symmetric distribution functions (d.f.s) on the real line $\mathbb{R}$, symmetric about 0. Throughout this thesis F is assumed to have a density $f \geq 0$. Let $\{\gamma_n, n \geq 0\}$ be a sequence of numbers in $[0,1]$ converging to 0 as $n \to \infty$. Define

$$(0.1) \qquad \beta_n(x) := (1-\gamma_n)\, I[x \geq 0] + \gamma_n L_n(x), \qquad x \in \mathbb{R},$$

where $I[A]$ denotes the indicator function of the set A. Let ϵ_j, $j = 0, \pm 1, \pm 2, \ldots$, be independent and identically distributed (i.i.d.) F random variables (r.v.s), with $E\epsilon_0^2 < \infty$. Let $v_{j,n}$, $0 \leq j \leq n$, be i.i.d. β_n r.v.s.

We consider the model in which one observes, at stage n, r.v.s $Y_{j,n}$, $0 \leq j \leq n$, satisfying

$$(0.2) \qquad Y_{j,n} = X_j + v_{j,n}, \qquad j = 0, 1, \ldots, n,$$

with $\{X_j\}$ obeying the autoregressive model of order one (AR(1)), viz.

$$(0.3) \qquad X_j = \rho X_{j-1} + \epsilon_j, \qquad |\rho| < 1, \qquad j = 0, \pm 1, \pm 2, \ldots,$$

where $\{X_j\}$ is stationary. Moreover, $\{X_j, j \leq n\}$ is assumed to be independent of $\{v_{j,n}, 0 \leq j \leq n\}$, $n \geq 0$. This chapter studies the problem of estimating ρ.

Denby and Martin (1979) called the model in (0.2) and (0.3) the additive effects outliers (A.O.) model. All the above assumptions on

$\{Y_j, 0 \le j \le n\}$, $\{X_j\}$, $\{v_{j,n}, 0 \le j \le n\}$ and $\{\epsilon_j\}$ will be referred to as the model assumptions. The assumptions on $\{v_{j,n}, 0 \le j \le n\}$ reflect the situation in which the outliers are isolated in nature. Isolated outliers are defined by Martin and Yohai (1986) as the outliers any pair of which are separated in time by a nonoutlier. Martin and Yohai (1986, page 796, Theorem 5.2 and Comment 5.1) also made the assumption of independence of the process $\{X_j, j \le n\}$ and $\{v_{j,n}, j = 0, 1, 2, \ldots, n\}$, $n \ge 0$.

In practice, an appropriate model for time series data with outliers may be difficult to specify. Fox (1972) and Martin and Zeh (1977) point out the importance of finding the difference between various types of outliers in order to effectively deal with them. The two types of outliers in time series analysis that have received considerable attention are A.O. and innovations outliers (I.O.). In the I.O. model one observes X_j of (0.3), and that large data points are consistent with the future and perhaps the past values. On the other hand, in the A.O. model outliers are generally not consistent with the past or future values of the unobservable process X_j. The additive effects outliers may occur due to measurement errors like key punch errors (Denby and Martin, 1979) or round off errors in which case L_n is taken to be uniform d.f. on the interval [–.5, .5] (Machak and Rose, 1984).

Denby and Martin (1979) studied the least squares estimator, M–estimators and a class of generalized M–estimators (GM–estimators) of ρ under the above models; they took F and L_n to be $\mathcal{N}(0,\sigma_\epsilon^2)$ and $\mathcal{N}(0,\sigma^2)$, respectively. Under their A.O. model all of these estimators have non–vanishing asymptotic biases with a possible reduction in biases for GM–estimators.

This chapter of the thesis studies the behavior of the class of minimum distance estimators of Koul (1986)(KL) defined under the A.O. model (0.2) and (0.3). To define this class of estimators, let h be any Borel measurable function from $\mathbb{R}$ to $\mathbb{R}$, H be any non-decreasing function on $\mathbb{R}$, define

$$(0.4)\qquad S_h(x,t) = n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1,n})\{I[Y_{j,n} \le x + tY_{j-1,n}] - \\ - I[-Y_{j,n} < x - tY_{j-1,n}]\}, \qquad x \in \mathbb{R}$$

and

$$(0.5)\qquad M(t) = \int S_h^2(x,t)\, dH(x), \qquad t \in \mathbb{R}.$$

Denote $\hat{\rho}_h(H)$ to be a measurable minimizer of M, if it exists. Then $\hat{\rho}_h(H)$ satisfies

$$(0.6)\qquad \inf_t M(t) = M[\hat{\rho}_h(H)].$$

KL studied this class of estimators under the I.O. model. Among other things, he proved that the estimator with $h(x) \equiv x$ has the smallest asymptotic variance within a certain class of estimators $\hat{\rho}_h(H)$.

Section 1 gives the assumptions that facilitate the small and large sample study of $\hat{\rho}_h(H)$. It also contains a proof of the existence of $\hat{\rho}_h(H)$. Section 2 discusses the asymptotic distribution of the proposed class of estimators. Theorem 2.1 uniformly approximates $M(t)$, $t \in \mathbb{R}$, by a quadratic function of t, uniformity taken over small closed neighborhoods of the true parameter ρ. Its proof uses the technique presented by KL (proof of Theorem 3.1) and Koul and DeWet (1983, proof of Theorem 5.1). The techniques of Koul and DeWet (1983, Corollary 5.1) and Koul (1985, Lemma 3.1, Theorem 3.1) are used to obtain an asymptotic approximation for $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$ in Theorem 2.2.

Throughout this thesis, the asymptotic bias of $\hat{\rho}_h(H)$ is defined as the mean of the asymptotic distribution of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$. Section 3 contains the study of the asymptotic normality of a suitably standardized $\hat{\rho}_h(H)$ and its asymptotic bias. Section 3 thus begins by reproducing in Theorem 3.1, the Central Limit Theorem (C.L.T.) for ℓ–mixing set of processes proved by Withers (1981, Theorem 2.1, and 1983). Lemma 3.2 gives a general method to verify that the set of processes involved in the approximation of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$, is α–mixing. Definitions of ℓ–mixing and α–mixing sets of processes are as in Withers (1981) and have been restated in this section for the sake of completeness. Using Ibragimov and Linnik (1975, Theorem 17.2.2), the Lemma 3.3 gives a general method to compute the asymptotic variance. Theorem 3.1, Lemma 3.2 and Lemma 3.3 are used to prove, in Theorem 3.4, the asymptotic normality of $\sqrt{n}[\hat{\rho}_h(H) - \rho]$, when appropriately centered. At this point, Remark 3.0 disscusses conditions under which the model assumption of symmetry of F or L_n's about 0, could be dropped in the study of the asymptotic behavior of $\hat{\rho}_h(H)$. Theorem 3.5 contains conditions under which $\hat{\rho}_h(H)$ has vanishing and nonvanishing asymptotic bias and Remark 3.1 states the corresponding asymptotic normality results.

As in KL, Remark 3.2 notes that the optimal estimator which minimizes the asymptotic variance is the one with $h(x) \equiv x$. For this estimator we see that ρ and the asymptotic bias of $\hat{\rho}_h(H)$ have the same sign.

Remark 3.3 gives a smaller set of sufficent conditions which imply the assumptions of Section 1. Remarks 3.4 and 3.5 discuss these assumptions when $H(x) \equiv x$ or H is bounded, respectively.

Remark 3.6 points out that all the assumptions of Section 1 are satisfied by $h(x) \equiv x$, H given by $dH = \{F(1-F)\}^{-1}dF$ and F either equal to the d.f. of a double exponential or $\mathcal{N}(0,\sigma^2)$.

Finally, using Martin and Yohai (1986, Definition 4.2), under some regularity conditions, the influence function of $\hat{\rho}_h(H)$ is computed in Remark 3.7. The influence function turns out to be proportional to the asymptotic bias of $\hat{\rho}_h(H)$.

Before proceeding further, observe that the process $\{X_j\}$ is stationary ergodic and X_{j-1} is independent of ϵ_j, $j \geq 1$. From the assumptions on $v_{j,n}$'s and X_j's it can be seen for each n, that the process $\{(X_j, v_{j,n}),$ $0 \leq j \leq n\}$ is stationary ergodic and hence so is $\{Y_{j,n}, 0 \leq j \leq n\}$. These observations will be used in the sequel repeatedly.

Notation. Throughout this thesis, by $o_p(1)$ $(O_p(1))$ is meant a sequence of r.v.s that converges to zero in probability (is tight or bounded in probability). Also, let Z_n be a r.v. with d.f. L_n, $n \geq 0$.

1. Assumptions and existence.

This section contains a list of assumptions that will be used subsequently. Using some of these assumptions it also contains a proof of the existence of $\hat{\rho}_h(H)$ of (0.6). We begin by stating the

Assumptions.

A_1: $n\gamma_n^2 = O(1)$, where $\gamma_n \in [0,1]$.

A_2: H is a non-decreasing continuous function such that

$$|H(x) - H(y)| = |H(-x) - H(-y)| \ \forall\ x, y \in \mathbb{R}.$$

<u>Note.</u> H generates a unique Lebesgue–Stieltjes messure, hence H will also be used to represent a measure in the sequel repeatedly.

A_3: (a) $0 < EX_0^2 < \infty$. (b) For some $\delta > 0$, $0 < E|h(X_0)|^{2+\delta} < \infty$.

A_4: $EX_0^2h^2(X_0) < \infty$.

A_5: For some $\delta > 0$, $\sup_n E \int |h(X_0+z)|^{2+\delta} dL_n(z) < \infty$.

A_6: $xh(x) \geq 0 \quad \forall\ x$ or $xh(x) \leq 0 \quad \forall\ x$.

A_7: $0 < \int f^k\ dH < \infty$, where $k = 1, 2$.

A_8: (a) $\sup_n \int Ef(x-Z_n)\ dH(x) < \infty$. (b) $\overline{\lim_n} \int Ef^2(x-Z_n)\ dH(x) < \infty$.

A_9: $\exists\ C_0 > 0$ such that $\int |f(x-u) - f(x)|\ dx \leq C_0|u|, \ \forall\ u \in \mathbb{R}$.

A_{10}: (a) $\lim_{s\to 0} \int E|X_0|h^2(X_0)[f(x+sX_0) - f(x)]\ dH(x) = 0$,

(b) $\lim_{s\to 0} \int EX_0^2h^2(X_0)[f(x+sX_0) - f(x)]^2\ dH(x) = 0$.

In all the assumptions to follow $0 \leq C_n \in \mathbb{R}$ are such that $C_n \to 0$.

A_{11}: (a) $\overline{\lim_n} \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\left[\int |X_0|h^2(X_0)f(x+sX_0-z)\ dL_n(z)\right] dH(x)\ ds < \infty$.

(b) $$\overline{\lim_n} \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\left[X_0^2 h^2(X_0) \int f^2(x+sX_0-z)\ dL_n(z)\right] dH(x)ds < \infty.$$

A_{12}: $$\sup_n \int E\left[\int |X_0+z|\,|h(X_0+z)|\ Ef(x+\rho z-jZ_n)\ \ dL_n(z)\right]\ dH(x) < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A_{13}: $$\overline{\lim_n} \int E\left[\int (X_0+z)^2 h^2(X_0+z) Ef^2(x+\rho z-jZ_n)\ dL_n(z)\right]\ dH(x) < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A_{14}: $$\overline{\lim_n} \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\left[\int |X_0+z|\, h^2(X_0+z) \{\int f(x+\rho z+s[X_0+z]-ju)\ dL_n(u)\}\cdot \cdot dL_n(z)\right]\ dH(x)\ ds < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A_{15}: $$\overline{\lim_n} \frac{1}{2C_n} \int_{-C_n}^{C_n} \int E\left[\int (X_0+z)^2 h^2(X_0+z) \int f^2(x+\rho z+s[X_0+z]-ju)\ dL_n(u)]\cdot \cdot dL_n(z)\right]\ dH(x)\ ds < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

A_{16}: $$\overline{\lim_n} \int Eh^2(Y_{0,n}) G_n(x+\rho v_{0,n})[1 - G_n(x+\rho v_{0,n})]\ dH(x) < \infty.$$

A_{17} $$\overline{\lim_n} \int E\left[\int h^2(X_0+z)[E\{F(x+\rho z-jZ_n)-F(x-\rho z-jZ_n)\}]^2 dL_n(z)\right] dH(x) < \infty,$$

holds with (a) $j = 0$ and (b) $j = 1$.

<u>Note</u>. In Remarks 3.2 – 3.6 various sets of sufficient conditions that imply the above assumptions are given.

Existence.

Lemma 1.1. Assume that A_2 and A_6 hold. Then either (i) $H(\mathbb{R}) = \infty$ or (ii) $H(\mathbb{R}) < \infty$ and $h(0) = 0$, implies the existence of $\hat{\rho}_h(H)$.

Proof. The proof will be given only for the case $xh(x) \geq 0 \ \forall x \in \mathbb{R}$; the proof in the case $xh(x) \leq 0 \ \forall x \in \mathbb{R}$ is exactly the same, with h replaced by $-h$.

Define

$$(1.1)\qquad \begin{aligned} c(x) &:= n^{-1/2}h(0) \sum_{j=1}^{n} I[Y_{j-1,n} = 0]\{I[Y_{j,n} \leq x] - I[-Y_{j,n} < x]\}, \quad x \in \mathbb{R}, \\ d &:= n^{-1/2} \sum_{j=1}^{n} |h(Y_{j-1,n})| I[Y_{j-1,n} \neq 0], \qquad b := \max_{1 \leq j \leq n} |Y_{j,n}|. \end{aligned}$$

Observe that $c(x) = 0$ for $|x| > b$ and hence c is H–integrable. Now rewrite

$$(1.2)\qquad \begin{aligned} S_h(x,t) = n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1,n}) I[Y_{j-1,n} \neq 0]\{I[Y_{j,n} \leq tY_{j-1,n} + x] - \\ - I[-Y_{j,n} < -tY_{j-1,n} + x]\} + c(x), \quad x, t \in \mathbb{R}. \end{aligned}$$

The first term on the r.h.s. of (1.2) is bounded by d. Hence

$$(1.3)\qquad c(x) - d \leq S_h(x,t) \leq d + c(x), \qquad t, x \in \mathbb{R}.$$

Further $xh(x) \geq 0$ implies

$$(1.4)\qquad S_h(x,t) \longrightarrow c(x) \pm d \quad \text{as} \quad t \longrightarrow \pm \infty.$$

Moreover, under A_2, H is continuous and by calculations similar to those in KL (equation (2.1) – (2.3))

$$(1.5)\qquad M(t) = n^{-1} \sum_i \sum_j h(Y_{i-1,n})h(Y_{j-1,n})[|H(Y_{i,n} - tY_{i-1,n}) - H(-Y_{j,n}+tY_{j-1,n})| - |H(Y_{i,n}-tY_{i-1,n}) - H(Y_{j,n}-tY_{j-1,n})|].$$

Hence M is continuous on $\mathbb{R}$. Now consider

Case (i). $H(\mathbb{R}) = \infty$. If $d = 0$ then from (1.3) $M \equiv \int c^2 dH$. Hence a trivial measurable minimizer exists. Now let $d > 0$. Since c and c^2 are H–integrable,

$$\int (c(x) \pm d)^2\, dH(x) = \infty.$$

Hence from (1.4) and the Fatou Lemma,

$$M(t) \longrightarrow \infty \quad \text{as} \quad t \longrightarrow \pm\infty.$$

This and the continuity of M ensure the existence of a measurable minimizer of M.

Case (ii). $H(\mathbb{R}) < \infty$. From (1.1), $h(0) = 0$ implies that $c \equiv 0$. Thus (1.3), (1.4) and the Dominated Convergence Theorem (D.C.T.) give

$$M(t) \longrightarrow d^2 H(\mathbb{R}) \quad \text{as} \quad t \longrightarrow \pm\infty.$$

This with (1.3) and the continuity of M ensure the existence of a measurable minimizer of M. □

2. Asymptotic approximation of $\hat{\rho}_h(H)$ when centered.

For stating the main results of this section we need some more notation. Let G_n denote the d.f. of $v_{j,n} + \epsilon_j$, $j \leq n$. Since $v_{j,n}$ and ϵ_j are independent,

$$(2.1)\qquad G_n(x) = (1-\gamma_n)F(x) + \gamma_n EF(x-Z_n), \qquad x \in \mathbb{R}.$$

A density of G_n is

$$(2.2) \qquad g_n(x) = (1-\gamma_n)f(x) + \gamma_n \; Ef(x-Z_n), \qquad x \in \mathbb{R}.$$

Define

$$(2.3) \qquad Q(t) = \int \Big[S_h(x,\rho) + n^{1/2}(t - \rho)\{a_n(x) + \underline{a}_n(x)\}\Big]^2 dH(x), \; t \in \mathbb{R},$$

where $a_n(x) = EY_0h(Y_0)g_n(x + \rho v_0)$ and $\underline{a}_n(x) = a_n(-x)$, $x \in \mathbb{R}$.

We shall first uniformly approximate M by Q, uniformity taken over small closed neighborhoods of ρ. Using this approximation we obtain the asymptotic approximation of $\hat{\rho}_h(H)$ in terms of the minimizer of Q.

From here on we shall suppress n in the r.v.s $v_{j,n}$, a_n and $Y_{j,n}$, etc., for the sake of convenience.

