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BIFURCATION OF SYMMETRIC
PLANAR VECTOR FIELDS

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Hyeong-Kwan Ju

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By

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ABSTRACT

BIFURCATION OF SYMMETRIC PLANAR VECTOR FIELDS

By

Hyeong–Kwan Ju

The pitchfork homoclinic bifurcation of symmetric planar vector fields and some codimension three bifurcation of symmetric planar vector fields with nilpotent linear part are studied.

The set of symmetric planar vector fields with an equilibrium point of pitchfork type and a symmetric homoclinic orbits at this equilibrium point is a codimension two submanifold. This is shown using Melnikov's integral around the homoclinic orbit and studying the asymptotic behavior near the equilibrium point in $\mathbb{R}^2$. The bifurcation diagram is also obtained.

The bifurcation diagrams of generic 3–parameter families of symmetric planar vector fields with linear nilpotent part is analyzed and described on the sphere using abelian integrals. These integrals generate the Picard–Fuchs equations which in turn gives the number of limit cycles. The topological equivalence of the bifurcation diagrams is shown to be determined by the number of limit cycles. This can then be used to determine the topological equivalence classes of the bifurcation diagrams.

To

Rockjune Ju
January 21, 1904 – June 13, 1969

He
always
wanted the world to be
a mathematical place
without mathematics,
and
whenever and wherever
he touched it,
it was.

He
always
tried to make his son to be
a strong man
without strength.

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CHAPTER 1. PRELIMINARIES.

In this chapter we will introduce basic definitions, some fundamental results concerning the structural stability on the plane, and the center manifold theorem. Theorems will be stated without proofs. We restrict our interests to the autonomous vector fields through the dissertation.

§1. Basic Definitions and Structural Stability on the Plane.

Let M be a two–dimensional smooth manifold and $\mathscr{X}^r(M)$ a set of C^r–vector fields on M. $x_0 \epsilon M$ is an <u>equilibrium point</u> of $f \epsilon \mathscr{X}^r(M)$ if $f(x_0) = 0$. Let $f \epsilon \mathscr{X}^r(M)$ and x_0 an equilibrium point of f in M. x_0 is <u>generic</u> if $\text{Re}\lambda \neq 0$ for $\lambda \epsilon \text{Spec}(Df(x_0)) = \{\lambda_1, \lambda_2\}$. x_0 is a <u>sink</u> (resp. <u>source</u>) if $\text{Re}\lambda_i < 0$ (resp. $\text{Re}\lambda_i > 0$) for $i = 1,2$, a <u>saddle</u> if $\lambda_1\lambda_2 < 0$, a <u>node</u> if $\lambda_1\lambda_2 > 0$, and a <u>focus</u> if $\text{Im}(\lambda_i) \neq 0$ for $i = 1,2$.

Let $\varphi(t;x)$ be a flow of $f \epsilon \mathscr{X}^r(M)$ with $\varphi(0;x) = x$. Then

$$\omega(f,x) = \bigcap_{s \epsilon \mathbb{R}} \bigcup_{t \geq s} \varphi(t;x), \quad \alpha(f,x) = \bigcap_{s \epsilon \mathbb{R}} \bigcup_{t \leq s} \varphi(t;x),$$

and are called ω– and α–limit set respectively. Γ is a <u>closed orbit of x</u> (<u>periodic orbit with period T</u>) if there is $T > 0$ such that $\varphi(t+T;x) = \varphi(t;x)$ for $t \epsilon \mathbb{R}$ (and T is a smallest positive such number possible). If Γ is a closed orbit whose α– and ω–limit set are only a point $\{x_0\}$ where x_0 is an equilibrium point of f, then Γ is called a <u>homoclinic orbit at x_0</u>. Let two points $\{x_0, y_0\}$ be equilibrium points in M with $x_0 \neq y_0$. Then Γ is called a <u>heteroclinic orbit between x_0 and y_0</u> if there exists an orbit Γ whose only α–limit set is $\{x_0\}$ and ω–limit set is $\{y_0\}$.

A family of vector fields $F:\mathbb{R}^m \to \mathscr{X}^r(M)$ is a <u>universal unfolding</u> of $f \in \mathscr{X}^r(M)$ if m is a minimal dimension of a stable family F with $F(0) = f$. For $f_1, f_2 \in \mathscr{X}^r(M)$, f_1 is <u>topologically equivalent</u> to f_2 (and we denote it $f_1 \sim f_2$) if there is a homeomorphism $h: M \to M$ such that for each $x \in M$, $\delta > 0$ and $0 < t < \delta$, $h(\varphi_1^t(x)) = \varphi_2^s(h(x))$ for some $\epsilon > 0$ with $0 < s < \epsilon$. Note that $\sim$ is an equivalence relation. If f is an element of an interior of a topological equivalence class, then f is called to be <u>structurally stable</u>.

Now let Λ be a manifold. We want to define the topological equivalence between two families of vector fields.

Suppose $F_i(i=1,2): \Lambda \to \mathscr{X}^r(M)$ are $\tilde{C}^r$. F_1 is <u>topologically equivalent</u> to F_2 if there exists a homeomorphism $g: \Lambda \to \Lambda$ and for each $\lambda \in \Lambda$, $F_1(\lambda)$ is topologically equivalent to $F_2(g(\lambda))$ in the previous sense. Also a family of vector fields F is <u>structurally stable</u> if F is an element of an interior of a topological equivalence class in $\tilde{C}^r(\Lambda, \mathscr{X}^r(M))$. $f \in \mathscr{X}^r(M)$ is called a <u>bifurcation point</u> if it is not structurally stable.

Next we want to define the bifurcation point of degree n by induction which is crucial in the understanding of the bifurcation diagram.

f is a <u>bifurcation point of degree 0</u> if it is structurally stable.

f is a <u>bifurcation point of degree 1</u> if (1) f is not a bifurcation point of degree 0, and (2) there is a neighborhood U of f such that for every g in U, $g \sim h$ for some h of degree 0, or $g \sim f$.

f is a <u>bifurcation point of degree 2</u> if (1) f is not a bifurcation point of degree 0 or 1, and (2) there is a neighborhood U of f such that for every g in U, $g \sim h$ for some h of degree 0 or 1, or $g \sim f$. Similarly we can define a bifurcation point of degree n.

From the above definitions we have the following theorem which is fundamental in the characterization of the bifurcation point of degree 1 on $M = \mathbb{R}^2$.

Theorem (1.1.1). A vector field f is a bifurcation of degree 1 in $\mathscr{X}^r(M)$, $r \geq 3$, if and only if there is a neighborhood U of f and a submanifold Λ of codimension one in U such that $h:\mathbb{R} \to \mathscr{X}^r(M)$ is continuous with $h(0) = f\epsilon\Lambda$ and $h(\alpha)\ \epsilon U-\Lambda$ for $\alpha \neq 0$ is structurally stable but $h(\alpha_1)$ is not topologically equivalent to $h(\alpha_2)$ if $\alpha_1\alpha_2 < 0$. For $h(0)\epsilon\Lambda$, only one of the following occurs:

(1) $h(0)\epsilon\Lambda$ has an elementary saddle–node at x_0 in M. There are no equilibrium points of $h(\alpha)$ near x_0 if $\alpha < 0$ and a saddle and a node near x_0 if $\alpha > 0$.

(2) $h(0)\epsilon\Lambda$ has an elementary focus at x_0. There is no periodic orbit of $h(\alpha)$ near x_0 if $\alpha < 0$ and a periodic orbit near x_0 if $\alpha > 0$.

(3) $h(0)\epsilon\Lambda$ has a periodic orbit γ which is stable from one side and unstable from the other. $h(\alpha)$ for $\alpha < 0$ has no periodic orbit near γ and $h(\alpha)$ for $\alpha > 0$ has two hyperbolic periodic orbit near γ.

(4) Trace $(\frac{\partial f}{\partial x}(x_0)) \neq 0$ and $h(0) = f\epsilon\Lambda$ has a homoclinic orbit γ at a saddle point x_0. $h(\alpha)$ for $\alpha < 0$ has a saddle near x_0 and no periodic orbit near γ, $h(\alpha)$ for $\alpha > 0$ has a saddle point and a unique hyperbolic periodic orbit near γ which coalesce as $\alpha \to 0^+$.

(5) There is a connection between distinct saddle points (heteroclinic orbit).

For the proof, see Sotomayor [16] or Andronov et al [1].

Remarks.

(i) Phase portraits are for (1) – (5) in the Theorem (1.1.1) are given in Figure 1.

(ii) This is the generic situation which arises in the case of one parameter families (codimension one) of vector fields.

(iii) Schecter [15] described the bifurcation of codimension two which occurs (1) and (4) simultaneously and showed the bifurcation diagram of saddle–node homoclinic bifurcation.

(iv) Let $\mathscr{X}_s^r(\mathbb{R}^2) = \{f\epsilon\mathscr{X}^r(\mathbb{R}^2) \mid f(-x) = -f(x) \text{ for } x\epsilon\mathbb{R}^2\}$. Then $f\epsilon\mathscr{X}_s^r(\mathbb{R}^2)$ implies $f(0) = 0$, and we have the following in Theorem (1.1.1) in a neighborhood of 0:

(6) There is a continuous map $h:\mathbb{R} \to \mathscr{X}_s^r(\mathbb{R}^2)$ such that $h(0) = f$ and $\text{Spec}(\frac{\partial f}{\partial x}(0)) = \{0,\lambda\}$, $\lambda \neq 0$, 0 is an only equilibrium point which is a saddle for $h(\alpha)$ if $\alpha < 0$, and there are two saddles and 0 is a node for $h(\alpha)$ if $\alpha > 0$. (See also Theorem (2.3.1) and Theorem in Appendix.)

The phase portraits for (6) are given in Figure 2.

(v) We are interested in the bifurcation diagram of an equilibrium point which occurs (4) and (6) simultaneously (pitchfork homoclinic bifurcation) and we will describe it in Chapter 2.

§2. Center Manifold Theorem for the Vector Fields.

Center manifold theorem is one of the most important and necessary techniques for the nonlinear analysis and the bifurcation problem. It provides us a benefit of the dimension reduction to a certain number (dimension of the eigenspace of eigenvalues whose real parts are zero). We introduce the center manifold theorem of the finite dimension for vector fields (we can get the same theorem for maps by the discretized version which is the outside of our interests here). In an obvious way it can be generalized to the infinite dimensional problem. For details and proofs, for example, see Carr [3], Chow and Hale [4] or Vanderbauwhede [19].

Let $f \epsilon \mathscr{X}^r(\mathbb{R}^n)$, $r \geq 1$, and $x(t;x_0)$ be a solution of $\dot{x} = f(x)$ with $x(0;x_0) = x_0$. We say that a set $A \subset \mathbb{R}^n$ is an <u>invariant</u> <u>manifold</u> of $\dot{x} = f(x)$ if for every $x_0 \epsilon A$, $x(t;x_0) \epsilon A$ for all $t \epsilon \mathbb{R}$. Also a set A is called a <u>local</u> <u>invariant</u> <u>manifold</u> of $\dot{x} = f(x)$ if there exists $\epsilon > 0$ such that for $x_0 \epsilon A$, $x(t;x_0) \epsilon A$ for all $t \epsilon (-\epsilon, \epsilon)$.

Let $x = 0$ be an equilibrium point of an $f \epsilon \mathscr{X}^r(\mathbb{R}^n)$, $r \geq 1$, and let the spectrum of $Df(0)$ be $\mathrm{Spec}(Df(0)) = SP_+ \cup SP_0 \cup SP_-$, where

$$SP_+ = \{\lambda \epsilon \mathrm{Spec}(Df(0)) \,|\, \mathrm{Re}\lambda > 0\}$$
$$SP_0 = \{\lambda \epsilon \mathrm{Spec}(Df(0)) \,|\, \mathrm{Re}\lambda = 0\}$$
$$SP_- = \{\lambda \epsilon \mathrm{Spec}(Df(0)) \,|\, \mathrm{Re}\lambda < 0\}.$$

Let E_+(resp. E_0, E_-) be the generalized eigenspace for SP_+(resp. SP_0, SP_-) so that $\mathbb{R}_n = E_+ \oplus E_0 \oplus E_-$. Now we state the theorem.

<u>Theorem (1.2.1)</u>. With above notations and assumptions, there exist local invariant manifolds W^u, W^c and W^s tangent to E_+, E_0 and $E\_$ at 0 respectively, W^u and W^s are C^r and unique, however, W^c is C^{r-1} and not necessarily unique.

In Theorem (1.2.1) W^c is called a <u>local</u> <u>center</u> <u>manifold</u>. Usually we prove the so–called global center manifold for the bounded vector field and then using a cut–off function in the neighborhood of the equilibrium point of the (not necessarily bounded) vector field and applying the global center manifold theorem, we prove the local center manifold. Proof of the existence of the global center manifold is required to use the implicit function theorem or a contracting mapping theorem.

CHAPTER 2. PITCHFORK HOMOCLINIC BIFURCATION.

In this chapter we will give a bifurcation diagram and its explanation. Next we will state formal assumptions for the pitchfork homoclinic bifurcation and describe main theorems that we should prove. Then proofs will be given.

§1. Introduction.

Let $f \in \mathscr{X}^r_s(D)$, r sufficiently large (will be determined later), and consider the following:

$$(2.1.1) \qquad \dot{x} = f(x), \quad x \in \mathbb{R}^2,$$

such that 0 is a pitchfork and there are a pair of homoclinic orbits in D, Γ and $-\Gamma$ which are stable. Then the set of all such vector fields is a codimension two submanifold with an appropriate smoothness in $\mathscr{X}^r(D)$, where D is a symmetric neighborhood of Γ (D is said to be symmetric if $-D = D$ in $\mathbb{R}^2$).

Suppose we have a two–parameter unfolding family of (2.1.1)

$$(2.1.2) \qquad \dot{x} = \tilde{f}(x,\alpha_1,\alpha_2)$$

where $\tilde{f}(\cdot,\alpha_1,\alpha_2) \in \mathscr{X}^r_s(D)$ for each $(\alpha_1,\alpha_2) \in \mathbb{R}^2$, such that $\tilde{f}(x,0,0) = f(x)$. We would like to find a computable condition on the transversality of the family (2.1.2) to $\mathscr{X}^r_s(D)$ at $(\alpha_1,\alpha_2) = (0,0)$.

If the transversality condition is satisfied, we have certain smooth nonsingular parameter coordinates changes $(\alpha_1,\alpha_2) \to (\tau_1,\tau_2) \to (\mu_1,\mu_2)$, preserving the origin, and let $\tilde{f}(x,\alpha_1,\alpha_2) = g(x,\tau_1,\tau_2) = f(x,\mu_1,\mu_2)$, then $\dot{x} = f(x,\mu_1\mu_2)$ has the bifurcation diagram of Figure 3 in a sufficiently small neighborhood of $(\mu_1,\mu_2) = (0,0)$. We have two curves, H_{o1} and H_e, in the left hand side of the pitchfork bifurcation curve $P(=\mu_2$–axis). They meet P at $(\mu_1,\mu_2) = (0,0)$ with quadratic tangencies. H_{o1} and H_{o2} (= the positive μ_1–axis) are homoclinic bifurcation curves of codimension one, while H_e is a heteroclinic bifurcation curve of codimension one. H_{o2} meets P transversally.

The phase portraits of $\dot{x} = f(x,\mu_1,\mu_2)$ in a neighborhood of $-\Gamma \cup \Gamma$ is as follows:

A. $\mu_1 = 0$ (The origin is a pitchfork.)

1. $\mu_2 = 0$: two homoclinic orbits at the origin (figure eight).
2. $\mu_2 > 0$: two stable closed orbits inside the stable manifolds.
3. $\mu_2 < 0$: one stable closed orbit surrounding the origin.

B. $\mu_1 > 0$ (The origin is a saddle.)

4. $\mu_2 > 0$: two stable closed orbits inside the stable manifolds.
5. $\mu_2 = 0$: two homoclinic orbits at the origin (figure eight).
6. $\mu_2 < 0$: one stable closed orbit enclosing the origin.

C. $\mu_1 < 0$ (The origin is a node, and there are two saddles in the opposite side of the origin.)

7. μ_2 above H_{o1}: two stable closed orbits inside the stable manifolds.
8. μ_2 on H_{o1}: two homoclinic orbits at saddles.
9. μ_2 between H_{o1} and H_e: the flow of the vector field with the initial x in the unstable manifolds of saddles tend to the origin as $t \to \infty$.

10. μ_2 on H_e: two heteroclinic orbits which join one saddle to the other.

11. μ_2 below H_e: unique closed orbit surrounding three equilibria.

Our techniques can be applied to the investigation of the vector field with double heteroclinic orbits joining a saddle–node equilibrium point and a saddle equilibrium point as in Figure 4.

We can extend our results to the more general problem by dropping "the symmetry condition" on the vector field which will lead to the codimension three problem. Also we can consider this on the higher dimensional manifold (of dimension greater than two) in a similar way as in Chow and Lin [5].

§2. Assumptions.

We consider a vector field $\dot{x} = f(x)$ with $f \in \mathscr{X}^r_s(D)$ where $r \geq 3$ for a moment and $D \subset \mathbb{R}^2$ a symmetric neighborhood of Γ, satisfying the following conditions at the origin in $\mathbb{R}^2$.

(I) $\quad \mathrm{Spec}(Df(0)) = \{0, -\lambda\},\ \lambda > 0.$

Let u be a right eigenvector of the eigenvalue 0 and w be a left eigenvector of the eigenvalue 0 such that $u \cdot w > 0$.

(II) $\quad w \cdot D^3 f(0)(u,u,u) > 0.$

(III) $\quad \dot{x} = f(x)$ has a homoclinic orbit Γ (hence $-\Gamma$) at 0 which is hyperbolic. (Here we assume that Γ is stable.)

A homoclinic orbit is <u>hyperbolic</u> if any Poincare return map from a transversal section of the homoclinic orbit into itself (if defined) doesn't have eigenvalues with absolute value 1 at a fixed point which is an intersection of the homoclinic orbit and the transversal intersection.

Remarks.

(i) $f \in \mathscr{X}^r_s(D)$ implies $w \cdot D^2 f(0)(u,u) = 0$.

(ii) Assumptions I and II imply that $\dot{x} = f(x)$ has a pitchfork at $x = 0$ with one negative eigenvalue. (See Appendix.)

(iii) We may assume that u is a tangent vector to Γ at 0. (Otherwise replace it by $-u$.)

For $z, w \in \mathbb{R}^2$, we denote $z \wedge w$ by $Jz \cdot w$ where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Note that $w \wedge z = -z \wedge w$ and Jz is a rotation of the vector z by $\pi/2$ to the positive direction in angle.

Now let v be a right eigenvalue of $Df(0)$ corresponding to $-\lambda$ such that v is tangent to Γ at 0. We may assume that $u \wedge v > 0$ without loss of generality since if not, one can consider the reflection of the vector field with respect to the x–axis $((x,-y) \to (x,y))$ or y–axis $((-x,y) \to (x,y))$.

Let $\dot{x} = \tilde{f}(x,\alpha_1,\alpha_2)$ be a two parameter family of vector fields on $\mathbb{R}^2$ such that $\tilde{f}(\cdot,\alpha_1,\alpha_2) \in \mathscr{X}^r_s(D)$ for each $(\alpha_1,\alpha_2) \in \mathbb{R}^2$ and $\tilde{f}(x,0,0) = f(x)$ as well as

(IV) $$w \cdot \left(D_x \frac{\partial}{\partial \alpha_1} \tilde{f}(0,0,0)(u)\right) > 0,$$

(V) $\tilde{f}$ is sufficiently smooth (at least C^{11}).

From Assumptions (II) and (IV), perturbation in the positive α_1 direction makes the origin a saddle from a pitchfork, while perturbation in the negative α_1 direction produces a new pair of saddles from the pitchfork point(0,0), which becomes a node.

§3. Statement of the Main Theorems.

We state the theorems whose proofs will be given in the next section except the proof of Theorem (2.3.1). First we state the following. Its proof will be shown in the Appendix.

Theorem (2.3.1). Under the assumptions (I), (II), and (IV), there is a C^{r-1} function $p(\alpha_2)$, with $p(0) = 0$, such that for (α_1,α_2) near (0.0) $\dot{x} = \tilde{f}(x,\alpha_1,\alpha_2)$ has an equilibrium of pitchfork at 0 if and only if $\alpha_1 = p(\alpha_2)$.

From the Theorem (2.3.1), we change the coordinates in the parameter space, say,

$$\tau_1 = \alpha_1 - p(\alpha_2)$$
$$\tau_2 = \alpha_2$$

and let

$$g(x,\tau_1,\tau_2) = \tilde{f}(x,\alpha_1(\tau_1,\tau_2),\ \alpha_2(\tau_1,\tau_2)).$$

Then

$$\dot{x} = g(x,\tau_1,\tau_2) \tag{2.3.1}$$

is C^{r-1}. $x = 0$ is an equilibrium of pitchfork in (2.3.1) if and only if

$\tau_1 = 0$. If $\tau_1 < 0$, 0 is a node of (2.3.1) and there are a pair of saddles near 0 but opposite sides of 0 each other, and if $\tau_1 > 0$, 0 is a unique equilibrium point which is a saddle.

Let x_0 be a point Γ and let $\varphi(t; x_0)$ be a flow of (2.3.1) with $\varphi(0; x_0) = x_0$ for $(\tau_1,\tau_2) = 0$. We will denote $\varphi(t; x_0)$ simply by $\varphi(t)$ if there is no confusion.

Let

$$(2.3.2) \qquad I_1 = \int_{-\infty}^{\infty} \exp\left(-\int_0^t \operatorname{div} g(\varphi(s),0,0)ds\right) g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial \tau_2}(\varphi(t),0,0)dt.$$

Then we have the following theorem.

Theorem (2.3.2).

(1) I_1 converges.

(2) If $I_1 \neq 0$, then there exists a C^{r-1} function $q(\tau_1)$, with $q(0) = 0$, such that for (τ_1,τ_2) near (0,0) (2.3.1) has a pair of homoclinic orbit at 0 if and only if $\tau_2 = q(\tau_1)$ for $\tau_1 \geq 0$, where $q(\tau_1) = m\tau_1 + o(\tau_1)$ for some constant $m \in \mathbb{R}$.

Theorem (2.3.2) gives us an analysis for the bifurcation on the right hand side of the pitchfork bifurcation curve. Once we have the transversality condition $(I_1 \neq 0)$, we can change the coordinates in the parameter space again as in the following.

Let

$$\mu_1 = \tau_1$$
$$\mu_2 = \tau_2 - q(\tau_1),$$

and let

$$f(x,\mu_1,\mu_2) = g(x,\tau_1(\mu_1,\mu_2),\ \tau_2(\mu_1,\mu_2)).$$

Then

(2.3.3) $\dot{x} = f(x,\mu_1,\mu_2)$

is a vector field of C^{r-1} since q is C^{r-1}.

In (2.3.3) 0 is an equilibrium point of pitchfork if and only if $\mu_1 = 0$. Also $\mu_1 > 0$ and $\mu_2 = 0$ if and only if 0 is a saddle and there exist a pair of homoclinic orbits at 0 near $-\Gamma\cup\Gamma$.

To consider the bifurcations on the left hand side of the pitchfork bifurcation curve ($\mu_1 = \tau_1 < 0$), we need the following.

Let

(2.3.4) $$I_2 = \int_{-\infty}^{\infty} \exp\left(-\int_0^t \operatorname{div} f(\varphi(s),0,0)ds\right) f(\varphi(s),0,0) \wedge \frac{\partial f}{\partial \mu_2}(\varphi(t),0,0)dt.$$

Then

Theorem (2.3.3).

(1) I_2 converges.

(2) If $I_2 \neq 0$, then there exists C^{r-4} curve $\ell(\mu_2)$, $\ell(0) = 0$, such that for (μ_1,μ_2) sufficiently near (0,0), (2.3.3) has a pair of homoclinic (resp. heteroclinic) orbits near $-\Gamma\cup\Gamma$ if and only if $\mu_1 = \ell(\mu_2)$ and $I_2 \cdot \mu_2 \geq 0$ (resp. ≤ 0) where $\ell(\mu_2) = m_1^2\mu_2^2 + o(\mu_2^2)$, $m_1 \neq 0$ a constant in $\mathbb{R}$.

In the expression of I_1 and I_2, since $g(x,0,0) = f(x,0,0)$, div $g(x,0,0)$ = div $f(x,0,0)$, and $\frac{\partial g}{\partial \tau_2}(x,0,0) = \frac{\partial f}{\partial \mu_2}(x,0,0)$, we have $I_1 = I_2$. So we denote $I_1 = I_2$ simply by I.

Next, we consider the transversality condition of the family of vector fields $\dot{x} = \tilde{f}(x,\alpha_1,\alpha_2)$ to the vector fields which satisfy the assumptions (I) – (V). Let D be a symmetric open neighborhood of (not necessarily $-\Gamma \cup \Gamma$, but) 0. We denote Σ_0 by the space of all symmetric vector fields in D with C^r–topology, $r \geq 11$, i.e., $\Sigma_0 = \mathscr{X}^r_s(D)$, $r \geq 11$. Let $\Sigma_1 = \{f \epsilon \Sigma_0 | f$ satisfies (I) and (II) at 0, and all other equilibria of f in D are hyperbolic$\}$. The Appendix says that Σ_1 is a C^{r-1} submanifold of codimension one in Σ_0. Also we let $\Sigma_2 = \{f \epsilon \Sigma_1 | f$ satisfies (III), and $-\Gamma \cup \Gamma \subset D\}$.

Theorem (2.3.4). The family $\dot{x} = \tilde{f}(x,\alpha_1,\alpha_2)$ under assumptions in the previous secton is transverse to Σ_2 at $(\alpha_1,\alpha_2) = (0.0)$ if and only if $I \neq 0$.

We are interested only in a neighborhood of $-\Gamma \cup \Gamma$. When the transversality condition $(I \neq 0)$ is satisfied, Theorems (2.3.1) – (2.3.4) and Poincare–Bendixon Theorem (plus the hyperbolicity of Γ) gives our diagram and its corresponding phase portraits of Figure 3.

§4. Proofs.

Without loss of generality we may consider the neighborhood of Γ by the symmetry property.

A. Proof of Theorem (2.3.2).

Assume that $\tau_2 \geq 0$, and consider the system

$$(2.4.1) \qquad \begin{aligned} \dot{x} &= g(x,\tau_1,\tau_2) \\ \dot{\tau}_1 &= 0 \\ \dot{\tau}_2 &= 0. \end{aligned}$$

Center manifold theorem shows that there exists a 3–dimensional C^{r-1} local center manifold W^c_{loc} at $(0,0,0)$ tangent to the center space in (2.4.1). Let φ be a flow of (2.3.1), and let

$$W = \bigcup_{t \in \mathbb{R}} \; \bigcup_{(x,\tau_1,\tau_2) \in W^c_{loc}} \varphi(t;x,\tau_1,\tau_2).$$

Then each (τ_1,τ_2)–section of W is a curve and $(0,0)$–section of W is $-\Gamma \cup \Gamma$. Let $W(\tau_1,\tau_2)$ be a (τ_1,τ_2)–section of W. Then for (τ_1,τ_2) sufficiently near $(0,0)$, $W(\tau_1,\tau_2)$ is a curve in the neighborhood of $-\Gamma \cup \Gamma$. Let L be a transversal line segment in $\mathbb{R}^2$ perpendicular to Γ at x_0. Then for (τ_1,τ_2) near $(0,0)$, $W(\tau_1,\tau_2)$ intersects L transversally near x_0. Hence there exists $x(\tau_1,\tau_2)$ such that $x(0,0) = x_0$ and $x(\tau_1,\tau_2) \in W(\tau_1,\tau_2) \cap L$. Also $x(\tau_1,\tau_2)$ is C^{r-1}. Now we have a C^{r-1} family of solution of (2.3.1) $\varphi^c(t;\, x(\tau_1,\tau_2),\tau_1,\tau_2)$ (simply $\varphi^c(t,\tau_1,\tau_2)$) for (τ_1,τ_2) small, such that $\varphi^c(0,\tau_1,\tau_2) = x(\tau_1,\tau_2)$, and so $\varphi^c(t,0,0) = \varphi^c(t;\, x_0,0,0) :\equiv \varphi(t)$, $\varphi^c(t,\tau_1,\tau_2) \in W(\tau_1,\tau_2)$ for $t \in \mathbb{R}$.

Note that $\varphi^c(t,0,\tau_2)$ is a branch of the local unstable manifold of the pitchfork 0 of $\dot{x} = g(x,0,\tau_2)$ and for $\tau_1 > 0$, $\varphi^c(t,\tau_1,\tau_2)$ is that of the saddle 0 of (2.3.1).

Next we define a parameter dependent C^{r-1} change of coordinates on $\mathbb{R}^2$

$$(2.4.2) \qquad y(x,\tau_1,\tau_2) = (y_1(x,\tau_1,\tau_2),\ y_2(x,\tau_1,\tau_2))$$

in such a way that for (x,τ_1,τ_2) near $(0,0,0)$

(1) $y(-x,\tau_1,\tau_2) = -y(x,\tau_1\tau_2)$,

(2) $y_2(x,\tau_1,\tau_2) \equiv 0$ for $(x,\tau_1,\tau_2)\epsilon(\tau_1,\tau_2)$–section of $W^c_{loc}(0)$, and

(3) $y_1(\varphi(t;\ x,\tau_1,\tau_2),\ \tau_1,\tau_2) =$ constant for every x in $\mathbb{R}^2$ near 0 with $y_1(x,\tau_1,\tau_2) =$ constant and for each fixed (τ_1,τ_2).

(2.4.2) is possible because of the symmetry property and the center manifold theorem. By the change of coordinates (2.4.2), (2.3.1) becomes the following C^{r-1} differential equation in y:

$$(2.4.3) \qquad \begin{cases} \dot{y}_1 = y_1\ a(y_1^2,\tau_1,\tau_2) \\ \dot{y}_2 = y_2\ b(y_1,y_2,\tau_1,\tau_2) \end{cases}$$

where a is independent of y_2, b is even in $y = (y_1,y_2)$. We also assume that

$$(2.4.4) \qquad D_xy(0)u = (1,0) \quad \text{and} \quad D_xy(0)v = (0,1).$$

$\tilde{f}(\cdot,\alpha_1,\alpha_2)\epsilon\mathcal{X}^r_s$ implies $g(\cdot,\tau_1,\tau_2)\epsilon\mathcal{X}^r_s$, and $w\cdot(D_x \frac{\partial}{\partial\alpha_1} \tilde{f}(0)u) = w\cdot(D_x \frac{\partial}{\partial\tau_1} g(0)u)$. So from our assumptions (I), (II), (IV) and (2.4.4), (2.4.3) becomes

$$(2.4.5) \qquad \begin{cases} \dot{y}_1 = \varphi_0(\tau_2)y_1^3(1+y_1^2\varphi_1(y_1,\tau_2)) + \tau_1y_1\varphi_2(y_1,\tau_1,\tau_2) \\ \dot{y}_2 = -\varphi_3(y_1,\tau_1,\tau_2)y_2(1+y_2\varphi_4(y_1,y_2,\tau_1,\tau_2)) \end{cases}$$

where

$\varphi_0(0)$, $\varphi_1(0)$, and $\varphi_3(0) = \lambda$ are positive, φ_1, φ_2 and φ_3 are even in y_1, and φ_4 is odd in y, $\varphi_4 = O(|y|)$. Let $v(\tau_1,\tau_2) = D_y x(0,\tau_1,\tau_2)(0,1)$. Then $v(0,0) = v$ and (2.3.1) has an invariant curve at $0 \epsilon \mathbb{R}^2$ tangent to $v(\tau_1,\tau_2)$. Note that this invariant curve is C^{r-1} in (τ_1,τ_2) and contains Γ for $(\tau_1,\tau_2) = (0,0)$.

In a similar way as in $\varphi^c(t,\tau_1,\tau_2)$, we can get a C^{r-1} family $\varphi^s(t,\tau_1,\tau_2)$, (τ_1,τ_2) small, such that $\varphi^s(t,\tau_1,\tau_2)$ is a solution of (2.3.1) with $\varphi^s(t,\tau_1,\tau_2) \to 0$ as $t \to \infty$ along the negative $v(\tau_1,\tau_2)$ direction and $\varphi^s(0,\tau_1,\tau_2) \epsilon L$ so that $\varphi^s(t,0,0) = \varphi(t)$. (See Figure 5.)

Define
$$d_1^c(\tau_1,\tau_2) = g(\varphi(0),0,0) \wedge [\varphi^c(0,\tau_1,\tau_2) - \varphi(0)]$$
$$d_1^s(\tau_1,\tau_2) = g(\varphi(0),0,0) \wedge [\varphi^s(0,\tau_1,\tau_2) - \varphi(0)], \text{ and}$$
$$d_1(\tau_1,\tau_2) = d_1^c(\tau_1,\tau_2) - d_1^s(\tau_1,\tau_2).$$

It is easy to see that $d_1(\tau_1,\tau_2) = 0$ if and only if there exist a pair of homoclinic orbits of (2.3.1) at 0.

We will show that $I_1 = \frac{\partial d_1}{\partial \tau_2}(0,0)$ since, if had shown it, by the implicit function theorem $I_1 \neq 0$ implies that there exists a C^{r-1} function q with $q(0) = 0$ such that $d_1(\tau_1,q(\tau_1)) = 0$ for $0 \le \tau_1 << 1$.

Hence
$$\frac{\partial d_1}{\partial \tau_1}(0,0) + \frac{\partial d_1}{\partial \tau_2}(0,0)\, q'(0) = 0, \text{ and so}$$
$$q'(0) = -\frac{\partial d_1}{\partial \tau_1}(0,0) \Big/ \frac{\partial d_1}{\partial \tau_2}(0,0) : \equiv m.$$

It follows that $q(\tau_1) = m\tau_1 + o(\tau_1)$, and our proof will be completed.

Claim. $I_1 = \frac{\partial d_1}{\partial \tau_2}(0,0)$.

Let $\rho^c_{\tau_2}(t) = g(\varphi(t),0,0) \wedge \frac{\partial\varphi^c}{\partial\tau_2}(t,0,0)$,

$\rho^s_{\tau_2}(t) = g(\varphi(t),0,0) \wedge \frac{\partial\varphi^s}{\partial\tau_2}(t,0,0)$.

Note that

$$\frac{\partial d_1}{\partial\tau_2}(\tau_1,\tau_2) = g(\varphi(0),0,0) \wedge (\frac{\partial\varphi^c}{\partial\tau_2}(0,\tau_1,\tau_2) - \frac{\partial\varphi^s}{\partial\tau_2}(0,\tau_1,\tau_2)),$$

so

$$\frac{\partial d_1}{\partial\tau_2}(0,0) = \rho^c_{\tau_2}(0) - \rho^s_{\tau_2}(0).$$

While, by definitions of φ^c and φ^s,

$$\frac{d}{dt}\frac{\partial\varphi^c}{\partial\tau_2}(t,0,0) = D_x g(\varphi(t),0,0)\frac{\partial\varphi^c}{\partial\tau_2}(t,0,0) + \frac{\partial g}{\partial\tau_2}(\varphi(t),0,0),$$

$$\frac{d}{dt}\frac{\partial\varphi^s}{\partial\tau_2}(t,0,0) = D_x g(\varphi(t),0,0)\frac{\partial\varphi^s}{\partial\tau_2}(t,0,0) + \frac{\partial g}{\partial\tau_2}(\varphi(t),0,0).$$

From these we have equations

$$\frac{d}{dt}\rho^c_{\tau_2}(t) = \operatorname{div} g(\varphi(t),0,0)\,\rho^c_{\tau_2}(t) + g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial\tau_2}(\varphi(t),0,0),$$

$$\frac{d}{dt}\rho^s_{\tau_2}(t) = \operatorname{div} g(\varphi(t),0,0)\rho^s_{\tau_2}(t) + g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial\tau_2}(\varphi(t),0,0),$$

and their solutions are

$$(2.4.6)_c \quad \rho^c_{\tau_2}(0) = \rho^c_{\tau_2}(t_1)\exp\int_{t_1}^{0}\operatorname{div} g(\varphi(t),0,0)\,dt$$
$$+ \int_{t_1}^{0}\exp(-\int_0^t \operatorname{div} g(\varphi(s),0,0)\,ds)\, g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial\tau_2}(\varphi(t),0,0)\,dt$$
$$:\equiv I_c(t_1) + II(t_1),$$

$$(2.4.6)_s \quad \rho^s_{\tau_2}(0) = \rho^s_{\tau_2}(t_1) \exp\left(-\int_0^{t_1} \operatorname{div} g(\varphi(t),0,0)\, dt\right)$$

$$+ \int_{t_1}^{0} \exp\left(-\int_0^{t} \operatorname{div} g(\varphi(s),0,0)\, ds\right) g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial \tau_2}(\varphi(t),0,0)\, dt$$

$$:\equiv I_s(t_1) + II(t_1)$$

respectively. We will show later that

$$\lim_{t\to -\infty} I_c(t) = 0 = \lim_{t\to\infty} I^s(t).$$

First, consider the following.

Lemma (2.4.1).

$$(2.4.7)_c \quad \lim_{t\to -\infty} \frac{\partial \varphi^c}{\partial \tau_2}(t,0,0) = 0 \text{ and}$$

$$(2.4.7)_s \quad \lim_{t\to\infty} \frac{\partial \varphi^s}{\partial \tau_2}(t,0,0) = 0.$$

Proof. Let

$$\tilde{\varphi}^c(t,\tau_2) = y(\varphi^c(t,0,\tau_2),0,\tau_2) := (y_1(t,\tau_2),0) \quad \text{for} \quad t << -1$$

then $y_1(t,\tau_2) > 0$ for $t << -1$ and $y_1(t,\tau_2)$ is a solution of the scalar equation

$$(2.4.8) \quad \frac{dz}{dt} = \varphi_0(\tau_2)z^3\,(1+z^2\varphi_1(z,\tau_2)).$$

(See equation (2.4.5).)

φ_0 and $z^2\varphi_1$ are C^{r-4}. Let

$$z^{-3}(1+z^2\varphi_1(z,\tau_2))^{-1} = z^{-3} + H_1(\tau_2)z^{-1} + H_2(z,\tau_2).$$

Then H_1 is C^{r-6} and H_2 is C^{r-7}, and hence $H_3(z,\tau_2) = \int_{z_0}^{z} H_2(s,\tau_2)ds$ is C^{r-7}.

Separation of variables in (2.4.8) gives

$$(2.4.9) \qquad -\tfrac{1}{2}z^{-2} + H_1(\tau_2)\ln z + H_3(z,\tau_2) = \varphi_0(\tau_2)t + H_4(\tau_2)$$

where $H_4(\tau_2)$ depends on the value of $y_1(t,\tau_2)$ at some $t = t_0$. Hence H_4 is C^{r-7}. From (2.4.9) we have

$$2z^2(1 - 2\,H_1(\tau_2)z^2\ln z - 2z^2H_3(z,\tau_2))^{-1} = -(\varphi_0(\tau_2)t + H_4(\tau_2))^{-1}.$$

Let $w = z^2$ and let $z > 0$.

Then

$$(2.4.10) \quad 2w(1-H_1(\tau_2)w\ln w - 2wH_3(\sqrt{w},\tau_2))^{-1} = -(\varphi_0(\tau_2)t + H_4(\tau_2))^{-1}.$$

Let $\phi(w,\tau_2)$ be the LHS of (2.4.10). Then by the assumption (V), ϕ is at least C^1 and $\frac{\partial\phi}{\partial w}(0^+,\tau_2) \neq 0$ for τ_2 near 0. By the implicit function theorem we can solve $\phi(w,\tau_2) = v$ for w, and let $w = \frac{v}{2} + R(v,\tau_2)$ be the solution to $\phi(w,\tau_2) = v$. Then R is C^1 and $R = O(v^2)$.

