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Ruhama Even

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Glenda Lappan
Major professor

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PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE AND UNDERSTANDING ABOUT MATHEMATICAL FUNCTIONS

By

Ruhama Even

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ABSTRACT

PROSPECTIVE SECONDARY MATHEMATICS TEACHERS' KNOWLEDGE AND UNDERSTANDING ABOUT MATHEMATICAL FUNCTIONS

By

Ruhama Even

The main goals of this study are 1) to identify important aspects of subject matter knowledge for teaching the concept and application of function, and 2) to describe kinds of knowledge prospective teachers have with respect to these aspects and to point to some of the limitations of their conceptions.

Six aspects of teachers' subject matter content knowledge about functions studied:

* What is a function?
* Representations
* Inverses and compositions
* Functions of the school curriculum
* Ways of approaching functions: point-wise, interval-wise, global and as entities
* Kinds of knowledge and understanding of function and mathematics

Two pedagogical content knowledge aspects were added:

* Teaching toward different kinds of knowledge and understanding of functions and mathematics
* Students' mistakes

Data was gathered in two phases. First, an open-ended questionnaire was administered to 152 prospective secondary mathematics teachers, before they started teaching. The second phase included interviews with ten subjects. The questionnaire and interview were developed for this study.

The results include description of the participants' subject matter knowledge and

understanding of function with respect to the aspects above. Lacking knowledge related to one aspect influenced the knowledge related to another aspect. These results show discrepancies between the participants' concept image and concept definition of function, difficulties with translations between different representations, lack of rich relationships between the informal meaning of inverse as "undoing" and the formal definition, and incomplete understanding of functions in the curriculum. Global approach seemed to be related to understanding of different representations, but not to understanding the meaning of graphs representing situations. Many interviewees were rule oriented. There was a tendency to use inductive reasoning as a proof. A tendency not to use modern terms when defining function for students -- seemed to be related to the incomplete concept image of function that the participants held and to teachers' tendency to provide students with rules. Explanations for students seemed to be related to subject matter content knowledge. Most of the subjects seemed to be aware of the common misconceptions students have and their sources. Subject matter content knowledge seemed to be related to the kinds of explanations provided for students' mistakes.

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INTRODUCTION

At first glance, Figure 1.1 may seem like an arbitrary collection of examples.

****

* $\cos(x)$

* $a_n = n^2 + 3n$

* $| x - y |$

* Finding the area of a given figure

* $\mu(n)$

* "The more people we get to help, the sooner we'll finish the work."

* $\{(1, 5), (4.3, -8), (\pi, 5)\}$

* Adding 4 to a given number

* Max (p - 2, 3q)

* $\int f(x)dx$

* Reflection of the plane about the x axis.

*

Distance (in km)	90	180	270	...
Time (in hrs)	1	2	3	...

*

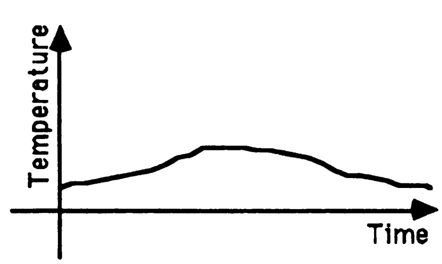

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Figure 1.1 -- A Common Thread?

However, there is a common thread that connects all the examples in Figure 1.1. The common thread is function. Functions are used in every branch of mathematics, as algebraic operations on numbers, transformations on points in the plane or in space, intersection and union of pairs of sets, and so forth. Function is a unifying concept in all mathematics. Relationships among phenomena in everyday life, such as the relationship between the speed of a car and the distance travelled, are functions.

Research on learning functions is needed

The concept of function has an important part in school curriculum. It is given prominence in almost every curriculum guide (e.g., *Academic Preparation in Mathematics*, 1985; Chambers et al., 1986; *Curriculum and Evaluation Standards for School Mathematics*, 1988; *Mathematics Framework for California Public Schools*, 1985; *Michigan Essential Goals And Objectives For Mathematics Education*, 1988; *Oregon Mathematics Concept Paper no. 2*, 1987). Several curriculum projects -- elementary and secondary -- use the concept of function as a central idea (e.g., the Comprehensive School Mathematics Program, the University of Chicago School Mathematics Project, Algebra with Computers -- Maryland).

While no one argues about the importance and centrality of the concept of function in school mathematics curriculum, few studies have been conducted on learning and teaching functions compared to studies in other content domains, such as operations with whole numbers or ratio and proportion. In the last decade, significant advances have been made in understanding students' cognitions in the above areas. But we still do not know as much about students' cognitions in learning about functions.

At this point we know that students have difficulties learning the concept of function. Studies of students' understanding of function show that this complex concept is not easy to comprehend. Many students cannot consider a function as it behaves over intervals or in a global way (Bell and Janvier, 1981; Janvier, 1978; Monk, 1988).

Thinking of functions as entities and performing operations on them is also hard for students (Lovell, 1971; Thomas, 1975). In the area of discrepancies between definition and image of the concept of function it was found that many students hold a linear prototypical image of functions (Dreyfus and Eisenberg, 1983; Markovits, Eylon and Bruckheimer, 1983). Students were found to reject piece-wise functions (Dreyfus and Eisenberg, 1987; Markovits et al., 1986; Vinner, 1983; Vinner and Dreyfus, in press). Students also expect graphs of functions to be 'reasonable', and functions to be representable by a formula (Vinner, 1983; Vinner and Dreyfus, in press). Students do not consider the constant function or the function obtained by composition of functions to be functions (Dreyfus and Eisenberg, 1987; Lovell, 1971; Markovits et al., 1986; Thomas, 1975; Vinner, 1983; Vinner and Dreyfus, in press).

The role of teachers and teachers' subject matter knowledge in learning

The process of learning is influenced directly by three major factors: The learner, the curriculum and the teacher. All of these are influenced by school and society. Most of the studies of learning mathematics focus on the learner and his/her cognition; studies about students' understanding of the concept of function are no exception. Understanding students' conceptions and how they think about a concept is an essential step in improving the learning of that concept. Another important step is providing a good curriculum, the aim of the "new math" movement, for example. But the third factor in the process of learning -- the mathematics teacher whose role is to help the learner achieve understanding of the subject matter -- has not been considered seriously until recently.

Teachers' subject matter knowledge influences their teaching. Teachers need to ask questions, stimulate discussions, suggest different points of view, make decisions about action. These activities and others are based on teachers' subject matter knowledge as well as on other kinds of knowledge, such as knowledge of general pedagogy, students and learning. Teachers' subject matter knowledge includes a personal understanding of the content, understanding not only that something is so but why it is so -- procedural and conceptual knowledge. It also includes knowledge about mathematics. Teachers also need to have pedagogical content knowledge as a part of their subject matter knowledge. This includes ways of representing and formulating the subject that make it comprehensible to others so that the teacher can help someone else develop an understanding of the content as well as a broader vision of the discipline (e.g., Ball, 1988; Shulman, 1986; Tamir, 1987). Teachers' subject matter knowledge is not the only factor that is related to instructional practice -- other factors influence that as well (Ball, in press). But it is one of the major components that influences teachers' decisions. A teacher who has solid mathematical knowledge -- both subject matter content knowledge and pedagogical content knowledge, is more capable of helping his/her students achieve meaningful learning.

This study -- integrating research on learning and teaching

Research on learning and learners, and research on teaching and teachers have been conducted separately from each other for a long time. There is little research in mathematics education that integrates knowledge from both bodies of work. But teaching cannot occur unless learning takes place and the process of learning is influenced directly by the teacher. Learning and teaching, learners and teachers are two parts of a whole. Recently, the desirability of research that integrates research on teaching and teachers and research on learning and learners has been realized. One of the major goals of the National Center for Research in Mathematical Sciences Education is to facilitate integrating research of this kind (Fennema, Carpenter, Brown, Peterson, Thompson, Post and Wearne, 1988).

The research reported in this dissertation is integrating research. The main goals of this study are 1) to contribute to the current discussion and analysis of subject matter knowledge for teaching by identifying important aspects of subject matter knowledge for teaching the concept and application of function, and 2) to describe kinds of knowledge prospective teachers have with respect to these aspects and to point to some of the limitations of their conceptions. The choice of aspects to be studied was based on integrated knowledge from several bodies of work: the role and importance of function in mathematics and in the mathematics curriculum; research and theoretical work on learning, knowledge and understanding of functions in particular and other mathematical concepts in general; and research and theoretical work on teachers' subject matter knowledge and its role in teaching.

As a result of this integration, six aspects of teachers' subject matter content knowledge about functions seemed to be very important to study:

* What is a function? (includes image and definition of the concept of function, univalent property of functions, and arbitrariness of functions).
* Different representations of function.
* Inverse function and composition of functions.
* Functions of the high school curriculum.
* Different ways of approaching functions: point-wise, interval-wise, global and as entities.
* Different kinds of knowledge and understanding of function and mathematics.

Each one of the above aspects is important and broad enough to serve as a base for a whole study. The decision to study them together, although this may lead to a less detailed picture of each aspect, was made so that a more complete picture of the prospective teachers' knowledge of function can be illustrated. The interrelations among different aspects, which constitute part of subject matter knowledge, are lost when each aspect is studied in isolation. Two additional pedagogical content knowledge aspects were also added as they seemed to be closely related to the previous aspects. These were:

* Teaching toward different kinds of knowledge and understanding of functions and mathematics.

* Students' mistakes -- what they do and why?

The choice of aspects was also based on the specific population to be studied. The participant subjects in this study were prospective secondary mathematics teachers in the last stage of their professional education. They were finishing or had already finished their mathematics methods class. Almost all of them were seniors, a few were juniors or postbaccalaureate students. This group was selected so that the description of their knowledge would reflect the knowledge teachers have gained during their college education, but before they begin teaching. The subjects came from eight mid-western universities.

Data was gathered in two phases. At the first phase, an open-ended questionnaire was administered to 152 subjects. Information gathered from a written questionnaire is sufficient for a general description of some facets of the subjects' knowledge, but is limited and sometimes hard to interpret. In order to overcome these difficulties, the second phase included interviews with ten additional subjects. By probing, asking subjects to explain what they did and why, asking for their reactions as future teachers to students' misconceptions and asking questions which are related to the questionnaire but require more general, longer or more thoughtful responses, a more accurate and detailed picture of the subjects' subject-matter content knowledge and pedagogical content knowledge was developed. The questionnaire and the interview were developed for this study.

The structure of this dissertation

The structure of this dissertation is as following:

Chapter 2 -- Learning and Teaching Functions -- provides the specific background about functions for this study. The first section of this chapter describes students' understanding (and misunderstanding) of the concept of function as reported in the literature. The limited conception of function that many of the students hold has interesting relationship with the historical development of function. The evolution of the concept of function is discussed in the next section. The influence of the development of the function concept on school curriculum is described in the following section which concludes this chapter.

Chapter 3 -- Meaningful Learning, Teaching for Understanding, and Teachers' Subject Matter Knowledge -- provides the general background about meaningful learning and teaching for understanding for this study as well as the specific background about teachers' subject matter knowledge. The first section of this chapter traces the calls for making change in the way teachers teach since the beginning of this century. Although today's calls for making change are not new, there is a difference between past and present conceptions of the construction of knowledge which has some promise in achieving meaningful learning. The difference between past and present views of knowledge acquisition occurred when researchers started to "look" at what is happening in the learner's head. Similar changes in research on teaching and teachers are described in the following section. One of the prominent new trends in research on teaching is studying teachers' subject matter knowledge. A discussion of what subject matter knowledge for teaching is concludes this chapter.

Chapter 4 -- The Study: Purpose and Design -- integrates the specific knowledge about functions from Chapter 2 with the general knowledge about teachers' subject matter knowledge from Chapter 3. The first section of this chapter discusses the identification of the different aspects of teachers' subject matter knowledge being studied and their importance. The second section describes the design of the study. It gives information about the background of the participating prospective teachers and describes the questionnaire and the interview that were developed especially for the purpose of this study. Procedure and data collection, and data analysis conclude this chapter.

The results of this study are reported in Chapters 5 through 7. The first three aspects -- what is a function?, different representations of functions, inverse function and composition of functions -- were chosen to be the main focus of the description of

the results. Each chapter is devoted to one of these aspects. Results about the other five aspects of teachers' subject matter knowledge are interwoven with the results of the main three aspects, making the whole picture more vivid and complete.

Chapter 5 -- What is a Function? -- describes the prospective teachers' subject matter content knowledge and pedagogical content knowledge in relation to what a function is. It starts with a description of the ways in which the participants chose to define function. Then, discrepancies between concept definition and concept image of function are discussed. The acceptance of the arbitrary nature of both the relationship between the two sets on which the function is defined and the sets themselves is described. The participants' understanding of the univalent property of function is also considered. The next part discusses ways in which the prospective teachers would explain what a function is to students. The chapter ends with the results of the participants' reactions to students' misconceptions in relation to the nature of function.

Chapter 6 -- Different Representations of Function -- examines the prospective teachers' knowledge and understanding of symbolic and graphic representations of functions. First, the connectedness between the two representations is discussed. The prospective teachers' tendency to treat each representation in isolation is presented by examining their treatment of a problem on quadratic functions. Then, the participants' knowledge and understanding of the translation from one representation to the other is examined. The tendency to use a point-wise approach instead of a more global view of functions and its implications is also discussed. Finally, remarks on the participants' pedagogical content knowledge in relation to graphic and symbolic representations are mentioned.

Chapter 7 -- Composition of Functions and Inverse Function -- is devoted to a description of the prospective teachers' knowledge and understanding of these two important sub-concepts of the concept of function. The chapter starts with a discussion of the prospective teachers' informal knowledge and understanding of the meaning of inverse function as undoing. Then, the participants' misunderstanding of composition of functions as a source of difficulties with inverse function is discussed. The next part describes problems with inverse function as a result of dealing with it on an informal level of "undoing" only, with no relation to the mathematical notion of inverse function. Unfamiliarity with the relationship between a function and its inverse in a graphical representation are discussed in the last part.

Chapter 8 -- Prospective Teachers' Knowledge and Understanding about Mathematical Functions -- concludes this work. It summarizes and discusses the main findings as well as the significance and implications of the study.

CHAPTER 2

LEARNING AND TEACHING FUNCTIONS

The concept of function appears everywhere. In everyday life functions can be used to represent relationships among phenomena, such as the relationship between the speed of a car and the cost of gas for a specific trip. Functions are basic to the mathematical sciences, to very advanced mathematics as well as elementary mathematics, to calculus and analysis as well as algebra and geometry. Mathematical operations, such as multiplication by 2, are functions, and so are geometrical transformations such as reflections. Function serves as a unifying concept for seemingly different topics. As the discipline of mathematics has grown, function has become one of the most important and fundamental topics in the mathematics program. Today function is a central concept in mathematics in general and in the high school and college mathematics in particular. The importance of the function concept is stated in almost every curriculum guide (e.g., *Academic Preparation in Mathematics* , 1985; Chambers et al, 1986; *Curriculum and Evaluation Standards for School Mathematics*, 1988; *Mathematics Framework for California Public Schools*, 1985; *Michigan Essential Goals And Objectives For Mathematics Education*, 1988; *Oregon Mathematics Concept Paper no. 2*, 1987).

The study of the function concept, as recommended by most curriculum guides, should start in the elementary curriculum by pattern discovery and recognition. Looking for regularities and describing the patterns that they find, will enable students to recognize and understand relationships in their observations of the world around them. This idea is not new. Smith, in 1925, stated that functions describing relationships between quantities that occur in the real world are valuable to all persons. Helping students develop flexibility in describing patterns and functions by using different representations such as a table, a formula, a graph, and a statement, and interpreting among these representations is also important.

The expectations are that at the end of high school, students will know the function concept in general, and also have a set of familiar examples that they can use. According to the Curriculum and Evaluation Standards for School Mathematics that the National Council of Teachers of Mathematics published in 1988, high school students should study functions so that all students can "model real-life phenomena with a variety of functions; represent and analyze relationships using tables, rules, and graphs; translate among tabular, symbolic, and graphical representations of functions; recognize that a variety of problem situations can be modeled by the same type of function; [and] analyze the effects of parameter changes on the graphs of functions" (p. 169). In addition, college-intending students should understand operations on and the general properties and behavior of classes of functions. The Standards as well as most curriculum guides emphasize understanding of function in different representations. Students should also be familiar with the behavior of specific types of functions including linear, quadratic, general polynomial, step, exponential, logarithmic, trigonometric, and piece-wise functions in different representations.

Do students meet these expectations? What difficulties do they have in studying function? The first section of this chapter describes students' understanding (and misunderstanding) of the concept of function as reported in the literature. The limited conception of function that many of the students hold has an interesting relationship with the historical development of function. The evolution of the concept of function is discussed in the next section. The influence of the development of the function concept on school curriculum is described in the section which concludes this chapter.

Students' Understanding of Functions

Studies of students' understanding of the concept of function show that this complex concept is not easy to comprehend. Many students develop misconceptions about functions. This section describes findings of research on students' learning, understanding, and difficulties with the concept of function. It starts with research on intuitions about functions. The next part describes research that aims at finding stages in the learning of functions. The ability of students to approach functions in different ways are described in the third part. The last part describes research on students' images of the concept of function and discrepancies between image and definition of the concept of function.

Intuitions about functions

The importance of intuitions in the development of understanding of mathematical concepts has been stressed by several people (e.g., Dreyfus and Eisenberg, 1981, 1982, 1984; Fischbein, 1978; Polya, 1957). *Academic Preparation in Mathematics* (1985) emphasizes the importance of having a strong intuitive understanding of various functions and their graphs. Dreyfus and Eisenberg argue that approaching a new topic intuitively will encourage transfer of learning, one of the main goals of the learning process. But because intuitions are to some extent personal, a systematic examination of their presence in students should precede the writing of curriculum materials. They developed a program designed to assess intuitions students have about function concepts in various settings. This way they hoped to obtain information about the sequencing of concepts that will encourage maximal vertical and horizontal transfer. They conceptualized a 3-dimensional function "block" (see Figure 2.1) to guide a systematic study for ascertaining a hierarchical ordering of intuitive function concepts. Along the x-axis of the block are function settings -- representations -- tables, ordered pairs, arrow diagrams, graphs, etc. Along the y-axis are function subconcepts (image, preimage, zeros, etc.), and along the z-axis are levels of abstractions and generalization (one, two, or several variables, discrete domain, etc.) of the concepts.

In research guided by this function block they found that high ability students tend to favor looking at functions in a graphical representation whereas low ability students prefer tabular or arrow diagram representations of functions. Very few studies were conducted in light of the function "block". Dreyfus and Eisenberg (1983) themselves

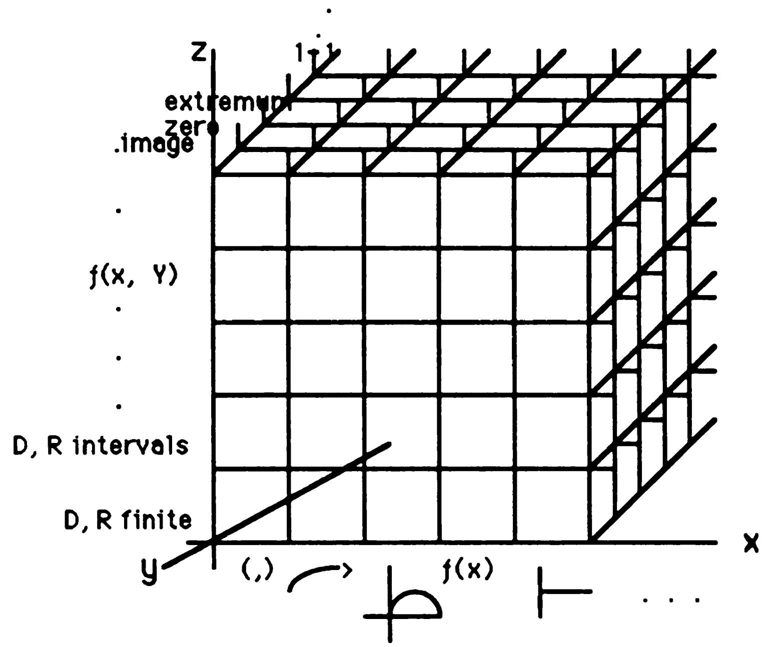


Figure 2.1 -- Dreyfus and Eisenberg's Function "Block"

reported later that several problems arose with this block model for studying functions. Among the problems were the following: (a) It missed the dynamic aspect of the function concept by looking at functions as static rules stated in various forms that "sent" objects from the domain to objects in the range; (b) The z-axis, "levels of abstraction", was not clearly defined and seemed to be multidimensional itself.

By using axes in their model of function "block", Dreyfus and Eisenberg seemed to assume that there might be hierarchical ordering of intuitive function concepts. An expansion of this idea is the idea of stages in the process of understanding functions. Attempts to study these stages are discussed in the next part.

Stages of understanding functions

Several researchers tried to define stages in the development of the understanding of functions by students. Piaget, Grize, Szeminska, & Vinh-Bang (1968/1977) studied young children ages 3-14, asking them to perform operations, and to predict for a given value of one variable the corresponding value of the other. From this research Piaget et al. proposed a developmental model of functions from infancy through early childhood to middle childhood.

Thomas (1975) and Orton (in Lovell, 1971) studied interrelations between learner and curriculum. They tried to determine stages of acquisition of the concept of function. Thomas also related the hypothesized stages to various aspects of Piagetian theory. He based his analysis on three components: concept identification, process, and operations. Concept identification refers to the ability to discriminate instances and non-instances of function and to formulate a correct criterion for making such discriminations. Process refers to the ability to work with various representations and names of functions in finding images, pre-images, domain, range, and sets of images. Operations refers to the ability to carry out operations on functions, with an indication that the result of the operation is understood to be a function.

The basic division of his stages hinges on the concept identification component. Subjects who can discriminate functions from non-functions and who can verbalize an adequate criterion for such discrimination are considered to be at a higher level relative to the attainment of the concept of function than are those who cannot, regardless of the subjects' performance on the other two components. "For if the subject does not recognize those distinctive entities which are functions, his performance on the other two components cannot be truly functional in nature" (Thomas, 1975).

Based on this assumption and the three basic components, Thomas found four stages:

Stage 1 is concrete-intuitive in nature, subjects are able to carry out processes associated with the function concept when they are basically arithmetic in nature or where assignments are given quite specifically as in an arrow diagram or in a table.

Stage 2 is preconceptual. Subjects have better command of the process component.

Images, pre-images, and sets of images are found, with adequate explanations in different representations.

Stage 3 is basic conceptual. Subjects can identify relations in several representational modes as functions or not functions and can give an adequate criterion for each such discrimination. In a higher level of this stage, subjects have generalized their understanding of process with respect to functions given in various representational modes.

Stage 4 is described as a stage at which the basic criteria for a relation to be a function have been mastered, at which the concept has been extended to different representations, and at which there is evidence given that true operations on functions as entities have been attained.

After attaining stage 4, the subjects are at the beginning level of the stage of operations on functions and in thinking of functions as entities.

Orton (in Lovell, 1971) found similar stages in his study. Both Thomas and Orton did not deal with the concept of function in the broad sense of functionality, nor with functional thinking like pattern discovery and relationships between phenomena. Rather, they dealt with the learning of a formal, contemporary concept of function as exemplified by the text materials of a specific curriculum. Therefore, the generalizability of these stages still needs to be studied. In addition, basing the stages on the concept identification component was based on a subjective assumption that one who can discriminate functions from non-functions (with an adequate explanation) is at a higher level than someone who cannot. This assumption was not studied, but rather was taken for granted. For Thomas and Orton someone who can deal with functions as entities but cannot discriminate functions from non-functions was at a lower level than someone who cannot work with functions as entities but can identify functions. Therefore, it is not clear how accurate the above stages are in representing developmental stages in students' understanding of the concept of function.

Being able to operate on functions as entities was considered to be a high stage of acquisition of the concept of function. Different situations require different ways of approaching and dealing with functions. Thomas and Orton refer to one such way in their stages -- approaching functions as entities -- but did not include other ways. Students' difficulties with different approaches to functions are described in the next part.

Ways of interpreting functions

The formal definition of function describes function as a special set of ordered pairs. But a very common use of functions is as a connection between two (or more) variables. When a function defines a relationship between variables, it is not approached as an entity but rather point-wise, interval-wise or in a global manner. Studies show that many students deal with functions point-wise i.e., they can only plot and read points, but cannot think of a function as it behaves over intervals (interval-wise) or in a global way (the behavior of the function as a whole). Janvier (1978) and Bell and Janvier (1981) found that secondary students are weak in their ability to interpret global graphical features. These students had no problems with reading and plotting points, but had difficulties dealing with the behavior of the function over intervals and could not interpret graphs representing situations whenever pictorial aspects conflict with correct meaning. Difficulties with dealing with a function interval-wise or in a global way were presented, for example, when the answer should have been an interval, but students gave specific points. In addition, intervals were obtained by referring to the axis, reading off the values and subtracting them, instead of reading the differences directly from the scale on the axis, without subtraction, or reading it more directly from the grid in the body of the graph. Difficulties with interpretations of graphs representing situations when pictorial aspects conflict with correct meaning were presented, for example, when greatest increase was interpreted as greatest value. Lovell (1971) describes similar results.

Monk (1988) says that if a function is given by a table, or we use its graph or formula as if it were only a table, reading particular numerical values of the independent variable as corresponding to particular numerical values of the dependent variable, then it is a very simple concept -- one that most of the students acquire in high school. In other words, students have no problems with point-wise understanding of functions. But, Monk claims, this is not the way the concept of function is used in calculus. The crucial question asked about function in calculus is about the behavior of the output variables as influenced by variation in the input variables. In order to answer such questions, students need what Monk calls "across time understanding." This is similar to what Bell and Janvier call interpretation of global graphical features. In this case, the definition of the tangent line to a graph and the concept of its slope, would be utterly meaningless to someone who could only look at the graph or the function a few specific points at a time. Monk studied beginning calculus college students. His findings are similar to those of Bell and Janvier. Students may have a confident and secure point-wise understanding of functions, but still have to struggle in order to use functions in an across-time manner.

Marnyanskii (1965) also mentions that students approach functions point-wise. He says that students think mainly in terms of individual images and are not used to thinking in terms of groups of objects. Davidson (in press) reports similar findings from his study of young children and their early function concepts. His analysis of solution types indicates a transition between 5 and 7 years, in which trial-and-error procedures (correspond in many cases to point-wise thinking) are replaced by anticipatory or inferential strategies (correspond to a global view of function). In anticipatory solutions, children even mentally combine two functions to produce another (correspond to operations on functions as entities).

Point-wise approach to functions seems to be easy for students while interval-wise, a global way of approaching functions or operating on functions as entities are harder. The following part describes another area of students' difficulties -- the image of the concept of function and discrepancies between this image and the definition of function.

Definition and image of the concept of function

Vinner (1983) defines concept image as the mental picture of this concept (i.e., the set of all 'pictures' that have ever been associated with the concept in the person's mind) together with the set of properties associated with the concept (in the person's mind). The image of a concept might be different for different people. Resnick and Ford (1981/1984) outlined 'correspondence' -- "the match of one's mental picture [what Vinner calls 'image'] with correct mathematical concepts" -- as one of the three criteria for well structured knowledge about mathematics.

Studies show that there are discrepancies between image and definition of the concept of function (Dreyfus and Vinner, 1982; Vinner, 1983; Vinner and Dreyfus, in press). They found that when asked to give a definition of a function, many of the college students gave primitive definitions (such as identifying function with equation or a rule). Some students used the modern Dirichlet-Bourbaki definition: A function is any correspondence between two sets which assigns to every element in the domain exactly one element in the range. But very few of the students acted according to this definition when asked various questions about functions. Discontinuity, split domain and exceptional point played a crucial role in the explanations for rejecting "something" from being a function, even when it met all the requirements in the textbook definition or the definitions the students gave. For example, many students did not regard piece-wise functions such as $f(x) = \begin{cases} 2x+1, \text{ if } x>1 \\ 3x-5, \text{ if } x\leq 1 \end{cases}$ as functions. Markovits et al (1986) found this thinking in 9th graders, Vinner (1983), Dreyfus and Eisenberg (1987), Vinner and Dreyfus (in press) - even with college students. Some students expect functions to be given by one rule. If two rules are given for two disjoint domains then the student might think that there are two functions. If the correspondence between the numbers looks arbitrary then the student might think that there are infinitely many functions, as if each number has its own rule of correspondence. Some students may accept that a function can be given by several rules relating to disjoint domains providing these domains are half lines or intervals, but may reject a function with a rule with one exception. In the same way they expect graphs of functions to be 'reasonable'. Students also think that functions, which are not algebraic, exist only if mathematicians officially recognize them, by giving them a name or denoting them by specific symbols.

Research shows that constant function (e.g., $f(x) = 4$) is also not considered to be a function (Dreyfus and Eisenberg, 1987; Markovits et al., 1986; Vinner, 1983; Vinner and Dreyfus, in press). Marnyanskii (1965) provides an explanation for that. He says that students think that when x changes, y has to change also. This belief contributes to the difficulties of dealing with the constant function.

The function obtained by composition of functions is not viewed by many students as a function. Rather, as Thomas (1975) found, composition is seen as the sequencing of assignments. Lovell (1971), too, found that students have difficulties with the composition of functions. This finding supports Thomas (1975) and Orton's (in Lovell, 1971) decision to include attainment of operations on functions as entities in a high level stage. Accepting the composition of functions as a function shows that students are at a higher conceptual level, where functions are conceived as entities and operations can be performed on them.

Many students hold a linear prototypical image of functions (Dreyfus and Eisenberg, 1983; Markovits et al., 1983, 1986). Markovits studied Israeli ninth graders who had been introduced to functions and had met the linear function in particular. She found that when asked to give examples of functions that satisfy certain conditions (e.g., "In the given coordinate system, draw a graph of a function such that the coordinates of each of the [given] points A, B represent a preimage and the corresponding image of the function."), most of the students chose linear functions or functions composed of straight lines, even in situations when it was wrong. For example, in the case where three points were given, the students connected them into a triangle which did not represent a function. One might expect that "linear" thinking would disappear after students meet various functions and become more knowledgeable about functions. But Dreyfus and Eisenberg (1983, 1987) found this kind of thinking in eleventh and twelfth graders and even in college students. Linear prototypical image of functions seems to be related to "linear thinking" in general and not to functions only. Karplus (1979), for example, found that students frequently use linear interpolation when solving problems involving graphs. He found that even in grade 12, the majority of students used linear interpolation when curved interpolation should have been used. An example was the case where students were given the following four heights and masses (in relation to spacecraft designs): (1m, 400kg), (4m, 100kg), (8m, 100kg), (16m, 240kg), and were asked to predict the masses for spacecraft designs that were 2m high, 12m high, and of the height that needed the least amount of metal. The students used linear interpolation even though the four pairs of data were not in a linear relationship.

The finding that "linear thinking" does not disappear after students meet various functions is similar to findings about other aspects of conception of functions. Marnyanskii (1965/1975) found the same phenomenon of similar naive conceptions of functions across groups of students with different experience, although these misconceptions were not related to "linear" thinking. He reported that students who had studied a large number of functions had almost the same conception of function as did students who had just begun studying elementary functions.

Markovits et al. (1983, 1986) and Dreyfus and Eisenberg (1983) also report that in cases where there were an infinite number of solutions (such as in the case where students had to draw a graph of a function that passes through two points), many students claimed that the linear function was the only solution. The reason the students gave was that "through two points there is only one straight line." Dreyfus and Eisenberg (1983) say that the same students, when given three collinear points in the plane, stated that an infinite number of graphs of functions could contain them, and they easily provided examples. It seems that for the students 'two points' were connected to 'only one line' even in cases where it was irrelevant. Their geometrical knowledge interfered with their ability to solve non-geometric problems.

In addition to the geometrical influence, there might be another reason for this adherence to the belief that there is only one solution. Piaget and Voyat's (1979) work shows a similar pattern in a different setting. They report their observations of this phenomena of belief in a limited number of solutions in situations where there is an infinite number of solutions. Children between 4 and 12 years of age were asked to decide what possible routes might be taken by a toy car between two given points in a room. The smallest children restricted themselves to straight lines. Those between 6 and 7 invented curving and zigzag routes, but limited in number. The older children (who had attained the level of formal operations) stated that there are as many routes as you want, no limit. It seems though, that the context of this problem might be much broader than functions only. The ability to handle the idea of more than one solution to a problem, not to mention an infinite number of solutions, which are arbitrary in nature, requires a lot of sophistication.

Summary and discussion

Not many studies have been conducted on students' learning of the concept of function. Some researchers tried to develop a theoretical framework to guide their studies. A model of function "block" was developed to guide a systematic study for ascertaining a hierarchical ordering of intuitive function concepts. This model was used in a few studies but appeared to be problematic as a model for studying functions. We still do not have enough information about the intuitions students bring with them to the study of function. Another attempt to develop a theoretical framework for studying functions was done by researchers who tried to determine stages of acquisition of the concept of function. These researchers did not deal with the concept of function in a broad sense. Rather, they dealt with the learning of a special way of presenting the concept of function in a specific curriculum. In addition, they based their construction of developmental stages on some questionable assumptions. So, the generalizability of the stages that were found is still not clear.

The main research findings of students' difficulties in learning function are in two areas: different ways of approaching functions and discrepancies between definition and image of the concept of function. Studies show that many students deal with functions point-wise but cannot think of a function as it behaves over interval or in a global way. Students also cannot interpret graphs representing situations whenever pictorial aspects conflict with correct meaning. Thinking of functions as entities and performing operations on them is also hard for students. In the area of discrepancies between definition and image of the concept of function it was found that many students hold a linear prototypical image of functions. Students were found to reject piece-wise functions. Students also expect graphs of functions to be 'reasonable', and functions to be representable by a formula. Students do not consider a constant function nor the function obtained by composition of functions as functions.

Expecting functions to have some regulation and rejecting functions that seemed to have an arbitrary nature was part of the definition of function when function was first introduced. Looking at the historical development of functions shed interesting light on students' difficulties and especially on the discrepancies between definition and image of the concept of function. The history of the concept of function is described in the next section.

Definition and Meaning of Functions:
A Historical View of the Development of Functions

(This section is based on Bennett, 1956; Bottazzini, 1986; Eves, 1983; Freudenthal, 1983; Hamley, 1934; Hight, 1968; Kline, 1972; Malik, 1980).

The concept of function has undergone an interesting evolution. Its history furnishes an example of the tendency of mathematicians to generalize and extend their concepts. We can see how mathematicians seek to find order by looking for ways to generalize across many situations with a common structure. The recognition that many situations involve quantities varying in some systematic way relative to each other was a big breakthrough in the development of mathematics.

Galileo (1564-1642) proposed a program for the study of motion that led to the investigation of a relation between two varying quantities. Many historians of mathematics consider it to be fundamental in arriving at the concept of function. At the middle of the 17th century, on the basis of physical motivations, curves described by motion or formula referring to motion, using the analytic geometry of Descartes, were included in investigations. A relation representable as an expression and its graph were accepted as mathematical objects. Before the function concept was fully recognized, most of the functions introduced during the 17th century were first studied as curves by means of motion. They dealt with problems such as the motion of a pendulum, the shape of a rope suspended from two fixed points, and motion along curved paths. Leibniz (1646-1716) is thought to be the first to use the term "function" in 1694. He used it to denote any quantity connected with a curve, such as the coordinates of a point on the curve, the slope of the curves, and the radius of curvature of the curve.

For Euler (1707-1783), the Bernoullis, and other mathematicians of the 18th century, a function was an analytic expression made up of variables and constants representing the relation between two variables with its graph having no "sharp corners" (this is usually referred to as Euler's definition). So, a general way to express a function $f(z)$ was an infinite series of the form: $Az^{\alpha} + Bz^{\beta} + Cz^{\gamma} + Dz^{\delta} + \cdots$. An accepted graph was, for example, a smooth curve of the form of Figure 2.2, while a "sharp corners" graph of the form of Figure 2.3 was not "eligible" to be a graph of a function. Euler also introduced the notation $f(x)$. The 18th century mathematicians believed that a function must have the same analytic expression throughout the domain. At the middle of that century the study of the shapes assumed by a homogeneous string held in tension

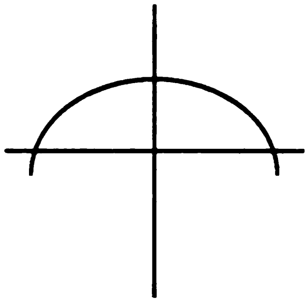


Figure 2.2 -- A Smooth Curve

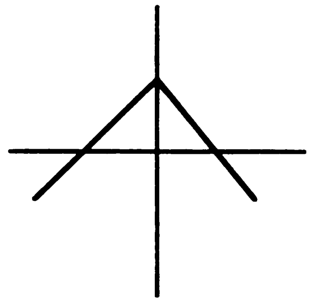


Figure 2.3 -- A "Sharp Corners" Graph

and placed in vibration in a plane, caused a serious discussion over the concept of function. In many cases it was impossible to describe the shape by a single expression. Euler, motivated probably by the physical nature of the problem, expanded the notion of "legal" solutions and included solutions that were still continuous according to the modern notion of continuity, but had to be expressed by a combination of several expressions. He even included solutions with "corners" whose derivative is undefined at the corners. During the latter part of the century, largely as a consequence of the controversy over the vibrating string problem, Euler and Lagrange allowed functions that have different expressions in different domains and used the word continuous where the same expression held and discontinuous at points where the expression changed form (though in the modern sense such a function could be continuous). While these mathematicians had to reconsider the concept of function, they did not arrive at any widely accepted definition. However, the gradual expansion in the variety and use of functions forced mathematicians to accept a broader concept.

Fourier's work shook the 18th century belief that all functions were "at worst" extensions of algebraic functions. The most common version of the history of the concept of function says that toward the middle of the 19th century, Dirichlet (1805-1859), while working on the convergence of "Fourier series", realized that the problem is difficult to study with the classical definition of function. So he re-defined a function y as a function of x if for any value of x in a given interval there corresponds a unique value of y. He added that it does not matter whether throughout this interval y depends upon x according to one law or more than one law or even whether the dependence of y on x can be expressed by mathematical operations. This definition is a very broad one and does not imply anything regarding the possibility of expressing the relationship between x and y by some kind of analytic expression. It stresses the basic idea of a relationship between two sets of numbers. But this definition was too general to be used in calculus and mathematicians, for almost a century, debated the question of how a function should be defined.

Bottazzini (1986) claims that Dirichlet was far from what today considered to be 'Dirichlet's concept of function'. He says that Dirichlet's fundamental idea was that every continuous function can be expanded as a Fourier series, and only Rieman (1826-1866) seriously dealt with 'pathological' functions.`

With the creation of analysis, 'Dirichlet's definition' received reinforcement. New discoveries about properties of continuous and discontinuous functions made the mathematicians realize that the rigorous study of functions extends beyond those used in the calculus and the usual branches of analysis where the requirement of differentiability usually restricts the class of functions. The study of functions was continued in the twentieth century and resulted in the development of a new branch of mathematics known as the theory of functions of a real variable. This rigorization of analysis did not go unopposed. The new, peculiar functions, violating laws deemed perfect, were looked upon as signs of anarchy and chaos which mocked the order and harmony previous generations had sought.

Then, the development of the concepts of metric space and topology led to the concepts of domain and range since it was realized that the properties of a function depend on the structure of these two sets. In 1917 Caratheodory defined a function as a rule of correspondence from a set A to real numbers, and in 1939 Bourbaki defined function as a rule of correspondence between 2 sets (not necessarily numbers). By the end of the first half of this century, the Dirichlet-Bourbaki definition of function had become the common definition in mathematical practice.

The most profound activity of twentieth century mathematics has been research on the foundations of mathematics. Set theory, the foundation for all mathematics, has naturally extended the concept of function to embrace relationships between any two sets of elements, be the elements numbers or anything else. In set theory a function f is defined to be any set of ordered pairs of elements such that if $(a,b)\varepsilon f$, $(c,d)\varepsilon f$, and $a=c$, then $b=d$. The set of all first elements of the ordered pairs is called the domain of the function, and the set of all second elements of the ordered pairs is called the range of the function. In other words, a function is a special subset of the Cartesian product of two sets, in which any element in the domain should be paired with one and only one element in the range. This last requirement was made, as Freudenthal (1983) pointed out, only because otherwise working with functions might require too much watchfulness. The new possibilities created by composing and inverting functions were hard to deal with without keeping functions univalent.

Mathematical activity during the nineteenth century expanded in several respects. The apparent ones are the vast expansion in subject matter and the opening of new fields as well as the extension of older ones. The number of mathematicians increased enormously as a consequence of the democratization of learning. But accompanying the explosion of mathematical activity, many branches became autonomous, each featuring its own terminology and methodology.

Today functions are everywhere in mathematics. MacLane (1986) gives many examples: Algebraic operations provide examples of functions of numbers. Geometric definitions produce trigonometric functions. The exponential function and its inverse, the logarithmic function, are also numerical functions. Functions of points in the plane or in space, such as rotation, reflection and translation, arise in geometry. In group theory, the *inverse* is a function from the group to itself. In a metric space, the *distance* is a real-valued function of pairs of points. In Boolean algebra, *intersection* and *union* are functions of pairs of sets. In geometry, *length* is a real-valued function of curves.

While the pure definition of function may make it look like a unifying concept, the segregation of mathematics into numerous distinct and unrelated divisions, disunifies it. Function has different labels: mapping, transformation, permutation, operation, functional, operator, sequence, morphism, etc. Each one uses different notation. *Function* is usually used when the set of values is numerical. *Mappings* and *transformations* come from geometry but are used also in algebra. *Permutation*, which is a one to one and onto mapping, is used mostly in group theory. *Morphism*, which is a function that preserves the group structure, is also used in group theory. *Sequence* is a mapping from N while *Operation* is used with certain simple standard functions. *Functional* is used when the domain is a set of functions, in order to avoid repetitions or ambiguities. When the range is also a set of functions, *operator* is used.

There are different function notations which makes function look like different concepts instead of one unifying concept. When algebraic operations are used to describe functions, the most common notation is f : x --> 2+x. Many functions that were recognized as such in the early days of functions, have specific names and use specific notation. These, for example, are the trigonometric functions: sin, cos, tan, etc, and also exp and log. Linear transformations are described, in many cases, by matrices which do not look like the common notation of functions at all. In set theory, set notation is used, such as $<x,y>$, $\in$, { }, etc.

Summary and discussion

Functions started as curves described by motions. They were described by analytic expressions. So a function was a "formula" or an "expression" representing the relation between two variables with its graph having no "sharp corners". This conception is similar to the image of the concept of function that students hold these days. Later, forced by the evolution of mathematics, "arbitrary" functions were introduced. At first, only the kind of dependence could be arbitrary, but later the sets themselves could also be arbitrary. Today the sets may include curves, surfaces, functions, etc. This change in the definition of function did not occur without reason. In mathematics, the most frequent reason for rejecting a traditional phrasing of a definition in favor of new wording is that it has been too restrictive to include situations of newly recognized importance, and not as inclusive as desired for modern research.

The development of the concept of function in mathematics influenced the way function was presented in school mathematics. The next section describes the ways function appears in school curriculum.

Functions in School Curriculum

The report of the National Committee on Mathematical Requirements (1923) had an apparent influence on American textbooks. The chapter "Function concept", written by Hedrick, emphasizes the importance of functional thinking in life. It also shows how the concept of functionality appears in every subject of the mathematical curriculum. Since that time the function concept appears in every secondary mathematics curriculum. Modern curriculum projects that use the concept of function as a central idea are the University of Chicago School Mathematics Project (UCSMP) directed by Zalman Usiskin, and Algebra with Computers directed by James Fey from the University of Maryland.

Until the first half of this century it was common to find "function" in texts used synonymously with "dependent variable", or a "quantity that varies", or as an "expression" or "formula". This was similar to Euler's definition from 1755. But at the beginning of the second half of this century the "new math" movement created some changes in school mathematics in general and in the way function was used in particular. This is described in the next part.

New math

One of the goals of the "new math" reform of the 1950s-1960s was to update the mathematics taught in school, reflecting developments in mathematics. Two outstanding accomplishments of the late nineteenth and early twentieth centuries were taking set theory as a foundation of mathematics and generalizing and giving succinct definitions in terms of set concepts and notation. In *Goals for School Mathematics* (Cambridge Conference on School Mathematics, 1963), function is considered as a simple concept from contemporary mathematical research that belongs in the curriculum not because it is modern, but because it is useful in organizing the material. Among many other changes in textbooks, the definition of functions has been changed to a modern one. SMSG (1960) defines function as follows: "Let A and B be sets and let be given a rule which assigns exactly one member of B to each member of A. The rule, together with the set A is said to be a function and the set A is said to be its domain. The set of all members of B actually assigned to members of A by the rule is said to be the range of the function." This definition, although not exactly the modern, formal mathematical definition, is closer to the modern one than to the old definition that was used in texts until that time.

Since then variations of the SMSG definition have been used in popular secondary math texts. In a popular text from the "new math" period, which is still being used today, Dolciani, Berman, & Freilach (1962) introduced "function" as a relation which assigns to each element of the domain one and only one element of the range. Twenty years later Dolciani, Wooton, & Beckenbach (1980), trying, perhaps, to be less formal, described a function as the following: "A function consists of two sets, D and R, together with a rule that assigns to each element of D exactly one element of R. Each element of R is assigned to at least one element of D. The set D is called the domain of the function, and R is called the range of the function." One paragraph later they become more formal and a function whose domain and range are sets of numbers, is described in the following way: "A function is a set of ordered pairs of numbers in which no two different ordered pairs have the same first coordinate." At the end of this page a relation is defined and immediately a function is defined as a special kind of relation: "A function is a relation in which each member of the domain is paired with exactly one element of the range." In a bit more than one page, the students meet the concept of function in three seemingly different ways, all of them are modern. In that introductory chapter, functions are represented in several forms: a table, a mapping diagram (arrow diagram), a set of ordered pairs, a graph and as a set of ordered pairs where the first elements belong to a certain set and the second elements are represented by an algebraic expression.