<u>Theorem 2.1</u>. Let all the model assumptions (0.1) – (0.3) hold. Further let $A_1 - A_4$, A_7, A_8 and $A_{10} - A_{17}$ hold. Then for any $0 < b < \infty$

$$E\left[\sup_{n^{1/2}|t-\rho|\le b} |M(t) - Q(t)|\right] = o(1).$$

<u>Proof</u>. The techniques used in here are as in KL. Define, $\forall$ x, t $\in \mathbb{R}$,

$$W(x,t) = \{n^{-1/2}\sum_{j=1}^{n} h(Y_{j-1})I[v_j-\rho v_{j-1}+\epsilon_j \le x+n^{-1/2}tY_{j-1}]\} - \mu(x,t)$$

(2.4) with

$$\mu(x,t) = n^{-1/2}\sum_{j=0}^{n-1} h(Y_{j-1})G_n(x+n^{-1/2}tY_{j-1}+\rho v_{j-1}).$$

Note that the j^{th} summand in $W(x,t)$ is conditionally centered, given (v_{j-1}, Y_{j-1}). From (0.4) and (2.4) we get

$$(2.5)\qquad S_h(x,n^{-1/2}t+\rho) = W(x,t) + W(-x,t) + \mu(x,t) + \mu(-x,t) - n^{-1/2}\sum_{j=1}^{n} h(Y_{j-1}).$$

From (0.5) and (2.5),

$$M(n^{-1/2}t+\rho) = \int\Big[W(x,t) - W(x,0) + W(-x,t) - W(-x,0) + S_h(x,\rho) + t[a(x) + a(-x)] + \mu(-x,t) - \mu(-x,0) - ta(-x) + \mu(x,t) - \mu(x,0) - ta(x)\Big]^2 dH(x).$$

From the above representation of $M(n^{-1/2}t+\rho)$, using (2.3), the Hölder inequality, the Transformation Theorem for integrals and A_2,

$$|M(n^{-1/2}t+\rho) - Q(n^{-1/2}t+\rho)|$$

$$(2.6)\qquad \le 8|W(t) - W(0)|_H^2 + 8|\mu(t) - \mu(0) - ta|_H^2 + 4(|S_h(\rho) + t[a + \underline{a}\,]|_H^2)^{1/2}\Big[(|W(t) - W(0)|_H^2)^{1/2} + (|\mu(t) - \mu(0) - ta|_H^2)^{1/2}\Big],$$

where $W(t)$, $S_h(\rho)$ and $\mu(t)$ are functions $W(x,t)$, $S_h(x,t)$ and $\mu(x,t)$ with their integrating variables suppressed and $|\ |_H^2$ denotes the square of the $L^2(H)$-norm. From (2.6) it suffices to prove:

(i) $\quad E \sup_{|t|\le b} |\mu(t) - \mu(0) - ta|_H^2 = o(1).$

(ii) $\quad E \sup_{|t|\le b} |W(t) - W(0)|_H^2 = o(1).$

(iii) $\quad \overline{\lim_n}\, E \sup_{|t|\le b} |S_h(\rho) + t[a + \underline{a}\,]|_H^2 < \infty.$

Proof of (i). Define

$$h^+(x) = h(x)I[xh(x) \ge 0] \quad \text{and} \quad h^-(x) = h(x) - h^+(x) \quad \forall\, x \in \mathbb{R}.$$

Replacing h with $h^\pm$ in each of the functions a and μ gives new functions, say $a^\pm$ and $\mu^\pm$.

From the inequality $(a+b)^2 \le 2a^2 + 2b^2$, a and b in $\mathbb{R}$,

$$|\mu^\pm(t) - \mu^\pm(0) - a^\pm(x)|_H^2$$

$$\le 2\int \Big[\mu^\pm(t,x) - \mu^\pm(0,x) - \frac{t}{n}\sum_{j=0}^{n-1} Y_j h^\pm(Y_j) g_n(x+\rho v_j)\Big]^2 dH(x) + \tag{2.7}$$

$$+\ 2t^2\int \Big[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^\pm(Y_j) g_n(x+\rho v_j) - a^\pm(x)\Big]^2 dH(x)$$

$$= I(t) + 2t^2 II, \qquad\qquad |t| \le b,$$

where $I(t)$ and II represent the first and the second integral on the r.h.s. of (2.7), respectively. From (2.2) and (2.4) we can rewrite

$$I(t) = \int\Big[n^{-1/2}\sum_{j=0}^{n-1} Y_j h^\pm(Y_j)\{1-\gamma_n\}\int_0^{n^{-1/2}t} [f(x+sY_j+\rho v_j) - f(x+\rho v_j)]\, ds +$$

$$+\ \gamma_n\iint_0^{n^{-1/2}t}[f(x+sY_j+\rho v_j-z)\ -\ f(x+\rho v_j-z)]\ ds\ dL_n(z)\}\Big]^2 dH(x)$$

$$\text{(2.8)}\qquad \le 4(1-\gamma_n)^2\int n^{-1/2}b\sum_{j=0}^{n-1} Y_j^2h^2(Y_j)\cdot$$
$$\cdot\int_{-n^{-1/2}b}^{n^{-1/2}b}[f(x+sY_j+\rho v_j)\ -\ f(x+\rho v_j)]^2 ds\ dH(x)\ +$$

$$+\ 4\gamma_n^2\int n^{-1/2}b\sum_{j=0}^{n-1} Y_j^2h^2(Y_j)\cdot$$
$$\cdot\int_{-n^{-1/2}b}^{n^{-1/2}b}\int [f(x+sY_j+\rho v_j-z)\ -\ f(x+\rho v_j-z)]^2 dL_n(z)\ ds\ dH(x).$$

Inequality (2.8) follows from the Cauchy Schwartz inequality and the moment inequality. Use (2.8), the Fubini Theorem, the stationarity of $\{(v_j,\ Y_j),\ 0 \le j \le n\}$, (0.1) and (0.2) to get

$$\operatorname*{Esup}_{|t|\le b} I(t) \le 4n^{1/2}(1-\gamma_n)^3 b\int_{-n^{-1/2}b}^{n^{-1/2}b}\int EX_0^2h^2(X_0)[f(x+sX_0)-f(x)]^2 dH(x)\ ds\ +$$

$$+\ 4n^{1/2}\gamma_n(1-\gamma_n)^2 b\int_{-bn^{-1/2}}^{bn^{-1/2}}\int E\Big[\int (X_0+z)^2h^2(X_0+z)\cdot$$
$$\cdot[f(x+s[X_0+z]+\rho z)\ -\ f(x+\rho z)]^2\ dL_n(z)\Big]\ dH(x)\ ds\ +$$

$$\text{(2.9)}$$

$$+\ 4n^{1/2}\gamma_n^2(1-\gamma_n)b\int_{-n^{-1/2}b}^{n^{-1/2}b}\int\Big[EX_0^2h^2(X_0)\int [f(x+sX_0-z)\ -$$
$$-\ f(x-z)]^2 dL_n(z)\Big]\ dH(x)\ ds\ +$$

$$+\ 4n^{1/2}\gamma_n^3 b \int_{-n^{-1/2}b}^{n^{-1/2}b} \int E\Big[\int (X_0+z)^2 h^2(X_0+z)\cdot$$

$$\cdot\Big[\int [f(x+s(X_0+z)+\rho z-u) - f(x+\rho z-u)]^2 dL_n(u)\Big]\ dL_n(z)\Big]\ dH(x)\ ds.$$

A_1 and the continuity property $A_{10}(b)$ show that the first term on the r.h.s. of (2.9) converges to zero. The remaining terms of (2.9) go to zero by A_1, A_4, $A_8(b)$, A_{11}, A_{13} and A_{15}. Thus

(2.10) $$E \sup_{|t|\le b} I(t) = o(1).$$

Now consider

$$EII = \int E\Big[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^{\pm}(Y_j) g_n(x+\rho v_j) - a^{\pm}(x)\Big]^2 dH(x)$$

(2.11) $$\le 2(1-\gamma_n)^2 \int E\Big[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^{\pm}(Y_j) f(x+\rho v_j) - $$
$$- EY_0 h^{\pm}(Y_0) f(x+\rho v_0)\Big]^2 dH(x) +$$

$$+\ 2\gamma_n^2 \int E\Big[\frac{1}{n}\sum_{j=0}^{n-1} Y_j h^{\pm}(Y_j) \int f(x+\rho v_j - z)\ dL_n(z) -$$
$$- EY_0 h^{\pm}(Y_0) \int f(x+\rho v_0 - z)\ dL_n(z)\Big]^2 dH(x).$$

The inequality (2.11) follows from (2.2) and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, a and b in $\mathbb{R}$. From A_1, A_4, $A_8(b)$, $A_{13}(b)$ and the stationarity of (v_j, Y_j), $0 \le j \le n-1$, the 2nd term in (2.11) goes to zero as $n \to \infty$ and the first term can be written as

$$
(2.12)\quad
\begin{aligned}
&2n^{-1}(1-\gamma_n)^2 \int \mathrm{Var}\{Y_0h^{\pm}(Y_0)f(x+\rho v_0)\}\ dH(x) + \\
&+ 4n^{-2}(1-\gamma_n)^2 \sum_{j=1}^{n-1} (n-j)\int \mathrm{Cov}\{Y_0h^{\pm}(Y_0)f(x+\rho v_0), \\
&\qquad\qquad , Y_jh^{\pm}(Y_j)f(x+\rho v_j)\}\ dH(x).
\end{aligned}
$$

The first integral in (2.12) can be written as

$$
\begin{aligned}
&(1-\gamma_n)EX_0^2h^{2\pm}(X_0)\int f^2(x)\ dH(x) + \\
&+ \gamma_n\int E\Big[\int (X_0+z)^2\{h^{\pm}(X_0+z)\}^2f^2(x+\rho z)\ dL_n(z)\Big]\ dH(x) - \\
&- (1-\gamma_n)^2[EX_0h^{\pm}(X_0)]^2\int f^2(x)\ dH(x) - \\
&- 2\gamma_n(1-\gamma_n)EX_0h^{\pm}(X_0)\int f(x)E\Big[\int (X_0+z)h^{\pm}(X_0+z)f(x+\rho z)\ dL_n(z)\Big]\cdot \\
&\qquad \cdot dH(x) - \\
&- \gamma_n^2 \int\Big[E\int (X_0+z)h^{\pm}(X_0+z)f(x+\rho z)\ dL_n(z)\Big]^2 dH(x),
\end{aligned}
$$

which in turn converges to $\mathrm{Var}[X_0h^{\pm}(X_0)]\int f^2dH$. This follows from A_1, A_4, A_7, $A_{13}(a)$, the moment inequality and the Hölder inequality. Hence the first term in (2.12) converges to zero. The second term in (2.12) can be written as

$$
4n^{-2}(1-\gamma_n)^4\sum_{j=1}^{n-1} (n-j)\mathrm{Cov}\{X_0h^{\pm}(X_0),\ X_jh^{\pm}(X_j)\}\int f^2dH +
$$

$$
\begin{aligned}
& + 4n^{-2}\gamma_n(1-\gamma_n)^3 \sum_{j=1}^{n-1} (n-j)\int \mathrm{Cov}\Big[X_0h^{\pm}(X_0)f(x), \\
&\qquad\qquad , \int (X_j+z)h^{\pm}(X_j+z)f(x+\rho z)\ dL_n(z)\Big]\ dH(x)\ + \\
(2.13)\quad & \\
& + 4n^{-2}\gamma_n(1-\gamma_n)^3 \sum_{j=1}^{n-1} (n-j)\int \mathrm{Cov}\Big[X_jh^{\pm}(X_j)f(x), \\
&\qquad\qquad , \int (X_0+z)h^{\pm}(X_0+z)f(x+\rho z)\ dL_n(z)\Big]\ dH(x)\ + \\
& \\
& + 4n^{-2}\gamma_n^2(1-\gamma_n)^2 \sum_{j=1}^{n-1} (n-j)\int \mathrm{Cov}\Big[\int (X_0+z)h^{\pm}(X_0+z)f(x+\rho z)\ dL_n(z), \\
&\qquad\qquad , \int (X_j+z)h^{\pm}(X_j+z)f(x+\rho z)\ dL_n(z)\Big]\ dH(x).
\end{aligned}
$$

By assumptions A_1, A_4, A_7, and $A_{13}(a)$ and the Hölder inequality, the second, third and the fourth terms in (2.13) go to zero. Since,

$$
(2.14)\qquad \mathrm{Var}\{n^{-1}\sum_{j=1}^{n} X_jh^{\pm}(X_j)\} =
$$
$$
n^{-1}\mathrm{Var}[X_0h^{\pm}(X_0)] + 2n^{-2}\sum_{j=1}^{n-1} (n-j)\ \mathrm{Cov}\{X_0h^{\pm}(X_0),\ X_jh^{\pm}(X_j)\},
$$

to prove that the first term in (2.13) goes to zero, from A_4, A_7 and (2.14) it suffices to prove that

$$
(2.15)\qquad \mathrm{Var}\{n^{-1}\sum_{j=1}^{n} X_jh^{\pm}(X_j)\} \to 0 \quad \text{as } n \to \infty.
$$

But from A_4 and the Stationary Ergodic Theorem

$$
(2.16)\qquad n^{-1}\sum_{j=1}^{n} X_jh^{\pm}(X_j) \to EX_0h^{\pm}(X_0) \quad \text{a.s..}
$$

Also, from A_4, the sequence $\{X_j h^{\pm}(X_j)\}$ is uniformly integrable of order 2, hence so is the sequence $\{n^{-1} \sum_{j=1}^{n} X_j h^{\pm}(X_j)\}$, which follows clearly from Chung (1974, exercise 9, p. 100). Thus, from (2.16), (2.15) follows, which in turn gives

(2.17) $\quad EII \to 0 \quad \text{as } n \to \infty.$

Thus (2.10) and (2.17) applied to (2.7) prove (i).

<u>Proof of (ii).</u> Replace the h in W by $h^{\pm}$ and call the new r.v. $W^{\pm}$.

Fix t in $[-b,b]$. Using the Fubini Theorem and the fact that the j^{th} summand in $W^{\pm}$ is conditionally centered, given (v_{j-1}, Y_{j-1}),

$$E|W^{\pm}(t) - W^{\pm}(0)|_H^2$$

$$(2.18) \quad = n^{-1}\int \sum_{j=1}^{n} E\Big[\{h^{\pm}(Y_{j-1})\}^2 E\Big[\{I[v_j+\epsilon_j \le x+n^{-1/2}tY_{j-1}+\rho v_{j-1}] - - I[v_j+\epsilon_j \le x+\rho v_{j-1}] - G_n(x+n^{-1/2}tY_{j-1}+\rho v_{j-1}) + + G_n(x+\rho v_{j-1})\}^2 \Big| (v_{j-1}, Y_{j-1})\Big]\Big] \, dH(x),$$

which can be dominated by

$$2\int E\{h^{\pm}(Y_0)\}^2 |G_n(x+n^{-1/2}tY_0+\rho v_0) - G_n(x+\rho v_0)| \, dH(x).$$

Using (2.1), the Fubini Theorem and representation of F in terms of its density, this term can be dominated by

$$2(1-\gamma_n)^2\int_{-n^{-1/2}b}^{n^{-1/2}b}\int E|X_0|h^2(X_0)f(x+X_0s)\ dH(x)\ ds\ +$$

$$+\ 2\gamma_n(1-\gamma_n)\int_{-n^{-1/2}b}^{n^{-1/2}b}\int E\Big[\int |X_0+z|h^2(X_0+z)f(x+\rho z+s[X_0+z])\cdot$$

$$\cdot\ dL_n(z)\Big]\ dH(x)\ ds\ +$$

(2.19)

$$+\ 2\gamma_n(1-\gamma_n)\int_{-n^{-1/2}b}^{n^{-1/2}b}\int E\Big[\int |X_0|h^2(X_0)f(x-z+X_0s)\ dL_n(z)\Big]\ \cdot$$

$$\cdot\ dH(x)\ ds\ +$$

$$+\ 2\gamma_n^2\int_{-n^{-1/2}b}^{n^{-1/2}b}\int E\Big[\int |X_0+z|h^2(X_0+z)\int f(x-u+[X_0+z]s+\rho z)\ \cdot$$

$$\cdot\ dL_n(u)\ dL_n(z)\Big]\ dH(x)\ ds.$$

From A_1, $A_3(a)$, A_4 and $A_{10}(a)$ the first term in (2.19) converges to zero. That the remaining terms also converge to zero, follows from A_1, $A_{11}(a)$ and A_{14}, giving

(2.20) $\quad E|W^{\pm}(t) - W^{\pm}(0)|_H^2 \to 0,\ t \in \mathbb{R}.$

Thus to complete the proof of (ii), use the monotone structure of $W^{\pm}$ and $\mu^{\pm}$, the compactness of $[-b,b]$, $\overline{\lim_n}\ E|a|_H^2 < \infty$ and (i), just as in Koul and Dewet (1983, p. 929, Theorem 5.1, proof of (ii)). The details are similar, hence deleted.

Proof of (iii). From (2.5) taking $t = 0$, we get

$$(2.21) \quad S(x,\rho) = W(x,0) + W(-x,0) + \\ + n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1})\{G_n(x+\rho v_{j-1}) - G_n(x-\rho v_{j-1})\}.$$

We shall now proceed to prove that

$$(2.22) \quad \overline{\lim_n}\ E|S(\rho)|_H^2 < \infty.$$

Using the fact that the summands in W are conditionally centered, the stationarity of the process (v_j,Y_j), $0 \le j \le n-1$ and the Fubini Theorem,

$$E \int W(0)^2\, dH$$

$$(2.23) \quad = \int En^{-1} \sum_{j=1}^{n} h^2(Y_{j-1})\{I[v_j - \rho v_{j-1} + \epsilon_j \le x] - G_n(x+\rho v_{j-1})\}^2\, dH(x)$$

$$= \int Eh^2(Y_0)\{I[v_1 - \rho v_0 + \epsilon_1 \le x] - G_n(x+\rho v_0)\}^2\, dH(x).$$

The lim sup of the r.h.s. of (2.23) is finite by A_1, $A_3(b)$ and A_{16}. Next, using the stationarity of (v_j,Y_j), $0 \le j \le n-1$,

$$E\int \{n^{-1/2} \sum_{j=0}^{n-1} h(Y_j)[G_n(x+\rho v_j) - G_n(x-\rho v_j)]\}^2\, dH(x)$$

$$(2.24) \quad = \int Eh^2(Y_0)[G_n(x+\rho v_0) - G_n(x-\rho v_0)]^2\, dH(x) +$$

$$+ \ 2n^{-1}\sum_{j=1}^{n-1}(n-j)\int E\Big[h(Y_0)[G_n(x+\rho v_0) - G_n(x-\rho v_0)]h(Y_j)\cdot$$
$$\cdot[G_n(x+\rho v_j) - G_n(x-\rho v_j)]\Big] \ dH(x).$$

The lim sup of the first term on the r.h.s. of (2.24) is finite by (0.1), A_1 and A_{17}. The expression inside the sum in the second term on the r.h.s. of (2.24) can be written as

$$(2.25) \quad (n-j)\gamma_n^2\int E\Big[\int h(X_0+z)[G_n(x+\rho z) - G_n(x-\rho z)] \ dL_n(z)\cdot$$
$$\cdot\int h(X_j+z)[G_n(x+\rho z) - G_n(x-\rho z)] \ dL_n(z)\Big] \ dH(x),$$

which follows from the independence of $\{X_j, \ j \le n\}$ and $\{v_j, \ 0 \le j \le n\}$ and the latter being i.i.d. β_n. Thus applying the Cauchy–Schwartz and the moment inequalities to the integrand in (2.25) and using the stationarity of $\{X_j\}$, the second term on the r.h.s. of (2.24) can be dominated by

$$(2.26) \quad (n-1)\gamma_n^2\int E\Big[\int h^2(X_0+z)[G_n(x+\rho z) - G_n(x-\rho z)]^2 dL_n(z)\Big]dH(x).$$

From A_1 and A_{17}, the lim sup of (2.26) is finite. Hence (2.22) holds. The proof of (iii) now follows from $\overline{\lim_n} \ E|a|_H^2 < \infty$, which in turn follows from (2.2), A_4, A_7, A_8(b) and A_{13}. This also completes the proof of the theorem. □

<u>Note</u>. In Theorem 2.1, the proof of (2.15) gives an alternate way to prove KL's equation (14), p. 1211.