From (2.4.10)

$$(2.4.11) \quad y_1^2 = -\tfrac{1}{2}(\varphi_0(\tau_2)t + H_4(\tau_2))^{-1} + R((\varphi_0(\tau_2)t + H_4(\tau_2))^{-1}, \tau_2).$$

So we have

$$y_1 = \frac{1}{\sqrt{2}}(-\varphi_0(\tau_2)t - H_4(\tau_2))^{-1/2} + R_1((\varphi_0(\tau_2)t + H_4(\tau_2))^{-1},\tau_2)$$

where $R_1(v,\tau_2) = O(v^{3/2})$.

So $\frac{\partial y_1}{\partial \tau_2}(t,\tau_2) \to 0$ as $t \to -\infty$ for τ_2 near 0. This implies that

$$\frac{\partial \tilde{\varphi}^c}{\partial \tau_2}(t,\tau_2) = (\frac{\partial y_1}{\partial \tau_2}(t,\tau_2),0) \to (0,0) \quad \text{as} \quad t \to -\infty \quad \text{and}$$

$$\frac{\partial \varphi^c}{\partial \tau_2}(t,0,0) = D_y x(\tilde{\varphi}^c(t,0),0,0) \frac{\partial \tilde{\varphi}^c}{\partial \tau_2}(t,0) \to D_y x(0,0,0)(0,0) = (0,0)$$

as $t \to -\infty$.

Hence we proved $(2.4.7)_c$.

Next, to prove $(2.4.7)_s$, let

$$\tilde{\varphi}^s(t,\tau_2) = y(\varphi^s(t,0,\tau_2),0,\tau_2) = (0,y_2(t,\tau_2)).$$

Since $\varphi^s(t,0,\tau_2)$ is tangent to $v(0,\tau_2)$ and $\varphi^s(t,0,\tau_2) \to 0$ as $t \to \infty$ for each τ_2 near 0, $\tilde{\varphi}^s(t,\tau_2) > 0$ for $t >> 1$, and $y_2(t,\tau_2)$ satisfies a differential equation

$$(2.4.12) \quad \frac{dz}{dt} = -\varphi_3(\tau_2)z(1+z^2\varphi_4(z,\tau_2))$$

where $\varphi_3(0) = \lambda$. φ_3 and $z^2\varphi_4$ are C^{r-2}.

(See equation (2.4.5)).

Note here that we set $\tau_1 = 0$ so that φ_3 and φ_4 may be different from those of (2.4.5). It is easy to get the following from (2.4.12)

$$(2.4.13) \quad z \exp H_5(z,\tau_2) = H_6(\tau_2) \exp(-\varphi_3(\tau_2)t)$$

where H_5 and H_6 are C^{r-3} and $H_6 > 0$. Since $\frac{\partial}{\partial z}(z \exp H_5(z,\tau_2))|_{z=0} \neq 0$, we can solve the equation

$$(2.4.14) \quad z \exp H_5(z,\tau_2) = v$$

for z if z and v are near 0. Let $z = R(v,\tau_2)$ be the solution to (2.4.14). Then R is C^{r-3} and

(2.4.15) $R(0,\tau_2) \equiv 0.$

Hence we get the following solution from (2.4.13)

(2.4.16) $y_2 = R(H_6(\tau_2)\exp(-\varphi_3(\tau_2)t),\ \tau_2).$

(2.4.16) and (2.4.15) imply that

$$\frac{\partial y_2}{\partial \tau_2}(t,\tau_2) \to 0 \quad \text{as} \quad t \to \infty.$$

From the definition of $\tilde{\varphi}^s$,

$$\frac{\partial \tilde{\varphi}^s}{\partial \tau_2}(t,\tau_2) = \left(0,\ \frac{\partial y_2}{\partial \tau_2}(t,\tau_2)\right) \to 0 \quad \text{as} \quad t \to \infty.$$

Now

$$\frac{\partial \varphi^s}{\partial \tau_2}(t,0,0) = D_y x(\tilde{\varphi}^s(t,0),0,0)\,\frac{\partial \tilde{\varphi}^s}{\partial \tau_2}(t,0) + \frac{\partial x}{\partial \tau_2}(\tilde{\varphi}^s(t,0),0,0) \to 0 \quad \text{as} \quad t \to \infty,$$

and we complete the proof of Lemma (2.4.1). □

Now back to the previous problem,

$$I_c(t) = g(\varphi(t),0,0) \wedge \frac{\partial \varphi^c}{\partial \tau_2}(t,0,0)\exp\int_t^0 \operatorname{div} g(\varphi(s),0,0)ds.$$

(2.4.17) $g(\varphi(t),0,0) \to 0$ as $t \to -\infty,$

(2.4.18) $\operatorname{div} g(\varphi(t),0,0) = -\lambda + O((\varphi_0(\tau_2)t + H_4(\tau_2))^{-1})$ as $t \to -\infty$

from (2.4.5) and (2.4.11). Hence from (2.4.17), (2.4.18), and $(2.4.7)_c$ of Lemma (2.4.1), we have

$$\lim_{t\to -\infty} I_c(t) = 0.$$

While, from the definition of $\tilde{\varphi}^s$, (2.4.15) and (2.4.16),

$$\varphi(t) = O(\exp(-\lambda t)) \quad \text{as} \quad t \to \infty,$$

$$\dot{\varphi}^s(t,0) = (0,-C\exp(-\lambda t) + o(\exp(-\lambda t)))$$

for $t >> 1$ and a constant $C > 0$.

Hence $g(\varphi(t),0,0) = \dot{\varphi}(t)$

$$= [D_x y(\varphi(t),0,0)]^{-1}\,\dot{\varphi}^s(t,0)$$

$$= \{[D_x y(0,0,0)]^{-1} + O(\exp(-\lambda t))\} \cdot (0,-C\exp(-\lambda t) + O(\exp(-\lambda t))).$$

Also $\operatorname{div} g(\varphi(t),0,0) = \operatorname{div} g(0,0,0) + D_x(\operatorname{div} g(0,0,0)) + O(\varphi(t)^2)$

$$= -\lambda + O(\varphi(t)) = -\lambda + O(\exp(-\lambda t)) \quad \text{as} \quad t \to \infty.$$

Now, as $t \to \infty$

$$I_s(t) = g(\varphi(t),0,0) \wedge \frac{\partial\varphi^s}{\partial\tau_2}(t,0,0)\exp[-\int_0^t \operatorname{div} g(\varphi(s),0,0)ds]$$

$$= -\frac{\partial\varphi^s}{\partial\tau_2}(t,0,0) \wedge g(\varphi(t),0,0)\exp[-\int_0^t(-\lambda + O(\exp(-\lambda s)))ds]$$

$$= -\frac{\partial\varphi^s}{\partial\tau_2}(t,0,0) \wedge \{[D_x y(0,0,0)]^{-1} + O(\exp(-\lambda t))\} \cdot$$

$$(0,-C\exp(-\lambda t) + O(\exp(-\lambda t)))\exp(\lambda t)\exp\int_0^t O(\exp(-\lambda s))ds.$$

Thus $\lim_{t\to\infty} I_s(t) = -0 \wedge [D_x y(0,0,0)]^{-1} \cdot (0,-C\exp\int_0^\infty O(\exp(-\lambda s))ds) = 0$

since the integral converges.

Now, back to $(2.4.6)_{c,s}$, the LHS of $(2.4.6)_{c,s}$ are constant, so as $t \to \infty$, II(t) and II(−t) converge, and

$$\rho^{c}_{\tau_2}(0) = \int_{-\infty}^{0} \exp(-\int_{0}^{t} \operatorname{div} g(\varphi(s),0,0)ds)\; g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial \tau_2}(\varphi(t),0,0)dt,$$

$$\rho^{s}_{\tau_2}(0) = \int_{0}^{\infty} \exp(-\int_{0}^{t} \operatorname{div} g(\varphi(s),0,0)ds)\; g(\varphi(t),0,0) \wedge \frac{\partial g}{\partial \tau_2}(\varphi(t),0,0)dt,$$

since $\lim_{t\to\infty} I_s(t) = 0 = \lim_{t\to -\infty} I_c(t)$.

Hence I_1 in (2.3.2) is convergent and is equal to $\rho^{c}_{\tau_2}(0) - \rho^{s}_{\tau_2}(0)$, and so equal to $\frac{\partial d_1}{\partial \tau_2}(0,0)$. So we proved Theorem (2.3.2). □

Remark. Theorem (2.3.2) gives us a homoclinic bifurcation curve of codimension 1: $\{\tau_1,\tau_2) \mid \tau_2 = q(\tau_1),\ \tau_1 > 0\}$. After change of coordinates in the parameter space

$((\tau_1,\tau_2) \to (\mu_1,\mu_2)\colon \mu_1 = \tau_1,\ \mu_2 = \tau_2 - q(\tau_1))$ (see page 12),

if we consider (2.3.3), for $\mu_1 > 0$ we have two structurally stable connected components. If $I_1 \cdot \mu_2 > 0$, then the phase portrait is topologically equivalent to that for region 4 in Figure 3, and if $I_1 \cdot \mu_2 < 0$, then it is topologically equivalent to that for region 6 in Figure 3. (Note that Figure 3 is described only the case $I_1 > 0$.)

Existence of limit cycles in phase portraits for region 4 and region 6 is immediate from the hyperbolicity of Γ, the Poincare–Bendixon Theorem plus the symmetry property of the vector field. We again emphasize that our interests are only in the neighborhood of $-\Gamma \cup \Gamma$.

B. Proof of Theorem (2.3.3).

Basically this proof is almost the same as the previous one. However, for $\mu_1 < 0$ new equilibrium points arise and this makes the proof complicated.

Through the proof, we assume that $\mu_1 \le 0$.

Again let us consider the system

$$\text{(2.4.19)} \quad \begin{aligned} \dot{x} &= f(x,\mu_1,\mu_2) \\ \dot{\mu}_1 &= 0 \\ \dot{\mu}_2 &= 0 \end{aligned}$$

which is C^{r-1}. Using the center manifold theorem in (2.4.19) and so on, we can define $\varphi^c(t,\mu_1,\mu_2)$ of C^{r-1} family. (See equations (2.4.1) and (2.4.2). Also see Figure 5.) [Note: we will use the same notations even through they may be different from those in the proof of Theorem (2.3.2).]

Under the same assumptions and change of variables (from x– to y–coordinates), finally we get to a C^{r-1} equation which is almost the same as (2.4.5):

$$\text{(2.3.20)} \quad \begin{cases} \dot{y}_1 = \varphi_o(\mu_2)y_1^3(1+y_1^2) + \mu_1 y_1 \varphi_2(y_1,\mu_1,\mu_2) \\ \dot{y}_2 = -\varphi_3(y_1,\mu_1,\mu_2)\; y_2(1+y_2\varphi_4(y_1,y_2,\mu_1,\mu_2)) \end{cases}$$

where $\varphi_1,\varphi_2,\varphi_3$ are even in y_1, $\varphi_4 = O(|y|)$ is odd in $y = (y_1,y_2)$, and $\varphi_o(0)$, $\varphi_2(0)$, $\varphi_3(0) = \lambda$ are positive.

Equilibria of (2.4.20) near $(0,0,0)$ consists of

$$E_1 = \{0\} \times \mathbb{R}^2 \quad \text{and}$$

$$E_2 = \{(y_1,0),\mu_1,\mu_2) \mid \mu_1 = \mu_1(y_1,\mu_2)\}$$

where $\mu_1 = \mu_1(y_1,\mu_2)$ is as follows:

$$\frac{\partial^2 \mu_1}{\partial y_1^2}(0,\mu_2) \ngtr 0 \quad \text{so that} \quad \mu_1 = -y_1^2(\psi_o(\mu_2) + y_1^2\psi_1(y_1^2,\mu_2))$$

for some $\psi_o(\mu_2) > 0$ and $\psi_1(y_1^2,\mu_2)$ by the implicit function theorem.

Note that $\psi_o(\mu_2) + y_1^2\psi_1(y_1,\mu_2)$ is C^{r-4}. For $\mu_1 < 0$, E_1 does not give us any further information concerning the topological qualitative features since $0 \epsilon \mathbb{R}^2$ is a node. So we will consider E_2 only.

For $\mu_1 \leq 0$, setting $\mu_1 = -\delta^2$, we have

$$(2.4.21) \quad \delta = y_1(\psi_o(\mu_2) + y_1^2\psi_1(y_1,\mu_2))^{1/2}.$$

So $\frac{\partial \delta}{\partial y_1}\Big|_{y_1=0} \neq 0.$

By the implicit function theorem, we can solve (2.4.21) for y_1, and we let the solution to (2.4.21) be

$$(2.4.22) \quad y_1 = \hat{p}(\delta,\mu_2).$$

Since (2.4.21) is odd in y_1, $\hat{p}$ in (2.4.22) is odd in δ, i.e., $\hat{p}(-\delta,\mu_2) = -\hat{p}(\delta,\mu_2)$. Also $\hat{p}(\delta,\mu_2) > 0$ if $\delta > 0$. $\varphi_o(0) > 0$ in (2.4.20). So (2.4.22) implies that the equilibrium point $(\hat{p}(\delta,\mu_2),0)$ of (2.4.20) with $\mu_1 = -\delta^2$ is a saddle if $\delta \neq 0$ and $(\hat{p}(0,\mu_2),0) = (0,0)$ is a pitchfork.

Now define

$$(2.4.23) \quad p(\delta,\mu_2) = x((\hat{p}(\delta,\mu_2),0), -\delta^2,\mu_2).$$

Then $p(\delta,\mu_2)$ is a C^{r-4} mapping such that for (δ,μ_2) near $(0,0)$,

$$p(\delta,\mu_2) = \begin{cases} (0,0) \text{ is a pitchfork of } \dot{x} = f(x,0,\mu_2), \\ \text{a saddle of } \dot{x} = f(x,-\delta^2,\mu_2) \text{ if } \delta > 0, \\ \text{another saddle of } \dot{x} = f(x,-\delta^2,\mu_2) \text{ if } \delta < 0. \end{cases}$$

(See Figure 5.)

From (2.4.23) $\frac{\partial p}{\partial \delta}(0,0) = D_y x(0,0,0)\ (\frac{\partial \hat{p}}{\partial \delta}(0,0),0)$, by the definition of u

$$u = (D_x y(0,0,0))^{-1}(1,0) = D_y x(0,0,0)\ (1,0).$$

Hence

$$(2.4.24)\qquad \frac{\partial p}{\partial \delta}(0,0) = \frac{\partial \hat{p}}{\partial \delta}(0,0)\ u.$$

Now (2.4.20) with $\mu_1 = -\delta^2$ has an invariant curve $\{(y_1,y_2) | y_1 = \hat{p}(\delta,\mu_2)\}$ at the equilibrium point $(\hat{p}(\delta,\mu_2),0)$. For $\delta = 0$ (so $\mu_1 = 0$), this invariant curve is the stable manifold of the pitchfork (0,0), and for $\delta \neq 0$, it is the stable manifold of the corresponding saddle $(\hat{p}(\delta,\mu_2),0)$.

Let $v(\delta,\mu_2) = D_y x((\hat{p}(\delta,\mu_2),0), -\delta^2,\mu_2)(0,1)$. Then $\dot{x} = f(x,-\delta^2,\mu_2)$ has an invariant curve at $p(\delta,\mu_2)$ tangent to $v(\delta,\mu_2)$ and this invariant curve is C^{r-4} in (δ,μ_2). For $(\delta,\mu_2) = (0,0)$, this invariant curve contains Γ.

Similarly (see the paragraph below (2.4.5)) we have a C^{r-4} family $\varphi^s(t,\delta,\mu_2)$, (δ,μ_2) small, such that $\varphi^s(t,\delta,\mu_2)$ is a solution of $\dot{x} = f(x,-\delta^2,\mu_2)$, $\varphi^s(t,\delta,\mu_2) \to p(\delta,\mu_2)$ as $t \to \infty$ along the negative $v(\delta,\mu_2)$ direction, $\varphi^s(0,\delta,\mu_2) \epsilon L$ and $\varphi^s(t,0,0) = \varphi(t)$.

$\varphi^s(t,0,\mu_2)$ is a branch of the stable manifold of the pitchfork $p(0,\mu_2) = (0,0)$ of $\dot{x} = f(x,0,\mu_2)$ and if $\delta \neq 0$, $\varphi^s(t,\delta,\mu_2)$ is a branch of the stable manifold of the corresponding saddle $p(\delta,\mu_2)$ of $\dot{x} = f(x,-\delta^2,\mu_2)$.

We define

$$d_2^c(\mu_1,\mu_2) = f(\varphi(0),0,0) \wedge (\varphi^c(0,\mu_1,\mu_2) - \varphi(0)),$$
$$d_2^s(\delta,\mu_2) = f(\varphi(0),0,0) \wedge (\varphi^s(0,\delta,\mu_2) - \varphi(0)).$$

From now on, we assume that $\delta \geq 0$, and define

$$d_2^{\pm}(\delta,\mu_2) = d_2^c(-\delta^2,\mu_2) - d_2^s(\pm\delta,\mu_2)$$

respectively. Hence $d_2^{\pm}$ is C^{r-4} since d_2^c is C^{r-1} and d_2^s is C^{r-4}.

From the definition,

$d_2^+(\delta,\mu_2) = 0$ (resp. $d_2^-(\delta,\mu_2) = 0$) if and only if there exists a pair of homoclinic (resp. heteroclinic) orbits of $\dot{x} = f(x,-\delta^2,\mu_2)$ at $p(\pm\delta,\mu_2)$(resp. from $p(\pm\delta,\mu_2)$ to $p(\mp\delta,\mu_2)$).

We will show the following:

(2.4.25) $\frac{\partial d_2^+}{\partial \delta}(0,0) < 0$ (resp. $\frac{\partial d_2^-}{\partial \delta}(0,0) > 0$) and

(2.4.26) $\frac{\partial d_2}{\partial \mu_2}(0,0) = I_2.$

Suppose we have shown (2.4.25) and (2.4.26). Then (2.4.25) and $I_2 \neq 0$ imply that $\{(\delta,\mu_2) | d_2^\pm(\delta,\mu_2) = 0\}$ is a C^{r-4} curve through $(0,0)$ of the form

$$\delta = m^\pm \mu_2 + o(\mu_2)$$

where

$$m^\pm = - I_2 / \frac{\partial d_2^\pm}{\partial \delta}(0,0) \neq 0.$$

(Note that $\delta = m^+ \mu_2 + o(\mu_2)$ and $-\delta = -m^- \mu_2 + o(\mu_2)$.)

Hence $\mu_1 = -(m^\pm)^2 \mu_2^2 + o(\mu_2^2) : \equiv \ell(\mu_2)$.

Also, $0 \leq \mathrm{sgn}(\delta) = \mathrm{sgn}(m^+ \mu_2) = \mathrm{sgn}(I_2 \cdot \mu_2)$, so $I_2 \cdot \mu_2 \geq 0$.

(Resp. $0 \geq \mathrm{sgn}(-\delta) = \mathrm{sgn}(-m^- \mu_2) = \mathrm{sgn}(I_2 \cdot \mu_2)$, so $I_2 \cdot \mu_2 \leq 0$.)

To show (2.4.25) and (2.4.26) , first we construct variational equations for $\varphi^c(t,-\delta^2 \mu_2)$ and $\varphi^s(t,\pm\delta,\mu_2)$.

Since

$$\frac{d}{dt}\frac{\partial\varphi^c}{\partial\mu_2}(t,0,0) = D_x f(\varphi(t),0,0)\frac{\partial\varphi^c}{\partial\mu_2}(t,0,0) + \frac{\partial f}{\partial\mu_2}(\varphi(t),0,0),$$

$$\frac{d}{dt}(\pm\frac{\partial\varphi^s}{\partial\delta}(t,0,0)) = D_x f(\varphi(t),0,0)(\pm\frac{\partial\varphi^s}{\partial\delta}(t,0,0)),$$

$$\frac{d}{dt}\frac{\partial\varphi^s}{\partial\mu_2}(t,0,0) = D_x f(\varphi(t),0,0)\frac{\partial\varphi^s}{\partial\mu_2}(t,0,0) + \frac{\partial f}{\partial\mu_2}(\varphi(t),0,0),$$

if we define

$$\rho^c_{\mu_2}(t) = f(\varphi(t),0,0) \wedge \frac{\partial\varphi^c}{\partial\mu_2}(t,0,0),$$

$$\rho^{s\pm}_\delta(t) = f(\varphi(t),0,0) \wedge (\pm\frac{\partial\varphi^s}{\partial\delta}(t,0,0)),$$

$$\rho^s_{\mu_2}(t) = f(\varphi(t),0,0) \wedge \frac{\partial\varphi^s}{\partial\mu_2}(t,0,0),$$

we have the following variational equations:

$$(2.4.27)_c \qquad \frac{d}{dt}\rho^c_{\mu_2}(t) = \operatorname{div} f(\varphi(t),0,0)\,\rho^c_{\mu_2}(t) + f(\varphi(t),0,0) \wedge \frac{\partial f}{\partial\mu_2}(\varphi(t),0,0),$$

$$(2.4.27)_{s\pm} \qquad \frac{d}{dt}\rho^{s\pm}_\delta(t) = \operatorname{div} f(\varphi(t),0,0)\,\rho^{s\pm}_\delta(t),$$

$$(2.4.27)_s \qquad \frac{d}{dt}\rho^s_{\mu_2}(t) = \operatorname{div} f(\varphi(t),0,0)\,\rho^s_{\mu_2}(t) + f(\varphi(t),0,0) \wedge \frac{\partial f}{\partial\mu_2}(\varphi(t),0,0).$$

If we solve the equations (2.4.27), we have the following:

$$(2.4.28)_c \qquad \rho^c_{\mu_2}(0) = \rho^c_{\mu_2}(t_1)\exp\int_{t_1}^{0}\operatorname{div} f(\varphi(t),0,0)dt$$
$$+ \int_{t_1}^{0}\exp[-\int_0^t \operatorname{div} f(\varphi(s),0,0)ds]\, f(\varphi(t),0,0) \wedge \frac{\partial f}{\partial\mu_2}(\varphi(t),0,0)dt,$$

$$(2.4.28)_{s\pm} \qquad \rho^{s\pm}_\delta(0) = \rho^{s\pm}_\delta(t_1)\exp[-\int_0^{t_1}\operatorname{div} f(\varphi(t),0,0)dt],$$

$$(2.4.28)_s \qquad \rho^s_{\mu_2}(0) = \rho^s_{\mu_2}(t_1)\exp[-\int_0^{t_1} \operatorname{div} f(\varphi(t),0,0)dt]$$

$$+ \int_{t_1}^{0} \exp[-\int_0^t \operatorname{div} f(\varphi(s),0,0)ds]\ f(\varphi(t),0,0) \wedge \frac{\partial f}{\partial \mu_2}(\varphi(t),0,0)dt.$$

Let $I_c(t_1)$ be the first term and $II(t_1)$ the second term in the RHS of $(2.4.28)_c$. Also let $I_s(t_1)$ be the first term in the RHS of $(2.4.28)_s$. Then

$$\rho^c_{\mu_2}(0) = I_c(t_1) + II(t_1),$$

$$\rho^s_{\mu_2}(0) = I_s(t_1) + II(t_1).$$

For a moment, we prove the following:

<u>Lemma (2.4.2)</u>.

$$(2.4.29)_c \qquad \lim_{t\to-\infty} \frac{\partial \varphi^c}{\partial \mu_2}(t,0,0) = 0,$$

$$(2.4.29)_{s\pm} \qquad \lim_{t\to\infty} \frac{\partial \varphi^{s\pm}}{\partial \delta}(t,0,0) = \pm \frac{\partial p}{\partial \delta}(0,0), \text{ and}$$

$$(2.4.29)_s \qquad \lim_{t\to\infty} \frac{\partial \varphi^s}{\partial \mu_2}(y,0,0) = 0.$$

<u>Proof</u>. For $(2.4.29)_c$, see the proof of $(2.4.7)_c$ in Lemma (2.4.1). (Note that $\delta = 0$ corresponds to $\mu_1 = -\delta^2 = 0$.) For $(2.4.29)_{s\pm}$, we define

$$(2.4.30) \qquad \tilde{\varphi}^{s\pm}(t,\delta,\mu_2) = y(\varphi^{s\pm}(t,\delta,\mu_2), -\delta^2,\mu_2) = (\hat{p}(\pm\delta,\mu_2),\ y_2(t,\pm\delta,\mu_2))$$

for $t >> 1$, for each $(\pm\delta,\mu_2)$ near $(0,0)$. By the relationship between the x–coordinates and the y–coordinates and from (2.4.20),

$$y_2(t,\pm\delta,\mu_2) > 0 \quad \text{for} \quad t >> 1,$$

and $y_2(\cdot,\pm\delta,\mu_2)$ satisfies a differential equation of the form

$$\text{(2.4.31)} \quad \frac{dz}{dt} = -\varphi_3(\pm\delta,\mu_2)\ z(1+z^2\varphi_4(z,\pm\delta,\mu_2)).$$

Here φ_3 and $z^2\varphi_4$ are C^{r-5}.

If we solve (2.4.31) by the separation of variables,

$$\text{(2.4.32)} \quad z\ \exp J_1(z,\pm\delta,\mu_2) = J_2(\pm\delta,\mu_2)\ \exp(-\varphi_3(\pm\delta,\mu_2)t)$$

for some J_1 and $J_2 > 0$, both C^{r-6}. Since $\frac{\partial}{\partial z}(z\ \exp J_1(z,\pm\delta,\mu_2))|_{z=0} \neq 0$ for $(\pm\delta,\mu_2)$ near $(0,0)$, by the implicit function theorem, we can solve

$$\text{(2.4.33)} \quad z\ \exp J_1(z,\pm\delta,\mu_2) = v$$

for z. Let $z = R(v,\pm\delta,\mu_2)$ be the solution to (3.4.33). Then R is C^{r-6} and

$$\text{(2.4.34)} \quad R(0,\pm\delta,\mu_2) \equiv 0.$$

Thus from (3.4.32), if we let $z = y_2$ and $v =$ the RHS of (3.4.32), then we get

$$\text{(2.4.35)} \quad y_2(t,\pm\delta,\mu_2) = R(J_2(\pm\delta,\mu_2)\ \exp(-\varphi_3(\pm\delta,\mu_2)t),\ \pm\delta,\mu_2).$$

(2.4.34) and (2.4.35) imply that

$$\pm \frac{\partial y_2}{\partial \delta}(t,\pm\delta,\mu_2),\ \frac{\partial y_2}{\partial \mu_2}(t,\pm\delta,\mu_2) \to 0 \quad \text{as} \quad t \to \infty.$$

From (2.4.30)

$$\frac{\partial \tilde{\varphi}^{s\pm}}{\partial \delta}(t,\delta,\mu_2) = (\pm \frac{\partial \hat{p}}{\partial \delta}(\pm\delta,\mu_2),\ \pm \frac{\partial y_2}{\partial \delta}(\pm\delta,\mu_2,t)).$$

So

$$\frac{\partial \tilde{\varphi}^{s\pm}}{\partial \delta}(t,\delta,\mu_2) \to (\pm \frac{\partial \hat{p}}{\partial \delta}(\pm\delta,\mu_2),0) \quad \text{and}$$

(2.4.36)

$$\frac{\partial \tilde{\varphi}^{s\pm}}{\partial \mu_2}(t,\delta,\mu_2) \to (\frac{\partial \hat{p}}{\partial \mu_2}(\pm\delta,\mu_2),0) \quad \text{as} \quad t \to \infty.$$

Since $\frac{\partial \varphi^{s\pm}}{\partial \delta}(t,0,0) = D_y x(\tilde{\varphi}^{s\pm}(t,0,0),0,0) \frac{\partial \tilde{\varphi}^{s\pm}}{\partial \delta}(t,0,0)$, from (2.4.20) and (2.4.36)

$$\lim_{t\to\infty} \frac{\partial \varphi^{s\pm}}{\partial \delta}(t,0,0) = D_y x(0,0,0)(\pm \frac{\partial \hat{p}}{\partial \delta}(0,0,),0) = \pm \frac{\partial p}{\partial \delta}(0,0).$$

Finally for $(2.4.29)_s$, we consider

$$\varphi^{s\pm}(t,\delta,\mu_2) = x(\tilde{\varphi}^{s\pm}(t,\delta,\mu_2),\ -\delta^2,\mu_2).$$

So $\frac{\partial \varphi^{s\pm}}{\partial \mu_2}(t,0,0) = D_y x(\tilde{\varphi}^{s\pm}(t,0,0),0,0)(\frac{\partial \hat{p}}{\partial \mu_2}(0,0),0) + \frac{\partial x}{\partial \mu_2}(\tilde{\varphi}^{s\pm}(t,0,0),0,0) \to$

$$D_y x(0,0,0)(\frac{\partial \hat{p}}{\partial \mu_2}(0,0),0) + \frac{\partial x}{\partial \mu_2}(0,0,0) = 0 \quad \text{as} \quad t \to \infty$$

since $\frac{\partial \hat{P}}{\partial \mu_2}(0,0) = 0$ and $x(0,0,\mu_2) \equiv 0$.

So we complete the proof of Lemma (2.4.2). □

By (2.4.30)

(2.4.37) $\quad \dot{\varphi}^{s\pm}(t,\delta,\mu_2) = D_x y(\varphi^{s\pm}(t,\delta,\mu_2),\ -\delta^2,\mu_2)\ \dot{\varphi}^{s\pm}(t,\delta,\mu_2)$

Let $\delta = \mu_2 = 0$ in (2.4.37). Since $\dot{\varphi}^{s\pm}(t,0,0) = \dot{\varphi}(t) = f(\varphi(t),0,0)$,

$$f(\varphi(t),0,0) = [D_x y(\varphi(t),0,0)]^{-1} \dot{\varphi}^{s\pm}(t,0,0).$$

From (2.4.30), (2.4.34) and (2.4.35), we obtain $\varphi(t) = O(\exp(-\lambda t))$ and

$$(2.4.38) \quad \dot{\varphi}^{s\pm}(t,0,0) = (0,-C \exp(-\lambda t) + O(\exp(-\lambda t))) \quad \text{as} \quad t \to \infty$$

where $C > 0$. Therefore

$$(2.4.39) \quad f(\varphi(t_1),0,0) = \{[D_x y(0,0,0)]^{-1} + O(\exp(-\lambda t))\} \cdot (0,-C \exp(-\lambda t) + o(\exp(-\lambda t))).$$

(2.4.38) also gives

$$\operatorname{div} f(\varphi(t),0,0) = -\lambda + O(\exp(-\lambda t)).$$

Hence

$$(2.4.40) \quad \exp[-\int_0^{t_1} \operatorname{div} f(\varphi(s),0,0)ds] = \exp(\lambda t_1)\cdot\exp \int_0^{t_1} O(\exp(-\lambda s))ds.$$

Then, from (2.4.39) and (2.4.40) we have

$$(2.4.41) \quad \lim_{t_1\to\infty} f(\varphi(t_1),0,0) \exp[-\int_0^{t_1} \operatorname{div} f(\varphi(s),0,0)ds] = [D_x y(0,0,0)]^{-1} \cdot (0,-C \exp \int_0^{\infty} O(\exp(-\lambda s))ds).$$

Now back to $(2.4.28)_{s\pm}$, by $(2.4.29)_{s\pm}$ of Lemma (2.4.2)

$$\rho_\delta^{s\pm}(0) = f(\varphi(t_1),0,0) \wedge (\pm \frac{\partial\varphi^{s\pm}}{\partial\delta}(t_1,0,0)) \exp [-\int_0^{t_1} \operatorname{div} f(\varphi(s),0,0)ds]$$

$$= -(\pm \frac{\partial P}{\partial\delta}(0,0)) \wedge \lim_{t_1\to\infty} f(\varphi(t_1),0,0) \exp[-\int_0^{t_1} \operatorname{div} f(\varphi(s)0,0,)ds]$$

where the limit exists by (2.4.41).

Note that

(1) from definitions of $d_2^{\pm}$ and $\rho_\delta^{s\pm}$,

$$\frac{\partial d_2^{\pm}}{\partial\delta}(0,0) = -\rho_\delta^{s\pm}(0),$$

(2) $\frac{\partial p}{\partial\delta}(0,0)$ is a positive multiple of v by (2.4.41) from similar assumption as (2.4.4),

(3) $\lim_{t_1\to\infty} f(\varphi(t_1),0,0)$ is a negative multiple of u.

By above (1), (2) and (3),

$$-\rho_\delta^{s+}(0) = \frac{\partial d_2^{+}}{\partial\delta}(0,0) \text{ (resp. } -\rho_\delta^{s-}(0) = \frac{\partial d_2^{-}}{\partial\delta}(0,0))$$ is a negative (resp. positive) since $u \wedge v > 0$ which shows (2.4.25). The proofs of

(2.4.42) $$\lim_{t\to -\infty} I_c(t) = 0 = \lim_{t\to\infty} I_s(t)$$

are immediate by Lemma (2.4.2), and are the same as those in Theorem (2.3.2). It is easy to see that (2.4.42) shows our final claim (2.4.26).

The only thing we have to notice is the smoothness of $\tilde{f}$ we need. In the proof $(2.4.29)_c$ in Lemma (2.4.2), since φ^s is C^{r-4} instead of C^{r-1}, we need the smoothness of C^{r-10} instead of C^{r-7}. So r has to be at least 11.

We finished our proof of Theorem (2.3.3). □

Remark. Theorem (2.3.3) gives us a homoclinic bifurcation curve of codimension 1: $\{(-(m^+)^2\mu_2^2,\mu_2)) | \mu_2 > 0\} = H_{01}$ and a heteroclinic bifurcation curve of codimension 1: $\{(-(m^-)^2\mu_2^2,\mu_2) | \mu_2 < 0\} = H_e$.

For $\mu_1 < 0$, we have three structurally stable connected components as in Figure 4 which is the case $I_2 > 0$.

Existence of limit cycles in regions 3, 7, and 11 of Figure 4 is again from the same reasons, hyperbolicity of Γ, Poincare–Bendixon Theorem, and the symmetry property of the vector field.

Note that $I_1 = I_2 = I$.

C. Proof of Theorem (2.3.4).

Consider a C^r–mapping $\Psi: U \subset \mathbb{R}^2 \to \mathcal{X}_s^r(D)$ with $\Psi(\alpha_1,\alpha_2) = \tilde{f}(\cdot,\alpha_1,\alpha_2)$ and $\Psi(0,0) = f(\cdot)$, where U is a neighborhood of 0.

We want to show that Ψ is transversal to Σ_2 at f if $I \neq 0$.

First, we will show that Σ_2 is a C^{r-1} submanifold of Σ_1 of codimension one.

Let $f\epsilon\Sigma_2$ with pitchfork 0 and let L be a line segment perpendicular to Γ as in the Theorems (2.3.2) and (2.3.3). For $g\epsilon\Sigma_1$ near f, 0 is again pitchfork and the stable and center manifolds of 0 are C^{r-1}–dependent on g. Thus their intersection with L are C^{r-1} function of g. Therefore the function $d(0,\mu_2)$ is C^{r-1} even though $d^{\pm}(\delta,\mu_2)$ is only C^{r-4}. $d(g) = 0$ if and only if $g\epsilon\Sigma_2$. It is easy to find a perturbation $f + \epsilon h$ in Σ_1 such that $\frac{d}{d\epsilon}\Big|_{\epsilon=0} d(f+\epsilon h) \neq 0$ and this says that Σ_2 is a

C^{r-1} submanifold of Σ_1 of codimension one. By Theorem (2.3.1) (or Appendix), Σ_1 is a C^{r-1} submanifold of $\Sigma_0 = \mathscr{X}^r_s(D)$ of codimension one.

Now let $\phi(\mu_1,\mu_2) = f(\cdot,\mu_1,\mu_2)$. Then it is enough to show that ϕ is transverse to Σ_2 at $(\mu_1,\mu_2) = (0,0)$ if and only if $I \neq 0$.

Since $\phi(0,\mu_2)\epsilon\Sigma_1$ for $|\mu_2| << 1$,

$$(2.4.43) \quad \frac{\partial\phi}{\partial\mu_2}(0,0) \text{ is tangent to } \Sigma_1.$$

Also, if $I \neq 0$, then we have

$$(2.4.44) \quad \begin{cases} \frac{\partial g}{\partial\tau_1}(\cdot,0,0) = \frac{\partial\phi}{\partial\mu_1}(0,0) - q'(0)\,\frac{\partial\phi}{\partial\mu_2}(0,0) \\ \frac{\partial g}{\partial\tau_2}(\cdot,0,0) = \frac{\partial\phi}{\partial\mu_2}(0,0) \end{cases}$$

from the transformation $(\tau_1,\tau_2) \to (\mu_1,\mu_2)$ $(\mu_1 = \tau_1,\ \mu_2 = \tau_2 - q(\tau_1))$ and

$$(2.4.45) \quad \begin{cases} \frac{\partial\Psi}{\partial\alpha_1}(0,0) = \frac{\partial g}{\partial\tau_1}(\cdot,0,0) \\ \frac{\partial\Psi}{\partial\alpha_2}(0,0) = -p'(0)\,\frac{\partial g}{\partial\tau_1}(\cdot,0,0) + \frac{\partial g}{\partial\tau_2}(\cdot,0,0) \end{cases}$$

from the transformation $(\alpha_1,\alpha_2) \to (\tau_1,\tau_2)(\tau_1 = \alpha_1 - p(\alpha_2),\ \tau_2 = \alpha_2)$.
By assumption (IV) and Theorem (2.3.1),

$$(2.4.46) \quad \frac{\partial\Psi}{\partial\alpha_1} \text{ is transversal to } \Sigma_1 \text{ at } f.$$

So (2.4.43), (2.4.44) and (2.4.45) imply that, if $I \neq 0$, (2.4.46) if and only if $\frac{\partial g}{\partial \tau_1}(\cdot,0,0)$ is transversal to Σ_1 if and only if $\frac{\partial \phi}{\partial \mu_1}(0,0)$ is transversal to Σ_1.

Next from the fact that $I = \frac{\partial d_1}{\partial \tau_2}(0,0) = \frac{\partial d_2}{\partial \mu_2}(0,0)$ and (2.4.43),

$I \neq 0$ implies $\frac{\partial \phi}{\partial \mu_2}(0,0)$ is transversal to Σ_2.

$I = 0$ implies $\frac{\partial d_1}{\partial \tau_2}(0,0) = \frac{\partial d_2}{\partial \mu_2}(0,0) = 0$ which says $\frac{\partial \phi}{\partial \mu_2}(0,0)$ is tangent to Σ_2.

This completes the proof. □

CHAPTER 3. GENERIC 3–PARAMETER FAMILIES OF SYMMETRIC PLANAR VECTOR FIELDS WITH NILPOTENT LINEAR PART.

We will study in this chapter the symmetric planar vector fields—which means the vector fields with the invariance by the rotation of an angle π with respect to the origin in the plane—with nilpotent linear part. Also we will classify the bifurcations of the generic 3–parameter families of those vector fields to be mentioned later.

§1. Introduction.

There are two kinds of k–jet normal form of the vector field with nilpotent linear part (see Guckenheimer and Holmes [10]).

$$(3.1.1)_a \quad \begin{cases} \dot{x} = y + O(|x,y|^{k+1}) \\ \dot{y} = \sum_{i=2}^{k} (a_i x^i + b_i x^{i-1} y) + O(|x,y|^{k+1}) \end{cases}$$

or

$$(3.1.1)_b \quad \begin{cases} \dot{x} = y + \sum_{i=2}^{k} a_i x^i + O(|x,y|^{k+1}) \\ \dot{y} = \sum_{i=2}^{k} b_i x^i + O(|x,y|^{k+1}) \end{cases} .$$

k–jet normal forms (3.1.1) can be shown easily using the normal form theory.

A. Nonsymmetric Case.

We assume that $a_2 b_2 \neq 0$ in (3.1.1). Hence terms of order 3 or higher can be neglected, and we can write it down simply

$$(3.1.2)_a \quad \begin{cases} \dot{x} = y \\ \dot{y} = ax^2 + bxy \end{cases}$$

or

$$(3.1.2)_b \quad \begin{cases} \dot{x} = y + ax^2 \\ \dot{y} = bx^2 \end{cases}$$

where $ab \neq 0$.

Bogdanov [2] analyzed $(3.1.2)_a$ and Takens ([17] and [18]) did $(3.1.2)_b$ independently.