Malik (1980) claims that the necessity of teaching the modern definition of function at school level is not at all obvious and pedagogical considerations were ignored while designing a modern course on functions. Freudenthal (1983) emphasizes that "from the fact that all mathematics can be reduced to set theory, one may not conclude that it should be done and even less that it is a didactic necessity and possibility." Malik says that Euler's definition of function from the 18th century covers all the functions used or required in a calculus course, and one never confronts a situation where one has to use the modern definition of function. Malik states that a student retains a concept only if it is used. If, as is the case with the function concept, only a particular form is used, the student unconsciously accepts the particular form as the definition. This is why the student understands function as a smooth relation between two varying quantities. Other studies, as described earlier (Markovits et al., 1986; Marnyanskii, 1965; Vinner and Dreyfus, in press), support Malik's claim when showing that while students are being taught the modern definition of function, the old one serves as the concept image. Malik concludes that in mathematics research a new concept receives recognition only when its relevance to the current phase of research is established. (The description of the history of the concept of function in the last section reinforces this conclusion.) Relevance, Monk (1987) says, should be taken into account when teaching a mathematics course as well.

The "new math" movement influenced not only the high school curriculum, but elementary school mathematics as well. Function, as a unifying concept, appealed to curriculum developers.

Function in elementary school mathematics

The importance of the concept of function in the K-6 mathematics curriculum is stressed in *Goals for Mathematics Education for Elementary School Teachers* (Cambridge Conference on Teacher Training, 1967). It suggested there that function (and set) will be used throughout the curriculum wherever natural examples and uses occur. By Grade 6, the word *function* (and set) and the idea behind it should be established firmly and correctly as natural parts of the pupil's mathematical language. Recommendations to incorporate function into the K-6 curriculum is also stated by the National Council of Teachers of Mathematics (Curriculum and Evaluation Standards for School Mathematics, 1988) and by many other curriculum guides.

Among the elementary programs which have the concept of function included in their mathematics curriculum are the Curriculum Development Association (CDA) Math Program, the Comprehensive School Mathematics Program (CSMP), the Madison Project and the Nuffield Project. All four programs speak of mathematics as the study of relations and present the concept of function within this orientation to mathematics.

CDA Math. Robert W. Wirtz, the director of CDA Math said: "The idea of a determining relationship is what mathematicians call a 'function'" (Wirtz, 1974). He added that this is probably "the most pervasive, unifying idea of all mathematics." Wirtz sees mathematics as a search for relationships that are not obvious and based his curriculum on that. The four basic operations of arithmetic are used in classifying and determining relationships, without specifying the mathematical definition of functions.

CSMP. The concept of function is one of the underlying mathematical idea in CSMP. Through a graphical language, the language of arrows, the child experiences thinking and writing about relations. In Teacher's guides to CSMP mathematics (1975, 1976) it is stated that we are always trying to establish, explore, and understand relations - in everyday life, in school, in science and also in mathematics. Studying mathematics means studying the relations between mathematical objects. Function is viewed as a specific kind of relation. It is defined using the language of arrows: K is a function if there is at most one arrow starting at each number. Functions are used in teaching the basic arithmetic operations, and later with projections, translations, magnifications, reflections and rotations. Children also work with composition of functions.

The Madison Project. The emphasis in this program is on the mathematical definition and understanding of what a function is. Robert B. Davis, the director, says that the study of functions is one of the most important topics in mathematics (Davis, 1967). The concept is presented as an equivalence class of formulas, a set of all formulas which produce the same table of numbers. In this project, a rule is used as a way to put numbers together and a formula -- as a way to use variables in writing a rule. "The function, then, refers to how we have paired up numbers in our table, and not to the rule by which we did it" (Davis, 1967). Function notation is also introduced. Children experience making up and guessing rules, working with tables, graphs and equations. Examples from physics are also used.

Nuffield Mathematics Project (Britain). Functions are used as an instrument of cause and effect. The students look for the independent variable in an experimental situation from physics, and try to predict the results if the situation is changed. The mathematical definition of function is not given.

Summary and discussion

Developments in mathematics in general and in the concept of function in particular influenced school mathematics in the second half of this century. Although the function concept appeared in secondary mathematics curriculum since the second quarter of this century, function was used then in a way similar to Euler's definition from the 18th century. Function, in a modern sense, was introduced to school curriculum during the "new math" reform of the 1950's-1960's. The modern concept of function appears today in every secondary school mathematics curriculum and in several elementary school mathematics curricula. Although function is defined in textbooks in a modern sense, students hold an old image of function, similar to the one from the 18th century when function was a dependent variable or an analytic expression whose graph has no "sharp points". Students have difficulties learning the concept of function.

Three major components influence directly the learning process: The learner, the curriculum and the teacher. All of these are influenced by school and society. Most of the studies of learning mathematics in general and functions in particular concentrate on the learner and his/her cognition. The studies mentioned earlier in this chapter about students' understanding of the concept of function are of this type. Finding students' misconceptions and how they think about a concept is an essential step in improving the learning of that concept. Another important step is providing a good curriculum. The "new math" movement aimed at that. Also Hamley (1934), Buck (1970), Felix (1970) and others gave suggestions of how and what to teach about functions in order to help the learner. But the third factor -- the mathematics teacher -- has not been considered until lately. In the next chapter this issue will be discussed.

CHAPTER 3
MEANINGFUL LEARNING
TEACHING FOR UNDERSTANDING
AND TEACHERS' SUBJECT MATTER KNOWLEDGE

Historical Perspective

Mathematics educators today call for making a change in the way teachers teach to emphasize teaching for understanding and meaningful learning (e.g., *Curriculum and Evaluation Standards for School Mathematics*, 1988; Davis, 1986; *Educational Technology Center*, 1988; Lampert, 1988; Lappan and Schram, to appear; Peterson, 1988; Resnick, 1987; Romberg, 1983; Schoenfeld, 1987). It is widely accepted in the math education community that, in general, meaningful learning is better than rote learning. Chambers et al. (1986), for example, recommend, in the *Wisconsin Guide to Curriculum Planning in Mathematics*, that teachers shift the focus of instruction from an emphasis on manipulative skills to an emphasis on developing concepts, relationships, structures and problem solving. The words "understanding" and "concepts" appear these days in almost every report that calls for change in the way teachers teach and students learn mathematics. This call for making a change is not new. The following section traces these calls for making a change to meaningful teaching and learning mathematics since the beginning of this century.

E.H. Moore

Moore (1903/1926), in his presidential address delivered before the American Mathematics society, called for an evolutionary change in the way mathematics was taught to make it more meaningful. To do that, Moore called for emphasizing the practical sides of mathematics in continuous relation with problems of physics, chemistry and engineering, so abstract mathematics would be connected with subjects which are naturally of interest to the students, and the results obtained by theoretic process would be capable of verification by laboratory process. Using this method would help the student feel that "mathematics is indeed itself a fundamental reality of the domain of thought, and not merely a matter of symbols and arbitrary rules and conventions."

Moore also made the suggestion to organize the algebra, geometry and physics of the secondary school into a thoroughly coherent four-year course, so students would see the connection among these subjects. Moore wanted to develop in every student the true spirit of research. Elements of mathematical routine, he said, should not be taught in an uninteresting way but should be attached to interesting problems. In teaching one thing the teacher must illuminate relations to other things.

Moore emphasized that the student should be independent of all authority in his/her thinking and therefore should be able to check his/her work by obtaining every result of importance by more than one method. He wanted the learning to be meaningful to the student. The teacher, he said,

> should lead up to an important theorem gradually in such a way that the precise meaning of the statement in question, and further, the practical, i.e., computational or graphical or experimental - truth of the theorem is fully appreciated; and furthermore, the importance of the theorem is understood, and, indeed, the desire for the formal proof of the proposition is awakened, before the formal proof itself is developed. Indeed, in most cases, much of the proof should be secured by the research work of the students themselves.

Moore's call did not create any significant change in the way mathematics was taught. This can, probably, be attributed to the impact associationist theories of drill and practice emphasizing isolated skill learning (Thorndike, 1922) had on American education in those days.

Brownell -- The Meaning Theory

Brownell (1935, 1947) opposed the emphasis on learning isolated skills that was so widespread in his days. He argued for emphasis on understanding, for meaningful instruction. He advocated the Meaning Theory which said that children should see sense in what they learned. This theory was closely related to Gestalt psychology (Jones and Coxford, 1970).

Arithmetic, according to the Meaning Theory, is "a closely knit system of understandable ideas, principles, and processes." The Meaning Theory took full account of the complexity of the first stages in arithmetical learning, adapted the pace of instruction to the difficulty of the learning, and provided activities and experiences for the child which would carry him/her to meaningful ideas of numbers. It emphasized discovery of relations between numbers, using, at first, concrete materials and eventually abstract numbers. "Number facts" were, in terms of the Meaning Theory, generalizations of number combinations. As such, their learning is arduous and time consuming. They cannot be learned all at once, but slowly, by abstracting likenesses and differences in many situations, by reacting to the number aspects of situations in steadily more mature ways.

According to the Meaning Theory, as opposed to associationist theories of drill and practice, rote memorization during the learning process should by all means be prevented. Only after numerous and varying experiences, when the child has reached generalization, the "fact" should be memorized.

The Meaning Theory emphasized relationships within the subject in order to make number sensible, and encouraged children to recognize relations between statements, to connect different topics, etc. In other words, it emphasized conceptual understanding. But the Meaning Theory did not change the general way students learned arithmetic from rote to meaningful. Another advocate of meaningful and conceptual learning was Van-Engen who based his theory on Piaget.

Van-Engen -- Conceptual learning

Van-Engen (1953) distinguished between two dimensions of meanings: syntactic and semantic. The syntactic dimension means emphasizing the use of a symbol (or word) in relation to other symbols. The semantic dimension means emphasizing the referents of the given symbol, the objects, other symbols, simple events or mental constructs that are represented by that symbol. Van-Engen says that drill is the syntactic dimension of meaning and should come only after the teacher has established a referent for the symbol in isolation. Only after that the teacher should establish the various shades of meaning of that symbol (or word) when used in context. While semantic meaning is a substitution process of symbol for object, or symbol for symbol, or symbol for concept, understanding, Van-Engen says, is an organizational process, integrating concepts, fitting a new idea into a conceptual structure already in the student's possession.

Abstraction and generalization are two important processes in conceptual learning. To achieve that Van-Engen suggests a sequence of three stages for formulating a concept, from concrete to semi-concrete to symbol. Van-Engen emphasizes, based on Piaget, that learning has its roots in actions or manipulations and not in "teaching by telling." This approach led to the "learning by discovery" during the "new math" reform in the 1950s and 1960s.

J.S. Bruner -- Discovery

The "discovery method" is most associated with Bruner. Although different people talked about it before, this method did not receive widespread consideration until after the publication of Bruner's *The Process of Education* in 1960 (Grossnickle and Reckzeh, 1973). Shulman (1970) describes Bruner as the discovery's prophet.

Bruner based his approach to discovery on Piaget's developmental theory. Discovery method is the opposite of authoritarian, lecture, or tell-how-to-do method of teaching. Bruner (1961) describes discovery as "a matter of rearranging or transforming evidence in such a way that one is enable to go beyond the evidence so reassembled to new insights." Bruner (1960) wants the student to discover mathematical ideas for him/herself since he believes that a person knows what he/she has discovered. His emphasis is not on the product discovered, but on the process of working. Bruner wants to teach the fundamental structure of the subject, but to do that not in a systematic, ordered approach. He prefers to have the learner begin with manipulation of materials or tasks that present problems (Bruner, 1966; Shulman, 1970). These problems may take the form of goals to be achieved in the absence of readily discernible means for reaching these goals; contradictions between sources of information of apparently equal credibility; or the quest for structure or symmetry in situations where such order is not readily apparent. The first step of discovery is a sensed incongruity or contrast. Contradictions will result in an intellectual discomfort. Then, in an attempt to resolve this disequilibrium, the learner will make a new discovery in the form of a reorganization of his/her understanding. So, instruction is a successive disequilibria and equilibria terminating in the attainment or discovery of a desired cognitive state.

Bruner, based on his interpretation of Piaget's developmental theory, describes the child as moving through three levels of representations as he/she learns (Bruner, Olver, & Greenfield, et al., 1966). According to the three levels, he suggests, as Van-Engen does, to use different representations of the subject matter - concrete, semi concrete and symbolic.

Gagne and Ausubel opposed Bruner's position (Shulman, 1970). They advocated approaches to instruction which were antitheses of discovery. Gagne, using behavioral objectives advocated guided learning. He began with a task analysis of the instructional objectives and then developed a very complex pyramid of prerequisites to the objectives. Then, by administering diagnosting pretests one could determine which had already mastered and what must be taught. Ausubel, like Gagne, emphasized the great importance of systematically guided exposition in the process of education. He advocatesd careful sequencing of instructional experiences so that any unit taught was clearly related to those that preceded it. Ausubel (1968) argues that discovery and meaningfulness are not the same thing. Shulman (1970) agrees with him. They point out that it is possible to discover facts which are intrinsically non-meaningful, and on the other hand, that material can be assimilated meaningfully into existing knowledge even if learnt from an expository presentation. Ausubel says that if the material is first obtained by discovery, that assimilation needs to take place.

Discovery is not well defined. Different people have different interpretations of discovery. Biggs (1971) describes five different types of Discovery Learning: Impromptu, Free exploratory, Guided, Directed, and Programmed. While Bruner's interpretation of "discovery" can be described as "free exploratory", Dienes who worked at the same time, used three types of "discovery": free exploratory, guided and directed discovery, in a very structured way.

Z.P. Dienes

Dienes (1960) based his theory on works of Piaget, Bruner and Bartlett. He emphasizes that mathematics should be regarded as a structure of relationships and not as an elaboration of a number of techniques. The formal symbolism, he said, is merely a way of communicating parts of the structure from one person to another. Learning mathematics means the apprehension of the structural relationships between concepts and their applications, together with symbolization, and the acquisition of the ability to apply the resulting concepts to real situations.

Dienes says that the problem of learning is essentially how to find a kind of "best fit" between the structure of the task and the structure of the person's thinking. Dienes, as Piaget, sees three stages in the formulation of a concept. To each corresponds a very different type of learning. He emphasizes freedom to experiment, with the ingredients of the concept available as play material, in the first, preliminary, play stage, which he also calls concept learning. At the second stage a certain degree of structured activity is desirable, but a great number of experiences, of varying structure, all leading to the concept should be provided to match different ways of thinking. In the third stage practice should be provided for the fixing and application of the concepts that have been formed.

Dienes says that students should work independently since the teacher cannot teach each one of the different students in class. He adds that authority would not allow inquiry to flourish. The desired classroom situation is one in which we shift the emphasis from teaching to learning, from our experience to the children's, from our world to their world. This kind of classroom situation was also desired by Biggs.

E.E. Biggs -- Learning by Doing

Biggs (Biggs and MacLean, 1969) wants the students to learn in an active, creative way using discovery. The aims of learning mathematics in this way are to free students to think for themselves; to provide opportunities to discover the order, pattern and relations which are the very essence of mathematics in the man-made world and in the natural world as well; and to train students in the necessary skills. The basic idea is that children learn better by touch and sensation in a child-centered classroom where the emphasis is not content but the use of the experiences of the children to build concepts and strategies. Children can learn mathematics by their own efforts and the teacher should plan the learning situations which will effect this (School Council, HMSO,1969).

Basing the approach on Piaget's theory, Biggs suggests that the child should be allowed to do things over and over again and thus reassure him/herself that what he/she learned is true. Since we do not really know what mathematical concepts, processes and facts individual children are ready to learn, teachers should provide a variety of relevant materials, encourage understanding, assist the growth from one stage to the next, detect this growth when it happens, and open the way ahead. Biggs recommends that children will meet situations which give rise to the basic operations before they are taught the written processes. In this way, the students will understand and appreciate the need for polishing and expanding their computational skills. Students should be helped to devise and refine their own methods.

R.R. Skemp -- Relational Understanding

Skemp is interested in how children learn mathematical concepts. He adds to the discussion of different ways of teaching for understanding the issue of different kinds of understandings. He distinguishes between two kinds of understandings (Skemp, 1976, 1979): relational understanding which is knowing both what to do and why, and instrumental understanding which is rules without reasons. He is against the latter since it usually involves a multiplicity of rules rather than fewer principles of more general application. Skemp says that although instrumental mathematics is easier to understand within its own context, the rewards are more apparent and immediate and one can get the right answer more quickly, relational mathematics has more advantages. He sees relational mathematics as more adaptable to new task, easier to remember, and can be effective as a goal in itself.

Learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can produce an unlimited number of plans for getting from any starting point within the schema to any finishing point. Skemp says that with relational understanding the means become independent of particular ends to be reached. Building up a schema within a given area of knowledge becomes an intrinsically satisfying goal in itself. The more complete a student's schema is, the greater his/her feeling of confidence in his/her own ability to find new ways of 'getting there' without outside help. But the schema is never complete. The process becomes self-continuing and self-rewarding.

Meissner (1983) says that relational understanding is an inevitable condition for

the development of conceptual skills, but is mostly intuitive and unconscious. Each student must construct internally his/her relational understanding. He/she has to re-invent ideas and relationships. The student needs a strong correspondence between external mathematics and his/her internal representation. Greeno (1978) says that the more feedback the student has and the more frequent the interaction is, the better is the 'correspondence'.

Today

Conceptual and procedural knowledge. As we can learn from the historic discussion of understanding and skill, meaningful and rote learning, calls for making a change in the way teachers teach to emphasize meaningful learning are not new. The idea of making the learning of mathematics meaningful by making connections among different concepts and topics in mathematics and between different subject matters is advocated these days (e.g., *Curriculum and Evaluation Standards for School Mathematics*; 1988; Lampert, 1986, 1988; Resnick and Ford, 1981/1984). This idea is not new, Moore (1903/1926) has already talked about it at the beginning of this century and so did others since then. These people either emphasized that basically mathematics is the study of patterns and relationships or explicitly talked about building up rich relationships.

People today seem to confuse meaningful and rote learning with conceptual and procedural knowledge. Hiebert and Lefevre (1986) attempted to define these different kinds of knowledge and learning. Conceptual knowledge is described as knowledge which is rich in relationships. It is a network of concepts and relationships (Bell, Costello, & Kuchermann, 1983). The learning of a new concept or relationship implies the addition of a node or link to the existing cognition structure, thus making the whole more stable than before. Meaning, say Hiebert and Lefevre, is generated as relationships between units of knowledge are recognized or created. So, conceptual knowledge must be learned meaningfully. Procedural knowledge, on the other hand, is made up of the formal language of mathematics and the algorithms for completing mathematical tasks. Procedural knowledge can be learned with or without meaning.

During preschool years, conceptual knowledge and procedural knowledge are closely related. But as students move through elementary and junior-high school, conceptual knowledge and procedural knowledge continue to develop along separate tracks (Hiebert and Lefevre, 1986). Many math educators today complain about the over-emphasis of school mathematics on procedural knowledge without any relation to conceptual knowledge and meaning. Students are asked to memorize a lot of facts and procedures without paying attention to understanding of concepts and their application. Many math educators say that the advanced technology we have now and will have in the future, will handle this aspect of performing procedures and algorithms and emphasis should be on concepts and understanding.

But our desire to achieve meaningful learning does not mean that we should aim only at one kind of knowledge, i.e., conceptual knowledge. While there is no reason to make students memorize procedures that are easily done by machines, some procedural knowledge is important. Nesher (1986) claims that the dichotomy between learning algorithms and understanding is a superficial and misleading dichotomy. She based her argument on two considerations: 1) research on mathematical performance does not inform us about the relationship between success in algorithmic performance versus success in understanding nor does it give evidence about the contribution of understanding to algorithmic performance, and 2) the possibility of teaching for understanding in mathematics without attending to the algorithmic and procedural aspects is questionable. Resnick and Ford (1984) add that memorization of certain facts and procedures is important not so much as an end in itself, but as a way to extend the capacity of the working memory. This can be done by developing automaticity of responding. When certain processes can be carried out automatically, without need for direct attention, more space becomes available in working memory for processes that do require attention.

Constructivism. The idea that knowledge is actively constructed by the learner and not passively received from the environment is also currently advocated in relation to meaningful learning (e.g., Cobb, 1986; Confrey, 1985; Steffe, von Glasersfeld, Richards, & Cobb, 1983; von Glasersfeld, 1984). Again, this idea is not completely new. The discovery method, which flourished during the 1950's and the 1960's, was also based on the assumption that the learner should discover mathematical ideas for it was believed that this is the only way the student will know them. Discovery of ideas and relationships by the student has been recommended to some degree since the beginning of this century.

Still, there is a difference between past and present conceptions of the construction of knowledge. The historic discussion of understanding and skill, meaningful and rote learning has dealt with each in the context of advocating instructional programs. The issue has been which should be emphasized during classroom instruction. It was, probably, assumed that knowledge will occur as a result of meaningful instruction. Only a few years ago it was realized that the acquisition of knowledge may not be isomorphic to or a direct result of instruction. Until lately, meaningful, for example, was considered to be objective (subject matter oriented). It was assumed that instruction which follows the structure of the subject matter (either by using discovery learning or guided learning, to mention the two extremes after the period of associationist theories of drill and practice) will cause meaningful learning. In the last few years, meaningful has started to be considered subjective (learner oriented). Recent studies (e.g., Ginsburg, 1983; Lesh & Landau, 1983; Hiebert, 1986) showed that children make sense of the subject matter in their own way, which is not always isomorphic or parallel to the structure of the subject matter or to the instruction. Constructivism, which is a rather new knowledge acquisition theory (e.g., Von Glasersfeld, 1984), took the idea of subjective meaning to the extreme. Radical constructivism involves two principles (Kilpatrick, 1987): (a) knowledge is actively constructed by the learner, not passively received from the environment; and (b) coming to know is a process that organizes one's experiential world, not a discovery of an independent, pre-existing world outside the mind of the knower. While almost all math educators these days agree with the first principle of constructivism, there is not such agreement on the second.

Cognitive research on learning made a difference between past and present views of knowledge acquisition, which influence the way we think about meaningful learning and understanding. The most profound difference is that instead of evaluating "meaning" by looking only at the subject matter, people are "looking" now at what is happening in the learner's head. It has been realized that we need to have a better understanding about learning and the learner in order to make learning meaningful and to help students achieve understanding.

The teacher's role is to help the learner achieve understanding of the subject matter. Changes in research on teaching and teachers have also occurred in recent years. These changes have the potential to contribute to making learning more meaningful. This will be discussed in the next section.

Teachers' Subject Matter Knowledge

A new trend in research on teaching

While cognitive studies that aimed at understanding of 'what is going on in the learner's head' were part of research in math education for quite sometime, this was not the case with research on teachers. Studies of teachers that were conducted in the last 20 years adopted an approach to teachers that was similar to the way math educators viewed the teachers' role in the classroom those days. The development of new math curricula during the 1960's and 1970's viewed the teachers' role as that of implementing curriculum developed by experts. Teacher-proof curricula, which were the extreme of this process, assumed that children can learn directly from ready-made curriculum materials while the teacher, instead of teaching, should adopt a role of manager and facilitator. This was most apparent during the period of individualized instruction (e.g., Erlwanger, 1973; NACOME, 1975). Process-product research and the later research on effective teaching were "content free" and tried to identify generic teacher behaviors that seemed to be effective (Brophy and Good, 1986; Gage, 1978). "Effective" teachers were identified by their students' scores on standardized tests. Without considering the content, the identified "effective" instructional behaviors were closely connected with the management of classrooms rather than with content pedagogy. Spending class time on task, as one of the findings of "process-product" research suggests is, of course, important. But the teacher still has to find an appropriate task.

Teachers' thoughts and beliefs have come into consideration in recent years (Clark and Peterson, 1986). This was the beginning of studies that aimed at understanding of 'what is going on in the teacher's head'. But these studies still focused primarily on generic cognitive processes that transcend the particularities of the subject matter. Recently, researchers realized that research on issues surrounding instruction within particular subject matter areas is badly needed (Brophy, 1986). Studying teachers' subject matter knowledge is one such issue. Analysis of subject matter knowledge for teaching in several subjects is being done at Stanford University, Pittsburgh's Learning Research and Development Center, and Michigan State University's National Center for Research on Teacher Education.

Why study teachers' subject matter knowledge?

The teacher has an important role in the process of learning. He/she helps students understand the subject matter. The teacher, as Romberg and Carpenter (1986) stated, should help students put new ideas into perspective with past and future ideas, and encourage pupil engagement and achievement. This cannot be done by a teacher who lacks subject matter knowledge. Attempts to ignore the teacher's role by designing curriculum materials that aim directly at the student, leaving the teacher the role of managering and facilitating only, proved to be unsuccessful (e.g., Erlwanger, 1973; NACOME, 1975). The teacher helps the student understand the subject matter by representing it in appropriate ways, by asking questions, suggesting activities and conducting discussions.

Teachers teach specific subject matter and one would assume that their knowledge of the subject matter influences the way they teach it - the quality of their instruction -- and therefore their students' learning. Interestingly, this common belief about the relationship between teachers' subject matter knowledge and students' learning is not supported by research findings.

Research, however, shows that teachers' subject matter knowledge influences their instructional practice. Wilson, Shulman and Richert (1987) describe a study of teacher knowledge from 1985 where Hashweh found that subject matter knowledge affected teachers' transformation of the curriculum, both in terms of the modifications that were made of the textbook materials and the representations that teachers used in their explanations of concepts and principles. Hashweh found that knowledgeable teachers were more likely to organize materials according to their understanding whenever the textbook's organization did not match that. They also were more likely to detect student misconceptions and to correctly interpret students' insightful comments. Hashweh observed a qualitative difference in the representations of the subject matter generated in planning to teach. The less knowledgeable teachers used representations that reflected their surface knowledge of the topic while the knowledgeable teachers used representations that reflected their deeper understanding of the topic. Although this study does not make connections between teacher knowledge and student learning, it does point to the different things teachers with different subject matter knowledge do in class. Thompson (1984) also found that teachers' views, beliefs, and preferences about mathematics (which are part of their subject matter knowledge) influence their instructional practice.

One of the reasons for the surprising lack of research findings about the relationship between teachers' subject matter knowledge and students' learning has to do with the way teachers' subject matter knowledge is defined by the researcher. In the earliest research on teaching in process-product research, among the teacher characteristics examined for relationship with student outcome were "measures" of teachers' subject matter knowledge like the number of courses taken in college and teachers' scores on standardized tests (Ball, in press; Wilson et al., 1987). These "measures" of teachers' subject matter knowledge, as in the case of evaluating students' learning, are problematic, and do not represent teachers' knowledge of the subject matter. The number of correct answers does not give enough information about the knowledge one has, nor does the number of courses one has taken.

Ball (in press) provides another reason for the lack of research findings that show relationship between teachers' subject matter knowledge and students' learning. She claims that the relationship between knowledge of mathematics and teaching is not linear. Teachers' subject matter knowledge is not the only component that influences instructional practice. Rather, there are other components that shape the way the teacher teaches. "In teaching, teachers' understandings and beliefs about mathematics <u>interact with</u> their ideas about the teaching and learning of mathematics, ideas about pupils, teachers, and the context of classrooms" (emphasis in the original). But while teachers' subject matter knowledge is not the only component that influences teaching, nevertheless, it has a major role in instructional practice.

A simplistic definition of teachers' subject matter knowledge by the number of college mathematics credits or by results on standardized tests has proved to be inadequate. The next part deals with the issue of what teachers' subject matter knowledge is. Other kinds of teachers' knowledge, such as knowledge of general pedagogy, students and learning, which are also important for teaching, are not discussed.

What is teachers' subject matter knowledge?

At the beginning of this century, Dewey (1904) suggested that the would-be teacher must have "mastery of subject matter from the standpoint of its educational value and use, or what is the same thing, the mastery of educational principles in their application to the subject matter which is at once the material of instruction and the basis of discipline and control" (p. 147). The would-be teacher must understand principles in the psychology, logic, and history of education. Dewey also wanted the teacher to be trained in the higher levels of intellectual method so s/he will have a sense of what adequate and genuine intellectual activity means. Dewey recognized that teachers' knowledge of the subject matter must go beyond a regular, personal knowledge of the subject matter.

Leinhardt and Smith (1985) describe teachers' knowledge of arithmetic in a similar way:

> Subject matter knowledge includes conceptual understanding, the particular algorithmic operations, the connection between different algorithmic procedures, the subset of the number system being drawn upon, understanding of classes of student errors, and curriculum presentation (p. 247).

Again, teachers' subject matter knowledge includes more than the regular subject matter knowledge.

Shulman (1986) analyzes teachers' subject matter knowledge in more detail, not specific to arithmetic in particular or mathematics in general. He adds an important component to the definition of teachers' subject matter knowledge -- pedagogical content knowledge. He describes the content knowledge that a teacher should have as consisting of three categories: Subject matter content knowledge, pedagogical content knowledge, and curricular knowledge:

a) Subject matter content knowledge. This means understanding the structure of the subject matter; why a particular proposition is worth knowing and how it relates to other propositions; understanding not only that something is so but why it is so; why a given topic is particularly central to a discipline whereas another may be somewhat peripheral.

b) Pedagogical content knowledge. This means subject matter knowledge for teaching; knowing the ways of representing and formulating the subject that make it comprehensible to others; understanding what makes the learning of specific topics easy or difficult, the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning.

c) Curricular knowledge. This means understanding the curricular alternatives available for instruction; familiarity with the topics and issues that have been and will be taught in the same subject area during the preceding and later years in school.

Tamir (1987), while organizing and extending the categories suggested by Shulman and Sykes in their report for the Carnegie Task Force On Teaching As A Profession (1986), came up with a framework for teacher's knowledge, again not specific to mathematics. Like Shulman, in addition to other categories, he has specific categories for subject matter and subject matter specific pedagogical. Tamir's framework distinguishes between knowledge and skill emphasizing that the teacher needs both. For example, the subject matter specific pedagogical category (includes examples from science teaching) appears as following:

5. SUBJECT MATTER SPECIFIC PEDAGOGICAL
   - 5.1 Student
     - 5.1.a Knowledge: Specific common conceptions and misconceptions in a given topic
     - 5.1.b Skills: How to diagnose a student conceptual difficulties in a given topic
   - 5.2 Curriculum
     - 5.2.a Knowledge: The pre-requisite concepts needed for understanding photosynthesis
     - 5.2.b Skills: How to design an inquiry oriented laboratory lesson
   - 5.3 Instruction (Teaching and management)
     - 5.3.a Knowledge: A laboratory lesson consists of three phases: pre-lab discussion, performance, and post-laboratory discussion
     - 5.3.b Skills: How to teach students to use a microscope
   - 5.4 Evaluation

5.4.a Knowledge: The nature and composition of the Practical Tests Assessment Inventory
5.4.b Skills: How to evaluate manipulation laboratory skills

Tamir, as the other scholars studying teachers' subject matter knowledge, suggests that teachers knowledge must go beyond knowledge of the facts and concepts of a domain. He emphasizes that teachers should be familiar with the substantive structure and the syntactic structure of the discipline (as described by Schwab, 1964). The first refers to theories, models, concepts and facts while the second focuses on the ways, means and processes by which these accepted truths have been established. Teachers also should be able to interrelate between their subject and other related subjects. Tamir considers subject matter specific pedagogical knowledge as the knowledge that distinguishes the professional teacher from others.

Ball (1988), refers to teachers' subject matter knowledge of mathematics. She, like Tamir, conceives knowledge of mathematics along two dimensions: Substantive knowledge and knowledge about mathematics. 'Substantive knowledge' refers to understandings of particular topics, procedures and concepts, and the relationships among these -- what is usually referred to as subject matter knowledge. 'Knowledge about mathematics' includes understandings about the nature of knowledge in the discipline, e.g., how truth is established, the relative centrality of different ideas, as well as what is conventional or socially agreed-upon in mathematics versus what is necessary or logical.

Ball (1988) sees pedagogical content knowledge as a product of pedagogical reasoning. Pedagogical reasoning, which was first used by Shulman (1987), is the process of integrating knowledge. When teachers represent mathematics they are influenced by what they know and believe across different domains of knowledge: mathematics, learning, learners, and context.

To summarize, all the above agree that teachers' subject matter knowledge should include a personal understanding of the content, emphasizing understanding not only that something is so but why it is so -- procedural and conceptual knowledge. Tamir in general and Ball in relation to mathematics, emphasize also knowledge about the discipline. Teachers also need to have pedagogical content knowledge as a part of their subject matter knowledge. This should include ways of representing and formulating the subject that make it comprehensible to others so that the teacher can help someone else develop an understanding of the content as well as about the discipline.

Defining teachers' subject matter knowledge not by the number of courses they have taken or their success on standardized tests, but by analyzing what it means to know mathematics for teaching, has some promise to contributing to meaningful learning. Teachers' subject matter knowledge is not linearly related to instructional practice -- other factors influence that a well. But it is one of the major components that influences teachers' decisions. A teacher who holds a solid mathematical knowledge -- both subject matter content knowledge and pedagogical content knowledge, is more able to use his/her knowledge to help achieve meaningful learning in his/her students.

Still, analyzing what it means to know mathematics for teaching in general does not give a definite answer to questions like: What does a teacher need to know in order to teach 'functions' to secondary students? What is important about the concept of function? Theoretical models of teachers' subject matter knowledge in general are not enough to answer this kind of question. The next chapter discusses some of the specific aspects about function that seem to be important for secondary teachers to have knowledge about. These aspects comprise the purpose of this study. The second section of the next chapter will include the methods by which these aspects have being studied.

CHAPTER 4
THE STUDY -- PURPOSE AND DESIGN

Function is a central concept in mathematics in general and in the high school mathematics curriculum in particular. But studies of students' understanding of functions point to difficulties in the learning of this concept and show that many students hold a very limited concept. Teachers should help students learn and teachers' subject matter knowledge play an important role in teaching for understanding. The main goals of this study are 1) to contribute to the current discussion and analysis of subject matter knowledge for teaching by identifying important aspects of subject matter knowledge for teaching functions, and 2) to describe the kinds of knowledge prospective teachers have with respect to these aspects and to point to some of the limitations of their conceptions. The main areas of teachers' subject matter knowledge selected for this study are subject matter content knowledge -- a common personal knowledge of the subject matter, and pedagogical content knowledge -- knowledge of how to help someone else develop an understanding of the subject matter. (The terms 'subject matter content knowledge' and 'pedagogical content knowledge' are taken from Shulman, 1986.) The first section of this Chapter discusses the identification of the different aspects of teachers' subject matter knowledge being studied and their importance. The second section describes the design of this study.

Purpose

When one wants to study knowledge about a specific piece of mathematics, one is confronted with many aspects that are related to the topic. This is especially true with complex and broad concepts such as the concept of function. Work that covers every aspect of the concept of function will give a comprehensive picture of the complexity and many facets of the concept, but will be very general and miss a lot of details of each aspect. Some researchers choose to concentrate on one aspect only, studying it deeply but ignoring other aspects. This kind of work gives a very detailed picture of the knowledge of one aspect of the concept but misses all the other aspects and therefore ends up giving a limited picture of the whole concept. Other researchers choose a manageable number of the important aspects of the concept and try to illustrate a general picture including details on some aspects. This picture will give less information and details on each aspect than the previous one but, on the other hand, will give a more complete view of the whole concept. There is room in research in the mathematics education field for each one of these methods depending on the objectives one wants to achieve. This study follows the last option.

What is subject matter knowledge for teaching functions?

According to recent theoretical work and research, one component of teachers' subject matter knowledge is a personal understanding of the content including procedural and conceptual knowledge (Ball, 1988; Leinhardt and Smith, 1985; Shulman, 1986; Tamir, 1987) and also knowledge about the discipline (Ball, 1988; Tamir, 1987). But what is a personal understanding of function? In spite of the importance and centrality of the concept of function in mathematics and school curriculum, there is not much theoretical work that identifies what it means to know functions.

Freudenthal (1983) and MacLane (1986) contributed a comprehensive discussion of the many facets of the concept of function. Analysis of research on understanding functions shows that researchers (Dreyfus and Eisenberg, 1981, 1982, 1983, 1984, 1987; Karplus, 1979; Markovits et al., 1983, 1986; Marnyanskii, 1965; Orton, in Lovell, 1971; Thomas, 1975; Vinner, 1983; Vinner and Dreyfus, in press) were mainly interested in three components of the concept of function which are usually emphasized in school curricula: concept identification, process, and operations. Concept identification refers to the ability to discriminate instances and non-instances of function and to formulate a correct criterion for making such discriminations. Process refers to the ability to work with various representations and names of functions in finding images, pre-images, domain, range, and sets of images. Operations refers to the ability to carry out operations on functions, with an indication that the result of the operation is understood to be a function. Some researchers were interested in another component of the concept of function, which is usually not emphasized in school curricula -- interpretations of graphs representing situations (Bell and Janvier, 1981; Janvier, 1978, Monk, 1988).

While school curriculum should be considered when deciding about the important aspects of subject matter knowledge for teaching function, consideration should also be given to the role of function in mathematics; to research and theoretical work on learning and teaching, knowledge and understanding of functions in particular and other concepts in general; and to the specific population being studied. This study deals with prospective secondary mathematics teachers. So the decision was made to concentrate on their knowledge of aspects that are addressed in the high school curriculum, before calculus is introduced. This, of course, does not mean that the subjects could not have used knowledge they gained in college courses. They probably did use it and were expected to do that. But the aspects being studied as well as the questions presented to the participants did not require any sophisticated university mathematics, i.e., limits, derivatives, integrals, etc. were not dealt with explicitly.

As a result of analysis of various bodies of knowledge, the following six aspects of subject matter content knowledge of function were identified as very important:

* What is a function? (includes image and definition of the concept of function, univalent property of functions, and arbitrariness of functions).
* Different representations of functions.
* Inverse function and composition of functions.
* Functions of the high school curriculum.
* Different ways of approaching functions: point-wise, interval-wise, global and as

entities.

* Different kinds of knowledge and understanding of function and mathematics.

These aspects seem to form the main facets of knowledge about functions for secondary teachers as elaborated further later. In addition, aspects of pedagogical content knowledge were also examined in relation to the content aspects. When one wants to study pedagogical content knowledge about a specific concept in mathematics, one is confronted again with the problem of having to deal with many aspects of this knowledge. The following two aspects seem to have a close relationship with the aspects of teacher subject matter content knowledge described earlier:

* Teaching toward different kinds of knowledge and understanding of functions and mathematics.

* Students' mistakes - what they do and why?

The reasons for choosing these aspects were based on integration of theoretical work on knowledge and understanding of mathematics concepts in general and functions in particular, the importance of functions in mathematics and in the high school mathematics curriculum, research on difficulties in learning functions, and the specific population studied. These reasons are described in the following.

What is a function?

The match of one's subjective mental picture of a specific concept with the correct mathematical concept is one criterion for evaluating well-structured knowledge about mathematics (Greeno, 1978; Resnick and Ford, 1981/1984; Vinner, 1983). As was indicated in Chapter 2, studies show discrepancies between image and definition of the concept of function (Dreyfus and Vinner, 1982; Vinner, 1983; Vinner and Dreyfus, in press). Many students hold a limited view of the concept of function as an expression or an equation only, which fits the old definition of function, 200 years ago, but does not fit the modern definition.

Freudenthal (1983) considers arbitrariness and univalence to be the essential feature of the concept of function as it has evolved in history. Therefore, when evaluating correspondence between concept image and concept definition of function, one has to pay attention to arbitrariness and univalence.

Arbitrary functions. The 18th century mathematicians struggled with the idea of arbitrary functions. It was not until the 19th century, when Dirichlet introduced the well known now as Dirichlet function which assigns to each rational number 1 and to each irrational number 0, that arbitrary functions started to be accepted. Later, in addition to the arbitrariness of the relationship between the variables, the variables themselves or the sets on which the function is defined were allowed to be any sets - arbitrary sets.

This study seeks to investigate the following questions: What is the image (or the images) of the concept of function prospective teachers hold? Are there discrepancies between these images and the definition of function? How do they define function? Are there relationships between their conception of function and the history of the development of function? Do prospective teachers think that functions can be arbitrary? If not, do they accept the arbitrary nature of functions to some degree but reject complete arbitrariness? What specific properties do they expect functions to have?

Univalent functions. In the definition of function there is a requirement that for each element in the domain there will be only one element (image) in the range. This requirement is emphasized in almost every text definition. For example,

"A function is a relation for which no two ordered pairs have the same first element" (Coxford and Payne, 1983/1987 -- a HBJ Algebra 1 textbook).

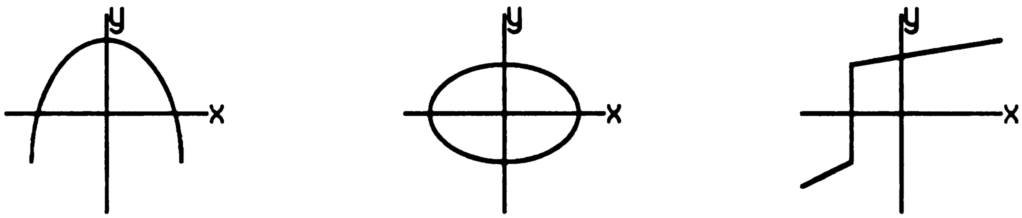
"A function is a relation in which different ordered pairs have different first coordinates" (Nichols, Edwards, Garland, Hoffman, Mamary, & Palmer, 1986 -- a Holt Algebra 2 textbook).

"Let D and R be two sets. A function from D to R is a rule that assigns to each member of D a unique member of R" (Dolciani, Sorgenfrey, Brown, & Kane, 1986 -- a Houghton Mifflin Algebra 2 textbook).

Usually, when students first meet functions they spend the first lessons learning to distinguish between functions and non-functions. For example,

> State whether the given set of ordered pairs is a function. When a relation is not a function, tell why it is not.
>
> 5. {(1,2), (0,1), (1,3), (2,1)}   6. {(0,0), (-1,-1), (2,2), (3,3)}
>
> (Coxford and Payne, 1983/1987, p. 178-9).

Or, "Which of the following are graphs of functions?"



(Keedy, Bittinger, Smith, & Orfan, 1894/1986). Similar exercises are given in most of the other textbooks which introduce functions (e.g., Nichols et al., 1986; Dolciani et al., 1986).

Thomas (1975) says that the ability to discriminate instances and non-instances of function and to formulate a correct criterion for making such discriminations is the base for understanding functions. The basic division of his stages in understanding functions hinges on the concept identification component. Subjects who can discriminate functions from non-functions and who can verbalize an adequate criterion for such discrimination are considered to be at a higher level relative to the attainment of the concept of function than are those who cannot, regardless of the subjects' performance on the other components. "For if the subject does not recognize those distinctive entities which are functions, his performance on the other two components cannot be truly functional in nature" (Thomas, 1975).

Why is this requirement so important for the understanding of function? Why should this be the cornerstone of the concept? What kinds of mistakes will one make if one does not distinguish a function from a non-function? After the first lessons of discriminating functions from non-functions, students usually do not have to use it for a long time. As the history of the development of the concept of function shows, univalence was not required at the beginning. Freudenthal (1983) attributes this requirement to mathematicians' desire to keep things manageable. Keeping track of meanings of multivalued symbols (such as $\sqrt{\ }$), and taking care and see that they have the same meaning in the same context is not easy and requires a lot of watchfulness. When one has to deal with differentials of orders higher than one, one has to distinguish independent from dependent variables, and then multivalued symbols become too messy. Advanced analysis of functions led to the restriction of functions to univalent functions only and this was generally accepted as the definition of functions.

This study seeks to investigate the following questions: What do prospective teachers think about the importance and use of the univalent requirement? Where do they think it came from? What do they think are the reasons that functions have to be univalent?

Different representations of functions

Representations play an important role in the understanding of a concept. Whenever we deal with a mathematical concept we deal with it in one of its representations. Lesh, Post and Behr (1987) identified five different types of representation systems that occur in mathematical learning: real scripts, manipulative models, static pictures, written symbols, and spoken language.

Function appears in different representations. The most common representations of function are formulae (written symbols), and graphs (static pictures). Other representations are arrow diagrams (static pictures), tables and sets of ordered pairs (written symbols), and situations from everyday life or other disciplines (real scripts). In higher mathematics, functions are often represented by a symbol only. Spoken (and written) language is also used sometimes to describe a function. While the pure definition of function may make it look like a unifying concept, its appearance in different representations makes function look like different concepts (e.g., Freudenthal, 1983; MacLane, 1986). Therefore, understanding function in one representation does not necessarily means understanding it in another representation. A more complete understanding of function means understanding it in different representations.

Lesh, Post and Behr (1987) say that "not only are these distinct types of representation systems important in their own right, but *translations* among them, and *transformations* within them, also are important". They, as well as Dufour-Janvier, Bednarz, and Belanger (1987) say that by dealing with a mathematical concept in different representations, one may grasp the common properties of the concept while ignoring the irrelevant characteristics that are imposed by the specific representation at hand.

Lesh et al. (1987) conclude that part of understanding an idea is being able to : (1) recognize this idea in a variety of different representations, (2) flexibly manipulate the idea within a given representation, and (3) translate the idea from one representation to another.

Janvier (1987) notices "that the translation 'table → formula' is often carried out as "table → graph → formula' and 'formula → graph' as 'formula → table → graph'". In other words, when students had to translate from table to formula, they translated first from table to graph and then from graph to formula. In the same manner, when they had to translate from formula to graph, they translated first from formula to table and then from table to graph. This may look as if the direct translation 'graph → formula' is easier than the direct translation 'formula → graph', but there is not enough in the literature to support that. Markovits (1982) found that ninth graders had difficulty in translating functions in both directions: from graphical representation to algebraic and vice versa. She also found that graphical representation was easier for students. But as Janvier (1978) and Bell and Janvier (1981) found, students have difficulties with the interpretation of graphs representing situations whenever pictorial aspects conflict with correct meaning.

This study seeks to investigate the following questions: What representation of function do prospective teachers prefer? Can they represent functions in different representations? Can they interpret functions in different representations? How flexible are they in translating from one representation to another? Where and why do they seem to encounter special difficulties?

Inverse function and composition of functions

Functions opened new opportunities which Freudenthal (1983) considers to be the cause for the success of functions. In addition to the typically algebraic operations of addition, subtraction, multiplication, division and raising to power, functions can also be composed and inverted. The ability to substitute functions into each other and invert them created new functions and helped with the study of differentials and integrals. Freudenthal attributes that to the explosive growth of the analysis: "the strength of the function concept is rooted in the new operations -- composing and inverting functions -- which create new possibilities" (p. 523). Therfore, understanding of the concept of function includes undrestanding of inverse function and composition of functions.

Composition of functions described by Freudenthal (1983) as created "a never before known wealth of new objects -- functions as wild as one wants to contrive" (p. 523). But studies (Lovell, 1971; Thomas, 1975) show that students have difficulties with the composition of functions, and the function obtained by composition of functions is not considered by many students as a function at all, but rather as the sequencing of assignments.