In order to prove the next theorem, analogous to the definition of $\hat{\rho}_h(H)$, define $\tilde{\rho}_h(H)$ with M replaced by Q. Using the definition of $\tilde{\rho}_h(H)$, (0.4), (2.3) and the fact that $S_h(\cdot,\rho)$ is even,

$$\sqrt{n}(\tilde{\rho}_h(H) - \rho) = -\int \frac{S_h(\rho)\,[\underline{a}+a]\,dH}{|a+\underline{a}|_H^2}$$

(2.27)

$$= -2\int \frac{S_h(\rho)a}{|a+\underline{a}|_H^2}\,dH.$$

Theorem 2.2. In addition to the assumptions of Theorem 2.1, let us assume that A_6 holds and also for each n let $\hat{\rho}_h(H)$ be a measurable minimizer of (0.5); then

$$\text{(2.28)} \quad n^{1/2}[\hat{\rho}_h(H) - \rho] = n^{1/2}[\tilde{\rho}_h(H) - \rho] + o_p(1).$$

Proof. The line of proof is as follows:

(i) For any $\eta > 0$ and $0 < z < \infty$ there exists a N and b, $0 < b < \infty$, depending on η and z such that

$$P(\inf_{|t|>b} M(n^{-1/2}t+\rho) \geq z) \geq 1 - \eta \quad \forall\, n \geq N.$$

(ii) $\sqrt{n}\,[\tilde{\rho}_h(H) - \rho] = O_p(1)$ and
$\sqrt{n}\,[\hat{\rho}_h(H) - \rho] = O_p(1)$.

(iii) $M[\hat{\rho}_h(H)] = Q[\hat{\rho}_h(H)] + o_p(1)$ and
$M[\hat{\rho}_h(H)] = Q[\tilde{\rho}_h(H)] + o_p(1)$.

Proof of (i) follows exactly as in Koul and Dewet (1983, Corollary 5.1) or Koul (1985, Lemma 3.1).

Proof of (ii) follows from (i), (2.22) and the reasoning given in Koul (1985, Theorem 3.1).

Proof of (iii) follows from (ii) and Theorem 2.1. From (iii) and (2.27) we get

$$(2.29)\qquad n[\hat{\rho}_h(H) - \tilde{\rho}_h(H)]^2\ |a+\underline{a}|_H^2 = o_p(1).$$

From (2.2), (2.3) and the symmetry of the function g_n w.r.t. to the y-axis, we get

$$|a+\underline{a}|_H^2 = 4(1-\gamma_n)^2[EX_0h(X_0)]^2 \int g_n^2\, dH +$$

$$(2.30)\qquad + \gamma_n^2\int \Big[E\int (X_0+z)h(X_0+z)[g_n(x+\rho z)+g_n(-x+\rho z)]\, dL_n(z)\Big]^2 dH(x) +$$

$$+ 4\gamma_n(1-\gamma_n)EX_0h(X_0) \int g_n(x)E\Big[\int (X_0+z)h(X_0+z)\cdot$$
$$\cdot [g_n(x+\rho z)+g_n(-x+\rho z)]\, dL_n(z)\Big]\, dH(x).$$

From A_1, A_2, A_4, A_7, A_8(b), A_{12}, the moment and the Hölder inequalities, the 2^{nd} and 3^{rd} terms on the r.h.s. of (2.30) go to zero and the 1^{st} term converges to

$$(2.31)\qquad 2q = 4[EX_0h(X_0)]^2\int f^2\, dH.$$

From A_3, A_6 and A_7, the r.h.s. of (2.31) is strictly greater than zero. Hence from (2.29) the proof is complete. □

Note. In order to study the limiting distribution of $n^{1/2}[\hat{\rho}_h(H) - \rho]$ we need to appropriately center (2.28). In view of the Theorem 2.2, for fixed h and H, the asymptotic behavior of $\hat{\rho}_h(H)$ will not be affected if for each $n \geq 0$, $\hat{\rho}_h(H)$ is replaced by any convex combination of the measurable minimizers of (0.5). Further note that the proofs of Theorems 2.1 and 2.2 only need $Eh^2(X_0) < \infty$ instead of $A_3(b)$.

3. Asymptotic Normality and Influence function of $\hat{\rho}_h(H)$.

In this section we apply Withers (1981), (1983) C.L.T. for ℓ-mixing sequence of arrays. We also discuss the sufficient conditions under which the asymptotic bias of $\hat{\rho}_h(H)$ is vanishing or nonvanishing. Finally we compute the influence function and show that it is directly proportional to the asymptotic bias of $\hat{\rho}_h(H)$. To study the limiting distribution of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$ when centered, from (2.27), (2.30), (2.31) and Theorem 2.2, we need only to study

$$(3.1) \qquad - q^{-1} n^{-1/2} \sum_{j=1}^{n} h(Y_{j-1}) \psi_n(Y_j - \rho Y_{j-1})$$

when centered, where $q > 0$ is as in (2.31) and

$$(3.2) \qquad \psi_n(x) = \int_{-\infty}^{x} a \, dH - \int_{-\infty}^{-x} a \, dH.$$

Let

$$(3.3) \qquad \begin{aligned} \mu_n &= Eh(Y_0)\psi_n(Y_1 - \rho Y_0) \quad \text{and} \\ \xi_{j,n} &= h(Y_{j-1})\psi_n(Y_j - \rho Y_{j-1}) - \mu_n, \quad 1 \leq j \leq n. \end{aligned}$$

From (0.1), (0.2) and (3.2),

$$
\begin{aligned}
\psi_n(y) &= (1-\gamma_n)EX_0h(X_0)\Big[\int_{-\infty}^{y} g_n\, dH - \int_{-\infty}^{-y} g_n\, dH\Big] + \\
(3.4) \qquad &+ \gamma_n\Big[\int_{-\infty}^{y} E\int (X_0+z)h(X_0+z)g_n(x+\rho z)\, dL_n(z)\, dH(x) - \\
&- \int_{-\infty}^{-y} E\int (X_0+z)h(X_0+z)g_n(x+\rho z)\, dL_n(z)\, dH(x)\Big].
\end{aligned}
$$

From (3.4), A_2, $A_3(b)$, A_4, A_5, A_7, $A_8(a)$, A_{12} and $\gamma_n \in [0,1]$, note that $\xi_{j,n}$, $1 \le j \le n$ are real valued r.v.s and $\mu_n < \infty$ for all $n \ge 0$.

For the sake of completeness we shall reproduce the following definitions and theorem from Withers (1981 and 1983) and use them to prove the asymptotic normality of $n^{-1/2}\sum_{j=1}^{n} \xi_{j,n}$.

Consider a series of random processes

$$
(3.5) \qquad \xi = \{\xi_n,\ n \ge 1\}, \quad \text{where} \quad \xi_n = \{\xi_{j,n},\ d_n \le j \le N_n\},
$$

which are not necessarily real (although possibly complex), defined on some probability space $(\Omega, \mathcal{A}, P)$, with d_n, N_n integers such that $-\infty \le d_n < N_n \le \infty$ and $N_n - d_n \to \infty$ as $n \to \infty$. For any (not necessarily real) r.v.s θ, ϕ we set

$$
(3.6) \qquad
\begin{aligned}
\mathcal{M}(\theta) &= \text{the } \sigma\text{-algebra generated by } \theta, \\
\alpha(\theta,\phi) &= \sup|P(I \cap J) - P(I)P(J)|,
\end{aligned}
$$

where sup is over all $I \in \mathcal{M}(\theta)$, $J \in \mathcal{M}(\phi)$. Define

$$\alpha_n(k) = \max_{d_n \le j \le N_n - k} \alpha(\{\xi_{d_n,n},\ldots,\xi_{j,n}\}, \{\xi_{j+k,n},\ldots,\xi_{N_n,n}\})$$

(3.7)

$$\alpha(k) = \max_{\{n:\ k \le N_n - d_n\}} \alpha_n(k), \quad 0 \le k < \infty.$$

The set of processes ξ is said to be α–mixing if $\alpha(k) \to 0$ as $k \to \infty$. Set

$$\ell_n(k,u) = \max_{d_n \le j \le N_n - k} \sup \left| \mathrm{Cov}\left[\exp\{iu \sum_{p=d_n}^{j} \delta_p \xi_{p,n}\},\ \exp\{-iu \sum_{p=j+k}^{N_n} \delta_p \xi_{p,n}\}\right] \right|$$

for u real, $0 \le k \le d_n - N_n$, $n \ge 1$, where sup is over $\{\delta_j = 0 \text{ or } 1\}$ and the covariance of complex r.v.s means that

$$\mathrm{Cov}(\theta,\phi) = E\theta\,\overline{\phi} - E\theta\,\overline{E\phi}.$$

Now set

(3.8) $$\ell(k,u) = \max_{\{n: k \le N_n - d_n\}} \ell_n(k,u), \quad 0 \le k < \infty, \ u \text{ real}.$$

The set of processes ξ is said to be ℓ–mixing if for all real u, $\ell(k,u) \to 0$ as $k \to \infty$. From Ibragimov and Linnik (1971, p. 307), as pointed out by Withers (1981), $\ell_n(k,u) \le 16\ \alpha_n(k)$, $0 \le k < \infty$, $\forall\ u \in \mathbb{R}$, thus

(3.9) $$\ell(k,u) \le 16\ \alpha(k), \quad 0 \le k < \infty, \quad u \text{ real}.$$

Henceforth assume $d_n = 1$ and $N_n = n$. Define

$$S_n(a,b) = \sum_{j=a+1}^{a+b} \xi_{j,n},\ 0 \le a,\ 1 \le b \le n-a,\ S_n = S_n(0,n),$$

$$\sigma_n^2 = \operatorname{Var} S_n \quad \text{and} \quad \tilde{c}_n(j) = \sup|\operatorname{Cov}(\xi_{d,n}, \xi_{m,n})|,\ 0 \le j < n,$$

where sup is over $\{d, m: |d - m| \ge j\}$,

$$\text{(3.10)} \qquad \tilde{c}(j) = \max_{\{n:j<n\}} \tilde{c}_n(j).$$

Theorem 3.1. (Withers, 1981, Theorem 2.1, 1983). Assume that the following hold: For some $\eta > 0$ and $\eta_1 \ge 0$, ξ satisfies the moment inequality

$$\text{(3.11)} \qquad \sup_{a,n} E|S_n(a,b)|^{2+\eta} = O(b^{1+\eta/2+\eta_1}) \quad \text{as} \quad b \to \infty.$$

ξ is ℓ-mixing and for all real u

$$\text{(3.12)} \qquad \ell(k,u) = o(k^{-\delta}) \quad \text{as} \quad k \to \infty, \quad \text{where} \quad \delta = 2\eta_1/\eta.$$

$$\text{(3.13)} \qquad \sigma_n^2 \longrightarrow \infty \quad \text{as} \quad n \longrightarrow \infty, \quad \varliminf_n \sigma_n^2/n > 0$$

and

$$\text{(3.14)} \qquad \sum_{j=0}^{\infty} \tilde{c}(j) < \infty.$$

Then

$$\sigma_n^{-1}(S_n - ES_n) \longrightarrow \mathcal{N}(0,1) \quad \text{in distribution.}$$

Lemma 3.2. Let θ_n, ω be Borel measurable functions from $\mathbb{R}^2$ to $\mathbb{R}$ and $\xi_{j,n} = \theta_n(Y_{j-1,n},Y_{j,n})$, $1 \le j \le n$, $n \ge 1$. Let $Y_{j,n} = \omega(X_j,v_{j,n})$, where $\{X_j,\ j = 0, \pm 1, \pm 2,..\}$ is a stationary process that is strongly α_X–mixing (Ibragimov and Linnik, 1971, Definition 17.2.1) and the sequence of independent r.v.s $\{v_{j,n},\ 0 \le j \le n\}$ is independent of $\{X_j,\ 0 \le j \le n\}$, $n \ge 0$. Then ξ, as in (3.5) with $d_n = 1$ and $N_n = n$, is strongly α_ξ–mixing with

$$\alpha_\xi \le \alpha_X. \tag{3.15}$$

Proof. In (3.6) let $\theta = (\xi_{1,n},\dots,\xi_{j,n})$, $\phi = (\xi_{j+k,n},\dots,\xi_{n,n})$, $I = [(\xi_{1,n},\dots,\xi_{j,n}) \in B_1]$ and $J = [(\xi_{j+k,n},\dots,\xi_{n,n}) \in B_2]$, where $B_1 \in \mathscr{B}(\mathbb{R}^j)$, the Borel σ–field, and $B_2 \in \mathscr{B}(\mathbb{R}^{n-j-k+1})$, $1 \le j \le n$, $2+j \le j+k \le n$. Let us suppress the n in $v_{j,n}$ and define

$$\phi_{1,j}: \quad \mathbb{R}^{j+1} \to \mathbb{R}^j$$
$$[x_0,x_1,\dots,x_j] \to [\theta_n(x_0,x_1),\dots,\theta_n(x_{j-1},x_j)],$$

$$T_{\underset{\sim}{v},j+1}: \quad \mathbb{R}^{j+1} \to \mathbb{R}^{j+1}$$

$$[x_0,x_1,\dots,x_j] \to [\omega(x_0,v_0),\dots,\omega(x_j,v_j)]$$

and $T_{\underset{\sim}{v}^*,n-j-k+2}$ by replacing in $T_{\underset{\sim}{v},j+1}$, $\underset{\sim}{v}$ by $\underset{\sim}{v}^*$ and j by n–j–k+1, where $\underset{\sim}{v} = (v_0, v_1,\dots,v_j)$ and $\underset{\sim}{v}^* = (v_{j+k-1},\dots,v_n)$. Then,

$$|P(I \cap J) - P(I)P(J)| =$$

$$= \Big| P\Big[\{\omega(X_0,v_0),\dots,\omega(X_j,v_j)\} \in \phi^{-1}_{1,j}(B_1), \\ \{\omega(X_{j+k-1},v_{j+k-1}),\dots,\omega(X_n,v_n)\} \in \phi^{-1}_{1,n-j-k+1}(B_2)\Big] -$$

$$- P\Big[\{\omega(X_0,v_0),\dots,\omega(X_j,v_j)\} \in \phi^{-1}_{1,j}(B_1)\Big] \cdot \\ \cdot P\Big[\{\omega(X_{j+k-1},v_{j+k-1}),\dots,\omega(X_n,v_n)\} \in \phi^{-1}_{1,n-j-k+1}(B_2)\Big]\Big|$$

$$(3.16) \quad \le E\Big[\Big|P\Big[(X_0,\dots,X_j) \in T^{-1}_{\underset{\sim}{v},j+1}\phi^{-1}_{1,j}(B_1),\ (X_{j+k-1},\dots,X_n) \in \\ \in T^{*-1}_{\underset{\sim}{v},n-j-k+2}\phi^{-1}_{1,n-j-k+1}(B_2)\Big] -$$

$$- P\Big[(X_0,\dots,X_j) \in T^{-1}_{\underset{\sim}{v},j+1}\phi^{-1}_{1,j}(B_1)\Big]\ P\Big[(X_{j+k-1},\dots,X_n) \in \\ \in T^{*-1}_{\underset{\sim}{v},n-j-k+2}\phi^{-1}_{1,n-j-k+1}(B_2)\ \Big]\Big|\ \Big|(v_0,\dots,v_j,v_{j+k-1},\dots,v_n)\Big].$$

Inequality (3.16) holds because for all $k \ge 2$, $(v_0,\dots,v_j,v_{j+k-1},\dots,v_n)$ is a sequence of independent r.v.s and independent of $(X_0,\dots,X_n)$. From the definition of strongly α–mixing sequence of stationary r.v.s, and the stationarity of $\{X_j\}$, we see that the r.h.s. of (3.16) can be bounded by α_X. Taking sup over all B_1 and B_2 in $\mathscr{B}(\mathbb{R}^j)$ and $\mathscr{B}(\mathbb{R}^{n-j-k+1})$ and then taking max twice as in (3.7), we get that (3.15) holds. □

<u>Note.</u> The proof of the Lemma 3.2 goes through even when ω and θ_n are replaced by ω_n and $\theta_{j,n}$ for each j and n.

Lemma 3.3. Define $\xi_{j,n}$ as in Lemma 3.2 satisfying all the conditions there. In addtion, let $\{v_{j,n}, 0 \le j \le n\}$ be identically

(3.17) distributed β_n, with $\gamma_n \in [0,1]$, $\gamma_n = o(1)$ and α_X satisfying, for any $\eta > 0$ $\sum_{j=1}^{\infty} \alpha_X(j)^{\eta} < \infty$.