Recently, some of codimension three problems concerning the planar vector fields with nilpotent linear part with degenerate singularity are appeared. For example, in $(3.1.2)_a$, Dumortier et. al. [7] worked for the case $a \neq 0$ and $b = 0$ (DRS–A), and Medved [13] and Dumortier et. al. [8] for the case $a = 0$ and $b \neq 0$ (Medved–B, DRS–B) with following versal unfoldings:

$$\begin{cases} \dot{x} = y \\ \dot{y} = x^2 + \epsilon_1 + y(\epsilon_2 + \epsilon_3 x \pm x^3) \end{cases} \qquad \text{(DRS–A)},$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 + \epsilon_2 x + \epsilon_3 x^2 \pm x^3 + xy \end{cases} \qquad \text{(Medved–B)},$$

$$\begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 + \epsilon_2 x \pm x^3 + y(\epsilon_3 + bx \pm x^2) \end{cases} \qquad \text{(DRS–B)}.$$

B. Symmetric Case

While parallel to the above, we can consider the unfolding of the symmetric vector fields with nilpotent linear part with degenerate singularity on the plane. In this case $a_i = b_i = 0$ for i even. So we assume that $a_3 b_3 \neq 0$ in (3.1.1). Then, similarly we have

$$(3.1.3)_a \quad \begin{cases} \dot{x} = y \\ \dot{y} = ax^3 + bx^2 y \end{cases}$$

or

$$(3.1.3)_b \quad \begin{cases} \dot{x} = y + ax^3 \\ \dot{y} = bx^3 \end{cases}$$

where $ab \neq 0$.

Carr [3] worked $(3.1.3)_a$ and again Takens [17] worked $(3.1.3)_b$.

For example, in $(3.1.3)_a$, if $a > 0$ and $b \neq 0$, the phase portrait near (0,0) is a degenerate saddle of codimension 2 and for $a < 0$ and $b \neq 0$, it is a degenerate focus of codimension 2. (See Figure 6.)

However, if $b = 0$ (and $a \neq 0$), we have to consider

(3.1.4) $$\begin{cases} \dot{x} = y \\ \dot{y} = ax^3 + \tilde{b}x^4y \end{cases}$$

where $a\tilde{b} \neq 0$. The phase portrait near (0,0) is a degenerate saddle or focus (depending on the sign of a) of codimension 3, and mostly we are concerned with these in this chapter.

If $a = 0$ and $b \neq 0$, we have to consider

(3.1.5) $$\begin{cases} \dot{x} = y \\ \dot{y} = \tilde{a}x^5 + bx^2y \end{cases}$$

where $\tilde{a}b \neq 0$.
In this case we have some difficulties which will be discussed later.

In studying the equation (3.1.4) with the case $a > 0$, basically we followed the similar ideas as Dumortier et.al.[7]. Since the type of the equations and its dynamical behaviors are similar. However, the equation (3.1.4) with the case $a < 0$ has produced many difficult problems in proving the existence of limit cycles and new phenomena occur including triple limit cycle bifurcation.

We will introduce the bifurcation diagram and the corresponding phase portraits with a short explanation for the equation (3.1.5).

§2. Versal Deformation ($a \neq 0$, $b = 0$).

First we study the versal deformation of (3.1.4).

Lemma (3.2.1). Any symmetric perturbation of (3.1.4) with small parameter μ can be transformed into the form

$$(3.2.1)_\pm \quad \begin{cases} \dot{x} = y \\ \dot{y} = \varphi_1(\mu)x + \varphi_2(\mu)y + \varphi_3(\mu)x^2y \pm x^3 + x^4y\, G(x,\mu) + y^2\Psi(x,y,\mu), \end{cases}$$

where $\mu \in \mathbb{R}^3$, $G(x,0) = 1$ and $\Psi(x,y,0) = 0$.

Remark. We can take a transformation in the parameters $\epsilon_i = \varphi_i(\mu)$, $i = 1,2,3$, such that $(3.2.1)_\pm$ becomes

$$(3.2.2)_\pm \quad \begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 x + \epsilon_2 y + \epsilon_3 x^2 y \pm x^3 + x^4 y G(x,\epsilon) + y^2\Psi(x,y,\epsilon) \end{cases}$$

where $G(x,0) = 1$ and $\Psi(x,y,0) = 0$.

Proof of Lemma (3.2.1). Let the following equations

$$(3.2.3) \quad \begin{cases} \dot{x} = y + \omega_1(x,y,\mu) \\ \dot{y} = ax^3 + \tilde{b}x^4y + \omega_2(x,y,\mu) \end{cases} \quad : \mu \in \mathbb{R}^n$$

be the perturbed system of (3.1.4), where for $i = 1,2$, $\omega_i(x,y,0) = 0$, ω_i are sufficiently smooth (say, C^∞) and symmetric (i.e., $\omega_i(-x,-y,\mu) = -\omega_i(x,y,\mu)$).

Without loss of generality, we may assume that $a = \pm 1$ in (3.2.3) depending on the sign of the original a in (3.1.4) and $\tilde{b} = 1$ since otherwise we can take scaling

$$x \to \frac{\tilde{b}^{1/3}}{|a|^{1/6}}\,x,\quad y \to \frac{\tilde{b}^{2/3}}{|a|^{5/6}},\quad t \to \frac{|a|^{2/3}}{\tilde{b}^{1/3}}\,t.$$

By a change of coordinates $p = x$, $q = y + \omega_1$, (3.2.3) becomes

$$(3.2.4)\qquad \begin{cases} \dot{p} = q \\ \dot{q} = p^3(a+\frac{\partial}{\partial y}\omega_1) + p^4 q(1+\frac{\partial}{\partial y}\omega_1) \\ \qquad + \{-p^4\omega_1(1+\frac{\partial}{\partial y}\,\omega_1) + \omega_2(1+\frac{\partial}{\partial y}\,\omega_1) + p(\frac{\partial}{\partial x}\,\omega_1)\}. \end{cases}$$

Let $\frac{\partial}{\partial y}\,\omega_1(x(p,q,\mu), y(p,q,\mu),\mu) = h_1(p,\mu) + qh_2(p,\mu) + q^2h_3(p,q,\mu)$ and

$$-p^4\omega_1(1+\frac{\partial}{\partial y}\,\omega_1) + \omega_2(1+\frac{\partial}{\partial y}\,\omega_1) + p(-\frac{\partial}{\partial x}\,\omega_1)$$
$$= \Psi_1(p,\mu) + q\Psi_2(p,\mu) + q^2\Psi_3(p,q,\mu)$$

for some h_i and $\Psi_i (i = 1,2,3)$.

Then $h_i = \Psi_i = 0$ at $\mu = 0$ $(i = 1,2,3)$. Hence the equation (3.2.4) is

$$(3.2.5)\qquad \begin{cases} \dot{p} = q \\ \dot{q} = [p^3(a+h_1) + \Psi_1] + q[p^4(1+h_1) + p^3h_2 + \Psi_2] + q^2\phi, \end{cases}$$

where $\phi = p^3h_3 + p^4(h_2 + qh_3) + \Psi_3$. By the symmetry property, we can say that

$$\Psi_1 = \varphi_1(\mu)p + \beta_1(p,\mu)p^3,$$
$$p^3h_2 + \Psi_2 = \varphi_2(\mu) + \varphi_3(\mu)p^2 + \beta_2(p,\mu)p^4$$

for some $\varphi_i (i = 1,2,3)$ and β_i $(i = 1,2)$. Let $F(p,\mu) = a + h_1 + \beta_1$ and $G(p,\mu) = 1 + h_1 + \beta_2$. Hence (3.2.5) is changed into the following form

$$(3.2.6)\quad \begin{cases} \dot{p} = q \\ \dot{q} = \varphi_1(\mu)p + \varphi_2(\mu)q + \varphi_3(\mu)p^2q + p^3F(p,\mu) + p^4qG(p,\mu) + q^2\phi. \end{cases}$$

By the Malgrange Preparation Theorem (see Chow & Hale [4], pg. 43), we get

$$\varphi_1(\mu)p + F(p,\mu)p^3 = [\tilde{\varphi}_1(\mu)p + \text{sgn}F(0,0)p^3]\ \theta(p,\mu),$$

where sgn $F(0,0) = a \neq 0$, $\theta(p,0) = 1 > 0$, and F and θ are even in p. Hence in (3.2.6),

$$\dot{q} = [\tilde{\varphi}_1(\mu)p \pm p^3 + \frac{\varphi_2(\mu)}{\theta(p,\mu)}\, q + \frac{\varphi_3(\mu)}{\theta(p,\mu)}\, p^2q + \frac{G}{\theta}\, p^4q + \frac{\phi}{\theta}\, q^2]\theta.$$

Again, let $u = p$, $v = q/\sqrt{\theta}$. Then

$$(3.2.7)\quad \begin{cases} \dot{u} = v\sqrt{\theta} \\ \dot{v} = [\tilde{\varphi}_1(\mu)u \pm u^3 + (\dfrac{\varphi_2(\mu)}{\sqrt{\theta}} - \dfrac{\dot{\theta}}{2\sqrt{\theta}})v + \dfrac{\varphi_3}{\sqrt{\theta}}\, u^2v + \dfrac{G}{\sqrt{\theta}}\, u^4v + \phi v^2]\sqrt{\theta}. \end{cases}$$

By the symmetry property, in (3.2.7)

$$\frac{\varphi_2}{\sqrt{\theta}} - \frac{\dot{\theta}}{2\sqrt{\theta}} = z_1(\mu)u^2 + z_3(u,\mu)u^4,\ \frac{\varphi_3}{\sqrt{\theta}} = z_4(\mu) + z_5(u,\mu)u^2.$$

Note that $z_i = 0$ at $\mu = 0$ $(i = 1,\ldots,5)$.

Let $\tilde{\varphi}_2(\mu) = z_1(\mu)$,

$\tilde{\varphi}_3(\mu) = z_2(\mu) + z_4(\mu)$,

$\tilde{G}(u,\mu) = z_3(u,\mu) + z_5(u,\mu) + G/\sqrt{\theta}$.

Then (3.2.7) is

$$(3.2.8)_\pm\quad \begin{cases} \dot{u} = v\sqrt{\theta} \\ \dot{v} = [\tilde{\varphi}_1(\mu)u + \tilde{\varphi}_2(\mu)v + \tilde{\varphi}_3(\mu)u^2v \pm u^3 + \tilde{G}(u,\mu)u^4v + \phi(u,v,\mu)v^2]\sqrt{\theta}. \end{cases}$$

In (3.2.8) $\tilde{G}(u,0) \neq 0$. (3.2.8) is topologically equivalent to (3.2.1). (For simplicity we denote $\tilde{\varphi}_i$ by φ_i (i = 1,2,3) in $(3.2.1)_\pm$.) In (3.2.8) $\phi(u,v,0) = 0$ and $\tilde{G}(u,0) = 1$. □

Note that $(3.2.2)_\pm$ is versal to

$$(3.2.9)_\pm \quad \begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 x \pm x^3 + \epsilon_2 y + \epsilon_3 x^2 y + x^4 y \end{cases}$$

(For this, see Section 5 of Bogdanov [2].) Hence we will study $(3.2.9)_+$ in §3 and $(3.2.9)_-$ in §4 instead of $(3.2.2)_+$ and $(3.2.2)_-$ respectively.

§3. The Case $a > 0$.

For reminding $(3.2.9)_+$, it is written down again:

$$(3.2.9)_+ \quad \begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 x + x^3 + y(\epsilon_2 + \epsilon_3 x^2 + x^4). \end{cases}$$

The equilibria in $(3.2.9)_+$ is determined by the equations $y = 0$ and $x(\epsilon_1+x^2) = 0$. Hence for $\epsilon_1 > 0$ a saddle (0,0), and for $\epsilon_1 < 0$ a focus (0,0) and saddles $(\pm\sqrt{|\epsilon_1|},0)$ are equilibria.

Let the RHS of $(3.2.9)_+$ be $f_\epsilon(x,y)$. Then

$$Df_\epsilon(x,0) = \begin{bmatrix} 0 & 1 \\ \epsilon_1+3x^2 & \epsilon_2+\epsilon_3 x^2+x^4 \end{bmatrix}.$$

It is immediate that $\{\epsilon_1 = 0\}$ is a pitchfork bifurcation surface. In $\{\epsilon_1 > 0\}$, the phase portrait is topologically constant and it is a saddle at

(0,0). In $\{\epsilon_1 < 0\}$, several bifurcations occur. We analyze the equation $(3.2.9)_+$ by drawing the trace of the bifurcation surfaces on the hemisphere $S = \{\epsilon = (\epsilon_1,\epsilon_2,\epsilon_3) | 0 < |\epsilon| = \epsilon_0 << 1, \epsilon_1 \leq 0\}$.

The bifurcation diagram of equation $(3.2.9)_+$ is a cone based on its trace in S. This trace on S consists of 4 curves:

(1) H_f: Hopf bifurcation curve ($\epsilon_2 = 0$),

(2) H_e: heteroclinic loop bifurcation curve,

(3) C: semistable limit cycle bifurcation curve, and

(4) P: pitchfork bifurcation curve ($\epsilon_1 = 0$).

The curve C joins a point h_2 on H_f to a point c_2 on H_e, and the curve C is tangent to H_f and H_e at these points respectively. (See Figure 7).

The curves H_f and H_e on S touch ∂S at b_1 and b_2 transversally and in the neighborhood b_1 and b_2 we have the degenerate saddle bifurcation of codimension 2 (see Figure 6(a)).

There exists a unique unstable closed orbit for the parameter in between H_f and H_e in the neighborhood of b_1 and a unique stable closed orbit in between H_f and H_e in the neighborhood of b_2. On $H_f - \{h_2\}$, a Hopf bifurcation of codimension 1 with appearance of an unstable (resp. stable) closed orbit by crossing the line b_1h_2 (resp. b_2h_2) from right (resp. left). On $H_e - \{c_2\}$, a symmetric heteroclinic loop bifurcation of codimension 1 occurs. By crossing the curve b_1c_2 of H_e from left, we have a pair of heteroclinic loops, then they are destroyed and unstable closed orbit appears. While by crossing the curve b_2c_2 of H_e from right, again we have a pair of heteroclinic loops and then stable closed orbit appears. The point h_2 (resp. c_2) corresponds to a Hopf (resp. a heteroclinic) bifurcation of codimension 2. The curves H_f and H_e intersect transversally at points b_1, b_2 and d.

The point d corresponds to the simultaneous Hopf bifurcation of codimension 1 and symmetric heteroclinic loop bifurcation of codimension 1. For parameter values in the curved triangle (dh_2c_2), there are exactly two closed orbits. One of them is stable inside an unstable closed orbit. These two closed orbits coalesce in a generic way when crossing the curve C from left. On C itself there exist a unique semistable limit cycle.

Theorem (3.3.1). Let $\Sigma = \partial S \cup H_f \cup H_e \cup C$ be a subset of S, where the semisphere and curves H_f, H_e and C are described above. The bifurcation diagram of $(3.2.9)_+$ in the ball $B_{\epsilon_0} = \{\epsilon \mid |\epsilon| \le \epsilon_0\}$ is a cone homeomorphic to $\{(s^2\eta, s^4\tau_0, s^2\tau_1) \mid s\epsilon[0,\epsilon_0], (\mu,\tau_0,\tau_1)\epsilon\Sigma\}$. The topological type of the phase portraits of equation $(3.2.9)_+$ in a fixed neighborhood of the (0,0) in $\mathbb{R}^2$ is constant in each connected component surrounded by the bifurcation surfaces (5 components: $R_1,\dots,R_5$, $R_1 \cap S = I,\dots,R_5 \cap S = V$), and is constant in each bifurcation surfaces (9 surfaces: $S_1,\dots,S_9$, $S_1 \cap S = 1,\dots,S_9 \cap S = 9$) and curves (5 curves: $C_1,\dots,C_5$, $C_1 \cap S = b_1,b_2,c_2,h_2$, and $C_5 \cap S = d$). (See Figure 8).

Proof will be given at the end of this section.

The main difficult problem is the determination of the number of limit cycles.

For this we use the blowing–up method as following for $\epsilon_1 < 0$:

$$(3.3.1)\qquad \begin{aligned} x &\to sx & \epsilon_1 &= s^2\eta \\ y &\to s^2y & \epsilon_2 &= s^4\tau_0 \\ t &\to t/s & \epsilon_3 &= s^2\tau_1 \end{aligned}$$

where $s > 0$. First we will study the neighborhood of $0\epsilon_1$–axis for $\epsilon_1 \leq 0$. Hence let $\eta = -1$ in (3.3.1).

Then the equation $(3.2.9)_+$ has the form

$$\text{(3.3.2)} \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 + s^3(\tau_o+\tau_1 x^2+x^4)y. \end{cases}$$

Let $\mu_o = s^3\tau_o$, $\mu_1 = s^3\tau_1$, $\mu_2 = s^3$, then (3.3.2) becomes

$$\text{(3.3.3)} \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 + (\mu_o+\mu_1 s^2+\mu_2 x^4)y \end{cases}$$

with $\mu_2 > 0$.

By the change of parameters $(\epsilon_1,\epsilon_2,\epsilon_3) \to (s,\tau_o.\tau_1) \to (\mu_2,\mu_o,\mu_1)$ and the change of variables $(x,y,t) \to (sx,s^2y,t/s)$, $(3.2.9)_+$ has the form (3.3.2) and (3.3.3). Note that the equation (3.3.3) has the equilibria $(0,0)$ and $(\pm 1,0)$ where $(0,0)$ is a focus and $(\pm 1,0)$ are saddles.

Coming back to the equation (3.3.2), if $s = 0$, it becomes a Hamiltonian system

$$\text{(3.3.4)} \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 \end{cases}$$

with the first integral

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$

The phase portrait of (3.3.4) is shown in Figure 9.

Every closed orbit surrounding (0,0) corresponds to a level curve γ_b: $H(x,y) = b$, $0 < b < 1/4$. On equilibria $(\pm 1,0)$ and heteroclinic loops joining these equilibria, $H(x,y) = 1/4$.

Now we consider (3.3.2) for small $s \neq 0$. Every closed orbit of (3.3.2) should intersect with the interval $U = \{(x,0) \mid 0 \leq x \leq 1\}$ (hence with $-U$) and enclosed the point $B = (0,0)$ since the point B has an index 1 for every s, τ_0 and τ_1. We define w_b for $b \in [0,1/4]$ as follows:

(1) $w_b \in -U \cup U$,

(2) $H(w_b) = b$.

Let $\lambda_s = ((\tau_0,\tau_1),s)$ and W^u be an upper branch of the unstable manifold of (3.3.2) at (1,0). We define a Poincare map $P_{\lambda_s}: U \to -U$ (or $P^{-1}_{\lambda_s}: -U \to U$) in the following way.

Let $b \in (0,1/4]$. Then we can choose $\bar{b} \in (0,1/4]$ such that the points w_b and $w_{\bar{b}}$ are successive intersection points of U and $-U$ respectively with an orbit so that (i) $P_{\lambda_s}: U \to -U$ is defined and $P_{\lambda_s}(w_b) = w_{\bar{b}}$ if $W^u \cap (-U) \neq \{\}$, or (ii) $P^{-1}_{\lambda_s}: -U \to U$ is defined and $P^{-1}_{\lambda_s}(w_{\bar{b}}) = w_b$ if $W^u \cap (-U) = \{\}$. (See Figure 10 (a) and (b).)

Let $\gamma(b,\lambda_s)$ be the orbit of (3.3.2) which joins the points w_b and $w_{\bar{b}}$. Hence $\gamma(b,\lambda_s)$ is defined for $b \in (0,1/4]$. Then we have a lemma.

<u>Lemma (3.3.2)</u>.

(1) Every closed orbit of (3.3.2) is expressed by the form $\gamma(b,\lambda_s)$ with $w_{\bar{b}} = -w_b$.

(2) A trajectory $\gamma = \gamma(b,\lambda_s)$ of (3.3.2) is a periodic orbit if and only if

(3.3.5) $$\int_\gamma \frac{dH(x,y)}{dt}\, dt = 0.$$

In particular γ is a heteroclinic orbit if and only if (3.3.5) is satisfied for $b = 1/4$.

(3) For $s > 0$, condition (3.3.5) is equivalent to

(3.3.6) $$F(b,\lambda_s) \equiv \int_{\gamma(b,\lambda_s)} (\tau_o+\tau_1 x^2+x^4)y\, dx = 0.$$

<u>Proof.</u>

(1) is obvious by the symmetry property

(2) $\int_\gamma \frac{dH(x,y)}{dt}\, dt = H(w_{\bar{b}}) - H(w_b)$ and

$\frac{\partial H(x,y)}{\partial x} = x(1-x^2) \neq 0$ for $x \neq 0$ and $|x| \neq 1$.

Hence $H(w_{\bar{b}}) = H(w_b)$ if and only if $w_{\bar{b}} = -w_b$ for $b\epsilon(0,1/4)$.

(3) is immediate since

$$\frac{dH(x,y)}{dt}\Big|_{(3.3.2)} dt = s^3(\tau_o+\tau_1 x^2+x^4)y^2\Big|_{(3.3.2)} dt$$
$$= s^3(\tau_o+\tau_1 x^2+x^4)ydx. \quad \square$$

Let $I_i(b) = \int_{\gamma_b} x^i y\, dx$, $i = 0,2,4$, where γ_b: $H(x,y) = b$.

We will consider $F(b,\lambda_s)$ as a perturbation of $F(b,\lambda_o) = F(b,(\tau_o,\tau_1))$. The function $F(b,\lambda_o)$ can be written explicitly by

(3.3.7) $$F(b,\lambda_o) = \tau_o I_o(b) + \tau_1 I_2(b) + I_4(b).$$

$$\begin{aligned} s^3F(b,\lambda_s) &= s^3(F(b,\lambda_o) + (F(b,\lambda_s) - F(b,\lambda_o))) \\ &= s^3(F(b,\lambda_o) + o(s)) \\ &= s^3\, F(b,\lambda_o) + o(s^3). \end{aligned}$$

Hence by the Lemma (3.3.2) (3),

$$F(b,\lambda_0) + \epsilon(s) = 0$$

where $\epsilon(s)$ is a smooth function in all variables such that $\epsilon(s) \to 0$ as $s \to 0$. The limiting position of the closed orbits is given by the solution of

$$(3.3.8) \qquad F(b,\lambda_0) = 0 \quad \text{for} \quad s \to 0.$$

From now on, we denote $F(b,\lambda_0) = F(b,0,(\tau_0,\tau_1)) = F(b,(\tau_0,\tau_1))$ simply by $F(b)$ if there is no confusion.

For a moment we study a Hopf bifurcation curve and a symmetric heteroclinic loop bifurcation curve in (τ_0,τ_1)–plane from (3.3.7).

<u>Lemma (3.3.3)</u>. The point (0,0) of (3.3.2) is stable (resp. unstable) if $\tau_0 < 0$ (resp. $\tau_0 > 0$). It has a Hopf bifurcation of order 1 along the line $\bar{H}_f = \{(\tau_0,\tau_1) | \tau_0 = 0\}$ except the point $\bar{h}_2 = (0,0)$ at which a Hopf bifurcation of order 2 occurs. Moreover, there are two limit cycles at (τ_0,τ_1) with $\tau_0 > 0$, $\tau_1 < 0$ around the point $(x,y) = (0,0)$.

<u>Proof</u>. Direct calculation of formulas of the Liapunov's focal values for (3.3.2). (For formulas, see Andronov et al [1], Medved [13].) □

Next, we want to change symmetric heteroclinic loops to a homoclinic loop by using symmetry property as follows:

For $x,y \in \mathbb{R}^2$, we define $x \sim y$ if and only if $x = y$ or $x = -y$. Let $R^{2*} = \mathbb{R}^2/\sim$ with a quotient topology and let $x^* = \{x,-x\}$. Let us regard our symmetric vector field

$$\text{(3.3.2)} \qquad \begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 + s^3 y(\tau_0 + \tau_1 x^2 + x^4) \end{cases}$$

in $\mathbb{R}^2$ as

$$\text{(3.3.9)} \qquad \begin{cases} dx^*/dt = y^* \\ dy^*/dt = x^* + (x^*)^3 + s^3 y^* (\tau_0 + \tau_1 (x^*)^2 + (x^*)^4) \end{cases}$$

in the new phase space $\mathbb{R}^{2*}$. Then (3.3.9) has only one saddle point $(1,0)^* = \{(\pm 1,0)\}$ and a pair of symmetric heteroclinic loops in (3.3.2) correspond to a homoclinic loop at $(1,0)^*$ in (3.3.9). $\mathbb{R}^{2*} - \{(0,0)^*\}$ has a 2–dimensional smooth manifold structure. Note that $\mathbb{R}^{2*}$ itself is not a manifold.

Now let $\mathbb{R}^2_+ = \{(x,y) \mid x>0\}$. Then the phase space $\mathbb{R}^{2*}$ of (3.3.9) can be thought of the half–plane $\mathbb{R}^2_+ \cup$ (y–axis/$\sim$). (See Figure 11.)

Then we can apply (3.3.7) $F(b,\lambda_0) = \tau_0 I_0(b) + \tau_1 I_2(b) + I_4(b)$ to Joyal and Rousseau (pg. 19 of [12]) on saddle quantity. (Also see Roussarie [14].)

<u>Lemma (3.3.4)</u>. The equation (3.3.2) has a heteroclinic loop bifurcation of order 1 along the curve $H_e \doteq \{(\tau_0,\tau_1) \mid \tau_0 + \tau_1/5 + 3/35 = 0\}$ except the point $\bar{c}_2 = (1/7, -8/7)$ at which a symmetric heteroclinic loop bifurcation of order 2 occurs. The curves H_f and H_e intersect transversally at the point $\bar{d} = (0, -3/7)$ which corresponds to a Hopf bifurcation of order 1 with asymmetric heteroclinic loop bifurcation of order 1 simultaneously. (See Figure 13).

Proof. See page 19 of Joyal and Rousseau [12], Roussarie [14], Joyal [11]. Note that the trace of the saddle point (1,0) is $\tau_o + \tau_1 + 1 = 0$ and $\frac{\partial F}{\partial b}(1/4,\bar{c}_2) \neq 0$. □

Next we study a semistable limit cycle bifurcation in (τ_o,τ_1)–space. Equation (3.3.7) defines a surface in the space $(s,(\tau_o,\tau_1))$. The following lemma eliminates the term $I_4(b)$ in (3.3.7).

Lemma (3.3.5). $I_4(b)$ can be expressed in terms of $I_o(b)$ and $I_2(b)$, and

(3.3.10) $7I_4(b) = 8I_2(b) - 4bI_o(b).$

Proof. $H(x,y) = b$ on γ_b. Hence

$$ydy + (x-x^3)dx = 0.$$

$$x^4ydx = y(2y^2+2x^2-4b)dx$$

$$= 2y^3dx + 2x^2ydx - 4bydx, \text{ and}$$

$$y^3dx = d(xy^3) - 3xy^2dy$$

$$= d(xy^3) - 3xy(x^3-x)dx$$

$$= d(xy^3) - 3x^4ydx + 3x^2ydx.$$

Hence

$$x^4ydx = 2/7\ d(xy^3) + 8/7\ x^2ydx - 4/7\ bydx.$$

Taking integration on γ_b gives (3.3.10). □

From (3.3.10), (3.3.7) becomes

(3.3.11) $F(b,(\tau_o,\tau_1)) = (\tau_o - 4b/7)I_o(b) + (\tau_1+8/7)\ I_2(b).$

Note that $I_0(0) = I_2(0) = 0$, $I_0(b) > 0$ for $b \in (0,1/4]$ and $I_2(b)/I_0(b) \to 0$ as $b \to 0$. So the degeneracy of the equation (3.3.11) at $b = 0$ can be removed by changing (3.3.11) into the following:

(3.3.12) $G(b)$ (instead of $G(b,(\tau_0,\tau_1))$) $= \tau_0 - 4b/7 - (\tau_1 + 8/7)P(b)$

where $P(b)$ is defined by

(3.2.13) $$P(b) = \begin{cases} -I_2(b)/I_0(b) & \text{for} \quad b \in (0,1/4] \\ 0 & \text{for} \quad b = 0. \end{cases}$$

Lemma (3.3.6). $P(b)$ defined by (3.3.13) satisfies the equation

(3.3.14) $4b(4b-1)\, P'(b) = 5P^2 + (8b+4)P + 4b.$

Proof. Let $\{l(b),u(b)\} = \gamma_b \cap \{y=0\}$ with $l(b) \leq u(b)$, and

$$J_i(b) = \int_{l(b)}^{u(b)} x^i \varphi(x,b)dx$$

where $\varphi(x,b) = [2b-x^2+x^4/2]^{1/2}$ for $x \in [l(b),u(b)]$.

Then $I_i(b) = 2J_i(b)$.

From the definition of J_i,

$$J_i'(b) = \int_{l(b)}^{u(b)} \frac{x^i}{\varphi(x,b)}dx.$$

So

(3.3.15) $$J_i(b) = \int_{l(b)}^{u(b)} \frac{x^i[\varphi(x,b)]^2}{\varphi(x,b)}dx = 2bJ_i'(b) - J_{i+2}'(b) + \tfrac{1}{2}J_{i+4}'(b).$$

Also, integration by parts gives us

(3.3.16) $J_i(b) = \frac{1}{i+1}\,[J'_{i+2}(b) - J'_{i+4}(b)]$ for all $i \geq 0$.

From (3.3.15) and (3.3.16), eliminating $J'_{i+4}(b)$, we have

$$(i+3)J_i = -J'_{i+2} + 4bJ'_i \quad \text{for all} \quad i \geq 0.$$

In particular,

(3.3.17) $$\begin{cases} 3J_o = -J'_2 + 4bJ'_o \\ 5J_2 = -J'_4 + 4bJ'_2. \end{cases}$$

From (3.3.10),

(3.3.18) $7J'_4(b) = 8J'_2(b) - 4J_o(b) - 4bJ'_o(b)$.

Plugging (3.3.18) into (3.3.17), we have

(3.3.19) $$\begin{cases} 3J_o = 4bJ'_o - J'_2 \\ 15J_2 = 4bJ'_o + (12b-4)J'_2. \end{cases}$$

Solving for J'_o and J'_2 in (3.3.19), we get

(3.3.20) $$\begin{cases} 4b(4b-1)J'_o = 4(3b-1)J_o + 5J_2 \\ 4b(4b-1)J'_2 = 4b\,J_o + 20bJ_2. \end{cases}$$

Hence

$$P'(b) = \frac{-J'_2J_o+J_2J'_o}{J_o^2} = -\frac{J'_2}{J_o} - \frac{J'_o}{J_o}P(b)$$

$$= \frac{5P^2+(8b+4)P+4b}{4b(4b-1)}. \qquad \square$$

Lemma (3.3.7): P(b) also satisfies:

(1) $\lim_{b\to 1/4} P(b) = -1/5$,

(2) $P'(b) < 0$ for $0 < b < 1/4$, $P'(0)$ $-1/2$ and $P'(b) \to -\infty$ as $b \to 1/4$.

Proof. P(b) is a solution of the differential equation (3.3.14) with initial condition $P(0) = 0$. We can interpret (3.3.14) into the form

$$\text{(3.3.21)} \quad \begin{cases} \dot{p} = -5P^2 - (8b+4)P - 4b \\ \dot{b} = 4b - 16b^2. \end{cases}$$

The graph of $P = P(b)$ is the heteroclinic orbit from the saddle (0,0) to the stable node $(1/4, -1/5)$ in the (b,P) – phase plane (See Figure 12.)

The equation

$$\text{(3.3.22)} \quad 5P^2 + (8b+4)P + 4b = 0$$

describes a locus on which the tangential direction of the vector field (3.3.21) is horizontal. The branch of hyperbola (3.3.22) above the line $P = -1/2$ denoted by γ is

$$\text{(3.3.23)} \quad P = \overline{P}(b) \equiv: (-(4b+2) + (16b^2+6b+4)^{1/2})/5.$$

The vector field (3.3.21) is transverse to γ and directed to the right of γ. Calculation from (3.3.14) and (3.3.23) gives

$$\lim_{b\to 0} P'(b) = -1/2 \quad \text{and} \quad \lim_{b\to 0} \overline{P}'(b) = -1.$$

Therefore, the graph of $P = P(b)$ is entirely above γ, i.e., $P'(b) < 0$ for $0 \leq b < 1/4$.

Also we can see that $P'(b) \to -\infty$ as $b \to 1/4$. □

We also need the following lemma.

Lemma (3.3.8). (See also Dumortier et al [7])

$$P''(b) < 0 \quad \text{for} \quad b \epsilon [0,1/4].$$

Proof. From (3.3.14) we get

$(32b-4)P' + 4b(4b-1)P'' = 10PP' + 8P + (8b+4)P' + 4$. So if we solve for P'',

$$(3.3.24) \quad P''(b) = [(10P-24b+8)P' + 8P + 4]/(16b^2-4b)$$

$$\lim_{b\to 0} P''(b) = \lim_{b\to 0} [(10P'-24)P'+(10P-24b+8)P''+8P']/(32b-4)$$
$$= [(-5-24)(-1/2)+8 \lim_{b\to 0} P''(b) + 8(-1/2)]/(-4).$$

(Note that $\lim_{b\to 0} P'(b) = -1/2$.)

Hence $\lim_{b\to 0} P''(b) = -7/8 < 0$.

Now let us suppose that P' has a zero on $[0,1/4)$ and let $b_o = \min \{b\epsilon[0,1/4) | P''(b) = 0\}$. Hence $P''(b) < 0$ for all $x\epsilon[0,b_o)$. Let

L be a tangent line of Γ at $(b_0, p(b_0))$, where Γ is the graph of $P(b)$. Since $P''(b_0) = 0$, the order of contact between L and Γ is at least 2. Let v be a vector orthogonal to L, and $L(u)$ be a linear parametrization of L. Also let

$$f_1(b,P) = 4b - 16b^2,$$
$$f_2(b,P) = -5P^2 - (8b+4)P - 4b, \text{ and}$$
$$f = (f_1,f_2).$$

Then the function $\psi(u) \equiv: < f(L(u)),v >$ has a zero of order at least 1 in u_0 with $L(u_0) = (b_0,p(b_0))$ where $<,>$ is a Euclidean inner product on $\mathbb{R}^2$. As $P''(b) < 0$ on $[0,b_0)$, the corresponding arc of Γ is situated below L. The line L cuts the line $\{b = 0\}$ at a point n_0 above $\alpha_0 = (0,0)$. f is directed downward at n_0. f is directed towards the half plane above L in the neighborhood of b_0 in L with $b < b_0$. Hence $\psi(u)$ must possess a zero at some $u_1(\neq u_0)$ with $L(u_1)\epsilon[n_0,m_0]$ where $n_0 = \{b = 0\} \cap \{L(u)\}$, and $m_0 = (b_0,p(b_0)) = \Gamma \cap \{L(u)\}$.

However, f is quadratic. So $\psi(u)$ is a polynomial of degree 2 in u.

$$\psi(u_0) = \psi'(u_0) = \psi(u_1) = \psi'(u_1) = 0 \quad \text{with} \quad u_0 \neq u_1$$

implies $\psi \equiv 0$, and Γ is a line segment. This contradicts to $P'(0) = -7/8 < 0$. □

Now we consider the problem of the semistable limit cycle bifurcation which is given by $\overline{C}$: $G(b) = G'(b) = 0$ in (τ_0,τ_1)–space.

<u>Lemma (3.3.9)</u>. $\overline{C}$ is a smooth curve which connects the points $\bar{h}_2$ on $\overline{H}_f$ and $\bar{c}_2$ on $\overline{H}_e$ (see Lemmas (3.3.3) and (3.3.4)), and which is tangent to

H_f and H_e at these points respectively. On $\overline{C}$ the semistable limit cycle bifurcation of the equation (3.3.2) occurs.

Proof. From (3.3.2)

$$G(b) = (\tau_o - 4b/7) - (\tau_1 + 8/7)P(b) \quad \text{on} \quad [0,1/4],$$

$$G'(b) = -4/7 - (\tau_1 + 8/7)P'(b), \text{ and}$$

$$G''(b) = -(\tau_1 + 8/7)P''(b).$$

If $G(b) = G'(b) = 0$, then $\tau_1 + 8/7 \neq 0$. Hence by the Lemma (3.3.8), $G''(b) \neq 0$. By the implicit function theorem, there exists $b = b(\tau_1)$ such that $G'(b(\tau_1)) = 0$, and so $\tau_o = \tau_o(b,\tau_1) = \tau_o(b(\tau_1),\tau_1)$ from $G(b) = 0$. Hence $\tau_o = \tau_o(\tau_1)$ is smooth and the semistable limit cycle bifurcation occurs on $\overline{C}$. From $G(b) = G'(b) = 0$, we get

$$(3.3.25) \quad \begin{cases} \tau_o = 4b/7 - \dfrac{4P(b)}{7P'(b)} \\ \tau_1 = -8/7 - \dfrac{4}{7P'(b)}. \end{cases}$$

Note that as $b \to 0$, $P(b) \to 0$, $P'(b) \to -1/2$, and $(\tau_o(b),\tau_1(b)) \to (0,0) = h_2$. Also as $b \to 1/4$, $P(b) \to -1/5$, $P'(b) \to -\infty$, and $(\tau_o(b), \tau_1(b)) \to (1/7, -8/7) = c_2$. From (3.3.25), $\dfrac{d\tau_o}{d\tau_1} = P(b)$ along the curve $\overline{C}$: $\tau_o = \tau_o(\tau_1)$. This implies that $\overline{C}$ is tangent to H_f and H_e at $\bar{h}_2$ and $\bar{c}_2$ respectively. □

Given (τ_o,τ_1), the number of limit cycles of equation (3.3.2) is determined by the number of roots of equation $G(b) = 0$ for $0 < b < 1/4$. If $\tau_1 + 8/7 = 0$, then $G(b) = 0$ if and only if $\tau_o = 4b/7$. So

$\tau_0 \epsilon (0,1/7)$ and $\tau_1 = -8/7$ at which $G(b) = 0$ has a unique solution (note that $G'(b) \neq 0$).

We suppose that $\tau_1 + 8/7 \neq 0$. Then

$$G(b) = (\tau_1 + 8/7)[A(b) - P(b))]$$

where $A(b) = (\tau_0 - 4/7b)/(\tau_1 + 8/7)$ which is linear in b. The roots of $G(b) = 0$ is the intersection of the straight line $P = A(b)$ and the curve $P = P(b)$ on the (b,P)–plane. Since $P''(b) < 0$, the graph of $P = P(b)$ is concave downward. P(b) is independent of τ_0 and τ_1, and $P(0) = 0$, $P(1/4) = -1/5$. A(b) depends on τ_0 and τ_1, and for $\tau_1 + 8/7 > 0$,

(a) $A(0) = 0$ (resp. > 0 or < 0)

$\leftrightarrow (\tau_0, \tau_1) \epsilon \overline{H}_f$(resp. is on the RHS, or LHS of $\overline{H}_f$).

(b) $A(1/4) = -1/5$(resp. $> -1/5$, or $< -1/5$)

$\leftrightarrow (\tau_0, \tau_1) \epsilon H_e$ (resp. is above, or below $\overline{H}_e$).

(c) The straight line $P = A(b)$ is tangent to the curve $P = P(b)$

$\leftrightarrow (\tau_0, \tau_1) \epsilon \overline{C}$ (i.e., $G(b) = G'(b) = 0$ for some $b \epsilon (0,1/4)$).

The bifurcation diagram of equation (3.3.2) is as in Figure 13 and the relative positions between the straight line $P = A(b)$ and the curve $P = P(b)$ is as in Figure 14.

Lemmas (3.3.3), (3.3.4) and the implicit function theorem provide us the following extended results from $s = 0$ to $s > 0$. (Also see Dumortier et. al. [7].)