This study seeks to investigate the following questions: What conception do prospective teachers have about composition of functions and inverse function? How do they view the relationship between a function and its inverse?

Functions from the high-school curriculum

The most common functions that students meet in high school includes linear, quadratic, general polynomial, exponential, and logarithmic. In addition, many students also meet trigonometric and rational functions. Every high school text on functions includes some or all of these specific functions. An emphasis on these functions is also stated in many curriculum guides (e.g., *Academic Preparation in Mathematics*, 1985; *Curriculum and Evaluation Standards for School Mathematics*, 1988). Therefore it is reasonable to assume that every high school teacher should have a good grasp of these specific functions in particular. These functions are also used as examples and illustrations of general ideas and theorems in college mathematics and are assumed to be part of each student's and teacher's repertoire.

This study seeks to investigate the following questions: What conception do prospective teachers have about these basic functions? Can they draw upon them freely and comfortably? How well do they know them including their different general and specific properties and the relations between them?

Different ways of approaching functions

Functions appear everywhere -- in everyday life and in every branch of mathematics. MacLane (1986) and Freudenthal (1983) give many examples of functions, such as algebraic operations, trigonometric functions, isometries in the plane, and morphisms. There are different uses of functions in the different divisions of mathematics. Sometimes we have to deal with functions point-wise, i.e., to plot, read or deal with discrete points of the function only. Reading values from a given graph is an example of a point-wise approach to functions. Other times we need to look at intervals, for example, when we deal with local extremum. There are also times when we have to consider the function in a global way, and look at its behavior. For example, when we want to sketch the graph of a function given in an algebraic form or when we want to find an extremum of a function which is defined on an infinite set. And there are times when we deal with functions as entities or objects without paying attention to their behavior. For example, when we define functions of functions. Each one of these ways of dealing with functions is very different from the others and being able to handle one of them does not imply the ability to handle another.

Studies (Bell and Janvier, 1981; Janvier, 1978; Lovell, 1971; Monk,1988) show that many students deal with functions point-wise i.e., they can only plot and read points, but cannot think of a function as it behaves over intervals or in a global way. Students have a confident and secure point-wise understanding of functions, but still have to struggle in order to use functions in an across-time manner.

This study seeks to investigate the following questions: How do prospective teachers approach functions? Can they use all these four different ways when they deal with functions? If they can use more than one way in order to solve a problem, which one will they prefer?

Different kinds of knowledge and understanding of function and mathematics

Knowledge and understanding of functions. Research on knowledge acquisition brought to a common distinction between two kinds of knowledge -- conceptual and procedural knowledge. According to Hiebert and Lefevre (1986), conceptual knowledge is knowledge that is rich in relationships. All pieces of information are linked to some network. Procedural knowledge is made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics, which implies only an awareness of surface features, not a knowledge of meaning. The other part consists of the algorithms, or rules, for completing mathematical tasks, executed in a predetermined linear sequence.

Conceptual knowledge must be learned meaningfully. Procedures may or may not be learned with meaning. Procedures that are learned with meaning are procedures that are linked to conceptual knowledge. But they can be acquired and executed even if they are learned by rote. Mathematical knowledge includes both kinds of knowledge and the relationships between them. When knowledge is used dynamically to solve a problem or perform some nontrivial task, it is the relationships between conceptual and procedural knowledge that become important (Silver, 1986). People are not competent in mathematics if either kind of knowledge is deficient or if they have been acquired but remain separate entities (Hiebert and Lefevre, 1986). When concepts and procedures are not connected, people may have a good intuitive feel for mathematics but not solve the problems, or they may generate answers but not understand what they are doing.

This study seeks to investigate the following questions: What conceptual and procedural knowledge do prospective teachers have about functions? What kind of relationship between them do they have? Do they try to conceptualize a problem first or do they just try to execute a procedure without looking at the meaning?

Knowledge about mathematics. Knowledge about mathematics includes understandings about the nature of mathematical knowledge, e.g., how truth is established, the relative centrality of different ideas, as well as what is conventional or socially agreed-upon in mathematics versus what is necessary or logical (Ball, 1988; Lampert, 1988; Schoenfeld, in press; Thompson, 1984). Knowledge of mathematics refers to understandings of particular topics, procedures and concepts, and the relationships among them (Hiebert and Lefevre, 1986; Skemp, 1976, 1979) -- what is usually referred to as subject matter knowledge. Knowledge of mathematics in general and of functions in particular interacts with knowledge about mathematics. Therefore, knowledge of functions cannot be studied in isolation, without paying attention to understandings about the nature of mathematical knowledge.

This study seeks to investigate the following questions: What is the prospective teachers' knowledge about mathematics? How does their knowledge about mathematics interact with their knowledge of functions? What do they count as a mathematical explanation? How do they decide whether their answers are correct?

Teaching toward different kinds of knowledge and understanding of function and mathematics

This part is the pedagogical content knowledge aspect of the previous part. Pedagogical content knowledge is the product of melding different domains of knowledge (Ball, 1988). The way teachers teach is based on their knowledge both of and about mathematics, view of learning, the learner, and the context. For example, a teacher who thinks that learning mathematics means memorizing rules and examples is likely to teach in a different way from a teacher who thinks that learning mathematics means actively constructing one's knowledge for and by oneself.

This study seeks to investigate the following questions: How do prospective teachers think about teaching a specific piece of mathematics? What will they emphasize, when confronted with a situation where they have to explain something to a student who does not understand? Do they think of aiming at teaching for understanding or are they rule oriented?

Students' mistakes -- what they do and why?

Studies of students' mistakes are common today in research in cognitive science. The information obtained from these studies helps us understand how students interpret what they learn and construct their own knowledge. Maurer (1987) says that it is important for teachers to know that there are systematic errors many students make, to be familiar with the most common types and to look for them in their classrooms. Teachers, he says, should focus on what students are doing and why, not just on the answers they produce. This attitude to teaching is a natural consequence of accepting a (even a partial) constructivist point of view to learning. Since, as von Glasersfeld (1987) says, "the teacher's role will no longer be to dispense 'truth' but rather to help and guide the student in the conceptual organization of certain areas of experience...[one of the things teachers need in order to do this is] an adequate idea of where the student is". Shulman (1986) and Tamir (1987) also include this aspect in the pedagogical content knowledge category of teachers' subject matter knowledge.

Knowing where the students are means knowing where they are in their knowledge of the concept which includes their "right" and "wrong" conceptions. Instead of trying to eliminate the mistakes without paying attention to their sources and meaning, teachers should, as Confrey (1987) says, "seek to build from these limited conceptions towards more sophisticated conceptions". Nesher (1987) also claims that teachers should be aware of students' mistakes in order to learn about students' understanding and then connect the new knowledge to the student's previous conceptual framework.

This study seeks to investigate the following questions: What do prospective teachers know about students' common mistakes when dealing with functions? What are their interpretations of students' mistakes when presented with them? What do they think the sources of these mistakes are? What are the relationships between research findings of students' mistakes (as described in Chapter 2) and prospective teachers' understanding of them?

Interrelations

The six aspects of subject matter content knowledge and the two aspects of pedagogical content knowledge described above are parts of teachers' subject matter knowledge. The aspects are not independent from one another, but rather they are interrelated. For example, solving a problem that deals with a quadratic function involves knowledge about functions of the high school curriculum but also requires knowledge of different representations of functions as well as different ways of approaching functions. It also depends heavily on different kinds of knowledge and understanding of function and mathematics. While each one of the aspects can be highlighted separately, ignoring the interrelations among them will create a misleading picture of teachers' subject matter knowledge. The interrelations among the aspects add another dimension to the analysis of subject matter knowledge, helping to create a more complete picture of the participants' knowledge.

Design

Population and sample

General background. The participant subjects in this study were prospective secondary mathematics teachers in the last stage of their professional education. They were finishing or had already finished their mathematics method class. Almost all of them were seniors, a few were juniors or postbaccalaureate students. This group was selected so that the description of their knowledge would reflect the knowledge teachers have gained during their college education, but before they start teaching. The subjects came from eight mid-western universities: Ohio State University, University of Wisconsin -- Madison, University of Iowa, Indiana University, Michigan State University, Central Michigan University, Western Michigan University, and Northern Michigan University. Five of the universities are Big Ten universities with large teacher education programs. The other three are state universities, previously teacher colleges, with an emphasis on teacher education. Seven of the universities participated in the first phase of the study and two in the second phase. The subjects of the first phase were the students enrolled in a math methods class. The selection of the universities for this phase was made according to the math methods instructors' cooperation and willingness to devote one hour of their class time to administering questionnaires to their students. The selection of the subjects for the second phase of the study was made according to their willingness to spend about three hours of their free time answering a questionnaire and being interviewed about functions. An attempt was made in the second phase to get a variety of students representing different groups in the prospective secondary mathematics teachers population by means of their responses, sex, age, academic background and grades. Table 4.1 presents the distribution of the first phase sample in each of the seven universities by sex and by age. (The order in which the universities appear was randomly selected.)

Table 4.1 -- Distribution of First Phase Subjects by Universities, by Sex and by Age

	Sex			Age					
Univ.	Male	Female	N/R	19-23	24-29	30-35	over 35	N/R	Total
1	6	7	2	8	2	3	--	2	15
2	14	12	--	20	5	1	--	--	26
3	16	18	1	22	9	1	2	1	35
4	18	14	--	20	8	1	3	--	32
5	11	13	1	20	3	1	--	--	25
6	2	2	1	3	--	--	--	2	5
7	6	8	--	12	2	--	--	--	14
Total	73	73	6	105	29	7	5	6	152

Note. N/R = No Response.

We can see that most of the subjects were in their early twenties, and the distribution of male/female is about the same. This was the case in general as well as at each university.

Table 4.2 presents the distribution of the second phase sample in each of the two universities by sex and by age. We can see again that most of the subjects were in their early twenties, and the distribution of male/female is the same.

Table 4.2 -- Distribution of Second Phase Subjects by Universities, by Sex and by Age

	Sex		Age				
Univ.	Male	Female	19-23	24-29	30-35	over 35	Total
1	3	4	5	2	--	--	7
2	2	1	1	1	1	--	3
Total	5	5	6	3	1	--	10

Academic background. About two thirds of the subjects in the first phase of the study had an over all college grade point average between 3.0 and 4.0 (on a scale of 0 to 4). The math grade point average was a little lower -- about half of the subjects in the first phase had a math grade point average between 3.0 and 4.0. Two thirds of the ten subjects in the second phase of the study had both their over all college and math grade point averages between 3.0 and 4.0.

The number of different college math courses the first phase subjects have taken ranged from three to fourteen where six to eight math courses were the most common -- an average of 7.6 math courses. (A sequence of courses on the same topic, and courses on the same topic but in different levels were considered as one course.) About one third of these subjects have taken an advanced calculus course. The number of different education classes taken by the first phase subjects ranged from one to seven where three to five were the most common -- an average of 3.5 education courses. More than two thirds of the subjects had had some field experience, usually pre-student teaching.

The second phase subjects have taken between four to eleven different math courses -- an average of 8.7 math courses. Two thirds have taken advanced calculus. The range of education courses they have taken was between one and six -- an average of four education courses. Here again two thirds of the subjects had had field experience, but in this case -- student teaching. Generally speaking, an average second phase subject had stronger mathematics and education background than an average first phase subject, especially by means of experiencing advanced calculus and student teaching.

Instrumentation

One of the problems in doing educational research of the kind of this study is the price one has to pay when one chooses an instrument and a sample size. In order to generalize the findings and be able to say something about the whole population, one needs to study quite a few subjects. An appropriate instrument for studying a large sample is a short written instrument. Information gathered through this kind of instrument is sufficient for a general description of some facets of the subjects, but is limited and sometimes hard to interpret. In order to get a better understanding of the meaning of the findings and a more detailed picture of what is really going on with each subject, one needs to spend more time with each subject, observing, listening and/or asking questions. Naturally, the amount of data with which one has to deal in this kind of study forces the number of subjects to be much smaller and the generalizability of the findings is questionable This study attempted to partially resolve this dilemma by using two phases. The first phase aimed at getting a general picture of the population's knowledge, and the second phase aimed at clarifying this picture and adding more details. The instruments used for that were a questionnaire for the first phase and the same questionnaire and an interview for the second phase. (Gorden, 1980, describes a combined use of a questionnaire and an interview as an appropriate research strategy when there is a need to complete or clarify the answers to the questionnaire.) Using the same questionnaire for both phases was assumed to contribute to the generalizability of the findings from the second phase by comparing the answers to the questionnaires in both phases. The questionnaire and the interview were developed for this study.

Questionnaire. In the first phase, an open-ended questionnaire of 45-60 minutes (appears in Appendix A) was administered to the subjects in their class. They were not asked to fill in their names. The questionnaire included math problems, addressing the different aspects of subject matter knowledge described earlier, and "students'" mistaken solutions to be analyzed. Both parts' problems were based on students' limited conceptions and systematic mistakes as described in the literature, learned in personal communication with scholars in math education, or known from personal experience with students.

The aim of this study was to describe the kind of knowledge prospective teachers have and to point to some of the limitations of their conceptions. The study did not aim at measuring performance and success of the topics: function and teaching function. Therefore, no attempt was made to create an instrument that would measure performance. In order to characterize people's subject matter knowledge, one needs, as Confrey (1987) says, to build a model of what they believe about the topic by "creating an outline of the boundaries of the belief system". The outline of these boundaries can be created only when the subject's responses differ from the ones expected by the researcher (either because they are wrong or because they are not rich as they can be). So, the subject matter problems presented to the subjects were chosen to be nonstandard problems. It was assumed that in non standard situations, the subject would reveal more clearly his/her conception's boundaries since it would be harder to try to recall a familiar ritual solution without understanding in this case. Since both the method of solution and the final answer reflect the subjects' knowledge, the prospective teachers were asked to explain their answers to each question on the questionnaire.

Both the math problems and "students'" mistaken solutions were designed to address the different aspects of teachers' subject matter knowledge described earlier. The first aspect -- what is a function? -- was mainly addressed in questions 1, 3, 4 and 11, although all the questions in the questionnaire touched upon this aspect to some degree. The questions were designed to explore prospective teachers' definitions and images of the concept of function, their understandings of the arbitrariness and univalence characteristics of function, and the relationship between their mental picture of function and the "historic" function from the eighteenth century as an expression or equation only.

The second aspect -- different representations of functions -- was again addressed by almost all the questions and mainly by questions 2, 6, 8, 9, 10 and 11. Two representations which considered to be the most common and useful ones were dealt with in depth: symbolic and graphic. Situations, set of ordered pairs and natural language also appeared. The questions were designed to explore prospective teachers' understanding of each one of the representations and also how flexible they are in translating from one to another, using different ways of approaching functions. The questions addressed also the relationships between the coefficients of an expression and the shape of the correspondence graph, and how the graph of bf(cx+a) + d relates to the graph of f(x).

The third aspect -- inverse function and composition of functions -- was addressed in questions 12, 13, 14 and 15. The questions were designed to explore the prospective teachers' knowledge and understanding of these two issues in different situations which can be approached in different ways: point-wise, interval-wise, global or as an entity.

In addition to the above three main aspects, four of the other five aspects were also addressed in the questionnaire ('teaching toward different kinds of knowledge and understanding of functions and mathematics' was not addressed). Knowledge about functions of the high school curriculum -- was mainly addressed by questions 5, 6, 7 and 14. The functions emphasized were linear, quadratic, sin, power, exponent and log.

Different ways of approaching functions: point-wise, interval-wise, global and as entities -- was addressed by most of the questions, and mainly by 2, 5, 6, 8, 9, 10 and 15. An attempt was made to design some questions that can be approached (correctly) in different ways, and some questions that are better approached by one way only, in order to discover the prospective teachers' preference and ability. Since the literature (Bell and Janvier, 1981; Janvier, 1978; Lovell, 1971; Monk, 1988) and a pilot study conducted for the purpose of this study show that usually students do not have problems with the point-wise approach to functions, emphasis was put on the other three: interval-wise, global , and entity.

Different kinds of knowledge and understanding of function and mathematics -- was also addressed by most of the questions, and mainly by 2, 5, 12, 13 and 15. An attempt was made to design the questions so that they can be approached either conceptually or procedurally, in order to learn better what prospective teachers know and what they can do.

Students' mistakes: what they do and why -- was addressed through students' mistaken solutions. Part of the questionnaire, questions 3, 7, 8, 11 and 14, was designed to explore what prospective secondary teachers think about students' mistakes in functions, to find what they think students do and why. The prospective teachers were presented with "students'" mistaken solutions to math problems on functions. They were asked first to decide if the student was right or wrong and to give reasons for their decision. In most cases they were also asked what they think the student had in mind when he/she answered that way. The mistaken solutions were again chosen according to students' limited conceptions and systematic mistakes as described in the literature, learned in personal correspondence with scholars in math education, or known from the researcher's personal experience with students. The following mistakes were addressed in the questionnaire:

(1) Linear prototypical image of functions and the tendency of students to be satisfied with the finding of one function that satisfy certain conditions while ignoring all the other infinite number of appropriate functions (e.g., Dreyfus and Eisenberg, 1983; Markovits et al., 1983, 1986) were addressed in question 3.

(2) Some students tend to relate proportionally the slope of a linear function to the angle formed by that line and the x-axis (as described in a personal communication with Avraham Arcavi, in August 1987, when he was visiting Schoenfeld' group at the University of California in Berkeley). This was addressed in question 7.

(3) Problems with interpretation of global graphical features whenever pictorial aspects conflict with correct meaning (Bell and Janvier, 1981; Janvier, 1978) were addresses in question 8.

(4) Discrepancies between definition and image of the concept of function (Dreyfus and Vinner, 1982; Vinner, 1983; Vinner and Dreyfus, in press) were addressed in question 11.

(5) Difficulties with exponential, log and root functions, and the relationships between them (personal experience with students) were addressed in question 14.

Before the questionnaire was used for this study it had been through several phases of piloting and modification. The piloting was done with prospective secondary mathematics teachers in their senior year and also with graduate students in mathematics and math education. Attention was given to the clarity of the questions asked, the quality of the answers given, and the time needed to complete the questionnaire. As suggested by Gorden (1980), interviews were also conducted with the pilot subjects as an exploratory tool in building a valid questionnaire. For this reason, experts in math education and mathematics were also consulted. Questions that were neither math problems nor did they ask for explanations of student's mistakes, were found to be not informative in a short written questionnaire. So some of the questions that required more general, longer or more thoughtful responses, were taken out of the questionnaire and were put in the interview.

Interview. In the second phase of the study an interview of about two hours was conducted with each subject (see Appendix B). Information gathered from a written questionnaire is sufficient for a general description of some facets of the prospective teachers' knowledge, but is limited and sometimes hard to interpret. In order to overcome these difficulties the second phase included interviews with ten subjects. By probing, asking subjects to explain what they did and why, asking for their reactions as teachers to students' misconceptions and asking questions which are related to the questionnaire but require more general, longer or more thoughtful responses, a more accurate and detailed picture of the subjects' subject matter content knowledge and pedagogical content knowledge may be developed. So the interview was used to clarify the answers to the questionnaire and at the same time as an instrument to gather information on questions that were too difficult to answer via a written questionnaire.

The interview consisted of three parts: (1) questions that did not appear on the questionnaire, (2) going through the answered questionnaire, and (3) card sort. Together, the three parts addressed the aspects of subject matter content knowledge and the pedagogical content knowledge described earlier with more emphasis on the latter. Here again, as with the questionnaire, the questions were based on students' limited conceptions and systematic mistakes as described in the literature, learned in personal communication with scholars in math education, or known from the researcher's personal experience with students. In addition, the interview questions were also based on what was learned from the analysis of the first phase questionnaires.

(1) Questions that did not appear on the questionnaire: This part (questions 1-6 in Appendix B) included some of the questions that were first planned to be on the questionnaire but had to be taken out since they required more general, longer, and thoughtful responses (such as questions 2 and 6) or because they needed a more flexible

structure that only an interview can provide (such as question 1). These were usually questions that dealt with some aspects of the prospective teachers' pedagogical content knowledge although they were also related to their subject matter content knowledge. This part of the interview included also questions about issues that emerged as important after analysis of the first phase questionnaires, such as the role of the vertical line test for graphs of functions (question 3) and the relationship between a graph of a function and the graph of its inverse (question 4).

(2) Going through the answered questionnaire: In this part (question 7) the interviewees were asked to reflect on their thinking, and to explain and clarify their answers to the questionnaire. In addition to their free, voluntary explanation, they were also probed in two different ways: uniform and non-uniform. The uniform probing included questions that were presented to all the subjects. Most of the questions were identified as important after analysis of the first phase questionnaires. They represented themes that appeared in many of the written answers (such as arbitrary versus specific functions), and meant to clarify ambiguities (such as the meaning of 'many' -- finite or infinite). The uniform probing included also presentations of ideas and misconceptions that were found in some of the answers (such as the confusion between 'further' and 'faster' on a given graph), asking the subjects first to relate and evaluate them, and then to explain what they think the "student" had in mind when he/she answered that way. The non-uniform probing included questions that were based on the specific answers each subject gave to the questionnaire. They were meant to clarify ambiguous answers and to discover specific dimensions that seemed important.

(3) Card sort: In this part (question 8) the interviewees were presented with a set of ten cards. On each card there was a statement that described a function. The subjects were then asked to sort the cards and order them. This card sort was meant to force the subjects to relate to different descriptions of function -- some of them are true for all functions, some of them only for some -- in order to get a better understanding of what a function is for them.

The three parts of the interview: questions that did not appear on the questionnaire, going through the answered questionnaire, and card sort, were basically used in that order. Since the process of reflection that takes place during the interview also causes learning (e.g., Confrey, 1987), questions that did not appear on the questionnaire were used before the discussion of the answered questionnaire. The first question from the questionnaire, about the definition of function, was discussed at the very end to eliminate influence on the card sort.

As with the questionnaire, before the interview was used for this study it had been through several phases of piloting and modification. The piloting was done with prospective secondary mathematics teachers in their senior year and also with graduate students in mathematics and math education. Attention was given to the clearness of the questions asked, the quality of the answers given, and the time needed to complete the interview. Expert validity was used again, by consulting experts in math education and mathematics.

Procedure and data collection

Data collection for this study was conducted from November 1987 to April 1988. The administration of the first phase questionnaires to the 152 subjects took place between November 1987 and January 1988, in a regular math methods class, by the regular instructor of that class. Data collection for the second phase -- the interviews -- was conducted from February to April 1988, and took place in two consecutive days. Each of the ten subjects answered the questionnaire first and was interviewed a short time -- usually one day -- after that. This was done so that the subjects would easily recall their answers to the questionnaire and the reasoning behind them. The researcher administered the questionnaires and conducted the interviews in the second phase.

The following data were collected from the whole sample of 162 subjects (152 subjects from the first phase and 10 from the second) in eight universities:

(1) General information. The following information was gathered from the subjects: gender, age, college grade point average in general and in mathematics, mathematics and education courses taken at the college level and minor. This information was collected in order to describe some characteristics of the prospective teachers.

(2) Questionnaire. This included non-standard open-ended math problems on function and students' mistaken solutions to problems to react and evaluate. The subjects were instructed to explain the decisions they made and the ways they chose to solve the problems. They were not allowed to use any resources and had 45-60 minutes to complete answering. This information was gathered in order to learn about the general knowledge and understanding of some aspects of functions and teaching functions prospective teachers have.

In addition, the following data were collected only from the second phase sample of ten subjects in two universities:

(3) Interview. This included questions on functions and teaching functions that did not appear on the questionnaire, "going over" the answered questionnaire, and a card sort task. Probing was an important component of the interview procedure, in order to understand why people said or did something. Some of the probes were standard -- a lot of "why?", "what do you mean by that?" and "can you give me an example?". Some of them were specific to the given situation. The interview sessions were audiotaped to insure an accurate record of what was said.

Data Analysis

Questionnaires. The analysis of the questionnaires aimed at getting a general picture of aspects of the prospective teachers' knowledge and understanding by means of the variety of different answers (correct and wrong) and also by studying the existence of patterns and dominant answers, with relation to the historical development of function and students' difficulties and limited conception as described in the literature.

The analysis began by surveying about 40 questionnaires out of the 152 -- a few from each of the seven universities of the first phase -- in order to gain an impression about the responses. This information served as a tool for improving the interview, on one hand, and for creating preliminary categories of responses for each question and sub question on the other hand. These preliminary categories were mostly created directly from the different responses for each question, with an attempt to avoid imposing any preconceptions of what the answers might or should be. The reason for choosing this way of preliminary analysis was "to let the data talk", i.e., to present an accurate picture of the different answers and to be open to unexpected directions.

These preliminary categories were then examined and modified consulting experts in math education. Similar categories were combined, and new categories that are considered to be important according to what is known about students' difficulties and limited conception of function as described in the literature or in relation to the historical development of function were added. At the end of this process, each of the questions and sub-questions of the questionnaire had a list of modified categories related both to the answers and the procedures used to obtain those answers and to the themes that emerged from the eight aspects of subject matter content knowledge and pedagogical content knowledge.

The new categories were used for the analysis of the questionnaire data. Sometimes, in the process of analysis, a need arose for adding a few categories. After refinement, the final categories were used to analyze all the questionnaires. No attempt was made to make the number of categories small in that stage, but rather to postpone that until viewing the preliminary results. Experts in math education were also consulted in that stage and when faced with problematic cases. This is the place to emphasize that in many cases an answer was categorized several times: by the final answer, the method used to get that answer, attention given to important issues (in relation to different aspects of subject matter knowledge), or common issues (to many of the questionnaires). *D-Base 3* for the microcomputer was used for summarizing and further investigation of the data, by creating frequency tables first for each question and sub-question and then across related questions.

Interviews. The analysis of the interviews aimed at clarifying the answers to the questionnaire, and getting a more detailed picture of the prospective teachers' knowledge and understanding of both subject matter content knowledge and pedagogical content knowledge. This, again, was done with relation to the historical development of function and students' difficulties and limited conception as described in the literature.

The interviews were transcribed in order to get an easy access to the data. The analysis began by listening to the taped interviews and editing the transcripts. The next steps were analyzing the interview transcripts by person, by question and by theme.

By person: Each person's answer to an interview question was summarized. In addition, other significant comments were recorded.

By question: A summary table for each interview question was created across subjects. Attention was given to several dimensions, such as the final answer, the method used to get that answer, mistakes made, ideas, preferences, ways of evaluating students' mistakes, and other significant comments.

By theme: The above analyses were used for thematic analysis. The themes correspond to the eight subject matter content knowledge and pedagogical content knowledge aspects being studied:

* What is a function?
* Different representations of functions.
* Inverse function and composition of functions.
* Functions of the high school curriculum.
* Different ways of approaching functions: point-wise, interval-wise, global and as entities.
* Different kinds of knowledge and understanding of function and mathematics.
* Teaching toward different kinds of knowledge and understanding of functions and

mathematics.

* Students' mistakes - what they do and why?

Every evidence related to the aspects was recorded while looking for patterns, relationships and contradictions.

Since the aspects of subject matter knowledge for teaching functions are interrelated, results about one aspect interacted, usually, with results about other aspects. Therefore, the first three aspects -- what is a function?, different representations of functions, inverse function and composition of functions -- were chosen to be the main focus of the description of the results. Each chapter is devoted to one of these aspects. Results about the other five aspects of teachers' subject matter knowledge are interwoven with results of the three main aspects, making the whole picture more vivid and complete. The results of these analyses are reported in Chapters 5 through 7. Reported first, in Chapter 5, are results about the aspect 'what is a function?'. Chapter 6 includes the results about the aspect 'different representations of functions' and Chapter 7 -- the results about the third main aspect 'inverse function and composition of functions'.

CHAPTER 5

WHAT IS A FUNCTION?

Function has a formal mathematical definition: Any set of ordered pairs of elements such that if (a, b) and (c, d) belong to the set and a=c, then b=d. This modern definition grew out of the movement to use set theory to develop rigorous foundations and structure for mathematics. A more workable conception of function is the Dirichlet-Bourbaki definition: A correspondence between two sets that assigns to each element from the first set exactly one element from the second set. As was indicated in Chapter 2, there are discrepancies between definition and image of the concept of function among high school and even college students. Many students hold a limited view of function as an expression or an equation only. This view fits the old definition of function, 200 years ago, but does not fit the modern definition. Today, the arbitrary nature and the univalent characteristic of function are considered to be the essential feature of the concept of function as it has evolved.

To prospective teachers, what is a function? How would they explain what a function is to students? What do they think of misconceptions that students have about the image of the concept of function -- both their nature and sources? This chapter deals with these questions. It describes the prospective teachers' subject matter content knowledge and pedagogical content knowledge in relation to what a function is. It starts with a description of the ways in which the participants chose to define function. Next, discrepancies between concept definition and concept image of function are discussed. The acceptance of the arbitrary nature of both the relationship between the two sets on which the function is defined and the sets themselves is described. Participants' understanding of the univalent property of function is examined as well. The next section of the chapter discusses ways in which the prospective teachers would explain what a function is to students. The chapter ends with the results of the participants' reactions to students' misconceptions in relation to the nature of function.

Discrepancies between Concept Definition and Concept Image of Functions

The definition of functions has changed during the last two centuries. While an 18th century function was an analytic expression, representing the relation between variables, with its graph having no "sharp corners" (referred to as Euler's definition), in the 20th century function came to be a subset of the Cartesian product of two sets in which each member of the domain is paired with exactly one element of the range, or a less formal definition: A correspondence between two sets that assigns to each element from the first set exactly one element from the second set (Dirichlet-Bourbaki). Two centuries ago functions were equations that described the relation between two variables using expressions. Today's definition of functions is less limited. Functions do not have to be graphable, nor be representable by equations, and their domain and range may be sets of objects other than numbers.

An informal and non-representative survey of mathematicians about what a function is was conducted for the purpose of this study. The survey yielded variations of the mathematical definition: A set of ordered pairs such that..., a mapping from one set to another such that..., a correspondence between two sets such that... .

When the prospective teachers were asked to define a function (question 1 from the questionnaire, see Appendix A), they gave a large variety of answers. To analyze those answers, an attempt was made to use Vinner and Dreyfus' (in press) six categories for function definition: correspondence (Dirichlet-Bourbaki definition); dependence relation (between two variables); rule (having some regularity, the domain and range usually not mentioned); operation (acts on a given number, generally by means of algebraic operations); formula (algebraic expression, equation); and representation (graph or meaningless identification with notation). However, these categories were found to be not useful for this study. Most of the definitions that the participants gave were not clear enough to make a definitive judgment, and it was difficult to determine the exact meaning of the words used. In addition, a large number of definitions seemed to belong to more than one category. A decision was made, therefore, to categorize the answers into more global categories. No attempt to evaluate the correctness of the definitions was made in this process of categorization.

Two main categories were selected for this analysis. One represents a modern view of function with an emphasis on arbitrary nature as one characteristic of functions. The other represents an older view of function with an emphasis on specifics and some regularity in the behavior of the function. The two categories are:

1. A modern definition which includes relation, correspondence and mapping, with some reference to set theory characteristics and the arbitrary nature of function. The following are examples of responses that were categorized into this category.

- "A function is a set of ordered pairs (x, y) that have different x values but may or may not have the same y value. Ex (2, 3) & (-2, 3) is a function."
- "A function is a one-to-one and onto correspondence that maps a number from one set onto a number from another set."
- "A function is a mapping from one set to another."

2. An old definition which includes dependence relation of variables, rule, operation and formula, usually with reference to variables and regularity of the function behavior. The following are examples of responses in this category.

- "It is $f(x)=x$ where value of y is placed into $f(x)$ & x is determined from it."
- "A function is a relationship between coordinates that meet certain requirements of smoothness, and...."
- "An operation on a set of numbers that gives out other numbers. For each number you operate on, a different value comes out."
- "An expression which yields an output or answer when given the appropriate input."

Table 5.1 summarizes the first phase subjects' definitions of function. In cases where definitions seemed to include some characteristics from both categories, they were categorized into the modern definition category.

Table 5.1 -- Distribution of Function Definitions of the First Phase Subjects.

Modern	Old	Other	N/R	Total
78	53	11	10	152

Note. N/R = No Response

Table 5.1 shows that about one third of the participants defined function in old terms and about one half in modern terms. However, the fact that one chooses a particular way to define function means neither that this is his/her only conception of function nor that this is even a component of his/her conception of functions. Defining function in modern terms does not necessarily reflect a modern image of the concept of function. One prospective teacher, Bert, illustrates this point in his example of functions. He suggested the use of different intervals in order to get different representations for $f(x)=x$. He was not aware that this change in domain and range would give him different functions rather than different representations of the same function. He did not consider domain and range as part of the function. For him, the function was only the expression that described the behavior of the function. At the same time, Bert gave a rather good and modern definition of function: "A function is a relation (a set of ordered pairs) such that for each value in the domain there corresponds exactly one value in the range." By changing the domain, the set of ordered pairs is also changed, and therefore the function does not remain the same. But Bert did not use his own definition, since it did not match his concept image of function.

Valerie also showed signs that her definition and her image of the concept of function were not the same. She defined function in a way which was close to the modern definition: "a 1-1 mapping of a set of points x onto y." But during a card sort task of statements that describe functions, she chose all ten statements as descriptions of all functions:

1. A function is a correspondence between 2 sets.
2. A function is a rule.
3. A function is a formula.
4. A function is a set of ordered pairs.
5. A function is a graph.
6. A function is a dependence of one variable on another.
7. A function is a relation.
8. A function is an equation.
9. A function is a relationship between 2 variables.
10. A function is a mapping from one set to another.

So, for Valerie, all functions were mappings -- which apparently points to a modern conception of functions, but at the same time she thought that all functions were equations, formulae and graphs, which points to an older and not so comprehensive conception of functions.

These results -- discrepancies between concept definition and concept image of function -- are similar to results of other studies on high school and college students (Malik, 1980; Markovits et al., 1986; Marnyanskii, 1965; Vinner and Dreyfus, in press) which showed that while students are being taught the modern definition of function, the old one serves as the concept image for these students. The results of this study show that this was the case with many prospective teachers. The limited concept image of function held by Bert, Valerie and others, did not, usually, include arbitrary functions. This is discussed in detail below.

Arbitrary nature of functions

The arbitrary nature of functions refers to both the relationship between the two sets on which the function is defined and the sets themselves. The first means that functions do not have to be described by any specific expression, follow some regularity, or be described by a graph with any particular shape. The function that describes the relationship between time and temperature is an example of this kind of function. Additional examples can be found in question 11 in the questionnaire (see Appendix A). Arbitrary nature of the two sets means that functions do not have to be defined on any specific set of objects; in particular, the sets do not have to be sets of numbers. Rotation of the plane is an example of this type of function since it is defined on points. Question 15 from the questionnaire also deals with this kind of function.

The prospective teachers' understanding of the arbitrary nature of the relationship between the two sets on which the function is defined is examined in detail below, including a discussion of their acceptance of the arbitrary nature of the sets themselves.

The role of equations in the definition and image of functions. Equation, expression and formula seemed to be the concept image of function for many of the participants. A large group of participants whose definitions of function were categorized into the second (old) category defined function as an equation, an algebraic expression, or a formula (33 out of 53).

- "A function is an equation with a one-to-one correspondence between the variables."
- "An equation which satisfies the following requirements..."
- "A function is a numerical expression that ..."

A strict categorization was employed to subjects' responses. Unless the words "equation", "algebraic (numerical) expression", or "formula" were mentioned, responses were not included.

Even people who did not define function as an equation, may have been conceiving function that way. This became apparent from the answers to question 4 from the questionnaire:

****

How are functions and equations related to each other?

****

In addition to the 33 subjects who defined function as equation (formula or expression), 26 subjects said that functions are equations or that rules for functions are equations (without any additional remarks that some functions may not be representable by equations).

__ "All functions can be written as equations, but not all equations are functions."

__ "They're the same thing."

__ "A function is really an equation."

Together, a large number of the participants (59 out of the 126 who answered both questions) showed signs of having equations as the concept image of functions. This tendency was also apparent from the examples of functions that the second phase subjects gave during the interview: 8 out of 10 started with equations. They seemed to know that functions can be described by different representations, but thought of equations as the "real" functions. It was not just that equations or expressions were the first representation the participants thought of, but most of them believed that <u>all</u> functions can be represented by using a formula. Bob, as well as six other second phase subjects (out of 10), expressed this belief.

> B: Yes, I think you could write all functions in terms of equations. It might be a trigonometric equation, like sin(x), but in every term the y value is going to be equal to some operation with x value.
>
> R: So you can always describe a function using a formula or an equation?
>
> B: I think so.

The belief that all functions can be described by a formula shows a limited understanding of function. It suggests that many of the participants in this study did not understand one of the most important characteristics of function in the 20th century -- the wide scope of arbitrary objects that cannot be described by an expression (or even by several expressions) but are included now under 'function'. A similar limited conception of function was held by some of the participants who expected graphs of function to be smooth and "nice", and others who accepted as functions only "known" objects (either to themselves or to mathematicians). This is discussed further in the next sections.

Graphs of functions should be "nice". Some of the prospective teachers expected graphs of functions to be "nice" and smooth. Signs of that were shown in some of the definitions of function which were given in question 1.

— "A function is a relationship between coordinates that meet certain requirements of smoothness, and...."

But having this requirement as part of the definition of function was rather rare. The feeling that graphs of functions should be "nice" was, usually, part of the concept image of function, not part of the definition. Brian, for example, when having to decide whether the following was a function (question 11(v)):

****

$$g(x) = \begin{cases} x, \text{ if } x \text{ is a rational number,} \\ 0, \text{ if } x \text{ is an irrational number.} \end{cases}$$

****

based his decision that it was a function on the correct definition of function: "There is an assignment of a single value to each number." But later, when he sketched the correspondence graph (see Figure 5.1):

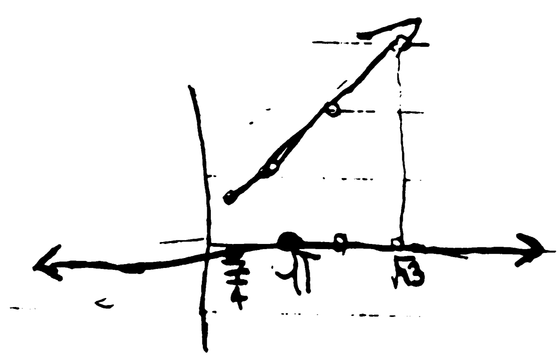


Figure 5.1 -- Brian's Graph of Question 11(v) from the Questionnaire.

he had a hard time deciding whether this graph was a graph of a function.

> I don't know if it's a function. It fits the criteria of mapping, but it does not look pretty. It's not really graphable. It might just not be. But this is a

discontinuous function. You're allowed to do discontinuous. There aren't sharp points. Oh, well...

Brian was not sure about discontinuous functions which looked very "weird". But he had no difficulties in deciding that the discontinuous graph in question 11(i) (see Figure 5.2) was a function: "Well, it passes the vertical line test". So his difficulty was

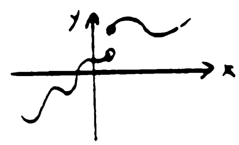


Figure 5.2 -- Question 11(i)' s Graph from the Questionnaire.

not discontinuity but "weirdness". Eighteen first phase subjects (out of the 130 who answered this) decided that Figure 5.2 was not a function because it was not continuous. For this group of people, discontinuity was the problem. They accepted as functions only continuous functions and rejected a large set of other functions.

Brian mentioned, while thinking about the graph in Figure 5.1, that "there aren't sharp points in the graph." Another kind of the"niceness" expectation seemed to be not to have "sharp points" or "sharp corners" in a graph (such as in Figure 5.3 which was presented to the participants during the interview).

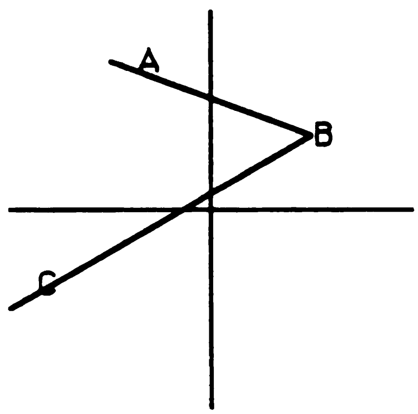


Figure 5.3 -- A graph Presented During the Interview

During the interview, Brian rejected the graph in Figure 5.3 as a function but not for the reason that some of the elements in the domain had more than one image. He said:

> ...when the function should be continuous, it shouldn't have any sharp breaks, where all of a sudden it just flips and reverses. Um, it doesn't pass the vertical line test, but a lot of functions don't do that.

Brian accepted discontinuous functions (like the one in Figure 5.2) as functions but rejected continuous functions which are non-differentiable. Tracy expected all functions to be continuous and smooth (differentiable) and used that as an argument to decide that the graph in Figure 5.3 was not a function.

> ...but it's not continuous at the point B. So the function is not continuous, so it's not a function. Plus, it does not pass the vertical line test either. So, it's two points that, you know, make it fail being a function.

Jenine also referred to the graph in Figure 5.3 as not continuous, but was willing to accept it as a function. This confusion between continuous and differentiable functions is similar to the debate about functions and continuous functions from the 18th century. At that time, only differentiable functions were accepted as functions. Later, continuous (in modern sense) but non-differentiable functions were also accepted. Those functions were called then discontinuous functions, referring not to discontinuity in the graph but rather to "discontinuity" in the expression that described the graph (several expressions instead of only one).

It is clear from the data that at least 18 first phase subjects expected graphs of functions to be continuous. It is not clear how many were willing to accept discontinuous graphs but still expected graphs not to be too weird or have "sharp corners" since only the interview data gave this information.

Functions are "known". Rejection of arbitrary functions was also detected in the answers people gave to question 3 from the questionnaire (see Figure 5.4). It seemed, first, that many participants understood that there were infinitely many arbitrary functions that pass through the given points.

__ "There are infinitely many functions, e.g., 2 parabolas $\cup$ or $\cap$ would work, or any 'curve' that goes through A & B that satisfies the definition of a function."

****

A student is asked to give an example of a graph
of a function that passes through the points A and B
(See fig.1).
The student gives the following answer
(See fig.2).
When asked if there is another answer the student says: "No."
If you think the student is right - explain why.
If you think the student is wrong - how many functions
which satisfy the condition can you find? Explain.

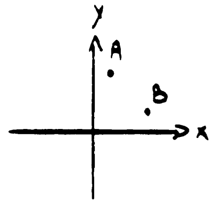


fig.1

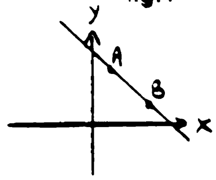


fig.2

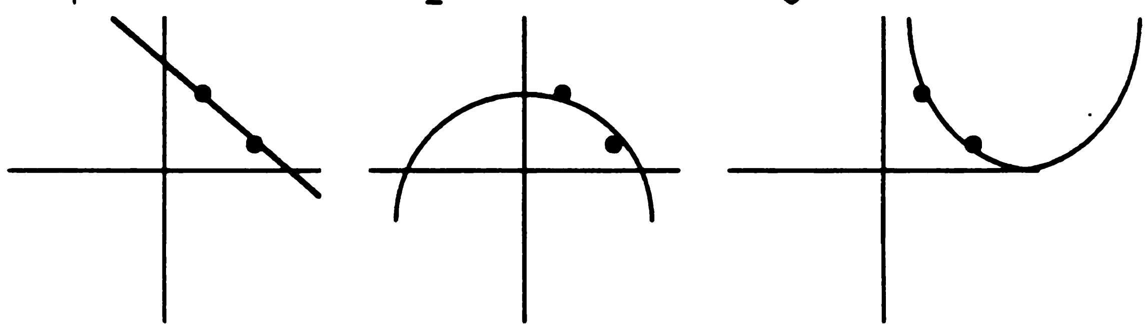
****

Figure 5.4 -- Question 3 from the Questionnaire

Only 12 of the participants said that the number of functions that pass through the two given points was finite.

— "I can think of three types offhand."

1 2 3



Still, many of those who said that the number of functions was infinite, explained it by referring to specific examples of functions. They did not use the characteristic of arbitrary functions.

— "There are infinite parabolas that would satisfy the conditions."

Table 5.2 summarizes the answers of the first phase subjects by the number of functions they thought could go through two points and by their explanation of their choice: whether they referred to arbitrary functions, or used specific examples of functions only in their explanation.

Table 5.2 -- Distribution of Answers by the Number and Kinds of Functions that Can Pass Through Two Points

	Infinite	Finite	N/R	Total
Arbitrary	35	0	5	40
Specific	25	12	20	57
N/R	41	0	14	55
Total	101	12	39	152

Note. N/R = No Response

Giving specific examples, without referring to arbitrary functions, does not necessarily mean that the person does not know this aspect of function. In order to investigate this further, the second phase subjects were asked how many functions can go through three non-linear points and were probed more about both questions. The results show three different ideas about this. Some people thought that there were only a finite number of specific functions that could go through two or three given points. Another group of people thought that the number of functions was infinite but they were all specific functions. The third group of people included those who seemed to understand that any object that satisfies the definition of a function, even if it is arbitrary and does not have any specific name, is a function.

Brian's answer illustrates the belief that only a finite number of specific functions pass through two or three non-linear points. He found two graphs that go through the two given points and, based on that, he guessed that there may be four functions altogether.

> I would guess just four. Just from the inference that I know two, there may be a few out there.

The belief that there were infinite specific functions that pass through two or three points is illustrated by Bert's answer. He suggested different parabolae as the functions that could go through the three points and then also added a horizontal line. Even after being pushed, during the interview, to find more functions, he could not come up with any. Some people, like Evelyne, found a broader class of functions and suggested, for example, an infinite number of polynomials. But, again, when asked if there were other functions that may work, they said that they could not think of any. This kind of answer seems to point to a belief that functions have specific names or shapes or are somehow "known" to be functions. By saying that they cannot think of any other function, the participants seemed to say that there was not any other function known to them.

The third group of participants included those who were aware that there were infinitely many arbitrary functions that pass through two or three points. Huey illustrates that by describing how by free drawing he can get infinitely many functions.

> H: Really any function value, anything that is a function...
> Infinite, because I could just draw a million curves.
>
> R: Do you have names for all these curves? Are those specific curves?
>
> H: No, I mean, you wouldn't be in, no, there's no specific names. You couldn't even write equations for all of them that would satisfy... but certainly for some of them, like parabola.