Further let θ and h, be real valued Borel measurable

(3.18) functions with θ defined on $\mathbb{R}^2$ and h on $\mathbb{R}$, such that $\theta_n(x,y) \le Ch(x)$ and $\theta_n(x,y) \to \theta(x,y)$ for each $x, y \in \mathbb{R}$, $0 < C \in \mathbb{R}$. Let for some $\delta > 0$,

$$(3.19) \qquad E|h(\omega(X_0))|^{2+\delta} < \infty \quad \text{and} \quad \sup_n E\int |h(\omega(X_0,z))|^{2+\delta} dL_n(z) < \infty,$$

where $\omega(\cdot) = \omega(\cdot,0)$; then

$$(3.20) \qquad n^{-1}\sigma_n^2 = n^{-1}\mathrm{Var}\sum_{j=1}^{n} \xi_{j,n} \to \tau^2,$$

where

$$(3.21) \qquad \tau^2 = \mathrm{Var}[\theta(\omega(X_0),\omega(X_1)] + 2\sum_{j=1}^{\infty} \mathrm{Cov}[\theta\{\omega(X_0),\omega(X_1)\},\theta\{\omega(X_j),\omega(X_{j+1})\}].$$

Proof. From the definition of $Y_{j,n}$'s and the conditions satisfied by X_j's and $v_{j,n}$'s, $\{Y_{j,n}, 0 \le j \le n\}$ is stationary, hence we can write

$$(3.22) \qquad n^{-1}\sigma_n^2 = \mathrm{Var}(\xi_{1,n}) + \frac{2}{n}\sum_{j=1}^{n-1} (n-j)\mathrm{Cov}[\theta_n(Y_{0,n},Y_{1,n}),\theta_n(Y_{j,n},Y_{j+1,n})].$$

From (0.1), the definition of $Y_{j,n}$'s and the conditions satisfied by X_j's and $v_{j,n}$'s,

$$(3.23) \qquad \mathrm{Var}(\xi_{1,n}) = (1-\gamma_n)^2 E\theta_n^2\{\omega(X_0),\omega(X_1)\} +$$

$$+ \gamma_n(1-\gamma_n)E\int \theta_n^2\{\omega(X_0),\omega(X_1,z)\}\, dL_n(z) +$$

$$+ \gamma_n\, E\int \theta_n^2\{\omega(X_0,z),\omega(X_1,v_{1,n})\}\, dL_n(z) - (E\xi_{1,n})^2,$$

which in turn converges to the first term on the r.h.s. of (3.21). The above convergence follows from $\gamma_n = o(1)$, (3.18), (3.19) and the D.C.T.. From (0.1), the definition of $Y_{j,n}$'s and the conditions satisfied by X_j's and $v_{j,n}$'s, for $j \geq 2$, we get

$$\mathrm{Cov}[\theta_n(Y_{0,n},Y_{1,n}),\ (Y_{j,n},Y_{j+1,n})]$$

$$= (1-\gamma_n)^4\mathrm{Cov}\Big[\theta_n\{\omega(X_0),\omega(X_1)\},\ \theta_n\{\omega(X_j),\omega(X_{j+1})\}\Big] +$$

$$+ \gamma_n(1-\gamma_n)^3\mathrm{Cov}\Big[\theta_n\{\omega(X_0),\omega(X_1)\},\ \int\theta_n\{\omega(X_j,z),\omega(X_{j+1})\}dL_n(z)\Big] +$$

$$+ \gamma_n(1-\gamma_n)^3\mathrm{Cov}\Big[\int \theta_n\{\omega(X_0,z),\omega(X_1)\}dL_n(z),\theta_n\{\omega(X_j),\omega(X_{j+1})\}\Big] +$$

$$+ \gamma_n(1-\gamma_n)^3\mathrm{Cov}\Big[\int\theta_n\{\omega(X_0),\omega(X_1,z)\}\, dL_n(z),\theta_n\{\omega(X_j),\omega(X_{j+1})\}\Big] +$$

$$+ \gamma_n(1-\gamma_n)^3\mathrm{Cov}\Big[\theta_n\{\omega(X_0),\omega(X_1)\},\int \theta_n\{\omega(X_j),\omega(X_{j+1},z)\}dL_n(z)\Big] +$$

$$+ \gamma_n^2(1-\gamma_n)^2\mathrm{Cov}\Big[\theta_n\{\omega(X_0),\omega(X_1)\},\ \iint \theta_n\{\omega(X_j,z),\omega(X_{j+1},u)\}\cdot$$
$$\cdot dL_n(z)\, dL_n(u)\Big] +$$

$$(3.24) \qquad + \gamma_n^2(1-\gamma_n)^2\mathrm{Cov}\Big[\int \theta_n\{\omega(X_0),\omega(X_1,z)\}\, dL_n(z),$$
$$,\int \theta_n\{\omega(X_j,z),\omega(X_{j+1})\}\, dL_n(z)\Big] +$$

$$+ \gamma_n^2(1-\gamma_n)^2 \mathrm{Cov}\Big[\iint \theta_n\{\omega(X_0,z),\omega(X_1,u)\}\ dL_n(z)\ dL_n(u), \\ , \theta_n\{\omega(X_j),\omega(X_{j+1})\}\Big] +$$

$$+ \gamma_n^2(1-\gamma_n)^2 \mathrm{Cov}\Big[\int \theta_n\{\omega(X_0,z),\omega(X_1)\}\ dL_n(z), \\ , \int \theta_n\{\omega(X_j),\omega(X_{j+1},z)\}\ dL_n(z)\Big] +$$

$$+ \gamma_n^2(1-\gamma_n)^2 \mathrm{Cov}\Big[\int \theta_n\{\omega(X_0,z),\omega(X_1)\}\ dL_n(z), \\ , \int \theta_n\{\omega(X_j,z),\omega(X_{j+1})\}\ dL_n(z)\Big] +$$

$$+ \gamma_n^2(1-\gamma_n)^2 \mathrm{Cov}\Big[\int \theta_n\{\omega(X_0),\omega(X_1,z)\}\ dL_n(z), \\ , \int \theta_n\{\omega(X_j),\omega(X_{j+1},z)\}\ dL_n(z)\Big] +$$

$$+ \gamma_n^3(1-\gamma_n) \mathrm{Cov}\Big[\int \theta_n\{\omega(X_0,z),\omega(X_1)\}\ dL_n(z), \\ , \iint \theta_n\{\omega(X_j,z),\omega(X_{j+1},u)\}\ dL_n(z)\ dL_n(u)\Big] +$$

$$+ \gamma_n^3(1-\gamma_n) \mathrm{Cov}\Big[\iint \theta_n\{\omega(X_0,z),\omega(X_1,u)\}\ dL_n(z)\ dL_n(u), \\ , \int \theta_n\{\omega(X_j,z),\omega(X_{j+1})\}\ dL_n(z)\Big] +$$

$$+ \gamma_n^3(1-\gamma_n) \mathrm{Cov}\Big[\int \theta_n\{\omega(X_0),\omega(X_1,z)\}\ dL_n(z), \\ , \iint \theta_n\{\omega(X_j,z),\omega(X_{j+1},u)\}\ dL_n(z)\ dL_n(u)\Big] +$$

$$+ \gamma_n^3(1-\gamma_n) \mathrm{Cov}\Big[\iint \theta_n\{\omega(X_0,z),\omega(X_1,u)\}\ dL_n(z)\ dL_n(u), \\ , \int \theta_n\{\omega(X_j),\omega(X_{j+1},z)\}\ dL_n(z)\Big] +$$

$$+ \gamma_n^4 \mathrm{Cov}\Big[\iint \theta_n\{\omega(X_0,z),\omega(X_1,u)\}\ dL_n(z)\ dL_n(u), \\ , \iint \theta_n\{\omega(X_j,z),\omega(X_{j+1},u)\}\ dL_n(z)\ dL_n(u)\Big].$$

From computations similar to those in (3.24), $\gamma_n = o(1)$, (3.18), (3.19) and applying the D.C.T., show that for $j = 1$

$$\text{(3.25)} \quad \mathrm{Cov}[\theta_n(Y_0,Y_1),\ \theta_n(Y_j,Y_{j+1})] \to \mathrm{Cov}[\theta\{\omega(X_0),\omega(X_1)\},\ \theta\{\omega(X_j),\omega(X_{j+1})\}].$$

Similarly, from (3.24), (3.25) holds for $j \geq 2$. In general, any covariance on the r.h.s. of (3.24) can be represented by

$$\mathrm{Cov}[\phi_{n2}(X_0,X_1),\ \phi_{n3}(X_j,X_{j+1})].$$

From (3.18) and (3.19) we see for each $n \geq 1$

$$\text{(3.26)} \quad \begin{aligned} &E|\phi_{n2}(X_0,X_1)|^{2+\delta} \leq c_1 < \infty \quad \text{and} \\ &E|\phi_{n3}(X_0,X_1)|^{2+\delta} \leq c_2 < \infty,\ 0 < c_1,\ c_2 \in \mathbb{R}. \end{aligned}$$

Applying Ibragimov and Linnik (1971, Theorem 17.22) to the sequence $\{X_j\}$ with $t = 1$, $\tau = j - 1$, $j \geq 2$, $\xi = \phi_{n2}(X_0,X_1)$, $\eta = \phi_{n3}(X_j,X_{j+1})$, and from (3.26),

$$\text{(3.27)} \quad |\mathrm{Cov}[\phi_{n2}(X_0,X_1),\ \phi_{n3}(X_j,X_{j+1})]| \leq C\ \alpha_X(j-1)^{\delta/(2+\delta)},$$

where $0 < C \in \mathbb{R}$ depends only on c_1, c_2 and δ. Thus from the conditions satisfied by α_X in (3.17), (3.25), (3.27) and the D.C.T. for counting measure, we see that the second term on the r.h.s. of (3.22) converges to the 2^{nd} term on the r.h.s. of (3.21). Hence (3.22) and the convergence of (3.23) to the appropriate limit imply that (3.20) holds. □

Theorem 3.4. (i) Under (3.17) – (3.19) and the assumption $\tau^2 > 0$

$$n^{-1/2} \sum_{j=1}^{n} \{\xi_{j,n} - E\xi_{j,n}\} \to \mathcal{N}(0,\tau^2) \quad \text{in distribution.}$$

(ii) Let $A_1 - A_{17}$ and all the model assumptions (0.1) to (0.3) hold. Then

$$\text{(3.28)} \qquad n^{1/2}[\hat{\rho}_h(H) - \rho + \mu_n q^{-1}] \to \mathcal{N}(0,\sigma_h^2) \quad \text{in distribution,}$$

where

$$\text{(3.29)} \qquad \sigma_h^2 = q^{-2}[EX_\theta h(X_0)]^2 Eh^2(X_0)E\psi^2(\epsilon_1),$$

$$\text{(3.30)} \qquad \psi(y) = \int_{-\infty}^{y} f\, dH - \int_{-\infty}^{-y} f\, dH \qquad \forall\ y \in \mathbb{R},$$

q is as in (2.31) and μ_n as in (3.3).

Proof. We shall prove (i) and then prove (ii) using a special case of (i). In view of Theorem 3.1 we shall first show that (3.11) holds for ξ as in (3.5) and (3.17). Let $\eta = \delta$ as in (3.19) and $\eta_1 = 1+\delta/2$ then

$$E\left|\frac{S_n(a,b)}{b}\right|^{2+\delta}$$

$$\text{(3.31)} \qquad \le E|\theta_n(Y_{0,n},Y_{1,n})|^{2+\delta}$$

$$\le C\{(1-\gamma_n)E|h\{\omega(X_0)\}|^{2+\delta} + \gamma_n E\int |h\{\omega(X_0,z)\}|^{2+\delta}\, dL_n(z)\}.$$

The above inequality follows from the definition of $Y_{j,n}$'s and the conditions satisfied by X_j's and $v_{j,n}$'s, (0.1), (3.18) and the Jensen inequality . From (3.19), (3.11) is satisfied. From the Lemma 3.2 and (3.9), (3.12) is satisfied. From Lemma 3.3 and $\tau^2 > 0$, (3.13) is satisfied. Since $\{Y_j,\ 0 \le j \le n\}$ is stationary, $\tilde{c}_n$ in (3.10) can be written as

$$\tilde{c}_n(j) = \sup|\mathrm{Cov}[\xi_{1,n}, \xi_{|d-m|+1,n}]|,$$

where sup is taken over $\{d, m: |d-m| \ge j\}$. From this, (3.10), the same argument as in (3.26) to (3.27) and (3.19), we get

$$\tilde{c}(j) \le 8C\alpha_X(j-1)^{\delta/(2+\delta)}, \quad \forall\ j \ge 2. \tag{3.32}$$

From the conditions satisfied by α_X in (3.17), (3.19) and (3.32), (3.14) is satisfied and hence the C.L.T. holds for ξ.

<u>proof of (ii).</u> We shall first show that the C.L.T. holds for ξ as in (3.3) and (3.5). Thus take in (i), $\theta_n(x,y) = h(x)\psi_n(y-\rho x)$, x, y $\in \mathbb{R}$, Y_j, X_j and $v_{j,n}$ as in the model assumptions (0.1) – (0.3) with $\omega(x,y) = x+y$.

Note X_j can be a.s. represented as $\sum_{k=0}^{\infty} \rho^k \epsilon_{j-k}$. Using A_3(a), A_9 and Pham and Tran (1985, Theorem 2.1) with $\delta = 2$, $A(k) = \rho^k$, we get that $\{X_j\}$ is strongly α_X–mixing with $\alpha_X(n) \le C_\rho|\rho|^{2n/3}$ $\forall$ n $\ge$ 1.

Also note from A_1, A_7, A_8(a) and (2.2)

$$\int_{-\infty}^{x} g_n\, dH \to \int_{-\infty}^{x} f\, dH \quad \text{uniformly in } x. \tag{3.33}$$

Thus the r.h.s. of (3.4) converges to $EX_0h(X_0)\psi(y)$, uniformly in y, by (3.33), A_1, A_3 and A_{12}. By A_7, ψ is bounded and hence ψ_n is uniformly bounded. Since F is symmetric about 0 and ψ is an odd, $E\psi(\epsilon_1) = 0$. Thus letting $\theta(x,y) = h(x)\psi(y-\rho x)EX_0h(X_0)$, $x, y \in \mathbb{R}$ in Lemmma 3.3 we see $\tau^2 = \sigma_h^2q^2$. From (3.29), (3.30), A_3 and A_7, $\sigma_h^2q^2 > 0$. We now see that all the conditions of (i) are satisfied. Hence C.L.T. holds for ξ as in (3.3), (3.5). Thus from (2.27), (2.30), (2.31) and Theorem 2.2, (ii) holds. □

Remark 3.0. One of the assumptions under which we have studied the asymptotic behavior of $\hat{\rho}_h(H)$ as an estimator of ρ is F and L_n's are symmetric about 0. One could generalize the results by making an attempt to discard this assumption. A careful study of the results shows that due to this change, the arguments involved in the proof of (2.22) fail. The proof of the Theorem 2.2 can be easily modified without this assumption of symmetry or any additional assumptions. In view of the Theorem 3.1, the Lemmas 3.2 – 3.3 and the Theorem 3.3(i) we see no addtional modifications are needed to incorporate this change. Thus it only remains to modify the arguments from (2.21) – (2.26). Note in (2.21) and (2.24) we need to replace $G_n(x-\rho v_0)$ by $1 - G_n(-x+\rho v_0)$, since G_n is not symmetric. Thus in view of (2.24) we will now need

$$\overline{\lim_n} \int Eh^2(Y_0)[G_n(x+\rho v_0) + G_n(-x+\rho v_0) - 1]^2 \, dH(x) < \infty .$$

Case (i). Let F be symmetric about 0 but L_n's need not be symmetric about 0. In this case (2.25) can be replaced by

$$(n-j)\gamma_n^2\Big[(1-\gamma_n)^2 Eh(X_0)h(X_j)\int [E\{F(x-Z_n)+F(-x-Z_n)-1\}]^2 dH(x) +$$

$$+ (1-\gamma_n)\int [E\{F(x-Z_n)+F(-x-Z_n)-1\}]\cdot$$
$$\cdot E\Big[h(X_0)\int Eh(X_j+z)[G_n(x+\rho z)+G_n(-x+\rho z)-1]\ dL_n(z)\Big]\ dH(x) +$$

$$+ (1-\gamma_n)\int [E\{F(x-Z_n)+F(-x-Z_n)-1\}]\cdot$$
$$\cdot E\Big[h(X_j)\int Eh(X_0+z)[G_n(x+\rho z)+G_n(-x+\rho z)-1]\ dL_n(z)\Big] dH(x) +$$

$$+ \int E\Big[\int h(X_0+z)[G_n(x+\rho z)+G_n(-x+\rho z)-1]\ dL_n(z)\cdot$$
$$\cdot\int h(X_j+z)[G_n(x+\rho z)+G_n(-x+\rho z)-1]\ dL_n(z)\Big]\ dH(x)\Big].$$

Thus in view of the arguments involved in (2.26) and under proper assumptions Theorem 2.1 holds.

Case (ii). Both L_n's and F need not be symmetric about zero. In this case we will need to assume for each $n \geq 1$,

$$Eh(Y_0)[G_n(x+\rho v_0)+G_n(-x+\rho v_0)-1] \equiv 0,$$

after that we can use the same technique as presented in Lemma 3.3 and some limit theorem to prove (2.22).

Theorem 3.5. Let all the model assumptions (0.1) – (0.3), A_1, A_2, A_4, A_5, A_7, $A_8(b)$, A_{13} and A_{16} – A_{17} hold; then

(a) If h is a continuous function on $\mathbb{R}$ and $Z_n \to Z$ in distribution and $\sqrt{n}\,\gamma_n \to \gamma_c$ then $\sqrt{n}\,\mu_n \to \mu$, where $0 < \gamma_c \in \mathbb{R}$ and

$$(3.34)\qquad \mu = \gamma_c[EX_0h(X_0)]\int f(x)E\int h(X_0+z)\{F(x-\rho z)-F(x+\rho z)\}dL_n(z)\ dH(x).$$

(b) If either $Z_n \to 0$ in distribution or $\sqrt{n}\,\gamma_n \to 0$ then $\sqrt{n}\,\mu_n \to 0$.