Lemma (3.3.10). Let K be a compact neighborhood of $\{H(x,y) \leq 1/4\} \cap \{|x| \leq 1\}$ in the (x,y)-plane, and D be a compact neighborhood of the curved region h_2dc_2 in (τ_0,τ_1)-plane. Then there exists $\alpha(D) > 0$ such that the bifurcation diagram of the equation (3.3.2) consists of three surfaces and three curves in $C(D) = (0,\alpha(D)) \times D$ which is as follows up to a diffeomorphism of C(D) equal to the identity at $s = 0$:

(1) $S_{H_f} = (0,\alpha(D)) \times (\overline{H}_f - \{\overline{h}_2\})$ is a surface of Hopf bifurcation of codimension 1,

(2) $S_{H_e} = (0,\alpha(D)) \times (\overline{H}_e - \{\overline{c}_2\})$ is a surface of heteroclinic loop bifurcation of codimension 1.

(3) $S_C = (0,\alpha(D)) \times \overline{C}$ is a surface of semistable limit cycle bifurcation of codimension 1,

(4) $(0,\alpha(D)) \times \{\overline{h}_2\}$ and $(0,\alpha(D)) \times \{\overline{c}_2\}$ are curves of Hopf and heteroclinic loop bifurcation of codimension 2 respectively.

(5) $(0,\alpha(D)) \times \{\overline{d}\} = S_{H_f} \cap S_{H_e}$ is a curve of simultaneous Hopf bifurcation and heteroclinic loop bifurcation.

Outside these bifurcation sets, the topological type of the phase portraits of the equation (3.3.2) is constant in K.

For $\overline{s}\epsilon(0,\alpha(D))$, we denote the intersection of the bifurcation diagram of equation (3.3.2) with the plane $\{(s,(\tau_0,\tau_1))|s = \overline{s}\}$ by $W_{\overline{s}}$. Then $W_{\overline{s}}$ has a cone structure (see Figure 15).

The bifurcation diagram for the equation $(3.2.9)_+$ with $\epsilon_1 < 0$ can be constructed from Lemma (3.3.10). The blowing-up (3.3.1) with $\eta = -1$ gives a transformation $\phi:(s,(\tau_0,\tau_1)) \to (\epsilon_1,\epsilon_2,\epsilon_3)$, and

(3.3.26) $\phi((0,\alpha(D)) \times D) = \{(-s^2,s^4\tau_0,s^2\tau_1) | s\epsilon(0,\alpha(D)),\ (\tau_0,\tau_1)\epsilon D\}.$

Let $E_{\epsilon_1}(D)$ be the RHS of (3.3.26). Then $E_{\epsilon_1}(D)$ is a cone in $(\epsilon_1,\epsilon_2,\epsilon_3)$–space around the axis $0\epsilon_1$ for $\epsilon_1 < 0$ (see Figure 15).

The bifurcation diagram of $(3.2.9)_+$ in $E_{\epsilon_1}(D)$ is the image of those sets described in Lemma (3.3.10) by the transformation and thus homeomorphic to cones based on $\bar{H}_f,\bar{H}_e,\bar{C},\bar{h}_2,\bar{c}_2$, and $\bar{d}$ with curves $s \to (s^2\eta,s^4\tau_0,s^2\tau_1)$ with $\eta = -1$, or equivalently $\epsilon_1 \to (\epsilon_1,\epsilon_1^2\tau_0,(-\epsilon_1)\tau_1)$ for $\epsilon_1 < 0$.

Now we will study the behavior of $(3.2.9)_+$ in a sector around $0\epsilon_3$–axis for $\epsilon_1 \leq 0$. In the blowing–up (3.3.1), we take $\tau_1 = \pm 1$ instead of $\eta = -1$. Since both cases $\tau_1 = 1$ and $\tau_1 = -1$ are similar, we will consider the case $\tau_1 = 1$. By (3.3.1) with $\tau_1 = 1$, equation $(3.2,9)_+$ becomes

(3.3.27) $$\begin{cases} \dot{x} = y \\ \dot{y} = \eta x + x^3 + s^3(\tau_0+x^2+x^4)y. \end{cases}$$

Let $s_1 > 0$ be fixed. Then for each $s\epsilon(0,s_1]$ we take a blowing–up again:

$$x \to rx \qquad \eta = -r^2$$
$$y \to r^2y \qquad \tau_0 = r^2\bar{\tau}_0$$
$$t \to t/r$$

Then (3.3.27) becomes

$$(3.3.28)\quad \begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 + s^3(r(\bar{\tau}_0+x^2)y + o(r^3)) \end{cases}$$

which is a perturbation of the Hamiltonian system $(r = 0)$

$$(3.3.29)\quad \begin{cases} \dot{x} = y \\ \dot{y} = -x + x^3 \end{cases}$$

with first integral

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$

As in Lemma (3.3.2), we have

$$H(w_{\bar{b}}) - H(w_b) = s^3 r(\tilde{F}(b,\bar{\tau}_0) + o(r))$$

where $\tilde{F}(b,\bar{\tau}_0) = \bar{\tau}_0 I_0(b) + I_2(b)$. Finally it leads to study $\tilde{F}(b),\bar{\tau}_0) = 0$, and get to the following conclusion:

Lemma (3.3.11). In the halfplane $\{\eta,\tau_0,\tau_1)\,|\,\eta \le 0,\ \tau_1 = 1\}$ there is a fixed compact subset B^+, diffeomorphic to a disk having a contact of order 1 with axis $0\tau_0$ at $(\eta,\tau_0,\tau_1) = (0,0,1)$, and such that for equation (3.3.27) the results of Carr–Takens (see page 54–81 and, in particular, Figure 3 on page 59 of Carr[3], also Guckenheimer and Holmes [10], Takens [17] and [18]) are valid for any $(\eta,\tau_0) \in B^+$ and any $s\in(0,s_1]$. (See Figure 16.) (3.3.1) with $\tau_1 = 1$ gives a mapping $(s,\eta,\tau_0) \to (\epsilon_1,\epsilon_2\epsilon_3)$ which maps $(0,s_1] \times B^+$ to $E^+_{\epsilon_3} = \{(s^2\eta,s^4\tau_0,s^2)\,|\,s\in(0,s_1],\ (\eta,\tau_0)\in B^+_0\}$. $E^+_{\epsilon_3}$ is a cone in $(\epsilon_1,\epsilon_2,\epsilon_3)$–space around $0\epsilon_3$–axis for $\epsilon_3 > 0$ based on B^+. The bifurcation diagram of (3.3.2) in $E^+_{\epsilon_3}$ consists of cones based on $\bar{\bar{H}}_f$, $\bar{\bar{H}}_e$ and $\{\bar{b}_1\}$ with generating curves $s \to (s^2\eta, s^4\tau_0, s^2)$. (See Figure 17). For $\tau_1 = -1$, we can get a cone $E^-_{\epsilon_3}$ around $0\epsilon_3$–axis for $\epsilon_3 < 0$ based on $\overline{H}_f$, $\overline{H}_e$ and $\{\overline{b}_2\}$ Similarly.

Proof. See Carr [3]. □

Proof of Theorem (3.3.1). Let $E^+_{\epsilon_3}$ and $E^-_{\epsilon_3}$ be the two cones from B^+ and B^- respectively as above. We can choose a compact set D in the (τ_0,τ_1)–plane to use Lemma (3.3.10) in such a way that (see Figure 18.)

(1) $E_{\epsilon_1}(D) \cup E^+_{\epsilon_3} \cup E^-_{\epsilon_3}$ contains a cone $C(M)$ based on a disc M in the hemisphere S,

(2) ∂M is tangent to ∂S at the point $b_1 = (0,0,\epsilon_0)$ and $b_2 = (0,0,-\epsilon_0)$,

(3) M contains the curves $S_{H_f} \cap S$, $S_{H_e} \cap S$, and $S_C \cap S$ where S_{H_f}, S_{H_e} and S_C are defined in Lemma (3.3.10).

Condition (3) is possible because the curve of Hopf bifurcation and the curve of heteroclinic loop bifurcation in $M \cap E^+_{\epsilon_3}$ are connected with the curves $H_f = S_{H_f} \cap S$ and $H_e = S_{H_e} \cap S$ respectively.

To show this, we consider the equations of curves H_f and H_e. From Lemmas (3.3.3) and (3.3.4), we have

(3.3.30) $$H_f: \begin{cases} \tau_0 = 0 \\ \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = \epsilon_0^2 \end{cases} \quad \text{and} \quad H_e: \begin{cases} \tau_0 = -1/5\tau_1 - 3/35 \\ \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = \epsilon_0^2. \end{cases}$$

From (3.3.1)

$$(3.3.31)\quad \begin{aligned} s &= (-\epsilon_1)^{1/2} \\ \tau_0 &= \epsilon_2/(-\epsilon_1)^2 \\ \tau_1 &= \epsilon_3/(-\epsilon_1), \end{aligned}$$

where $\epsilon_1 < 0$. Substituting (3.3.31) into (3.3.30) we obtain

H_f: $\epsilon_1^2 + \epsilon_3^2 = \epsilon_0^2$ and H_e: $\epsilon_1^2 + (\frac{\epsilon_1\epsilon_3}{5} - \frac{3\epsilon_1^2}{35})^2 + \epsilon_3^2 = \epsilon_0^2$ respectively.

Let $\epsilon_1 \to 0$. Then $\epsilon_3 \to \pm\, \epsilon_0$.

This implies that if $\epsilon_1 \to 0$, then the curve H_f and H_e tend to the points b_1 and b_2 in ∂S, hence that H_f and H_e are connected with the Hopf bifurcation curve and the heteroclinic loop bifurcation curve in $M \cap E^{\pm}_{\epsilon_3}$.

Thus we can choose D and M satisfying the condition (1), (2) and (3). The conclusion of the Theorem, for ϵ near $(0,0,\pm\epsilon_0)$ with $\epsilon_1 < 0$, follows from Lemma (3.3.10) and Lemma (3.3.11), and for $\epsilon_1 > 0$ or $\epsilon_1 = 0$ but $\epsilon \neq (0,0,\pm\epsilon_0)$ are obvious. □

§4. The Case $a < 0$

$(3.2.9)_-$ is as follows:

$$(3.2.9)_- \quad \begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 x - x^3 + y(\epsilon_2 + \epsilon_3 x^2 + x^4). \end{cases}$$

The equilibria in $(3.2.9)_-$ is determined by the equation $y = 0$ and $x(\epsilon_1 - x^2) = 0$. Hence for $\epsilon_1 < 0$ $(0,0)$ is a focus, for $\epsilon_1 > 0$ $(0,0)$ is a saddle and $(\pm\sqrt{\epsilon_1}, 0)$ are foci.

Let the RHS of $(3.2.9)_-$ be $f_\epsilon(x,y)$. Then

$$Df_\epsilon(x,0) = \begin{bmatrix} 0 & , & 1 \\ \epsilon_1 - 3s^2 & , & \epsilon_2 + \epsilon_3 x^2 + x^4 \end{bmatrix}.$$

It is immediate that $\{\epsilon_1 = 0\}$ is a pitchfork bifurcation surface. We will consider two cases ($\epsilon_1 > 0$ and $\epsilon_1 < 0$) and by the same way as in the previous section, we will use a hemisphere section to be easy to understand the bifurcation diagram.

Next, we combine the above two results to get our complete bifurcation diagram on the sphere.

First let us consider the case $\epsilon_1 > 0$.

Let

$$S = \{(\epsilon_1, \epsilon_2, \epsilon_3) = \epsilon \mid 0 < |\epsilon| = \epsilon_0 << 1\},\ S^+ = S \cap \{\epsilon_1 \geq 0\}.$$

The bifurcation diagram of $(3.2.9)_-$ is a cone based on its trace with S. This trace on S consists of 4 curves:

(1) H_f: Hopf bifurcation curve,

(2) H_o: (symmetric) homoclinic loop bifurcation curve.

(3) $C = C_1 \cup C_2 \cup C_3 \cup C_4$: semistable limit cycle bifurcation curve,

and

(4) P: pitchfork bifurcation curve $(\epsilon_1 = 0)$.

The curve C_1 joins a point a on H_f to a point e on H_o, and it is tangent to H_f and H_o at these points respectively. Also the curve C_2 joints a point b_1 on P to a point e on H_o, and it is tangent to H_o at e, however, intersects with P transversally at b_1. The curve C_3 (resp. C_4) joins a point f to a point b_2(resp. b_3) on P. C_3(resp. C_4) intersects with

P transversally at b_4 (resp. b_3) and C_3 meets C_4 tangentially at f. (See Figure 19.) In the neighborhood of b_1 and b_2 we have the degenerate focus bifurcation of codimension 2. (See Figure 6 (b).)

The point a (resp. e, f) corresponds to a Hopf (resp. homoclinic loop, triple limit cycle) bifurcation of codimension 2. At points b_3,c,d,g and h two corresponding bifurcations occur simultaneously.

Theorem (3.4.1). Let $\Sigma = (S \cap \{\epsilon_1 = 0\}) \cup H_f \cup H_o \cup C$ be a subset of S^+. The bifurcation diagram of (3.2.9)_ in the ball

$$B^+ = \{\epsilon = (\epsilon_1,\epsilon_2,\epsilon_3) | \ |\epsilon| \leq \epsilon_o, \epsilon_1 \geq 0\}$$

is a cone hemeomorphic to

$$\{(s^2\eta, s^4\tau_o, s^2\tau_1) | s\epsilon[0,\epsilon_o], \ (\eta,\tau_o,\tau_1)\epsilon\Sigma\}.$$

The topological type of the phase portraits of equation (3.2.9)_ in a neighborhood of (0,0) in $\mathbb{R}^2$ is constant in each connected component surrounded by the bifurcation surfaces (10 components: $R_1,\dots,R_{10}$, $R_1 \cap S^+ = I,\dots,R_{10} \cap S^+ = X$) and is constant in each bifurcation surfaces (19 surfaces. $S_1,\dots,S_9$, $S_1 \cap S^+ = ab_\infty,\dots,S_{19} \cap S^+ = eg$) and curves (10 curves: $C_1,\dots,C_{10}$, $C_1 \cap S^+ = \{b_1\},\dots,C_{10} \cap S^+ = \{b_3\}$). (See Figure 19 (a)–(d).)

Proof will be given later.

Next let us consider the case $\epsilon_1 < 0$. Let $S^- = S \cap \{\epsilon_1 \leq 0\}$. As before the trace of bifurcations on S^- consists of 3 curves:

(1) H_f: Hopf bifurcation curve,

(2) C: semistable limit cycle bifurcation curve, and

(3) P: pitchfork bifurcation curve ($\epsilon_1 = 0$).

In this case ($\epsilon_1 < 0$), the Hopf bifurcation curve is described by $\epsilon_2 = 0$ and connects b_1 and b_2 at which Carr–Takens bifurcation of codimension 2 occur. C is tangent to H_f at k and connects k on H_f and b_3 on P. Note that b_1, b_2 and b_3 in Figure 20 (a) are same points as those in Figure 19 (a) respectively.

<u>Theorem (3.4.2)</u>. Let $\Sigma = (S \cap \{\epsilon_1 = 0\}) \cup H_f \cup C$ be a subset of S. The bifurcation diagram of $(3.2.9)_-$ in the ball $B^- = \{\epsilon = (\epsilon_1,\epsilon_2,\epsilon_3) | \; |\epsilon| \leq \epsilon_0, \; \epsilon_1 \leq 0\}$ is a cone homeomorphic to $\{(s^2\eta, \; s^4\tau_0, \; s^2\tau_1) | s\epsilon[0,\epsilon_0], \; (\eta,\tau_0,\tau_1)\epsilon\Sigma\}$. The topological type of the phase portraits of equation $(3.2.9)_-$ in a neighborhood of (0,0) in $\mathbb{R}^2$ is constant in each connected component surrounded by the bifurcation surfaces (3 components: R_1,R_2,R_3, $R_1 \cap S = I$, $R_2 \cap S = II$, $R_3 \cap S = III$) and is constant in each bifurcation surfaces (6 surfaces: $S_1,\ldots,S_6$, $S_1 \cap S = b_1k$, $S_6 \cap S = b_1b_3$) and curves (4 curves: $C_1,\ldots,C_4$, $C_1 \cap S = \{k\},\ldots,C_4 \cap S = \{b_3\}$). (See Figure 20 (a)–(d)).

For the proof of this Theorem, see the proof of Theorem (3.4.1).

A. $\epsilon_1 > 0$.

We use the blowing–up technique (3.3.1) for $\epsilon_1 > 0$. First we investigate the behavior of $(3.2.9)_-$ in a neighborhood of $0\epsilon_1$–axis for $\epsilon_1 \geq 0$. Hence let $\eta = 1$ in (3.3.1).

Then the equation (3.2.9) has the form

$$(3.4.1)\qquad \begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + s^3(\tau_0+\tau_1 x^2+x^4)y. \end{cases}$$

Equation (3.4.1) has the equilibria (0,0) and (±1,0) where (0,0) is a saddle and (±1,0) are foci. If $s = 0$, (3.4.1) becomes a Hamiltonian system

$$(3.4.2)\qquad \begin{cases} \dot{x} = y \\ \dot{y} = x-x^3 \end{cases}$$

with the first integral $H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$.

The phase portrait of (3.4.2) is shown in Figure 21.

Closed orbits surrounding $A = (-1,0)$ or $C = (1,0)$ (type 1) correspond to level curves γ_b: $H(x,y) = b$, $-1/4 < b < 0$ and those surrounding $A,B = (0,0)$ and C at the same time (type 2) correspond to level curves γ_b: $H(x,y) = b$, $0 < b < \infty$. $\gamma_{-1/4} = \{A,C\}$. γ_0 corresponds to a level curve of a figure–eight homoclinic orbit.

Now we consider (3.4.1) for small $s \neq 0$. Every closed orbit of (3.4.1) should intersect with the interval $U = \{(x,0)\,|\,x\geq 1\}$ and/or $-U$. Since closed orbits of type 1 enclosing A is a $1-1$ correspondence to those of type 1 enclosing C, we only consider the latter.

We define w_b for $b\epsilon[-1/4,\infty)$ as follows:

(a) $w_b \epsilon U$, (b) $H(w_b) = b$.

Let $\lambda_s = ((\tau_0,\tau_1),s)$ and W^s(resp. W^u) be a stable (resp. unstable) manifold of (3.4.1) at B in the right half–plane. We define a Poincare map

$P_{\lambda_s} : U \to U$ in the following way:

Let $\{w_{bs}\} = W^s \cap U$ and $\{w_{bu}\} = W^u \cap U$. For $b \epsilon [-1/4,\infty)$, we can choose $\bar{b} \epsilon [-1/4,\infty)$ such that the point w_b and $w_{\bar{b}}$ are successive intersection points of U with an orbit so that

$P_{\lambda_s} : U \to U$ is defined and $P_{\lambda_s}(w_b) = w_{\bar{b}}$, $P_{\lambda_s}(w_{bs}) = w_{bu}$ if

$H(w_{bs}) < H(w_{bu})$, and

$P^{-1}_{\lambda_s} : U \to U$ is defined and $P^{-1}_{\lambda_s}(w_{\bar{b}}) = w_b$, $P^{-1}_{\lambda_s}(w_{bu}) = w_{bs}$ if

$H(w_{bu}) < H(w_{bs})$. (See Figure 22 (a) and (b).)

Let $\gamma(b,\lambda_s)$ be the orbit of (3.4.1) which joins the points w_b and $w_{\bar{b}}$. Hence $\gamma(b,\lambda_s)$ is defined for $b \epsilon (-1/4,\infty)$. Then we have the lemma.

Lemma (3.4.3).

(1) Every closed orbit of (3.4.1) is expressed by the form $\gamma(b,\lambda_s)$ with $w_{\bar{b}} = w_b$.

(2) A trajectory $\gamma = \gamma(b,\lambda_s)$ of (3.4.1) is a periodic orbit if and only if

$$(3.4.3) \qquad \int_\gamma \frac{dH(x,y)}{dt} dt = 0.$$

In particular γ is a homoclinic orbit if and only if (3.4.3) is satisfied for $b = 0$.

(3) For $s > 0$, condition (3.4.3) is equivalent to

$$(3.4.4) \qquad F(b,\lambda_s) \equiv: \int_{\gamma(b,\lambda_s)} (\tau_0 + \tau_1 x^2 + x^4)\, y dx = 0.$$

$P_{\lambda_s}: U \to U$ in the following way:

Let $\{w_{bs}\} = W^s \cap U$ and $\{w_{bu}\} = W^u \cap U$. For $b\epsilon[-1/4,\infty)$, we can choose $\bar{b}\epsilon[-1/4,\infty)$ such that the point w_b and $w_{\bar{b}}$ are successive intersection points of U with an orbit so that

$P_{\lambda_s}: U \to U$ is defined and $P_{\lambda_s}(w_b) = w_{\bar{b}}$, $P_{\lambda_s}(w_{bs}) = w_{bu}$ if $H(w_{bs}) < H(w_{bu})$, and

$P_{\lambda_s}^{-1}: U \to U$ is defined and $P_{\lambda_s}^{-1}(w_{\bar{b}}) = w_b$, $P_{\lambda_s}^{-1}(w_{bu}) = w_{bs}$ if $H(w_{bu}) < H(w_{bs})$. (See Figure 22 (a) and (b).)

Let $\gamma(b,\lambda_s)$ be the orbit of (3.4.1) which joins the points w_b and $w_{\bar{b}}$. Hence $\gamma(b,\lambda_s)$ is defined for $b\epsilon(-1/4,\infty)$. Then we have the lemma.

Lemma (3.4.3).

(1) Every closed orbit of (3.4.1) is expressed by the form $\gamma(b,\lambda_s)$ with $w_{\bar{b}} = w_b$.

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(3) For $s > 0$, condition (3.4.3) is equivalent to

$$(3.4.4) \qquad F(b,\lambda_s) \equiv: \int_{\gamma(b,\lambda_s)} (\tau_0+\tau_1x^2+x^4)\, ydx = 0.$$

Proof. See the proof of Lemma (3.3.2). □

Let $I_i(b) = \int_{\gamma_b} x^i y dx$, $i = 0,2,4$, where γ_b: $H(x,y) = b$.
We will consider $F(b,\lambda_s)$ as a perturbation of $F(b,\lambda_o)$. The function $F(b,\lambda_o)$ can be written explicitly by

(3.4.5) $$F(b,\lambda_o) = \tau_o I_o(b) + \tau_1 I_2(b) + I_4(b).$$

$$\begin{aligned} s^3 F(b,\lambda_s) &= s^3(F(b,\lambda_o) + (F(b,\lambda_s) - F(b,\lambda_o))) \\ &= s^3\ (F(b,\lambda_o) + O(s)) \\ &= s^3\ F(b,\lambda_o) + o(s^3). \end{aligned}$$

Hence by (3) of the Lemma (3.4.3),

$$F(b,\lambda_o) + \epsilon(s) = 0$$

where $\epsilon(s)$ is a smooth function in all variables such that $\epsilon(s) \to 0$ as $s \to 0$. The limiting position of the closed orbits is given by the solution of

(3.4.6) $$F(b,\lambda_o) = 0 \quad \text{for} \quad s \to 0.$$

From now on, we denote $F(b,(\tau_o,\tau_1)) \equiv: F(b,\lambda_o) = F(b,(\tau_o,\tau_1),0)$. (Also $F(b)$ instead $F(b,(\tau_o,\tau_1))$ if no confusion.)

Equation (3.4.6) defines a surface in the $(s,(\tau_o,\tau_1))$–space. The following lemma eliminates the term $I_4(b)$ in (3.4.5).

Lemma (3.4.4).

(3.4.7) $$7I_4(b) = 8I_2(b) + 4bI_o(b).$$

Proof. See the proof of Lemma (3.3.5). □

From (3.4.7), (3.4.5) becomes

(3.4.8) $F(b,(\tau_o,\tau_1)) = (\tau_o+4b/7)I_o(b) + (\tau_1+8/7)I_2(b).$

Note that $I_o(-1/4) = I_2(-1/4) = 0$, $I_o(b) > 0$ for $b > -1/4$ and $I_2(b)/I_o(b) \to 1$ as $b \to -1/4$. So we can change (3.4.8) into the following:

(3.4.9) $G(b)$ (instead $G(b,(\tau_o,\tau_1))) = \tau_o + 4b/7 + (\tau_1+8/7)P(b)$

where $P(b)$ is defined by

(3.4.10) $$P(b) = \begin{cases} I_2(b)/I_o(b) & \text{for } b > -1/4 \\ 1 & \text{for } b = -1/4 \end{cases}.$$

Lemma (3.4.5). $P(b)$ defined in (3.4.10) satisfies

(3.4.11) $4b(4b+1)P'(b) = 5P^2 + (8b-4)P - 4b.$

Proof. See the proof of Lemma (3.3.6). □

Lemma (3.4.6). $\lim_{b\to\infty} P(b) = \infty.$

Proof. (Carr [3]) Let $J_i(\beta) = \int_0^\beta x^i y\, dx$ where $\beta > 0$ is a maximum of the solutions to $\beta^4 - 2\beta^2 = 4b$ and $y = (2b+x^2-x^4/2)^{1/2}$. Then

$P(b) = J_2(\beta)/J_o(\beta)$. Let $x = h\beta$.

Then

$$J_i(\beta) = \beta^2 \int_0^1 (\beta h)^i \varphi(h)dh, \ i = 0,2,$$

where

$$\varphi(h) = [\frac{\beta^2}{2}(1-h^4) + (h^2-1)]^{1/2}.$$

Since $g(h) \leq g(1/\beta)$ for $0 \leq h \leq 1$, $J_o(\beta) \leq C_1\beta^3$ for some constant $C_1 > 0$. Also we have $J_2(\beta) \geq C_2\beta^5$ for some constant $C_2 > 0$. Hence $\lim_{b\to\infty} P(b) = \lim_{\beta\to\infty} J_2(\beta)/J_o(\beta) = \infty$. □

<u>Lemma (3.4.7)</u>. $P(b)$ also has the following properties:

(1) $\lim_{b\to 0} P(b) = 4/5$,

(2) There exists $b^* > 0$ such that

$P'(b) < 0$ for $b\epsilon[-1/4,b^*) - \{0\}$ and

$P'(b) > 0$ for $b > b^*$.

Also b^* satisfies: $P(b^*) > 1/2$, $P''(b^*) > 0$.

<u>Proof</u>. Rewrite (3.4.11) into the form

$$\text{(3.4.12)} \quad \begin{cases} \dot{b} = 4b(4b+1) \\ \dot{P} = 5P^2 + (8b-4)P - 4b \end{cases}$$

where '$\cdot$' $= \frac{d}{dt}$.

Since $P(-1/4) = 1$, the graph of $P = P(b)$ for $b\epsilon[-1/4,0]$ is the heteroclinic orbit between the saddle $(-1/4,1)$ and the node $(0,4/5)$ in the (b,P)-plane. (See Figure 23.) Hence $\lim_{b\to 0} P(b) = 4/5$. We denote two branches of the

hyperbola $5P^2 + (8b-4)P - 4b = 0$ by $\overline{P}(b)$ (upper) and $\overline{\overline{P}}(b)$ (lower).

$b = 0$, $b = -1/4$, $P = \overline{P}(b)$ and $P = \overline{\overline{P}}(b)$ divide (b,P)–plane into 9 regions in each of which $\frac{dP}{db}$ has a constant sign. We denote 4 regions among them (which are interested in) by A,B, C, and D as in Figure 23.

First one can show that

$$\lim_{b\to-1/4} P'(b) = -1/2, \quad \lim_{b\to 0} P'(b) = -\infty,$$
$$\lim_{b\to-1/4} \overline{P}'(b) = -1, \quad \text{and} \quad \lim_{b\to 0} \overline{P}'(b) = -3/5.$$

Hence the graph of $P = P(b)$ must stay in region A for $-1/4 < b < 0$ and enter into region D for $0 < b << 1$. In regions A and D, $\frac{dP}{db} < 0$. But $P(b) \to \infty$ as $b \to \infty$ and $\overline{P}(b) \to 1/2$ as $b \to \infty$.

Thus there exists $b^* > 0$ such that $P(b^*) = \overline{P}(b^*)$ (hence $P'(b^*) = 0$) and $P'(b) > 0$ for $b > b^*$. Since $\overline{P}'(b) < 0$ and $\overline{P}(b) \to 1/2$ as $b \to \infty$, $P(b^*) = \overline{P}(b^*) > 1/2$. By (3.4.11),
$4b(4b+1)P'' = (10P-24b-8)\,P' + 8(P-1/2)$. So we have
$4b^*(4b^*+1)P''(b^*) = 8(P(b^*)-1/2) > 0$ which implies $P''(b^*) > 0$. □

<u>Lemma (3.4.8)</u>. $P(b) \sim c\sqrt{b}$ as $b \to \infty$ for some $c > 0$.

<u>Proof</u>. From (3.4.11) $P'(b) = (5P^2+(8b-4)P-4b)/(4b(4b+1))$, we easily see that it can not be that $P(b) \sim ce^{lb}$ as $b \to \infty$ for any $c,l\in\mathbb{R}-\{0\}$. Hence let $P(b) \sim cb^r$ $(c\neq 0)$ as $b \to \infty$ for some $r\in\mathbb{R}$. Also let
$A = 5P^2 + (8b-4)P - 4b$, $B = 4b(4b+1)$.
$A/B = \{5(P/b)^2 + (8-4/b)(P/b)-4/b\}/\{4(4+1/b)\}$.
$P'(b) \sim crb^{r-1}$ as $b \to \infty$.

(1) If $r > 1$, $A/B \sim \frac{5c^2b^{2r-2}}{16}$ as $b \to \infty$.

Hence $crb^{r-1} = \frac{5c^2b^{2r-2}}{16}$.

This implies $r - 1 = 2r - 2$ and thus $r = 1$ which contradicts to $r > 1$.

(2) If $r = 1$, $A/B \sim \frac{5c^2+8c}{16}$ as $b \to \infty$.

Hence $c = \frac{5c^2+8c}{16}$. Since $c \neq 0$, $c = 8/5$, i.e., $P(b) \sim 8b/5$ as $b \to \infty$.

(a) $P'(b) \to 8/5^+$ as $b \to \infty$.

Since $P'(b^*) = 0$, there must be $\tilde{b} > b^*$ such that $P''(\tilde{b}) = 0$ and $P'(\tilde{b}) > 8/5$ but $P'''(\tilde{b}) \leq 0$. (See Figure 24 (a).) Note that $P'(b) > 0$ for $b > b^*$ by Lemma (3.4.7). $P' = A/B$ implies $P'B + P''B' = A'$ and so $P'''B + 2P''B' + P'B'' = A''$. Hence at $b = \tilde{b}$, $BP''' = (10P'-16)P'$. Since $P'(\tilde{b}) > 8/5$, $P'''(\tilde{b}) > 0$, which gives a contradiction to $P'''(\tilde{b}) \leq 0$. Thus $P'(b) \leq 8/5$ for any $b \in (b^*,\infty)$, and there is no local maximum since, if there is, then there must be a local minimum point $\tilde{\tilde{b}} \in (b^*,\infty)$, so $0 < P'(\tilde{\tilde{b}}) < 8/5$, $P''(\tilde{\tilde{b}}) = 0$, $P'''(\tilde{\tilde{b}}) \geq 0$, however, $BP''' = (10P'-16)P' < 0$ at $b = \tilde{\tilde{b}}$ so that $P'''(\tilde{\tilde{b}}) < 0$ which is a contradiction to $P'''(\tilde{\tilde{b}}) > 0$.

(b) $P'(b) \to 8/5^-$ as $b \to \infty$.

Since there is no local maximum of $y = P'(b)$ on (b^*,∞), $y = P'(b)$ is monotonically increasing. Also we can get a contradiction easily if we assume that there is an inflection point in the graph of $y = P'(b)$ on (b^*,∞). So $P''(b) > 0$ on (b^*,∞). (See Figure 24 (b).)

$BP'' = (10P-24b-8)P' + 8P-4$

$$\text{(3.4.13)} \quad \frac{B^2P''}{2} = (5P-12b-4)\,(5P^2+(8b-4)P-4b) + (4P-2)\,(16b^2+4b).$$

Let $f(b,P)$ be the RHS of (3.4.13) and let $x = P/b$. Then

$$\frac{f(b,P)}{b^3} = (5x-12-4/b)(5x^2+(8-4/b)x-4/b) + (4x-2/b)(16x^2+4/b)$$

$$= x(5x+4)(5x-8) - (4/b)(18x^2-3x-12) + (8/b^2)(2x+1).$$

Since for $b >> 1$, $x(b) = \frac{P(b)}{b}$ is less than $8/5$ but near $8/5$, we have $x(5x+4)(5x-8) < 0$, and $-(4/b)(18x^2-3x-12) + (8/b^2)(2x+1) < 0$. Thus $f(b,P) < 0$ for $b >> 1$. From (3.4.13) this implies that $P'' < 0$ for $b >> 1$, a contradiction.

(c) $P'(b) = 8/5$, $b \geq b_o$ for some $b_o >> 1$.

Then $P^{(n)}(b_o) = 0$ for $n = 2,3,....$ (See Figure 24 (c).)

From $BP'' = (10P-24b-8)P'+8P-4$, at $b = b_o$

$$0 = (10P-24b-8)(8/5) + 8P-4$$

which gives $P = 8b/5 + 7/10$.

While from $P'(b) = A/B$, we have

$$25P^2 + 20(2b-1)P - 4b(32b+13) = 0.$$

If we solve the following system

$$\begin{cases} P = (8b/5) + 7/10 \\ 25P^2 + 20(2b-1)P - 4b(32b+13) = 0, \end{cases}$$

then the solution is $(b_o,P(b_o)) = (-1/4, 3/10)$, which also gives a contradiction because $b_o > b^* > 0$.

By (1) and (2), $r < 1$.

In this case, $A/B \sim \frac{8c}{16} b^{r-1}$ as $b \to \infty$.

Hence $\frac{8c}{16} b^{r-1} = crb^{r-1}$, and so

$$r = 1/2. \qquad \square$$

<u>Corollary (3.4.9)</u>. $\lim_{b\to\infty} P'(b) = 0.$

<u>Proof</u>. Easy from Lemma (3.4.8). □

<u>Lemma (3.4.10)</u>. If $P''(b) = 0$ for $b > b^*$, then $P'''(b) \neq 0$.

<u>Proof</u>. $P''(b) = 0$ gives $BP'''(b) = 2(5P'(b)-8)P'(b)$. Suppose $P''(b) = 0 = P'''(b)$ for some $b > b^*$. Then $P'(b) = 8/5$.

From $BP'' = (10P-24b-8)P' + 8P-4$, we have

$$P = 8b/5 + 7/10.$$

From $P' = A/B$,

$$25P^2 + 20(2b-1)P - 4b(32b+13) = 0.$$

There is no solution on (b^*,∞) which satisfies both above. Hence if $P''(b) = 0$ for some $b > b^*$, then $P'''(b) \neq 0$. □

<u>Lemma (3.4.11)</u>. There exists $b^{**}\epsilon(b^*,\infty)$ such that $P''(b) > 0$ for $b\epsilon(0,b^{**})$ and $P''(b) < 0$ for $b\epsilon(b^{**},\infty)$, but $P'''(b^{**}) \neq 0$.

<u>Proof</u>. Existence follows from Lemmas (3.4.7) and (3.4.8) since $P''(b^*) > 0$ and $P''(b) < 0$ for $b >> 1$. Note that from

$$BP'' = (10P-24b-8)P' + 8P-4$$
$$= (10(P-4/5) - 24b)P' + 8(P-1/2),$$

on $(0,b^*)$, since $1/2 < P < 4/5$ and $P' < 0$, we have $P'' > 0$. This says, inflection points exist in (b^*,∞). To complete the proof, we will show the uniqueness: i.e., $P''(b_1) = P''(b_2) = 0$, b_1, $b_2\epsilon(b^*,\infty)$ implies $b_1 = b_2$.

If $P''(b) = 0$ at more than one point, then there must exist at least three points (and an odd number of points) since $P''(b^*) > 0$ and $P''(b) < 0$

for $b >> 1$ and $P'''(b) \neq 0$ if $P''(b) = 0$ (Lemma (3.4.10)).

Suppose $P''(b_i) = 0$, $i = 1,2,3$ with $b_3 > b_2 > b_1 > b^*$ and $P''(b) \neq 0$ for all $b \in (0,b_3) - \{b_1,b_2\}$. Then $P'''(b_1) < 0$, $P'''(b_3) < 0$ and $P'''(b_2) > 0$ (see Figure 25).

At $b = b_2$,

$$0 < BP'''(b_2) = (10P'(b_2) - 16)P'(b_2)$$

which implies that $P'(b_2) > 8/5$. On the other hand, at $b = b_i (i=1,3)$,

$$0 < P'(b_i) < 8/5.$$

But this contradicts because $y = P'(b)$ has local maxima at $b = b_1$ and b_3 and a local minimum at $b = b_2$ so that $P'(b_1)$, $P'(b_3) > P'(b_2)$. Hence $b_1 = b_2 = b_3$. □

Corollary (3.4.12). $0 < P'(b) < 8/5$ on (b^*,∞).

Proof. Lemma (3.4.10) and Lemma (3.4.11) $P'''(b^{**}) < 0$, so $0 < P'(b^{**}) < 8/5$.

$y = P'(b)$ has a miximum at $b = b^{**}$ and $P'(b) > 0$ on (b^*,∞).

Hence $0 < P'(b) < 8/5$ for all $b > b^*$. □

Lemma (3.4.13). $P''(b) < 0$ for $b \in (-1/4,0)$.

Proof. See the proof of Lemma (3.3.8). □

Now it is time to investigate bifurcation curves in (τ_0,τ_1)–plane. The Hopf bifurcation curve H_f of order 1 in (3.4.1) is given by the equation $G(-1/4) = 0$ and $G'(-1/4) \neq 0$, that is, H_f: $\tau_0 + \tau_1 + 1 = 0$ except $(\tau_0,\tau_1) = (-1,0) = \bar{a}$ where the Hopf bifurcation of order 2 occurs since

$G(-1/4) = G'(-1/4) = 0$ but $G''(-1/4) \neq 0$ at $\bar{a}$. (Andronov et.al. [1]). Also the symmetric homoclinic loop bifurcation curve H_0 of order 1 in (3.4.1) is H_0: $\tau_0 + 4\tau_1/5 + 32/35 = 0$ except $(\tau_0,\tau_1) = (0,-8/7) = \bar{e}$ where the symmetric homoclinic loop bifurcation of order 2 occurs. (Roussarie [14], Joyal et.al. [11]).

Clearly H_f and H_0 intersect transversally at the point $\bar{c} = (-4/7,-3/7)$. (See Figure 26).

The semistable limit cycle bifurcation curve $\overline{C}$ is given by the equation $G(b) = G'(b) = 0$ for $b\epsilon(-1/4,\infty) - \{0,b^*,b^{**}\}$. $\overline{C}_1$ is a smooth curve which connects the points $\bar{a}$ on H_f and $\bar{e}$ on H_0, and which is tangent to H_f and H_0 at these points respectively.

$$\overline{C} = \bigcup_{i=1}^{4} \overline{C}_i$$

where $\overline{C}_1$: $G(b) = G'(b) = 0, \quad b\epsilon(-1/4,0)$,

$\overline{C}_2$: $G(b) = G'(b) = 0, \quad b\epsilon(0,b^*)$,

$\overline{C}_3$: $G(b) = G'(b) = 0, \quad b\epsilon(b^*,b^{**})$,

and $\overline{C}_4$: $G(b) = G'(b) = 0, \quad b\epsilon(b^{**},\infty)$.

The behavior of H_f, H_0 and $\overline{C}_1$ are same as H_f, H_e and $\overline{C}$ in §3 ($\overline{C}_1$ corresponds to $\overline{C}$ in the previous section.) So we concentrate our attention to $\overline{C}_i(i=2,3,4)$. The (τ_0,τ_1)-plane is divided into 10 regions by curves H_f, H_0, $\overline{C}_i(i=1,2,3,4)$ (see Figure 26). Let

$Cl(H_f) \cap Cl(\overline{C}_1) = \{\bar{a}\}, Cl(H_f) \cap Cl(H_0) = \{\bar{c}\}, Cl(H_f) \cap Cl(\overline{C}_2) = \{\bar{d}\}$,

$Cl(H_0) \cap Cl(\overline{C}_1) = \{\bar{e}\}, Cl(\overline{C}_3) \cap Cl(\overline{C}_4) = \{\bar{f}\}, Cl(H_0) \cap Cl(\overline{C}_4) = \{\bar{g}\}$,

$Cl(H_f) \cap Cl(\overline{C}_4) = \{\bar{h}\}$.