Most people answered the same way in both the two and the three point cases. But Brian, who started with a finite number of specific functions in the first case, changed his way of thinking in the second case, probably because of the thinking that he did. He decided that the number of functions could be infinite. He started with infinite parabolas and then added examples of non-functions,

> I'd put down hyperbolas, um, cubic, um, I think if you really tried hard, you could probably get some complicated...An ellipse will fit it, circle maybe not. Those are the ones I can think of right off hand.

It may seem that people who know that an infinite number of functions can go through two or three points, understand much more about functions than people who do not. But Brian's case tells us that there may not be such a big difference between the two. (There may be a big difference in their conception of infinity, though.) The biggest difference seems to lie between those who understand the arbitrariness characteristic of function and those who do not, between those who expect functions to have specific names or shapes and those who do not.

Bob was a special case. He was the only participant that gave one specific function as an answer to the case with the three points, and then, in the two point case, he said that there were infinitely many arbitrary functions. All the other nine second phase subjects used the same kind of functions, either specific or arbitrary, in both cases. Being able to use arbitrary functions in one case and not in the other seems strange. The explanation for that may be that Bob did understand the arbitrariness property, but he was convinced that three points define a function:

> I think there's one [function]. I would think it would be something like, I think it's going to be an x-cube term. Something with a zig in it, like that. Only because it seems to me, a long time ago, an instructor told me that any three specific points define a function.

Three non-collinear points do determine a parabola, so Bob may have confounded information about parabolas with general ideas about all functions. For him, three points define a function. This mistaken information, about which he was so sure, led him to ignore the arbitrary property and to assert that there was only one specific function that passes through the three points. But, in the case of two points only, he was free of that misleading information, and therefore could use his knowledge about functions.

Summary of the arbitrary nature of the relationship. We have seen that the participants in this study revealed several facets of misunderstanding what a function is. Many of them ignored the arbitrary nature of the relationship between the two sets on which the function is defined in one way or another. Some expected functions to always be representable by an expression. Others expected all functions to be continuous. Still others accepted only "reasonable" graphs, etc. Having one misconception does not necessarily mean that all the others are also included. Tracy, for example, thought that all functions should have smooth and continuous graphs (differentiable functions). But at the same time she understood the arbitrary nature of functions. This is illustrated in her answer about the number and nature of functions that pass through two points:

> As many as you could...Any graph you can get to go through those two points that's continuous and pass the vertical line test will work.

The participants' conception of function was limited in many cases to specific functions only. However, all of them (but one) found non-linear functions that pass through two points. These results are different from studies that report on high school and college students that hold a linear prototypical image of functions (Dreyfus and Eisenberg, 1983; Markovits et al., 1983, 1986). This difference may be attributed to the longer and richer experience with functions that the participants in this study had. They were all in their last stage of their college education, while the participants in the other studies were 9th, 11th and 12th graders and college freshmen. So the subjects in this study had encountered more functions and had dealt with functions over a longer period of time. This richer experience with functions might have helped to eliminate "linear" thinking. One of the explanations that the other studies provide for the "linear" thinking has to do with geometrical influence. When students learn that two points define a line, they may tend to overgeneralize this idea. When two points are given, those students claim that there is only one function that passes through these two points -- a linear function. Therefore the time that passed since the participants in this study learned geometry, could also be a factor in their "non-linear" thinking.

Thus far we have discussed the results related to the arbitrary nature of the relationship between the two sets on which the function is defined. The following section deals with some aspects of the participants' understanding of the arbitrary nature of the sets themselves.

Functions are defined on numbers only. The participants in this study seem to think of domain and range of functions as sets of numbers only. Some of the first phase subjects (22 out of 142 who answered this) included that explicitly in their definition of function:

— "A function is a means to compute values by substituting numbers for variables and for every number put into the function you get a value out."

— "A function is a mapping from one set called the domain to a set called the range in which each number in the domain is mapped to only one number in the range."

But implicitly the assumption that functions are defined on numbers only was much more widespread. Conception of functions as equations only implies, in many cases, a conception of domain and range as sets of numbers. Many of the participants thought that all functions can be described by formulae and implicitly thought of domain and range as limited to numbers only.

While a conception of function as defined on numbers only may be sufficient for studying calculus, other branches of mathematics and fields other than mathematics use functions that are defined on different sets. Outside mathematics, functions usually describe relationships or cause and effect phenomena. Brian complained of not being able to use functions that way as a result of looking at functions as defined on numbers only.

> I'd like to see something of the idea of the function as a relationship, but I'm stuck in numbers right now and cannot get out of it.

In mathematics functions are sometimes used as objects and serve as elements of domain and range. Abstraction is very powerful in mathematics. Abstraction of the notion of function from being defined on sets of numbers to a broader one where function can be defined on any set of objects (even functions) was a very powerful idea in the historical development of function in mathematics. Question 15 dealt with functions as entities or objects where a function was defined on functions:

****

Consider the set of functions whose domain and set of images are all the real numbers. K assigns to each pair of such functions, their composition.

a) Is K a function? Explain. *b) is K^{-1} a function? Explain.*

****

The domain of the function K in question 15 was a set of ordered pairs of functions, and the range was a set of functions. Each function in the range was a composition of two functions from the domain. A function may be described as a composition of several pairs of functions (e.g., the function $h(x)={}^5\sqrt{x+3}$ in question 13 is a composition of several pairs of functions, such as $f(x)=x+3$ and $g(x)={}^5\sqrt{x}$; $f(x)=x+9$ and $g(x)={}^5\sqrt{x-6}$; $f(x)=x$ and $g(x)={}^5\sqrt{x+3}$). Therefore, K was not a 1-1 function, and did not have an inverse. From the way K was defined, one could interpret K as a function (if one understood the definition of K as K: $(f, g) \rightarrow f \cdot g$), and K not as a function (if one understood the definition of K as K: $(f, g) \rightarrow f \cdot g$ and also K: $(f, g) \rightarrow g \cdot f$. In this case, the pair (f, g) has two images -- $f \cdot g \neq g \cdot f$ usually -- and therefore K is not a function). Both answers were considered correct. The following is an example of a correct explanation for a decision that K is a function, and the next -- for a decision that K is not a function

— "Yes. K: $f, g \rightarrow f \cdot g$."

— "No. $K(f(x), g(x)) = f(g(x))$ or $g(f(x))$. $f(g(x)) \overset{!}{\neq} g(f(x))$."

Only 66 of the first phase subjects (out of 152) tried to answer question 15. This small number may be attributed to lack of time to finish the questionnaire since this was the last question. But from the nature of the answers and from the interviews it became apparent that the reason for not answering that question was also rooted in its high difficulty level.

A vast majority of those who answered part (a) of question 15 decided that K was a function (57 out of 66). Many of the answers were hard to interpret. Not many (less then 10) seemed to base their decision that K was/was not a function, on a correct understanding of the definition of K.

A large group of those who provided an explanation for their decision (18 out of 48), seemed to think of K as the composition itself, instead of thinking of the compositions as images of K.

— "Yes, the composition ftn [function] maps values onto other values K does this."

— "Yes. A composition of functions is a function."

The rest of the participants either said that they did not understand/know/remember this or gave answers that were impossible to interpret.

Recognizing that a composition of functions is also a function is not an easy step for students, as was indicated in chapter 2. But accepting the composition as a function is easier than identifying K as a function, since the composition is defined on numbers while K was not.

Treating functions as elements of domain and range seemed to be the main obstacle that prevented the participants from understanding the definition of K. Katie, who correctly identified the domain of K as a set of ordered pairs of functions and the range as a set of functions, explained that she was not sure about her answer because of the idea of a function defined on functions:

> K: The domain of the function is the set of all functions from the real numbers to the real numbers. I'm not real sure on 15, so I'm not going to claim that I'm right.
>
> R: What did you find difficult?
>
> K: ...The idea of domain and range, the idea of function on the domain and range, and the idea of a function on that... And if, as in my case, you're not real positive on the definition of a function on a function going into the second level. It gets kind of hazy. And the idea of how that would apply. It took me several times going back and forth trying to figure out how I could apply that.
>
> R: The idea of having functions as --

K: As the domain. Yeah. Which pretty sure was covered quite a bit in algebra, like 337. But again, I dropped that course. I intend to take it eventually.

This idea of function of functions probably caused some other people, who understood the definition of K, to say that they were not sure if K was a function. One of the first phase subjects, for example, seemed to understand the definition of K. He/she wrote: "Let $f: \Re \to \Re$, $g: \Re \to \Re$, $K: f \times g \to f \cdot g$." But then, when he had to decide whether K was a function, he/she wrote: "Not sure."

Identifying K with the composition of two functions seemed to help people handle seemingly contradictory information. On one hand, the domain should be (pairs of) functions. On the other hand, they thought of domain of function as a set of numbers. In order to resolve this uncomfortable situation, they ignored part of the information and used only the part that made sense to them. Jenine illustrates this point.

J: Um, I guess I was trying to say, see, the domain was the real numbers, the set of. I was really confused on how could a set of functions be in the domain of the real numbers. Was it the, I just couldn't picture the set. And so, therefore, I found it very difficult to determine whether it was a function or whether it has an inverse function.

R: What do you think now? What do you think the domain is?

J: Well, it says that the set of functions is the domain. So the domain of K has to be the real numbers, because those are the functions, you're working with those functions whose, their domain has to be K, I mean, real numbers. Domain of K has to be the real numbers. But I still...

R: Okay. How about the range of K?

J: (Pause). Well, I would assume it's the reals, because the real in composition with another real function or something with another real number. I would assume it would be in the reals.

Katie and Jenine struggled to understand the definition of K. Katie managed to understand it, while Jenine could not overcome her difficulties. Both of them realized that there was something unusual in the definition of K. Some people did not recognize the complexity of the situation, either because they did not see anything unusual with having a function defined on functions, or because they completely ignored this complicated information. Bob illustrates the first kind.

> Yes, it is a function. It is taking pairs of functions and composing them. For any two functions there is a composition.

Later he gave an example and explained why K^{-1} was not a function. It was clear that he understood the definition of K and had no problems handling this kind of functions.

The other group of people who did not seem to have difficulties with K were people who ignored the complicated information. They seemed to look for some key words -- real numbers, composition -- and simply decided that K was a composition of two functions with the real numbers as its domain and range. Bert was one of these.

> B: I think that it would be a function.
>
> R: Why do you think so?
>
> B: Composition of functions are functions themselves, I think...
>
> R: Okay, so what is the domain of K?
>
> B: Domain is the set of integers or real numbers.
>
> R: Okay.
>
> B: So the domain is the reals -- of K, and then of K inverse would be, the range is the reals. Since it's the reals for both of the domain and the range, it would seem like it would stay that way.
>
> R: The domain and the range of K are the real numbers. Is that what you're saying?
>
> B: I think that's what it says. Domain and the set of images are all the real numbers.
>
> R: Okay. Can you give me an example of how K may work? How do you see K?
>
> B: Okay. When I see, K is actually combining the two functions into one and then taking the same domain and spitting out a range.
>
> R: What do you mean by that?
>
> B: Well, if the domain of the two functions, I mean, whatever the domain of the two functions is, it is going to be the domain of K. So that's how I see it. Because K is the combination of two functions. This would be K, the composition.

To summarize, dealing with functions that are defined on functions seemed to be difficult for most of the participants in this study. This was true even for the few who could handle it. People seemed to expect functions to be defined on numbers only. Consequently they changed the given information to match their knowledge about functions.

Discussion. The participants in this study seemed to reject arbitrary functions. This finding is surprising at first since many modern texts introduce functions by using arbitrary functions such as the function that assigns to each person his/her mother (no formula, no numbers). But this finding should not surprise us. After the first introductory lesson, almost all the functions that high school and sometimes even college students meet are the kind that have a "nice" graph and can be described by an expression. So the students' concept image of function is determined by the functions they meet and not by the modern definition of function which emphasizes the arbitrary nature of functions But the participants in this study were not "just" any students. They were in the last stage of their studying i.e., they had already taken most of their math classes. Do we want math majors at the end of the 20th century to have a limited concept of function, similar to the one from the 18th century? The concept of function has been changed since the 18th century not because someone arbitrarily decided to change it. It has been changed because new discoveries in mathematics created the need for change in the definition. New discoveries created new branches of mathematics which also led to changes in the definition of function. Math majors who have an 18th century concept of function are also deprived from the mathematics that has been developed since than, which is based on a more modern conception of function. What makes it even more serious is that the subjects in this study were prospective secondary math teachers. Should teachers have a more complete conception of the central concept in the high school curriculum? Can we expect teachers to be able to teach according to the modern definition of function, as it now appears in modern texts, while their conception of function is more restricted, more primitive?

If one accepts the arbitrary nature of functions, then one does not expect functions to be expressed in any specific way, have any specific graph or shape, or be defined on any specific set. The decision whether or not a mathematical object is a function is based then on the other essential feature of modern functions -- the univalent property. In order to decide whether the graph in Figure 5.2 was a function, for example, 46 first phase subjects (out of the 130 who answered this) used the univalent property of functions. The knowledge and understanding of this property is discussed in the next section.

Univalent functions -- why?

In the modern definition of function there is a requirement that for each element in the domain there will be only one element (image) in the range. This requirement is attributed to mathematicians' desire to keep things manageable. Keeping track of meanings of multivalued symbols (such as $\sqrt{}$), and taking care to see that the symbols have the same meaning in the same context is not easy and requires a lot of watchfulness. When one has to deal with differentials of orders higher than one, one has to distinguish independent from dependent variables and then multivalued symbols become too messy. Advanced analysis of functions led to the restriction of functions to univalent functions only, and this was generally accepted as the definition of functions. Today, this requirement is considered an integral and important part of the concept of function.

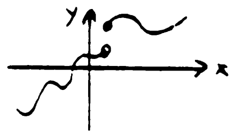
Familiarity with univalent functions. Almost half of the prospective teachers (69 out of 142 first phase subjects who answered this) remembered this requirement and included it in their definition of function.

— "A function is a relation such that a number in the domain can only be matched to one number in the range."

Many of the participants also used that to answer question 11(a) from the questionnaire (see Figure 5.5), as an argument for deciding whether something was a function or not.

****

A student marked all the following as non-functions. (R is the set of all the real numbers, N is the set of all the natural numbers).

(i) 

(ii) $f : R \dashrightarrow R$, $f(x) = 4$

(iii) $g : N \dashrightarrow R$

(iv) A correspondence that associates 1 with each positive number, -1 with each negative number, and 3 with zero.

(v) $g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$

(vi) $(1,4), (2,5), (3,9)$

For each case decide whether the student was right or wrong. Give reasons for each one of your decisions.

****

Figure 5.5 -- Question 11(a) from the Questionnaire

Table 5.3 presents the number of first phase subjects who used that requirement as a criterion for rejecting or accepting each one of the six parts of question 11 as a function. The first column shows the number of people who used the univalent property to accept the object as a function. The second column presents those who used it to decide that the given was not a function. The third column is the sum of the first and the second columns, representing the number of people who used the requirement (in one way or the other). The fourth column gives the total number of people who answered each part. We can see from Table 5.3 that between one third and one half of the participants used the requirement of having only one image for each element in the domain as a criterion for checking whether an object was a function, for all the parts of question 11 but (iii), where this requirement was irrelevant. (The other methods used for deciding whether something was a function, were based on a limited concept image of functions, such as discontinuity or familiarity with the specific function at hand: "This is a constant function.")

Table 5.3 -- Distribution of the Use of the Univalent Property in Question 11.

	Used the argument			
	Function	Non-function	Total	Total answered
part i	41	5	46	130
part ii	38	9	47	126
part iii	8	6	14	112
part iv	39	20	59	114
part v	40	13	53	109
part vi	46	1	47	116

Many participants were familiar with the univalent property and its use as a criterion for telling whether something was a function. Many also thought that it was important for students to know about the requirement of having only one image for each element in the domain. A large group of first phase subjects (59 out of 130 who answered this) mentioned the requirement in their explanation to students of what a function is. Many of them (26 out of 59) suggested the "vertical line test" as their explanation, sometimes as the only explanation.

— "By graphing the function and doing the vertical line test, a line never crosses the graph more than once.

— " -- Vertical line test.
-- Each x gets mapped to only one y.
-- y can be used more than once for different x's but the same x can't be mapped to 2 different y's."

The emphasis the participants put on the requirement of having only one image for each element in the domain, the wide scope of its use in the definitions of function they gave and in their decisions of accepting or rejecting something as a function, point to the importance this requirement played in the prospective teachers' conception of function. The interviews revealed that the univalent property was considered to be a very important characteristic of functions since it distinguishes between a function and a relation. Katie, for example, said

> ...it's something that important and you're trying to make a distinction between that and relations. It is almost vital that somebody understand why this is a function and this is not. Another thing might be a relation but it's not a function.

In addition, all eight second phase subjects (out of 10) who were familiar with the vertical line test, claimed that the test was important since it helped identify graphs of functions. Evelyne illustrates this.

> The basic thing is, if you take, you draw a vertical line from any point, the vertical line should always intersect once [with the graph]. And that will let you know: This is the graph of a function. If you have a circle it will intersect twice, so this is not a graph of a function. I would say that it would help them to identify graphs of functions. So in that sense, I think it will be good for them to know that.

We can conclude that many of the prospective teachers were familiar with the univalent property of functions and considered it to be important. The next section presents results about the participants' understanding of the reasons for the use of univalent functions and the kinds of explanations they provided for the importance of the univalent property.

Why univalent functions? Many of the participants thought that the univalent property of functions is important. When asked during the interview to explain its importance, the subjects gave two kinds of immediate responses: to distinguish between relations and functions (2 subjects), and "I don't know" (8 subjects). Tracy illustrates the first response.

> Well, if you don't have that, you have a relationship. So, it differentiates a relationship from a function.

Valerie illustrates the other response.

> I don't know why. I don't know why there should be one. It's the way I always learned though.

After being pressed to think of an explanation for this requirement (why is it important to distinguish between functions and relations?), they provided a larger variety of answers. Four people still said that they did not know. Two of them, who during the interview showed that they did not understand the requirement, claimed that they did not even think that it was important for functions. Brian, for example, said "I think that it's an overgeneralized tool." The other two thought that it was important but did not know the reason.

Three people tried to use everyday life, engineering or science as the source of this requirement, seeing no connection to pure mathematics. They claimed that in those areas there was a need to have one and definite answer. Bob, for example, said

> But, there's a lot of different professions that use functions, especially engineering and sciences. If you could get more than one y value for an x value you put in, you're not getting a complete answer. If I were a scientist or an engineer and I wanted to test a variable, when I put a value of that variable I want to get one answer. I don't want two answers. Because two answers tells me there's two possible ways, and if I'm designing a bridge and I want to know where the stress point is, I don't want to put in a number and find out there's two when there's really - and I know there's only one. But because of the way the function is set up, it allows for two.
>
> I think the reason it's set up like that is, when you start applying a function in everyday applications you're dealing with two variables. The one variable is dependent on the other. It's going to give you some kind of useful data that you can use. If you get two answers it's not helping. You need one answer, one unique answer. And by restricting a function to look like this, you're only going to get one answer. I think that might be why.

While all that was said in the above is true -- we do want one unique answer in many cases -- it puts the importance of getting a unique answer on the applications and not on the necessity of the mathematical analysis itself.

The remaining three people thought that the importance of the requirement was rooted in mathematics. Two of them provided explanations for that. Valerie was the only one who came close to the historical explanation of keeping things manageable. She said:

> V: I suppose it makes it easier for all the other things you do, like for finding inverses, you'd have a more difficult time with it otherwise, probably.
>
> R: What do you mean by that?
>
> V: Well, let's see. If you took like, if it weren't a one to one relationship, you'd have problems like with a graph that looked like that, because then how are you going to, for each x that you put in how are you going to come up with both those answers? You could do it if you were taking positive and negative roots and things like that. But that could be very confusing to come out with two answers for every number you put in, or three or four. What if your function looked like this, you'd be there all day trying to deal with that. So I suppose it's just an easier way of doing things, making it more orderly. I suppose there's probably another reason why they do that.

Tracy gave a similar reason. But while Valerie talked about the necessity of this requirement in keeping things manageable, Tracy had a more definitive view of the importance to mathematics. It was as if she thought the existing theorems of mathematics dictated this requirement and not that this requirement allowed the theorems to be the way they are.

> Well, I would imagine it has to come from mathematicians. Different theories that you're using have to have different limitations on whether or not they're going to work. It makes it a more specific case.

Bert, who also thought that the univalent requirement came from mathematics, described the origin of the requirement as arbitrary.

> It seems like that would be, whoever decided to call that a function just made it one of the requirements.
> I would just think, that would be, whoever decided to call it a function just decided: if it looks like a graph, like this, and has only one, and I'm going to call that a function.

Some serious questions are raised by the fact that, without prompting, none of the subjects could come up with a reasonable explanation for the need for the property of univalence. This requirement is presented to our students as one of the most important characteristics of functions and, as can be seen from this section, this is what many of them think. They know that it distinguishes between relations that are not functions and those which are. But, usually, they are not told why it is important to distinguish between these two groups. Mathematics teachers, usually, do not explain what it is that you can do with functions that you cannot do with relations which are not functions. This approach may contribute to making mathematics looks like an arbitrary collection of rules and definitions -- an approach that Bert seems to hold. Not being aware of the use or need of the requirement was, probably, the reason that some people remembered it backwards,

__ "An equation which satisfies the following requirements: for every member of the range, there is one and only one member of the domain which satisfies the equation and corresponds to the member of the range."

Some thought of it as a 1-1 property,

__ "A function is a one-to-one, on-to correspondence between two sets."

Not understanding the use of univalent functions was probably one of the reasons for forgetting the univalent requirement even though this requirement is emphasized in every beginning course on function. Jenine, for example, was completely unaware of the univalent requirement at the time of the interview. She never used it in her answers to the questionnaire -- neither in her definition of function nor in her decisions to accept or reject something as a function. She did not know what the "vertical line test" was, even though she remembered hearing about it. Function for her was a relationship: "...a function is an easy concept, because it is just a relationship." When asked about the need for the univalent requirement at the end of the interview she said she was not sure if her definition was correct

> You see that's what. No, because I guess my definition is much more general and that was one of the things that I was trying to figure out. Was a function, was it a requirement that it was onto and one to one, or were there functions that were not one to one and onto?

Jenine seemed to be concerned about understanding meanings -- for herself and for students in general. She talked a lot about the importance of understanding and complained about the algorithmic teaching and learning common to mathematics. Function as relationship suited her approach while the unexplained univalent property did not and therefore, probably, it was ignored and forgotten.

Contradiction between the univalent requirement and the concept image of function. Some people used "circles" and "ellipses" as examples of graphs of functions in question 3. Familiarity with those shapes was, probably, the reason for this mistake. For some people, familiarity with a mathematical object was the criterion for accepting it as a function. These people belong to the group of subjects who used "known" as a criterion for functions. Accepting only familiar functions as functions makes a very limited repertoire of functions. In addition, this approach sometimes causes an acceptance of non-functions as functions. Having a concept image of functions as equations with nice graphs makes it reasonable to accept the familiar circles and ellipses as functions. This is especially true since circle, ellipse, as well as parabola and hyperbola belong to the same family of conic sections. Parabola is a function so why should not a circle be one?

Some of the people who held this wrong image of the concept of function, were also aware of the univalent requirement for functions. There seemed to be a contradiction between two pieces of information about function that they had. On one hand, they remembered learning about the univalent requirement in relation to function. On the other hand, they were familiar with circles and ellipses -- their equations and graphs -- and therefore included them as part of their concept image of function. Brian's case illustrates this situation.

Brian used the univalent requirement in his definition of function:

— "A thing which maps every element in a domain set onto another **unique** element in the range set."

Then, in an alternate version of this definition for a student who does not understand it, he, again, emphasized the univalent requirement:

__ "For every number you put into a function you get **only one** number back out." In addition to memorizing the requirement, Brian knew how to use it. He correctly based his decisions whether the objects in question 11 were functions by using the univalent requirement. For example, he decided that $g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number} \\ 0, & \text{if } x \text{ is an irrational number} \end{cases}$

was a function because "there is an assignment of a single value to each number." He also used the "vertical line test" to support his decision to accept the graph in 11(i) (see Figure 5.2) as a function. Brian seemed to know that functions must be univalent and he understood how to use that requirement in the process of deciding whether a mathematical object was a function.

The first crack in this ideal picture occurred when Brian was asked to explain the "vertical line test" to which he referred as important to teach to students. He said

> Like if I was going to have... Well. Uh, a circle is a function, but a circle doesn't pass the line test. Oops, I made a mistake. But, um. Well, for linear functions it's okay, I guess. But I'll just use it in the beginning to give students an idea of 'yes, this is a function', 'no, this isn't a function', give the idea of one to one correspondence. That's really as much as I know and understand functions.

Brian understood the definition of function but at the same time he thought that a circle was a function. He did not have any problem to use the univalent property until he was confronted with a contradiction: "A circle is a function, but a circle doesn't pass the line test." Brian solved the contradiction by deciding that the "vertical line test" was appropriate only for linear functions, but then he ignored the contradiction and continued to talk about the use of the "vertical line test" for deciding if something was a function. Later, he was asked if he would accept the following graph (see Figure 5.3) as a

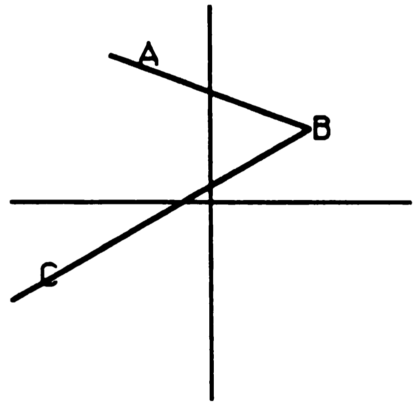


Figure 5.3 -- A graph Presented During the Interview

function. The contradiction arises again: "It doesn't pass the vertical line test, but a lot of functions don't do that." Brian was asked to give an example of a function that pass through the three points A, B and C in Figure 5.3. He sketched a graph of a function and said: "It would pass the vertical line test."

Brian seemed to hold two contradictory pieces of knowledge about functions. Throughout his answers to the questionnaire and the interview he kept talking about and appropriately using the univalent property for functions. At the same time he said that not all functions should be univalent and gave examples of such "functions". Not only that his concept image of function included only specific functions and not arbitrary ones, it also included "functions" that did not satisfy his own definition.

The solution for this uncomfortable situation was to go with the concept image. Brian decided that only some specific functions are univalent. He explained, for example, that functions and equations were the same thing "except that **certain** functions pass the vertical line test." For him, the univalent property was a sufficient condition for a function, not a necessary one. Therefore, if an equation or a graph satisfied the requirement of having only one image for each element in the domain, and if that equation or graph matched the concept image of function that Brian had, then that object was a function for him. But there still were other "functions" which did not satisfy that requirement such as circles and ellipses.

In Brian's case, the univalent requirement was meaningless. What was the point of checking to see if there was only one image for each element in the domain, if there are also functions that are not univalent? Interestingly, Brian did not ignore the property like Jenine, but rather tried to attach meaning to it. So he decided that univalence helps to understand what a function is, when one first begins to study functions.

At the end of the interview Brian explained his thoughts about the univalent requirement, admitted that he did not understand the use of that requirement, and made clear that he decided to use his concept image of function as a criterion for functions.

> It makes it a lot clearer up to a point. I've never been able to get it right in my own brain as to what it exactly is supposed to be. A function, well, early on it's a good idea to have one to one correspondence. Because really it's easy to see 'that's a function', 'that's not a function'. Or to get the idea, it relates the idea of a function to an equation really well. But when you start moving out into concepts like cause-effect or things like that, it tends to break down. Like I have a problem trying to make an ellipse and call it a function based on my definition. It's great for the vertical line test. For linear functions it's perfect. For some others -- quadratic -- it's beautiful. But it just breaks down in a certain point.
>
> It works for linear functions. That's what it's there for. I think it's an over generalized tool.

Brian's case vividly illustrates how people may seem to understand the univalent requirement for function, know how to use it, but still, when it contradicts with their concept image of function, they will use the latter as a criterion. More awareness and sensitivity to this situation, an emphasis on making sure that people understand the need for univalent functions, can help in making the concept image of function closer to the concept definition.

Summary and discussion

The participants in this study seemed to hold a very limited concept image of function. A strong tendency to identify functions with equations or expressions was detected among the prospective teachers as well as the expectations that functions should have specific "nice" graphs and shapes. Appreciation of the arbitrary nature of function seemed to be missing. Even if, as some might claim, having a more complete image of the concept of function is not important for the functions taught in secondary school, the teacher is expected to teach the modern definition of function. The prospective teachers' incomplete conception of function is problematic and may contribute to the cycle of discrepancy between concept definition and concept image of functions in students, keeping their concept image of function similar to the one from the 18th century.

Having an old and limited concept image of function, which basically includes functions that are described by equations, means that the arbitrary nature of function -- an essential feature of the concept of function as it has evolved in history -- is completely ignored. Rejection of arbitrary functions, which follows the historical development of functions, was apparent not only from the identification of functions and equations, but also from the ways some of the participants accepted or rejected graphs of functions, and by their acceptance of functions only those which were "known". Some of the participants followed the historical development of the definition of continuous functions when they accepted as functions only differentiable functions (continuous functions in 18th century terms) or when they called non-differentiable functions -- discontinuous functions.

Most of the participants knew about the univalent property of functions and considered it to be important. But almost none of them could explain why it is important and how functions became to be defined that way. The fact that so few recognized that the need for the requirement is rooted in mathematics itself points to a limited understanding of function and its related domains.

So far we have discussed what a function was for the participants in this study. Since the subjects were prospective teachers, it is of interest to know how they thought about explaining what a function was to students, what they thought should be emphasized and what they would deemphasize. This is discussed in the next section. The literature, as was indicated in Chapter 2, describes some common misconceptions that students have. The prospective teachers' understanding of those specific misconceptions are also discussed in the next section.

Function Definitions for Students

The prospective teachers were asked to define a function (question 1 from the questionnaire, see Appendix A). Then, they were presented with a situation where a student did not understand that definition. They were asked to give an alternate version of the definition, that might help the student understand. Three trends were detected in the answers:

1) The number of people who used modern terms like relation, mapping and correspondence in a definition for students dropped compared to the number of people who used a modern definition in the first part.

2) Many more people used graphs and the "vertical line test" to illustrate what a function was for students than for the first part.

3) Another very popular illustration for students of what a function is was to describe it as a machine or as a black box.

Modern definition versus old definition

The participants' definitions (or explanations) of function for students were analyzed in a similar manner to the analysis that was employed to the definitions they gave in the first part, when they were not asked to refer to students. Definitions were categorized as modern if function was defined as mapping, correspondence or relation with some reference to set theory characteristics and the arbitrary nature of function. Definitions were categorized as old if function was described as a dependence relation of variables, rule, operation and formula, usually with reference to variables and regularity of the function behavior. Table 5.4 summarizes the function definitions that were given in both parts by the first phase subjects. In cases where definitions seemed to include some characteristics from both categories, they were categorized into the modern definition category. The first row presents the function definitions given in the first part of question 1. The second row presents the function definitions given for students in the second part of question 1.

Table 5.4 shows that many prospective teachers who used modern terms when defining a function, decided not to do that when approaching a student with difficulties. Valerie, for example, defined function as "a 1-1 mapping of a set of points x onto y."

Table 5.4 -- Distribution of Function Definitions and Function Definitions for Students.

	Modern	Old	Other	N/A	Total
Function definitions	78	53	11	10	152
Function definitions for students	27	67	36	22	152

Then, for the student, Valerie chose not to use the term 'mapping' and defined function as an operation:

> You are taking a group of numbers. You perform some operation on the numbers (such as multiplying, etc.). This gives you a second group of numbers. The operation you did is called a function.

She explained that she changed her definition for the student since she assumed a student wouldn't understand what a mapping is and would not be able to understand what was it that she was talking about:

> If I'm talking to a student, then they probably don't understand a lot. They don't understand my definition at all. You know, it's going to be: 'What's a mapping? What are we talking about? What's going on?'

Valerie, as well as other 7 second phase subjects (out of 10) decided to use more familiar situations and terms for students. They thought that the use of mapping, relation and correspondence as well as domain and range, would not be understandable for students. So they chose to describe function as an operator, equation, graph or machine without reference to domain and range. Another interviewee, Katie, who started with function as a set of ordered pairs, chose to simplify it to students by describing it as a mapping. Although she continued to use modern terms in both cases, mapping is less formal and easier to understand than the most formal and rigorous definition as a set of ordered pairs. From the 10 second phase subjects, only Tracy did not give a modern definition for the first part but used operation first and then equation as an explanation for students.

The prospective teachers tended not to use modern terms such as relation, mapping and correspondence as well as domain and range, when defining function for students. They tried to describe what a function is by using terms that are more familiar to students. Choosing more familiar terms such as equation or graph seem to be a reasonable pedagogical decision, especially since the modern definition of function is problematic for many students. But the decision of how to describe what a function is for students was not based on pedagogical arguments alone, it was also based on the concept image of function that the prospective teachers themselves held. Most of the participants did not hold a modern concept image of function. Therefore, it is not surprising that when describing what a function is for students, many chose to use their concept image of function and tended not to use modern terms. So even if the decision not to use a modern definition for students may seem a good pedagogical decision, it should probably be attributed to the incomplete concept image of function that the prospective teachers held and not to mature pedagogical reasoning.

Many of the prospective teachers decided not to use modern terms in their definition of function to students. Instead, they tried to find ways to help the student understand what a function is. The next two sections describe the approaches they used.

<u>Function as a machine or a black box</u>

While only 2 first phase subjects (out of 142 who answered this) defined a function as a machine or a black box, 30 people chose to use it as an explanation for students:

— "I would tell the student to think of a function as a machine; when we put something in, we get something out."

— "A black box that has an input and an output → for each value in you get a value out the other side."

—

A function machine is a common illustration of function. Most of the interviewees remembered hearing about it when they were introduced to functions. In a non-representative survey among mathematicians, two opposite opinions about a machine as an illustration of function were given. The arguments in favor of a machine emphasized the importance of conceiving function as an object, an entity. A machine and a black box are objects. Therefore they illustrate an important feature of function. They also illustrate the arbitrary nature of function, since contrary to equations, for example, input and output, as well as the relationship between them, can be anything, arbitrary.

The arguments against a machine as an illustration for function referred to the inability to look at the behavior of the function in that way. You cannot see the relationship between the domain and the range. All you get from a machine is an illustration to the most formal and rigorous definition of function -- a set of ordered pairs (input and output), or to the easiest, but not very helpful, point-wise approach to function, when you look at one pair at a time. But you do not get to see the behavior of the function along its domain. Machine seems to fit a view of function as an entity which is used in advanced mathematics and is very hard for students to comprehend, or a point-wise approach which is a simplistic approach that most students tend to use without difficulties. Machine does not help with an approach that deals with the global characteristics of function, when one is interested in the global behavior of a function, as is the case in calculus, for example.

The interviewees were asked to give their opinion about the illustration of function as a machine or a black box. As in the case of the mathematicians, some of the prospective teachers liked the idea of a machine and some did not.

The arguments in favor of a machine as an illustration for function were based on the idea that it served as a simple, concrete representation for the concept of function. Three participants (out of 10) were in favor. Bert illustrates how he used a machine to simplify the notation of function.

> I have two people I am tutoring. They just, once they get to the chapters on functions, they don't realize that it's not hard. I mean, they make it a lot harder than it is. The different, like if you represent a function, f(x) is some function, it just blows their minds. It's a lot easier to say that this f takes an x, give it an x and it changes it in some way. And that's the thing, that it's equal to and then spits it out and gives you a whole other value. So I think it puts it in simpler terms for a student. So I think that it would probably be the way I'd do it too.
>
> R: What exactly does it simplify?
>
> B: Probably just representation of a function. I'm thinking of a specific case. This girl that I'm tutoring, you know, switching the function from f(x) and then the next problem be in terms of g(x), and the next would be h(x). Um, she just, she never made the jump between those. It doesn't matter how you represent it. And so, I just told her 'every time it says one of those things, think of that as a machine, where you do put it in and it spits out some other variable. Not variable, some other number'. And so, I think, that's how it puts it in simple terms. It replaces, maybe, the variable representation.

But most of the second phase subjects (7 out of 10) did not like the idea of representing a function as a machine. Valerie opposed the "magical" characteristic of a machine, the idea that you do not see how the two sets relate to each other:

> It's a way of thinking about it at first, starting about seeing that. It kind of makes the idea a little bit more magical though. I hate to do that, when you say to the students to take these numbers and put them in this machine and

> the machine just pops out an answer, and there you go. I hate starting them out thinking that way, because it's something that they're going to do, something they're going to deal with.

Bob referred also to the problem of approaching functions point-wise instead of looking at their global behavior.

> I think the idea of just putting numbers into a box or a machine and getting another number is going to lead students to try to solve functions and do their graphs by taking some value for x and making a table, and getting a value for y and plotting the points. And never understand the whole idea of a function, never understand how a function maps from one segment to another, how the inverse function maps back. Students like that will do fantastic in concrete problems and examples but they'll never be able to deal with the abstract.

Two prospective teachers claimed that they never understood the idea of a machine as a representation for function. Katie said: "That was hard and I never understood what they meant."

A function machine or a black box is usually used when functions are introduced. This is probably the reason for the large number of prospective teachers who used it as an explanation of what a function is for students. An illustration of function as a machine may be helpful when function notation is introduced, as some of the prospective teachers suggested, since the notation $f(x)$ -- f of x -- and its meaning are very difficult for students at the beginning. It is probably not a good illustration for function as an object, as was suggested by one mathematician who critiqued the questionnaire. Otherwise, the participants in this study would not have so many difficulties dealing with function as an entity, as we have seen. It is also not appropriate for everybody -- some people do not see the connection between machines and functions. While a function machine may be helpful for function notation, it is not helpful for a study of function behavior. As was indicated in chapter 2, students have difficulties dealing with functions in a global way, but have little difficulties approaching functions point-wise. The function machine idea does not help people overcome those difficulties, as was mentioned by many of the second phase subjects.

Function machine, as many other representations, is not an answer to all the learning difficulties that students may have with function. As many other teaching aids, it has some advantages and some disadvantages. By using a machine to illustrate function, one may help students understand function notation but might cause, at the same time, difficulties with approaching function in a global manner. Teachers should be aware of the strengths and weaknesses of a machine as an illustration for function and based their decision to use it on that. The second phase subjects did not seem to have all that knowledge. They either liked the idea or disliked it. But they did not seem to weigh advantages against the disadvantages or appropriate time versus inappropriate time for using a machine.

The Vertical Line Test

Another way that the prospective teachers chose to explain to students what a function is, was by using the "vertical line test" for graphs of functions. While only 3 first phase subjects (out of 142 who answered this) used that in their definition of function, 26 used the "vertical line test" as an explanation for students. Many of those who used it did not explain what a function is. They just gave the students a test.

— "Vertical line test shows a visual representation of functions."

— "By graphing the function and doing the vertical line test, a line never crosses the graph more than once."

The prospective teachers considered the univalent property of functions to be very important. But this cannot explain the large number of people who decided to use the "vertical line test" in their explanation of what a function is for students. Actually, fewer people mentioned that requirement for students. While 69 first phase subjects mentioned the univalent property of functions in their definition of function, 54 mentioned it for students.

An explanation for the extensive use of the "vertical line test" in the explanations for students, sometimes without any reference to what a function is at all, seem to be rooted in teachers' tendency to provide students with rules. The students then can follow the rules, and get the right answers without understanding. Katie illustrates this.

> For instance, if you have a function that is a parabola, the student is told that this is a function, but they don't, they still don't know what that means. If they're told to figure out whether it's a function or not, using the definition, they probably wouldn't be able to do it. If they know the vertical line test works, even if they don't know why it works, they can see right away why this is a function, because they can go through with a ruler or a straight edge and vertically go across the function, looking for places where there are two points.

It seems that a large group of prospective teachers chose to give the students a rule, even if the students do not understand why it works and what the purpose for this checking. The most important thing was, as Brian said, "I think it's a really quick way to tell without having to do a lot of analysis." While easy procedures are important, math educators today complain about the overemphasis of school mathematics on procedural knowledge with no connection to meaning. Making the "vertical line test" an explanation of what a function is, is an example of that.

Not all the prospective teachers supported the attitude of giving a quick rule without understanding. Mike, for example, emphasized the importance of understanding why the rule works.

> That's one way. I think it's important to give other ways to the students. What determine a function. To understand the idea of a function. I think the line test is good just as long as they don't get confused by it, and understand what it really represents. If they can understand they can see a function and put a line there, they know it's a function because wherever you put a vertical line it only hits one spot. But, again, they might not understand why it's a function or why it isn't a function.
>
> Well, you should use it as long as you can make them, the students, understand why they can use it. It is because each value of the domain can only have one distinct function value. And that's why, it shows why with a vertical line. It shows the distinctness.

Reactions to Students' Mistakes

Evaluation of students' work is an on-going task for a teacher. The teacher has to decide whether a student's answer is correct. If it is incorrect or incomplete, the teacher has to help the student understand what is wrong or what is missing. Knowledge about common misconceptions will help the teacher understand the reasons for the student's mistakes, and therefore will help him/her make knowledgeable decisions about appropriate actions.

Teacher's decision about whether an answer is correct is based on his/her content knowledge. A teacher who thinks, for example, that all functions should be continuous, cannot evaluate correctly students' answers on this topic. So, teachers' own subject matter content knowledge is a necessary requirement. But knowing if a student is right or wrong is not enough. A teacher should be able to anticipate sources for common mistakes.

The participants of this study were presented with several students' answers and were asked to evaluate them as right or wrong. In cases where they decided that an answer was wrong, they were asked what they thought the reasons for the mistakes were. Many first phase subjects did not answer the second part. Therefore, the analysis of the suggested reasons for the mistakes is mostly based on the answers from the second phase subjects. Only answers that were based on correct knowledge of the content were analyzed.

Most of the second phase subjects were aware of the difficulties that students have with the univalent property of functions. Seven (out of 8 who knew this property) mentioned this as one of the hard things about the concept of function. Difficulties with the understanding of this property were considered by some of the participants as the source for the mistaken student decision that the function in question 11(ii) $f\colon \Re \to \Re$, $f(x) = 4$, is not a function (6 out of 10 second phase subjects, and 9 out of 51 first phase subjects who answered this). They assumed that the student was confusing the definition of function. Bert, for example, said

> He might have confused that each value of x has only one value y, and might confuse that with the y's and x's because the y is the same for every x.

Other participants had different explanations for the mistake. Some thought that having the same image for all the elements in the domain made the student decide that *f* was not a function (3 out of 10 second phase subjects, and 11 out 51 first phase subjects). Bob illustrates that.

> Because this is a constant function. That might throw off some kids. They might say: 'Well, how can every value of x be a 4?'

Some people (8 first phase subjects out of 51) attributed the mistake to the lack of any formula. Having a number instead of an expression might seem strange.

Another explanation had to do with notation difficulties. Huey used this explanation in addition to the confusion about the univalent property of functions. Jenine used it as the only explanation for all three non-graphic functions -- 11(ii), (iv) and (v) (see Figure 5.5) -- that she chose as student mistakes. While all of the above explanations sound reasonable as sources for student mistakes in relation to a constant function, and basically agree with the literature, most of the participants thought of only one of them. This was also the case with the other "student mistakes" cases. The reason for this 'only one explanation' is not clear. It might be that the participants were not aware that there may be several sources for the same mistake or it may be that there was not enough probing during the interview. Still, subject matter content knowledge seems to be related to the limited explanations provided. Jenine, who used the same explanation all over again -- notation difficulties -- could not have thought of some of the other explanations, since her knowledge of function was very limited. She did not know that functions have to be univalent. Therefore, she could not have used it as an explanation for the mistake. On the other hand, people who expect all functions to be described by a formula, but also accept some other "known" functions, such as a constant function, might be more sensitive to students' expectation for a formula. So, teachers' subject matter knowledge is important not only for evaluating students' answers as right or wrong, but also for understanding the sources of the mistakes.

The participants in this study mentioned some other sources for students' mistaken decisions that functions are non-functions. Discontinuity was mentioned a lot. Eight second phase subjects (out of 9) and 30 first phase subjects (out of 53 who answered this) used discontinuity as an explanation in the case of the graph in 11(i) (see Figure 5.5). Discontinuity was also used to explain mistakes in the other discontinuous functions in 11(iv), (v) and (vi) (see Figure 5.5). The case of 11(vi) where the set $\{(1, 4), (2, 5), (3, 9)\}$ was given, was recognized to be problematic because of the limited domain. The participants were also aware of the influence that students might have from geometry, and used that to explain some of the mistakes that involved points and straight lines. Overall, the participants seemed to be aware of the common misconceptions that students have. However, since many of the first phase subjects did not explain what they thought about the sources of the mistakes, it is not clear how many of them where really aware of common misconceptions.

CHAPTER 6
DIFFERENT REPRESENTATIONS OF FUNCTIONS

Representations play an important role in the understanding of a concept (Davis, 1984; Lesh, Post and Behr, 1987). Whenever we deal with a mathematical concept we deal with it in one of its representations. The most common representations of function are formulae (written symbols), and graphs (static pictures). Other representations are arrow diagrams (static pictures), tables and sets of ordered pairs (written symbols), and situations from everyday life or other disciplines (real scripts). In higher mathematics, functions, as entities, are often represented by a symbol. Spoken (and written) language is also used sometimes to describe a function. Part of understanding an idea (according to Lesh, Post and Behr, 1987) is being able to recognize this idea in a variety of different representations, flexibly manipulate the idea within a given representation, and translate the idea from one representation to another. The National Council of Teachers of Mathematics also recommends emphasizing translation among different representations of function (*Curriculum and Evaluation Standards for School Mathematics*, 1988).

This chapter examines the prospective teachers' knowledge and understanding of symbolic and graphic representations of functions. First, the connectedness between the two representations will be discussed. The prospective teachers' tendency to treat each representation in isolation will be presented by examining their treatment of a problem on quadratic functions. Then, the participants' knowledge and understanding of the translation from one representation to the other will be examined. The tendency to use a point-wise approach instead of a more global view of functions and its implications will also be discussed. Finally, remarks on the participants' pedagogical content knowledge in relation to graphic and symbolic representations will be mentioned.