<u>Proof</u>. From (2.27) and (3.1) – (3.3),

$$\sqrt{n}\,\mu_n = E\int S_h(\rho)a\ dH.$$

For large enough n we shall justify the interchange of expectation and integral above using the Fubini Theorem. Consider

$$(3.35)\qquad \int E|S_h(\rho)||a|\ dH \le \left[\int ES_h^2(\rho)\ dH\right]^{1/2}\left[\int a^2\ dH\right]^{1/2}.$$

Inequalitity (3.35) follows from the Hölder and the moment inequalities. That the lim sup of the r.h.s. of (3.35) is finite follows from (2.22) and the same reasoning as in (2.30) and (2.31). Thus, by the stationarity of the process (v_{j-1}, Y_{j-1}), the independence of (v_{j-1}, Y_{j-1}) and $v_j + \epsilon_j$, $1 \le j \le n$, and (0.1), for large enough n, we get

$$(3.36)\qquad \sqrt{n}\,\mu_n = \sqrt{n}\gamma_n\int a(x)E\int h(X_0+z)[G(x-\rho z) - G(x+\rho z)]\ dL_n(z)\ dH(z).$$

Using (2.1) and (2.2), (3.36) can be rewritten as

$$
\begin{aligned}
&\sqrt{n}\ \gamma_n(1-\gamma_n)^3 EX_0h(X_0)\int f(x)E\int h(X_0+z)[F(x-\rho z) - F(x+\rho z)]\cdot \\
&\qquad\qquad \cdot\ dL_n(z)\ dH(x)\ + \\
&+ \sqrt{n}\ \gamma_n^2(1-\gamma_n)^2[EX_0h(X_0)]\int EF(x-Z_n)E\int\ \cdot \\
&\qquad\qquad \cdot\ h(X_0+z)[F(x-\rho z) - F(x+\rho z)]\ dL_n(z)\ dH(z)\ + \\
(3.37)\quad &+ \sqrt{n}\ \gamma_n^2(1-\gamma_n)\int\left[E\int (X_0+z)h(X_0+z)g_n(x+\rho z)\ dL_n(z)\right]\cdot \\
&\qquad\qquad \cdot\left[E\int h(X_0+z)[F(x-\rho z) - F(x+\rho z)]\ dL_n(z)\right] dH(x)\ + \\
&+ \sqrt{n}\ \gamma_n^2\int a(x)E\int h(X_0+z)E[F(x-\rho z-Z_n) - F(x+\rho z-Z_n)]dL_n(z)dH(x).
\end{aligned}
$$

The fourth term in (3.37) converges to zero by A_1, $A_{17}(b)$, the same reasoning as in (2.30), (2.31) and the Hölder inequality. The third term in (3.37) converges to zero by A_1, A_{13}, $A_{17}(a)$, the Hölder and the moment inequalities. That the second term in (3.37) converges to zero follows from A_1, A_4, $A_8(b)$, $A_{17}(a)$, the Hölder and the moment inequalities.

For each $x \in \mathbb{R}$,

$$
\begin{aligned}
&E\int h(X_0+z)[F(x-\rho z) - F(x+\rho z)]\ dL_n(z) \\
(3.38)\quad & \\
&\to E\int h(X_0+z)[F(x-\rho z) - F(x+\rho z)]\ dL(z),
\end{aligned}
$$

where L is d.f. of the r.v. Z; we get (3.38) from A_5, $Z_n \to Z$ in distribution and the D.C.T..

The proof of (a) and (b) now follows from the convergence of the first term in (3.37) to the appropriate quantity which itself follows from A_1, A_4, A_5, A_7, (3.38) and the D.C.T..

Remark 3.1. From Theorems 3.4 and 3.5 (a) we see

$$\sqrt{n}\,[\hat{\rho}_h(H) - \rho] \to \mathscr{N}(-\mu\, q^{-1},\ \sigma_h^2) \quad \text{in distribution}$$

and from Theorems 3.4 and 3.5 (b) we see

$$\sqrt{n}\,[\hat{\rho}_h(H) - \rho] \to \mathscr{N}(0,\ \sigma_h^2) \quad \text{in distribution.}$$

Remark 3.2. KL has shown that the function $h(x) \propto x$, $x \in \mathbb{R}$ minimizes σ_h^2. Let $\hat{\rho}_x(H)$ be the estimator corresponding to $h(x) \equiv x$, and measure H. Then, the asymptotic bias of $\hat{\rho}_x(H)$ under Theorem 3.5 (a) looks like

$$\left[2EX_0^2\int f^2 dH\right]^{-1} \gamma_c\int f(x)EZ[F(x+\rho Z) - F(x-\rho Z)]\, dH(x),$$

which has the same sign as the sign of ρ.

Remark 3.3. Consider

S_1: The function defined by $q_1(u) = \int \left[f(x+u) - f(x)\right]^2 dH(x)$, $u \in \mathbb{R}$, is bounded and continuous at 0.

S_2: The function defined by $q_2(u) = \int f(x+u)\, dH(x)$, $u \in \mathbb{R}$, is bounded and continuous at 0.

S_3: $\sup_n E\int (|X_0|+|z|)^2 h^2(X_0+z)\, dL_n(z) < \infty$.

S_4: $\overline{\lim_n}\, EZ_n^2 < \infty$.

Then under S_1, S_2, S_3 and S_4, we can show that the assumptions in Section 1 reduce to $A_1 - A_7$, A_9 and A_{16}.

Proof. A_8(b) follows from S_1, A_7, the Fubini Theorem and the inequality

$$(3.39)\qquad \int f^2(x+u)\, dH(x) \le 2\int [f(x+u) - f(x)]^2\, dH(x) + 2\int f^2\, dH.$$

A_8(a) follows from S_2 and the Fubini Theorem. A_{10} follows from the Fubini Theorem, A_3(b), A_4 and the Bounded Convergence Theorem. A_{12} follows from the Fubini Theorem, S_2, S_3 and the moment inequality. A_{13} follows from the Fubini Theorem, S_1, S_3, A_7 and (3.39). A_{11}(a) and A_{14} follow from the Fubini Theorem, boundedness in S_2, S_3, A_3(b), A_4, A_5 and S_4. A_{11}(b) and A_{15} follow from the Fubini Theorem, boundedness in S_1, S_3, A_4, S_4, A_5, A_7 and (3.39). Further, A_{17}(b) can be bounded by

$$\int E\int h^2(X_0+z)\left[E\left|\int_{-\rho|z|}^{\rho|z|} f(x+u-Z_n)\, du\right|\right]^2 dL_n(z)\, dH(x),$$

which in turn is

$$(3.40)\qquad \le 2\rho \int E\int h^2(X_0+z)|z|E\int_{-\rho|z|}^{\rho|z|} f^2(x+u-Z_n)\, du\, dL(z)\, dH(x),$$

using the moment inequality. That the lim sup of the r.h.s. of (3.40) is finite follows from the Fubini Theorem, S_1, S_3, A_7 and (3.39). A_{17}(a) follows similarly as above.

Remark 3.4. In the case when $H(x) \equiv x$, all the assumptions of Section 1 reduce to A_1, S_3, S_4, $A_3 - A_7$ and A_9.

Proof. Note that A_2 and S_2 are trivially satisfied. In view of A_7 and the translation invariance of the Lebesgue measure, to prove S_1 we need only prove that $\int f(x)f(x+u)\ dH(x)$ is bounded as a function of u and is continuous at 0. Boundedness follows easily by the Hölder inequality and the translation invariance of the Lebesgue measure. From A_7, $f \in L^2(H)$, hence from Rudin (1974, Theorem 3.14), we have for any $\eta > 0\ \exists$ a continuous function ϕ_η vanishing outside a compact set such that

$$(3.41) \qquad |\phi_\eta - f|_H < \eta.$$

Consider

$$(3.42) \qquad \begin{aligned} &\left|\int f(x)f(x+s)\ dH(x) - \int f(x)f(x+t)\ dH(x)\right| \\ &\leq \left|\int f(x)\left[f(x+s) - \phi_\eta(x+s)\right]\ dH(x)\right| + \\ &+ \left|\int f(x)\left[\phi_\eta(x+s) - \phi_\eta(x+t)\right]\ dH(x)\right| + \\ &+ \left|\int f(x)\left[\phi_\eta(x+t) - f(x+t)\right]\ dH(x)\right|. \end{aligned}$$

The continuity of $\int f(x)f(x+u)\ dH(x)$ as a function of u now follows from (3.41), (3.42), A_7, uniform continuity of the function ϕ_η, the Hölder inequality and the translation invariance of the Lebesgue measure.

We shall now prove that A_{16} holds. Note A_{16} holds if A_{16}^* holds, where

A_{16}^*: (a) $\int F(x)[1 - F(x)]\ \ dH(x) < \infty$.

(b) $\overline{\lim_n} \int EF(x-jZ_n)\{1 - EF(x-Z_n)\}\ \ dH(x) < \infty$, $j = 0, 1$.

(c) $\overline{\lim_n} \int E\left[\int h^2(X_0\pm z)F(x+\rho z)[1-EF(x+\rho z-Z_n)]\ dL_n(z)\right]\ dH(x) < \infty.$

(d) $\overline{\lim_n} \int E\left[\int h^2(X_0+z)EF(x+\rho z-jZ_n)[1-EF(x+\rho z-jZ_n)]dL_n(z)\right]\ dH(x)$

$< \infty\ ,\ j = 0,\ 1.$

Since F is continuous and $E|\epsilon_1| < \infty$, we have

$$(3.43)\qquad \int_0^{\infty} [1 - F(x)]\ dx < \infty \quad\text{and}\quad \int_{-\infty}^{0} F(x)\ dx < \infty.$$

Thus

$$\int F(x)[1 - F(x)]\ dx \le \int_0^{\infty} [1 - F(x)]\ dx + \int_{-\infty}^{0} F(x)\ dx < \infty.$$

Using the Fubini Theorem, the translation invariance of the Lebesgue measure and A_5, we see that $A^*_{16}(d)$ with $j = 0$ follows from $A^*_{16}(a)$ and also to prove $A^*_{16}(b)$ with $j = 1$ is the same as to prove $A^*_{16}(d)$ with $j = 1$. $A^*_{16}(b)$ with $j = 1$ can be rewritten as

$$\iiint F(x)[1 - F(x-z+u)]\ dx\ dL_n(z)\ dL_n(u)$$

$$(3.44)\qquad = \iiint_0^{\infty} F(x)[1 - F(x-z+u)]\ dx\ dL_n(z)\ dL_n(u)\ +$$

$$+ \iiint_{-\infty}^{0} F(x)[1-F(x-z+u)]\ dx\ dL_n(z)\ dL_n(u).$$

The second term in (3.44) is bounded by the 2nd term in (3.43). Now consider the first term in (3.44), which can be written as

$$\iiint_0^\infty F(x)[1 - F(x-z+u)]I[u \geq z] \, dx \, dL_n(z) \, dL_n(u) +$$

(3.45)

$$+ \iiint_0^\infty F(x)[1 - F(x-z+u)]I[u < z] \, dx \, dL_n(z) \, dL_n(u).$$

The first term in (3.45) is bounded by the first term in (3.43). Rewrite the second term in (3.45), using change of variable and splitting the range of integration, as

$$\iiint_{u-z}^0 F(x+z-u)[1 - F(x)]I[u < z] \, dx \, dL_n(z) \, dL_n(u) +$$

(3.46)

$$+ \iiint_0^\infty F(x+z-u)[1 - F(x)]I[u < z] \, dx \, dL_n(z) \, dL_n(u).$$

The first term in (3.46) is bounded by $2E|Z_n|$. Hence from S_4 we get that the lim sup of the first term in (3.46) is finite. The second term in (3.46) is bounded by the first term in (3.43); consequently $A_{16}^*(b)$ and $A_{16}^*(d)$ with $j = 1$ hold. By the Fubini Theorem, the symmetry of L_n's about 0 and the translation invariance of the Lebesgue measure, $A_{16}^*(c)$ can be written as

$$\overline{\lim_n} \, E\{\int h^2(X_0+z) \, dL_n(z)\} \int F(x)[1 - EF(x-Z_n)] \, dH(x),$$

which is the same as proving $A_{16}^*(b)$ with $j = 0$, in view of A_5.

Proceeding exactly as in (3.44) – (3.46) and using (3.43), S_4 and A_5, we get that $A^*_{16}(c)$ holds.

Remark 3.5. In case H generates a finite measure, all the assumptions of Section 1 reduce to A_1 – A_7, A_9, $A_{10}(b)$, $A_{11}(b)$, A_{15} and, $A_8(b)$ and A_{13} with lim sup replaced by sup.

Proof. From A_5 and the fact that H generates a finite measure, we see A_{16} and A_{17} are easily satisfied. Using the Hölder inequality, A_5 and $A_{15}(b)$, we see $A_{14}(b)$ holds. Using the Hölder inequality, A_5 and $A_{15}(a)$, we see $A_{14}(a)$ holds. Using the Hölder inequality, $A_3(b)$ and $A_{11}(b)$, we see $A_{11}(a)$ holds. Using the moment inequality and A_{13} with lim sup replaced by sup, we see A_{12} holds. Using $A_3(b)$, $A_{10}(b)$ and the Hölder inequality, we see $A_{10}(a)$ holds. Using $A_8(b)$ with lim sup replaced by sup and the moment inequality we see $A_8(a)$ holds.

Remark 3.6. For H given by

$$dH = \frac{dF}{F(1-F)}, \quad \text{where} \quad f(x) = 2^{-1}\exp\{-|x|\}, \quad x \in \mathbb{R},$$

[Note here we could take f to be the density function of a $\mathcal{N}(0,\sigma^2)$.] we shall show that in view of Remark 3.3, the assumptions in Section 1 reduce to A_1, $A_3(b)$, S_3, A_4 – A_6, and

$$S_5\colon\ \overline{\lim_n}\ E\exp|Z_n| < \infty,$$
$$S_6\colon\ \overline{\lim_n}\ E\int h^2(X_0+z)\exp\{|\rho z|\}\ dL_n(z).$$

<u>Proof</u>. Note

$$F(x) = 1 - 2^{-1}\exp\{-x\} \quad \text{if} \quad x \geq 0$$
$$= 2^{-1}\exp\{x\} \quad \text{if} \quad x < 0.$$

Let $x, u \in \mathbb{R}$, and note

$$\text{(3.47)} \qquad \frac{\exp\{-k|x|-j|x+u|\}}{F(x)[1-F(x)]} \leq \exp\{-(k-1)|x|\}[2 - \exp\{-|x|\}]^{-1},$$

where $k, j = 1, 2$. Since the r.h.s. of (3.47) is integrable over the real line, applying the D.C.T. to the functions in (3.47), we get S_1, S_2 and A_7 hold. Let $x, u \in \mathbb{R}$. From the inequality

$$\left|\frac{e^{-|x-u|}}{2} - \frac{e^{-|x|}}{2}\right| \leq 2^{-1}|u|\left[\exp\{-|x-u|\} + \exp\{-|x|\}\right]$$

and the translation invariance of the Lebesgue measure, we see

$$\int |f(x-u) - f(x)| \, dx \leq |u|.$$

Since A_{16}^* implies A_{16}, we shall verify A_{16}^* holds. A_2 and $A_{16}^*(a)$ are trivially satisfied from definition of H. $A_3(a)$ follows trivially from the a.s. representation of X_0 as $\sum_{j=0}^{\infty} \rho^j \epsilon_{-j}$. S_5 implies S_4 holds. Using A_5, S_5 and S_6, lengthy calculations show that $A_{16}^*(b)$ – (d) hold.

If in addition to the restrictions of this remark we assume $h(x) \equiv x$, then the assumptions in Section 1 reduce to A_1 and S_5. For these assumptions to be satisfied we could choose Z_n to be $\mathcal{N}(0,\sigma^2)$ and $\gamma_n = n^{-1/2}$ for all n.

Remark 3.7 (Influence function). We shall define a functional T on a subset, say P_0, of the set of all stationary ergodic measures on $(\mathbb{R}^{-\infty,\infty}, \mathcal{B})$, where $\mathbb{R}^{-\infty,\infty}$ is a collection of all sequences of the type $\underset{\sim}{y} = (\ldots, y_{-1}, y_0, y_1, y_2, \ldots)$ and $\mathcal{B}$ is the Borel σ-field on $\mathbb{R}^{-\infty,\infty}$. Proceeding as in Martin and Yohai (1986, equations 3.1 – 3.3), the functional T, $\mu \in P_0$, is defined as $T(\mu)$ satisfying the equation

$$(3.48)\qquad \frac{d}{dt}\int \left[\int h(y_0)\{I[y_1 \le x+ty_0] - I[-y_1 < x-ty_0]\}\, d\mu(\underset{\sim}{y})\right]^2 dH(x) = 0,$$

where P_0 is the set of all stationary ergodic measures on $(\mathbb{R}^{-\infty,\infty}, \mathcal{B})$, such that the integral in (3.48) exists and is differentiable. Further, $T(\mu)$ minimizes the double integral in (3.48) as a function of t, and of all the minimizers it is defined as the one with the smallest magnitude. Under the assumptions A_2, A_6 and following the same argument as in Section 1, we see that (3.48) has at least one solution which minimizes the double integral in (3.48), as a function of t. For the sake of completeness, let us repeat the definition of the time-series influence functional as in Martin and Yohai (1986, Definition 4.2). For T as in equation (3.48) and μ_y^γ the measure generated by the process (0.2) where $\gamma_n = \gamma$ and $L_n = L \ \forall\, n$, the influence functional (IF) of T is defined as

$$(3.49)\qquad \mathrm{IF} = \lim_{\gamma \to 0} \frac{T(\mu_{\underset{\sim}{y}}^{\gamma}) - T(\mu_{\underset{\sim}{y}}^{0})}{\gamma}$$

provided the limit exists.