Lemma (3.4.14).

The curves $\overline{C}_i(i=2,3,4)$ satisfy the following properties (see Figure 26).

(1) $\overline{C}_i(i=2,3,4)$ are smooth curves.

(2) $\overline{C}_2$ is tangent to $\overline{H}_0$ at $\overline{e} = (0,-8/7)$.

(3) On $(0,b^*)$, the slope of $\overline{C}_2$ is decreasing monotonically and tends to $-1/P(b^*)$, as $b \to b^{*-}$.

(4) $\overline{H}_f$ and $\overline{C}_2$ intersect transversally at $\overline{d}$.

(5) There is an $\overline{f} = (\tau_0^{**},\tau_1^{**})$ in (τ_0,τ_1)–plane such that
$G(b^{**}) = G'(b^{**}) = G''(b^{**}) = 0$ at $(\tau_0,\tau_1) = \overline{f}$.

(6) $\tau_1^{**} < -8/7$ and $\tau_0^{**} + 4\tau_1^{**}/5 + 32/35 < 0$.

(7) On (b^*,b^{**}), the slope of $\overline{C}_3$ is increasing monotonically and tends to $-1/P(b^*)$ as $b \to b^{*+}$.

(8) On (b^{**},∞), the slope of $\overline{C}_4$ is increasing monotonically and tends to 0 as $b \to \infty$.

(9) $\overline{C}_3$ and $\overline{C}_4$ are tangent at $\overline{f}$.

(10) $\overline{C}_4$ intersects transversally with $\overline{H}_f$ at $\overline{g}$ and with $\overline{H}_0$ at $\overline{h}$ respectively.

(11) $\overline{C}_3 \cap \overline{H}_0 = \{\}$.

Proof.

For (1) see the proof of Lemma (3.3.9).

Next, from $G(b) = G'(b) = 0$, we have

$$\tau_0 = -\frac{4}{7}b + \frac{4P(b)}{7P'(b)}$$

$$\tau_1 = -\frac{4}{7P'(b)} - 8/7.$$

Hence

$$\text{(3.4.14)} \quad \frac{d\tau_1}{d\tau_o} = \frac{d\tau_1}{db} / \frac{d\tau_o}{db} = -\frac{1}{P(b)}.$$

By (3.4.14), we have properties (2), (3), (4), (7), (8) and (9). (5) is immediate from Lemma (3.4.11).

(6) $\tau_o^{**} = -\frac{4}{7} b^{**} + \dfrac{4P(b^{**})}{7P'(b^{**})}$

$\tau_1^{**} = -\dfrac{4}{7P'(b^{**})} - 8/7.$

Hence $\tau_1^{**} < -8/7$ since $P'(b^{**}) > 0$.

Now $\tau_o^{**} + \frac{4}{5}\tau_1^{**} + 32/35 < 0$ is equivalent to

$b^{**}P'(b^{**}) - P(b^{**}) + 4/5 > 0.$

Let $f(b) = bP'(b) - P(b) + 4/5.$

$f(b^*) = 4/5 - P(b^*) > 0$

$f'(b) = bP''(b) > 0$ on (b^*, b^{**}) by Lemma (3.4.11).

Hence $f(b^{**}) > 0$. In fact,

$$\text{(3.4.15)} \quad f(b) > 0 \text{ on } [b^*, b^{**}].$$

(10) is clear from (6) and (8). (Note that $\overline{C}_4$ is concave upward.)

(11) Suppose $\overline{C}_3 \cap \overline{H}_o \neq \{\}$ and let $(\tilde{\tau}_o, \tilde{\tau}_1) \epsilon \overline{C}_3 \cap \overline{H}_o$.

Then $\tilde{\tau}_o + \frac{4}{5}\tilde{\tau}_1 + 32/35 = 0$ since $(\tilde{\tau}_o, \tilde{\tau}_1) \epsilon \overline{H}_o$.

Also $\tilde{\tau}_o = -\frac{4}{7}\tilde{b} + \dfrac{4P(\tilde{b})}{7P'(\tilde{b})}$ and $\tilde{\tau}_1 = -\dfrac{4}{7P'(\tilde{b})} - 8/7$

for some $\tilde{b} \epsilon (b^*, b^{**})$ since $(\tilde{\tau}_o, \tilde{\tau}_1) \epsilon \overline{C}_3$.

Hence $\overline{C}_3 \cap \overline{H}_o \neq \{\}$ is equivalent to

$\tilde{b}P'(\tilde{b}) - P(\tilde{b}) + 4/5 = 0$ for some $\tilde{b} \epsilon (b^*, b^{**})$.

However, from (3.4.15),

$f(b) = bP'(b) - P(b) + 4/5 > 0$ on (b^*, b^{**}).

Hence $\overline{C}_3 \cap \overline{H}_0 = \{\}$. □

Now we consider the number of limit cycles. For a given (τ_0, τ_1), the number of limit cycles of equation (3.4.1) is determined by the number of roots of equation $G(b) = 0$ for $b > -1/4$. The roots of $G(b) = 0$ for $b \epsilon (-1/4, 0)$ correspond to the limit cycles of type 1 (two limit cycles for each root; one surrounds the point $A = (-1,0)$ and the other surrounds the point $C = (1,0)$, see Figure 21) and those for $b \epsilon (0, \infty)$ correspond to the limit cycles of type 2 (one for each root which surrounds A, $B = (0,0)$ and C simultaneously).

If $\tau_1 + 8/7 = 0$, then $G(b) = 0$ if and only if $\tau_0 = -4b/7$. Hence for $\tau_0 \epsilon (-\infty, 1/7) - \{0\}$ (i.e. $b \epsilon (-1/4, \infty) - \{0\}$) and $\tau_1 = -8/7$, $G(b) = 0$ has a unique root since $G'(b) = 4/7 \neq 0$ and for $\tau_0 \epsilon (1/7, \infty)$ and $\tau_1 = -8/7$, $G(b) = 0$ has no root.

We suppose that $\tau_1 + 8/7 \neq 0$, and rewrite

$G(b) = \tau_0 + \frac{4}{7} b + (\tau_1 + 8/7)P(b)$ into the form

$$G(b) = (\tau_1 + 8/7)[P(b) - A(b)]$$

where $A(b) = -(\tau_0 + 4b/7)/(\tau_1 + 8/7)$.

For given (τ_0, τ_1), A is linear in b. Again the root of $G(b) = 0$ is the intersection of the straight line $P = A(b)$ and the curve $P = P(b)$ on the (b,P)–plane. The curve $P = P(b)$ is concave downward $(P''(b) < 0)$ on $(-1/4, 0) \cup (b^{**}, \infty)$ and concave upward $(P''(b) > 0)$ on $(0, b^{**})$. Thus points $(0, 4/5)$ and $(b^{**}, P(b^{**}))$ in (b,P)–planeare inflection points. $P'(b) < 0$ on $(-1/4, b^*)$ and $P'(b) > 0$ on (b^*, ∞) so that the point $(b^*, P(b^*))$ is an extreme point. A(b) depends on τ_0 and τ_1 while P(b) does not. Recall that $P(-1/4) = 1$, $P(0) = 4/5$.

A(b) has the following properties;

(1) $A(-1/4) = 1$ (resp. $A(-1/4) > 1$, $A(-1/4) < 1$) if and only if $(\tau_0,\tau_1)\epsilon \bar{H}_f$ (resp. is below $\bar{H}_f$, is above $\bar{H}_f$).

(2) $A(0) = 4/5$ (resp. $A(0) > 4/5$, $A(0) < 4/5$) if and only if $(\tau_0,\tau_1)\epsilon \bar{H}_0$ (resp. is below $\bar{H}_0$, is above $\bar{H}_0$).

(3) The straight line $P = A(b)$ is tangent to the curve $P = P(b)$ for $b\epsilon(-1/4,\infty) - \{0,b^*,b^{**}\}$ if and only if $(\tau_0,\tau_1)\epsilon \bigcup_{i=1}^{4} \bar{C}_i$.

The relative positions between the straight line $P = A(b)$ and the curve $P = P(b)$ are as in Figure 27 (a) – (h). In the strict sense, I to X in Figure 26 are different from those in Figure 19, however, we will use the same notations for convenience.

For $\bar{g}$ and $\bar{h}$ in (τ_0,τ_1)–plane, we have associated straight lines, say, $\bar{g} \leftrightarrow P = \bar{A}(b)$ and $\bar{h} \leftrightarrow P = \bar{\bar{A}}(b)$. Let $\bar{b}$ (resp. $\bar{\bar{b}}$) be the solution of the system

$$\begin{cases} \bar{A}(b) = P(b) \\ \bar{A}'(b) = P'(b) \end{cases} \quad \text{resp.} \quad \begin{cases} \bar{\bar{A}}(b) = P(b) \\ \bar{\bar{A}}'(b) = P'(b). \end{cases}$$

It is easy to see that $b^* < b^{**} < \bar{b} < \bar{\bar{b}} < \infty$ (see Figure 27 (g) and (h)).

Lemma (3.4.15).

Let K be a compact neighborhood of $\{H(x,y) \leq \bar{\bar{b}}\}$ and let D be a compact neighborhood of the curved region $\bar{a}\ \bar{e}\ \bar{g}\ \bar{h}\ \bar{d}\ \bar{c}$ and a curve $\bar{f}\ \bar{g}$ in (τ_0,τ_1)–plane. Then there exists $\alpha(D) > 0$ such that the bifurcation diagram of the equation (3.4.1) can be described in $C(D) = (0,\alpha(D)) \times D$ as follows (up to a diffeomorphism of $C(D)$ equal to the identity of $s = 0$):

(1) $S_{H_f} = (0,\alpha(D)) \times \bar{H}_f$ is a surface of Hopf bifurcation of codimension 1.

(2) $S_{H_o} = (0,\alpha(D)) \times \overline{H}_o$ is a surface of homoclinic loop bifurcation of codimension 1.

(3) $S_{C_i} = (0,\alpha(D)) \times \overline{C}_i$ (i=1,2,3,4) is a surface of semistable limit cycle bifurcation of codimension 1.

(4) $(0,\alpha(D)) \times \{\overline{a}\}$, $(0,\alpha(D)) \times \{\overline{e}\}$, and $(0,\alpha(D)) \times \{\overline{f}\}$ are curves of Hopf, homoclinic loop, and triple limit cycle bifurcations of codimension 2 respectively.

(5) $(0,\alpha(D)) \times \{\overline{c}\} = \mathrm{Cl}(S_{H_f}) \cap \mathrm{Cl}(S_{H_o})$

$(0,\alpha(D)) \times \{\overline{d}\} = \mathrm{Cl}(S_{H_f}) \cap \mathrm{Cl}(S_{C_2})$

$(0,\alpha(D)) \times \{\overline{g}\} = \mathrm{Cl}(S_{H_o}) \cap \mathrm{Cl}(S_{C_4})$

$(0,\alpha(D)) \times \{\overline{h}\} = \mathrm{Cl}(S_{H_f}) \cap \mathrm{Cl}(S_4)$

are curves of the corresponding two simultaneous bifurcations.

Outside these bifurcation sets in C(D), the topological type of the phase portraits of the equation (3.4.1) is constant in K.

Proof. The paragraph after Lemma (3.4.13) plus the implicit function theorem applied to $F(b,\lambda_s) = s^3(\tau_o I_o(b) + \tau_1 I_2(b) + I_4(b)) + o(s^3)$ give the proof. For details, see Dumortier et al [7]. □

The blowing–up (3.3.1) with $\eta = 1$ gives a transformation ϕ: $(s,(\tau_o,\tau_1)) \to (\epsilon_1,\epsilon_2,\epsilon_3)$, and

(3.4.16) $\phi((0,\alpha(D)) \times D) = \{(s^2,s^4\tau_o,s^2\tau_1) \mid s\epsilon(0,\alpha(D)),\ (\tau_o,\tau_1)\epsilon D\}$.

Let $E_{\epsilon_1}(D)$ be the RHS of (3.4.16). The bifurcation diagram of (3.2.9)_ in

$E_{\epsilon_1}(D)$ is the image of those described in Lemma (3.4.15) by the transformation and thus homeomorphic to cones based on H_f, H_0, $\bar{C}$, and $\bar{a} - \bar{h}$ with curves $s \to (s^2\eta, s^4\tau_0, s^2\tau_1)$ with $\eta = 1$, or equivalently $\epsilon_1 \to (\epsilon_1, \epsilon_1^2\tau_0, \epsilon_1\tau_1)$ for $\epsilon_1 > 0$.

Now we will consider the behavior of $(3.2.9)_-$ in a sector around $0\epsilon_3$–axis. For this we take $\tau_1 = \pm 1$ instead of $\eta = 1$ in (3.3.1). Since both cases $\tau_1 = 1$ and $\tau_1 = -1$ are similar, we will consider only the case $\tau_1 = 1$ like in §3.3. By (3.3.1) with $\tau_1 = 1$, equation $(3.2.9)_-$ becomes

(3.4.17) $$\begin{cases} \dot{x} = y \\ \dot{y} = \eta x - x^3 + s^3(\tau_0 + x^2 + x^4)\, y. \end{cases}$$

Let $s_1 > 0$ be fixed. Then for each $s \epsilon (0, s_1]$ we take a blowing–up again:

(3.4.18) $$\begin{matrix} \eta = r^2 & & x \to rx \\ & \text{and} & y \to r^2 y \\ \tau_0 = r^2\bar{\tau}_0 & & t \to t/r. \end{matrix}$$

Then (3.4.17) becomes

(3.4.19) $$\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 + s^3(r(\bar{\tau}_0 + x^2)y + o(r^3)) \end{cases}$$

which is a perturbation of the Hamiltonian system $(r = 0)$

(3.4.20) $\begin{cases} \dot{x} = y \\ \dot{y} = x - x^3 \end{cases}$

with the first integral $H(x,y) = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^4}{4}$. As in Lemma (3.4.3), we have

$$H(w_{\bar{b}}) - H(w_b) = s^3 r(\tilde{F}(b,\bar{\tau}_0) + o(r))$$

where $\tilde{F}(b,\bar{\tau}_0) = \bar{\tau}_0 I_0(b) + I_2(b)$.

Lemma (3.4.16).

In the plane $\{(\eta,\tau_0,\tau_1) | \tau_1 = 1\}$ there is a fixed compact neighborhood B^+ of $(\eta,\tau_0,\tau_1) = (0,0,1)$ such that for equation (3.4.19) the results of Carr–Takens are valid for any $(\eta,\tau_0) \epsilon B^+$ and any $s \epsilon (0,s_1]$. (See Figure 28.)

Proof. Carr [3]. □

Blowing–up (3.3.1) with $\tau_1 = 1$ (i.e., $\epsilon_1 = \eta s^2$, $\epsilon_2 = \tau_0 s^4$, $\epsilon_3 = s^2$) gives a mapping $(s, \eta,\tau_0) \to (\epsilon_1,\epsilon_2,\epsilon_3)$ which maps $(0,s_1] \times B^+$ to

$$E^+_{\epsilon_3} = \{(s^2\eta, s^4\tau_0, s^2) | s\epsilon(0,s_1], (\eta,\tau_0)\epsilon B^+\}.$$

$E^+_{\epsilon_3}$ is a cone in $(\epsilon_1,\epsilon_2,\epsilon_3)$–space around $0\epsilon_3$–axis $(\epsilon_3 > 0)$ based on B^+.

The bifurcation diagram of (3.4.1) in $E^+_{\epsilon_3}$ consists of cones based on $\bar{H}_f$, $\bar{H}_0$, $\bar{C}_2$, $\tilde{P}$, $\tilde{H}_f$ and $\{\bar{b}_1\}$ with generating curves $s \to (s^2\eta, s^4\tau_0, s^2)$. For $\tau_1 = -1$, we can get a cone $E^-_{\epsilon_3}$ around $0\epsilon_3$–axis $(\epsilon_3 < 0)$ based on $\bar{H}_f$, $\bar{H}_0$, $\bar{C}_3$, $\tilde{P}$, $\tilde{H}_f$ and $\{\bar{b}_2\}$ similarly.

Now we consider the case $\epsilon_1 = 0$. In $(3.2.9)_-$, if $\epsilon_1 = 0$, we have the following equation

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x^3 + (\epsilon_2+\epsilon_3 x^2+x^4)y. \end{cases}$$

After blowing-up ((3.3.1) with $\eta = 0$), we get

$$(3.4.21) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -x^3 + s^3(\tau_0+\tau_1 x^2+x^4)y. \end{cases}$$

If $s = 0$ in (3.4.21), it becomes a Hamiltonian system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x^3 \end{cases}$$

with the first integral

$$(3.4.22) \quad H(x,y) = \frac{y^2}{2} + \frac{x^4}{4} .$$

As before, we get a similar lemma of Lemma (3.4.3) and lead to solve

$$(3.4.23) \quad F(b,\lambda_0) = \tau_0 I_0(b) + \tau_1 I_2(b) + I_4(b) = 0.$$

where $I_i(b) = \int_{\gamma_b} x^i y dx$, γ_b: $H(x,y) = b$ in (3.4.22).

(All the same notations will be used again.)

<u>Lemma (3.4.17)</u>.

$$(3.4.24) \quad 7I_4(b) = 4b\ I_0(b).$$

Proof. Omit. □

Hence by (3.4.24) we get a new form in $F(b,\lambda_0)$ from (3.4.23).

(3.4.25) $$F(b,\lambda_0) = (\tau_0+\frac{4b}{7})\ I_0(b) + \tau_1 I_2(b).$$

Let

(3.4.26) $$P(b) = \begin{cases} I_2(b)/I_0(b) & \text{for } b > 0 \\ 0 & \text{for } b = 0 \end{cases}$$

and from (3.4.25) and (3.4.26) we define

(3.4.27) $$\begin{aligned} G(b,\lambda_0) &= \frac{1}{I_0(b)} F(b,\lambda_0) \quad (\text{its limit at } b = 0) \\ &= \tau_0 + \frac{4b}{7} + \tau_1 P(b) \end{aligned}$$

(simply write $G(b)$ instead $G(b,\lambda_0)$).

Lemma (3.4.18). $P(b)$ defined in (3.4.26) satisfies

(3.4.28) $$P'(b) = \frac{P}{2b}.$$

Proof. Immediate from $\begin{cases} 3I_0(b) = 4b\ I_0'(b) \\ 5I_2(b) = 4b\ I_2'(b). \end{cases}$ □

By (3.4.28),

(3.4.29) $P(b) = \ell\sqrt{b}$, $b \geq 0$, ℓ is a constant independent of b.

If $b = 1/4$, $\ell = P(1/4) = \int_{\gamma_1} x^2 y dx / \int_{\gamma_1} y dx$

$$= \int_{-1}^{1} x^2(1-x^4)^{1/2} dx / \int_{-1}^{1} (1-x^4)^{1/2} dx.$$

For the semistable limit cycle bifurcation, if we solve

$$\begin{cases} G(b) = \tau_0 + \frac{4b}{7} + \tau_1 P(b) = 0 \\ G'(b) = \frac{4}{7} + \tau_1 P'(b) = 0, \end{cases}$$

then we get

$$\begin{cases} \tau_0 = \frac{4}{7} b \\ \tau_1 = -\frac{8\sqrt{b}}{7\ell}. \end{cases}$$

Hence

(3.4.30) $$\frac{\epsilon_3^2}{\epsilon_2} = \frac{\tau_1^2}{\tau_0} = \frac{16}{7\ell^2} .$$

We restrict (3.4.30) to a sphere S (actually $S \cap \{\epsilon_1 = 0\} = \{\epsilon_2^2+\epsilon_3^2=\epsilon_0^2\}$) so that the solution is exactly $(0,\alpha_2,\alpha_3)$ where $\alpha_2 = -\frac{8}{7\ell^2} + ((\frac{8}{7\ell^2})^2 + \epsilon_0^2)^{1/2}$ and $\alpha_3^2 = \frac{16}{7\ell^2}\alpha_2$ $(\alpha_3 < 0)$. We let $b_3 = (0,\alpha_2,\alpha_3)$.

The Hopf bifurcation of order 1 occurs at $b_1 = (0,0,\epsilon_0)$ and $b_2 = (0,0,-\epsilon_0)$. $(G(0) = 0 \neq G'(0) \rightarrow \tau_0 = 0 \rightarrow \epsilon_2 = 0)$.

So $S \cap \{\epsilon_1 = 0\} = (b_1,b_2) \cup (b_2,b_3) \cup (b_3,b_1) \cup \{b_1,b_2,b_3\}$.

In (3.4.27), $G(b) = 0$ has only one generic solution if $\epsilon_2 < 0$. So on

(b_1,b_2), there is one limit cycle. If $\epsilon_3^2 < \frac{16}{7\ell^2}\epsilon_2$ or (ϵ_2,ϵ_3) is in the first quadrant, $G(b) = 0$ has no solution. So on (b_1,b_3), there is no limit cycle. If $0 < \frac{16}{7\ell^2}\epsilon_2 < \epsilon_3^2$ and $\epsilon_3 < 0$, $G(b) = 0$ has two generic solutions. So on (b_2,b_3) there are two limit cycles.

Those limit cycles are generic on $(b_1,b_2) \cup (b_2,b_3) \cup (b_1,b_3)$ so that they still persist if ϵ_1 is a nonzero small number such that $\epsilon = (\epsilon_1,\epsilon_2,\epsilon_3)$ is in S.

Let $R \subset S$ be a compact neighborhood of b_3 diffeomorphic to a closed disk and let D_o (resp. D_1,D_2) $\subset S$ be a compact neighborhood of part of (b_1,b_3) (resp. (b_1,b_2), (b_2,b_3)) around the circle $\{\epsilon_1 = 0\} \cap S$ in such a way that (1) limit cycle(s) persist on D_o (resp. D_1, D_2), and (2) $(\bigcup_{i=0}^{2} D_i) \cup E^+_{\epsilon_3} \cup E^-_{\epsilon_3} \cup R \supset \{\epsilon_1 = 0\} \cap S$ as in Figure 29. We choose a compact set D in Lemma (3.4.15) such that union of $E_{\epsilon_1}(D) \cap S$, D_o, D_1, D_2, $E^+_{\epsilon_1} \cap S$, $E^-_{\epsilon_3} \cap S$ and R covers hemisphere $\{\epsilon_1 \geq 0\} \cap S$. (See Figure 29.)

Let $H_f = \phi(S_{H_f}) \cap S$, $H_o = \phi(S_{H_o}) \cap S$, $C_i = \phi(S_{C_i}) \cap S$ (i=1,2,3,4). Then we will show that the curves of Hopf bifurcation, homoclinic loop bifurcation and semistable limit cycle bifurcation (C_2 and C_3) in $E_{\epsilon_1}(D) \cap S$ are connected with those in $E^+_{\epsilon_3} \cap S$ (and thus H_f, H_o and C_2 with b_1) and in $E^-_{\epsilon_3} \cap S$ (and thus H_f, H_o and C_3 with b_2).

We have

$$H_f: \quad \left[\begin{array}{l} \tau_o + \tau_1 + 1 = 0 \\ \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = \epsilon_0^2,\ \epsilon_1 > 0 \end{array}\right.$$

and

$$H_o: \begin{cases} \tau_o + 4\tau_1/5 + 32/35 = 0 \\ \epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 = \epsilon_o^2,\ \epsilon_1 > 0 \end{cases}$$

by the transformation (3.3.1)

$$s = \sqrt{\epsilon_1},\ \tau_o = \epsilon_2/\epsilon_1^2,\ \tau_1 = \epsilon_3/\epsilon_1,\ \epsilon_1 > 0.$$

Let $\epsilon_1 \to 0$ on H_f and H_o. Then $\epsilon_2 \to 0$. Hence $\epsilon_1 \to 0$ implies $(\epsilon_1,\epsilon_2,\epsilon_3) \to (0,0,\pm\epsilon_o) = b_1$ or b_2.

Now let $\overline{C}_2$: $\tau_1 = f(\tau_o)$.

As b varies from 0 to b^* (i.e., τ_o varies from 0 to $-\infty$), $\frac{d\tau_1}{d\tau_o} = f'(\tau_o)$ varies from $-5/4$ to $-1/P(b^*)$ and $f'(\tau_o)$ is monotonically decreasing on $(0,b^*)$.

$$-\frac{\tau_o}{P(b^*)} < f(\tau_o) < -5\tau_o/4 - 8/7.$$

Since on the straight lines $\tau_1 = -\frac{\tau_o}{P(b^*)}$ and $\tau_1 = -\frac{5}{4}\tau_o - 8/7$, if ϵ_1 tends to 0, then ϵ_2 tends to 0 and $\tau_o \to -\infty$ and thus $\epsilon_3 \to \epsilon_o$. Hence by squeezing, $\epsilon_2 \to 0$ and $\epsilon_3 \to \epsilon_o$ as $\epsilon_1 \to 0$, i.e. as $\epsilon_1 \to 0$, $(\epsilon_1,\epsilon_2,\epsilon_3)$ on C_2 tends to b_1. Similarly, as $\epsilon_1 \to 0$, $(\epsilon_1,\epsilon_2,\epsilon_3)$ on C_3 tends to b_2.

Next we will show that C_4 in $E_{\epsilon_1}(D) \cap S$ is connected with b_3 in R.

Note that $\overline{C}_4$: $(\tau_o,\tau_1) = (-\frac{4}{7}b + \frac{4P}{7P'},\ -\frac{4}{7P'} - 8/7)$ for $b \in (b^{**},\infty)$.

From $\epsilon_2/\epsilon_1^2 = \tau_o = -\frac{4}{7}b + \frac{4P}{7P'}$ and $\frac{\epsilon_3}{\epsilon_1} = \tau_1 = -\frac{4}{7P'} - 8/7$,

as $b \to \infty$ $P'(b) \to 0^+$, $\tau_1 \to -\infty$ and $\epsilon_3/\epsilon_1 \to -\infty$,

and

as $b \to \infty$ $\tau_o \to \infty$, $\epsilon_2/\epsilon_1^2 \to \infty$ and $\epsilon_1^2/\epsilon_2 \to 0$

since $\tau_o = -\frac{4}{7}b + \frac{4P}{7P'} \sim 4b/7$ as $b \to \infty$.

$$\frac{\tau_1^2}{\rho_o} = \frac{\epsilon_3^2}{\epsilon_2} = \frac{\epsilon_o^2-\epsilon_1^2-\epsilon_2^2}{\epsilon_2} = \frac{\epsilon_o^2}{\epsilon_2} - \frac{\epsilon_1^2}{\epsilon_2} - \epsilon_2$$

and

$$\frac{\tau_1^2}{\tau_o} = \frac{(-\frac{4}{7P^{\tau}} - \frac{8}{7})^2}{(-\frac{4}{7}b + \frac{4P}{7P^{\tau}})} \to \frac{16}{7c^2} \quad \text{as } b \to \infty \text{ where } c > 0 \text{ is a constant in}$$

Lemma (3.4.8).

Hence $\epsilon_1 \to 0$ but ϵ_2 and $\epsilon_3 \not\to 0$ as $b \to \infty$. $(\epsilon_1,\epsilon_2,\epsilon_3)$ on C_4 does not tend to b_1 or b_2 as $b \to \infty$ since $\epsilon_2 \not\to 0$. The only other point at which the number of limit cycle change on $\{\epsilon_1 = 0\} \cap S$ is b_3. Thus $(\epsilon_1,\epsilon_2,\epsilon_3)$ on C_4 must tend to b_3 as $\epsilon_1 \to 0$ and so it must be that $c = \ell$.

We have proved Theorem (3.4.1).

B. $\epsilon_1 < 0$.

As in the case $\epsilon_1 > 0$, we use (3.3.1) with $\eta = -1$ to investigate the behavior of $(3.2.9)_-$ in a neighborhood of $0\epsilon_1$–axis for $\epsilon_1 \le 0$. Hence the equation $(3.4.31)_-$ has the form

$$(3.4.31) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -x - x^3 + s^3(\tau_o+\tau_1 x^2+x^4)y. \end{cases}$$

Equation (3.4.31) has the equilibrium point (0,0) which is a focus. If $s = 0$, (3.4.31) becomes a Hamiltonian system

$$(3.4.32) \quad \begin{cases} \dot{x} = y \\ \dot{y} = -x - x^3 \end{cases}$$

with the first integral

$$H(x,y) = \frac{y^2}{2} + \frac{x^2}{2} + \frac{x^4}{4},$$

and the phase portrait of (3.4.32) is shown in Figure 30.

Closed orbits surrounding $(0,0)$ correspond to level curves γ_b: $H(x,y) = b$, $b \in (0,\infty)$. Now we consider (3.4.31) for small $s \neq 0$. Every closed orbit of (3.4.31) should intersect with the interval $U = \{(x,0) | x \geq 0\}$. We define w_b for $b \in [0,\infty)$ as follows:

(a) $w_b \in U$ (b) $H(w_b) = b$.

Let $\lambda_s = ((\tau_0,\tau_1),s)$ and we define a Poincare map $P_{\lambda_s}: U \to U$ by a successive intersection points of U with an orbit in an obvious way. (See Figure 31.)

Let $\gamma(b,\lambda_s)$ be the orbit of (3.4.31) which joins the point w_b and $w_{\bar{b}}$. Hence $\gamma(b,\lambda_s)$ is defined for $b \in (0,\infty)$. Then we have the lemma.

<u>Lemma (3.4.19)</u>.

(1) Every closed orbit of (3.4.31) is expressed by the form $\gamma(b,\lambda_s)$ with $w_{\bar{b}} = w_b$.

(2) A trajectory $\gamma = \gamma(b,\lambda_s)$ of (3.4.31) is a periodic orbit if and only if

$$(3.4.33) \quad \int_\gamma \frac{dH(x,y)}{dt}\, dt = 0.$$

(3) For $x > 0$, condition (3.4.33) is equivalent to

(3.4.34) $F(b,\lambda_s) \equiv: \int_{\gamma(b,\lambda_s)} (\tau_o+\tau_1 x^2+x^4)y\ dx = 0.$

Proof. See the proof of Lemma (3.3.2). □

Following the same procedure as Subsection A, it reduced to solve (see the paragraphs after Lemma (3.4.2).)

(3.4.35) $F(b,\lambda_o) = \tau_o I_o(b) + \tau_1 I_2(b) + I_4(b) = 0$

where $I_i(b) = \int_{\gamma_b} x^i y\ dx$, γ_b: $H(x,y) = b$.

Lemma (3.4.20).

(3.4.36) $7I_4(b) = 4bI_o(b) - 8I_2(b)$.

Proof. Similar calculation of the Lemma (3.3.5). □

From (3.4.36), (3.4.35) becomes

(3.4.37) $F(b,(\tau_o,\tau_1)) = (\tau_o + 4b/7)\ I_o(b) + (\tau_1 - 8/7)I_2(b).$

Note that $I_o(0) = I_2(0) = 0$, $I_o(b) > 0$ for $b > 0$ and $I_2(b)/I_o(b) \to 0$ as $b \to 0$. So we can change (3.4.37) as follows:

(3.4.38) $G(b,(\tau_o,\tau_1)) = \tau_o + 4b/7 + (\tau_1 - 8/7)P(b)$

(we will denote $G(b,(\tau_o,\tau_1))$ simply by $G(b)$) where

$$(3.4.39)\quad P(b) = \begin{cases} I_2(b)/I_o(b) & \text{for} \quad b > 0 \\ 0 & \text{for} \quad b = 0. \end{cases}$$

Lemma (3.4.21). P(b) defined in (3.4.39) satisfies

$$4b(4b+1)\, P'(b) = -5P^2 + (8b-4)P + 4b$$

Proof. See the proof of Lemma (3.3.6). □

Lemma (3.4.22). P(b) also satisfies:

(1) $\lim_{b\to 0} P'(b) = 1/2$ and $P'(b) > 0$ for $b > 0$

(2) $\lim_{b\to 0} P''(b) = -7/8$ and $P''(b) < 0$ for $b > 0$

hence the graph of $P = P(b)$ is concase downward.

(3) $P(b) \sim m\sqrt{b}$ as $b \to \infty$ for some positive constant m.

So

$$\lim_{b\to\infty} P(b) = \infty \quad \text{and} \quad \lim_{b\to\infty} P'(b) = 0.$$

Proof. (1) and (2) are easy calculations from the system of equations

$$\begin{cases} \dot{p} = -5P^2 + (8b-4)P + 4b \\ \dot{b} = 4b(4b+1). \end{cases}$$

which is gotten by Lemma (3.4.21).

See Lemma (3.3.7) and Lemma (3.3.8) for details.

For (3), see Lemma (3.4.8). □

Now we will compute the bifurcation curves of the equation (3.4.31) in (τ_0,τ_1)–plane. The Hopf bifurcation curve H_f of order 1 in (3.4.31) is

H_f: $\tau_0 = 0$ except $(\tau_0,\tau_1) = (0,-16/7) = \bar{k}$

where the Hopf bifurcation of order 2 occurs. (Compute the Liapunov's focal values in (3.4.31).) The semistable limit cycle bifurcation curve $\bar{C}$ is given by the equation $G(b) = G'(b) = 0$ for $b \epsilon (0,\infty)$ and is

$$C: (\tau_0,\tau_1) = \left(-\frac{4}{7}b - \frac{4P}{7P'}, -\frac{4}{7P'} + 8/7\right), \; b\epsilon(0,\infty).$$

(See Figure 32.)

Lemma (3.4.23).

(1) The curve $\bar{C}$ is smooth.

(2) $\bar{C}$ is tangent to H_f at $\bar{k} = (0,-16/7)$.

(3) As $b \to \infty$, the slope $(\frac{d\tau_1}{d\tau_0})$ of $\bar{C}$ is increasing monotonically and tends to 0.

Proof. See Lemma (3.4.14). □

It works through on the semistable curve almost the same as the case $\epsilon_1 > 0$. For the number of limit cycles, we consider the relative positions of the curve $P = P(b)$ and the straight line $P = A(b)$ shown in Figure 33, where $A(b)$ is such that

$$G(b) = (\tau_1 - 8/7)(P(b) - A(b))$$

so

$$A(b) = -\frac{\tau_0 + 4b/7}{\tau_1 - 8/7}.$$

We can get a similar version of Lemma (3.3.15) and, after secondary blowing–up, Lemma (3.3.16), then using the results for $\epsilon_1 = 0$ ((3.4.21) – (3.4.30)) and the same idea as before ($\epsilon_1 > 0$), we can prove that $(\epsilon_1,\epsilon_2,\epsilon_3)$ on C tends to b_3 as $\epsilon_1 \to 0$ and $m = \ell$. We omit the details because the steps are routine repeated arguments of Subsection A.

§5. The Case: $b \neq 0$.

We can get the versal deformation of (3.1.5) as follows:

Lemma (3.5.1). Any symmetric perturbation of (3.1.5) with small parameter μ can be transformed into the form

$$(3.5.1)_\pm \quad \begin{cases} \dot{x} = y \\ \dot{y} = (\varphi_1(\mu) + \varphi_3(\mu)x^2 \pm x^4)x + (\varphi_2(\mu) + G(x,\mu)x^2 + y\phi(x,y,\mu))y, \end{cases}$$

where $\mu \in \mathbb{R}^3$, $G(x,0) = |\tilde{a}|^{1/2}$ and $\phi(x,y,0) = 0$.

Proof. By an appropriate scaling, we can change (3.1.5) into the following form:

$$(3.5.2) \quad \begin{cases} \dot{x} = y \\ \dot{y} = \alpha x^5 + \beta x^2 y \end{cases}$$

where $\alpha = \mathrm{sgn}(\tilde{a})$, $\beta = |\tilde{a}|^{1/2}$.

Using (3.5.2) to follow the same steps as in the proof of Lemma (3.2.1), we get $(3.5.1)_\pm$. □

Remark. We can take a transformation $\varphi = (\varphi_1,\varphi_2,\varphi_3)$: $\mu \mapsto \epsilon$ in the parameter space so that $(3.5.1)_\pm$ becomes

$$(3.5.3)_\pm \quad \begin{cases} \dot{x} = y \\ \dot{u} = (\epsilon_1+\epsilon_3x^2\pm x^4)x + (\epsilon_2+G(x,\mu)x^2+y\phi(x,y,\epsilon))y \end{cases}$$

where $G(x,0) = |\tilde{a}|^{1/2}$ $(:\equiv c)$, $\phi(x,y,0) = 0$.

$(3.5.3)_\pm$ is versal to

$$(3.5.4)_\pm \quad \begin{cases} \dot{x} = y :\equiv f_1(x,y,\epsilon) \\ \dot{y} = (\epsilon_1+\epsilon_3x^2\pm x^4)x + (\epsilon_2+cx^2)y :\equiv f_2(x,y,\epsilon). \end{cases}$$

Note that $\mathrm{div}(f) = \frac{\partial f_1}{\partial x}(x,y) + \frac{\partial f_2}{\partial y}(x,y) = \epsilon_2 + cx^2$.

Here we are not able to control cx^2, hence $(3.5.4)_\pm$ can't be regarded as a perturbation of the Hamiltonian system. This implies that it's likely hard to apply to abelian integrals and Picard–Fuchs equations. So probably we have to approach from other directions.

There is a recent paper by Dangelmayr et.al [6] which introduces an equation for a laser with saturable absorber. After using a center manifold theorem and a normal form theorem, they reduced it to the form (3.1.5) with a leading fifth order term. It described also the bifurcation diagrams and phase portraits of $(3.5.4)_-$ without a detailed analysis (Figure 1 (a), (b) and Figure 2 in [6]).

However, the conjecture of the bifurcation diagrams has a couple of mistakes, one is a saddle–node homoclinic bifurcation (Sechecter [15]) and the other is a pitchfork homoclinic orbit in the symmetric vector field (Chapter 2 of this thesis).

We will describe about $(3.5.4)_-$ briefly. For simplicity we assume that $c = -1$. First, equilibria of $(3.5.4)_-$ is determined by the equations $y = 0$, $(\epsilon_1+\epsilon_3 s^2 - x^4)x = 0$. Hence $\{\epsilon_1 = 0\}$ is a pitchfork bifurcation surface and $\{\epsilon_3^2 + 4\epsilon_1 = 0, \epsilon_3 > 0\}$ is a saddle–node bifurcation surface. Also $\{\epsilon_2 = 0, \epsilon_1 < 0\}$ is a Hopf bifurcation of order 1. The number of equilibrium points are as follows. (See Figure 34.)

(1) $\epsilon_1 > 0$: 3

(2) $-\epsilon_3^2/4 < \epsilon_1 < 0$ and $\epsilon_3 > 0$: 1

(3) $(\epsilon_1 < 0$ and $\epsilon_3 < 0) \cup (\epsilon_1 < -\frac{\epsilon_3^2}{4}$ and $\epsilon_3 > 0)$: 5.

Bifurcation diagram and phase portraits (Figure 35) are based on numerical results, Chapter 2 of this thesis, and Schecter [15]. (Also, of course referred Dangelmayr et.al. [6].)

We divide three cases depending on the sign of ϵ_3;

(i) Case 1: $\epsilon_3 > 0$

(ii) Case 2: $\epsilon_3 = 0$

(iii) Case 3: $\epsilon_3 < 0$,

and codimension two bifurcations are explained in each case.

(i) Case 1: $\epsilon_3 > 0$.

For a small but fixed $\epsilon_3 > 0$, we can change $(3.5.4)_-$ into the following form

(3.5.5) $$\begin{cases} \dot{x} = y \\ \dot{y} = \mu_1 x + \mu_2 y + x^3(1-x^2) + x^2 y \end{cases}$$

by an appropriate scaling, where $\mu_1 = \epsilon_1/\epsilon_3^2$, $\mu_2 = \epsilon_2/\epsilon_3$.
(3.5.5) gives us a local behavior near $(\mu_1,\mu_2) = (0,0)$ (i.e., $0\epsilon_3$–axis ($\epsilon_3 > 0$); $(5)_+$ in Figure 35). Also we can see that Takens–Bogdanov bifurcation occurs at $(\mu_1,\mu_2) = (-1/4,1/2)$
(i.e., $\epsilon = (-\epsilon_3^2/4,\epsilon_3/2,\epsilon_3)$; (1) in Figure 35), Hopf–saddle node bifurcation at $(\mu_1,\mu_2) = (-1/4,0)$ (i.e., $\epsilon = (-\epsilon_3^2/4,0,\epsilon_3)$; (9) in Figure 35), Hopf–pitchfork bifurcation at $(\mu_1,\mu_2) = (0,1)$ (i.e., $\epsilon = (0,\epsilon_3,\epsilon_3),\epsilon_3>0$; (7) in Figure 35).