Connectedness between Symbolic and Graphic Representations
The Case of a Quadratic Function

Sometimes it is easier to solve an equation by looking at the corresponding function (if it exists) and relating the solutions to the graphical representation. For example, the solutions of $a_n x^n + \ldots + a_1 x + a_0 = 0$ are the x-intercepts of the graph of the function $f(x) = a_n x^n + \ldots a_1 x + a_0$. Flexibility in moving from one representation to another allows one to see rich relationships, to develop a better conceptual understanding and strengthen the ability to solve problems. We especially would want to see this flexible translation from one representation to another in the case of dealing with basic common functions from the high school curriculum such as the quadratic function. So, when given a familiar algebraic expression in a problem situation it is desirable for students (and teachers) to be able to make connections between the expression and its corresponding function's graphical representation.

When asked the following (question 2 in the questionnaire, see Appendix A):

****

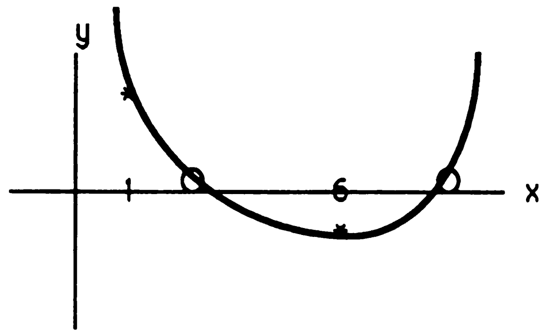
If you substitute 1 for x in $ax^2 + bx + c$ (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number. How many real solutions does the equation $ax^2 + bx + c = 0$ have? Explain.

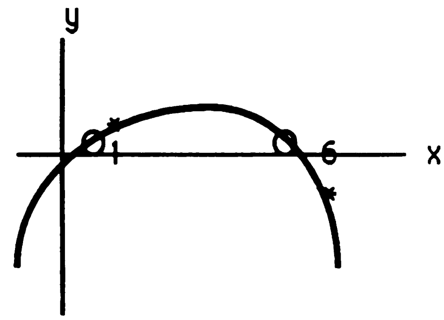
****

45 subjects (out of 127 who answered this question) gave the correct numerical answer: 2. But only 18 subjects (14%) got this answer by referring to the graph of a quadratic function and using the Intermediate Value Theorem:

— "Two. The graph is a parabola by its very nature. If it is positive and crosses the x-axis, it must cross it again."

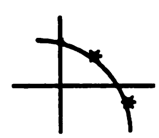
— "2. This is a parabola. the graph will be either $\cup$ or $\cap$ depending on if a is + or −. If x = 1, then y is above the x-axis. If x = 6, y is below the x-axis. It therefore must cross the x-axis. The parabola is symmetric, so there will be 2 x-axis intercepts."





Additionally eight subjects made the connection between symbolic representation and graphic representation and used the intermediate value theorem. But they did not consider the special given function -- quadratic -- and therefore found only one real solution.

__ " 1 real solutions - because it will cross the x-axis in one place."



A large group of subjects (19) claimed that the number of real solutions is two since a polynomial of degree 2 has two real solutions.

__ "2 real solutions because it has an order of 2."

__ "2 because it is of degree 2."

These people seemed to misuse a theorem that talks about the number of roots that a polynomial has. While every polynomial of degree n has n roots, some of the roots may not be real. Even when all are real, some may be identical, e.g., 1 is the only real root of $f(x)=x^2-2x+1$.

About the same number of subjects (20) just "played" with inequalities, trying to manipulate them in order to come up with something but without reaching any conclusion. They did not make any attempt to look at another representation of the problem.

__ "$a + b + c > 0 \quad 36a + 6b + c \leq 0$"

— "$a + b + c > 0$ $\quad 36a + 6b + c < 0$
the solution is real when $\sqrt{b^2-4ac}$ is real" $\quad x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

— "$a + b + c > 0$ $\quad 36a + 6b + c < 0$ $\quad a + b + c > 36a + 6b + c$

$$\begin{aligned} 0 &> 35b + 5a \\ -35a &> 5b \\ -7a &> b \\ a &< -1/7b \end{aligned}$$"

Another group either lost track of what was it that they were looking for or did not understand what solutions mean and considered a, b, c to be variables. They reached the conclusion that there is an infinite number of solutions:

— "Since $x = 1$ we have the equation $a + b + c = 0$. There are infinitely many real numbers which satisfy this equation."

— "$(a + b + c) > 0$ $\quad (36a + 6b + c) < 0$ $\quad a < 0$.
(∞) many - dependent on changes in b and c."

— "$a + b + c > 0$ $\quad 36a + 6b + c < 0$. Infinite, because if $b = c = 1$ then an infinite number values for a that work."

More than 1/4 of the prospective teachers gave answers which don't make sense at all, such as ∞ and even 3, 4 or 5 as the number of solutions of a quadratic equation. Although they probably "knew" that a quadratic equation has at most two solutions, they were unable to use that knowledge in a situation that requested dealing with seemingly conflicting information. So they chose to deal with one aspect of the problem only -- a, b, and c -- without verifying that the answer was reasonable. Table 6.1 shows the distribution of numerical answers and methods of solution used by those who answered the question.

A lack of rich relationships and connectedness between different representations of the same function and flexibility in using different methods for solving a problem, seem to prevent many of the prospective teachers from relating the given equation: $ax^2+bx+c=0$, to a graphical representation of the function $f(x) = ax^2+bx+c$. Eighty percent of the first phase subjects and 7 (out of 10) second phase did not make the connection. Eisenberg and Dreyfus (1986) report similar findings when in a coursewhich stressed the graphical method of solving inequalities, only 5% of the

Table 6.1 - Distribution of Numerical Answers and Methods Used to Find the Number of Real Solutions.

Method of solution	# of solutions 1	2	∞	other	No answer	Total N
Intermediate Value & Graph	8	18	—	—	2	28
Inequalities or Equalities	1	3	4	2	20	30
2nd degree polynomial	1	19	—	—	4	24
a, b, c are variables	—	—	8	1	—	9
Other	2	2	12	1	6	23
No Explanation	2	3	7	1	25	38
Total N	14	45	31	5	57	152

college students opted for the graphical solution on the exam.

All the 7 second phase subjects who did not solve the problem were asked during the interview if they could use graphs to do that. For three of them this simple hint was enough to make the connection between the two representations which allowed them to use the Intermediate Value Theorem and solve the problem. Bob, for example, tried to solve the problem first by using Descartes' rule that relates the number of changes in signs to the number of solutions. After being asked if he could use graphs he said,

> If x=1, ax^2... is somewhere up here, but I really don't know where. And when x=6 it's down here, so you know that between here and here, it crosses the x-axis at least once -- one real root... . So, only one root. Oh, no! That's not true! There's two. It's a parabola because of the x^2. It comes down here to a negative value for x [=] 6. It got to come back up somewhere... . There will be two.

Being able to use the graphic representation so quickly shows that they had the needed knowledge but lacked the connections between different pieces to make it accessible Still four second phase subjects were unable to use the suggestion to try graphs and did not make the connection between the two representations. It seems that the symbolic representation dominated their thinking and they were unable to change the way they thought about the problem.

Summary and Discussion

When given a problem that involved a quadratic expression in one representation (symbolic), most people tried to solve it by using that representation only, even if the use of another representation (graphic) could have been much easier and more appropriate. The quadratic function is a very fundamental and basic function in the high school curriculum. The prospective teachers have studied about and used it since they were in high school. They were, probably, very familiar with it in both symbolic and graphic representations. Still, seeing a quadratic expression did not immediately bring to mind the graphic representation. Not having good connections between the two representations in the case of such a familiar function, points, probably, to separation between the two representations in the participants' minds in general. This makes it very difficult for a person to translate from one representation to another. These difficulties will be discussed in the next section.

Translation between Two Representations: Symbolic and Graphic

The ability to identify and represent the same thing in different representations and the flexibility to move between representations, broaden and deepen one's understanding and empower one's ability to solve problems. When asked to give examples of functions, symbolic representation (expressions) was the dominant representation of functions for the second phase subjects in this study. All of the subjects also used graphs as another representation. The examples they used were very simple ($y=3x+2$, $y=x^2$, $y=4$, etc.); the participants did not have any problem with translating the function from one representation to the other. But the relationships between these two representations were not always clear to the participants. Two functions from the high school curriculum -- quadratic function and sine function -- were used to learn more about the prospective teachers' knowledge. Given the shape and position of the graph of a quadratic function, the participants had difficulties deciding what sign the coefficients of the quadratic expression should have and why. The participants also had difficulties relating expressions and graphs of trigonometric functions that were based on the sine function. These will be discussed in the next sections.

Quadratic function -- Relationships between coefficients and shape of graph

The general form of a quadratic function is $f(x) = ax^2 + bx + c$. Changes in a, b, and c, force the graph -- the parabola -- to change, and vice versa -- the shape of the graph determines the parameters a, b, and c. Question 5 in the questionnaire and its probing during the interview were meant to discover what the prospective teachers think about these relationships.

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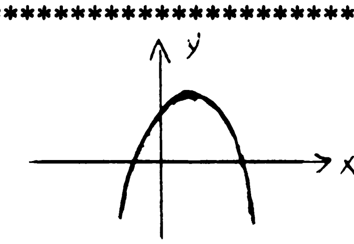
This is the graph of the function $f(x)=ax^2+bx+c$. State whether a, b and c are positive, negative or zero. Explain your decision.



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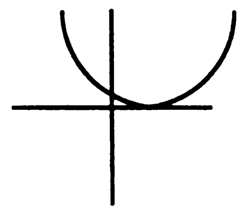
The relationships between the signs of "a" and "c", and the graph are rather simple. "a" is positive (negative) if and only if the graph opens upward (downward). "c" is the y-intercept. Therefore, "c" is positive (negative) if and only if the y-intercept is positive (negative). The relationship between the sign of "b" and the graph are not that simple. One way to approach this is to look at the x-coordinate of the vertex, which is $\frac{-b}{2a}$.

The sign of "b" can be determined by using the sign of the x-coordinate of the vertex together with the sign of "a." Another method is to look at the derivatives.

"a" and the graph. Almost all of the participants (112 out of 124 who answered this) knew that when the graph of a quadratic function looks like $\cap$, "a" (in the equation) should be negative. A vast majority (79) stated this rule as their explanation.

— "a is negative. If it was positive it would be like this  Ex. $y = x^2$."

— "$a < 0$ since the graph opens downward."

Still others (14 people) tried to explain why this rule holds. They used the argument that when $|x|$ becomes very large, x^2 becomes the dominant term in the quadratic function and therefore the values of the function become either infinitely negative (if "a" is negative) or infinitely positive (if "a" is positive).

— "a is negative because as x gets larger the ax^2 term becomes dominant and the function goes negative."

— "a is negative: otherwise the x^2 term would dominant for large x and make $f(x)$ positive."

Some of the participants (5 people) used more sophisticated knowledge of derivatives (either the first or the second) in order to show why "a" has to be negative.

— " $f'(x) = 2ax + b$. a is negative because $2ax + b = 0$ is a critical point. One can solve for a showing $-\frac{b}{2x}$ where x is positive. [He/she showed earlier that "b" was positive]."

— "a is negative. The 2nd derivative of this function would give us a. (some multiple). And for downward concavity, a $\Rightarrow$ negative."

Overall almost all the participants remembered the relationship between the sign of "a" and the shape of the graph (opens either down or up). Only 12 subjects thought that "a" was either positive, zero, or could be anything, when the graph was like $\cap$. Although a rule was usually stated as an explanation for choosing "a" to be negative ("opens down"), the interviews revealed different justification for this rule. Bert used the same argument that some of the first phase subjects used. He started by stating the rule:

> Okay, this is, um, basically there are some rules involving this type of equation. And, um, a lot of way it's usually taught is to memorize the rules and that's probably what I've done. And I remembered the rules and the coefficient of the x^2 term is negative and it opens downward and when it's positive it opens upwards. So this opens downward so it must be negative.

Bert admitted that he just remembered a rule. But, when presented with a situation where a student asks him to explain why the rule works, he referred to the argument that x^2 was the dominant term, and added:

> Um, (pause) probably because this [ax^2] would have to be eventually the dominant term. In this function, and...
>
> Eventually it's going to be larger than any of these terms. Also it will regulate whether the graph dips or rises and this one [the graph] obviously dips so this one [ax^2] would have to be negative. You would have to make the graph fall instead of rise.
>
> I mean the "a" regulates whether it rises or falls as well as distinguishes. I think that's how I would probably try and explain it.

Bert, who admitted first of memorizing the rule, was also able to provide an explanation why the rule should work. Two more second phase subjects (out of 10) also provided explanations during the interview. One was similar to Bert's, the other involved transformations. Katie analyzed the situation using rotations about the axes (or, actually, plane reflections). She started with the graph of $f(x) = x^2$ in her explanation, and tried to relate the other quadratic functions' graphs to this one.

> Start from graphing several parabolas, using basic $y = x^2$ so $f(x) = x^2$. And showing how that changes going back to the translations and having them realize they're essentially multiplying the y values by a negative so they are rotating the whole thing about the x-axis. And that what forces it to point down, the opening down.

> Try and connect that in their minds as far as how a negative coordinate, a negative coefficient of the x^2 is going to force this parabola to rotate downward.

The rest of the second phase subjects (7 out of 10) did not provide any convincing explanation. Two participants admitted to not being sure whether they even remembered the rule correctly, so they just guessed the answer. Valerie stated the rule correctly and then said in her interview:

> I am not...It goes one way or the other. It's either "a" is negative, it goes this way. Or if "a" is positive it goes that way. They're reversed. I am pretty sure it's this way but I am not positive.

Some other people who also were not sure first that they remembered the rule correctly, admitted to use a method of trying some numbers, in order to decide whether "a" should be positive or negative. Jenine, for example, said,

> Okay. This one I did okay. I know that's a quadratic, which is a parabola. Basically what I did was I thought of, I looked at the graph and said, okay, now what happens? First I took the simple case. If $f(x)$ just equals x^2. And when that happened I said, okay, 0 and it's -1, 1, so then we want it to go the other way. So I said, okay, it's got to be negative and I tested that out, you know, I test my thinking out with plugging in specific numbers. That's the way it works best for me.

Although most of the participants (in both phases) knew about the relationship between "a" in the quadratic expression and the graph, and stated a rule for that, the interviews revealed that some of them just remembered the rule without understanding why it works. Some supported their memorization with numerical examples. Others had different reasoning to explain why this works. They did it either by using precalculus or calculus ideas of limits and derivatives, or by using $f(x) = x^2$ as a basic function and reflecting it about the x-axis by multiplying it by a negative number.

Many of the participants did not provide a complete explanation for the relationship between the sign of "a" and the graph. They just stated the rule. Even when presented with a situation where a student asked them <u>why</u> the rule works, they still did not explain why, but rather suggested that the student be given several examples of positive and negative "a"'s. Jenine illustrates that.

> Give them a whole set of these, interchanging negatives and positives, and start out with the first one where $y=x^2$. And then, later on, give them $y=-x^2$. And then have them see if they can find the pattern.

It seems that Jenine and others were satisfied with checking several examples. They did not look for a reason. Is it because usually they are not asked to explain? their teachers do not try to make them understand? Katie described it this way:

> That's a good question. Um, it is basically one of the things that they [the students] are given, they are told, that it is negative if it opens down.

The tendency to justify a rule by checking some specific examples will be further illustrated in the cases of the relationships between "b" and "c", and the graph.

"c" and the graph. Most of the participants (108 out of the 118 who answered this) said correctly that "c" had to be positive. About one half of them explained their choice correctly by reading from the graph that $f(0) > 0$, and since $f(0) = c$, they concluded that $c > 0$.

__ "c is positive. (y intercept is positive)."

__ "c -> positive (When $x = 0$, $f(x)$ positive) ($f(x) = c$)."

Bob also explained it in his interview.

> Ok, from there I know that when x is equal to 0 then I know that there is no "ax^2" term and there is no "bx" term, there's only "c." And here is (0, c) or $f(0)$. And since that's above the x-axis, it's positive.

Others tried to use some vertex characteristics to conclude that "c" has to be positive. This was usually done by refering to the vertex location either in general terms by simply stating that "c" is positive because of the vertex location, or in more specific terms by describing its location (sometimes, as in the case of the following last example, with "non standard" mathematical terms such as 'positive point').

__ "c is positive because of where the vertex is."

__ "c is positive because the vertex > 0 on y-axis."

__ "c is positive because the vertex is above the x-axis."

__ "c + the vertex is in the first quadrant."

__ "c -- positive (the end point of the parabola is positive)."

Brian illustrates that. He said first that "c" was positive because of the vertex location. When asked to explain his statement, he added:

> The vertex of the parabola was about, was in a positive quadrant. That was as close as I could come. The x value was positive so I said the "c" value was positive.

Most of the explanations that used vertex location as an argument, used that in a static way. Bert used it dynamically. He talked about a vertical movement of the vertex in his explanation.

> Okay, and then the "c." That just moves the vertex up or down and then it's positive if it moves it up and negative if it goes down.

Although the location of the vertex has (almost) nothing to do with the sign of "c" (see, for example, Figure 6.1), Bert's explanation points to the idea that the use of the location

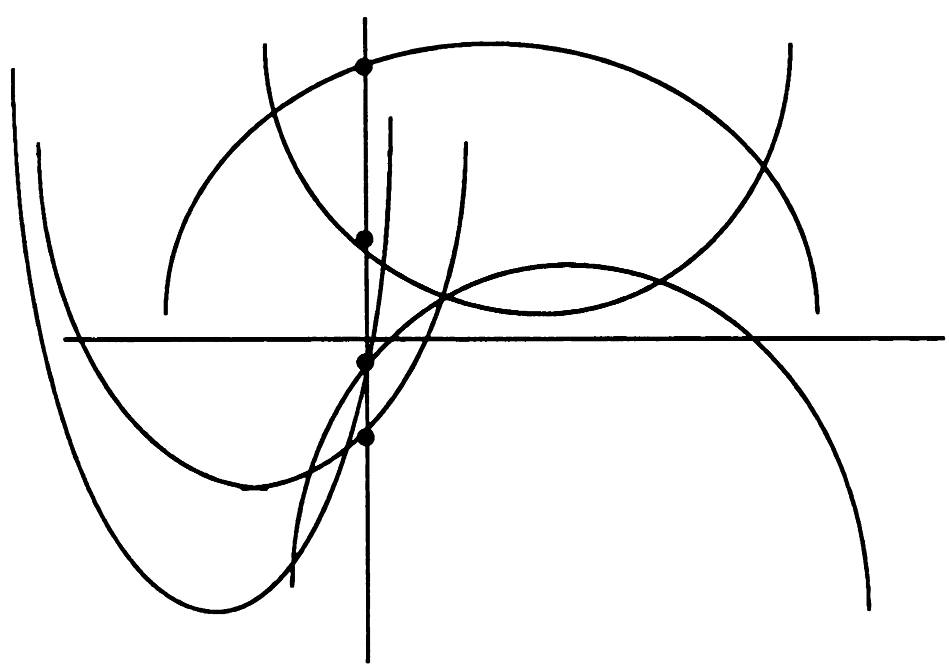


Figure 6.1 -- Negative and Positive c-Value for Different Vertex Locations.

of the vertex was actually similar to shifting the graph of $f(x) = x^2$ (or $f(x) = -x^2$) in either a positive x or a positive y direction (or a combination of the two). Some of the participants used this shifting explicitly in their explanation.

— "c is positive: shift right."

— "c is positive (shift up from x axis)."

Most of the people who used the "shifting" argument chose either the vertical shifting or the horizontal shifting in their explanation. Jenine was not sure first:

> ...and then here we can see that at the origin, but we know that it's not at the origin, that it's over to the right. That means that... and then we also said that it was, instead of being at the origin, it was up above from it, so we went to the right and up, then "b" and "c" are positive.

When asked if she could be more specific about "c", she tried some examples (e.g., $y=x^2+1$) and decided that the sign of "c" was influenced directly by a vertical translation.

> Okay, so by having the "c" positive it would move it up and then by having the "b" positive it would move it to the right.

After being asked if her conclusion holds for any quadratic function, she graphed $y=-x^2+1$ and concluded:

> (Pause). Okay, I didn't think about that, I'll have to try [graphing $y=-x^2+1$]. No, okay, for "c" it doesn't matter, it would still hold.

Again, the examples in Figure 6.1 show that neither shifting up nor shifting to the right guarantee positive "c" (or negative, as some claimed). The graph of $f(x) = ax^2 + bx + c$ is $|c|$ units higher (or lower) than the graph of $f(x) = ax^2 + bx$ if and only if "c" is positive (or negative). But the location of the vertex can still be in either quadrant for any "c." The "c" term determines only the y-intercept and not the location of the vertex. The latter is determined by the three parameters -- a, b and c. Only when the function is of the simpler form: $f(x) = ax^2 + c$, that the location of the vertex above the x-axis guarantees a positive "c." Jenine and others made their conclusion based on checking these simpler cases and therefore got it wrong.

Some of the participants claimed that "c" was negative. They usually used the vertex location or graph shifting as their explanation for their choice. Others said that "c" can be either positive or negative or "it does not matter." The following Table 6.2 summarizes the first phase subjects' answers about the sign of "c." Notice that 51 participants answered correctly including an explanation that "c" was positive because the y-intercept was positive.

Table 6.2 -- Distribution of Answers by Choice of the Sign of "c" and by Explanation.

	Reasoning				
	y-intercept	vertex location/ graph shifting	other	no explanation	total
"c" is positive	51	34	11	12	108
"c" is negative	__	3	1	__	4
"c" can be pos. or negative	__	__	__	3	3
doesn't matter	__	__	2	1	3
no answer	__	__	__	34	34
total	51	37	14	50	152

Although a large number of participants (108) correctly said that "c" was positive, many of them (45) based their conclusion on wrong arguments. They overgeneralized a rule about the relationship between "c" and graph shift, based on checking limited examples.

"b" and the graph. The relationships between the signs of "a" and "c", and the graph of a quadratic function are straight forward -- a maximum (minimum) parabola means a positive (negative) "a", and a positive (negative) y-intercept means a positive (negative) "c." But there is not such straight forward relationship between the quadratic graph and "b."

Most of the participants (72 out of the 104 who answered this) decided correctly that "b" was positive in the given graph. This decision was based on different arguments. Some people correctly used the x-coordinate of the vertex.

— "$b > 0$. $\frac{-b}{2a} = x$ at vertex. This x is > 0. $\Rightarrow \frac{-b}{2a} > 0$
since $a < 0$, b must be > 0."

Others used correctly the first derivative, together with the vertex location or the y-intercept.

— "b is positive since its derivative $2ax + b = 0$ (only when $x > 0$ [point to the vertex]), so $2ax$ is negative [He/she found earlier that "a" was negative], $\therefore$ b is positive ($= |2ax|$)."

— "The slope at $x = o$ is positive; therefore, b is positive. ($f'(x) = 2ax + b$, $f'(0) = 2a(0) + b$).

Another method that some people used had a hidden assumption about where 1 is on the x-axis. Probably because of the way parabolae are usually graphed, several people assumed that 1 is close to the x-coordinate of the vertex. Bob did that, realizing during his interview that he might be assuming too much, but did not change his answer.

> Now, I know that since, if I look at the graph of 1, that's going to be a+b+c. Well, I know that "c" is positive, so the sum of it will make up to here [point to the vertex].This is "c" [point to the y-intercept], and I know that "a" is negative, so "a" is going to decrease its [a+b+c] value. Since at 1, which is somewhere out here [point to the vertex] it is greater than 0 and this goes up beyond 0, I know that - I guess I'm assuming that 1 is in here somewhere, that it's not out here, which I guess it could be, but... At any rate, because this is going up above the zero amount then "b" has to be positive to increase its value.

While Bob's method could have been correct if 1 were where it was assumed to be, other participants based their conclusions about the sign of "b" on wrong methods.

Graph shifting or the location of the vertex to the right of the y-axis or above the x-axis, without any other consideration, is not enough to determine the sign of "b." Thirty people from the first phase of the study used this translation argument in their answer.

__ "b is positive -- shifted to the right."

__ "b is positive since the vertex of the parabola is to the rt. of the y-axis."

__ "b positive -- vertex above x axis."

Bert illustrates that. He decided that "b" regulates the axis of symmetry, and therefore shifts the graph in a horizontal direction While "b" does have an influence on the axis of symmetry since its equation is $y = -\frac{b}{2a}$, it is "a" and "b" together that determine it.

Bert and others simplified the situation and ignored the role of "a" completely, adopting a rule from a different context of how to move a graph of a function.

> B: "b" term. The coefficient of the x. That regulates the axis of symmetry. How it's shifted right or left.
>
> When it's negative it shifts to the right and when it's positive, it involves an absolute value. And when it's positive it shifts to the left. And so this obviously shifted to the right on the y axis so I said that "b" is negative.
>
> R: Does it work for all parabolas: maximum and minimum?
>
> B: Yeah, I think so. Yeah, that's what I thought. May be I am wrong.

In many cases people assumed that "b" and "c" have similar roles. One is responsible to moves the graph (or the vertex) in one direction and the other -- in a perpendicular direction. Jenine, for example, said:

> Okay, so by having the "c" positive it would move it up and then by having the "b" positive it would move it to the right.

Although many people cited those translation rules for "b" and "c" in their answers, some seemed to be not sure about them during the interviews. Valerie wrote in her answer to the questionnaire: "b > 0: since max of function is on the positive side of x." But later said:

> V: What else did I say? Oh, then I tried putting in numbers to see what would happen numerically. I thought I remembered learning that but I

wasn't sure.

R: Learning what?

V: The maximum point of the function. I remembered something about that. But I wasn't sure. So I wrote it down.

R: Is it also true when you have a minimum?

V: I'm not sure. I think I remember learning it that way. I thought that's the way they taught it. I think so.

R: Do you have any idea why it works?

V: Well, I'm sure if I played around with it for a while I'd figure it out, if it's true. I don't know if it is or not.

R: What do you mean by playing with equations?

V: Trying things out with that. Working with specific examples. Making up some a's, b's and c's and try them, see what happen.

The following Table 6.3 summarizes the first phase subjects' answers about the sign of "b." Since all the explanations under the "other" category were wrong -- for example, "Opens downwards", "Parabola shouldn't have linear x-term", or "Most y-values are negative" -- we can conclude that most people who got correctly that "b" was positive, got it by using wrong methods.

The case of "b" was the hardest, since there is not a straight forward relationship between the graph and "b." Still, as was the case with "a" and "c", the participants seemed to look for a simple rule by checking a limited number of examples. Since there is not a simple rule for the case of "b", they invented one.

Table 6.3 -- Distribution of Answers by Choice of the Sign of "b" and by Explanation.

	Reasoning					
	x-coordinate of vertex	1st/2nd derivative	vertex location/ graph shifting	other	no explanation	total
"b" is positive	3	5	31	19	14	72
"b" is negative	2	—	4	12	1	19
"b" is zero	—	—	—	1	4	5
"b" can be pos or negative	—	—	—	1	5	6
doesn't matter	—	—	—	1	1	2
no answer	—	—	—	—	48	48
total	5	5	35	34	73	152

Summary and discussion. Most of the participants knew about the relationship between "a" and the graph, but not all of them could explain it. Many had only partial understanding of the relationship between "c" and the graph and they overgeneralized simpler cases to all quadratic functions. Still many of the participants made up a simple relationship between "b" and the graph which does not exist but seemed to match nicely the overgeneralized rule for "c."

A method of checking a very limited number of simple examples was used by many of the participants. Investigation of a situation by checking specific cases is a very powerful strategy in mathematics. Many discoveries were done by inductive reasoning. Looking at specific cases helps with the understanding of the situation. But there are two problems in the way the prospective teachers used that strategy. First, the number of examples and their variety were too limited. Checking, for example, only quadratic functions of the form $f(x) = ax^2 + c$, and then concluding that "c" regulates the shifting of the graph up and down, is not a good strategy. Some examples of the general form of a quadratic equation, $f(x) = ax^2 + bx + c$, should have been also checked. Secondly, inductive reasoning is a very powerful strategy for discovering the rule, and it might also help in seeing why the rule should hold, but it is not enough as an explanation for the existing of the rule. In other words, checking examples in not a proof. The participants in this study were math majors and minors in their last stage of their studying. The fact that so many of them made conclusions and used explanations for students that were based only on checking examples without making sure that all possibilities were covered or using deductive reasoning, seems to point to lack of understanding of what counts as an explanation, and what ways are considered appropriate and acceptable in mathematics for transforming a conjecture to a theorem, i.e., what is acceptable as a proof.

The National Council of Teachers of Mathematics recommends that in grades 9-12, the mathematics curriculum should include principles of inductive and deductive reasoning. Students should experience the making a conjecture by generalizing from a pattern or observations made in particular cases (inductive reasoning) and then test the conjecture by constructing either a logical verification or a counter example (deductive reasoning) (*Curriculum and Evaluation Standards for School Mathematics*, 1988). The prospective teachers in this study accepted rules by using inductive reasoning only, without any attempt to construct logical verifications. This raises a question: Are they ready to help their students learn that "deductive reasoning is the method by which the truth of a mathematical assertion is finally established", as emphasized in the Standards?

The quadratic function is a special and important case of the functions used in high school mathematics. If one understands the relationship between "a" in the quadratic expression and the graph rather than only memorizing the rule, one has the ability to generalize it to related relationships between the leading coefficient of any polynomial and its graph. So understanding the relationship between the role of "a" in the symbolic representation of a quadratic function and in the graphic representation is very powerful. The relationship between "b" and the quadratic graph are specific to the quadratic function only. But the role of "c" in determining the y-intercept of the quadratic graph holds for all polynomials. So, again, this knowledge is very powerful for the understanding of the relationship between the symbolic and graphic representations of a family of functions. Many prospective teachers in this study seemed to lack this powerful knowledge.

In the case of the quadratic function many of the participants used a wrong variation of a correct rule about the role of "c" in relation to the graph. "c" determines the number of units the graph of $f(x)+c$ is above or below the graph of $f(x)$. The next section discusses an extension of this idea.

Relationships between $f(x)$ and $af(bx+c) + d$: From graph to expression and vice versa

A powerful mathematical idea is that of "classes of functions", "types of functions". This allows generalization of a few basic ideas to a large group of instances. Rather than dealing with each new function in isolation, these ideas allow one to look at common elements or behaviors of functions. One such idea is the relationships between the graph of the function $f(x)$ and the graph of the family of functions: $af(bx+c) + d$. The ability to predict how a specific change in the symbolic representation will influence the graphic representation of a function and vice versa, broadens the understanding of the behavior of families of functions, and helps with the understanding of the translation process between symbolic and graphic representations. The importance of understanding the relationship between the graph of the function: $f(x)$, and the graph of the function: $af(bx+c) + d$, is also emphasized by the National Council of Teachers of Mathematics (*Curriculum and Evaluation Standards for School Mathematics*, 1988).

Question 6 from the questionnaire dealt with the problem of graphing sin(x+1) when the graph of sinx was given, and finding the expressions for the functions whose graphs were variations of the graph of sinx.

Change in graph as a result of change in expression. Sketching the graph of a function g such that $g(x)=f(x+1)$, when the graph of $f(x)$ is given, can be done by shifting the graph of $f(x)$ one unit to the left. This is so since if (x',y') belongs to f then $(x'-1,y')$ belongs to g. Sketching the graph of g can also be done by shifting the two axes one unit to the right. Actually, no consideration to specific characteristics of the function at hand -- in our case, the sin function -- should be given.

Most of the participants (112 out of the 136 who answered this) seemed to follow a rule of translation. About half of them (45) shifted the graph of sinx correctly to the left, about half (55) to the right, and about 10% (12) shifted upwards. It is clear that those who shifted up did not have a good understanding of the relationship between the graphs of $f(x)$ and $f(x+1)$, since the change, as a whole, should not be in a vertical direction. They may not distinguish between $f(x)+1$ and $f(x+1)$, which shows difficulties with understanding notation and symbolic representation of functions. They also might not understand the rules of translations, and rather just memorized them wrongly.

Although a large number (100) of the participants knew that the graph of sin(x+1) could be obtained from the graph of sinx by a horizontal translation, most of them did not seem to understand the reasoning behind this rule and therefore made the shift in the wrong direction. Jenine, for example, shifted the graph to the right. When asked about it she said: "To the right because you added 1 and it was a positive 1." She considered the superficial characteristic of having "+1" in the expression but not the meaning of that, and therefore shifted the graph in the positive x direction instead of the negative direction.

Even those who correctly shifted the graph to the left did not always do that by analyzing the situation in a general manner. Rather they referred to the specific given function by checking several points and then connecting them in order to sketch the graph. Bert admitted to not remembering the appropriate rule, so he checked some points (and wrongly used degrees instead of radians).

> Well, I used the x as the degree so the sin of 91 is just a little bit less than 1. So I plotted points. I didn't, I couldn't remember what, you know, when you vary these things, what happens to graphs.
> I just basically took some of the values at 1, and -1, to see what would happen. And I know that they are less than 1 or greater than -1 there. And I just kind of sketched the graph between those points.

While this approach is sufficient for the specific situation, it shows that the person who uses it does not have a more global understanding of the general idea.

Not considering the problem in general terms lead to guesses of how the graph of sin(x+1) might look. Valerie, for example, explained her choice (she sketched the graph of -sinx): "Well, I just kind of guessed. It looked nice." Guessing what the graph might look like instead of using a translation of the given graph may also be an explanation for the graphs of $\sin\left(\frac{x}{2}\right)$, 2sinx, $\frac{3}{4}$sinx and sin2x, that about 10% (of the 136 first phase subjects who answered this), gave as their answer.

Table 6.4 summarizes the subjects' answers by the graph they sketched for sin(x+1). It also summarizes the second phase subjects' use of general analysis or specific points (there are not enough data on the latter from the first phase subjects). We can see that most of the second phase subjects who answered correctly needed to go through the process of checking points. Only one participant answered correctly without finding points for sin(x+1) and then connecting them to a curve.

The proportion of the second phase subjects who correctly shifted to the left was higher than the proportion of the first phase subjects who did that (0.6 versus 0.3). From the background data we know that overall the second phase subjects had a little stronger mathematical background. Therefore, we can assume that most of the participants in this study did not approach the problem in a general manner but mostly memorized a rule (indicated by the large number of people who shifted to the wrong direction -- right or up). Some analyzed the specific situation by checking some points, and some tried to guess what the graph might look like.

Table 6.4 -- Distribution of Answers by Graph for sin(x+1) and by Method.

	First phase	Second phase (Used/Did not use specific points)
Shift left	45	6 (5/1)
Shift right	55	2 (1/1)
Other	36	1 (0/1)
No Answer	16	1 (0/1)
Total	152	10

Change in expression as a result of change in graph. By comparing the graphs of Fig.2 and Fig.3 from question 6 in the questionnaire (see Figure 6.2) to the given graph of sinx, one can find the expressions that describe those functions. The

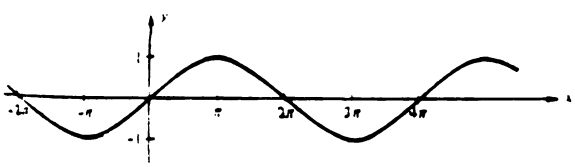


fig.2

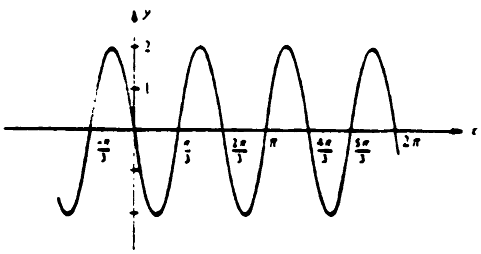


fig.3

Figure 6.2 -- Fig.2 and Fig.3 from Question 6

expression that corresponds to the first graph is sin $\frac{x}{2}$. This expression was found by 50 first phase subjects (out of 131 who answered this). Katie correctly explained why she chose it.

> I realized that we have basically multiplied the x values by 2 which going back to translations and stretches and shrink again, means that you would have sin $\frac{x}{2}$.

A large number of participants (37 from the first phase) also realized that "the x values are multiplied by two", meaning that if (x',y') belongs to the original function sinx, then (2x',y') belongs to the new function. But they interpreted that as if the new expression should be sin(2x) instead of sin $\frac{x}{2}$.

About 10% made a mistake similar to the one that occurred in the case of sin(x+1), when some of the participants confused a horizontal change in the graph with a vertical change, not understanding the relationship between these changes and changes in the symbolic representation. So people chose $\frac{1}{2}$ sinx (5 people) or 2 sinx (5 people). They had problems in distinguishing between two kinds of relationships: the relationship between the graph of f(ax) and the graph of f(x) (a horizontal stretch), and the relationship between the graph of af(x) and the graph of f(x) (a vertical stretch). The four people who chose sinx+c as their answer had a similar difficulty.

The participants in this study suggested 23 different expressions for the first graph in Figure 6.2 -- an average of 5.7 participants per answer. The number of different expressions suggested by the participants for the second graph in Figure 6.2 was much larger -- 72 different expressions, an average of 1.7 participants per answer. The reason for this large number of answers was probably the complexity of the relationship between the second graph and the graph of sinx. While the relationship between the first graph and the graph of sinx were rather simple and included only one component --

stretching the graph twice as much in a horizontal direction and therefore dividing x by 2 -- the relationship between the second graph and the graph of sinx was much more complicated. There were several ways to analyze the relationship between this graph and the graph of sinx. All of them included more than one component. One way was to keep sin as part of the new expression. This way there were three components in the relationship: shrink in a horizontal direction, stretch in a vertical direction, and reflection about either the x-axis or the y-axis (or translation in a horizontal direction). In order to find an appropriate expression, one should recognize all those components and then translate them into the corresponding changes in the expression sinx. Bob explained his correct choice of -2sin3x.

> B: Number 3 is very similar. First of all they've changed, instead of starting up, it starts down. So that tells me there's a negative coefficient in front of the equation itself. It's still a sin curve. The amplitude is twice as much as before, so that tells me there's also two there, so it's -2sin of something. Now again, it's just like before. I'm looking at from valley to peak, back down. And on this original curve from 0 to π I get one peak. Ok, here one, two, three. So I'm going to say that the $\sin(3\pi)$ up here is -2 times the $\sin(3\pi)$ on this graph is equal to this point here at π. So that's the equation I came up.

-2sin3x was the most common answer among the correct answers (14 out of 15 correct answers from the first phase subjects -- 121 answered this). The other correct answer used a reflection about the y-axis -- 2sin(-3x). No one used cos correctly although 9 subjects tried to include that in their answer.

For many of the participants, the situation of dealing with three components of the relationship in order to get a solution, was too complex. They ignored some of the components and concentrated on one or two only. Those people got expressions such as -2sinx, -sin3x, 2sinx and 2cosx.

All the people who found a correct expression to describe the second graph in Figure 6.2 also found a correct expression for the simpler graph -- the first graph (but one, who seemed to misread the scale). Those who answered correctly the first, simpler case but were wrong in the second, more complicated case, usually ignored one of the three components of the change. They chose expressions such as -sin(3x) -- missing the change in the amplitude, 2sin(3x) -- missing the change in the direction. Still some ignored more components and also added wrong ones such as sin3x -- missing both changes: amplitude and direction, and sin2x -- missing all the correct components and adding a wrong one.

The participants used two methods in their search for expressions to describe the given graphs. One of the methods can be described as point-wise approach, i.e., the participants checked some specific points on the graph of sinx and on the other graph. Based on these findings, they suggested an expression. The other method was based on a more global investigation of the behavior of the given graph in relation to the graph of sinx.

Katie illustrates how she analyzed both cases using a global approach:

> I realized that we have basically multiplied the x values by 2 which going back to translations, and stress and shrink again, means that you would have sin of x over 2. Um, the other graph there, similarly, I did remember a little about amplitude, etc. and realized that we were multiplying our y values by 2 to get it that tall, and we, it has been not shifted, but it has been rotated about an axis, so instead of, this could have been written two ways, and I chose to do the rotation about the y axis and put the negative inside, so I wrote negative and it's 3 because, again, we're dividing the x values by 3, so that forces it to have a 3x inside. That also could have been written the negative of 2 sin of 3x. Negative of 2 sin of the quantity 3x. Because that would just rotate it about the x-axis instead of about the y. The sin function, being the way it is, works that way.

Bert illustrates the use of the point-wise approach. He checked the values at some points of the first case: "I figured if sin of 2π, if x is 2π, then it's the sin of π which is 0."

The interviews showed a trend that people were successful in the more complicated case if they found the simpler case expression by using a general analysis of the situation rather than by checking few points. Three people (out of 10) used general analysis in the first case. Two of them also find correctly the second expression, while the third person wrote 2sin3x which is very close to a correct answer. Four people found the first expression by checking some points. Only one of them succeeded in finding also the second expression. Three people who did not find the first expression, also did not find the second. No one managed to find the complicated expression by testing and checking some points. This could only be used with the simple, one-component case. Those who used the general analysis for the simpler case succeeded also with the second one. But almost all of those who tried to solve the first case by checking points, could not solve the second case correctly although they tried to use a combination of general analysis and checking points. So people who can easily and freely use a general analysis of changes in graphs seem to have a better and more powerful understanding of the relationships between the graphic and the symbolic representation, than people who prefer to check some local and specific characteristics.

Was it a trig problem? Some of the participants considered the problem of sketching the graph of sin(x+1) by using the given graph of sinx, as a specific problem related to trigonometric functions and not to functions in general. They apologized for not remembering trigonometry so well since it has been a long time since they worked with it. Valerie, for example, said:

> Yeah, I never remember the sin function. I never remember that. We used it in physics before and I can't ever remember that function.
> I can never remember if the sin function passes through zero or is it the cosine function or how they start. I never remember that, because when you learn sin function you learn the cosine with it and you learn just all these ideas with it, and just, I never could understand it, could never remember it.

The problem itself was a general one, since the same process of analysis could have been employed to any other function. But some basic understanding of the nature of trigonometric functions, such as their domain and range, was needed in order to employ the general method. Lack of this basic understanding may interfere in the process of solution, as we can learn from the following.

Radians or degrees serve as either domain or range of each trigonometric function, and real numbers as the other set (range or domain). When working with these functions, the characteristics of the domain and range become important. Radians are usually referred to by using multiples or parts of π, e.g., π, 2π, $\frac{\pi}{2}$. Degrees are usually referred to by using factors and multiples of 360, e.g., 360, 720, 90, 30. Still, any real number may be used to describe either radians or degrees. In question 6, the subjects had to use the given graph of sinx (where x was radians) to sketch the graph of sin(x+1). The new graph had to cross the x-axis in -1. About one forth of the participants who shifted the graph of sinx in a horizontal direction, marked that crossing point wrong on the x-axis (see, for example, Figure 6.3). The common mistake was to mark it much closer to 0 than one third of $-\pi$ (or π), although $1 \approx \frac{\pi}{3}$. Another was stating that the graph of sin(x+1) is the same as the graph of sinx.

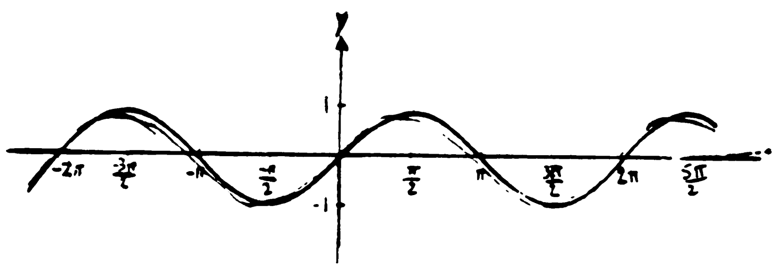


Figure 6.3 -- 1/-1 Marked Wrong on the X-Axis.

The source of these mistakes seems to be mixing radians and degrees together. Bert did that. He referred to π as 180 and to one-half π as 90. Then he substituted 90 for x in x+ and talked about 91.

> B: Okay, I, um, I took the basic graph that was given and I assumed that this was a degree, but I may not be right. Is that a degree or radian?
>
> R: Actually you have to decide.
>
> B: Well, I used the x as the degree so the sin of 91 is just a little bit less than 1 so I plotted points for... .
> ...Okay, I took x = 90. Okay. x = 180. So, these correspond to these in radians [point to $\frac{\pi}{2}$ and π]. So the sin of 91 is slightly less than 1.

Bert said that he used the x-axis as degrees. But at the same time he was aware that the x-axis was marked as radians. Still, he referred to $\frac{\pi}{2}$ as 90 degrees and marked 91 very close to it. Jenine was also not sure whether the 1 means radians or degrees and chose degrees.

> Okay. That one I was really unsure about. And the problem came with adding 1 and, you know, thinking about, um, different if it was in degrees, or if it was in radians. So I got...
>
> Degrees or radians. And it would make a difference, so, because if it was in radians then it would have been much easier to see because, then in radians 1 is, um, 1 is, now I can't..., but that's where I got confused, whether it was adding one degree. If you added one degree it wouldn't have made that much difference...
>
> Okay. What I decided to do was, a sin function. Since the number, sin goes, you know that sin goes from -1 to 1, the sin curve. So that's not going to change. So the only thing that changes is where it is along, along the graph. So since adding 1, if it was a degree, it wouldn't change it very much to add one degree. So I just moved the same shape of the curve just along the line.

Both Bert and Jenine were not sure whether 1 was in degrees or radians, even though the x-axis in the given graph was clearly in radians. One of the reasons for the difficulties might be that radians usually appear as a combination of π and numbers, while degrees as whole numbers. As Valerie said,

> It was a guess. I was trying to think about it. I couldn't remember how it would be, like how that would be represented in x+1, what 1 exactly represents. Because you're working with radians, and so I didn't really understand how that went together. Because here you are working with π and all that. So I wasn't sure.

Another reason for the difficulty people had with thinking of 1 in radian terms might be difficulties in thinking of relationship between two radian quantities, when one of them is not a combination of π and real numbers. Jenine, for example, got completel confused when she tried to think of 1 in relation to π, by converting both to degrees.

> If it was in radians, radians, and that's where I was trying to think, um, if it was radians you'd have like, π radians and 180 degrees. So, 1, you could think about it as, it's not exactly but it's around 60 degrees, since π is 3.1. Then we'd be adding 60 degrees which is really adding, π over, π divided by 3.41 would be 180 divide by 1/4, would be the degrees..., and converting it into degrees, then that way I could....

Degrees and radians were not the only ways subjects perceived the number 1 in x+1. Some people thought of it as one π which represents one complete part of the graph. Brian, for example, (and 7 first phase subjects), gave the following as the graph of sin(x+1) (see Figure 6.4).