In (3.48) taking $\mu = \mu_y^\gamma$, then computing the innermost integral and differentiating within the integral sign w.r.t. t under some regularity conditions, we get $T^\gamma \equiv T(\mu_y^\gamma)$ satisfies

$$
\begin{aligned}
(3.50)\quad & \int \Big[(1-\gamma)Eh(X_0)\{G(x+[T^\gamma-\rho]X_0) - G(x-[T^\gamma-\rho]X_0)\} + \\
& + \gamma E\int h(X_0+z)\{G(x+[T^\gamma-\rho](X_0+z)+\rho z) - G(x-[T^\gamma-\rho](X_0+z)-\rho z)\}\, dL(z)\Big]\cdot \\
& \cdot\Big[(1-\gamma)EX_0h(X_0)\{g(x+[T^\gamma-\rho]X_0) + g(x-[T^\gamma-\rho]X_0)\} + \gamma E\int h(X_0+z)(X_0+z)\cdot \\
& \cdot\{g(x+[T^\gamma-\rho](X_0+z)+\rho z)+g(x-[T^\gamma-\rho](X_0+z)+\rho z)\}dL(z)\Big]\ dH(x) = 0.
\end{aligned}
$$

Note in (3.50) G and g both depend on γ. Setting $\gamma = 0$ in (3.50) we see $T^0 \equiv T(\mu_y^0)$ satisfies

$$
\begin{aligned}
(3.51)\quad & \int Eh(X_0)\{F(x+[T^0-\rho]X_0) - F(x-[T^0-\rho]X_0)\}\cdot \\
& \qquad \cdot EX_0h(X_0)\{f(x+[T^0-\rho]X_0) + f(x-[T^0-\rho]X_0\}\ dH(x) = 0.
\end{aligned}
$$

Assuming the derivative of F exists and equals f we see from A_6 that $T^0 = \rho$ is the only solution to (3.51) which makes the double integral in (3.48) with $\mu = \mu_y^0$ minimum. Thus differentiating the l.h.s. of (3.50) under the integral sign w.r.t. γ, then setting $\gamma = 0$, solving for $\frac{\partial T^\gamma}{\partial \gamma}\big|_{\gamma=0}$, we get

$$
IF = q^{-1}EX_0h(X_0)\int f(x)E\int h(X_0+z)\{F(x+\rho z)-F(x-\rho z)\}\ dL(z)dH(x).
$$

Note that $IF = -\gamma_c^{-1}$(asymptotic bias of $\sqrt{n}\,[\hat{\rho}_h(H) - \rho]$), where the bias is computed under the assumptions of Theorem 3.5 (a).

CHAPTER 2

0. Introduction and Summary.

Heathcote and Welsh (1983)(H–W) proposed a class of minimum distance estimators $\hat{\rho}_n(s)$ of the vector $\underset{\sim}{\rho}$ in an autoregressive model of order k, defined so as to minimize

$$M_n(\underset{\sim}{t},s) = - s^{-2}\log\left|(n-k)^{-1}\sum_{j=k+1}^{n} \exp\{is(X_j - \underset{\sim}{X}_{j-1}'\underset{\sim}{t})\}\right|^2$$

as a function of $\underset{\sim}{t}$, where $\underset{\sim}{X}_{j-1}' = (X_{j-1},X_{j-2},\ldots,X_{j-k})$.

Here we shall study the behavior of this class of estimators of $\rho \in (-1,1)$, when $k = 1$, under the A.O. model (1.0.2) – (1.0.3). The model assumptions of this chapter exclude the assumption F, L_n's symmetric about 0 and $E\epsilon_0^2 < \infty$, but assume the first moment of ϵ_0 exists and $E\epsilon_0 = 0$.

Section 1 contains the definition of a class of minimum distance estimators $\hat{\rho}_n(s)$, $s \in \mathscr{S}$, of ρ where $\mathscr{S}$ is a compact set of the type $[-b,-a] \cup [a,b]$, $0 < a < b$.

The main result in Section 2 is stated under Theorem 2.5, wherein it is proved that the sup norm distance between $\hat{\rho}_n(s)$ and ρ converges to zero in probability or almost surely, the sup being taken over $\mathscr{S}$. This convergence in probability (almost surely) is referred to as uniform (strong) consistency of the estimators. The idea of the proof of Theorem 2.5 is taken from Csörgö (1983). Lemmas 2.1 to 2.4 are used to prove this result. Lemma 2.1 uniformly approximates $\exp\{- s^2M_n(t,s)\}$, under the model (1.0.2) – (1.0.3), by $\exp\{- s^2M_n(t,s)\}$ with $Y_{j,n}$ replaced by

X_j, where uniformity is taken over all $s \in \mathscr{S}$ and t in some compact set K containing the true value ρ in its nonempty interior. Remark 2.1 uses Lemma 2.1 to give sufficient conditions under which ρ is the unique global minimum of the limit of $M_n(t,s)$ for each $s \in \mathscr{S}$. Lemma 2.2 contains a standard result. For the sake of completeness, Lemma 2.3 gives a Glivenko–Cantelli type result for empirical d.f. of stationary ergodic random vectors from Gaenssler and Stute (1976). Lemma 2.4 finds the uniform limit of $\exp\{-s^2 M_n(t,s)\}$ under the A.O. model for $(s,t) \in \mathscr{S} \times K$.

Section 3 contains the discussion on the weak convergence of the process $\{\sqrt{n}\,[\hat{\rho}_n(s) - \rho],\ s \in \mathscr{S}\}$ and its asymptotic bias. This section begins by obtaining an approximation for $\sqrt{n}\,[\hat{\rho}_n(s) - \rho]$ using the Taylor series expansion. Theorem 3.1 gives the uniform limit of the coefficient of $\sqrt{n}\,[\hat{\rho}_n(s) - \rho]$ in the above expansion. Using this theorem, the techniques as in H–W and the results from Billingsley (1968), Theorem 3.2 proves the weak convergence of the process $\{\sqrt{n}\,[\hat{\rho}_n(s) - \rho],\ s \in \mathscr{S}\}$, when appropriately centered, to a Gaussian process in $C(\mathscr{S})$–space. The finite dimensional distribution convergence of the above process is proved by using the C.L.T. for ℓ–mixing set of processes of Withers (1981 and 1983) and Billingsley (1968). Remark 3.1 observes that the asymptotic bias of the process $\hat{\rho}_n(s)$ converges uniformly to zero if either $\sqrt{n}\,\gamma_n = o(1)$ or $\sqrt{n}\,\gamma_n = O(1)$ and $Z_n = o_p(1)$ and that it converges uniformly to a function if $Z_n \to Z$ in probability and $\sqrt{n}\,\gamma_n \to \gamma$, where Z_n is the r.v. associated with L_n and Z is a r.v.. The corresponding weak convergence results for the process $\{\sqrt{n}\,[\hat{\rho}_n(s) - \rho],\ s \in \mathscr{S}\}$ under the above observations are also stated in this remark.

1. Definition of a class of estimators.

Define

$$\mathscr{S} := [-b, -a] \cup [a,b], \quad 0 < a < b,$$

$$\text{(1.1)} \qquad M_n(t,s) = -s^{-2} \log \Big| n^{-1} \sum_{j=1}^{n} \exp\{is[Y_{j,n} - Y_{j-1,n}t]\} \Big|^2 \quad \text{if } s \in \mathscr{S}$$

$$= \frac{1}{n} \sum_{j=1}^{n} (Y_{j,n} - Y_{j-1,n}t)^2 \qquad \text{if } s = 0.$$

Let K be a compact set containing the true parameter ρ in its nonempty interior. Then $\hat{\rho}_n(s)$ for each $s \in \mathscr{S} \cup \{0\}$ denotes a measurable minimizer of $M_n(\cdot,s)$ when restricted to K. For M_n as in (1.1), $\hat{\rho}_n$ can be uniquely defined to be sample continuous on $\mathscr{S}$. Further $\hat{\rho}_n(s)$ satisfies

$$\text{(1.2)} \qquad \inf_{t \in K} M_n(t,s) = M_n(\hat{\rho}_n(s),s).$$

Note that $\hat{\rho}_n(0)$ is the least squares estimator which has been studied in detail by Denby and Martin (1979) under the A.O. model; hence we shall not allow s to be zero.

2. Uniform (strong) consistency of $\hat{\rho}_n(s)$, $s \in \mathscr{S}$.

In this section we prove uniform (strong) consistency of the estimators $\hat{\rho}_n(s)$, $s \in \mathscr{S}$. The idea of the proof for this result is taken from Csörgö (1983) and H–W.

Lemma 2.1. For each n, let $Y_{j,n} = X_{j,n} + v_{j,n}$, $j = 0,1,\ldots,n$ be r.v.s. Let $\kappa_n = n^{-1}\sum_{j=1}^{n} |v_{j,n}| \wedge 1$. Let $K_n \subset \mathbb{R}^2$ be such that,

$$c_{1,n} = \sup_{(t,s)\in K_n}\{|s|\}, \quad c_{2,n} = \sup_{(t,s)\in K_n}\{|st|\} \quad \text{and} \quad c_{1,n} + c_{2,n} \le c < \infty.$$

Then

$$\sup_{(t,s)\in K_n}\left|n^{-1}\sum_{j=1}^{n}\left[\exp\{is[Y_{j,n} - tY_{j-1,n}]\} - \exp\{is[X_{j,n} - tX_{j-1,n}]\}\right]\right| \le [(c+2) \vee 4]\kappa_n. \tag{2.1}$$

Proof. The Triangle inequality and $|e^{it} - e^{is}| \le |t - s| \wedge 2$, $t, s \in \mathbb{R}$, show that the l.h.s. of (2.1) can be bounded by

$$n^{-1}\sum_{j=1}^{n}[\{c_{1,n}|v_{j,n}| + c_{2,n}|v_{j-1,n}|\} \wedge 2]. \tag{2.2}$$

The expression inside the sum in (2.2) is bounded by

$$[c_{1,n}|v_{j,n}| \wedge 2] + [c_{2,n}|v_{j-1,n}| \wedge 2].$$

The r.h.s. of (2.2) is now dominated by

$$[(c_{1,n} \vee 2) + (c_{2,n} \vee 2)]\kappa_n.$$

From which the lemma follows. □

Note. Under condition:

$$(2.3) \qquad \begin{array}{l} \text{(a)} \quad \kappa_n \to 0 \text{ in probability} \\ \qquad \text{or} \\ \text{(b)} \quad \sum_{n=0}^{\infty} E\kappa_n < \infty, \end{array}$$

l.h.s. of (2.1) converges to zero in probability or a.s., the latter follows from the Markov inquality and the Borel–Cantelli Lemma applied to κ_n. In the case when $v_{j,n}$ are β_n distributed for any d.f. L_n in (1.0.1), $E\kappa_n = n^{-1}(n+1)\gamma_n E\{|Z_n| \wedge 1\}$, and thus $\gamma_n = o(1)$ or $\Sigma\, \gamma_n < \infty$ become sufficient conditions in (a) or (b). Further note from here on we shall supress the n in the random variables (r.v.s) $Y_{j,n}$ and $v_{j,n}$.

Remark 2.1. We shall now present the sufficient conditions for the unique true value ρ to be the global minimum of the limit of $M_n(t,s)$ for each $s \in \mathscr{S}$. Fix $\delta > 0$ and $s \in \mathscr{S}$. From (1.1) we get

$$(2.4) \qquad M_n(\rho\pm\delta,s) - M_n(\rho,s) = -s^{-2}\log\Big[|n^{-1}\sum_{j=1}^{n}\exp\{is[Y_j - (\rho\pm\delta)Y_{j-1}]\}|^2/$$
$$/|n^{-1}\sum_{j=1}^{n}\exp\{is[Y_j - \rho Y_{j-1}]\}|^2\Big].$$

The r.h.s. of (2.4) can be written as

$$-s^{-2}\log\Big[\Big|n^{-1}\sum_{j=1}^{n}[\exp\{is[Y_j - (\rho\pm\delta)Y_{j-1}] - \exp\{is[X_j - (\rho\pm\delta)X_{j-1}]\} +$$

$$(2.5) \qquad + \exp\{is[X_j - (\rho\pm\delta)X_{j-1}]\}\Big]\Big|^2 \Big/ \Big|n^{-1}\sum_{j=1}^{n}\Big[\exp\{is[Y_j - \rho Y_{j-1}]\} -$$

$$- \exp\{is[X_j - \rho X_{j-1}]\} + \exp\{is[X_j - \rho X_{j-1}]\}\Big|^2\Big].$$

From Lemma 2.1 and the Stationary Ergodic Theorem (S.E.T.), (2.5) converges in probability (a.s.) to

$$(2.6) \qquad - s^{-2} \log\{|\phi_{X_1 - (\rho\pm\delta)X_0}(s)|^2/|\phi_{\epsilon_1}(s)|^2\},$$

where ϕ_X denotes the characteristic function of a r.v. X. Under the assumptions $|\phi_{\epsilon_1}(s)| > 0$ and $|\phi_{X_0}(s)| < 1$, $(2.6) > 0 \quad \forall\, s \in \mathscr{S}$. Thus for sufficiently large n, with large probability (with probability 1), a minimum is achieved at the true value ρ. If the distribution of ϵ_1 is infinitely divisible then $|\phi_{\epsilon_1}(s)| > 0 \quad \forall\, s \in \mathscr{S}$ is satisfied. Also, if the distribution of ϵ_1 is not lattice type then $|\phi_{X_0}(s)| < 1 \quad \forall\, s \in \mathscr{S}$ follows from $\phi_{X_1}(s) = \phi_{X_0}(\rho s)\phi_{\epsilon_1}(s)$ and Chung (1974, Theorem 6.4.7).

<u>Lemma 2.2</u>. If $\phi_n(\underset{\sim}{t})$ are characteristic functions on $\mathbb{R}^k$ such that $\phi_n(\underset{\sim}{t}) \to \phi(\underset{\sim}{t})$ for each $\underset{\sim}{t} \in \mathbb{R}^k$, then for any compact subset K of $\mathbb{R}^k$

$$\sup_{\underset{\sim}{t}\in K} |\phi_n(\underset{\sim}{t}) - \phi(\underset{\sim}{t})| \to 0.$$

<u>Proof</u>. The proof follows from Ash (1974, p. 333, Theorem 3.2.9). □

Lemma 2.3. Let $X_1, X_2, \ldots$ be a sequence of strictly stationary and ergodic random vectors taking values in $\mathbb{R}^k$. Then

$$(2.7) \qquad P(\overline{\lim_n} \sup_{-\infty < x_1, x_2, \ldots, x_k < \infty} |F_n(x_1,\ldots,x_k) - F(x_1,\ldots,x_k)| = 0) = 1,$$

where $F_n(x_1,x_2,\ldots,x_k)$ is the joint empirical distribution function based on $X_1,\ldots,X_n$ and $F(x_1,\ldots,x_k)$ is the joint distribution of the random vector X_1.

Proof. See Stute and Schumann (1980) and Gaenssler and Stute (1976). □

To state the next lemma, let

$$(2.8) \qquad D_n(t,s) = \left| \frac{1}{n} \sum_{j=1}^{n} \exp\{is(Y_j - tY_{j-1})\} - \phi_{\epsilon_1}(s)\phi_{X_0}(s[\rho - t]) \right|, \quad (t,s) \in K \times \mathscr{S}.$$

Lemma 2.4. Let $\{X_j\}$ be as in (1.0.3) with ϵ_j's i.i.d.. Let $v_{j,n}$'s be as in Lemma 2.1. Then

(a) under (a) of (2.3)

$$(2.9) \qquad \sup_{(t,s) \in K \times \mathscr{S}} D_n(t,s) \to 0 \text{ in probability}$$

and

(b) under (b) of (2.3), (2.9) holds with convergence in probability replaced by a.s. convergence.

Proof. We shall give the proof of (b). The proof of (a) follows similarly. From the triangle inequality we get

$$(2.10)\quad D_n(t,s) \le \left| n^{-1}\sum_{j=1}^{n} \exp\{is(Y_j - Y_{j-1}t)\} - n^{-1}\sum_{j=1}^{n} \exp\{is(X_j - X_{j-1}t)\}\right|$$

$$+ \left| n^{-1}\sum_{j=1}^{n} \exp\{is(X_j - X_{j-1}t)\} - \phi_{\epsilon_1}(s)\, \phi_{X_0}(s[\rho - t])\right|.$$

That the first term on the r.h.s. of (2.10) goes to zero a.s. follows from condition (b) of (2.3). Since $\{(X_{j-1},\epsilon_j)\}$ is a stationary ergodic sequence, Lemma 2.3 yields

$$1 = P\Big(\sup_{x_1, x_2 \in \mathbb{R}} \left| n^{-1}\sum_{j=1}^{n} I[\epsilon_j \le x_1,\ X_{j-1} \le x_2] - F_{\epsilon_1}(x_1)F_{X_0}(x_2)\right| \to 0\Big)$$

$$(2.11)\quad \le P\Big(\sup_{(s_1, s_2)\in K_1 x K_2} \left| \sum_{j=1}^{n} \exp\{is_1\epsilon_j + is_2X_{j-1}\} - \phi_{\epsilon_1}(s_1)\phi_{X_0}(s_2)\right| \to 0\Big),$$

where K_1 and K_2 are any compact subsets of $\mathbb{R}$. The inequality (2.11) follows from the Continuity Theorem and Lemma 2.2. In particular, taking $K_1 = \mathscr{S}$ and $K_2 = \{s(\rho - t):\ s \in \mathscr{S},\ t \in K\}$ in (2.11), we get

$$(2.12)\quad P\Big(\sup_{(t,s)\in K x \mathscr{S}} \left| n^{-1}\sum_{j=1}^{n} \exp\{is(X_j - X_{j-1}t)\} - \phi_{\epsilon_1}(s)\phi_{X_0}(s[\rho - t])\right| \to 0\Big) = 1.$$

Now the lemma follows from (2.10) and (2.12). □

Next, set

$$\text{(2.13)} \quad M(t,s) = -\, s^{-2}\log|\phi_{\epsilon_1}(s)\phi_{X_0}(s[\rho-t])|^2, \quad (t,s) \in K \times \mathscr{S}.$$

Note that

$$\text{(2.14)} \quad M(t,s) \geq M(\rho,s) \ \forall\ s \quad \text{and hence} \quad \inf_{t \in K} M(t,s) = M(\rho,s) \ \forall\ s.$$

Theorem 2.5. Let X_j's and $v_{j,n}$'s be as in Lemma 2.4 with

$$\text{(2.15)} \quad |\phi_{\epsilon_1}(s)| > 0 \quad \text{and} \quad 0 < |\phi_{X_0}(s[\rho-t])| < 1, \quad s \in \mathscr{S} \text{ and}$$
$$t \in K - \{\rho\}.$$

Then the following statements hold:

(a) Under (a) of (2.3),

$$\text{(2.16)} \quad \sup_{s \in \mathscr{S}} |\hat{\rho}_n(s) - \rho| \to 0 \quad \text{in probability.}$$

(b) Under (b) of (2.3), (2.16) holds with probability convergence replaced by almost sure convergence.