(ii) Case 2: $\epsilon_3 < 0$.

As above, we have the following:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu_1 x + \mu_2 y - x^3(1+x^2) - x^2 y, \end{cases}$$

where $\mu_1 = \epsilon_1/\epsilon_3^2$, $\mu_2 = -\epsilon_2/\epsilon_3$.
On $0\epsilon_3$–axis ($\epsilon_3 < 0$), we have a Takens–Carr bifurcation ($(5)_-$ in Figure 35) which is only a codimension two bifurcation.

(iii) Case 3: $\epsilon_3 = 0$.

Symmetric version of cusp bifurcation occurs at $(0,\epsilon_2,0)$ ((6) in Figure 35). In this case ($\epsilon_3 = 0$), Hopf, homoclinic and semistable bifurcation curves are tangent to the pitchfork bifurcation curve.

There are nine different codimension two bifurcation occurring in (3.5.4)_. (See Figure 35).

(1) Takens–Bogdanov bifurcation.

This bifurcation has an equilibrium point whose linearization is nilpotent. The equations defining the locus of Takens–Bogdanov bifurcations for a family $\dot{x} = f(x,\lambda)$, $x \in \mathbb{R}^2$ are $f(x_0,\lambda) = 0$,
trace $D_x f(x_0,\lambda) = \det D_x f(x_0,\lambda) = 0$.

A normal form for the Takens–Bogdanov bifurcation is

$$\begin{cases} \dot{x} = y \\ y = \epsilon_1 + \epsilon_2 s \pm x^2 + xy. \end{cases}$$

(See Bogdanov [2] and Figure 36.)

(2) Saddle–node homoclinic bifurcation.

The vector field of this bifurcation has a homoclinic orbit at an equilibrium point whose linear part has a simple zero eigenvalue and one nonzero eigenvalue, and the center direction is a saddle–node codimension one bifurcation. A homoclinic bifurcation meets the saddle–node bifurcation with quadratic tangency. (See Schecter [15] and Figure 37.)

(3) Homoclinic bifurcation of order 2.

This is a homoclinic bifurcation at a saddle point where the Jacobian derivative has a trace 0 and a certain property is satisfied.

The semistable bifurcation curve approaches the homoclinic bifurcation curve tangentially with infinite order contact. (See Roussarie [14], Joyal [11] and Figure 38.)

(4) Pitchfork homoclinic bifurcation.

(See Chapter 2 of this thesis and Figure 39.)

$(5)_\pm$ Takens–Carr bifurcation.

If a symmetric vector field has a nilpotent linear part at 0, then it occurs with two distinct qualitative behaviors (depending on the sign of coefficient in x^3 in the normal form) whose normal form is as follows:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \epsilon_1 x + \epsilon_2 y \pm x^3 + x^2 y \end{cases}$$

(See Carr [3] and Figure 40 (a), (b).)

(6) Pitchfork saddle–node bifurcation.

If a pitchfork bifurcation occurs at (0,0) and a saddle–node bifurcation occurs at an equilibrium point splitted from (0,0) in the symmetric vector field, we have a pitchfork saddle–node bifurcation. The normal form is as follows:

$$\begin{cases} \dot{x} = \epsilon_1 x + \epsilon_2 x^3 - x^5 \\ \dot{y} = \alpha y \ (\alpha \neq 0). \end{cases}$$

(See Chapter 6 of Golubitsky and Schaeffer [9] and Figure 41.)

The following are codimension two bifurcation curves where two codimension one bifurcation curves meet transversally each other:

(7) Hopf pitchfork bifurcation

(8) Pitchfork semistable bifurcation, and

(9) Hopf saddle–node bifurcation.

APPENDIX

Theorem. Let $0 \in \mathbb{R}^2$ be a pitchfork of $f \in \mathscr{X}^r_s(\mathbb{R}^2)$, $r \geq 3$. Then there is a neighborhood B of f in $\mathscr{X}^r_s(\mathbb{R}^2)$, N a neighborhood of 0 in $\mathbb{R}^2$ and α: B → R, C^{r-1} function such that

(1) $g \in B$ has a pitchfork as a unique equilibrium point in N if and only if $\alpha(g) = 0$. If $\alpha(g) < 0$, g has three equilibrium points, origin is node, other two, both generic, are saddles. If $\alpha(g) > 0$, g has one saddle point, origin in N.

(2) $\alpha(f) = 0$ and $d\alpha_f \neq 0$.

Before proving it, first we introduce some notations and definitions. Let M be a 2-dimensional smooth manifold. Let $f \in \mathscr{X}^r(M)$ such that $f(p) = 0$ for some $p \in M$, and $f = \sum_{i=1}^{2} f_i \frac{\partial}{\partial x_i}$. Then define $Df_p: D_pM^2 \to D_pM$ by $Df_p(v) = [g,f](p)$ where $g(p) = v$ for some $g = \sum_{i=1}^{2} g_i \frac{\partial}{\partial x_i} \in \mathscr{X}^r(M)$, and $[\cdot,\cdot]$ is a Lie bracket.

Also define

$\Delta(f,p) = \mathrm{Det}(Df_p)$

$\sigma(f,p) = \mathrm{Trace}\ (Df_p)$.

Let $r \geq 3$ and $\mathrm{Spec}(Df_p) = \{\lambda_1 = 0, \lambda_2\}$ and T_i, a corresponding eigenspace of Df_p and π_i: $D_pM \to T_i$ projection $(i = 1,2)$.

For $v \in T_1$, $v \neq 0$, we define $\Delta_1(f,p,v)v$ by $\pi_1[g,[g,f]](p) = \Delta_1(f,p,v)v$, and likewise

$\Delta_2(f,p,v)$ by $\pi_1[g,[g,[g,f]]](p) = \Delta_2(f,p,v,v)v$.

Then $\Delta_1(i=1,2)$ does not depend on g. P is called a <u>saddle-node</u> of f if $\Delta_1(f,p,v) \neq 0$ for some $v \neq 0$.

If $f \in \mathscr{X}_s^r(\mathbb{R}^2)$ and $\Delta_2(f,0,v,v) \neq 0$ for some $v \neq 0$, 0 is called a <u>pitchfork</u> of f.

Let u be the vector on $D_p M^2$ such that $\pi, v = uv$ and f^i, u_i, v^i components of f,u,v respectively with respect to a coordinate system (x_1,x_2) around p. Then

(1) $\Delta_1(f,p,v) = u[g,[g,f]](p) = \sum_{i,j,k} \frac{\partial^2 f^i}{\partial x_j \partial x_k}(p) v^j v^k u_i.$

(2) Δ_1 does not depend on g, i.e., $u[g,[g,f]](p) = u[\tilde{g},[\tilde{g},f]](p)$ for every $\tilde{g} \in \mathscr{X}^r(M)$ with $\tilde{g}(p) = g(p) = v$.

(3) Let $f \in \mathscr{X}_s^r(\mathbb{R}^2)$, $g \in \mathscr{X}^r(\mathbb{R}^2)$. Then

$$\Delta_2(f,p,v,v) = u[g,[g,[g,f]]](p) = \sum_{i,j,k,l} \frac{\partial^3 f^i}{\partial x_j \partial x_k \partial x_l}(p)\, v^j v^k v^l u_i.$$

(4) Likewise, Δ_2 does not depend on g.

<u>Proof of Theorem</u>. Let $x = (x_1,x_2)$ be a coordinate system around 0 such that $x(0) = 0$ and $\frac{\partial}{\partial x_i}(0) \in T_i (i=1,2)$. Then $f = \sum_{i=1}^{2} f^i \frac{\partial}{\partial x_i}$ satisfies

$$\frac{\partial f^1}{\partial x_1}(0) = \frac{\partial f^1}{\partial x_2}(0) = \frac{\partial f^2}{\partial x_1}(0) = 0,\ \frac{\partial f^2}{\partial x_2}(0) = \sigma(f,0),$$

$$\frac{\partial^2 f^i}{\partial x_j \partial x_k}(0) = 0 \text{ for every } i,j,k = 1,2 \text{ since } f \in \mathscr{X}_s^r(\mathbb{R}^2),$$

$$\frac{\partial^3 f^1}{\partial x_1^3}(0) = \Delta_2(f,0,\frac{\partial}{\partial x_1}(0),\frac{\partial}{\partial x_1}(0)).$$

That is,

$$\begin{cases} f^1(x_1,x_2) = \Delta_2 x_1^3 + b x_1^2 x_2 + c x_1 x_2^2 + d x_2^3 + M_1(x_1,x_2) \\ f^2(x_1,x_2) = \sigma x_2 + \alpha x_1^3 + \beta x_1^2 x_2 + \gamma x_1 x_2^2 + \delta x_2^3 + M_2(x_1,x_2) \end{cases}$$

where $M_i(x_1,x_2) = o(|x|^2)$.

Assume $\sigma(f,0) < 0$ and $\Delta_2(f,0,\frac{\partial}{\partial x_1}(0), \frac{\partial}{\partial x_1}(0)) > 0$.

Let N_o and B_o be neighborhoods of 0 and f respectively such that for $g = \sum_{i=1}^{2} g^i \frac{\partial}{\partial x_i} \epsilon B$ the following hold on N_o:

(1) $\frac{\partial g^2}{\partial x_2} < 0$,

(2) $\Delta_2(g,0,v_g,v_g) = \sum_{i,j,k,l} \frac{\partial^3 g^i}{\partial x_j \partial x_k \partial x_l} v_g^j v_g^k v_g^l u_i^g > 0$

where

$$v_g^1 = 1, \; v_g^2 = -\left(\frac{\partial g^2}{\partial x_2}\right)^{-1}\left(\frac{\partial g^2}{\partial x_1}\right)$$

$$u_1^g = \left(1 + \left(\frac{\partial g^2}{\partial x_2}\right)^{-2} \frac{\partial g^2}{\partial x_1} \frac{\partial g^1}{\partial x_2}\right)^{-1}$$

$$u_2^g = -\frac{\partial g^1}{\partial x_2}\left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \cdot u_1^g,$$

(3) $\sigma(g) = \frac{\partial g^1}{\partial x_1} + \frac{\partial g^2}{\partial x_2} < 0$.

Existence of such N_o and B_o can be guaranteed because of the continuity since f satisfies (1), (2), and (3) at 0.

Take $v_g = \sum_{i=1}^{2} v_g^i \frac{\partial}{\partial x_i}$, $w_g = \sum_{i=1}^{2} w_g^i \frac{\partial}{\partial x_i}$, and $u^g = \sum_{i=1}^{2} u_i^g \, dx_i$

where

$$w_g^1 = \left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \frac{\partial g^1}{\partial x_2}, \; w_g^2 = 1.$$

If $0 \epsilon N_o$ is an equilibrium point of g and so $\Delta(g,0) = 0$, then $v_g(0)$ is an eigenvector corresponding to $\lambda = 0$, $w_g(0)$ is an eigenvector corresponding to $\lambda = \sigma(g,0)$, $u^g(0)$ is the covector such that $u^g(v_g) = 1$ and $u^g(w_g) = 0$.

Hence by (2) and (3), non-generic point $0 \epsilon N_o$ of $g \epsilon B_o$ is such that $\sigma(g,0) < 0$ and $\Delta_2(g,0,v_g,v_g) > 0$, i.e., 0 is a pitchfork.

Define $F: B_o \times N_o \to \mathbb{R}$ by $F(g,(x_1,x_2)) = g^2(x_1,x_2)$. F is C^r since g^2 is and it is an evaluation map. $F(f,(0,0)) = 0$, $\frac{\partial F}{\partial x_2}(f,(0,0)) = \frac{\partial f^2}{\partial x_2}(0,0) = \sigma(f,0) < 0$. By implicit function theorem, there exist $B_1 \times I_1$ a neighborhood of $(f,0)$ and I_2 a neighborhood of 0, $I_i \subset \mathbb{R}$ $(i=1,2)$ and $F_1: B_1 \times I_1 \to I_2$ a unique C^r–function such that

(*1) $\quad F_1(f,0) = 0,\ F(g,(x_1,F_1(g,x_1))) \equiv 0$ for every $(g,x_1) \epsilon B_1 \times I_1$.

Define $F_2: B_1 \times I_1 \to \mathbb{R}$ by

(*2) $\quad F_2(g,x_1) = g^1(x_1,F_1(g,x_1))$.

From (*1) and (*2), we have

(4) $\quad \frac{\partial F_2}{\partial x_1} = \left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \Delta(g)$

(5) $\quad \frac{\partial^2 F_2}{\partial x_1^2} \equiv 0$ in $B_1 \times I_1$

(6) $\quad \frac{\partial^3 F_2}{\partial x_1^3} = (u_1^g)^{-1}\ \Delta_2(g,0,v_g,v_g) > 0.$

The computations of (4), (5), and (6) will be later.

Define $\alpha: B_o \to \mathbb{R}$ by

$$\alpha(g) = \frac{\partial F_2}{\partial x_1}(g,0) = \frac{\partial g^1}{\partial x_1}(0,F_1(g,0)).$$

Then α is C^{r-1} and $\alpha(g)$ is the minimum of $\frac{\partial F_2}{\partial x_1}(g,x_1)$ at $x_1 = 0\ \epsilon I_1$ by (5) and (6). $\frac{\partial F_2}{\partial x_1}(g,x_1)$ is even in x_1.

(i) Case 1: $\alpha(g) > 0$.

There is a unique equilibrium point $(0,0)\epsilon N$ which is saddle since

$\alpha(g) > 0$ if and only if $\frac{\partial f_2}{\partial x_1}(g,0) > 0$ if and only if

$\Delta(g)(x_1) < 0$ at $x_1 = 0$ by (4) and (1).

(ii) Case 2: $\alpha(g) = 0$.

$\alpha(g) = 0$ if and only if $\frac{\partial F_2}{\partial x_1}(g,0) = 0$ if and only if

$\Delta(g)(x_1) = 0$ at $x_1 = 0$. Hence (0,0) is a pitchfork by (6) since g is symmetric.

(iii) Case 3: $\alpha(g) < 0$

$\alpha(g) < 0$ if and only if $\frac{\partial F_2}{\partial x_1}(g,0) < 0$ if and only if

$\Delta(g)(x_1) = 0$ at $x_1 = 0$. Hence (0,0) is node. Let zeros of $F_2(g,x_1)$ be $x_1 = 0$, $\gamma(g)$, and $-\gamma(g)$.

That is,

(0,0) and $(\pm\gamma(g),\ F_1(g,\pm\gamma(g)))$ are zeros of g.

$\frac{\partial F_2}{\partial x_1}(g,\pm\gamma(g)) > 0$ if and only if $\Delta(g)(\pm\gamma(g)) < 0$.

Hence $(\pm\gamma(g),\ F_1(g,\pm\gamma(g)))$ are saddle.

Next,

$$\alpha(f) = \frac{\partial F_2}{\partial x_1}(f,0) = \frac{\partial f_1}{\partial x_1}(0,F_1(f,0)) = \frac{\partial f^1}{\partial x_1}(0,0) = 0.$$

For $h = \sum_{i=1}^{2} h_i \frac{\partial}{\partial x_1} \epsilon\ \mathscr{X}^r_s(\mathbb{R}^2)$,

$$
\begin{aligned}
d\alpha_f(h) &= \lim_{\epsilon\to 0} \frac{\alpha(f+\epsilon h)-\alpha(f)}{\epsilon} \\
&= \lim_{\epsilon\to 0} (1/\epsilon)\ \alpha(f+\epsilon h) \\
&= \lim_{\epsilon\to 0} (1/\epsilon)[\frac{\partial (f+\epsilon h)^1}{\partial x_1}(0,F_1(f+\epsilon h,0))] \\
&= \lim_{\epsilon\to 0} (1/\epsilon)\ \frac{\partial f^1}{\partial x_1}(0,F_1(f+\epsilon h,0)) + \lim_{\epsilon\to 0} \frac{\partial h^1}{\partial x_1}(0,F_1(f+\epsilon h,0)) \\
&= \frac{\partial^2 f^1}{\partial x_1 \partial x_2}(0,0)\ \frac{\partial F_1}{\partial \epsilon}(f+\epsilon h,0)\Big|_{\epsilon=0} + \frac{\partial h^1}{\partial x_1}(0,0) \\
&= \frac{\partial h^1}{\partial x_1}(0,0).
\end{aligned}
$$

Hence $d\alpha_f \neq 0$. □

Computations of (4), (5), and (6).

For $(g,x_1)\ \epsilon B_1 \times I_1$, $g^2(x_1,F_1(g,x_1)) \equiv 0$.

So $\frac{\partial g^2}{\partial x_1} + \frac{\partial g^2}{\partial x_2}\frac{\partial F_1}{\partial x_1} = 0$ and then

$$\frac{\partial F_1}{\partial x_1} = -\frac{\partial g^2}{\partial x_1}\left(\frac{\partial g^2}{\partial x_2}\right)^{-1}.$$

$$
\begin{aligned}
(4)\quad \frac{\partial F_2}{\partial x_1} &= \frac{\partial g^1}{\partial x_1} + \frac{\partial g^1}{\partial x_2}\frac{\partial F_1}{\partial x_1} \\
&= \frac{\partial g^1}{\partial x_1} + \frac{\partial g^1}{\partial x_2}\left(-\frac{\partial g^2}{\partial x_1}\right)\left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \\
&= \left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \Delta(g).
\end{aligned}
$$

$$
\begin{aligned}
(5)\quad \frac{\partial^2 F_2}{\partial x_1^2} &= \frac{\partial^2 g^1}{\partial x_1^2} + \frac{\partial^2 g^1}{\partial x_1 \partial x_2}\frac{\partial F_1}{\partial x_1} + \left(\frac{\partial^2 g^1}{\partial x_1 \partial x_2} + \frac{\partial^2 g^1}{\partial x_2^2}\frac{\partial F_1}{\partial x_1}\right)\left(\frac{\partial F_1}{\partial x_1}\right) + \frac{\partial g^1}{\partial x_2}\frac{\partial^2 F_1}{\partial x_1^2} \\
&= \frac{\partial^2 g^1}{\partial x_1^2} + 2\,\frac{\partial^2 g^1}{\partial x_1 \partial x_2}\frac{\partial F_1}{\partial x_1} + \frac{\partial^2 g^1}{\partial x_2^2}\left(\frac{\partial F_1}{\partial x_1}\right)^2 + \frac{\partial g^1}{\partial x_2}\frac{\partial^2 F_1}{\partial x_1^2} = 0
\end{aligned}
$$

since

$$\frac{\partial^2 g^i}{\partial x_j \partial x_k} = 0 \quad (i,j,k = 1,2),$$

$$\frac{\partial^2 g^2}{\partial x_1^2} + \frac{\partial^2 g^2}{\partial x_1 \partial x_2} \frac{\partial F_1}{\partial x_1} + \left(\frac{\partial^2 g^2}{\partial x_1 \partial x_2} + \frac{\partial^2 g^2}{\partial x_2^2} \frac{\partial F_1}{\partial x_1}\right) \frac{\partial F_1}{\partial x_1} + \frac{\partial g^2}{\partial x_2} \frac{\partial^2 F_1}{\partial x_1^2} = 0, \text{ and}$$

$$\frac{\partial g^2}{\partial x_2} \neq 0.$$

$$(6) \quad \frac{\partial^3 F_2}{\partial x_1^3} = \frac{\partial^3 g^1}{\partial x_1^3} + \frac{\partial^3 g^1}{\partial x_1^2 \partial x_2} \frac{\partial F_1}{\partial x_1} + 2\left(\frac{\partial^3 g^1}{\partial x_1^2 \partial x_2} + \frac{\partial^3 g^1}{\partial x_1 \partial x_2^2} \frac{\partial F_1}{\partial x_1}\right) \frac{\partial F_1}{\partial x_1}$$

$$+ 2 \frac{\partial^2 g^1}{\partial x_1 \partial x_2} \frac{\partial F_1}{\partial x_1} \frac{\partial^2 F_1}{\partial x_1^2} + \left(\frac{\partial^3 g^1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 g^1}{\partial x_2^3} \frac{\partial F_1}{\partial x_1}\right) \left(\frac{\partial F_1}{\partial x_1}\right)^2$$

$$+ 2 \frac{\partial^2 g^1}{\partial x_2^2} \frac{\partial F_1}{\partial x_1} \frac{\partial^2 F_1}{\partial x_1^2} + \left(\frac{\partial^2 g^1}{\partial x_1 \partial x_2} + \frac{\partial^2 g^1}{\partial x_2^2} \frac{\partial F_1}{\partial x_1}\right) \frac{\partial^2 F_1}{\partial x_1^2} + \frac{\partial g^1}{\partial x_2} \frac{\partial^3 F_1}{\partial x_1^3}$$

$$= \frac{\partial^3 g^1}{\partial x_1^3} + 3 \frac{\partial^3 g^1}{\partial x_1^2 \partial x_2} \frac{\partial F_1}{\partial x_1} + 3 \frac{\partial^3 g^1}{\partial x_1 \partial x_2^2} \left(\frac{\partial F_1}{\partial x_1}\right)^2 + \frac{\partial^3 g^1}{\partial x_2^3} \left(\frac{\partial g^2}{\partial x_1}\right)^3$$

$$+ \frac{\partial g^1}{\partial x_2} \frac{\partial^3 F_1}{\partial x_1^3}.$$

Since
$$\frac{\partial F_1}{\partial x_1} = -\left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \frac{\partial g^2}{\partial x_1} \quad \text{and}$$

$$\frac{\partial^3 F_1}{\partial x_1^3} = -\left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \left[\frac{\partial^3 g^2}{\partial x_1^3} + \frac{\partial^3 g^2}{\partial x_1^2 \partial x_2} \left(-3 \left(\frac{\partial f^2}{\partial x_2}\right)^{-1} \frac{\partial g^2}{\partial x_1}\right)\right.$$

$$\left. + 3 \frac{\partial^3 g^2}{\partial x_1 \partial x_2^2} \left(\frac{\partial g^2}{\partial x_2}\right)^{-1} \left(\frac{\partial f^2}{\partial x_1}\right)^2 - \frac{\partial^3 g^2}{\partial x_2^3} \left(\frac{\partial g^2}{2 x_2}\right)^{-3} \frac{\partial g^2}{\partial x_1}\right],$$

we have

$$\frac{\partial^3 F_2}{\partial x_1^3} (u_1^g) - \Delta_2(g, v_g, v_g) = 0.$$

FIGURES

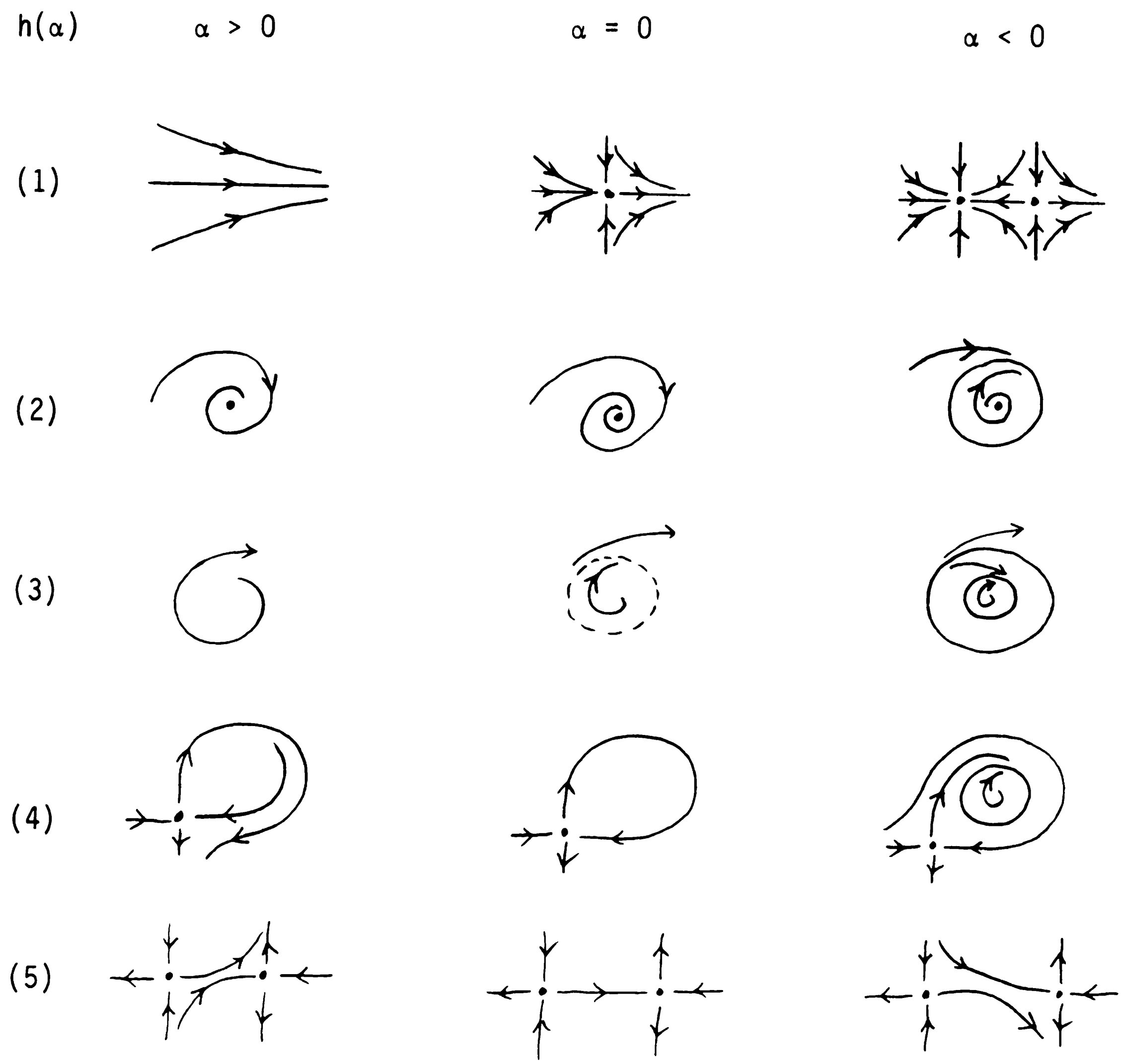


Figure 1

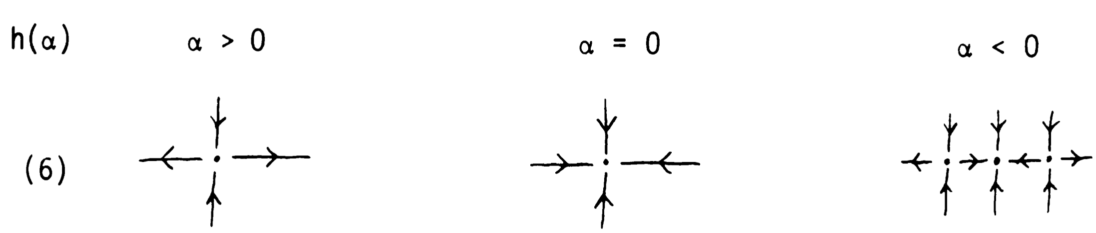


Figure 2

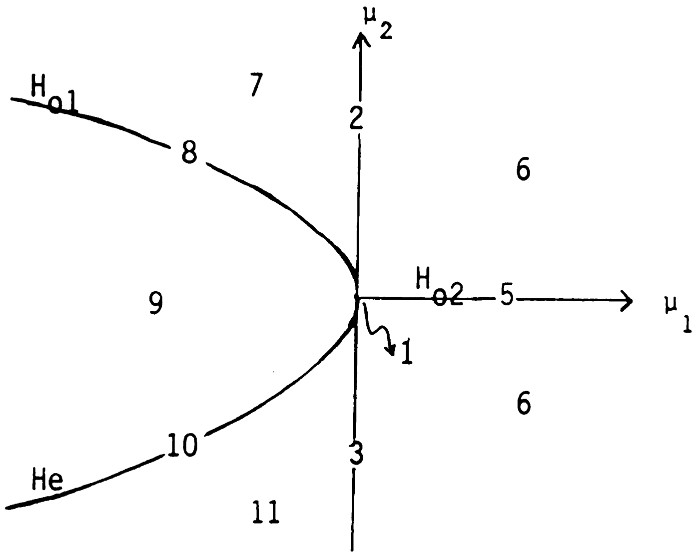
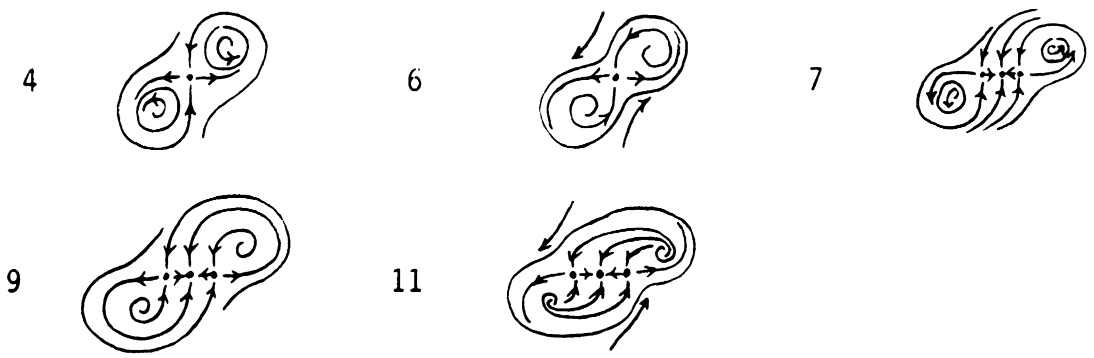
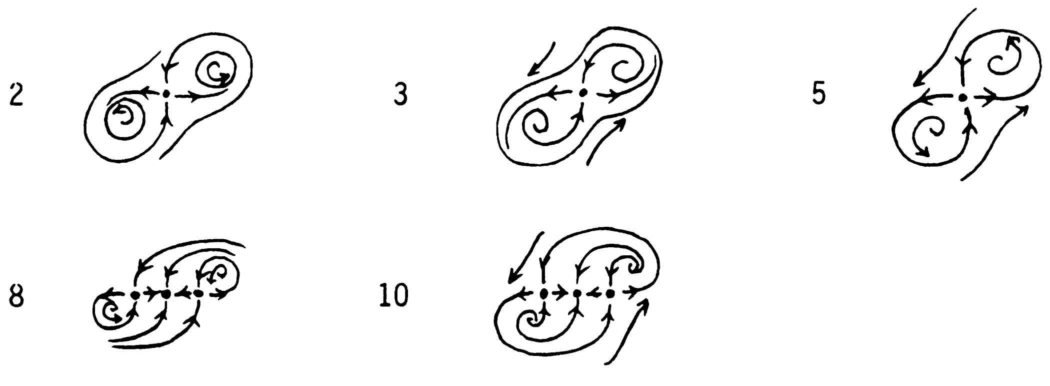


Figure 3 (a)

(b) Codimension 0:



(c) Codimension 1:



(d) Codimension 2:

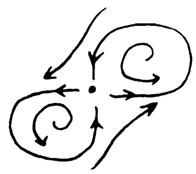
1. 

Figure 3

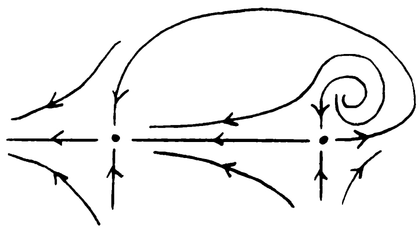
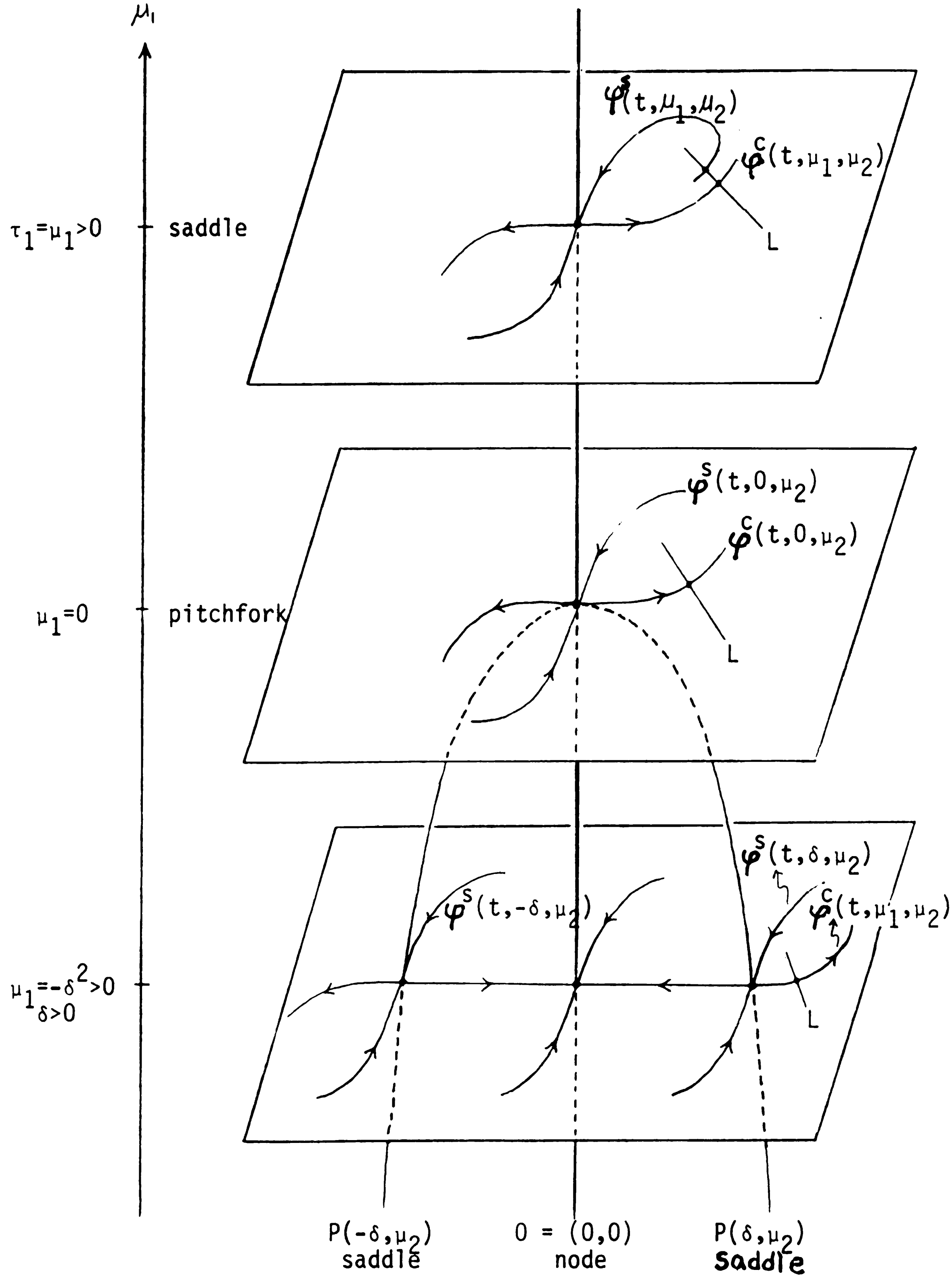
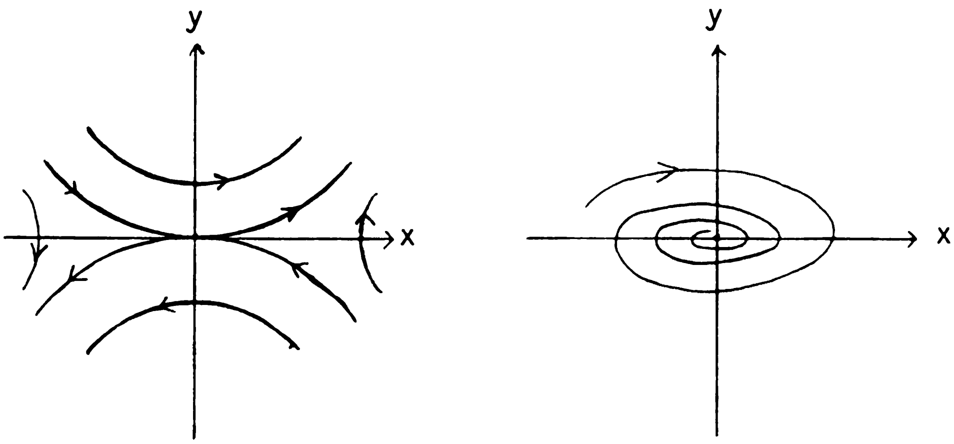


Figure 4



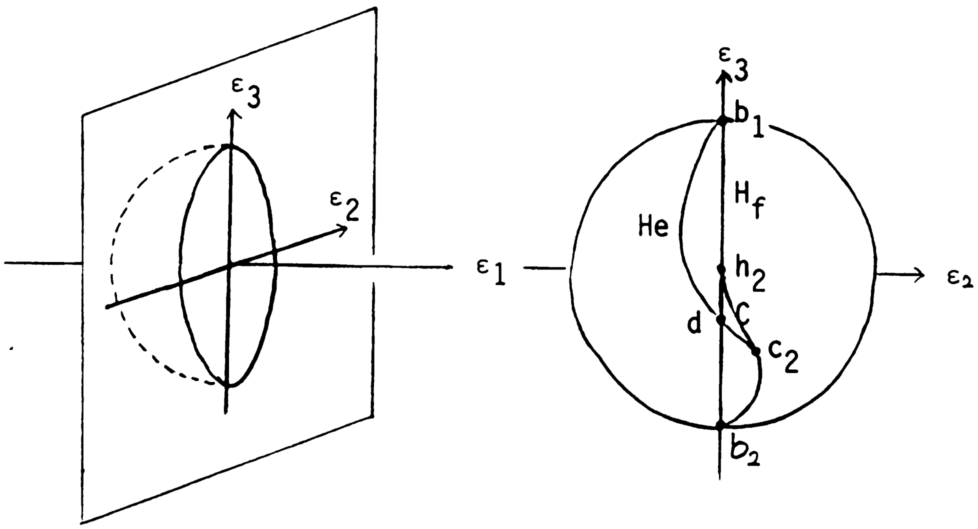
μ_1
$\tau_1=\mu_1>0$
saddle
$\varphi^s(t,\mu_1,\mu_2)$
$\varphi^c(t,\mu_1,\mu_2)$
L
$\mu_1=0$
pitchfork
$\varphi^s(t,0,\mu_2)$
$\varphi^c(t,0,\mu_2)$
L
$\mu_1=-\delta^2>0$
$\delta>0$
$\varphi^s(t,-\delta,\mu_2)$
$\varphi^s(t,\delta,\mu_2)$
$\varphi^c(t,\mu_1,\mu_2)$
L
$P(-\delta,\mu_2)$
saddle
$0=(0,0)$
node
$P(\delta,\mu_2)$
saddle

Figure 5



(a) degenerate saddle (b) degenerate focus

Figure 6



(a) The parameter space $\varepsilon=(\varepsilon_1,\varepsilon_2,\varepsilon_3)$ on S

(b) The trace of the bifurcation diagram on S ($\varepsilon_1 \leq 0$)