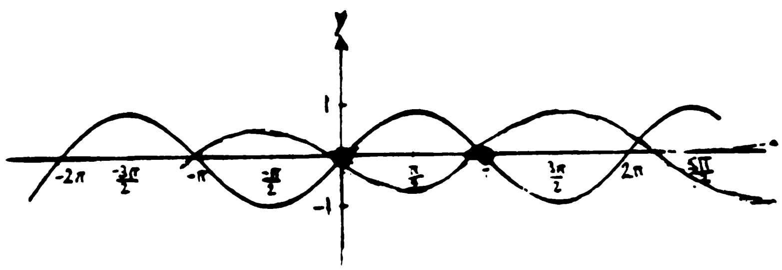


Figure 6.4 -- Brian's Graph of sin(x+1)

Brian explained how he got the graph.

> The sin of x+1 looks exactly like the sin of x, except that I added 1, so I have to shift the graph here left to right by one point or number 1. So I said, okay, find on that graph, sinx, a place where the sin of x equals, or actually find the sin of 1. All right, I messed. Bracket, 1, bracket [writes sin(1)].
>
> That's what I was looking for. And I said, that should correspond to point where, on the graph of the sin of x, to the sin of 0. Okay, so I just had to shift it, and it turned out that the sin of 0, is 0. So it crosses the x-axis and I said okay, that same cross or that same height on the x-axis, the same y-coordinate, is going to be used for the sin of 1.
>
> Oh, I shifted it 0+1, so it's shifted to the right. The graph moved 1 full or actually 1/2 cycle to the right, I think.
>
> But I shifted it a whole unit to the right. I assumed that π was the sin of 1, and then I shifted...

It seems that Brian considered π as 1, so 1 became a unit that measures the distance between two zeros of the function sinx. A similar explanation might also suit some of the cases where people claimed that the graph of sin(x+1) was the same as the graph of sinx. They might have considered the whole cycle, or 2π, to be 1.

The prospective teachers seemed to have difficulties dealing with radians. They were not sure about the meaning of a whole number in relation to what they considered to be an adequate domain for a trigonometric function. Not having a good understanding of domain and range of trigonometric functions means difficulties in working with these functions. But trigonometric functions are part of the high school curriculum. So prospective secondary teachers should be familiar with them. Teachers need to feel comfortable and familiarity with the functions they teach. Even if the problems are actually problems about functions in general and use trigonometric functions just as a case, as in problem 6, difficulties with trigonometric functions prevent the solver from reaching a correct solution. This also causes students to approach those problems as problems in trigonometry instead of problems about functions in general. Familiarity with basic functions, such as trigonometric and polynomials functions, can help not only in the understanding of those specific functions but also with the understanding of functions in general.

Summary and discussion. Overall, the prospective teachers had difficulties translating from symbolic representation to graphic representation and vice versa when dealing with two basic functions from the high school curriculum: quadratic and sine functions. The participants seemed to base their conclusions on checking of a few examples. This was the case when they made up rules and explanations about the relationship between the shape of a parabola and the signs of the coefficients of the quadratic expression. This was also the case when they used a point-wise approach to decide about the shape of the graph of sin(x+1) or about the expressions that describe given graphs.

Most of the people who correctly sketched the graph of sin(x+1) needed to go through the process of finding points for sin(x+1) and then connecting them to a curve. While this approach was sufficient for the specific situation, the choice to use it instead of analyzing the situation in a more general way, shows a lack in understanding of the more global characteristics.

The close relationship between understanding and the global approach to functions became apparent when the analysis of the answers to question 6 also revealed a trend that people were successful in the more complicated case, if they found expressions for the simpler case by using a general analysis of the behavior of the function rather than by checking a few points. It seems that in cases where both point-wise and global approaches to function can be used, people who can easily and freely use a global analysis of change have a better and more powerful understanding of the relationships between graphic and symbolic representations, than people who prefer to check some local and specific characteristics.

The tendency to use point-wise approach to functions seems to be related to difficulties in translation between symbolic and graphic representation. The next section investigates further the participants use of point-wise and global behavior to approach functions.

Point-Wise Approach and Global View of Functions

Functions can be approached and dealt with in four different ways: Point by point, interval by interval, globally, and as entities or objects. Neither one of these ways is appropriate for all situations. But sometimes, when more than one way can be used, some ways are more appropriate than others. Many students deal with functions point-wise i.e., they can only plot and read points, but cannot think of a function as it behaves over intervals or in a global way (Bell and Janvier, 1981; Janvier, 1978; Lovell, 1971; Monk, 1988). The next section describes ways the participants in this study chose to approach different problems which would be better approached by analyzing the function as a whole and not point-wise.

Graphing

Graphing is usually taught first by making a chart of some values of x and y (usually small whole numbers and their inverses, which are easy to deal with), plotting the points and then connecting them in order to produce a smooth curve. While this method is not hard to learn, it emphasizes a point-wise approach to functions, which in many cases is less powerful than a method that emphasizes graphing based on a more global analysis of the behavior of a function. For example, graphing a quadratic function that has (-100, 78) as a vertex by plotting several points near (0, 0) will not produce a very informative graph. More than that, graphing a function that is discontinuous at x=0.3 by plotting several points with whole number coordinates, and then connecting them to make a smooth curve will produce the wrong graph. The second phase subjects were asked to graph two different functions. Results of the ways they approached these tasks are presented in the following sections.

The case of $f(x) = \frac{1}{x^2-1}$. The graph of the above function is not continuous at 1 and -1. More than that, $\lim_{x \to 1^+} f(x) = \infty$ and $\lim_{x \to 1^-} f(x) = -\infty$. A similar behavior of the function occurs near -1. Approaching the graphing of this function by substituting some values of x which are close to 0 into the expression and then plotting the points and connecting them to a smooth curve, will not produce the graph of the function.

The 10 interviewees were asked how they would explain to a student in algebra 2 how to graph the function. Half of the participants started their explanations by suggesting the use of the point-wise approach. The other half suggested to look first for undefined points -- an approach that pays attention to the behavior of the function. Mike illustrates the point-wise approach.

M: I would just have them plot points and see the shape.

R: Any specific points?

M: Not really...You want to start with smaller numbers to make it easy. And then, positive, negative...Only 3, 4 points.

Bob illustrates the other approach.

Well, the first thing I would do is say to him, we are assuming that they understand that anything divided by 0 is an undefined term. Then, I would ask him, 'at what value of x is x^2-1 equal 0?' And it will be two: 1 and -1. And I'd make sure that they understood that at those two points that the function doesn't exist.

And then I'd say, 'well, what's it going to look like?' You have to touch three different places: when x is less than 1, when x is between 1 and -1. And when x is greater than 1.

And, okay, from there I'd probably have them draw a graph. First, I would find the points, 1, and -1. And from that point I'd just draw a vertical asymptote, because there's no defined function value there. Then, at values less than -1, when you square them, they are going to be larger than 1. But when you subtract 1 from them, it could still be very small. So, when you divide a number by a small number, it will be very large. So, it would probably start way up here like this, and as it comes out like this.

Anything greater than 1 is going to do the same thing. If the value was 1.00001, that's a very small number - 1 divided by the difference would be a very small number. So again, it's going to start up here and come around and go like that.

At points in between they're all going to be fractions and when you square them, they become smaller. But the closer you get to 0, the larger 1 divided by x^2 is going to be. So I think that probably it will look something like this [drawing]. I found where 0 will be. So, it looks like this.

Some of the people, who first suggested the use of the method of plotting some easy numbers, discovered, while working on the function, that it was undefined at 1 and -1. So they added that attention should be paid to undefined points. But they did not think of this problem when they started their explanation. If the problematic point was not so easy to discover, it seems that they would be satisfied with the plotting points approach.

Interestingly, all these people have had calculus and other advanced courses in mathematics. So, all of them should have known that analysis of some characteristics of the function to be graphed are important. They also should have known that some points are more important than others, so producing good graphing cannot be based on the choice of numbers that are easy to compute. Easy calculation is especially not a good argument at the end of the 1980's, when calculators and computers are so widespread. The National Council of Teachers of Mathematics also recommends avoiding graphing by hand using tables of values (*Curriculum and Evaluation Standards for School Mathematics*, 1988). But it seems that some of the prospective teachers had a strong tendency to use it as the graphing method.

The case of $g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$ This function is a variation of the famous Dirichlet function which is considered to be a breakthrough in the development of the concept of function. The graph of the above function does not look like a graph of a function. It looks like the graphs of y = x and y = 0 together (see Figure 6.5) since both sets -- the rational and the irrational numbers -- are dense. Therefore, it might seem that the function does not pass the vertical line test i.e., it is not a function.

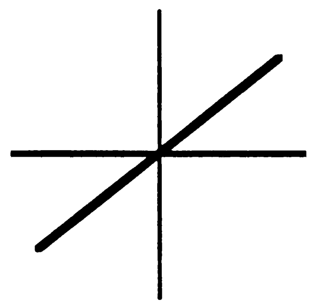


Figure 6.5 -- Correct Graph of $g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$

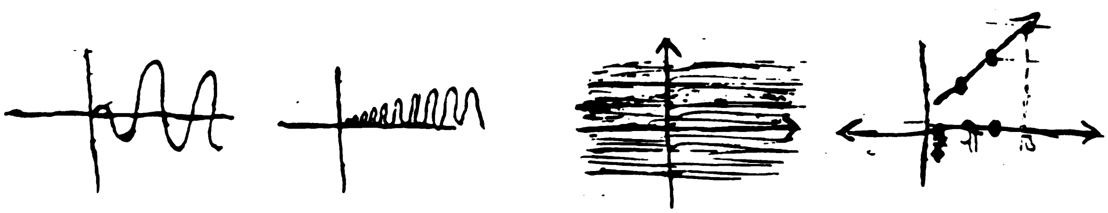
The 10 interviewees were asked to graph the function. One half of them admitted of seeing something similar in their course of study. Katie, for example said: "It's a classic in 424. They like to use that to show discontinuity." These people correctly sketched the graph as in Figure 6.5, or a slight variation of it. Most of the people who knew how to graph the function added that actually one can and one can't sketch the graph since it will look as if it is not a function. Mike illustrates this point.

There's so many infinite many rational numbers and infinite many irrational numbers and what you get is a function - something that looks like it isn't a function.

Like there'd be so many dots - it's just like for all the rational numbers, it's just graphing this top part - g(x)=x [points to the diagonal line in the graph], while irrational numbers, you get something like this [points to the horizontal line y=0 in the graph]. And it would really be a bunch of dots because there will be spaces where the irrational numbers are.

It wouldn't get lined and so in that sense you can't really graph it. Because you can't draw infinite points.

Still, 4 people suggested different graphs (see Figure 6.6) and one other subject



Tracy's Jenine's Evelyn's Brian's

Figure 6.6 -- "Graphs" of $g(x) = \begin{cases} x, \text{ if } x \text{ is a rational number,} \\ 0, \text{ if } x \text{ is an irrational number.} \end{cases}$

said that it was impossible to graph the function. Three of these people seemed to approach the problem of graphing the function point-wise. They started from 0 and then tried to sketch the graph point by point, as if the set of real numbers and/or the set of irrational numbers is countable. Tracy, illustrates this point.

1 is going to be 1, π is going to be 0. e is going to be 0, 2 is going to be 2 (pause). Well, it's going to be smooth everywhere except where you get to an irrational point and then you're going to have a sharp point which is not going to be continuous. Wherever there's an irrational number it's not going to be continuous.

It's going to curve up until it gets to, like if there's an irrational number between 0 and 1 it's going to go down to the irrational number, and then it's going to, we've got our negative numbers too. Maybe it's like this [points to the graph -- see Figure 6.6], except it's not going to be smooth because every time you hit zero it's going to come straight down and then it will

> have to go to the next rational or irrational number. I'm being really general about this.

Tracy thought of the real numbers as a countable and even discrete set, as if one starts from 0 and keeps going to the next number; as if one goes through several numbers until one hits an irrational number. She held misconception about the structure of real numbers which suited the point-wise approach to graphing that she used. Interestingly, Jenine, who said: "Between any two rational numbers there is an irrational number", and therefore seemed to have a good understanding of the structure of the set of real numbers, continued to say: "If you started at (0, 0), to me it would look, I would think you did something like that [points to her graph in Figure 6.6]." So, Jenine also held a conception of the real numbers as a discrete set. Her conception of the idea that between any two rationals there is an irrational number, seemed similar to the idea that between any two even numbers there is an odd number, ignoring completely the density of the sets. Again, having this conception of the real numbers suits a point-wise approach to graphing that she used.

Two people who got wrong graphs seemed to approach the graphing of the function in a global way. Brian had difficulties with the irrational numbers similar to Tracy's and Jenine's. He correctly considered the rational numbers as a dense set, but assumed that the irrational numbers were discrete. So he described his graph (see Figure 6.6) as "a straight line, a diagonal line with holes in it...The rational number block should go diagonally, horizontally a series of dots." Not like Tracy and Jenine, Brian did not try to connect one point to the "next". So his approach was more a consideration of the behavior of the function and not so point-wise. Evelyn also seemed to look at the behavior of the function but made a wrong interpretation of that.

Summary and discussion. A large group of the interviewees approached graphing by using a point-wise approach: plotting some points (usually ones that were easy to calculate) and connecting them to a smooth curve. This method is not recommended for graphing since it ignores important characteristics of the function and therefore might end up with a graph which is either wrong or not informative. The use of a point-wise approach to graphing also revealed misconceptions about the structure of number systems. Several participants held a wrong, limited view of the real numbers as a countable set and the irrational numbers as a discrete set of numbers. The real numbers are used as domain and range of many functions in the high school and college curriculum. Having a wrong conception of the structure of the sets which serve as domain and range leads to wrong understanding of the function itself, as was presented in the wrong graphs given by the participants. Understanding function is interrelated with the understanding of its domain and range.

So far we discussed different ways for graphing function. The next section describes ways of interpreting given graphs.

Interpretations of graphs

Graphs are helpful in describing real-world situations. In many cases, where a symbolic representation cannot be used to describe a problem situation, a graphic representation can be used. Janvier (1978), and Bell and Janvier (1981) found that secondary students are weak in their ability to interpret global graphical features. These students had no problems with reading and plotting points. But they also used a point-wise approach when a global way of interpreting function should have been used or was faster and more appropriate. For example, when the answer should have been an interval, students gave specific points. Also, intervals were obtained by referring back to the axis, reading off the values and subtracting them, instead of reading the differences directly from the scale on the axis, without subtraction, or reading it more directly from the grid in the body of the graph. In addition, students could not interpret graphs representing situations whenever pictorial aspects conflict with correct meaning. For example, they interpreted greatest increase as greatest value. Lovell (1971) describes similar results. The next part describes ways the participants in this study chose to interpret graphs describing situations.

Point-wise and global interpretation of graphs. Question 10 from the questionnaire (see Figure 6.7) was easier to the participants than question 6 (see

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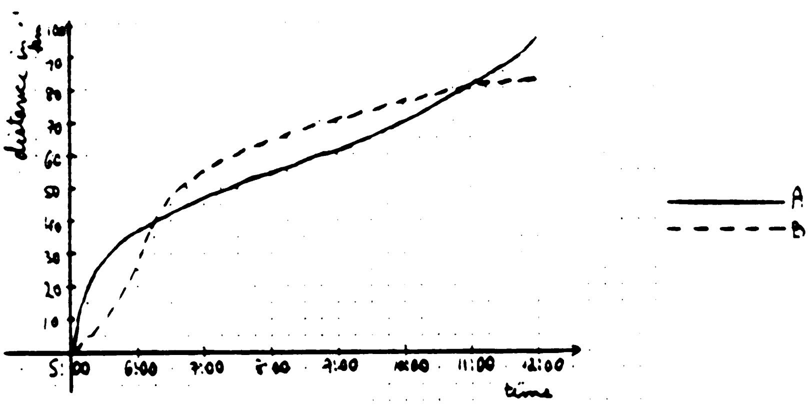
This graph describes the distances A and B travelled.



a) For what times had A travelled further than B? Explain.

b) At what time did A travel the fastest? Explain.

c) Who travelled faster between (i) 5:30 and 6:00? (ii) 8:00 and 9:00?

d) Who travelled more km between (i) 6:30 and 8:00? (ii)10:00 and 11:00?

****

Figure 6.7 -- Question 10 from the Questionnaire

Appendix A) since there was no need to translate functions from one representation to another, but interpret a graphic representation. Part (a) of question 10 was rather easy. Questions like this can even be found in some middle school texts. Therefore, it was not surprising that most of the first-phase subjects (101 out of the 141 who answered this) considered the behavior of the two functions, and correctly found that the times when A had travelled further than B were 5:00 - 6:15 (or 6:00, as some said) and 11:00 - on (or

12:00, as some said). A common explanation was based on the relative heights of the lines. Huey illustrates that.

> Well, for the first hour and 15 minutes, excluding that 15 minutes, A travelled farther than B. It's just because the distance here is more.... Because at 5:00 they're both at 0, but at 5:15 he's went 20 kilometers, B went about 5. And so, anytime, just by looking at the chart you can see, anytime, the dark, solid line is above the dotted line, means A went farther than B. So, from 5:00 to 6:15 and from 11:00 on.

Some people (17 first phase subjects) got only one interval as an answer -- 5:00 - 6:15, or 11:00 - on, instead of two. This might have happened because they did not look at the behavior of the functions as a whole, but only at some parts -- an interval-wise approach rather than a global one. The data do not provide enough information to conclude that.

Although not many, there were still some people (10 first phase subjects out of 141 who answered this) who approached the problem point-wise. They gave whole number answers: 6:00, 6:00 & 12:00, and 5:00 & 6:00 & 12:00, although they, usually, used the same explanations that the more global answers -- intervals -- used. Jenine, for example, said

> All you had to do was look at the different times. 5:00 they start out the same. At 6:00, B had travelled approximately 30 and A had travelled 40. So at 6:00 they travelled... Then, I did the same thing with 7:00.
>
> And since the curve was below, A was below B, for all those distances, since the shorter distances went up increasingly, then I knew that the next time, the only time I had to look for where they were going, had gone farther, was where the two lines crossed again. So, I didn't have to check all them. And then, where they met, that meant they had travelled the same distance, and then A went farther, so that had to be 12:00.

On one hand, Jenine seemed to look at the behavior of the functions, when she said that she did not have to check the times until the lines met again. On the other hand, she referred to whole number hours only, ignoring the rest completely. Valerie's method was similar, even though she used intervals instead of just points. She chose to check some specific intervals, and actually answered a different question: At what whole hour blocks did A travel faster?

> I just looked right on the graph - I said, like, take the block from 5:00 to 6:00. B went, like, 25 kilometers, and A went, like, 37 kilometers. I just did that, looking at the graph. Which one was going further in, like, hour blocks.
>
> And then I checked for the next, 6:00 to 7:00, 7:00 to 8:00. After I got down to, like, number c, letter c, I was thinking, ooh, may be I should have checked half hour. But I decided to go with just hour blocks instead.
>
> ...I looked at 9:00 and I said they were at 60 for A, and then by 10:00 A had gone to 70, so they had gone 10 kilometers. And B had gone from about 70 to 75, so B has only gone, like, five kilometers. So, I did in hour blocks, but you could do different. You could do it half hour or as marked even here. You can see the dots. You could do it 15 minutes blocks.

Doing what Valerie suggested, and using different intervals, could lead to different answers than the ones she got. This did not seem to bother her. Valerie and Jenine did not analyze the two functions as a whole, but rather checked some specific points (although in two different methods). This non-global point of view led them to wrong answers.

Between 5:00 to 5:25, the graph of A looks almost like a straight line. Therefore, for part (b) of question 10, answers such as 5:00, 5:15, 5:30, 5:00 - 5:15 and 5:00 - 5:30, could all be considered correct. 5:00 - 5:30 was the most "popular" answer together with the answer 5:00 - 6:00. The explanations used to support the time chosen, were usually based on the steepness of the graph or the slope, i.e., they looked at the behavior of the function:

- "The first 15 minutes -- steepest slope here."
- " 5:00 and 5:30 The graph is almost vertical compared w/B."
- "5:00 - 6:00. Obvious from graph (slope)."

Some people seemed to use a similar approach by explaining that the speed was the greatest when the distance increased a lot over a short period of time.

- "5:15 here the distance is increasing a great deal in regard to time."
- "Between 5:00 and 6:00 because he travelled the farthest in the shortest period of time."

But using increase in distance versus change in time may also be similar to a more point-wise approach where some kind of kilometer counting was used.

__ "From 5:00 - 6:15. A travelled 40 miles in 75 min's (the graph increased the greatest amount then."

__ "5:00 - 6:00 → travelled an ave. of 40 km per hr -- the fastest."

Table 6.5 presents the distribution of the first phase subjects' time choices and explanations. The first explanation -- steepness of graph or slope -- is based on the function behavior, the second -- kilometer count -- on point-wise approach. The third explanation may be based on either one of the first two.

Table 6.5 -- Distribution of Answers to Question 10b.

	5:00	5:15 (or 5:30)	5:00-5:15 (or 5:30)	5:00-6:00 (or 6:15)	Other	N/A	Total
Steepness of graph or slope	9	3	29	20	6	1	68
Km count	0	1	2	7	-	-	10
$r=\frac{d}{t}$	0	1	6	8	3	-	18
Other	1	0	2	3	4	-	10
N/A	2	2	10	5	10	17	45
Total	12	7	49	43	23	17	152

Note. N/A = No Answer

Two different kinds of approach guided the answers to this question. One was to look at the behavior of the function (the graph of A) as a whole, and find the point or the interval on the x-axis that corresponded to the steepest part on the graph. Most of the people who chose 5:00, 5:15, 5:30 and used "steepness" for explanation, and some of the people who used "most distance over a period of time" or chose 5:00 - 5:15 (5:30), belong to this category. This approach seemed to dominate the answers to this question.

The other approach was to decide on some time intervals and choose one of them as the answer. Most of the people who chose 5:00 - 6:00, or used kilometer counting, and some of the people who chose 5:00 - 5:30 (5:15) or used "most distance over a period of time", belong to this category.

The data do not provide enough information on the relative number of people who used either approaches, since after choosing the answer by one approach, the participant may choose to explain it either by referring to the behavior of the function or by providing actual speed or distance (kilometer count). So our report on the first phase subjects is based on the explanations people chose to provide. The interviews give a better description, even though not complete, of the ways people solved the problem. Katie illustrates the first kind of approach which looked at the behavior of the graph and also used it as her explanation. She chose one point, 5:00.

> I approached it kind of along the lines of calculus, which was instantaneous speed will be the slope of the curve. So, certainly it would be A, starting at 5:00.

Tracy used the same idea of steepness, both for her method of finding the answer and as an explanation. But she chose a time interval, 5:00 - 5:30.

> I would say right around, right in the beginning between 5:00 and 5:30. Because the curve is almost going straight up at that point. After that it starts to level off.

Valerie, like Tracy, chose 5:00 - 5:30. But she did not explain it by referring to the behavior of the function as a whole, but rather by counting kilometers for specific time intervals.

> For A, in the first half hour. They went like 30 kilometers per half hour, and they never did that again for a speed for a half hour. You never saw them doing that many kilometers in just a half hour.

When asked how she knew that this was the case, she said "Because of the slope. It was such a steep slope." So, Valerie did not completely ignore the behavior of the function, but she did decide to look at half hour blocks. Her approach was different from the one that looked at the function as a whole. Valerie also chose to check the actual number of kilometers to support her explanation.

While Valerie's case could be considered as a borderline -- she did get a correct answer by choosing half hour blocks, so she might have been aware of the behavior of the function when she chose those specific intervals -- Mike's choice of 5:00 - 6:00 clearly ignored the behavior of the function as a whole. He explained his choice by providing kilometer count:

> From 5:00 to 6:00 A travelled 40 km. From 6:00 to 11:00 A only travelled another 40 km.

When asked how he knew that he had to choose that specific interval, he, like Valerie, added comments about the behavior of the graph.

> The graph here is increasing quite rapidly, the distance is increasing quite rapidly, the time is not increasing quite rapidly.

But around 5:30 the distance stopped increasing so rapidly. This was not considered by Mike, since he decided to look at hour intervals.

Summary and discussion. The participants in this study were presented with several problems where they had to translate from symbolic representation to graphic representation and vice versa or where they needed to interpret graphs (questions 6, 10 and 11(v) from the questionnaire and question 8 from the interview -- see Appendices A and B). In all of those problems the subjects could have used a point-wise approach or a global view of the behavior of the function. The latter was more appropriate to use in the given situations. Most people were not consistent in their choice of method, but rather interchanged methods in different problems or used a combination of the two approaches in one problem. Only one person consistently used one approach throughout. Katie used a global approach and looked only at the behavior of the functions in all the problems. The mixed use of the two approaches (as was shown, for example, in part (b) of question 10, but occurred in all the parts of question 10) shows that people were familiar with both methods. But in many cases they used less appropriate knowledge.

People tended to use a point-wise approach when translating from one representation to another. But when the translation process was too complicated for a point-wise approach, the subjects used a combination of the two methods. People tended to use a global approach when interpreting graphs representing situations whenever pictorial aspects did not conflict with correct meaning. For example, when graph A was above graph B also meant that A went further than B. The next section describes how the participants in this study interpreted graphs representing situations where pictorial aspects conflicted with correct meaning.

Meaning of Graphs. Analysis of the questionnaire data that dealt with interpretation of graphs in relation to given situations (questions 8, 9 and 10 from the questionnaire) shows that most of the participants did not get distracted by seemingly conflict pictorial aspects. Still, about 15% of the first phase subjects seemed to be influenced by superficial characteristics of the graphs and therefore chose a wrong answer. During the interviews it became apparent that some of those who did choose the correct answer and correctly interpret the given graphs, were willing sometimes to accept a wrong answer. This will be discussed later.

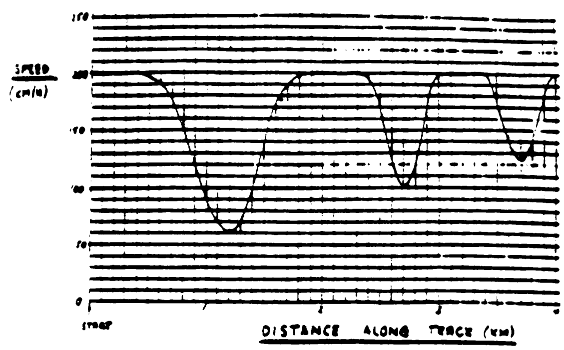
Circuits A and B in question 8 (see Figure 6.8) had a similar shape to the graph but much more than three curves as should have been interpreted from the graph. This similarity in shape was enough for 22 first phase subjects (of the 151 who answered this) to wrongly choose either circuit A or B, as the circuit which had produced the given graph. Katie, for example, chose circuit B. She explained her choice by referring to the superficial similarity that circuit B had with the graph: "...if you do cut this and roll it out, it does look very similar to the graph."

****

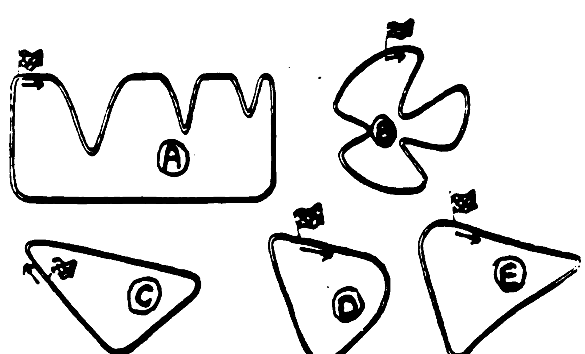
High-school students were asked the following question (taken from ***The Language of Graphs*** *by Malcolm Swan):*

A Motorcycle Race

The graph drawn below, shows how the speed of a motorcycle varied as it completed the second lap of a race.



Which of the circuits below would have produced this graph?



a) Which circuit has produced the graph? How did you decide?

b) For each of the four circuits you rejected, briefly explain what might make a student choose them.

****

Figure 6.8 -- Question 8 from the Questionnaire

In question 9 (see Figure 6.9) most of the first phase subjects (124 out of 147 who answered this) correctly chose graph 2 as Tim's and graph 6 as Donna's. Table 6.6 summarizes their choices.

****

Tim and Donna live 1 km from their school. Usually they walk together to school. Yesterday they both left their houses 10 minutes before school started. Tim started to walk but Donna was afraid to be late and started to run. After a while Tim realized that although he tried to walk faster and faster, he had to run if he did not want to be late, and started to run. At the same time, Donna became tired and had to walk instead of run. They both reached school exactly on time. Which of the following graphs is Tim's and which one is Donna's?

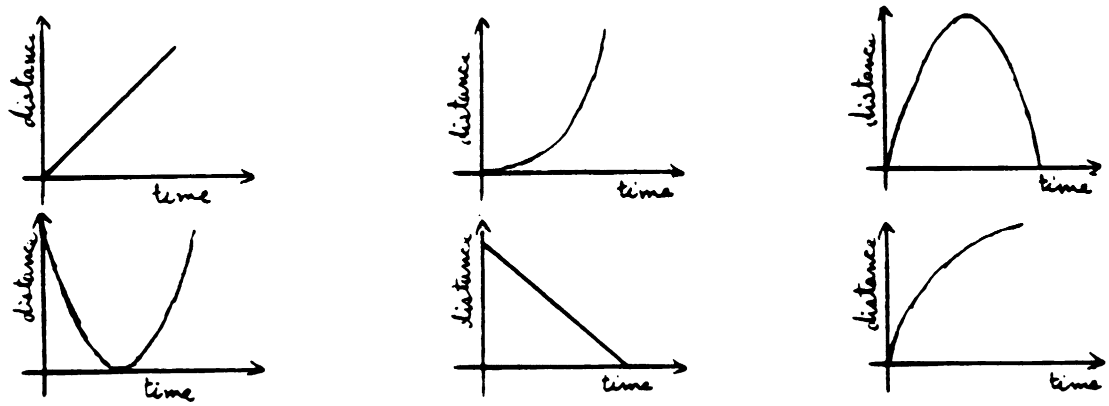


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Figure 6.9 -- Question 9 from the Questionnaire

Table 6.6 -- Distribution of Choices of Graphs for Tim and Donna.

	Donna's graph							
Tim's graph	1	2	3	4	5	6	No answer	Total
1	1	-	-	-	2	-	-	3
2	-	-	11	-	3	124	-	138
3	-	-	-	-	-	-	-	0
4	-	-	2	-	-	-	-	2
5	-	-	-	-	-	-	-	0
6	-	3	1	-	-	-	-	4
No answer	-	-	-	-	-	-	5	5
Total	1	3	14	0	5	124	5	152

We can see from Table 6.6 that graph 3 was problematic for about 10% of the participants who wrongly chose it as Donna's. Interestingly, the corresponding graph 4 was chosen as Tim's graph by two subjects only. But graphs 3 and 4 were not really symmetric. By changing the label of the y-axis from 'distance' to 'rate', graph 3 became appropriate for Donna while graph 4 remained inappropriate for Tim. Bob illustrates this point.

> B: Ok. They both start at the same time. I think this one here has to be Tim [points to graph 2], because he starts out walking, and then slowly increases the speed and then starts to run. And so as he starts, his distance is going to be fairly, well, directly related to time. But the more he gets to here, it's going to be much greater distance for a shorter time, when he starts running. Donna, on the other hand, I think is this one [points to graph 3]. She's going to start out running which means a slow acceleration. She gets up the speed, she's going to cover more and more distance. Here she's getting tired and she's slowing down. So this is Tim's and this is Donna's.

When talking about Tim, Bob clearly talked about distance versus time -- the way the graphs were prepared. He did the same thing while starting to talk about Donna, but when he reached the "top" of the graph, he got confused by the superficial characteristics of the graph, and interpreted the decrease as showing a rate decrease. Actually, he used two labels for the y-axis: first, the correct label -- distance, and then -- rate. Evelyn had the same problem of interpreting graphs while ignoring the labels on the axes. But she sticked to one label -- rate -- and did not switch between the two. She chose graph 5 for Donna.

> But the thing about Donna was that she started at a very fast speed and the only one that started at a fast speed... However, she slowed down but didn't pick her speed up again.

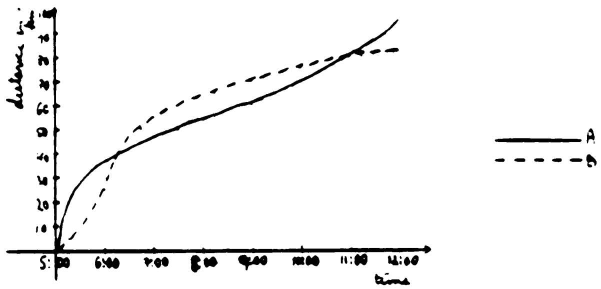
This interpretation of graph 5 could have been correct if the y-axis was labeled as "rate" instead of "distance." Having this interpretation in mind, Evelyn could not decide between graphs 1 and 2 for Tim (since both of them were appropriate if the y-axis were "rate") and made her final choice graph 1 so that Tim's graph would be the opposite of Donna's.

The interviews revealed that this misinterpretation of graphs and the labels attached to the axes was more widespread than it seemed just from the questionnaire data. Some other people from the first phase might have chosen graph 2 for Tim while thinking, like Evelyn, that graph 1 was also appropriate. Tracy, for example, correctly chose graph 2 for Tim as her answer to the questionnaire. But when talking about it in her interview she explained that graphs 1 and 6 were also appropriate: " ...depending on the rate you wanted him to walk in, how you wanted him to increase."

This phenomenon of attaching meaning to a graph by ignoring what is given and considering surface characteristics only was also apparent in some of the answers to part c of question 10 from the questionnaire (see Figure 6.10). Most of the first phase subjects (103 out of 139 who answered this) correctly said that B travelled faster between 5:30 and 6:00, and that between 8:00 and 9:00 both A and B travelled at the same speed. Table 6.7 summarizes their answers.

****

This graph describes the distances A and B travelled.



c) Who travelled faster between (i) 5:30 and 6:00? (ii) 8:00 and 9:00?

****

Figure 6.10 -- Part c of Question 10 from the Questionnaire

Table 6.7 -- Distribution of Answers to Part c of Question 10.

	Between 8:00 and 9:00				
Between 5:30 and 6:00	A	B	Same	N/R	Total
A	0	14	9	0	23
B	8	5	103	0	116
Same	0	0	0	0	0
N/R	0	0	0	13	13
Total	8	19	112	13	152

Note. N/R = No Response.

We can see from Table 6.7 that about 10% of the first phase subjects said that A travelled faster between 5:30 and 6:00, and B -- between 8:00 and 9:00. An additional 10% said one or the other. It seems that the source of this mistake was, again, ignoring the meaning of the graph and concentrating on surface characteristics -- in this case, having the graph of A higher than the graph of B in the first time period and vice versa in the second period. (The relative altitude of the graphs has nothing to do with this problem. Rather their relative steepness determines their relative speeds and therefore -- who travelled faster.) Tracy illustrates this point when she explains her choice of A as the one who travelled faster between 5:30 and 6:00: ."..because the dark curve was above the dotted curve." Tracy seemed to concentrate on superficial characteristics of graphs while ignoring meanings.

Summary and discussion. The questionnaire data suggested that about 15% of the participants could not interpret graphs representing situations whenever pictorial aspects conflicted with correct meaning. But the interview data revealed that the problem was more widespread than that. People who made correct interpretations were also willing to accept wrong interpretations of the same graphs as well.

As indicated earlier in this chapter, it seems that in cases where both point-wise and global approach to function can be used, people who can easily and freely use a global analysis of changes have a better and more powerful understanding of the relationships between graphic and symbolic representations, than people who prefer to check some local and specific characteristics. It is tempting to generalize this conclusion to a similar statement about close relationship between the use of global view of function and better understanding of the meaning of graphs. But apparently this is not the case. Some people oversimplified the global approach and used it without really understanding the meaning of the graph. Tracy's case illustrates this.

Tracy used a global approach in all the parts of question 10 (where a point-wise approach could have also been used, and was actually used by many participants). As long as pictorial aspects did not conflict with correct meaning, she had no difficulties and seemed to handle the global approach well. But, in part c (see Figure 6.10), for example, she claimed that A travelled faster between 5:30 and 6:00 and B -- between 8:00 and 9:00, although the correct answers should have been B first and "the same" second. She explained her choice by referring to the relative heights of the two graphs: "Because the dark curve was above the dotted curve." While this is true, it has nothing to do with the question 'who travelled faster?'. The relative steepnesses of the two graphs determine that. Tracy kept using the same method when she said in part (d) that B travelled more kilometers, both between 6:30 - 8:00, and 10:00 - 11:00, since "the B graph was above A." B, of course, travelled less kilometers between 10:00 and 11:00, since A managed to pass B at 11:00. Tracy seemed to be able to approach a function globally, but without understanding the meaning that it convey. This points to problems with translation of function from a situation to a graph and vice versa. Tracy showed this kind of difficulties when she chose, in question 9 (see Figure 6.9), graphs 1, 2 and 6 for Tim, and graphs 3 and 5 for Donna. She did look at the behavior of the functions but did not consider their meaning at all. For her, surface characteristics, such as higher, or goes down, were enough to make a conclusion. So, the ability to use a global approach is not sufficient for understanding meanings.

Teaching Functions in Different Representations

Graphing

The 10 interviewees were asked to explain to a student in algebra 2 how to graph the function $f(x) = \frac{1}{x^2-1}$. One half of the participants started their explanations by suggesting the use of point plotting by hand using tables of values. They also suggested to obtain the table of values by substituting numbers that are easy to calculate into the expression. This approach is not recommended for graphing since it results in many cases with graphs which are not informative (e.g., when graphing near the origin while the interesting part of the graph is somewhere else), or graphs which are completely wrong (e.g., when the points of discontinuity are ignored). The other half of the interviewees suggested to look first for undefined points -- an approach that pays attention to the behavior of the function. Some added explicitly the investigation of asymptotes.

The second phase subjects were also asked if they knew of some less appropriate methods of graphing that students tend to use. Not surprising, those who chose the use of point-wise approach could not come up with students' less appropriate method. Four of the five who suggested the use of an approach that considered the behavior of the function were aware of the point-wise approach to graphing. Katie thought that plotting points was less appropriate since it was not efficient and not informative enough.

> Finding ten million different points. That's the most common approach. Because once they learn how to find points, they'll find twenty or so points and plot them, and try graphing that way.
>
> It is certainly less efficient. Once you get into very complicated graphs, it is taking more time and also may not give you an exact idea what the graph should look like, because you could find 20 points here, and still not know anything about lesser areas.

Bob was concern with getting wrong graphs by using the method of plotting points.

> Plugging in, picking points and making a table. If this is x, then this is y, and so on. And then, graphing them, just plotting these ordered pairs.
>
> I think in this situation it is [less appropriate]. Because they could plot a lot of points, but in something like this [points to the given function], you know, they could maybe plot a point so that 0 would be there. At 1 they are not going to get any answer. But may be at 2, it's going to be 1 over [substitutes 2 in the function and gets 1/3]. And they might try to connect these lines through these asymptotes. I think it's important to realize that there are two positions here for which there is no value.

Bert suggested looking at the behavior of the function in order to graph it, by finding first undefined points. He recognized that the standard method students use is plotting points. But he did not see a difference between his method and the other one.

> I think that the standard method of approaching graphing like this is to have

> the students make some kind of T with the x's and y's. And have -1, -2, 0, 1, 2, and then see what the y value is. And this is really a kind of fail safe way to graph those things. So, I have never seen anybody graph something differently, you know, just plugging points in, having a function spit them out. Spit the y values out.
>
> That's what I usually see a student do. Finding some ordered pairs and plotting points.
>
> I think it's the same method. I'm sure. Hopefully they'd see this is undefined and plug these points in and get values for these and plot them.

Bert did not see a difference between finding arbitrary ordered pairs and looking for undefined points. He hoped the students would find the critical points while plugging numbers. This was a case of a prospective teacher who held a piece of subject-matter content knowledge, but could not transform that knowledge enough to make it a part of his pedagogical content knowledge.

The National Council of Teachers of Mathematics recommends to reduce attention to paper-and-pencil graphing of equations by point plotting and to graphing by hand using tables of values. On the other hand, computer graphing to develop conceptual understanding should receive increased attention (*Curriculum and Evaluation Standards for School Mathematics*, 1988). Only one participant -- Tracy -- out of the 10 second phase subjects, suggested the use of computer graphing.

> I think computers are the best because then you can see exactly right. A lot of time you spend doing this by hand is tedious and frustrating, and you lose interest with why you're even doing this in the first place. I know that's what happen to me when I was in school.
>
> And if you have a computer, you can put that function into. You can even design a program so that may be you'd want them to be able to determine where it's undefined, before hand. So, if that is something you are looking for, and then have them put the program in, and have the computer plot all these points and show that. And then, they can spend more time learning how it behaves and where it goes and actually seeing it.
>
> I know that would have helped me a lot when I was in learning about this, but I spent so much time making dots and little graphs that I didn't really care to know why I was doing that. I'd really advocate using a computer to learn graphic technic.

Tracy seemed to use her unpleasant experience with graphing to improve her pedagogical content knowledge. Her suggestions of ways to use computers for graphing are similar to the Standards' suggestions and those of other math educators today. But she was the only one out of 10 who suggested the use of computers. So, it is not clear that the prospective teachers are aware of the advantages and ready to implement this new approach to graphing.

Explanations for 'why'

Explanations for students that were suggested by the prospective teachers seemed to be closely related to their subject matter knowledge. The 10 second phase subjects were presented with a situation where a student asked them why "a" in the quadratic equation had to be negative if the parabola was concave. People who treated the relationship between the graph and "a" as a rule to memorize, suggested giving the student several examples of graphs of quadratic functions with positive and negative "a"s. Jenine illustrates that.

> I think that interesting or the best way to teach this is when you're having students graph them. Give them a whole set of these, interchanging negatives and positives, and start out with the first one where $y = x^2$. And then, later on, give them $y = - x^2$. And then have them see if they can find the pattern.
>
> Because I think that's where they will, instead of memorizing 'if "a" is positive, it's this way, if "b" is positive, it's this way', or instead of every time having them test it out, make them say, 'okay, let's see. I know there's a relationship' and find that relationship. And once they know one, they can know what will happen, and the same thing is changing, changing of "a", whether it's increasing or decreasing in size of the parabola.
>
> Have them make several different graphs. And then cut them out and try to put them in groups. And then having the kids split those groups up and the kids would group them differently. Some would, probably, group all the ones that are down the same and all the ones that are up, where some would do the ones on the origin, the ones to the right of the origin, the ones to the left.
>
> And by that way, kids can say, 'okay, what was the common in these to make it a group? What was common in this to make it a group?'. And then, as a class, I think that they could come up with it. And then they would remember it. By reading it in a book I don't think they will.

Jenine, and others who suggested a similar activity, described a nice exploration for students. The above activity can help students use inductive reasoning, look for pattern and find the rule that describes the relationship between the sign of "a" and the shape of the quadratic graph. Jenine's conclusion that by experiencing this activity, the students will remember the rule is, probably, right as well. But the student's question was not whether the rule exist. The student asked <u>why</u> the rule exist. Suggesting the student to try several examples to check to see if the rule holds instead of helping him/her to see reason(s) for the relationship, is like telling the student that inductive reasoning is the method by which the truth of a mathematical assertion is established. So, Jenine's way of explanation was wrong mathematically. While it is important that students experience activities such as the one that Jenine described and use inductive reasoning to discover rules and relationships, it is also important that they understand that checking examples does not answer the question 'why does the rule hold?' (A similar recommendation is also made by the National Council of Teachers of Mathematics in *Curriculum and Evaluation Standards for School Mathematics*, 1988).

In order for teachers to be able to help students appreciate the distinction between inductive and deductive reasoning, the teachers themselves should appreciate that. The prospective teachers who suggested the use of inductive reasoning as an explanation for the existence of a rule did not seem to appreciate that distinction. Teachers should also know why the rule hold. Their procedural knowledge should not be separated from meanings and conceptual knowledge. Otherwise, as was in the case of the relationship between "a" and the shape of the parabola, they would not be able to help their students understand.

Reactions to students' mistakes

Evaluation of students' work is an on-going task for a teacher. The teacher has to decide whether a student's answer is correct. If it is incorrect or incomplete, the teacher has to help the student understand what is wrong or what is missing. Knowledge about common misconceptions will help the teacher understand the reasons for the student's mistakes, and therefore will help him/her make knowledgeable decisions about appropriate actions.

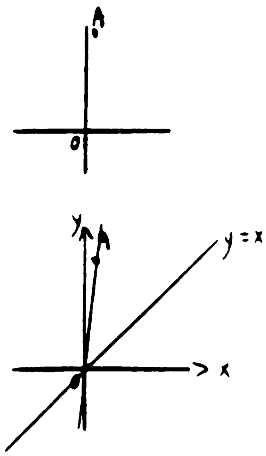
Teacher's decision whether an answer is correct is based on his/her content knowledge. A teacher who, for example, interprets graphs by using superficial characteristics only and ignoring meanings, cannot evaluate correctly students' answers on this topic. So, teachers' own subject matter content knowledge is a necessary requirement. But knowing if a student is right or wrong is not enough. A teacher should also be able to anticipate sources for common mistakes.

The participants of this study were presented with several students' answers, and were asked to evaluate them as right or wrong. In cases where they decided that an answer was wrong, they were asked what they think the reasons for the mistakes were.

Question 7 from the questionnaire (see Figure 6.11) resembled a common mistake

****

A student was asked to find the equation of a line that goes through A and the origin O. She said: "Well, I can use the line y=x as a reference line. The slope of line AO should be about twice the slope of the line y=x, which is 1. So the slope of line AO is about 2, and the equation is about y=2x, let's say y=1.9x."



What do you think the student had in mind? Is she right? Explain.

****

Figure 6.11 -- Question 7 from the Questionnaire

that students make when they learn about linear function. Students, sometimes, think that the slope of a linear function varies directly with the angle between the line and the

x-axis. More than half of the participants who answered the question (61 out of 107 first phase subjects) thought that the above mistaken way of thinking was the source of the student's mistake in question7. Huey illustrates this.

> The girl thinks that because the y = x line, which is a 45 degree angle, whether she knows that's a 45 degree angle or not, and the y axis, the y axis is twice that, 90 degrees. So she thinks that that, see, this has slope 1 [points to the graph of y = x], so this one has slope of 2 [points to the y-axis]. I can see where she made a mistake, so she says this [the slope of line AO] is just a little bit less than this [the "slope" of the y-axis], so that's going to be 1.9. That's what she did.