Proof. The proof of (b) is as in Csörgö (1983, p. 345, Theorem 4.1). We shall give the proof of (a). From (1.1) and (2.13),

$$|M_n(t,s)-M(t,s)| \leq C_0\log[\{D_n(t,s)/|\phi_{\epsilon_1}(s)|^2|\phi_{X_0}(s[\rho-t])|^2\} + 1],$$

where $C_0 = \sup\{s^{-2}: s \in \mathscr{S}\}$. Since

$$\inf_{(s,t) \in \mathscr{S} \times K} |\phi_{\epsilon_1}(s)|^2|\phi_{X_0}(s[\rho-t])|^2 > 0,$$

Lemma 2.4 and the above inequality imply that

$$(2.17)\qquad \sup_{(t,s)\in K\times\mathcal{S}} |M_n(t,s) - M(t,s)| = o_p(1).$$

From (1.2), (2.14) and (2.17)

$$(2.18)\qquad \sup_{s\in\mathcal{S}}\left|M_n(\hat{\rho}_n(s),s) - M(\rho,s)\right| = \sup_{s\in\mathcal{S}}\left|\inf_{t\in K} M_n(t,s) - \inf_{t\in K} M(t,s)\right| = o_p(1).$$

For $\delta > 0$, let $K(\delta) = K - \{t:|t-\rho| < \delta\}$. Then, (2.17) also implies

$$(2.19)\qquad \sup_{s\in\mathcal{S}}\left|\inf_{t\in K(\delta)} M_n(t,s) - \inf_{t\in K(\delta)} M(t,s)\right| = o_p(1).$$

Suppose that

$$(2.20)\qquad \sup_{s\in\mathcal{S}}|\hat{\rho}_n(s) - \rho| \text{ does not converge to zero in probability.}$$

Then $\exists$ $\eta_0, \eta_1 > 0$ and a sequence of integers $n_k \uparrow \infty$ such that

$$(2.21)\qquad P(\sup_{s\in\mathcal{S}}|\hat{\rho}_{n_k}(s) - \rho| > \eta_1) > \eta_0.$$

Let $\eta = 3^{-1}\inf_{s\in\mathcal{S}}|M(\rho,s) - \inf_{t\in K(\eta_1)} M(t,s)|$. From assumption (2.15), $\eta > 0$.

From (2.18), (2.19) and (2.21), $\exists\ k_0(\eta,\eta_0)$ such that $\forall\ k > k_0(\eta,\eta_0)$

$$\eta_0 < P\Big[\sup_{s\in\mathscr{S}}|M_{n_k}(\hat{\rho}_{n_k}(s),s) - M(\rho,s)| < \eta,\ \sup_{s\in\mathscr{S}}|\hat{\rho}_{n_k}(s)-\rho| > \eta_1\Big] + \eta_0/2,$$

$$< P\Big[\sup_{s\in\mathscr{S}}|M_{n_k}(\hat{\rho}_{n_k}(s),s) - M(\rho,s)| < \eta,\ \sup_{s\in\mathscr{S}}|\hat{\rho}_{n_k}(s)-\rho| > \eta_1,\ ,\ \sup_{s\in\mathscr{S}}|\inf_{t\in K(\eta_1)} M_{n_k}(t,s) - \inf_{t\in K(\eta_1)} M(t,s)| < \eta\Big] +$$

$$+ P\Big[\sup_{s\in\mathscr{S}}|M_{n_k}(\hat{\rho}_{n_k}(s),s) - M(\rho,s)| < \eta,\ \sup_{s\in\mathscr{S}}|\hat{\rho}_{n_k}(s) - \rho| > \eta_1,\ ,\ \sup_{s\in\mathscr{S}}|\inf_{t\in K(\eta_1)} M_{n_k}(t,s) - \inf_{t\in K(\eta_1)} M(t,s)| \geq \eta\Big] + \eta_0/2$$

$$\text{(2.22)}\quad < P\Big[\sup_{s\in\mathscr{S}}|M_{n_k}(\hat{\rho}_{n_k}(s),s) - M(\rho,s)| < \eta,\ \sup_{s\in\mathscr{S}}|\hat{\rho}_{n_k}(s) - \rho| > \eta_1,\ ,\ \sup_{s\in\mathscr{S}}|\inf_{t\in K(\eta_1)} M_{n_k}(t,s) - \inf_{t\in K(\eta_1)} M(t,s)| < \eta\Big] + 3\eta_0/4.$$

From the definition of η, the first term on the r.h.s. of (2.22) is zero, which leads to a contradiction. Therefore (2.20) must be false and hence the result. □

<u>Note</u>. The proof of Theorem 2.5 does not use the existence of f nor does it use any of the moments of ϵ_0 or Z_n. Note that under the assumptions ϵ_j's i.i.d. and $|\rho| < 1$, $\{X_j\}$ of (1.0.3) is invertible and strictly stationary ergodic.

3. Weak convergence of the process $\sqrt{n}\,(\hat{\rho}_n(s) - \rho)$, $s \in \mathscr{S}$.

In this section we prove the weak convergence of $\sqrt{n}\,[\hat{\rho}_n(s) - \rho]$ as a $C(\mathscr{S})$ valued random element. The idea of the proof for this result is taken from H–W. The C.L.T. given by Withers (1981 and 1983) has been used to prove its finite dimensional distribution convergence. We also discuss the behavior of its asymptotic bias.

Recall from (1.1) that to minimize $M_n(t,s)$ w.r.t. t is equivalent to maximizing $U_n^2(t,s) + V_n^2(t,s)$, where for $(t,s) \in K \times \mathscr{S}$,

$$U_n(t,s) = n^{-1} \sum_{j=1}^{n} \cos(s[Y_j - tY_{j-1}]),$$

$$V_n(t,s) = n^{-1} \sum_{j=1}^{n} \sin(s[Y_j - tY_{j-1}]) \tag{3.1}$$

and

$$U_n \equiv V_n \equiv 0, \text{ otherwise.}$$

Let $\quad m_n \equiv 2^{-1}\frac{\partial}{\partial t}(U_n^2 + V_n^2)$. Then

$$m_n(t,s) = -\frac{1}{ns}\sum_{j=1}^{n} Y_{j-1}\{U_n \sin(s[Y_j - tY_{j-1}]) - V_n \cos(s[Y_j - tY_{j-1}])\}, \quad (t,s) \in K \times \mathscr{S},$$
$$= 0, \qquad \text{otherwise.} \tag{3.2}$$

By the Taylor series expansion

$$m_n(\hat{\rho}_n(s),s) = m_n(\rho,s) + \frac{\partial}{\partial t} m_n(t,s)\Big|_{t=\bar{\rho}(s)}[\hat{\rho}_n(s) - \rho], \tag{3.3}$$

where

$$|\bar{\rho}_n(s) - \rho| \le |\hat{\rho}_n(s) - \rho|, \; s \in \mathscr{S}. \tag{3.4}$$

Theorem 3.1 below shows that $\frac{\partial}{\partial t} m_n(t,s)\Big|_{t=\hat{\rho}(s)}$, uniformly in s, converges in probability to a negative number. Hence

$$(3.5)\qquad \sup_{s\in\mathscr{S}} |m_n(\hat{\rho}_n(s),s)| = o_p(1).$$

Thus from (3.3) and (3.5) we see that in order to prove the weak convergence of $\sqrt{n}\,(\hat{\rho}_n(s) - \rho)$, it suffices to study the weak convergence of the process $m_n(\rho,s)$, $s \in \mathscr{S}$. Before stating the next theorem, note that

$$(3.6)\qquad \frac{\partial}{\partial t} m_n(t,s) = s^{-2}\Big\{U_n(t,s)\frac{\partial^2}{\partial t^2}U_n(t,s) + V_n(t,s)\frac{\partial^2}{\partial t^2}V_n(t,s) + \left[\frac{\partial}{\partial t}V_n(t,s)\right]^2 + \left[\frac{\partial}{\partial t}U_n(t,s)\right]^2\Big\}, \qquad t \in K,\ s \in \mathscr{S},$$

$$(3.7)\qquad \begin{aligned} \frac{\partial}{\partial t} U_n(t,s) &= -\,sn^{-1}\sum_{j=1}^{n} Y_{j-1}\sin(s[Y_j - tY_{j-1}]),\\ \frac{\partial}{\partial t} V_n(t,s) &= sn^{-1}\sum_{j=1}^{n} Y_{j-1}\cos(s[Y_j - tY_{j-1}]),\\ \frac{\partial^2}{\partial t^2} U_n(t,s) &= -\,s^2n^{-1}\sum_{j=1}^{n} Y_{j-1}^2\cos(s[Y_j - tY_{j-1}]),\\ \frac{\partial^2}{\partial t^2} V_n(t,s) &= -\,s^2n^{-1}\sum_{j=1}^{n} Y_{j-1}^2\sin(s[Y_j - tY_{j-1}]), \end{aligned}$$

for all $(t,s) \in K \times \mathscr{S}$. From here on it will be understood that sup is taken over all $s \in \mathscr{S}$, unless specified otherwise.

<u>Theorem 3.1</u>. In addition to the assumptions of Theorem 2.5(a) and all the model assumptions (1.0.1) – (1.0.3), assume $\overline{\lim_n}\, EZ_n^2 < \infty$.

Then

$$(3.8)\qquad \sup_{s\in\mathscr{S}}\left|\frac{\partial}{\partial t}\, m_n(t,s)|_{t=r(s)} + |\phi_{\epsilon_1}(s)|^2 EX_0^2\right| = o_p(1),$$

with $r = \hat{\rho}_n$ or $\bar{\rho}_n$.

<u>Proof</u>. Throughout this proof we shall need sup of each of the following random functions

$$(3.9)\qquad |U_n|,\ |V_n|,\ \left|\frac{\partial}{\partial t}\, U_n\right|,\ \left|\frac{\partial}{\partial t}\, V_n\right|,\ \left|\frac{\partial^2}{\partial t^2}\, U_n\right|,\ \left|\frac{\partial^2}{\partial t^2}\, V_n\right|$$

taken over all $(t,s) \in K \times \mathscr{S}$, to be bounded in probability, which is evident from (3.1), (3.7), $E\epsilon_0^2 < \infty$, the S.E.T. and

$$(3.10)\qquad E\, n^{-1} \sum_{j=1}^{n} v_{j-1}^2 = \gamma_n EZ_n^2 \to 0,$$

which in turn follows from the $\overline{\lim_n}\, EZ_n^2 < \infty$.

We shall now prove (3.8) with $r = \bar{\rho}_n$; the proof for $r = \hat{\rho}_n$ is exactly the same. From the triangle inequality the expression inside the sup on the l.h.s. of (3.8) can be bounded by

$$\left|\frac{\partial}{\partial t}\, m_n(t,s)|_{t=\bar{\rho}(s)} - \frac{\partial}{\partial t}\, m_n(t,s)|_{\substack{t=\bar{\rho}(s)\\ Y=X}}\right| +$$

$$(3.11)\qquad + \left|\frac{\partial}{\partial t}\, m_n(t,s)|_{\substack{t=\bar{\rho}(s)\\ Y=X}} - \frac{\partial}{\partial t}\, m_n(t,s)|_{\substack{t=\rho\\ Y=X}}\right| +$$

$$+ \left|\frac{\partial}{\partial t}\, m_n(t,s)|_{t=\rho} + \left|\phi_{\epsilon_1}(s)\right|^2 EX_0^2\right|$$

Here, and in what follows, $|_{\substack{t=\rho \\ Y=X}}$ means that in the given expression replace Y_j by X_j and t by ρ etc. The first term in (3.11) can be dominated by

$$
\begin{aligned}
C_0\Bigg[& \left|\frac{\partial^2}{\partial t^2} U_n(t,s)|_{t=\bar\rho(s)} - \frac{\partial^2}{\partial t^2} U_n(t,s)|_{\substack{t=\bar\rho(s)\\ Y=X}}\right| + \\
& + \left|U_n(\bar\rho(s),s) - U_n(\bar\rho(s),s)|_{Y=X}\right| \left|\frac{\partial^2}{\partial t^2} U_n(t,s)|_{\substack{t=\bar\rho(s)\\ Y=X}}\right| + \\
(3.12) \quad & + \left|\frac{\partial^2}{\partial t^2} V_n(t,s)|_{t=\bar\rho(s)} - \frac{\partial^2}{\partial t^2} V_n(t,s)|_{\substack{t=\bar\rho(s)\\ Y=X}}\right| + \\
& + \left|V_n(\bar\rho(s),s) - V_n(\bar\rho(s),s)|_{Y=X}\right| \left|\frac{\partial^2}{\partial t^2} V_n(t,s)|_{\substack{t=\bar\rho(s)\\ Y=X}}\right| \\
& + \left|\left[\frac{\partial}{\partial t} U_n(t,s)|_{t=\bar\rho(s)}\right]^2 - \left[\frac{\partial}{\partial t} U_n(t,s)|_{\substack{t=\bar\rho(s)\\ Y=X}}\right]^2\right| + \\
& + \left|\left[\frac{\partial}{\partial t} V_n(t,s)|_{t=\bar\rho(s)}\right]^2 - \left[\frac{\partial}{\partial t} V_n(t,s)|_{\substack{t=\bar\rho(s)\\ Y=X}}\right]^2\right|\Bigg].
\end{aligned}
$$

That the sup norms of the 2^{nd}, 4^{th}, 5^{th} and 6^{th} terms in (3.12) go to zero in probability follows from (3.1), (3.7), (3.9), (3.10), the Lipschitz property of the sine and cosine functions, the S.E.T. and

$$(3.13) \quad \sup_{s\in\mathscr{S}} |\bar\rho(s)| = O_p(1),$$

which in turn follows from Theorems 2.5(a) and (3.4). As the sine and cosine share similar properties, it now remains only to show that the sup norm of the 3rd term in (3.12) goes to zero in probability in order to prove that the sup norm of (3.12) goes to zero in probability. Accordingly

$$\left|\frac{\partial^2}{\partial t^2} V_n(t,s)\big|_{t=\bar{\rho}(s)} - \frac{\partial^2}{\partial t^2} V_n(t,s)\big|_{\substack{t=\bar{\rho}(s)\\ Y=X}}\right|$$

$$\leq s^2 n^{-1}\left|\sum_{j=1}^{n} \sin(s[Y_j-\bar{\rho}(s)Y_{j-1}])[Y_{j-1}^2-X_{j-1}^2]\right| + \tag{3.14}$$

$$+ s^2 n^{-1}\left|\sum_{j=1}^{n} X_{j-1}^2\{\sin(s[Y_j-\bar{\rho}(s)Y_{j-1}]) - \sin(s[X_j-\bar{\rho}(s)X_{j-1}])\}\right|.$$

Let $C_1 = \sup\{s^2\}$ and $C_2 = \sup\{|s|^3\}$; then the r.h.s. of (3.14) can be dominated by

$$C_1\left[2n^{-1}\sum_{j=1}^{n}|X_{j-1}v_{j-1}| + n^{-1}\sum_{j=1}^{n} v_{j-1}^2\right] +$$

$$+ C_2\left[n^{-1}\sum_{j=1}^{n} X_{j-1}^2|v_j| + |\bar{\rho}(s)|n^{-1}\sum_{j=1}^{n} X_{j-1}^2|v_{j-1}|\right]. \tag{3.15}$$

This follows from (1.0.2), the fact that the sine function is bounded by 1 and Lipschitz of order 1, with constant 1. That the sup norm of the expression (3.15) goes to zero in probability follows from (1.0.1), (3.10), (3.13), $E\epsilon_0^2 < \infty$, $\overline{\lim_n}\, EZ_n^2 < \infty$, the Markov inequality applied to each of the averages in (3.15) and the independence of $\{X_j, j \leq n\}$ and $\{v_j, 0 \leq j \leq n\}$. This completes the proof that the sup norm of the first term of (3.11) is $o_p(1)$. We shall now show that the sup norm of the second term in (3.11) is $o_p(1)$. It can be dominated by

$$
\begin{aligned}
& C_0\left|\frac{\partial^2}{\partial t^2} U(t,s)\Big|_{\substack{t=\bar{\rho}(s)\\ Y=X}} U_n(\bar{\rho}(s),s)\Big|_{Y=X} - \frac{\partial^2}{\partial t^2} U_n(t,s)\Big|_{\substack{t=\rho\\ Y=X}} \cdot \right. \\
& \qquad\qquad \left. \cdot U_n(\rho,s)\Big|_{Y=X}\right| + \\
& + C_0\left|\frac{\partial^2}{\partial t^2} V_n(t,s)\Big|_{\substack{t=\bar{\rho}(s)\\ Y=X}} V_n(\bar{\rho}(s),s)\Big|_{Y=X} - \right. \\
& \qquad\qquad \left. - \frac{\partial^2}{\partial t^2} V_n(t,s)\Big|_{\substack{t=\rho\\ Y=X}} V_n(\rho,s)\Big|_{Y=X}\right| + \\
(3.16) \quad & + C_0\left|\left[\frac{\partial}{\partial t} V_n(t,s)\right]^2\Big|_{\substack{t=\bar{\rho}(s)\\ Y=X}} - \left[\frac{\partial}{\partial t} V_n(t,s)\right]^2\Big|_{\substack{t=\rho\\ Y=X}}\right| + \\
& + C_0\left|\left[\frac{\partial}{\partial t} U_n(t,s)\right]^2\Big|_{\substack{t=\bar{\rho}(s)\\ Y=X}} - \left[\frac{\partial}{\partial t} U_n(t,s)\right]^2\Big|_{\substack{t=\rho\\ Y=X}}\right|.
\end{aligned}
$$

From (3.4), (3.7), (3.9), the Lipschitz property of the sine and cosine functions, Theorem 2.5 (a) and the S.E.T., the sup norm of the third and terms in (3.16) goes to 0 in probability. Since the sine and cosine functions satisfy similar properties, to prove that the sup norm of the expression in (3.16) converges to zero in probability we only need now prove that the sup norm of the 2^{nd} term in (3.16) converges to zero in probability. It can be dominated by

$$
\begin{aligned}
& C_0\left|\left[\frac{\partial^2}{\partial t^2} V_n(t,s)\Big|_{\substack{t=\bar{\rho}(s)\\ Y=X}} - \frac{\partial^2}{\partial t^2} V_n(t,s)\Big|_{\substack{t=\rho\\ Y=X}}\right] V_n(\bar{\rho}(s),s)\Big|_{Y=X}\right| + \\
(3.17) \quad & + C_0\left|\left[V_n(\bar{\rho}(s),s)\Big|_{Y=X} - V_n(\rho,s)\Big|_{Y=X}\right] \frac{\partial^2}{\partial t^2} V_n(t,s)\Big|_{\substack{t=\rho\\ Y=X}}\right|.
\end{aligned}
$$

From (3.1), (3.4), (3.9), the Lipschitz property of the sine function and Theorem 2.5 (a), we get that the sup norm of the 2^{nd} term in (3.17) goes to zero in probability. Let $s^* = \sup\{|s|\}$. The sup norm of the first term in (3.17) can be dominated by

$$(3.18)\qquad C_0 n^{-1}\sum_{j=1}^{n} X_{j-1}^2\Big[\{s^*|X_{j-1}|\sup_{s\in\mathscr{S}}|\hat{\rho}_n(s)-\rho|\}\wedge 2\Big];$$

this follows from (3.1), (3.4) and the sine function being bounded by 1 and the inequality $|\sin(s)-\sin(t)| \le |s-t| \wedge 2$, $t, s \in \mathbb{R}$. It remains to prove that (3.18) goes to zero in probability. Let $\epsilon > 0$ be arbitrary, then $\forall\ n > n_0(m)$

$$(3.19)\qquad P\Big(n^{-1}\sum_{j=1}^{n} X_{j-1}^2\Big[\{s^*|X_{j-1}|\sup_{s\in\mathscr{S}}|\hat{\rho}_n(s)-\rho|\}\wedge 2\Big] > \epsilon\Big)$$

$$\le P\Big(n^{-1}\sum_{j=1}^{n} X_{j-1}^2\Big[\{s^*|X_{j-1}|\sup_{s\in\mathscr{S}}|\hat{\rho}_n(s)-\rho|\}\wedge 2\Big] > \epsilon,\ \sup_{s\in\mathscr{S}}|\hat{\rho}_n(s)-\rho| \le m^{-1}\Big) + m^{-1}$$

$$\le \epsilon^{-1}EX_0^2[\{s^*|X_0|m^{-1}\}\wedge 2] + m^{-1}.$$

This follows from (2.16), the Markov inequality and the stationarity of $\{X_j\}$. In (3.19), taking limit as $n \to \infty$ and then $m \to \infty$, $E\epsilon_0^2 < \infty$ and the D.C.T. give (3.18) converges to zero in probability.