Figure 7

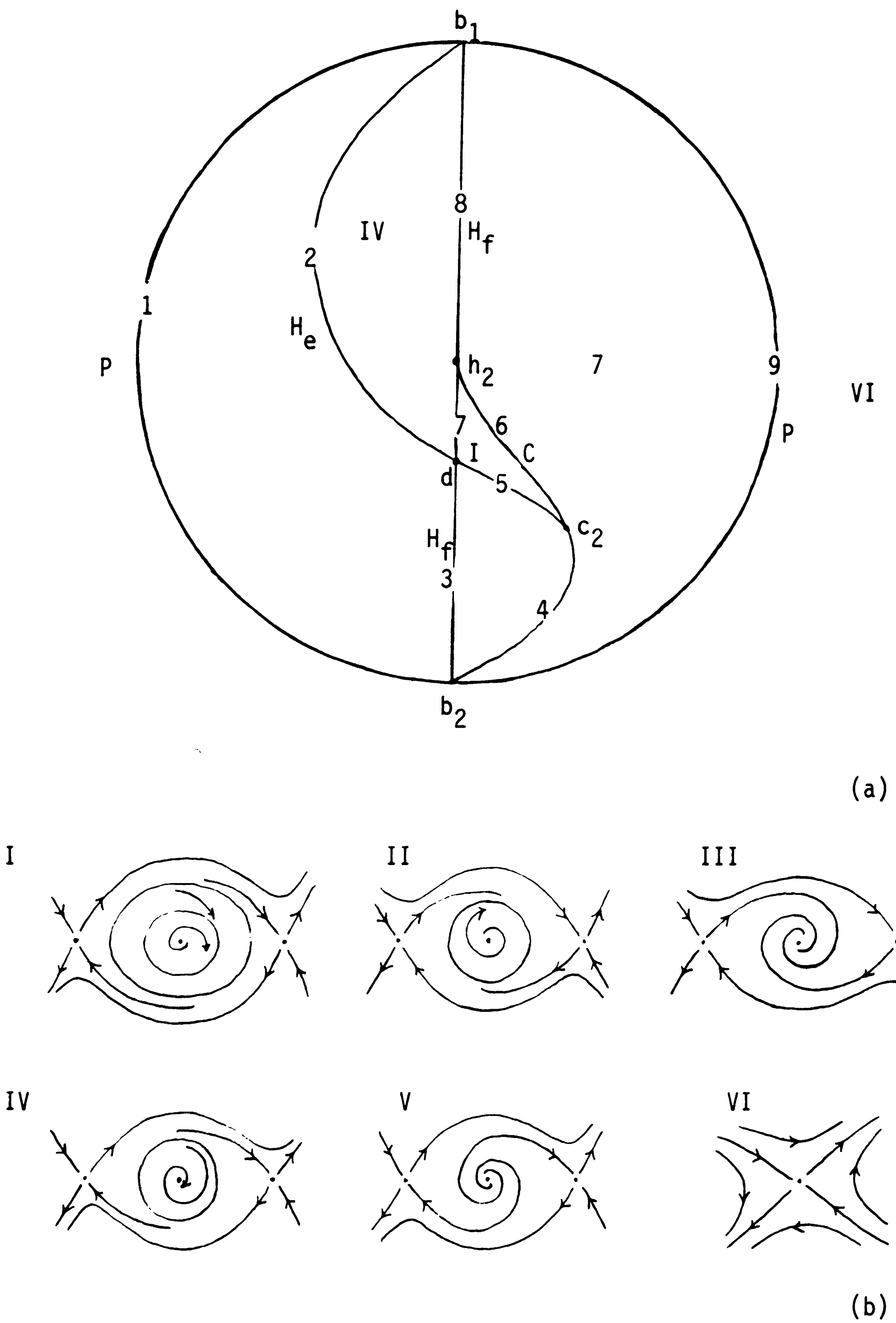
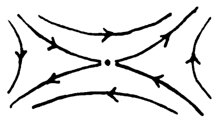


Figure 8: Phase portraits of codimension 0

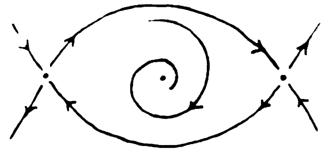
1 and 9

pitchfork



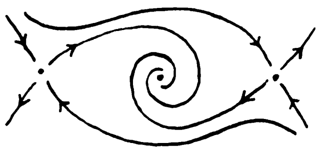
2

heteroclinic



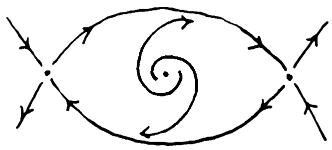
3

Hopf



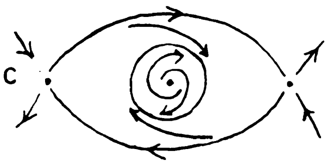
4

heteroclinic



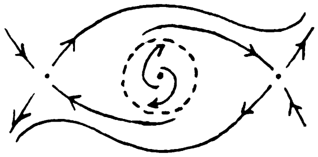
5

heteroclinic



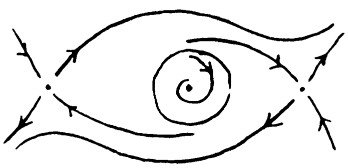
6

semistable



7

Hopf



8

Hopf

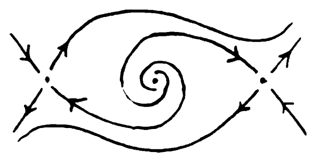
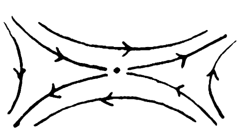


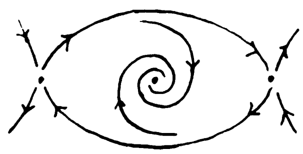
Figure 8: Phase portraits of codimension 1 (c)

b_1 and b_2:



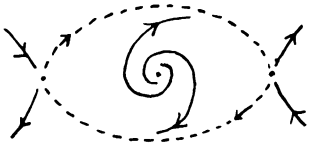
degenerate saddle of codim 2

d:



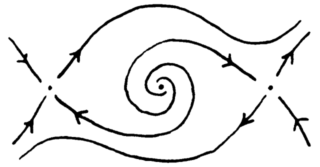
heteroclinic - Hopf

C_2:



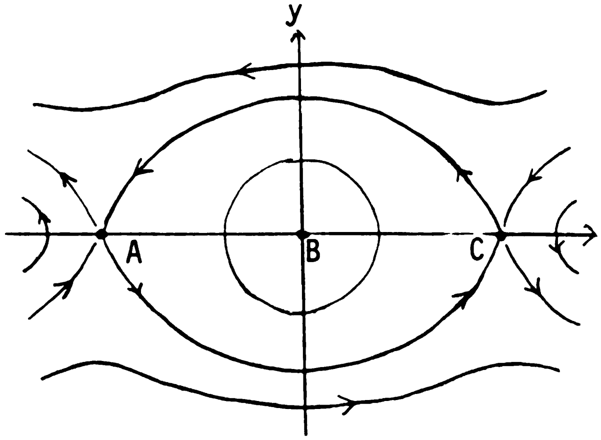
heteroclinic loop of codim 3

h_2:



Hopf of codim 2 (d)

Figure 8: Phase portraits of codimension 2

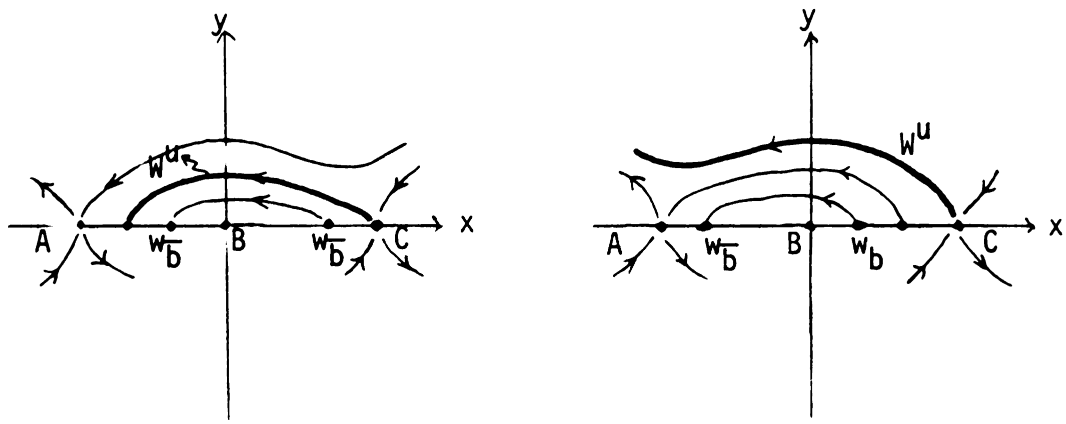


A = (-1,0)

B = (0,0)

C = (1,0)

Figure 9
Level curves of (3.3.4)



(a) $P_{\lambda_S}(w_b)=w_{\bar{b}}$ $(W^u \cap (-U) \neq \{\})$

(b) $P^{-1}_{\lambda_s}(w_{\bar{b}})=w_b(W^u \cap (-U)=\{\})$.

Figure 10

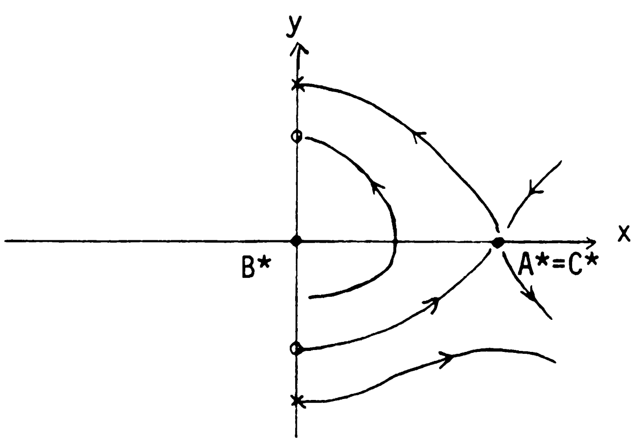


Figure 11

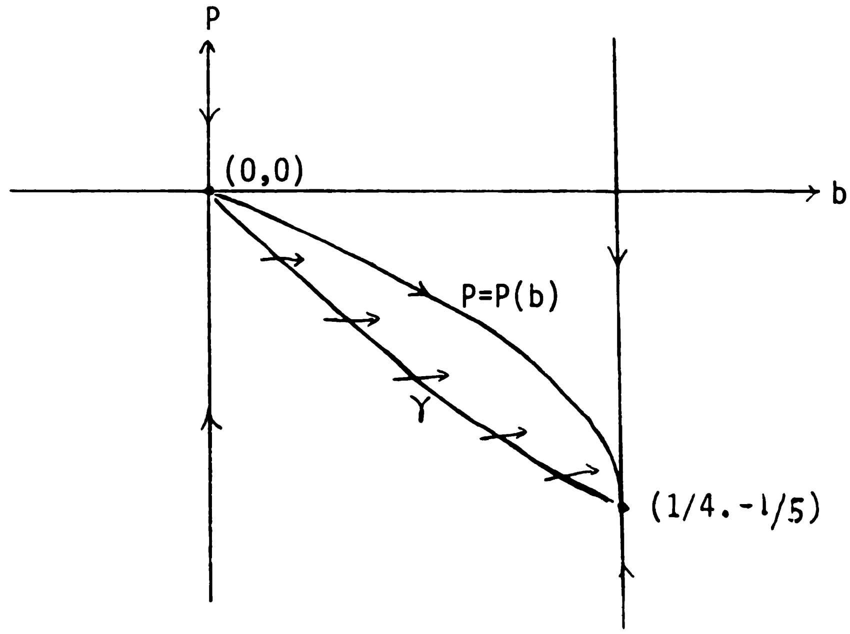


Figure 12: Phase portrait of (3.3.21) in (b,P)-plane

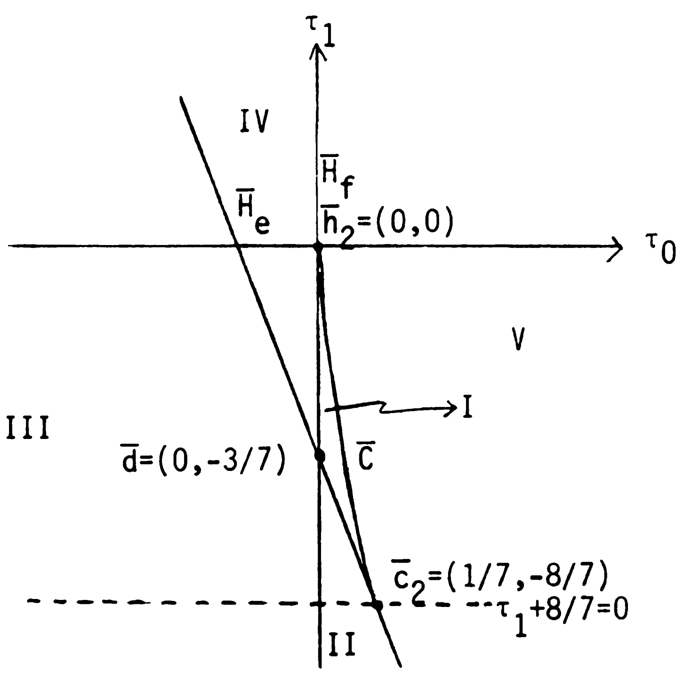
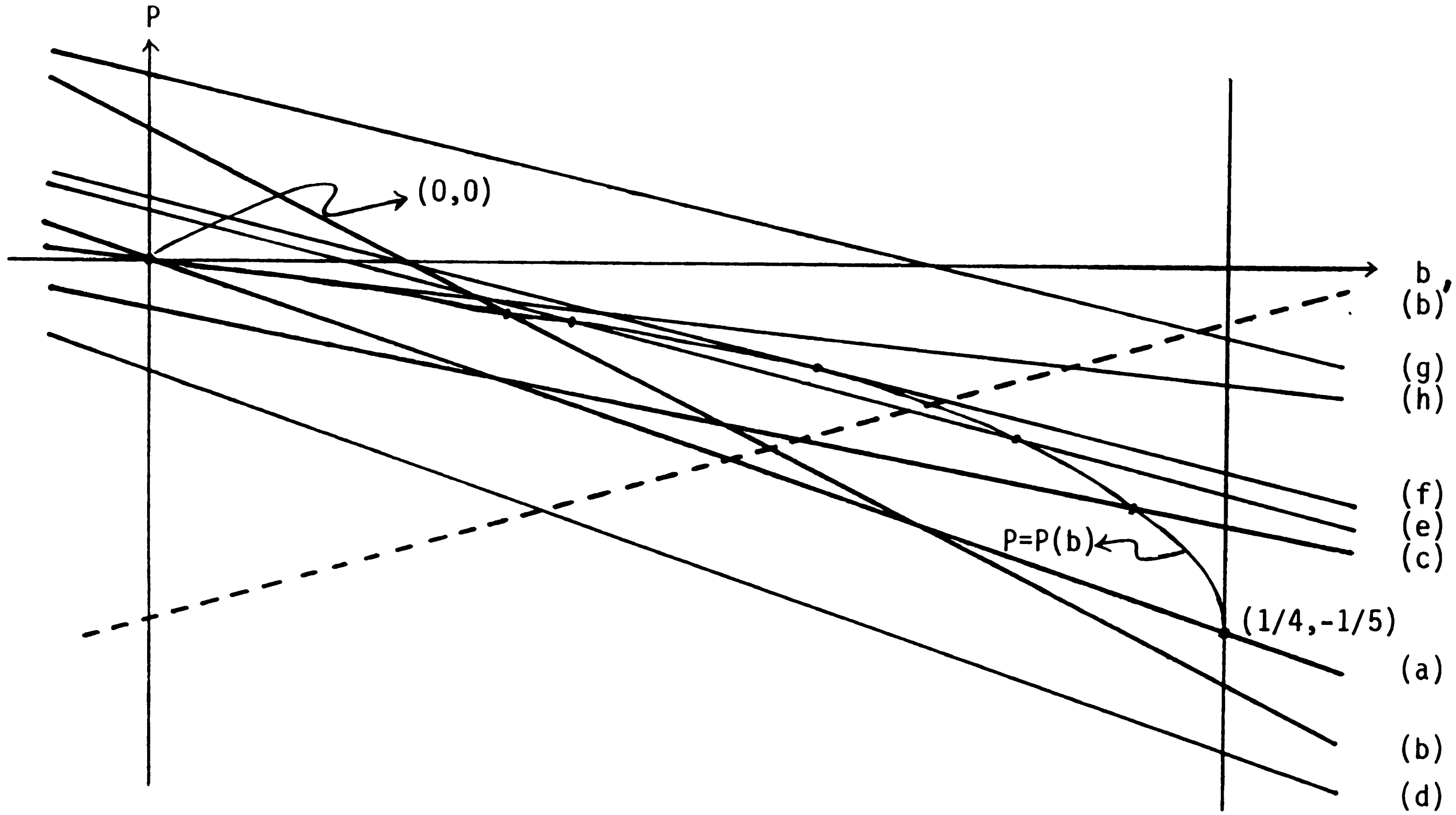
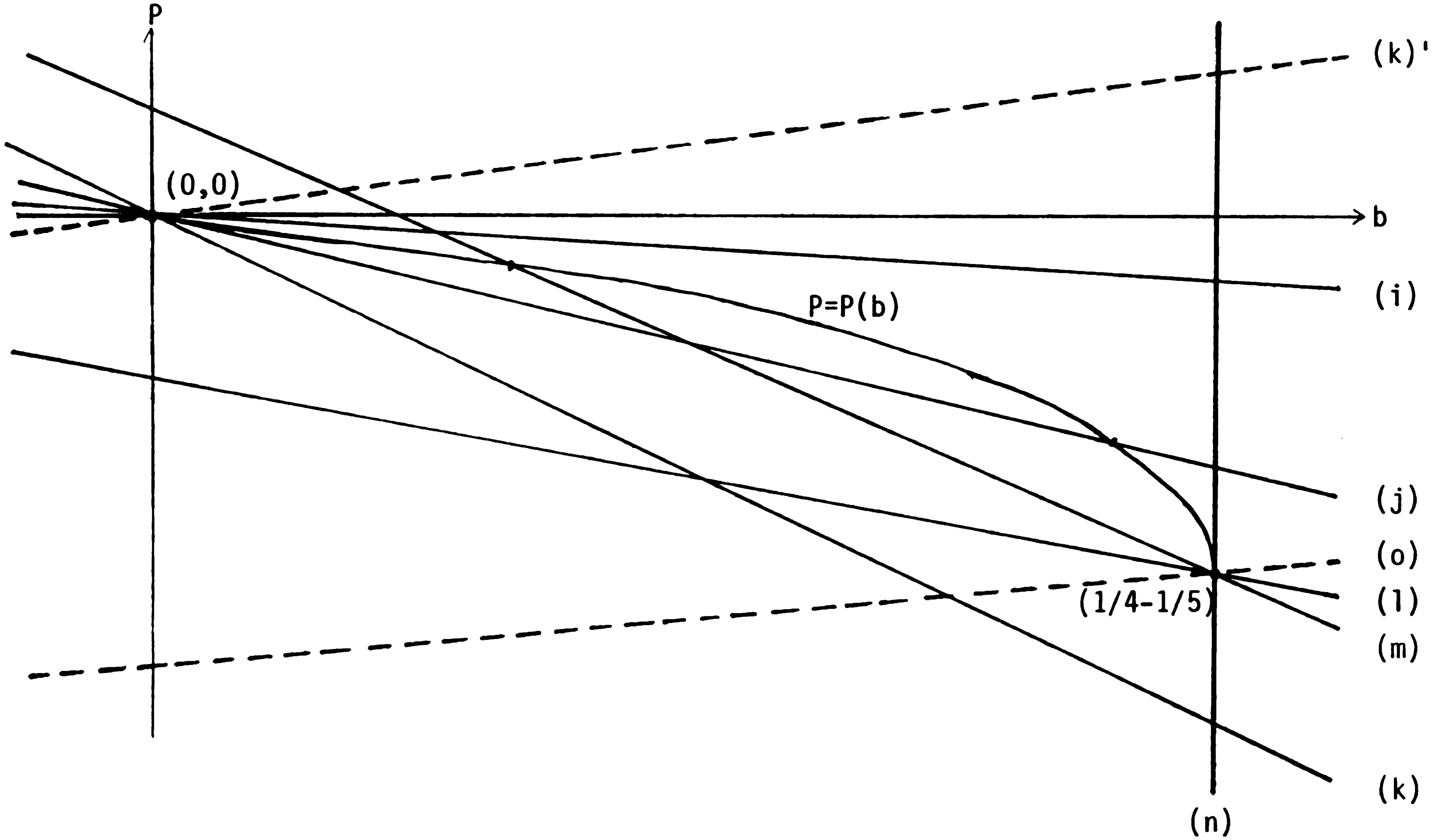


Figure 13: The bifurcation diagram of equation (3.3.2) in (τ_0, τ_1)-plane for small s>0



(a) $(\tau_0,\tau_1)=\bar{d}$ (b) $(\tau_0,\tau_1)\varepsilon II$ ((b)') (c) $(\tau_0,\tau_1)\varepsilon IV$ (d) $(\tau_0,\tau_1)\varepsilon III$

(e) $(\tau_0,\tau_1)\varepsilon I$ (f) $(\tau_0,\tau_1)\varepsilon\bar{C}$ (g) $(\tau_0,\tau_1)\varepsilon V$ (h) $(\tau_0,\tau_1)=\bar{h}_2$



(i) $(\tau_0,\tau_1)\varepsilon\bar{H}_f$ above $\bar{h}_2$

(j) $(\tau_0,\tau_1)\varepsilon\overline{h_2 d}$

(k) ((k)') $(\tau_0,\tau_1)\varepsilon\bar{H}_f$ below $\bar{d}$

(l) $(\tau_0,\tau_1)\varepsilon\bar{H}_e$ above $\bar{d}$

(m) $(\tau_0,\tau_1)\varepsilon\overline{dc_2}$

(n) $(\tau_0,\tau_1)=\bar{c}_2$

(o) $(\tau_0,\tau_1)\varepsilon\bar{H}_e$ below $\bar{c}_2$

Figure 14: The relative positions of the curve P=P(b) and the straight line P=A(b) (for $\tau_1+8/7>0$, for $\tau_1+8/7<0$)

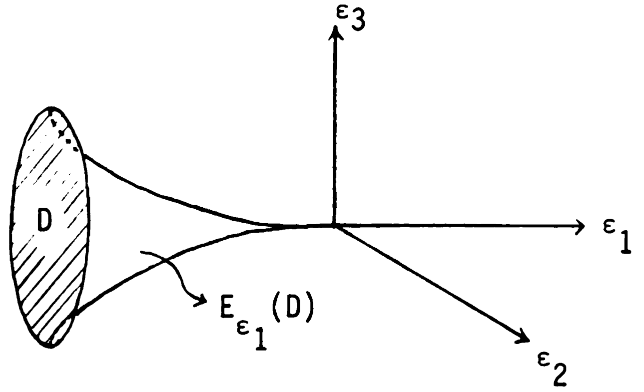


Figure 15

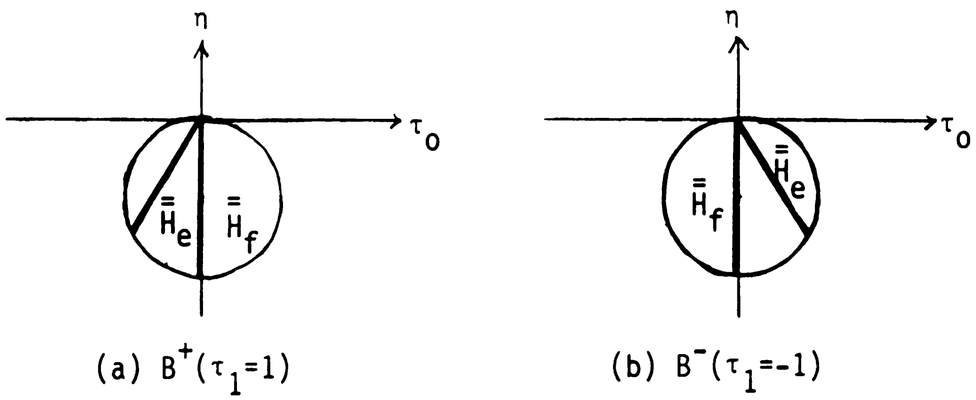


Figure 16

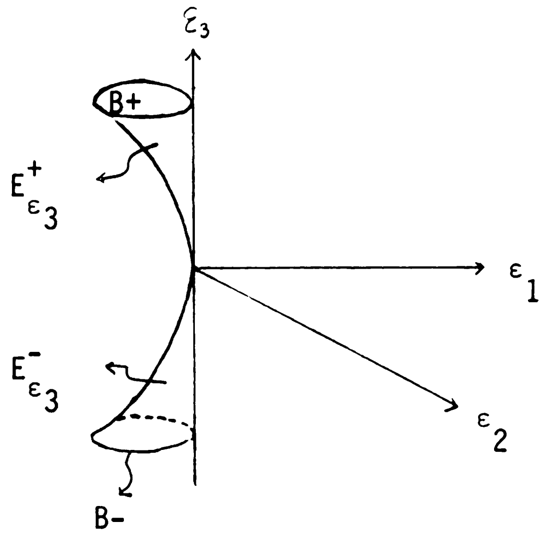


Figure 17

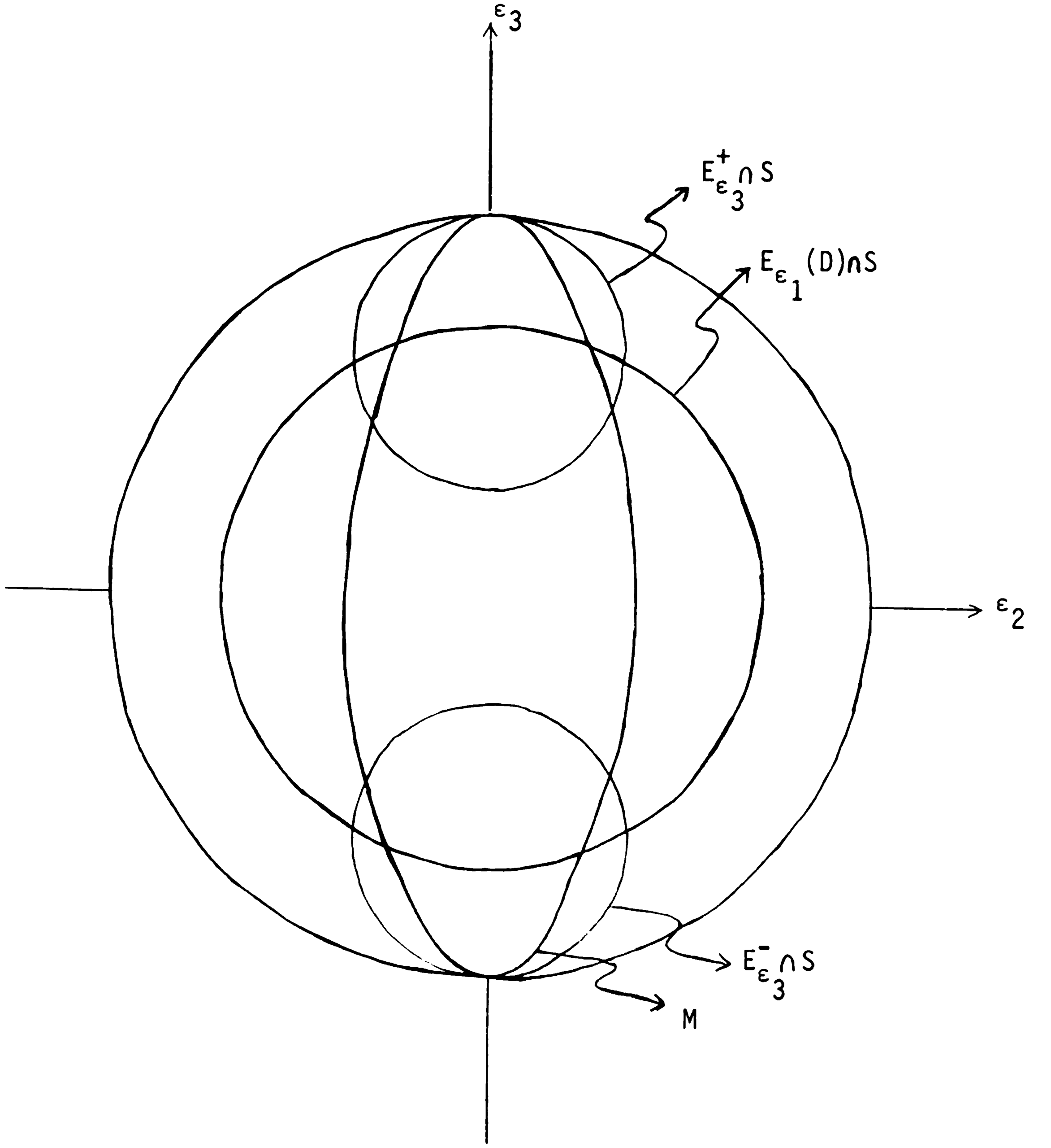

ε_3
$E^+_{\varepsilon_3} \cap S$
$E_{\varepsilon_1}(D) \cap S$
ε_2
$E^-_{\varepsilon_3} \cap S$
M

Figure 18

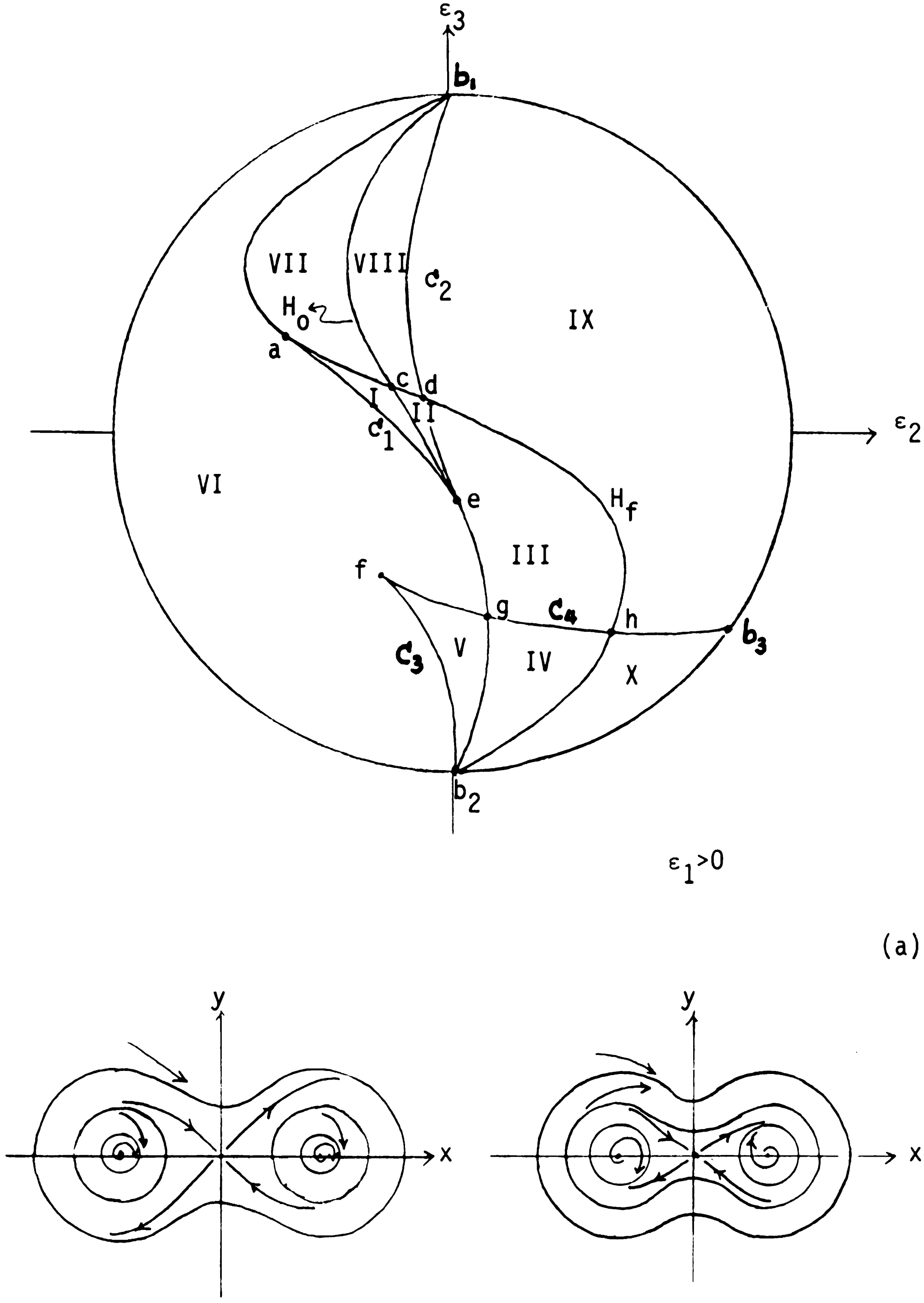
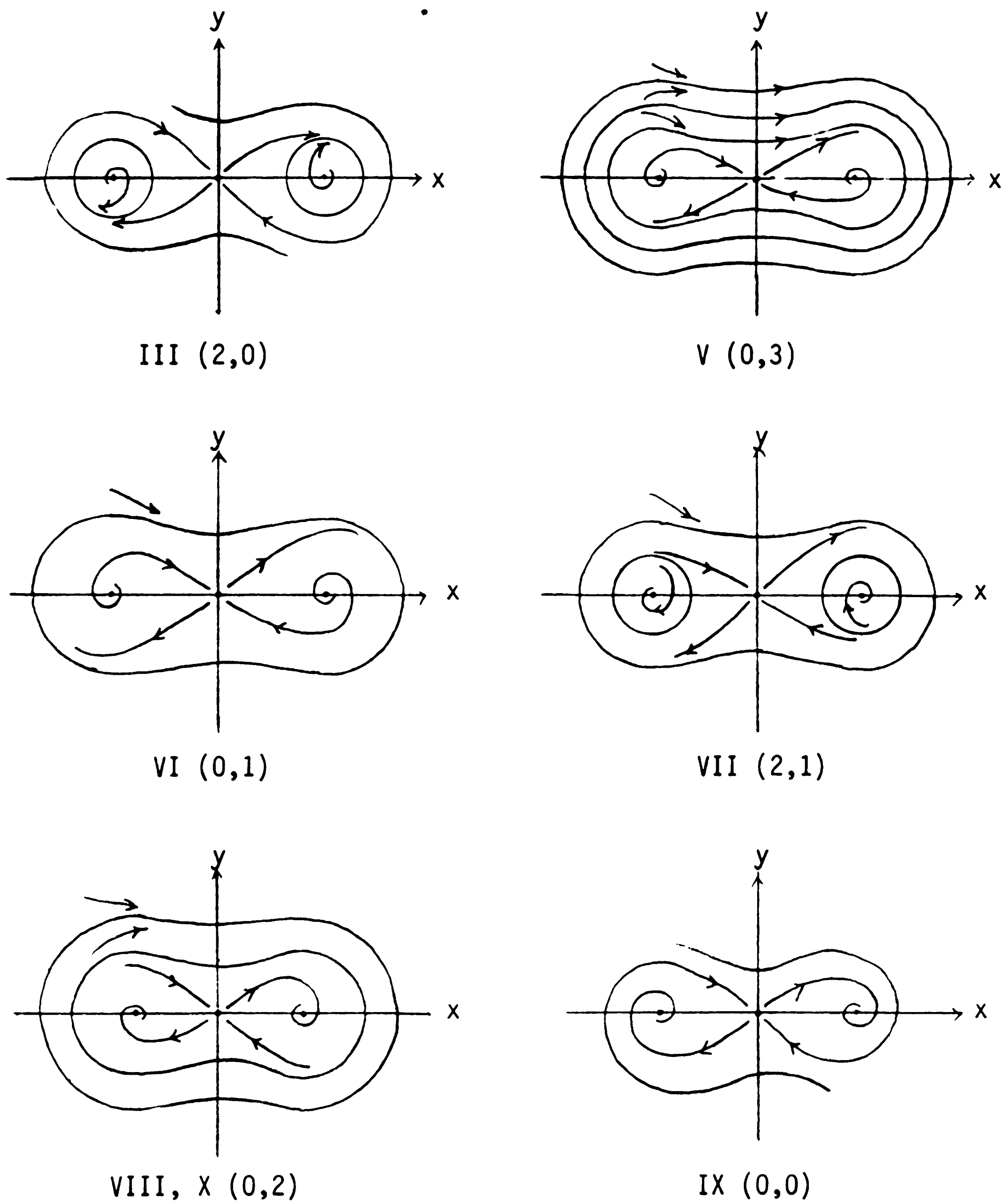

ε3
b1
VII
VIII
c2
IX
H0
a
c
d
I
II
c1
ε2
VI
e
Hf
III
f
g
c4
h
b3
c3
V
IV
X
b2
ε1>0
(a)
y
x
y
x
I (4,1)
II, IV (2,2)

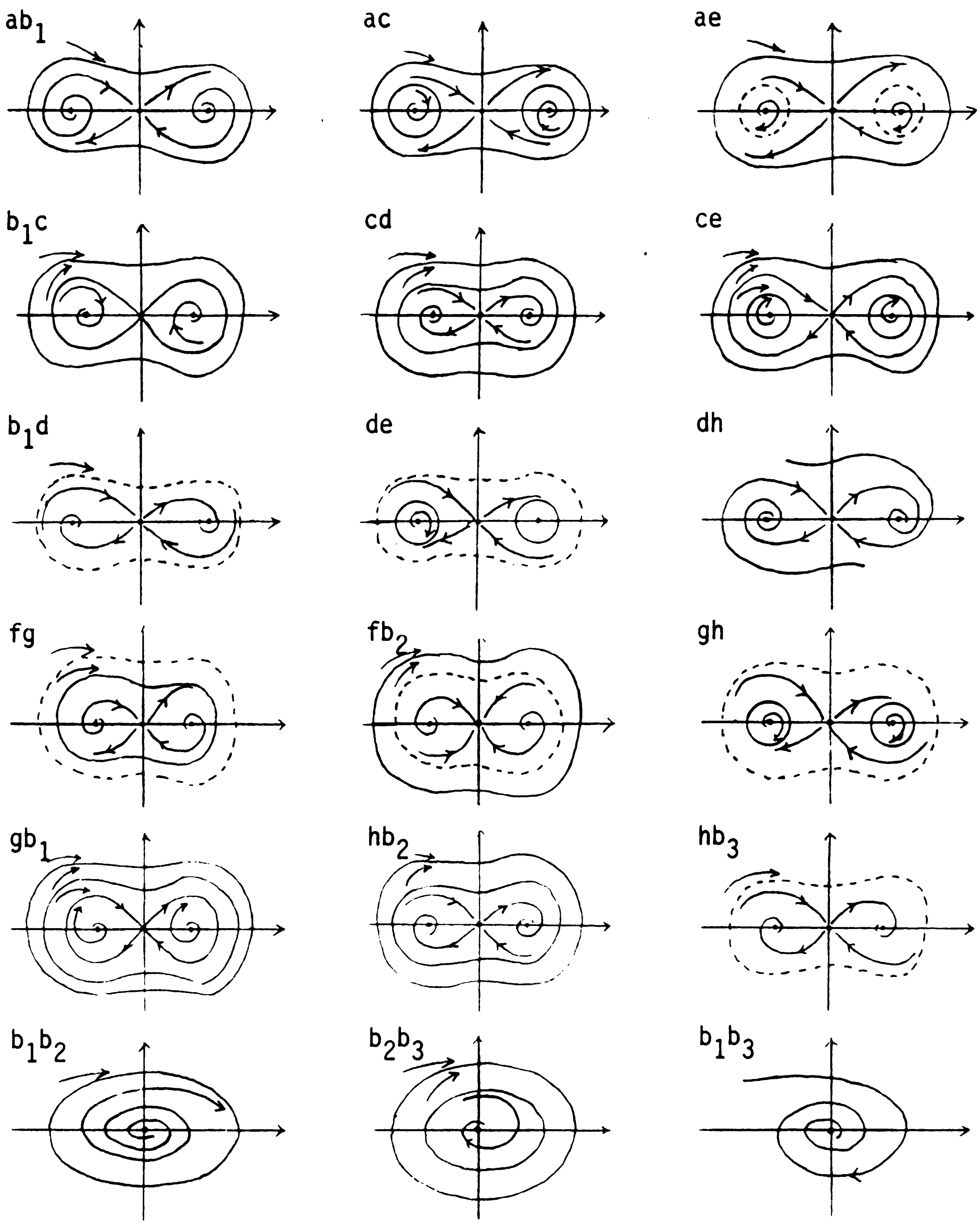
Figure 19

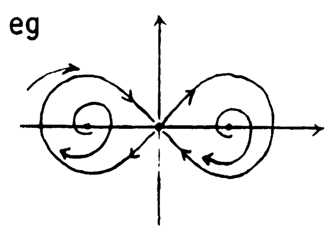


(m,n) m = no. of inner limit cycles.0,2,4

n = no. of outer limit cycles,0,1,2,3

Figure 19: Phase portraits of codimension 0 (b)