Still, 46 first phase subjects (out of 107 who answered this) provided different explanations. A large group -- 28 people -- ignored the specific process the student used and described what the student did as an estimation.

- "She had the right idea but she was off in her gross approximation of y=2x."
- "No -- the slope of AO is not necessarily twice that of y=x. She must be careful in her estimation."
- "She was thinking that the slope needed to be steeper which is good. She thought using a decimal would make the graph tighter."

While estimation can serve as a description of what the student did, it is not helpful for the teacher to be satisfied with this general description. Telling the student that she has to be careful in her estimation will not help her realize that her assumption about the linear relationship between slope and angle was wrong. In order to help the student overcome mistakes, the teacher needs to be more sensitive and have more specific knowledge about the nature and sources of the mistakes.

Assuming that one unit on the x-axis equals one unit on the y-axis, one can estimate the slope of the given line AO by comparing Δy's with the corresponding Δx's. The slope approximated this way is 9. The 10 second phase subjects were asked if they could estimate the equation of line AO. Only half of them said they could do that. The others said that they did not have enough information about the units used. Interestingly, although all five who tried to estimate the slope of line AO, realized that the student was wrongly using angles to estimate a slope, they themselves seemed to use angles in their own estimation. Bert, for example said

> It's almost no slope. (Pause.) It's going to be just a big number over a little number. That's all I can say about it. The slope is going to be a real big number. I'd say 100. I don't know. It's just going to be a really big number.

Bert knew that slope is "rise over run". But he did not try to use this knowledge. He looked at the graph and assumed that the result of dividing vertical change by horizontal change would be very big. Huey tried to find the ratio of vertical change to horizontal change. He counted and found a slope of 15. But this number was counter his intuition.

> Um, the slope would probably be (pause), may be about y = 15x. I might be wrong. I am probably wrong. Because I know the vertical line has an infinite slope. It's a pretty steep slope.

The prospective teachers recognized the wrong use of angles for estimating slopes but some of them were not free of that mistaken way of thinking when the situation involved non-familiar linear graphs.

Most of the participants were aware of the difficulties that students have with interpretation of graphs representing situations whenever pictorial aspects conflict with correct meaning. In question 8, for example (see Appendix A), almost all of the prospective teachers recognized that students might wrongly choose circuits A and B because they resembled the shape of the given graph. The prospective teachers were also sensitive to the common mistake of choosing a higher graph as represent further, faster, etc. Jenine illustrates this in her explanation to students' mistaken choice of A in question 10 (see Appendix A) as the one who travelled faster between 5:30 and 6:00.

> Okay, probably, one reason I would guess that they would choose A would be because the graph of A is on top and the graph of B is on the bottom. And a lot of times that means bigger, better and more and therefore faster. And it would mean they weren't really thinking about 'this is a curve for that', where I would think probably their misinterpretation is coming from. They saw that it was a bigger distance but they didn't take into, you know, it was up higher, but not how much it curved at that time.

recognizing students' mistakes is based on teachers' subject matter knowledge. A teacher who does not recognize a mistake as a mistake or who is not sure about his/her own knowledge cannot think of sources of students' misunderstanding.

CHAPTER 7

COMPOSITION OF FUNCTIONS AND INVERSE FUNCTION

Two functions can be composed if the set of images of one function is the domain of the other function. Composition of functions is both an operation and the result function. Many students see composition as the sequencing of assignments only and do not view the function obtained by composition as a function. The composition of a function f with a function g is written as $f \cdot g$.

The notion of inverse function is based on composition of functions. A function g is said to be an inverse function of a function f if and only if $f \cdot g = g \cdot f = I$ (I denotes the identity function). The symbolic notation for the inverse of f is f^{-1}. Informally, inverse function can be thought of as undoing the effect of applying a certain function. Every "doing" has "undoing" -- at least theoretically. In a similar manner, every function has an inverse. But this inverse is not always a function. Only one-to-one functions have an inverse function. However, if a function has an inverse function, this inverse is unique. The graphs of a function and its inverse have an interesting relationship. They are symmetric to each other with respect to the line y=x. This relationship exists since if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}, and the two points (a, b) and (b, a) are symmetric in respect to the line y=x.

Functions opened new opportunities in mathematics. In addition to the typically algebraic operations of addition, subtraction, multiplication, division and raising to power, functions can also be composed and inverted. The ability to substitute functions into each other and invert them created new functions and helped with the study of differentials and integrals. "The strength of the function concept is rooted in the new operations -- composing and inverting functions -- which create new possibilities" (Freudenthal, 1983).

Since composition of functions and inverse function are so important, this chapter is devoted to a description of the prospective teachers' knowledge and understanding of these two important sub-concepts of the concept of function. The chapter starts with a discussion of the prospective teachers' informal knowledge and understanding of the meaning of inverse function as undoing. Then, the participants' misunderstanding of composition of functions as a source of difficulties with inverse function is discussed. The next part describes problems with inverse function as a result of dealing with it on an informal level of "undoing" only, with no relation to the mathematical notion of inverse function. Unfamiliarity with the relationship between a function and its inverse in a graphical representation are discussed in the last part.

Meaning of inverse function as undoing

When working on problems and answering questions that dealt with inverse function, many prospective teachers seemed to ignore or overlook the meaning of an inverse function as "undoing" what the function does. The participants were asked the following question (question 12 from the questionnaire):

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Given $f(x) = 2x - 1$ *and* $f^{-1}(x) = \frac{x + 10}{2}$. *Find* $(f^{-1} \circ f)(512.5)$. *Explain.*

****

Using the idea of "undoing", the answer to this question is immediate. Without making any calculations with the specific given functions, one can conclude that no matter what the functions are, the inverse function undoES what the function does and therefore the answer will be the element that we started with. In our case, we will get 512.5 back. More formally: $(f^{-1} \cdot f)(512.5) = 512.5$, since $\forall x, (f^{-1} \cdot f)(x) = x$.

Less than half of the participants (53 out of the 123 who answered this question) based their answer on the above idea. They used the meaning of inverse function.

__ "512.5 since they are inverses of each other, $f^{-1} \cdot f(x) = x$."

__ " $(f^{-1} \cdot f)(512.5) = (512.5)$ since $f^{-1} \cdot f = i$ where i is the identity."

— " $(f^{-1} \cdot f)(512.5) = (512.5)$ When you put in a value and then put it in to the inverse function you'll get back the original value."

Although using the inverse property was sufficient, half of the participants who used this argument (26 out of 53) added calculations of some sort. For example:

— "$f^{-1}(f(x)) = \frac{2x - 10 + 10}{2} = x = 512.5$ $(f \cdot f^{-1})$ will always equal i."

— It should be 512.5 if these are inverse because you're undoing with f^{-1} that you did w/f. Check[:]
$f(512.5) = 2(512.5) - 10 = 1025 - 10 = 1015$
$f^{-1}(1015) = \frac{1025}{2} = \underline{512.5}$."

The interviews and some of the explanations in the questionnaires made clear that there were several reasons for using unneeded calculations together with an explanation that was based on the meaning of inverse function. One reason was that the solver did not consider the meaning of an inverse function until confronted with the original number: 512.5 as the result, realizing that the result of the execution of the calculation should have been known from the beginning. Another reason was that the solver knew and was able to use the inverse property, but either felt uncomfortable not using all the given data (the specific functions at hand) or felt that stating a property (or a definition) was not enough to be considered as an explanation. This attitude points to misunderstanding of what counts as an explanation in mathematics. The following excerpts from the interviews illustrate these two ways of thinking.

Valerie's solution to the question illustrates the use of calculations to support the use of the inverse property. She wrote in her questionnaire the following:

" $\frac{2(512.5) - 10 + 10}{2} = \frac{2(512.5)}{2} = 512.5$ All that has been done is taking the inverse of a function." When asked to explain her answer during the interview she said that she used the calculations as a way of explanation.

> You're taking the inverse of a function - you're taking this number and applying a function to it and then taking the inverse of it, so it's going to come out the same answer.
>
> So I didn't know if you wanted me to show it or just explain, so I just wrote it out. You would take the 512.5 and put it into this function, and you

> would take 2 times it and subtract 10 from it, and then you would place that value that you would come out with into the inverse function, replacing the x. So you would just add the 10 back to it and divide by 2.
>
> I wasn't sure what explanation you wanted.

Valerie, and others, could not decide whether using the "inverse property" was enough as an explanation so they added calculations. Using the concept of inverse function without any procedure did not seem sufficient to them. Is it because the emphasis we put on procedures in the study of mathematics and the de-emphasis of concepts and reasoning? Valerie said that she did not know how the researcher wanted her to answer, as if she did not have the authority of deciding whether her answer was right or wrong.

Jenine's answer illustrates how some people did not consider the meaning of the problem until they finished their calculations. She used both calculations and the inverse property in her answer to the question. When asked about it she admitted in the interview that she followed the mechanics of plugging numbers in first, without thinking about the concept of inverse function. This came only later.

> Okay. The first what I did, not thinking, I just started plugging numbers in. Just one of the things that I really don't like when kids do, I think you need to... As I said before if I actually thought about what was happening and realizing that you take the composite of f and its inverse, you'd get the same number you started out with and that would have avoided.

Seeing the expression $f \cdot f^{-1}$ did not make Jenine, and other people who used the same way of solution, think about the concept of inverse function and its meaning, but rather made them use the procedure of composition of functions. The following example illustrates this point further.

Mike, who did not use the inverse property in his answer to the questionnaire, was asked to explain his work (all he did on his questionnaire was to give instructions of how to find $f(512.5)$ and then to plug that result into f^{-1} to find the answer, without really doing that). When pressed, during the interview, to find the specific answer, he realized that he got 512.5 back. His reaction is a vivid example of the case in which the solver did not think of the inverse function conceptually until confronted with a

surprising result.

> M: Find inverse or $f^{-1}(512.5)$, and then you can call it x, c here, and then you can plug c into $f(x)$ and then you get your answer. It's sort of f^{-1}, then f.
>
> R: Ok. And you said you get the answer which will be...
>
> M: (Thinking.)
>
> R: Ok. So what was the answer?
>
> M: I didn't figure out the answer.
>
> R: Can you figure it out right now?
>
> M: Ok. (Figuring answer.) $f^{-1}(512.5) = \frac{512.5 + 10}{2}$...
>
> And then you're going to do f of this, and I said this equals c. Take f of... (works on the calculations and gets back the number 512.5). I guess I should have known that you are going to get... since you get the same number. I didn't realize it happens before (looks embareced).
>
> R: Ok. And why are you supposed to get the same answer?
>
> M: I just worked it out. (Laughs at himself.) Because you're just taking a function and then taking its inverse. So... I should have known that.

It is clear that Mike, Jenine and others "knew" the inverse property. But, still, they did not draw upon their conceptual knowledge but rather approached the problem by using an unnecessary procedural knowledge.

About half of the participants (62) did not refer to the concept of inverse function and its "undoing" meaning at all in their answer. They just went ahead and attempted to calculate the answer. One half of those who used calculations only (31 people), used them correctly:

— "$f(x) = 2 \cdot 512.5 - 10 = 1015$

$f^{-1}(f(x)) = \frac{f(x) + 10}{2} = \frac{1015 + 10}{2} = \frac{1025}{2} = 512.5$"

The questionnaire data do not provide enough information whether people who used calculations correctly were aware of the notion of inverse function and just did not mention it or whether their thinking was only mechanical. But this issue became clearer in cases where the participants (31 first phase subjects) made a calculation mistake or did only part of the calculations. Some of the first phase subjects (7 people) made a calculation mistake and got a number different from the original one (e.g., 517.5 instead of 512.5):

— "512.5 × 2 = 1035.0 $f^{-1}(1025) = \frac{1025}{2} =$ **517.5**."

In such cases it was clear that the meaning of inverse function was not considered (or understood), otherwise the calculations would have been checked as described by Jenine in her interview:

> And what happened was that I went through and did it and I messed up on my calculations and then went through and started talking about it and I thought, 'well, it has to be the same number and it's not'. And then I went back and said, 'okay, what did I do wrong?', because it has to work out. I found out I made a multiplication mistake.

The same conclusion was also valid in cases where people (11 first phase subjects) did only part of the calculations and left that as the answer:

— "$f^{-1} \cdot f(x) = f^{-1}f(x) = f^{-1}(2x - 10) = \frac{2x - 10 + 10}{2}$."

— "$f^{-1}(f(512.5)) = f^{-1}(1015) = \frac{1025}{2}$."

Another kind of answer that seems to relate to people who did not consider or misunderstood the meaning of inverse function was an explanation of how to do the composition without any numerical answer to the question asked (4 people).

— "Apply (512.5) to that [$f(x)$] take the result and plug it into that [f^{-1}]."

— " $(f^{-1} \cdot f)(x) = f^{-1}(f(x)) =$
Just plug in x to f, then plug in that answer to f^{-1} & you will have the answer."

The tendency to execute the calculations of $(f \cdot f^{-1})(512.5)$, instead of the use of the concept of inverse function, may also be related to naive conception of composition of functions. Accepting composition of functions as the sequencing of assignments only and not as a function in itself, as is often done by students (Lovell, 1971; Thomas, 1975), may lead to the use of the unnecessary composition procedure.

Table 7.1 summarizes the ways in which the first phase subjects answered question 12. The rows present the number of people who used each of the different methods. The columns describe the correctness and completeness of the use of the method.

Table 7.1 -- Distribution of Answers to Question 12 on Inverse Function by the Method Used and by Its Correctness Use.

	Correct	Incorrect	Not complete	No Answer	Total
Inverse property only	27	7	1	__	35
Inverse property & calculations	26	0	0	__	26
Calculations only	31	17	14	__	62
No answer	__	__	__	29	29
Total	84	24	15	29	152

So far we described participants' answers that were based on a correct interpretation of the concept of composition of functions. As we can see from table 7.1, some of the participants used the inverse property incorrectly. Those people and some of the participants who used calculations incorrectly, interpreted wrongly composition of functions. The next section describes difficulties with the concept of inverse function as a result of misunderstanding composition of functions.

"Meaning" of composition of functions as multiplication

Composition of functions is sometimes referred to in mathematics as multiplication, although it is different from multiplication of numbers. The notation f^{-1} for the inverse function, which many students interpret as f raised to the negative 1, does not help to distinguish between the two. Some of the subjects in this study confused composition with multiplication. For example, about 10% of those who answered question 12 in the questionnaire, related to the inverse function in multiplication terms. Some of them simply multiplied f and f^{-1}:

$$(2x - 10)\left(\frac{x+10}{2}\right) = \left(\frac{2x^2 + 10x - 100}{2}\right) = x^2 + 5x - 50 = (x+10)(x-5)$$

and then took the time to substitute 512.5 in $x^2 + 5x - 50$ (by hand) and got 34043.75.

Other people who confused composition with multiplication did not do any calculations but rather tried to use the inverse property conceptually. But they misunderstood the meaning of I as the identity function. Since the role of the identity function in composition of functions is the same as the role of 1 in multiplication of numbers, 1 is sometimes used to denote the identity function in mathematics. Some of the participants seemed to be confused by that. They thought of 1 (the identity function) as the number 1. This confusion lead them to think that the identity function always results in 1.

— "**1.** The mapping of a function with its inverse will always be the identity."

— "**1** since f^{-1} is the inverse of f."

— "512.5 The inverse of $f(x)$ **times** $f(x) = 1$. <u>Always</u> So **1**·(512.5) = 512.5."

— "512.5. Since you are using f^{-1} and f they **multiply** to the identity which is like **multiplying** 512.5 and **1**."

— "$f(x) \bullet f^{-1}(x) = 1$ for f^{-1} is f **reciprocal**."

The identity function was treated in those cases as the number 1. Then, instead of applying a function (the identity) to a number (512.5), the subjects multiplied 1 by that number. Bert did that. He wrote $f \cdot f^{-1} = 1$ and explained:

> B: I'd still say that this was the identity, and then just multiply by this [512.5], so the answer would be this [512.5].
>
> R: So this [1] is the identity. And then you multiply.
>
> B: Right. So it would be the identity times this and it would be that. A case of multiplication.

The tendency of some of the participants to treat inverse function as a function that has to be multiplied by f in order to get 1, was presented in several occasions. Illustration for that are Bert's choice of $g(x) = \frac{1}{x}$ as the inverse function of $f(x) = x$, and Mike's rejection of both the Log and the Root functions as inverse functions of $f(x)=10^x$ (see question 14 in Appendix A). Mike claimed that $\frac{1}{10^x}$ was the inverse function. Interestingly, he did not use the multiplicative interpretation of composition of functions when he answered question 12. There Mike used the correct interpretation of composition of functions. This seems like a contradictory behavior. The reason for it might be the presence of the composition sign "•" in question 12 and its absence in question 14, where the inverse function of $f(x)=10^x$ should have been found. When seeing the operation sign of composition, Mike was able to use his knowledge about composition. But when asked about inverse function without having the operation sign of composition in front of him, his multiplicative image of inverse function dominated his thinking.

Looking at the inverse function as "undoing" what the function "does", with a correct interpretation of composition of functions, was helpful in solving problem 12 in particular and in understanding the concept of inverse function in general. But limiting one's conception of inverse function to this "naive conception" only, results in mathematical difficulties. This will be discussed in the next section.

Undoing as naive conception of inverse function

Each "doing" has "undoing", at least theoretically. But this analogy cannot be applied to function and inverse function. Not every function has an inverse function -- only one-to-one functions do (every function does have an inverse though). Therefore, using "undoing" as the only criterion for inverse function, will result in an incomplete conception of inverse function. This problem occurred, for example, in answering question 15b from the questionnaire, where the participants had to decide whether K^{-1} was a function.

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Consider the set of functions whose domain and set of images are all the real numbers.
K assigns to each pair of such functions, their composition.
a) Is K a function? Explain.   *b) is K^{-1} a function? Explain.*

****

K^{-1} was not a function since a function could be obtained by composing different pairs of functions. For example, the function $\sqrt[5]{x+3}$ may be obtained as a composition of the two functions $f(x) = x+3$ and $g(x) = \sqrt[5]{x}$, or the two functions $f(x) = x+1$ and $g(x) = \sqrt[5]{x+2}$, etc; the identity function may be obtained by composing any function and its inverse.

Only 62 first phase subjects (out of 152) tried to answer question 15(b). This was probably due to its high difficulty level. Most of the participants did not understand the definition of K as a function that is defined on functions and therefore could not decide whether K^{-1} was a function. Still, some of the remarks that the participants made seem to point to a naive conception of inverse function as "undoing" only. Some of the participants claimed that inverses of functions are functions and therefore K^{-1} was a function.

__ "Yes, the inverse of a function is a function."

__ "If [']a['] is a function then so is its inverse."

A reasonable explanation to these statements can be the view of inverse function as "undoing" only. This was illustrated in Bert's interview:

> I think K inverse would be a function too. I think if K is a function, again this is out of memory, recall ability, I haven't thought about this stuff enough, but um, I think that K inverse would have to be a function, you know, you're just undoing what you've done, so you have to, you did it to make a function, and you have to undo it to make a function.
>
> It seems like you can undo everything you do. So I think that everyone does have an inverse function. It seems to me.

Only a few of the participants considered the one-to-one aspect when answering this question. Valerie did. She used her answer to question 13(b) (see Appendix A) as an illustration. There she found that there was more than one pair of functions that could be composed and give the function $\sqrt[5]{x+3}$. So she used this idea as an illustration of why more than one pair of functions may lead the same composition, making the inverse not a function.

> Ok. Yeah. I said that it could be, the inverse could be a function but it doesn't necessarily have to be. Because you end up with some value here eventually if they're the same - you take f(g)(x), you're going to come up with some representation; if you use those two [the two functions she gave in question 13] then you end up with it being the $\sqrt[5]{x+3}$, but if we took two other functions it would also yield that same thing. And then taking the inverse, we would end up with different... but it might not turn out that way.

Interestingly, even some of those who seemed to appreciate the one-to-one property needed for inverse functions, seemed to also be influenced by the "undoing" property and therefore hold a hidden assumption that "undoing" guarantees one and only one pre-image. Katie, for example, showed in her answer to question 13(b) that there exist several pairs of functions with the same composition: $\sqrt[5]{x + 3}$. But, when having to decide whether K (in question 15), which assigns pairs of functions to their composition, has an inverse function, she ignored her previous finding and decided that there was only one pair of functions for each composition.

> It would be a function, because you take the specific composition back to a specific ordered pair.

Looking at an inverse function as "undoing" only was probably partial responsible for the mistaken choice of the xth root of 10 as the inverse function of 10^x in question 14 in the questionnaire (see Appendix A). Since exponential and power functions are important functions in the high school curriculum, a detailed description of the participants' knowledge and understanding of them with relation to conception of inverse function as "undoing", as was presented in their answers to question 14, appears in the next section.

Exponent, Log and Root: Relationship, differences and similarities

Power function (e.g., $f(x) = x^3$) and exponential function (e.g., $f(x) = 3^x$) look similar. But the argument of a power function is the base while the argument of an exponential function is the exponent. This similar appearance completely disappears when the inverses of the two functions are considered. Root (which is actually also a power function) is the inverse function of an odd power function (e.g., $f(x) = \sqrt[3]{x}$, since $\sqrt[3]{x^3}=x$), while log is the inverse function of an exponential function (e.g., $f(x) = \log_3 x$, since $\log_3 3^x=x$). An even power function does not have an inverse function since it is not a one-to-one function.

Question 14 in the questionnaire dealt with the relationships between these four functions.

****

A student said that there are 2 different inverse functions for the function $f(x) = 10^x$:
One is the root function and the other is the log function. Is the student right? Explain.

****

Less than half of the first phase subjects (63 out of 152) tried to answer this question. Since this question was near the end of the questionnaire, many students may not have had time to complete it. But the few respondents could also be attributed to the fact that this question was one of the hardest questions on the questionnaire. The high level of difficulty of this question was apparent during the interviews.

The term "root function", which is the inverse function of a power function, is not used very often. So the participants had to decide about the meaning they wanted to attach to it. The most common description of the root function by the participants was the xth root of 10 or just the xth root (without specifying 'of what'). These subjects overgeneralized the idea of a root function, such as the square root: $f(x) = \sqrt[2]{x}$, or, in general, $f(x) = \sqrt[n]{x}$ (where n is a parameter), to an exponential function in the first plac -- $f(x) = \sqrt[x]{10}$, and to an incorrect use of variables and parameters in the second place $f(x) = \sqrt[x]{\ }$. Both "functions'" descriptions meant: take the xth root of what you have (which was 10^x) but neither description was appropriate. Tracy used $\sqrt[x]{x}$ in the same manner, checking both the log and the "root" functions.

"$f^{-1}(x) = \log x$: $\log(10^x) = x\log 10 = x$ -- correct

$f^{-1}(x) = \sqrt[x]{x}$: $(10^x)^{\frac{1}{x}} = 10$ -- incorrect."

Tracy used correctly the algorithm for checking whether a function is an inverse function but she did not really use her own definition of a root function. She composed the two functions and checked to see if she got the identity function $f(x)=x$ as the result. She explained why the root was not an inverse: ."..you're not going to get x back out of it, so that's how I determined it." Tracy used her procedural knowledge of inverse functions and therefore correctly chose log as the inverse function of $f(x)=10^x$.

But the root function appealed to many of the participants (23 first phase subjects out of 63 who answered this) who used their naive conceptual knowledge of what an inverse function was. These people used the idea of "undoing" as their interpretation of inverse function. The xth root of 10 seemed to them to "undo" what 10^x does: In order to get 10^x, one starts with 10 and then raises it to the xth power. By taking the xth root of 10^x, one gets 10 back. Bert's answer illustrates that:

> My understanding is we're trying to undo whatever is done in $f(x)$. It would seem like it would be one over. So it would be the xth root of 10 to the xth [writes: $\sqrt[x]{10^x}$]. So it does seem like it would be legitimate.

So, the "undoing" conception of inverse function mislead some of the participants in their search of the inverse function of $f(x)=10^x$. About one third of the participants who answered this question thought of "undoing" in this case as of taking the xth root of 10^x and getting 10 back. Bob composed log with $f(x)=10^x$ and got x, and then composed the xth root with 10^x and got 10. He then accepted both functions as the inverse function, even though in the second time he got 10 instead of x. This "feeling" that an inverse function gives back what you started with (10 in our example, instead of x) lead many others to wrongly conclude that root was the inverse function of $f(x)=10^x$.

Accepting the root function as an inverse function because of its "undoing" appealing created a dissonance: many of the participants also remembered from previous study that log was the appropriate inverse function and inverse function was unique. To solve this uncomfortable situation, about one third of those who answered stated that the log function and the root function were both inverse functions for the given function: $f(x)=10^x$, since they were the <u>same function</u>.

— "Yes. log in base 10, $\sqrt[x]{}$ work the same way."

— "No. log base 10 and the xth root of 10 are the same functions."

Jenine used a similar argument. She wrote: "I believe that there is only one function. The root function and the log function are just two different ways of representing the same function." In her interview she explained what she meant by that and described her understanding of what a root function is.

> J: Um, what I was thinking was, okay, log, what a log is, is, log is a power and that's what a root is, a root is a power of a fraction.
>
> R: What is a root fraction? How would you describe it?
>
> J: A root function? Like $f(x) = \sqrt[2]{x}$. Or...
>
> R: And in our case, where we have $f(x) = 10^x$?
>
> J: (pause). Okay, this would be, well if x is a fraction then it would be, <u>if x</u> was, if x was 1/2, that would be the square root, $10^{1/2}$, would be the $\sqrt[2]{10}$. And 10^0 equals 1/10, anything greater is just, it's to that power. So, um, when I think about roots I think of them as like just being less than 0.
>
> R: Um, so you're saying that basically they're not different from each

other.

J: Right, it's just a different way of expressing the same thing because a log is a way to express a power. It's, um, log a to the x of b [writes: $\log_a x=b$] means that, I know I get this wrong, when I think about it, but it's, um, a to the power of b equals x [writes $a^b=x$]. Cause this is telling you which power. Cause this is a power. "a" to the, that's what it is, equals x, so it's just a log, log is a way to express powers.

R: Um, why did you decide that they were the same? Did you see it somewhere?

J: That they were the same? I think when I really saw that they were the same thing was when I ended up tutoring people, when I came up here. I never did officially, but just helping people on my floor. That's when. I could always do it, you know, I could do log problems, power problems. But when I had to explain it to somebody, I had to be able to understand it mentally so I could explain it to them and hopefully get them to understand the relationship.

Jenine seems to think of root function, in this case, as the xth root of 10 ($\sqrt[x]{10}$) and of log as a root, or a power, a different way to describe powers. This wrong conception of log did not interfere with her ability to successfully solve regular log problems, as she recalled, since these problems usually require only procedural knowledge of logarithms. Valerie also did not seem to understand the difference between log and root. She chose log because she was taught this way, but also root which seemed correct to her.

V: Ok. I said that the log function would be the logical one I would think to take, just because if you want to find like x, if you want to find the x value, you take the log of it and then that's x. So that would be the logical one to do because that's the one you are taught way back. But then you could do it using the root function as long as you took the 10th, um, the xth root.

R: And what do you mean by that? What do you mean by the root function?

V: I thought of it as being like the idea of a square root, but it couldn't be a square root. But if the kid said the root function, he's right as long as he specifies by that he means like x.

R: Ok. Do you mean that there are two inverse functions, the log and the root function?

V: Yes, that's what it would turn out to be. Two different ways of phrasing it, because the root function vs. the log function would be seen differently, but they would be the same.

It seems as if Valerie and others did not distinguish between exponential function and power function. They thought of root as the inverse function for $f(x)=10^x$. Since they remembered learning that log was the appropriate inverse for the given function, and they might also remembered that there was only one inverse function (as 10 out of the 63 who answered this explicitly said), they assumed that log and root were two different names, or ways of writing, for the same function, as Bob said:

> In any group or in anything there's only one inverse for each, each element has a unique inverse. That's what makes it a group and that's what let you go through all the functions. I think in base 10 the roots, since, the logs are based on base ten, the root function and the log function are probably the same function.

Not all of the participants solved the dilemma of remembering that log was the inverse function but feeling that root was also appropriate by joining the two into one function. Two of the participants decided that the log was the inverse function and the root was the inverse operation. Brian was one of them. At first he seemed to understand what log was, but later in the interview he got confused and had difficulties sorting things out.

> B: I don't know. I think log is the inverse.
>
> R: What was your answer [referring to his answer on the questionnaire]? I'm not sure I understood it.
>
> B: I say that he's right as far as he knows. Cause, maybe he's not even close.
>
> R: What is the root function?
>
> B: The root, the root I understand is the (pause). It's just the inverse, it's the inverse of the power. And for the log, the log says what exponent do I need to get to this. I don't know. I think they are both right, but I don't remember. See, the root function is more of a multiplicative inverse and the log function is the true inverse. Sure.
>
> R: What do you mean by that?
>
> B: What do I mean by that? I don't know. It sounded good at a time. Well, you can run the formula backwards and get what your exponent is. Get the log function (pause). I really don't know whether the student was correct or not. I don't remember enough about it because I don't use it everyday, it's not part of my common knowledge.

R: Would you say that these two can be two different inverse functions?

B: I think they are two different inverse functions. They're inverses. They take one thing and you can perform a procedure, get... using the log or square root, the root function, to get backward to your original number. Well, actually they are inverses of, you know, you take an exponent and you take the root of the exponent. That's fine. You get back the number. It looks like what we did a couple of days ago. And a log function, I don't remember enough about, that's frustrating.

R: Okay.

B: But I think that they are both inverses, but, well, yeah, they are two different inverses and that what the student asked. Composition, I get lost when we do composition.

The results show that the relationships between exponential, log, power and root functions were not clear to many of the participants. The problem seems to be rooted in the distinction between power and exponential, log and root functions; in understanding what root function is; and in overgeneralization the meaning of inverse function as "undoing". Many of the participants treated both power and exponential functions as power function. They also did not have a good understanding of the log function and treated it as a root. The National Council of Teachers of Mathematics recommends that college-intending students develop a thorough understanding of specific functions including polynomial (a general case of a power function), exponential and logarithmic (*curriculum and Evaluation Standards for School Mathematics*, 1988). Can teachers who do not have a thorough understanding of these functions teach them according to these standards?

As we have seen so far, perceiving inverse function as "undoing" is powerful on one hand but is not enough for dealing with all aspects of the concept of inverse function on the other hand. "Undoing" is also not very useful when dealing with a function and its inverse in a graphical representation, since in this representation the 'operational' aspect of function does not stand out as it does in a symbolic representation, using equation or formula. Findings related to the relationship between a function and its inverse in the graphical representation are discussed in the next section.

Graphical representation of inverse function

The graph of a function and the graph of its inverse have a close relationship. If (a,b) is a point on the graph of a function f then (b,a) is a point on the graph of f^{-1}. This makes the two graphs symmetric to each other with respect to the line y = x. This simple relationship can help in producing the graph of an inverse function when the graph of a function is known. Understanding of this relationship also show a better understanding of the concept of inverse function in a different representation.

The following interview question was designed to explore whether the prospective teachers know this property, and if they do, can they explain it.

****

How would you show the relationship between a graph of a function and the graph of its inverse to a student?

****

More than half of the interviewees (6 out of 10) did not know anything about the symmetric relationship between a graph of a function and the graph of its inverse although most of them knew a lot about inverse functions in a symbolic representation. Some of them just said they did not know about this relationship and suggested the use of another representation, usually symbolic. Two tried to graph several examples in order to find some relationship. This was a bit problematic for Bert who started with $f(x) = x$ and decided that the inverse was $f^{-1}(x) = \frac{1}{x}$ (confusion between composition and multiplication, and misunderstanding of the meaning of the identity function). Bert sketched the two graphs (see Figure 7.1). But the only conclusion he could come up with was that the graphs were different.

Jenine also tried some graphs. She emphasized first how important it was for the kids to understand what an inverse was, by using the chart method. Starting with

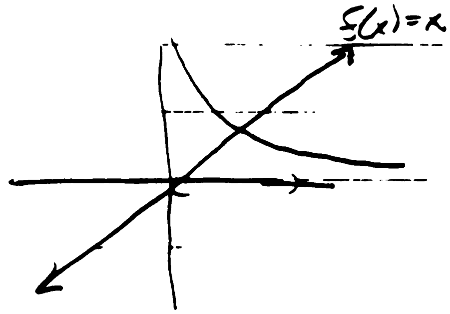


Figure 7.1 -- Bert's graphs of inverse function

$f(x)=x$, as Bert did, she quickly realized that $f^{-1}(x) = x$, and decided to try another function that may provide more information. She chose $f(x) = x+2$ and nicely made a chart for the three functions: f, f^{-1}, $f^{-1} \cdot f$ (see Figure 7.2).

x	f(x)	$f^{-1}(f(x)$
1	3	1

Figure 7.2 -- Jenine's chart for inverse function

Then she explained the relationship between the numbers. But when it came to the graphs of these three functions, she did not make a connection between the symbolic and graphic representations. Jenine did not realize that the graphical relationship was basically the same as the one in the chart that she made (see Figure 7.2). So, instead of discussing the reverse order in each pair, she got caught in an irrelevant visual characteristic of her specific example.

> J: Okay, let's use this case. I started out again. We have f(x) is x + 2. Then the inverse of f(x) is x - 2. So actually three column chart would be easiest. We chose x so that x = 1. Then f(x)=3. Then f inverse of this is just 1. So they can see that. You start out with x, and the composite is that. Then you can ask students to graph it. This would be x + 2, this would be...[graphing $f(x) = x + 2$, $f(x) = x - 2$ and $f(x) = x$.] This way

they can see that there is a relationship between these two lines.

R: What kind of relationship would you emphasize?

J: I'd have the students look at it. 'What do you see?' And they would probably say the lines are parallel. And I would ask, 'well, how do you know that they're parallel? I mean, you can go on look, but are we ever sure? Can you prove to me?' And I would hope that they would be able to say that, okay, here the slope is 1 and the slope here is 1. So if they have the same slope, then they will have, then they will be parallel. Cause that's what the definition of parallel is. And so, that's the kind of concept I would want them to be thinking about. It's putting the relationship of these functions on the graph and what it means. And you know, you know, it's always the characteristics of what is the b. You know, why does it cross here and why does it cross there? It shows there is a relationship between these two functions. And then have them see this. The composite of... 'What is an inverse? What does it mean in terms of graph?' So if I plot x and f^{-1}, you know, if I let $x = f^{-1}(f(x))$, that means that...

Jenine had the three graphs in front of her. She noticed that they were parallel to each other. But they were parallel only because of the specific function that she chose as an example. This was not something that could hold for any function and not only for any linear function. So, next, Jenine was confronted with the problem of having to deal with non-linear functions. In this case, the parallel idea does not hold, of course. She gave up, explaining that she had to do more research on the relationships between functions and their inverses because it was not something she had spent a lot of time thinking about.

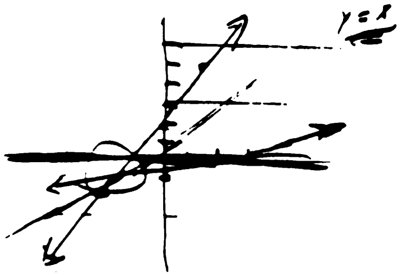
Jenine, in her chart, used the idea of reversed coordinates of points of f in order to get points of f^{-1} (see Figure 7.2). But she was not aware of the meaning of this idea in a graphic representation, and therefore could not find the symmetrical relationship between a function and its inverse in a graphical representation. Brian, on the other hand, seemed to remember something about symmetrical relationship in a graphical representation. He even knew that the coordinates had to be reversed in order for that to happen, but could not take the next step and explain why the coordinates have to be reversed when dealing with a function and its inverse, nor was he sure that this would always happen.

Okay, I'll try and remember how we do an inverse of a graph. Okay.

[working on finding an inverse to $y = 3x + 2$ in the following way:

$$x = 3y + 2$$
$$\tfrac{1}{3}(x - 2) = y$$
$$y = \frac{x - 2}{3}$$

Then he sketches the graph of $y = 3x + 2$ and the graph of $y = \frac{x-2}{3}$ on the same coordinate system and adds the graph of $y = x$. He explains:]



I'd probably, first assume that the person can create the inverse function. Um, you know, then I could go through and do the inverse (pause). Something (pause). They probably will intersect, the lines. It's been a while. But I would show graphically by having them sit down and say, here, you know, graph the function and then graph its inverse and then show there's a plane, I think, there should be a plane of symmetry.

R: Why should there be a plane of symmetry?

B: It just seems like it to me, I might be grabbing the wrong bit of information out of something. Those are reflections and I'm not sure if they are reflections. Oh, well.

R: Okay.

B: I lost, I don't have the tech[nic] to handle it right now.

R: Okay.

B: But, I'd have to start with it graphically, and start looking for relationships like: 'Are the coordinates reversed? Is there a point where...?' If I could find that point of intersection, these two functions got together, then I might find out that that does lie in one plane, you know, I'd compare the coordinates and see what happens.

Brian was not sure about the symmetry he saw and about the reversed coordinates idea, although the latter is the basic idea of inverse function in an 'ordered pairs' representation.

Huey started with the ordered pairs representation and later used it for the graphic representation. He started with a set of ordered pairs (Figure 7.3a) and then reversed the order (Figure 7.3b):

2	3
4	5
5	6

(a)

3	2
5	4
6	5

(b)

Figure 7.3 -- Huey's Ordered Pairs

When asked about a way this relationship can be represented in a graph, he sketched the two functions (f and f^{-1}), added the graph of x = y and said that another way to show it would be to use the mirror image about x = y. And he explained why it worked:

> Okay. Hopefully then the ordered pairs is back and you can show: here is (3,2), here is (2,3), they are the same distance from x = y. Or, if that didn't work and they still didn't know why the x = y worked, we'd have to go back to the definition of inverse. Go back and see that x's and y's are switched.

Overall, the relationship between the graph of a function and the graph of its inverse were not known to most of the interviewees. Even those who were willing to try and find some relationship during the interview were not able to do that. Many of them knew much more about functions and their inverses in a symbolic representation, but were unable to connect representation. Understanding an idea means being able to recognize this idea in a variety of different representations, flexibly manipulate the idea within a given representation, and translate the idea from one representation to another. Lack of ability to translate the relationship between a function and its inverse from an ordered pairs representation to a graphic representation shows limited understanding of the relationship between a function and its inverse.

Summary and discussion

Inverse function and composition of functions, as any other concept, cannot be understood in one simplistic way only. Understanding these sub-concepts of the concept of function requires understanding the general meaning as well as the formal mathematical definition. Knowledge about them includes conceptual knowledge as well as procedural knowledge in different representations. "Undoing" is an informal meaning of inverse function which captures the essence of the definition. The importance of this informal meaning is also recognized by the National Council of Teachers of Mathematics who recommends that all students explore the concept of inverse function informally as a process of undoing the effect of applying a given function, while the precise definition of inverse function and composition of functions be reserved for college intending students (*Curriculum and Evaluation Standards for School Mathematics*, 1988). But most of the participants did not approach inverse function by using the inverse property. Rather, many people used their procedural knowledge about composition of functions in a mechanical way, without any attempt to approach the problem conceptually. This was the case even with people who "knew" the "undoing" characteristic of inverse function. Although they had the needed knowledge, those people did not use it.

Using the inverse property when possible points to a better understanding of inverse function. Only interviewees who used the (correct) inverse property without adding any unnecessary calculations of some sort, when was possible (in question 12), correctly chose the log function as the only inverse of $f(x)=10^x$ (in question 14). So the use of unnecessary calculation, in addition to a conceptual solution, might reveal an incomplete understanding of the concept of inverse function. The kind of data gathered for this study through the questionnaire does not allow for a more general and definite conclusion.

Understanding inverse function as "undoing" only is not enough. This term is too vague and not precise. "Undoing" made many people choose the root function as the inverse function of an exponential function. It also made many people think that every function has an inverse function, while ignoring the one-to-one requirement. So, a solid understanding of the concept of inverse function cannot be limited to "undoing" only. In cases where the conceptual knowledge was limited to "undoing" only, people who used their procedural knowledge succeeded better. It seems that correct procedural knowledge can help in monitoring "wrong" conceptual knowledge.

Understanding an idea means being able to deal with it in different representations. While most of the participants were able to deal with inverse function in symbolic representation (even though in a rather mechanical way), they did not make connection between representations. They did not know about the relationship between a function and its inverse in a graphic representation. One explanation for not being able to make connections among representations is that the prospective teachers did not understand the relationship between a function and its inverse in symbolic representation. They just managed to deal better with symbolic representation because they had the appropriate procedural knowledge. This was shown in some cases where subjects knew how to compose a function and its inverse on one hand, but when having no procedure to perform, thought of the relationship between the two functions in multiplicative terms. Another explanation for the lack of connection between representations is lack of knowledge about graphic representation which prevented them from dealing with the relationship between the graph of a function and the graph of its inverse.

Exponential and logarithmic functions as well as power (as a special case of polynomial) and root (power) functions are common as illustrations of theorems and properties in mathematics. They are used as specific cases to clarify general properties. These functions are also an important part of the high school mathematics curriculum. Most of the participants did not seem to have a good understanding of them. They did not understand the difference between exponential and power functions and thought that taking the log and taking the root were the same thing. In such a case it is not clear how these functions can clarify theorems and properties.

One cannot understand function without understanding inverse function and composition of functions. The prospective teachers in this study seem to have a fragile knowledge about these important sub-concepts of the function concept. In order for them to be able to help their students know more about function, they need to have an informal conception as well as more formal knowledge.

CHAPTER 8
PROSPECTIVE TEACHERS' KNOWLEDGE AND UNDERSTANDING ABOUT MATHEMATICAL FUNCTIONS

Purpose of the Study

Function is one of the most important concepts in mathematics in general and in the high school curriculum in particular. The concept of function appears everywhere: in everyday life as well as in every branch of mathematics. Although function is so important, the body of research conducted on learning and teaching function is not large. Existing studies of students' understanding of the concept of function show that this complex concept is not easy to comprehend. Many students develop misconceptions about functions (Bell and Janvier, 1981; Dreyfus and Eisenberg, 1983, 1987; Dreyfus and Vinner, 1982; Janvier, 1978; Lovell, 1971; Markovits et al., 1983, 1986; Marnyanskii, 1965; Monk, 1988; Thomas, 1975; Vinner, 1983; Vinner and Dreyfus, in press).

Understanding students' conceptions and how they think about a concept is an essential step in improving the learning of that concept. Another important step is providing a good curriculum. The third major factor in the process of learning is the mathematics teacher whose role is to help the learner achieve understanding of the subject matter. The role that teachers' subject matter knowledge plays in teaching has come into consideration recently (Ball, in press; Leinhardt and Smith, 1985; Shulman, 1986, 1987; Wilson et al., 1987). Teachers' subject matter knowledge is one of the components that influences teachers' decisions (Thompson, 1984). A teacher who has solid mathematical knowledge for teaching -- both subject matter content knowledge and pedagogical content knowledge -- can use his/her knowledge to teach for understanding and is more capable of achieving meaningful learning in his/her students.

One of the goals of this study was to contribute to the current discussion and analysis of subject matter knowledge for teaching by identifying important aspects of subject matter knowledge for teaching functions. The choice of aspects was based on integrated knowledge from several bodies of work: the role and importance of function in mathematics and in the mathematics curriculum; research and theoretical work on learning, knowledge and understanding of functions in particular and other mathematical concepts in general; and research and theoretical work on teachers' subject matter knowledge and its role in teaching.

As a result of this integration, six aspects seemed critical components of subject matter content knowledge for teaching functions:

* What is a function? (includes image and definition of the concept of function, univalent property of functions, and arbitrariness of functions).

* Different representations of functions.

* Inverse function and composition of functions.

* Knowledge about functions of the high school curriculum.

* Different ways of approaching functions: point-wise, interval-wise, global and as entities.

* Different kinds of knowledge and understanding of function and mathematics.

The first aspect -- what is a function? -- has to do with the correspondence or the match of the subjects' mental picture of function with the correct mathematical concept (Greeno, 1978; Resnick and Ford, 1981/1984; Vinner, 1983; Vinner and Dreyfus, in press), emphasizing the essential feature of the concept of function as it has evolved in history -- the arbitrary nature and univalence of functions (Freudenthal, 1983). The second aspect emphasizes recognition of the same idea in different representations, flexible manipulation of an idea within a given representation, and translation of an idea from one representation to another (Dufour-Janvier, Bednarz, and Belanger, 1987; Lesh, post and Behr, 1987), using the most common representations of function -- formulas and graphs. The third aspect was chosen since the strength of the function concept is rooted in the new operations that came with functions -- composing and inverting functions (Freudenthat, 1983). The fourth aspect includes a basic repertoire of functions from the high school curriculum, such as linear, quadratic, general polynomial, exponential, logarithmic, trigonometric and rational functions. It emphasizes having a good grasp of these specific functions in particular and understanding of general ideas and theorems as illustrated by the basic functions. The fifth aspect includes understanding and preference for different ways of approaching functions: point-wise, interval-wise, global, and as entities (Bell and Janvier, 1981; Janvier, 1978; Monk, 1988). The sixth aspect includes conceptual and procedural knowledge of functions as well as meanings and understanding (Hiebert and Lefevre, 1986; Skemp, 1976, 1979). It also includes knowledge about the nature of mathematical knowledge (Ball, 1988; Lampert, 1988; Schoenfeld, in press; Thompson, 1984) and the interactions between knowledge about mathematics and knowledge of function.

A second goal of this study was to describe kinds of knowledge prospective teacher subjects of this study had with respect to the above aspects and the interrelations among them, and to point to some of the limitations of their conceptions. Two additional pedagogical content knowledge aspects were also added as they seemed to be closely related to the previous aspects. These were:

* Teaching toward different kinds of knowledge and understanding of functions and mathematics.

* Students' mistakes -- what they do and why?

The first aspect includes teaching with emphasis on conceptual or procedural knowledge, meaning or rote learning, teaching for understanding or emphasis on following rules. The second aspect includes knowledge about students' common mistakes in function and their sources.