H–W proved in their Theorem 2 that the sup norm of the third term in (3.11) goes to zero a.s.. This completes the proof of (3.8) as well. □

Next, set $u(s) = E\cos[s\epsilon_1]$ and $v(s) = E\sin[s\epsilon_1]$, $s \in \mathbb{R}$. Also, by the random elements X_n and Y_n satisfying $X_n(s) = Y_n(s) + \bar{o}_p(1)$ we shall mean $\sup_{s\in\mathscr{S}} |X_n(s) - Y_n(s)| = o_p(1)$.

Theorem 3.2. Let the assumptions of Theorem 3.1 hold. Also let

(a) $\int |f(x-u)-f(x)|\, dx < C|u|$, $u \in \mathbb{R}$, for some $0 < C \in \mathbb{R}$,

(b) $\mathrm{Var}[\cos(s\epsilon_1)] > 0$, $\mathrm{Var}[\sin(s\epsilon_1)] > 0$,
$\mathrm{Var}[u(s)\sin(s\epsilon_1) - v(s)\cos(s\epsilon_1)] > 0$, $s \in \mathscr{S}$ and

(c) $\sup_n E|Z_n|^{2+\alpha} < \infty$, $0 < E|\epsilon_1|^{2+\alpha} < \infty$, $\alpha > 0$,

hold. Then

$$n^{1/2}[\hat{\rho}_n(\cdot) - \rho + \mu_n(\cdot)] \tag{3.20}$$

converges weakly in $C(\mathscr{S})$ to a Gaussian process with mean 0 and covariance

$$(EX_0^2)^{-1}[|\phi(s)|\ |\phi(t)|]^{-2}(st)^{-1}h(t,s),$$

where

$$\begin{aligned} 2h(t,s) = {} & u(s-t)[u(s)u(t) + v(s)v(t)] + \\ & + u(s+t)[v(s)v(t) - u(s)u(t)] + v(s-t)[v(s)u(t) - u(s)v(t)] - \\ & - v(s+t)[u(s)v(t) + v(s)u(t)] \end{aligned}$$

and

$$\begin{aligned} \mu_n(s) = {} & s^{-1}\left[\frac{\partial}{\partial t} m_n(t,s)\big|_{t=\bar{\rho}(s)}\right]^{-1}\Big[E\cos[s(\epsilon_1+v_1-\rho v_0)]\ \cdot \\ & \cdot\ Ev_0\sin[s(\epsilon_1+v_1-\rho v_0)] - E\sin[s(\epsilon_1+v_1-\rho v_0)]Ev_0\cos[s(\epsilon_1+v_1-\rho v_0)]\Big]. \end{aligned} \tag{3.21}$$

<u>Proof</u>. From (3.1), (3.3) and (3.5) – (3.7) we get

$$n^{1/2}(\hat{\rho}_n(s) - \rho)s \frac{\partial}{\partial t} m_n(t,s)|_{t=\bar{\rho}(s)}$$

$$= -n^{-1/2} \sum_{j=1}^{n} Y_{j-1}\{U_n(\rho,s)\sin(s[Y_j-\rho Y_{j-1}]) - V_n(\rho,s)\cos(s[Y_j-\rho Y_{j-1}])\} + \bar{o}_p(1)$$

$$= -n^{-1/2} \sum_{j=1}^{n} Y_{j-1}\Big[\{E\cos[s(\epsilon_1+v_1-\rho v_0)]\}\sin[s(\epsilon_j+v_j-\rho v_{j-1})] - \{E\sin[s(\epsilon_1+v_1-\rho v_0)]\}\cos[s(\epsilon_j+v_j-\rho v_{j-1})]\Big] -$$

$$(3.22)\qquad - n^{-1} \sum_{j=1}^{n} Y_{j-1} \{\sin[s(\epsilon_j+v_j-\rho v_{j-1})] - E\sin[s(\epsilon_1+v_1-\rho v_0)]\} \cdot \sqrt{n}\,\{U_n(\rho,s) - E\cos[(\epsilon_1+v_1-\rho v_0)]\} +$$

$$+ n^{-1} \sum_{j=1}^{n} Y_{j-1}\{\cos[s(\epsilon_j+v_j-\rho v_{j-1})] - E\cos[s(\epsilon_1+v_1-\rho v_0)]\} \cdot \sqrt{n}\,\{V_n(\rho,s) - E\sin[s(\epsilon_1+v_1-\rho v_0)]\} -$$

$$- \{n^{-1} \sum_{j=1}^{n} Y_{j-1}\}\Big[E\sin[s(\epsilon_1+v_1-\rho v_0)] \cdot \sqrt{n}\{U_n(\rho,s) - E\cos[s(\epsilon_1+v_1-\rho v_0)]\} - E\cos[s(\epsilon_1+v_1-\rho v_0)]\sqrt{n}\{V_n(\rho,s) - E\sin[s(\epsilon_1+v_1-\rho v_0)]\}\Big] + \bar{o}_p(1).$$

We shall now proceed to prove that the sup norm of the second, third and fourth terms in (3.22) converge to zero in probability. To achieve this we shall prove

$$(3.23)\qquad \sqrt{n}\,\{U_n(\rho,s) - E\cos[s(\epsilon_1+v_1-\rho v_0)]\},$$
$$\sqrt{n}\,\{V_n(\rho,s) - E\sin[s(\epsilon_1+v_1-\rho v_0)]\},\ s \in \mathscr{S},$$

converges weakly in $C(\mathscr{S})$ to a Gaussian process,

(3.24i) $\sup|n^{-1}\sum_{j=1}^{n} Y_{j-1}\{\sin[s(\epsilon_j+v_j-\rho v_{j-1})] - E\sin[s(\epsilon_1+v_1-\rho v_0)]\}| = o_p(1)$

and

(3.24ii) $\sup|n^{-1}\sum_{j=1}^{n} Y_{j-1}\{\cos[s(\epsilon_j+v_j-\rho v_{j-1})] - E\cos[s(\epsilon_1+v_1-\rho v_0)]\}| = \bar{o}_p(1).$

In view of (3.22) – (3.24), to study the weak convergence of $n^{1/2}[\hat{\rho}_n - \rho + \mu_n)]$ it suffices to study the weak convergence of the first term in (3.22) when centered.

Proof of (3.23). Denote

$$\xi_{j,n}(s) = \cos[s(\epsilon_j+v_j-\rho v_{j-1})] - E\cos[(\epsilon_1+v_1-\rho v_0)] \quad \forall\, s \in \mathscr{S}.$$

Since $Y_j - \rho Y_{j-1}$ and $Y_k - \rho Y_{k-1}$ are independent for all $|j-k| \geq 2$ we see $\alpha(k) = 0 \ \forall\, k \geq 2$. Also, $s \in \mathscr{S}$, because $\gamma_n \to 0$

(3.25) $n^{-1}\sigma_n^2 = E\xi_{1,n}^2(s) + 2(n-1)n^{-1}E\xi_{1,n}(s)\xi_{2,n}(s) \to \mathrm{Var}[\cos(s\epsilon_0)],$

which is positive because of the assumption (b), σ_n^2 as in Theorem 1.3.1. The remaining conditions of Theorem 1.3.1 are trivially satisfied.

Hence from Billingsley (1968, p. 49) the finite dimensional

(3.26) distributions of $n^{-1/2}\sum_{j=1}^{n} \xi_{j,n}(s)$ converge to that of a Gaussian process. Also, for any $s, t \in \mathscr{S}$

(3.27) $E|n^{1/2}\sum_{j=1}^{n} \xi_{j,n}(s) - n^{-1/2}\sum_{j=1}^{n} \xi_{j,n}(t)|^2 =$

$$\mathrm{Var}[\xi_{1,n}(s)-\xi_{1,n}(t)] + \frac{(n-1)}{n}\mathrm{Cov}[\xi_{1,n}(s)-\xi_{1,n}(t),\ \xi_{2,n}(s)-\xi_{2,n}(t)].$$

From the Cauchy–Schwartz inequality and the Lipschitz property of the cosine function the r.h.s. of (3.27) is dominated by $C|t-s|^2$, where $0 < C \in \mathbb{R}$.

Thus from Billingsley (1968, Theorem 12.3) the process

(3.28) $n^{-1/2}\sum_{j=1}^{n} \xi_{j,n}(s)$ is tight, and its weak convergence to a Gaussian limit in $C(\mathscr{A})$–space follows from Billingsley (1968, Theorem 8.1).

The weak convergence of the second process in (3.23) to a Gaussian limit in $C(\mathscr{A})$–space follows similarly.

Proof of (3.24). The l.h.s. of (3.24i) without the sup can be dominated by

$$(3.29)\quad \begin{aligned} &2n^{-1}\sum_{j=1}^{n}|v_{j-1}| + \Big|n^{-1}\sum_{j=1}^{n}X_{j-1}\{\sin[s(\epsilon_j+v_j-\rho v_{j-1})] - \sin[s\epsilon_j]\}\Big| + \\ &+ \Big|n^{-1}\sum_{j=1}^{n}X_{j-1}\sin[s\epsilon_j]\Big| + \Big|n^{-1}\sum_{j=1}^{n}X_{j-1}\Big|. \end{aligned}$$

That the first term in (3.29) goes to zero in probability follows from the assumption (c) and $\gamma_n = o(1)$. From the Lipschitz property of the sine function, the Cauchy–Schwartz inequality, the S.E.T., assumption (c) and $\gamma_n = o(1)$, sup norm of the second term in (3.29) converges to zero in probability. That the sup norm of the third term in (3.29) converges to zero a.s. follows from H–W (Lemma 3.1). The last term in (3.29) converges a.s. to 0 by the S.E.T.. The proof of (3.24ii) is similar.

It remains to study the weak convergence of the first term in (3.22) when centered. To that effect let

$$\xi_{j,n}(s) = Y_{j-1}\Big[\{E\cos[s(\epsilon_1+v_1-\rho v_0)]\}\sin[s(\epsilon_j+v_j-\rho v_{j-1})] -$$

$$- \{E\sin[s(\epsilon_1+v_1-\rho v_0)]\}\cos[s(\epsilon_j+v_j-\rho v_{j-1})]\Big] - s\frac{\partial}{\partial t} m_n(t,s)\big|_{t=\bar{\rho}(s)}\mu_n(s).$$

We shall first prove the finite dimensional distributions convergence of $n^{-1/2}\sum_{j=1}^{n} \xi_{j,n}(\cdot)$ using Theorem 1.3.4. Take

$$\theta_n(x,y) = x\Big[\{E\cos[s(\epsilon_1+v_1-\rho v_0)]\}\sin[s(y-\rho x)] - \{E\sin[s(\epsilon_1+v_1-\rho v_0)]\}\cdot \cdot\cos[s(y-\rho x)]\Big],$$

$$\theta(x,y) = x\{u(s)\sin[s(x-\rho y)] - v(s)\cos[s(x-\rho y)]\}, \qquad x,\ y \in \mathbb{R},$$

$h \equiv x$, X_j, Y_j, v_j as in the model assumptions with $\omega(x,y) = x+y$, $x, y \in \mathbb{R}$, in Theorem 1.3.4. Since $X_j = \sum_{k=0}^{\infty} \rho^k \epsilon_{j-k}$ a.s., using assumptions (a) and (c) and Pham and Tran (1985,Theorem 2.1) with $\delta = 2$, $A(k) = \rho^k$, gives $\{X_j\}$ to be strongly α-mixing with

$$\text{(3.30)} \qquad \alpha(k) \le C_\rho|\rho|^{2k/3}, \quad \forall\ k \ge 2, \quad \text{for some } C_\rho > 0.$$

By assumption (b) and (c), $\tau^2 = EX_0^2\,\mathrm{Var}[u(s)\sin(s\epsilon_0) - v(s)\cos(s\epsilon_0)] > 0$ for each $s \in \mathscr{S}$. Thus all the conditions of Theorem 1.3.4 are satisfied. Hence the C.L.T. holds for ξ as defined above, for each $s \in \mathscr{S}$. Now using the argument as in (3.26) we get the required finite dimensional distributions convergence. Since for all i and j with $|i-j| \geq 2$, ϵ_j, v_j and v_{j-1} are independent of $\{(v_k, Y_k),\ k = i-1, i\}$,

$$\text{(3.31)} \qquad \mathrm{Cov}[\xi_{i,n}(t),\ \xi_{j,n}(s)] = 0 \quad \forall\ s, t \in \mathscr{S}.$$

Using (3.31), the same argument as in (3.27) and (3.28), we get $n^{-1/2} \sum_{j=1}^{n} \xi_{j,n}(\cdot)$ converges weakly in $C(\mathscr{S})$–space to a Gaussian process with mean 0 and covariance $EX_0^2 h(t,s)$. Thus from (3.3) – (3.5), (3.8), Billingsley (1968, Theorem 4.1) and (3.22), we get (3.20). □

Remark 3.1. From (3.8), the assumption $\sqrt{n}\ \gamma_n = O(1)$ and simple computations using (1.0.1), we can see that $\sqrt{n}\ \mu_n$ in Theorem 3.2 can be replaced by

$$\text{(3.32)} \qquad \nu_n(s) = -\, s^{-1} |\phi_{\epsilon_1}(s)|^{-2} [EX_0^2]^{-1} \Big[E\cos[s(\epsilon_1 + v_1 - \rho v_0)] \cdot \\ \cdot Ev_0 \sin[s(\epsilon_1 + v_1 - \rho v_0] - E\ \sin[s(\epsilon_1 + v_1 - \rho v_0)] Ev_0 \cos[s(\epsilon_1 + v_1 - \rho v_0)] \Big].$$

Note that $\nu_n(s)$ represents the asymptotic bias of $n^{1/2}(\hat{\rho}_n(s) - \rho)$. Consider the following assumptions:

(a) $\sqrt{n}\ \gamma_n = o(1)$.

(b) $\sqrt{n}\ \gamma_n = O(1)$ and $Z_n \to 0$ in probability.

(c) $\sqrt{n}\ \gamma_n \to \gamma$ and $Z_n \to Z$ in probability.

Using (1.0.1) and the continuity of the sine and cosine functions, one concludes that under (a) or (b), $\sup_{s\in\mathscr{S}} |\nu_n(s)| \longrightarrow 0$ and hence $\sqrt{n}\,\mu_n$ in Theorem 3.2 can be replaced by 0. Using the Lipschitz property of the sine and the cosine functions, the condition (c) implies that $\sup_{s\in\mathscr{S}} |\nu_n(s) - \mu(s)| \longrightarrow 0$, where

$$
\begin{aligned}
(3.32)\quad \mu(s) = -s^{-1}|\phi_{\epsilon_1}(s)|^{-2}[EX_0^2]^{-1}\Big[&\{\gamma E\int z\sin[s(\epsilon_1-\rho z)]\, dL(z)\}\Big[u(s) + \\
&+ 2\gamma\, E\int \cos[s(\epsilon_1+2^{-1}(1-\rho)z)]\cos[s2^{-1}(1+\rho)z]\, dL(z)\Big] - \\
&- \{\gamma E\int z\cos[s(\epsilon_1-\rho z)]\, dL(z)\}\Big[v(s) + 2\gamma E\int \sin[s(\epsilon_1+2^{-1}(1-\rho)z)\cdot \\
&\cdot \cos[s2^{-1}(1+\rho)z]\, dL(z)\Big]\Big].
\end{aligned}
$$

Consequently $\sqrt{n}\,\mu_n(s)$ in Theorem 3.2 can be replaced by $\mu(s)$.

<u>Note</u>. If f is a double exponential or $\mathscr{N}(0,\sigma^2)$ density, Z_n is such that $\overline{\lim_n}\, E|Z_n|^3 < \infty$ and $\gamma_n = o(1)$, then simple calculations show that all the conditions of Theorem 3.2 are satisfied. □□

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