The six aspects of subject matter content knowledge and the two aspects of pedagogical content knowledge described above are parts of subject matter knowledge for teaching functions. The aspects are not independent from one another, but rather they are interrelated. While each one of the aspects can be highlighted separately, ignoring the interrelations among them will create a misleading picture of teachers' subject matter knowledge. To avoid a presention of the results as if each aspect is independent, and in order to highlight the interrelations among the aspects, the first three aspects -- what is a function?, different representations of functions, inverse function and composition of functions -- were chosen to be the main focus of the description of the results. Results about the other aspects were interwoven with the results about the first three aspects, emphasizing the interaction among all the aspects. The interrelations among the three main aspects and the other five aspects helped to create a more complete picture of the participants' subject matter content knowledge.

The choice of aspects was also based on the specific population to be studied. The participant subjects in this study were prospective secondary mathematics teachers in the last stage of their professional education. They were finishing or had already finished their mathematics methods class. Almost all of them were seniors, a few were juniors or postbaccalaureate students. This group was selected so that the description of their knowledge would reflect the knowledge teachers have gained during their college education, but before they begin teaching. The subjects came from eight mid western universities.

Summary and Discussion of Main Findings

The main foci of the description of the results were three aspects of subject matter knowledge for teaching function. However, each one of these aspects is interrelated with each one of the other five aspects mentioned above. For example, the aspect 'different representations' is interrelated with the aspect 'different kinds of knowledge and understanding of function and mathematics'. Ignoring the ways these different aspects are interwoven would give a simplistic picture of a much more complicated situation. Chapters 5-7, which reported the results of this study, emphasized each of the three main aspects, but included relationships with the other aspects as well. By doing so, the interrelations among the different facets of subject matter knowledge for teaching functions were apparent. While results about the three main aspects and the interrelations among the various aspects were highlighted in the three results chapters, results related to the other five aspects were less apparent. Therefore, the summary and discussion of the main findings in this chapter will include results related to each one of the three main aspects as well as each one of the other five aspects. This way, the contribution of each aspect will be more apparent although the interrelations among will, to some extend, be implicit.

What is a function?

The arbitrary nature of functions. The participants in this study revealed several facets of misunderstanding what a function is. Many of them ignored the arbitrary nature of the relationship between the two sets on which the function is defined in one way or another. Some expected functions to always be representable by an expression. Others expected all functions to be continuous. Still others accepted only "reasonable" graphs, etc. These findings are similar to findings from other studies about college freshmen (Vinner, 1983; Vinner and Dreyfus, in press). Dealing with functions that are defined on functions was also difficult for most of the participants in this study. People seemed to expect functions to be defined on numbers only. However, a person having one misconception did not necessarily have all the other types of misconceptions also.

Having an old and limited concept image of function, which basically includes functions that are described by equations, means that the arbitrary nature of function -- an essential feature of the concept of function as it has evolved in history (Freudenthal, 1983) -- is completely ignored. Rejection of arbitrary functions, which follows the historical development of functions, was apparent not only from the identification of functions and equations, but also from the ways some of the participants accepted or rejected graphs of functions, and by the acceptance of functions only those which were "known". Some of the participants followed the historical development of the definition of continuous functions when they accepted as functions only differentiable functions (continuous functions in 18th century terms) or when they called non-differentiable functions -- discontinuous functions.

The participants in this study seemed to reject arbitrary functions. This finding is surprising at first since many modern texts introduce functions by using arbitrary functions such as the function that assigns to each person his/her mother (no formula, no numbers). But this finding should not surprise us. After the first introductory lesson, almost all the functions that high school and sometimes even college students meet are the kind that have a "nice" graph and can be described by an expression. So the students' concept image of function is determined by the functions they meet and not by the modern definition of function which emphasizes the arbitrary nature of functions.

The participants in this study were not just any students. They were in the last stage of the study required for the mathematics major. Do we want math majors at the end of the 20th century to have a limited concept of function, similar to the one from the 18 century? The concept of function has been changed since the 18th century not because someone arbitrarily decided to change it. It has been changed because new discoveries in mathematics created the need for change in the definition. New discoveries created new branches of mathematics which also led to changes in the definition of function. Math majors who have an 18th century concept of function are also deprived of understanding mathematics developed since then, which is based on a more modern conception of function. What makes it even more serious is that the subjects in this study were prospective secondary math teachers. Should teachers have a more complete conception of the central concept in the high school curriculum? Can we expect teachers to be able to teach according to the modern definition of function, as it now appears in modern texts, while their conception of function is more restricted, more primitive? The participants' incomplete conception of function is problematic and may contribute to the cycle of discrepancies between concept definition and concept image of functions in students, as described in the literature (Dreyfus and Vinner, 1982; Vinner, 1983; Vinner and Dreyfus, to appear), keeping the students' concept image of function similar to the one from the 18th century.

Univalent functions. Most of the participants knew about the univalent property of functions and considered it to be important. But almost none of the second phase subjects could explain why it is important and how functions came to be defined that way. The fact that so few recognized that the need for the requirement is rooted in mathematics itself points to a limited understanding of function and its related domains.

Some serious questions are raised by the fact that, without prompting, none of the second phase subjects could come up with a reasonable explanation for the need for the property of univalence. This requirement is presented to the students as one of the most important characteristics of functions and, as this study tells us, this is what many of them think. They know that it distinguishes between relations that are not functions and those which are. But, usually, students are not told why it is important to distinguish between these two groups. Missing are explanations and examples of what is it that one can do with functions that one cannot do with relations which are not functions. This approach may contribute to making mathematics looks like an arbitrary collection of rules and definitions -- an idea that some of the subjects seemed to hold. Not being aware of the use or need of the requirement was, probably, the reason that some people remembered it backwards or completely forgot it.

Some people used "circles" and "ellipses" as examples of graphs of functions. Familiarity with those shapes seemed to be the reason for this mistake. For many participants, familiarity with a mathematical object was the criterion for accepting it as a function. These people belong to the group of subjects who used "known" as a criterion for functions. Accepting only familiar functions as functions makes a very limited repertoire of functions. In addition, this approach sometimes causes an acceptance of non-functions as functions. Having a concept image of functions as equations with nice graphs makes it reasonable to accept the familiar circles and ellipses as functions. This is especially true since circle, ellipse, as well as parabola and hyperbola belong to the same family of conic sections. Parabola is a function so why should not a circle be one?

Some of the people who held this image of the concept of function were also aware of the univalent requirement for functions. There seemed to be a contradiction between two pieces of information about function that they held. On one hand, they remembered learning about the univalent requirement in relation to function. On the other hand, they were familiar with circles and ellipses -- their equations and graphs -- and therefore included them as part of their concept image of function. These people seemed to understand the definition of function. They had no difficulties using the univalent property in the process of deciding whether a mathematical object was a function until they were confronted with a contradiction: a mathematical object that they considered as a function did not pass the univalent requirement. One way to solve this uncomfortable situation was to go with the concept image, to decide that only some specific functions are univalent. More awareness and sensitivity to this situation, an emphasis on making sure that people understand the need for univalent functions, can help in making the concept image of function closer to the concept definition.

Different representations of functions

Translation between two representations: expression and graph. Most of the prospective teachers in this study did not seem to have good connections between different representations of function. They had difficulties translating from symbolic representation to graphic representation and vice versa when dealing with two basic functions from the high school curriculum: quadratic and sine functions and also when dealing with inverse function. A trend to base conclusions on checking of a few examples was detected among the second phase subjects. This was the case when they made up rules and explanations about the relationship between the shape of a parabola and the signs of the coefficients of the quadratic expression. This was also the case when they used a point-wise approach to decide about the shape of a specific graph or about the expressions that describe given graphs.

Interpretation of graphs representing situations. The questionnaire data suggested that most of the participants did not get distracted by seemingly conflicting pictorial aspects. Only about 15% of the first phase subjects seemed to be influenced by superficial characteristics of the given graphs and therefore chose a wrong answer. These results are much better than results of other studies which were conducted with school pupils (Bell and Janvier, 1981; Janvier, 1978). But the interview data revealed that the problem of misinterpretation was more widespread than it first seemed to be. People who made correct interpretations were also willing to accept wrong interpretations of the same graphs as well. The most common mistake was to ignore the labels on the axes and "attach" new meanings to the axes and therefore to the graphs. For example, when given a distance versus time graph, some people got confused by the superficial characteristics of the graph, and interpreted a decrease in the graph as showing a rate decrease.

Inverse function and composition of functions

Inverse function and composition of functions, as with any other concept, cannot be understood in one simplistic way only. Understanding these sub-concepts of the concept of function requires understanding the general meaning as well as the formal mathematical definition. Knowledge about them includes conceptual knowledge as well as procedural knowledge in different representations. "Undoing" is an informal meaning of inverse function which captures the essence of the definition. The importance of this informal meaning is also recognized by the National Council of Teachers of Mathematics who recommends that all students explore the concept of inverse function informally as a process of undoing the effect of applying a given function, while the precise definition of inverse function and composition of functions be reserved for college intending students (*Curriculum and Evaluation Standards for School Mathematics*, 1988). But most of the participants did not approach inverse function by using the inverse property. Rather, many people used their procedural knowledge, a step by step algorithm for composition of functions in a mechanical way, without any attempt to approach the problem conceptually. This was the case even with some of the people who "knew" the "undoing" characteristic of inverse function. Although they had the needed knowledge, those people did not use it. Having the appropriate knowledge but being reluctant to use it is also reported by Schoenfeld (1985, 1986, in press) in case of geometry.

Understanding inverse function as "undoing" only is not enough. This term is too vague and not precise. "Undoing" made many subjects choose the root function as the inverse function of an exponential function. It also made many think that every function has an inverse function, ignoring the one-to-one requirement. So, a solid understanding of the concept of inverse function cannot be limited to "undoing" only. In cases where the conceptual knowledge was limited to "undoing" only, subjects who used their procedural knowledge succeeded better. It seems that correct procedural knowledge can help in monitoring "wrong" conceptual knowledge.

Understanding an idea means being able to deal with it in different representations. While most of the participants were able to deal with the inverse function in symbolic representation (even though in a rather mechanical way), most of the second phase subjects did not make connections between representations. They did not know about the relationship between a function and its inverse in a graphic representation. One explanation for not being able to make connections among representations is that the prospective teachers did not understand the relationship between a function and its inverse in symbolic representation. They just managed to deal better with symbolic representation because they had the appropriate procedural knowledge. This was shown in some cases where subjects knew how to compose a function and its inverse on one hand, but when having no procedure to perform, thought of the relationship between the two functions in multiplicative terms. Another explanation for the lack of connection between representations is lack of knowledge about graphic representation which prevented them from dealing with the relationship between the graph of a function and the graph of its inverse.

One cannot understand function without understanding inverse function and composition of functions. The prospective teachers in this study seemed to have a fragile knowledge about these important sub-concepts of the function concept. In order for them to be able to help their students know more about function, they need to have an informal conception as well as more formal knowledge.

Functions from the high school curriculum

Quadratic function. Given the shape and position of the graph of a quadratic function, there was evidence that participants had difficulties deciding what sign the coefficients of the quadratic expression $ax^2 + bx + c$ should have and why. Most of the participants knew about the relationship between "a" and the graph but not all of the second phase subjects could explain it. Many had only partial understanding of the relationship between "c" and the graph and they overgeneralized simpler cases to all quadratic functions, claiming that "c" regulates the shifting of the graph ($y = x^2$) up and down or even horizontally. Still many of the participants made up a simple relationship between "b" and the graph which does not exist but seemed to match nicely the overgeneralized rule for "c." They claimed that "b" regulates the shifting of the graph horizontally (or sometimes vertically). Many of the participants thought that "b" and "c" have similar roles.

The quadratic function is a special and important case of the functions used in high school mathematics. If one understands the relationship between "a" in the quadratic expression and the graph rather than only memorizing the rule, one has the ability to generalize it to related relationships between the leading coefficient of any polynomial and its graph. So understanding the relationship between the role of "a" in the symbolic representation of a quadratic function and in the graphic representation is very powerful. The relationship between "b" and the quadratic graph are specific to the quadratic function only. But the role of "c" in determining the y-intercept of the quadratic graph holds for all polynomials. So, again, this knowledge is very powerful for the understanding of the relationship between the symbolic and graphic representations of a family of functions. Many prospective teachers in this study seemed to lack this powerful knowledge.

Trigonometric functions. The questionnaires and interviews revealed difficulties in dealing with radians. The second phase subjects were not sure about the meaning of a whole number in relation to what they considered to be an adequate domain for a trigonometric function. Not having a good understanding of domain and range of trigonometric functions means difficulties in working with these functions. But trigonometric functions are part of the high school curriculum. So prospective secondary teachers should be familiar with them. Teachers need to feel comfortable and familiar with the functions they teach. Even if the problems are actually problems about functions in general and use trigonometric functions just as a case, difficulties with trigonometric functions prevent the solver from reaching a correct solution. This also causes students to approach those problems as problems in trigonometry instead of problems about functions in general. Familiarity with basic functions, such as trigonometric and polynomial functions, can help not only in the understanding of those specific functions but also with the understanding of functions in general.

Exponential, log, power and root functions. The National Council of Teachers of Mathematics recommends that college-intending students develop a thorough understanding of specific functions including polynomial (a general case of a power function), exponential and logarithmic (*Curriculum and Evaluation Standards for School Mathematics*, 1988). Exponential and logarithmic functions as well as power (as a special case of polynomial) and root (power) functions are common as illustrations of theorems and properties in mathematics. They are used as specific cases to clarify general properties. These functions are also an important part of the high school mathematics curriculum.

Most of the participants did not seem to have a good understanding of these functions. The results show that the relationships between exponential, log, power and root functions were not clear to many of the participants. The problem seems to be rooted in the distinction between power and exponential, log and root functions; in understanding what root function is; and in overgeneralization of the meaning of inverse function as "undoing". Many of the participants did not seem to understand the difference between exponential and power functions and treated both functions as power functions. They also did not have a good understanding of the log function and thought that taking the log and taking the root were the same thing. In addition, many of the participants used their naive conceptual knowledge of what an inverse function was when working with exponential, log and root functions. These people used the idea of "undoing" as their interpretation of inverse function. Then, for example, the xth root of 10 seemed to them to "undo" what 10^x does: In order to get 10^x, one starts with 10 and then raises it to the xth power. By taking the xth root of 10^x, one gets 10 back. Therefore, they decided that the root function is the inverse function of the exponential function instead (or, in addition to) the log function.

Different ways of approaching functions

Point-wise for graphing. A large group of the interviewees approached graphing by using a point-wise approach: plotting some points (usually ones that were easy to calculate) and connecting them to a smooth curve. This method is not recommended for graphing since it ignores important characteristics of the function and therefore might end up with a graph which is either wrong or not informative.

Interestingly, all the subjects have had calculus and other advanced courses in mathematics. So, all of them should have known that analysis of some characteristics of the function to be graphed are important. They also should have known that some points are more important than others, so producing a good graph cannot be based on the choice of numbers that are easy to compute. Easy calculation is especially not a good argument at the end of the 1980's, when calculators and computers are so widespread. The National Council of Teachers of Mathematics also recommends avoiding graphing by hand using tables of values (*Curriculum and Evaluation Standards for School Mathematics*, 1988). That some of the prospective teachers had a strong tendency to use the point-wise approach as the graphing method is noteworthy. The reason for the extensive use of a point-wise approach seems to be the difficulties the participants had with thinking of a function as it behaves over intervals or in a global way, difficulties that are also reported in other studies (Janvier, 1978; Bell and Janvier, 1981; Monk, 1988).

Global versus point-wise. The participants in this study were presented with several problems where they had to translate from symbolic representation to graphic representation and vice versa or where they needed to interpret graphs. In all of those problems the subjects could have used a point-wise approach or a global view of the behavior of the function. The latter was more appropriate to use in the given situations. Most of the second phase subjects were not consistent in their choice of method, but rather interchanged methods in different problems or used a combination of the two approaches in one problem. The mixed use of the two approaches shows that people were familiar with both methods. But in many cases they used less appropriate knowledge.

The second phase subjects tended to use a point-wise approach when translating from one representation to another. But when the translation process was too complicated for a point-wise approach, the subjects used a combination of the two methods. The second phase subjects tended to use a global approach when interpreting graphs representing situations whenever pictorial aspects did not conflict with correct meaning.

There seemed to be a close relationship between understanding of function and the global approach to functions. The questionnaire and the interview included problems constructed so that both point-wise and global approach to function could be used. Participants in the second phase of the study, who could easily and freely use a global analysis of change, had a better and more powerful understanding of the relationships between graphic and symbolic representations, than participants who preferred to check some local and specific characteristics. But this conclusion cannot be generalized to a similar statement about close relationship between the use of global view of function and better understanding of the meaning of graphs representing situations. Some of the second phase subjects oversimplified the global approach and used it without really understanding the meaning of the graph. So the ability to use a global approach is not sufficient for understanding meanings.

Different kinds of knowledge and understanding of function and mathematics

Conceptual and procedural knowledge. Many of the second phase subjects were rule oriented. They tried to recall rules in order to solve problems such as in the case of translating from one representation to another instead of analyzing the situation conceptually. When they could not remember a relevant rule, they tried to invent one by checking a few cases and then used the "rule". They seemed to rely on procedural knowledge without connections to conceptual knowledge and meanings. These results are similar to those reported by Ball (1988).

Sometimes, when their conceptual knowledge was not solid enough, relying on correct procedural knowledge led the participants to the correct answer while relying on a limited conceptual knowledge led other people to the wrong answer. For example, using "undoing" as a criterion for inverse function, led many people to accept the root function as the inverse function of the exponential function. But people who used the procedure and checked to see if $(f \cdot g)(x) = x$, got correctly that the root function was not the inverse.

Many of the participants seemed to lack rich relationships which characterize conceptual knowledge. In many cases they had the needed knowledge but lacked the connections between different pieces to make it accessible. Evidence for lack of connectedness were, for example, in the case of finding the number of solutions of a given quadratic equation, dealing with the univalent property of functions, and in the case of the graphical relationship between a function and its inverse. Having the appropriate knowledge but being reluctant to use it is also reported by Schoenfeld (1985, 1986, in press). Ball (1988) also reports about lack of connectedness in her study of prospective teachers' knowledge.

Knowledge about mathematics. A method of checking a very limited number of simple examples was used by many of the participants in the second phase of this study. Investigation of a situation by checking specific cases is a very powerful strategy in mathematics. Many discoveries were made by inductive reasoning. Looking at specific cases helps with the understanding of the situation. But there are two problems in the way the prospective teachers used that strategy. First, the number of examples and their variety were too limited. Secondly, inductive reasoning is a very powerful strategy for discovering the rule, and it might also help in seeing why the rule should hold, but it is not enough as an explanation for the existence of the rule. In other words, checking examples is not a proof. The fact that so many of the second phase subjects made conclusions and used explanations for students that were based only on checking examples without making sure that all possibilities were covered or using deductive reasoning, seems to point to lack of understanding of what counts as an explanation, and what ways are considered appropriate and acceptable in mathematics for transforming a conjecture to a theorem, i.e., what is acceptable as a proof.

The National Council of Teachers of Mathematics recommends that in grades 9-12, the mathematics curriculum should include principles of inductive and deductive reasoning. Students should experience the making of a conjecture by generalizing from a pattern or observations made in particular cases (inductive reasoning) and then test the conjecture by constructing either a logical verification or a counter example (deductive reasoning) (*Curriculum and Evaluation Standards for School Mathematics*, 1988). Many of the prospective teachers in the second phase of this study accepted rules by using inductive reasoning only, without any attempt to construct logical verifications. This raises a question: Are they ready to help their students learn that "deductive reasoning is the method by which the truth of a mathematical assertion is finally established", as emphasized in the Standards?

Knowledge of structure of number systems. The use of a point-wise approach to graphing revealed misconceptions about the structure of number systems. Several participants held a wrong, limited view of the real numbers as a countable set and the irrational numbers as a discrete set of numbers. They tried to sketch the graph of the function that assigns x to every rational number, and 0 to every irrational number, point by point; as if one starts from 0 and keeps going to the next number; as if one goes through several numbers until one hits an irrational number. Having a conception of the real numbers as a countable set and the irrational numbers as a discrete set of numbers suits a point-wise approach to graphing that many of the participants used.

The real numbers are used as domain and range of many functions in the high school and college curriculum. Having a wrong conception of the structure of the sets which serve as domain and range leads to wrong understanding of the function itself, as was presented in the wrong graphs given by many participants in the second phase of the study. Understanding function is interrelated with the understanding of its domain and range.

Teaching toward different kinds of knowledge and understanding of function and mathematics

Function definitions for students. The prospective teachers tended not to use modern terms such as relation, mapping and correspondence, as well as domain and range, when defining function for students. They tried to describe what a function is by using terms that are more familiar to students. Choosing more familiar terms such as equation or graph seem to be a reasonable pedagogical decision, especially since the modern definition of function is problematic for many students (Vinner, 1983; Vinner and Dreyfus, in press). But the decision of how to describe what a function is for students was not based on pedagogical arguments alone, it was also based on the concept image of function that the prospective teachers themselves held. Most of the participants did not hold a modern concept image of function. Therefore, it is not surprising that when describing what a function is for students, many chose to use their concept image of function and tended not to use modern terms. So even if the decision not to use a modern definition for students may seem a good pedagogical decision, it should probably be attributed to the incomplete concept image of function that the prospective teachers in this study held and not to mature pedagogical reasoning.

The number of prospective teachers subjects who used modern terms like relation, mapping and correspondence in a definition for students dropped compared to the number of people who used a modern definition when they were not asked to refer to students. Instead, they tried to find ways to help the student understand what a function is. A very popular illustration for students of what a function is was to describe it as a machine or as a black box.

A function machine or a black box is usually used when functions are introduced. This is probably the reason for the large number of prospective teachers who used it as an explanation of what a function is for students. While a function machine may be helpful for function notation, it is not helpful for a study of function behavior or dealing with functions in a global way, as was mentioned by many of the second phase subjects.

Function machine, as many other representations, is not an answer to all the learning difficulties that students may have with function. As many other teaching aids, it has some advantages and some disadvantages. By using a machine to illustrate function, one may help students understand function notation but might cause at the same time difficulties with approaching function in a global manner. Teachers should be aware of the strengths and weaknesses of a machine as an illustration for function and base their decision to use it on that. The second phase subjects did not seem to weigh strengths and weaknesses when asked what they thought about a machine as an illustration for function. They either liked the idea or disliked it. But they did not seem to weigh advantages against the disadvantages or appropriate time versus inappropriate time for using a machine. Similar results about prospective teachers who hold a simplistic view of representations for teaching mathematics rather than sophisticated knowledge are also reported by Ball (1988).

The use of the function machine explanation is very widespread. One generation of teachers gives it to the next generation. Teachers tend to follow their teachers' footsteps unless they have developed a different repertoire. Therefore, prospective teachers should be more familiar with the reasoning for using/not using a machine as an illustration for function and the appropriate/inappropriate places for using it. The decision to use it should be based on pedagogical reasoning and not on the fact that this is what their teachers did before them.

Rule orientation. Another way that many of the prospective teachers in this study chose to explain to students what a function is was by using the "vertical line test" for graphs of functions. Many of those who used it did not explain what a function is. They just gave the student a test. Most of the prospective teachers in this study considered the univalent property of functions to be very important. But this cannot explain the large number of people who decided to use the "vertical line test" in their explanation of what a function is for students. Actually, fewer people mentioned that requirement for students than when they gave a definition of function without relating it to students. An explanation for the extensive use of the "vertical line test" in the explanations for students, sometimes without any reference to what a function is at all, seems to be rooted in teachers' tendency to provide students with rules. The students then can follow the rules, and get the right answers without understanding. It seems that a large group of prospective teachers subjects chose to give the student a rule, even if the student does not understand why it works and what the purpose is for this checking. While easy procedures are important, math educators today complain about the overemphasis of school mathematics on procedural knowledge with no connection to meaning (e.g., *Curriculum and Evaluation Standards for School Mathematics*, 1988; Davis, 1986; *Educational Technology Center*, 1988; Lampert, 1988; Peterson, 1988; Resnick, 1987; Romberg, 1983; Schoenfeld, 1987). Making the "vertical line test" alone an explanation of what a function is, is an example of procedural knowledge with no connection to meaning. Although many, not all the prospective teachers subjects supported the attitude of giving a quick rule without understanding. Some of them emphasized the importance of understanding why the rule works.

Explanations for 'why?'. Explanations for students that were suggested by the prospective teachers in this study seemed to be closely related to their subject matter knowledge. People who treated a specific relationship as a rule to memorize suggested, as an explanation of why the relationship exists, giving the student several examples. Suggesting that the student try several examples to check to see if the rule holds, instead of helping him/her to see reason(s) for the relationship, is like telling the student that inductive reasoning is the method by which the truth of a mathematical assertion is established. While it is important that students use inductive reasoning to discover rules and relationships, it is also important that they understand that checking examples does not answer the question 'why does the rule hold?' (This is also recommended by the National Council of Teachers of Mathematics in *Curriculum and Evaluation Standards for School Mathematics*, 1988).

In order for teachers to be able to help students appreciate the distinction between inductive and deductive reasoning, the teachers themselves should appreciate that. The prospective teachers who suggested the use of inductive reasoning as an explanation for the existence of a rule did not seem to appreciate that distinction. Teachers should also know why the rule holds. Their procedural knowledge should not be separated from meanings and conceptual knowledge. Otherwise, they would not be able to help their students understand.

Students' mistakes - what they do and why?

Overall, the participants seemed to be aware of the common misconceptions that students have and their sources. They provided explanations which sounded reasonable as sources for student mistakes, and basically agreed with the literature. Subject matter content knowledge seemed to be related to the explanations provided. People whose knowledge of functions was very limited seemed to have difficulties providing explanations for students' mistakes. Accepting all relations as functions, for example, caused difficulties in explaining students' rejection of some functions, attributing all difficulties to functional notations while ignoring a variety of other sources for mistakes. On the other hand, people who, for example, expected all functions to be described by a formula, but also accepted some other "known" functions, such as a constant function, seemed to be more sensitive to students' expectation for a formula. So, teachers' subject matter knowledge is important not only for evaluating students' answers as right or wrong, but also for understanding the sources of students' mistakes.

Still, having the appropriate subject matter knowledge implicitly, i.e., being able to apply the knowledge, did not guarantee enough awareness of that piece of knowledge to make it a part of pedagogical content knowledge. For example, knowing a correct procedure for graphing did not imply an ability of evaluating appropriateness of other procedures.

Significance and Implications

The description of prospective teachers' knowledge and understanding about mathematical functions should be of interest to several parties: teacher educators, mathematics educators and mathematics teachers at the university level as well as at the high school level. The implications of the results of this study are theoretical and practical. They have to do with both goals of the study -- the identification of aspects about the concept of function that are important for secondary teachers to know, and the description of the knowledge prospective teachers have with respect to the above aspects and the limitations of their conceptions. In addition, there are methodological implications.

Six aspects of subject matter content knowledge of function were identified as very important for secondary mathematics teachers. The choice of aspects was based on integrated knowledge from several bodies of work: the role and importance of function in mathematics and in the mathematics curriculum; research and theoretical work on learning, knowledge and understanding of functions in particular and other mathematical concepts in general; and research and theoretical work on teachers' subject matter knowledge and its role in teaching. But the choice of aspects was also based on the subjective values and personal knowledge of the researcher. The six identified aspects, together with the reasons for their choice, can serve as a starting point for a discussion of what do secondary mathematics teachers need to know about functions in order to be able to teach well.

Mathematics educators today are concerned with the way mathematics is taught. They call for making a change in the way teachers teach to emphasize teaching for understanding and meaningful learning (e.g., *Curriculum and Evaluation Standards for School Mathematics*, 1988; Davis, 1986; *Educational Technology Center*, 1988; Lampert, 1988; Lappan and Schram, to appear; Peterson, 1988; Resnick, 1987; Romberg, 1983; Schoenfeld, 1987). The kind of knowledge that prospective secondary teachers have about the central topic of the high school curriculum, as was described in this study, makes questionable their ability to teach mathematics for understanding. Lack of connectedness and even contradictions among different pieces of knowledge, weak and fragile knowledge of the basic functions from the high school curriculum, lack of flexibility in translating from one representation to another, use of memorized rules instead of understanding, use of inductive reasoning as if it was a proof or an explanation, are just some of the findings that may impede the teachers' ability to teach for understanding.

Current reform efforts in teacher education (e.g., Carnegie Task Force on Teaching as a Profession, 1986; *Tomorrow's teacher: A report of the Holmes Group*, 1986) point to the inadequate professional education teachers get from both subject matter and education courses. However, improving teacher education programs should be based on knowledge of the present situation. This study provides needed information about the limitations of the existing programs in the preparation of secondary teachers to teach function. Changes should be made so that secondary math teachers will understand better the relationships between different representations of function, have a better match between their concept image and concept definition of function, understand inverse function and composition of functions, put more emphasis on understanding than rote learning, be well acquainted with basic functions from the high school curriculum, etc.

This study, as a by product, provides knowledge of the function concept of college mathematics students at their last stage of their college education. Most of the subjects were senior math majors, half of them with a math grade point average between 3.0 and 4.0. Still, they presented a very limited understanding of important aspects of the concept of function. Mathematics teachers at the university level can use these findings to make some changes in math courses so that better understanding of the concept of function will be developed. High school teachers should also be concerned with the results of this study. They can help them understand students' difficulties in learning function.

The methodological implications of this study are concerned with the combination use of a questionnaire and an interview. This combination proved to be adequate and fruitful for the purpose of this study. The use of a questionnaire was helpful in getting a relatively large number of responses which provided a general picture of the prospective teachers' knowledge. The interview clarified this picture and added more details.

Recommendation for Future Research

This study contributes one piece to our understanding of the complex process of learning and teaching function. There is much more to be learned about this process. The present study was limited in its potential for finding answers to a number of important questions. The prospective teacher subjects were treated as one group even though they came from eight different universities. Consequently, it was not possible to examine whether differences in teacher education programs influence their conceptions of function and teaching functions. Studies are needed to examine whether different programs of preparation of prospective secondary math teachers produce different conceptions of function with relation to the aspects of understanding function identified in this study.

Another important question in need of investigation is whether and what changes occur in knowledge about function as the teacher goes through different stages. Therefore, replications of this study should be conducted with different populations: high school seniors, prospective secondary teachers in different stages of their professional education, novice teachers, and experienced teachers. Information gathered from these studies will help us understand what knowledge the teacher may gain through teaching. This will help in making informed decisions about possible changes in teacher education programs for pre service as well as in service teachers.

This study was limited in the pedagogical content knowledge aspects being studied. More aspects should be added to expand the scope of the study. These should include such aspects as evaluation of curriculum materials, the role of function in the K-12 curriculum, ideas about what students need to know about function at the end of high school, and ways of helping students understand function.

Other issues for investigation have to do with the role that subject matter knowledge plays in teaching. This study suggests that there are differences between prospective teachers' conceptions of function and their thinking about teaching functions. What are the relationships between what teachers know and what they do? What different teachers know and do in class when they teach functions, and how this relates to what their students know about functions? In order to investigate the above research questions, a component of participant observation should be added to this study. Documentation of teachers' instructional practice and students' learning followed by stimulated recall in addition to the instrumentation developed for this study are necessary in order to gain knowledge about the relationships between teachers' conceptions and instructional practice and their students' understanding of function.

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APPENDIX A
THE QUESTIONNAIRE

APPENDIX A

THE QUESTIONNAIRE

1. a) Give a definition of a function.

   b) A student says that he/she does not understand this definition.
   Give an alternate version that might help the student understand.

2. If you substitute 1 for x in ax^2+bx+c (a, b and c are real numbers), you get a positive number. Substituting 6 gives a negative number.
How many real solutions does the equation $ax^2+bx+c=0$ have? Explain.

3. A student is asked to give an example of a graph of a function that passes through the points A and B (See fig.1).

   The student gives the following answer (See fig.2).

   When asked if there is another answer the student says: "No".
   If you think the student is right - explain why.
   If you think the student is wrong - how many functions which satisfy the condition can you find? Explain.

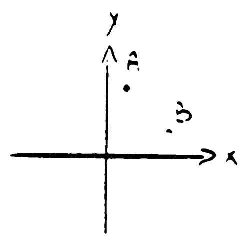


fig.1

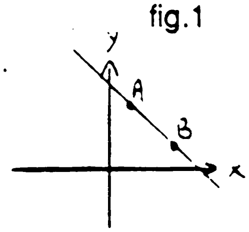
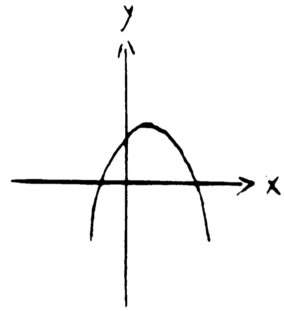


fig.2

4. How are functions and equations related to each other ?

5. This is the graph of the function $f(x)=ax^2+bx+c$. State whether a, b and c are positive, negative or zero. Explain your decision.



6. This is the graph of sin x (fig.1).
   a) Sketch the graph of sin (x +1) on the same coordinate system.

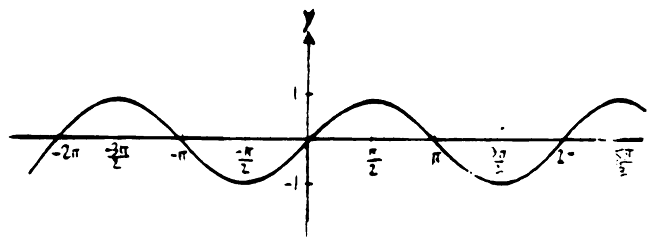


fig.1

   b) Give algebraic expressions for the functions whose graphs are shown in fig.2 and fig.3.

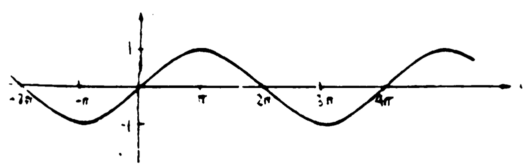


fig.2

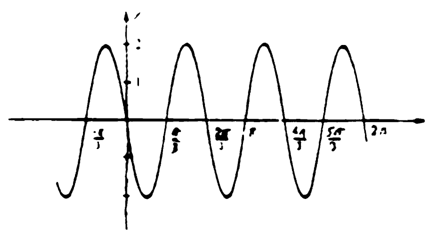
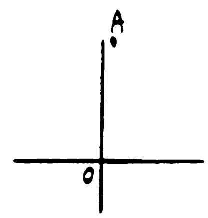
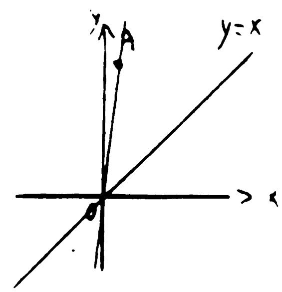


fig.3

7. A student was asked to find the equation of a line that goes through A and the origin O.

   She said: "Well, I can use the line $y=x$ as a reference line. The slope of line AO should be about twice the slope of the line $y=x$, which is 1. So the slope of line AO is about 2, and the equation is about $y=2x$, let's say $y=1.9x$." What do you think the student had in mind? Is she right? Explain.

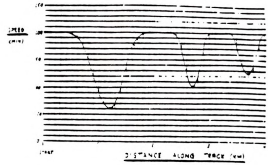




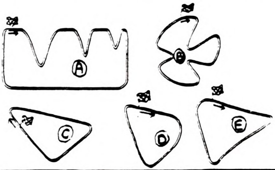
8. High-school students were asked the following question (taken from *The Language of Graphs* by Malcolm Swan):

A Motorcycle Race

The graph drawn below, shows how the speed of a motorcycle varied as it completed the second lap of a race.



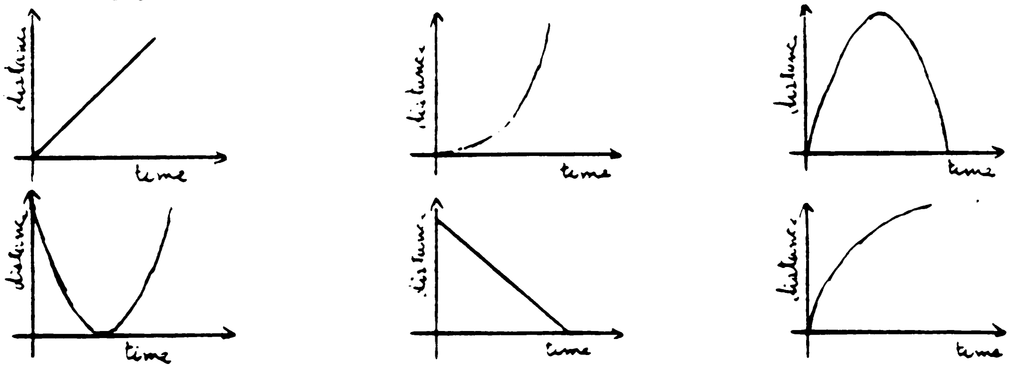
Which of the circuits below would have produced this graph?



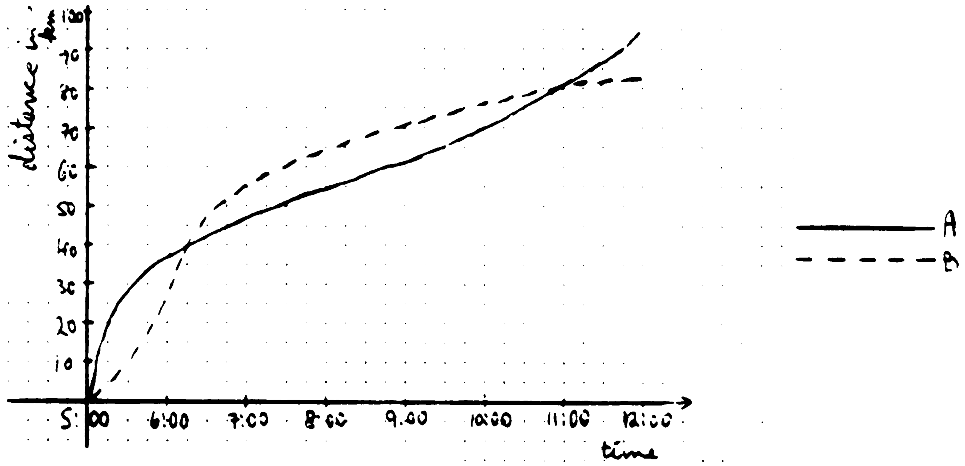
a) Which circuit has produced the graph? How did you decide?

b) For each of the four circuits you rejected, briefly explain what might make a student choose them.

9. Tim and Donna live 1 km from their school. Usually they walk together to school. Yesterday they both left their houses 10 minutes before school started. Tim started to walk but Donna was afraid to be late and started to run. After a while Tim realized that although he tried to walk faster and faster, he had to run if he did not want to be late, and started to run. At the same time, Donna became tired and had to walk instead of run. They both reached school exactly on time. Which of the following graphs is Tim's and which one is Donna's?



10. This graph describes the distances A and B travelled.



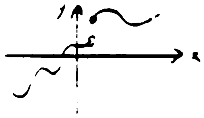
a) For what times had A travelled further than B? Explain.

b) At what time did A travel the fastest? Explain.

c) Who travelled faster between
   (i) 5:30 and 6:00?
   (ii) 8:00 and 9:00?

d) Who travelled more km between
   (i) 6:30 and 8:00?
   (ii) 10:00 and 11:00?

11. A student marked all the following as non-functions.
(R is the set of all the real numbers, N is the set of all the natural numbers).

(i) 

(ii) $f : R \longrightarrow R$
$f(X) = 4$

(iii) $g : N \longrightarrow R$

(iv) A correspondence that associates 1 with each positive number, -1 with each negative number, and 3 with zero.

(v) $$g(x) = \begin{cases} x, & \text{if } x \text{ is a rational number,} \\ 0, & \text{if } x \text{ is an irrational number.} \end{cases}$$

(vi) $\{(1,4), (2,5), (3,9)\}$

a) For each case decide whether the student was right or wrong. Give reasons for each one of your decisions.

(i) Right/wrong because

(ii) Right/wrong because

(iii) Right/wrong because

(iv) Right/wrong because

(v) Right/wrong because

(vi) Right/wrong because

b) In cases where you think the student was wrong, try to explain what the student was thinking that could cause the mistake.

12. Given $f(x) = 2x-10$ and $f^{-1}(x) = \dfrac{x+10}{2}$. Find $(f^{-1} \circ f)(512.5)$. Explain.

13. Given $(f \circ g)(x) = f(g(x)) = \sqrt[5]{x+3}$.
    a) Find f and g that satisfy this condition.

    b) Are there more than one answer to part a ? Explain.

14. A student said that there are 2 different inverse functions for the function $f(x) = 10^x$: One is the root function and the other is the log function. Is the student right? Explain.

15. Consider the set of functions whose domain and set of images are all the real numbers. K assigns to each pair of such functions, their composition.
    a) Is K a function? Explain.

    b) is K^{-1} a function? Explain.

Background Questions

1. ___ male ___ female

2. Age range: ___19-23 ___ 24-29 ___ 30-35 ___ over 35.

3. What is your over all college grade point average?

4. What is your over all college grade point average in mathematics?

5. Please check the **mathematics** courses you have taken at the college or university level.
___Differential calculus, # credits___, semester/quarter.
___Integral calculus, # credits___, semester/quarter.
___Differential equations, # credits___, semester/quarter.
___Advanced calculus, # credits___, semester/quarter.
___Euclidean geometry, # credits___, semester/quarter.
___Projective geometry, # credits___, semester/quarter.
___Number theory, # credits___, semester/quarter.
___Matrix/Linear algebra, # credits___, semester/quarter.
___Algebra (groups, rings, fields), # credits___, semester/quarter.
___Topology, # credits___, semester/quarter.
___Probability, # credits___, semester/quarter.
___Combinatorics, # credits___, semester/quarter.
___History of mathematics, # credits___, semester/quarter.
___Vector & Tensor analysis, # credits___, semester/quarter.
___Numerical analysis, # credits___, semester/quarter.
___Logic
___Other: ________, # credits___, semester/quarter.

6. Please check the **education** courses you have taken at the college or university level.
___Educational psychology, # credits___, semester/quarter.
___Methods of teaching secondary math, # credits___,semester/quarter.
___Social foundations, # credits___, semester/quarter.
___Pre student teaching field experience,#credits___,semester/quarter
___Student teaching, # credits___, semester/quarter.
___Other: ________, # credits___, semester/quarter.

7. What is your minor?
How many courses have you taken in your minor?

APPENDIX B
INTERVIEW QUESTIONS

APPENDIX B
INTERVIEW QUESTIONS

1. Give an example of a function.

   Can you represent it in a different way? (A different representation?)

   (Another way?)

   (Show me all the different ways that you can represent this function.)

   (Is there a representation of a function that you could not use in this case?)

2. What about the function concept is easy for students?

   (Why is it easy? Anything else?)

   What about the function concept is hard for students?

   (Why is it hard? Anything else?)

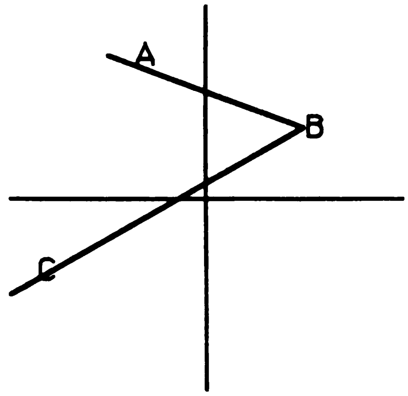
3. Is it important to teach the line test for graphs of functions to students? Why?

   (What is the line test? What would you teach to your students? How would you teach that? Can you give me an example?)

4. How would you show the relationship between a graph of a function and the graph of its inverse to a student?

   (Can you give me an example? How would you respond if a student asks you why does it work?)

5. When asked to draw a graph of a function that passes through the points A, B and C, a student gave the following answer:



What do you think about this answer?

(Is it correct? Why?

If yes - Are there other correct answers? Give me examples. Why do you think the student gave this answer?

If no - What would you like to see as a correct answer? Give me examples. Why do you think the student gave this answer?)

6. Explain to a student in algebra 2 how to sketch the graph of

$$y = \frac{1}{x^2 - 1}$$

(Can you illustrate what you expect the student to do? Are there other ways which are more/less appropriate? Can you give examples? Why are they more/less appropriate?)

7. Go through the questionnaire with the interviewee, asking him/her to explain what they did. For each students' misconceptions that they identified as a misconception, ask them to explain why would a student answer in this way. Starting with question 2, ask in addition the following:

*2)-Can you use graphs to solve this question? (If they didn't)

*3)-How many functions? Ask for examples, try to make clear if they are special only or also arbitrary.

How do you know that there is an infinite number of functions?

*4)-Are all functions equations? What do you mean?

-Are all equations functions? What do you mean?

-Can any function be represented by an algebraic expression? a formula?

*5)-What is important to teach students about the relationship between a, b and c and the graph?

(Why is $a<0$ when the graph looks like ? What will happen to the graph if we change c? Why? What will happen to the graph if we change b? Why? If we change a?)

*7)-Can you estimate the equation of the line?

*10)-Many students answered in c: A and then B. Why do you think they answered this way? (Ask that only if they didn't).

*11)-Do you think there is a formula to describe the graph in (i)?

-Many students said that (iii) is a function because N is included in R. Others said that (iii) is not a function because there are more numbers in R then in N. What do you think?

-In (v) some students said that g is not a function because 0 is a rational number. What do you think?

-Can you graph (v)?

-Many students said that (vi) is just a set of points and not a function. Others said that it is not a function since 3 should go with 6. What do you think?

*12)-Some students said that since $f\text{o}\, f^{-1}=1$, then $(f\text{o}\, f^{-1})(x)=1\cdot x=x$. What do you think?

*14)-What is the root function?

-How did you decide which one is the inverse?

-Why are/aren't they the same?

*15)-What are the domain and range of K?

-Give examples of how it works.

-Does every function have an inverse function?

8. Present a card sort (of different descriptions of functions):

A function is a set of ordered pairs.

A function is a graph.

A function is a correspondence between 2 sets.

A function is a mapping from one set to another.

A function is a rule.

A function is a formula.

A function is a relationship between 2 variables.

A function is a dependance of one variable on another.

A function is a relation.

A function is an equation.

-Ask the subject to sort the cards.

-Ask the subject to sort the cards to statements that describe all functions, some functions, and not a description of functions at all.

-Ask the subject to order the relevant cards according to the way students are usually introduced to functions, a developmental ordering.

9. *1)-What did you assume the student didn't understand?

-Many prospective teachers said they would describe a function to their students as a black box or a machine. You put a number in, the machine changes it, and gives you an output. What do you think about this approach?

-Why is there, in the definition of functions, the requirement of having only one image for each element in the domain?