(c)

Figure 19: Phase portraits of codimension 1

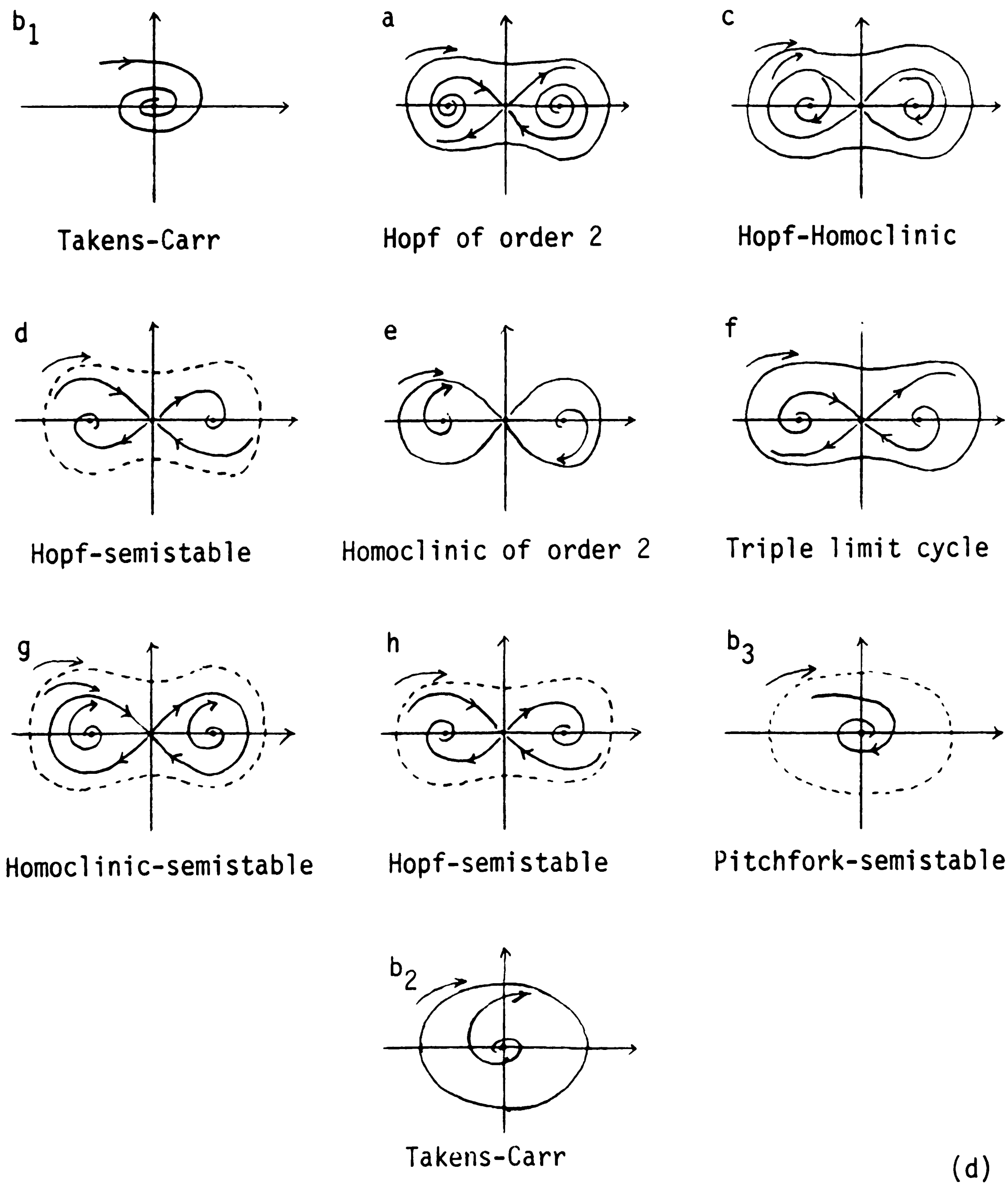
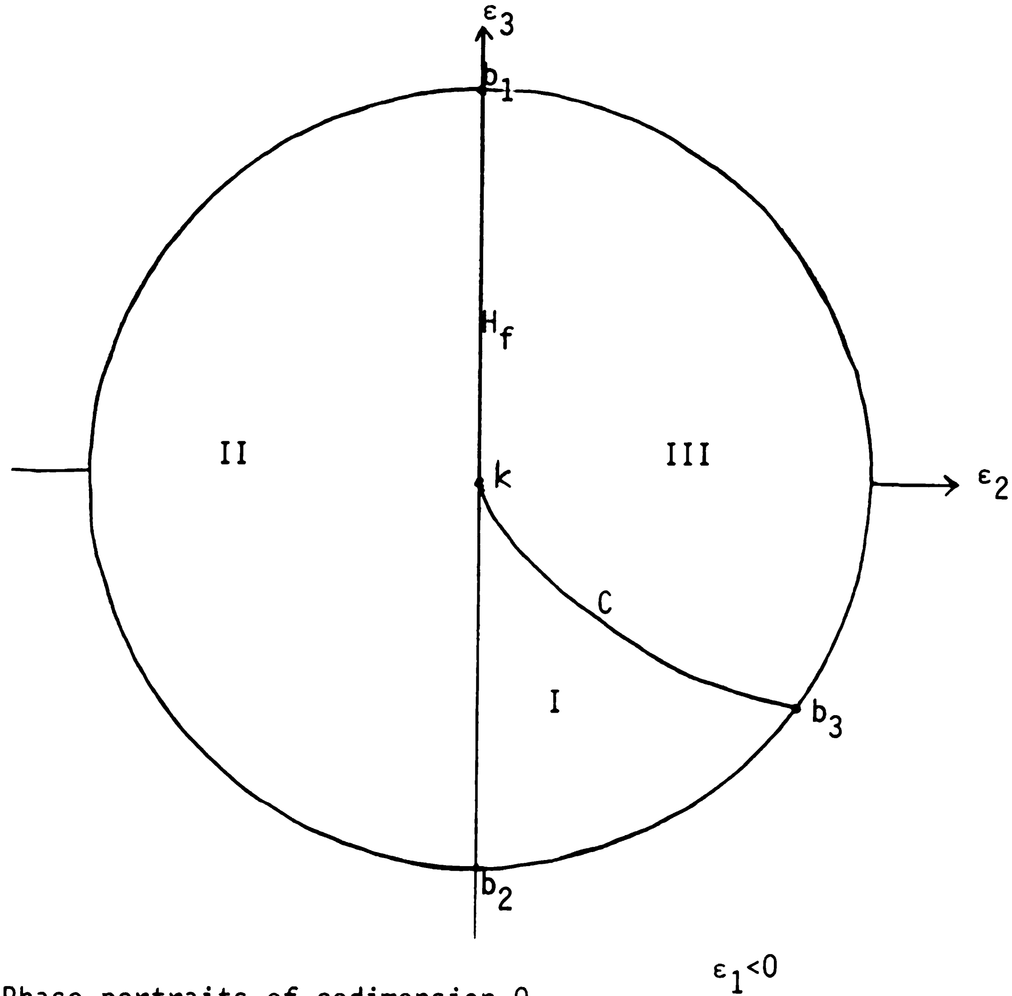
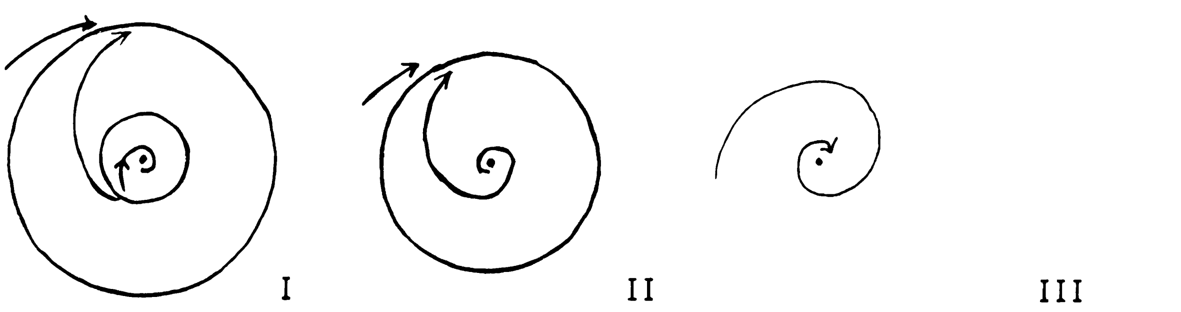


Figure 19: Phase portraits of codimension 2

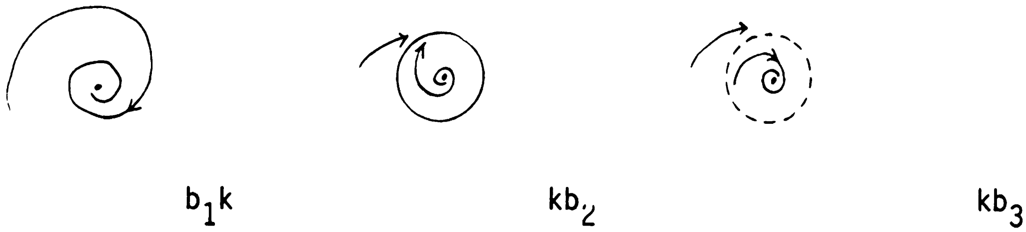
(a)



(b) Phase portraits of codimension 0

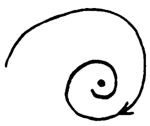


(c) Phase portraits of codimension 1



For b_1b_2, b_2b_3 and b_1b_3, See Figure 19 (c)

(d) phase portraits of codimension 2



Hopf of order 2

For b_1, b_2 and b_3, see Figure 19 (d)

Figure 20

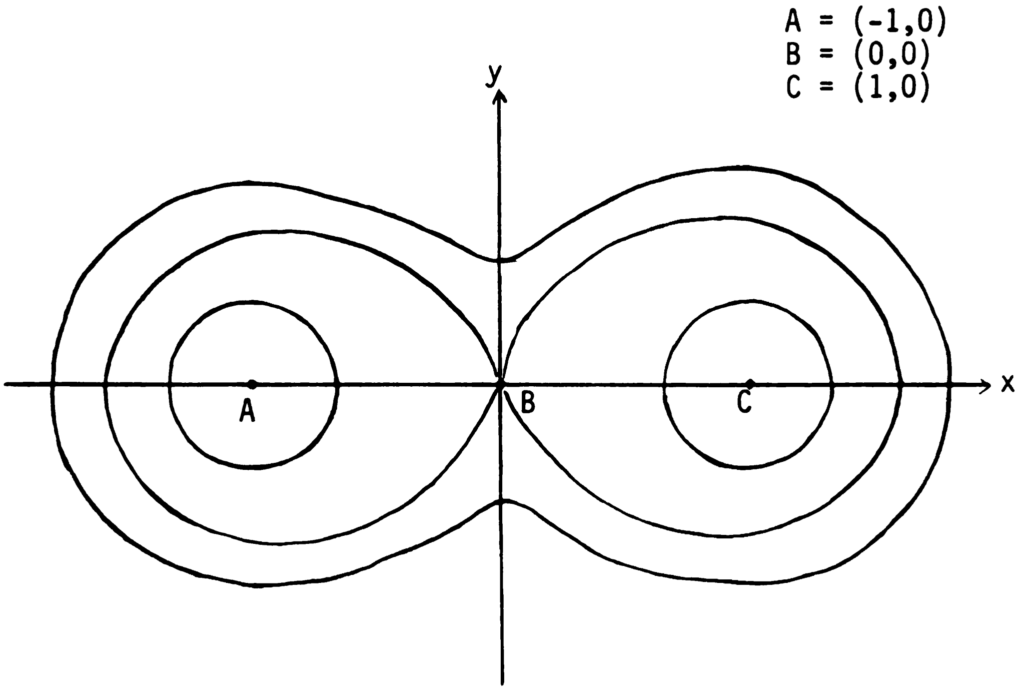


Figure 21: Level curves of (3.4.2)

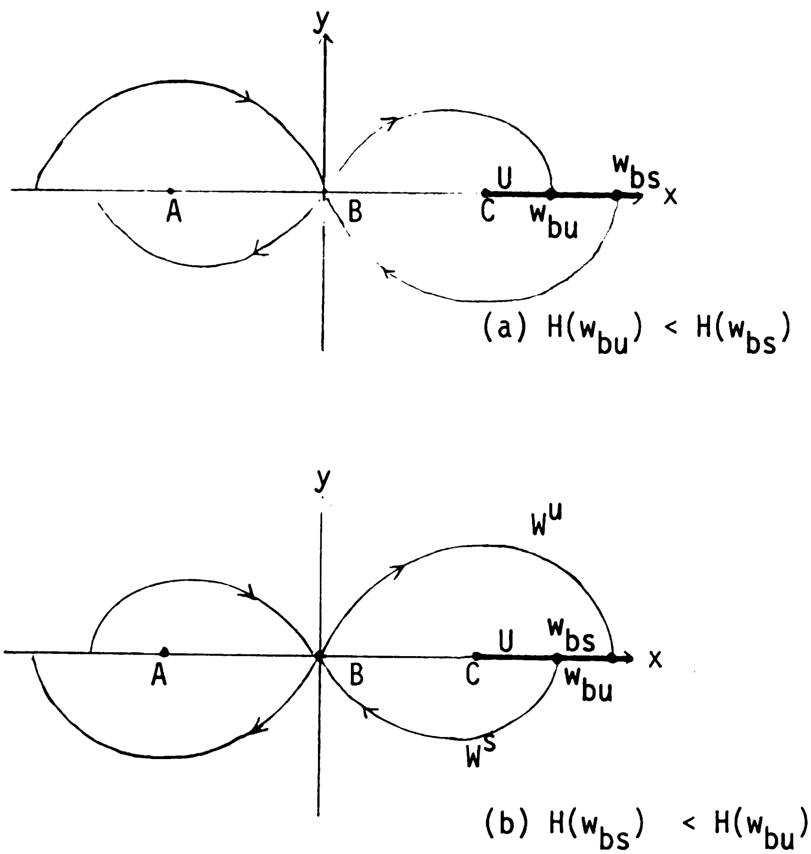


Figure 22

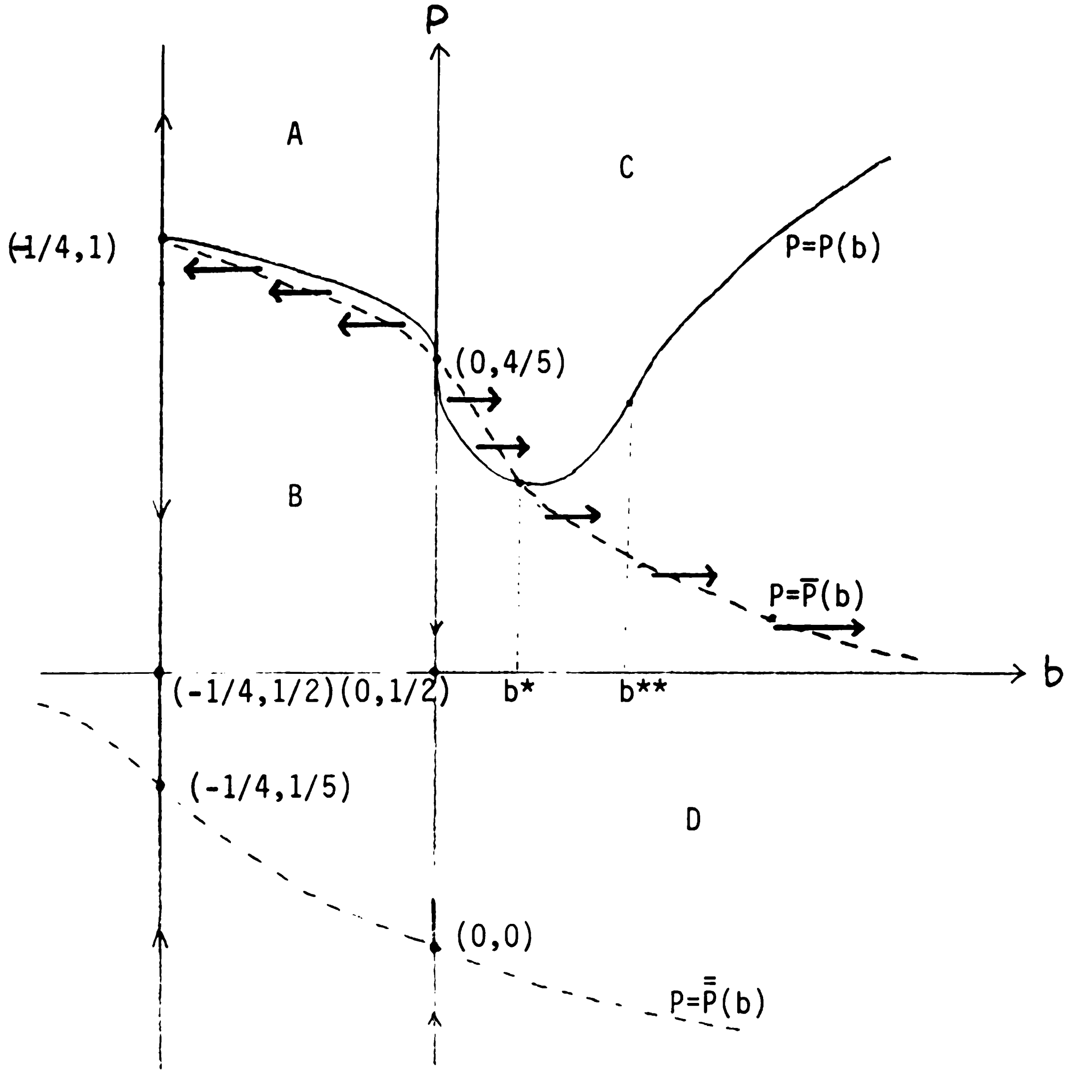
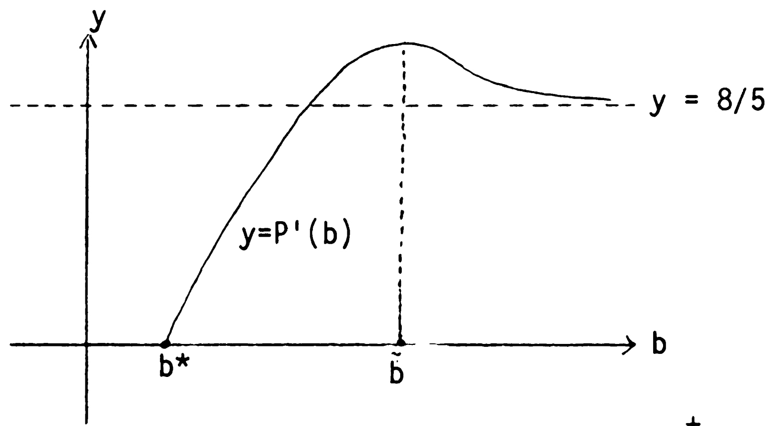
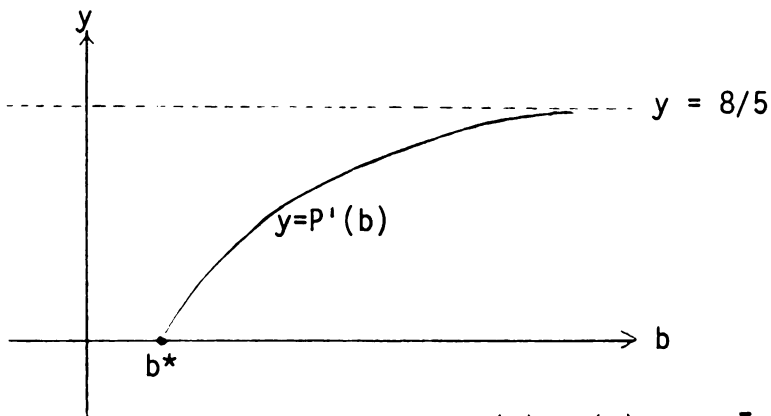


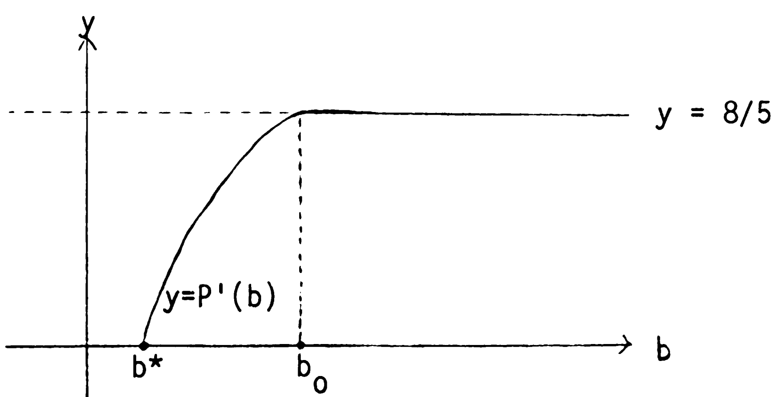
Figure 23: The graph of P(b)
(P'(b*) = P"(b**) = 0)



(a) $P'(b) \to 8/5^{+}$



(b) $P'(b) \to 8/5^{-}$



(c) $P'(b) \equiv 8/5$ for $b >> 1$

Figure 24

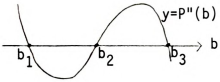


Figure 25

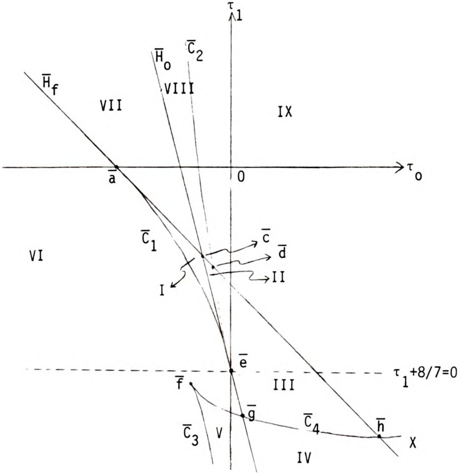
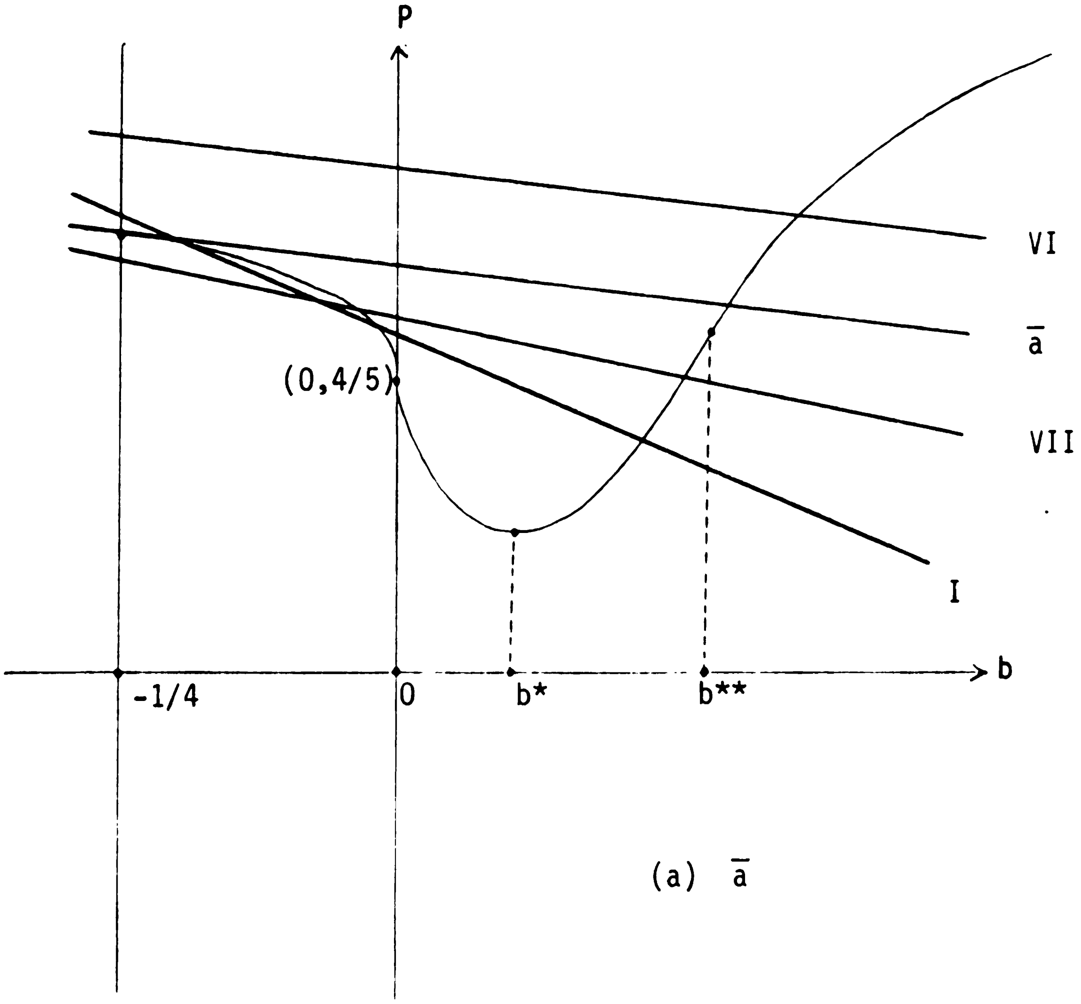
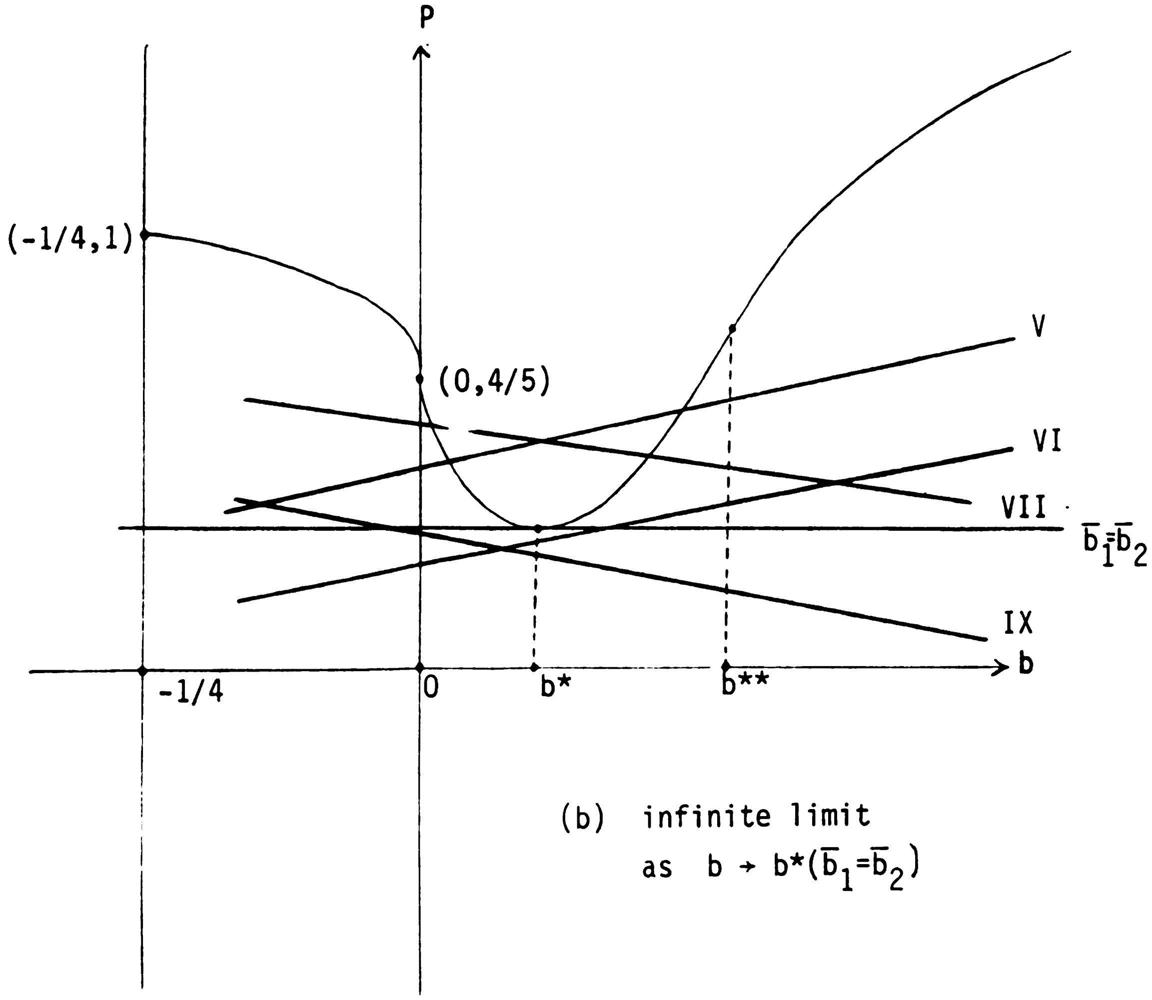


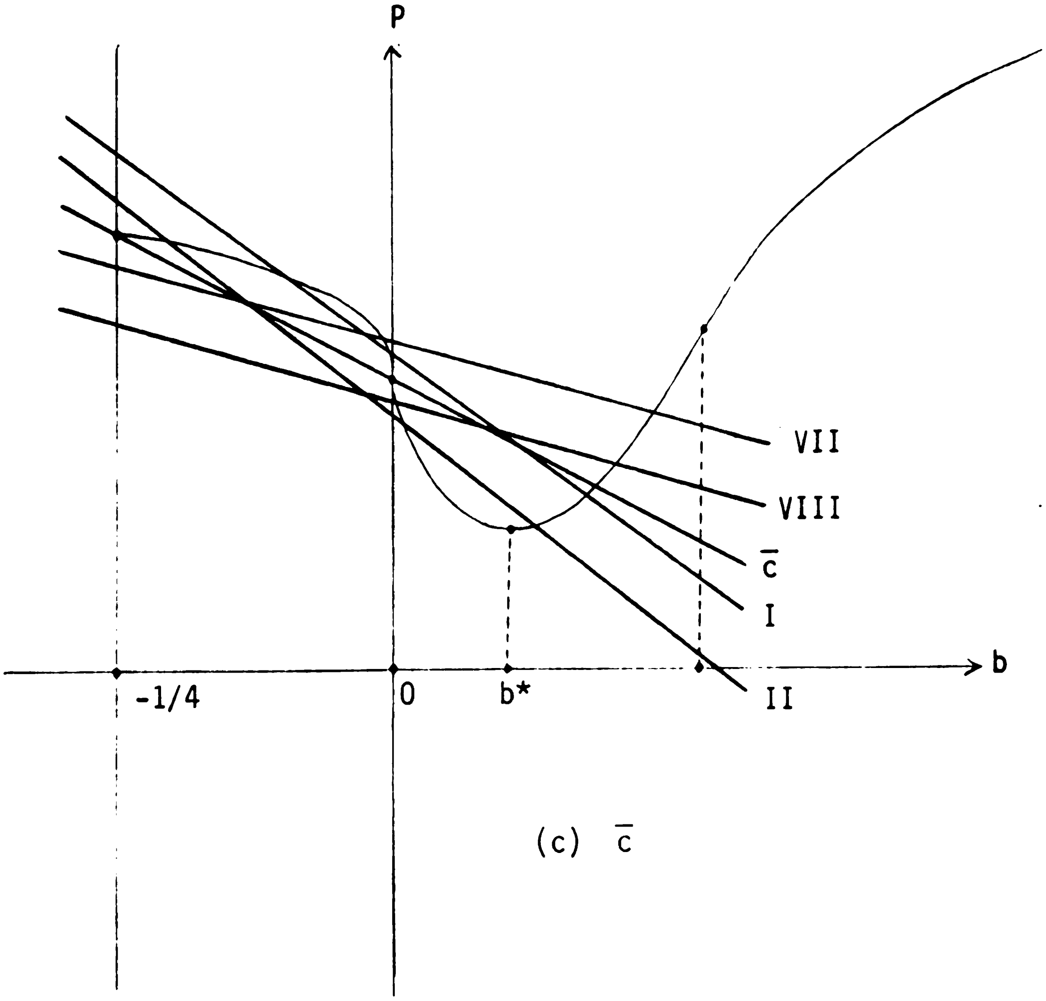
Figure 26



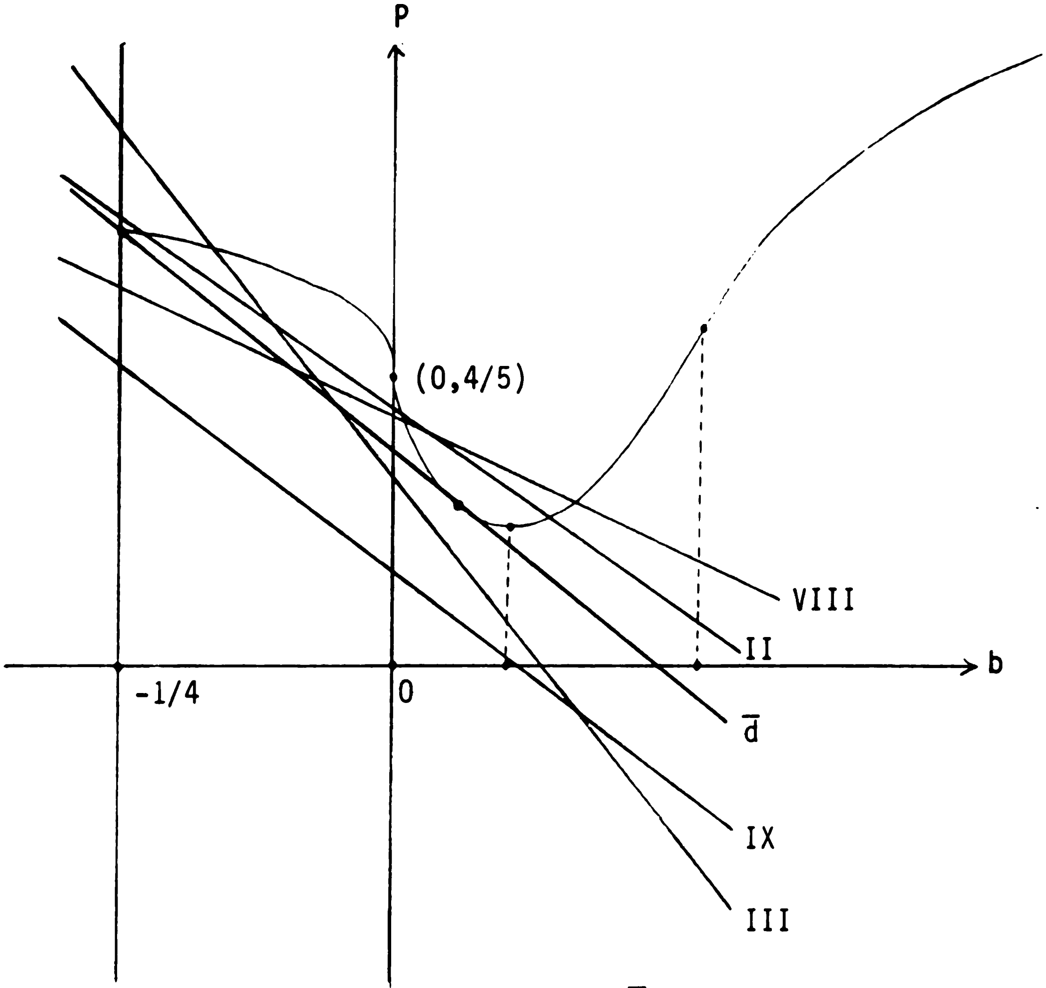
(a) $\bar{a}$



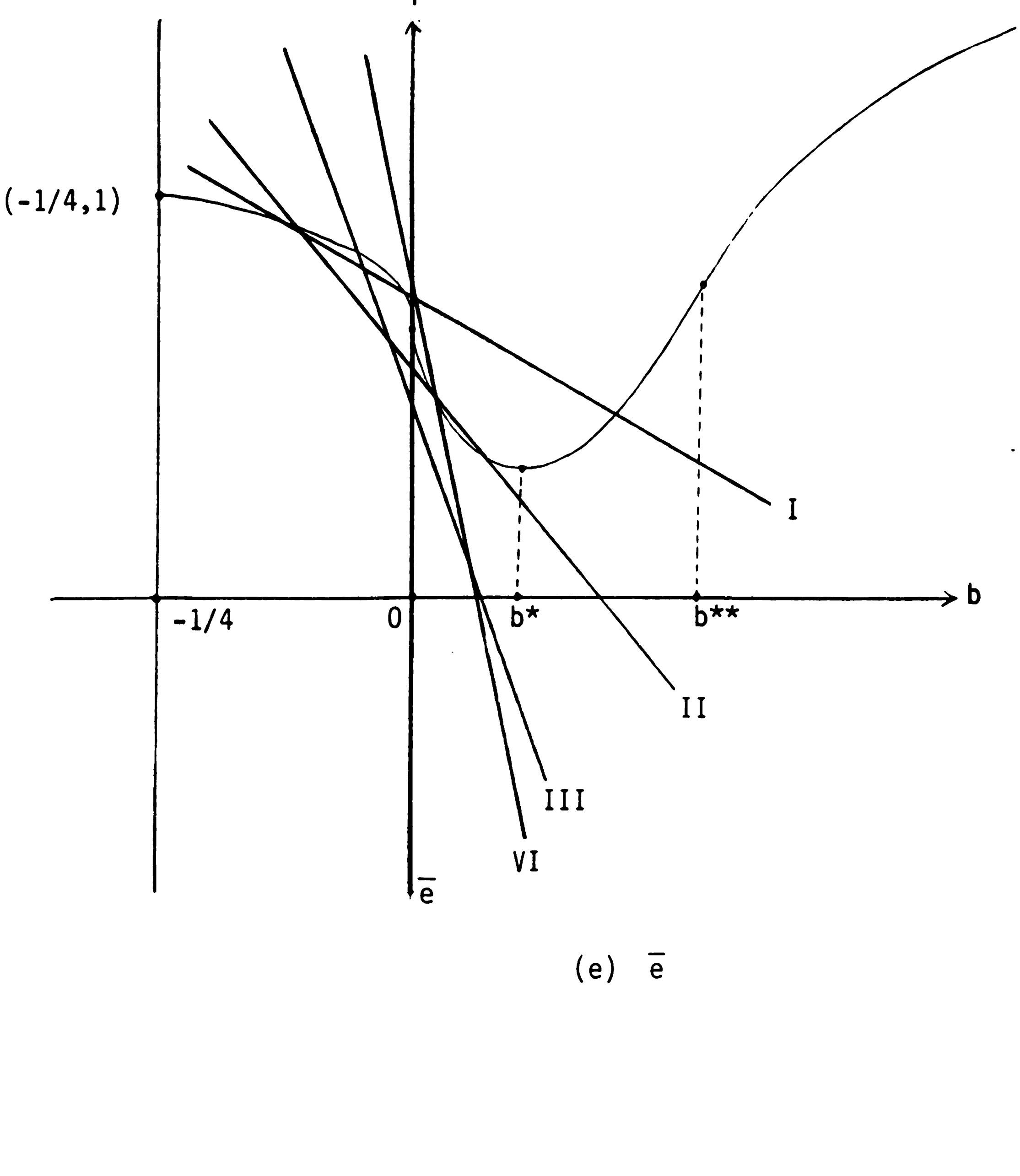
(b) infinite limit
as $b \to b^*(\bar{b}_1=\bar{b}_2)$



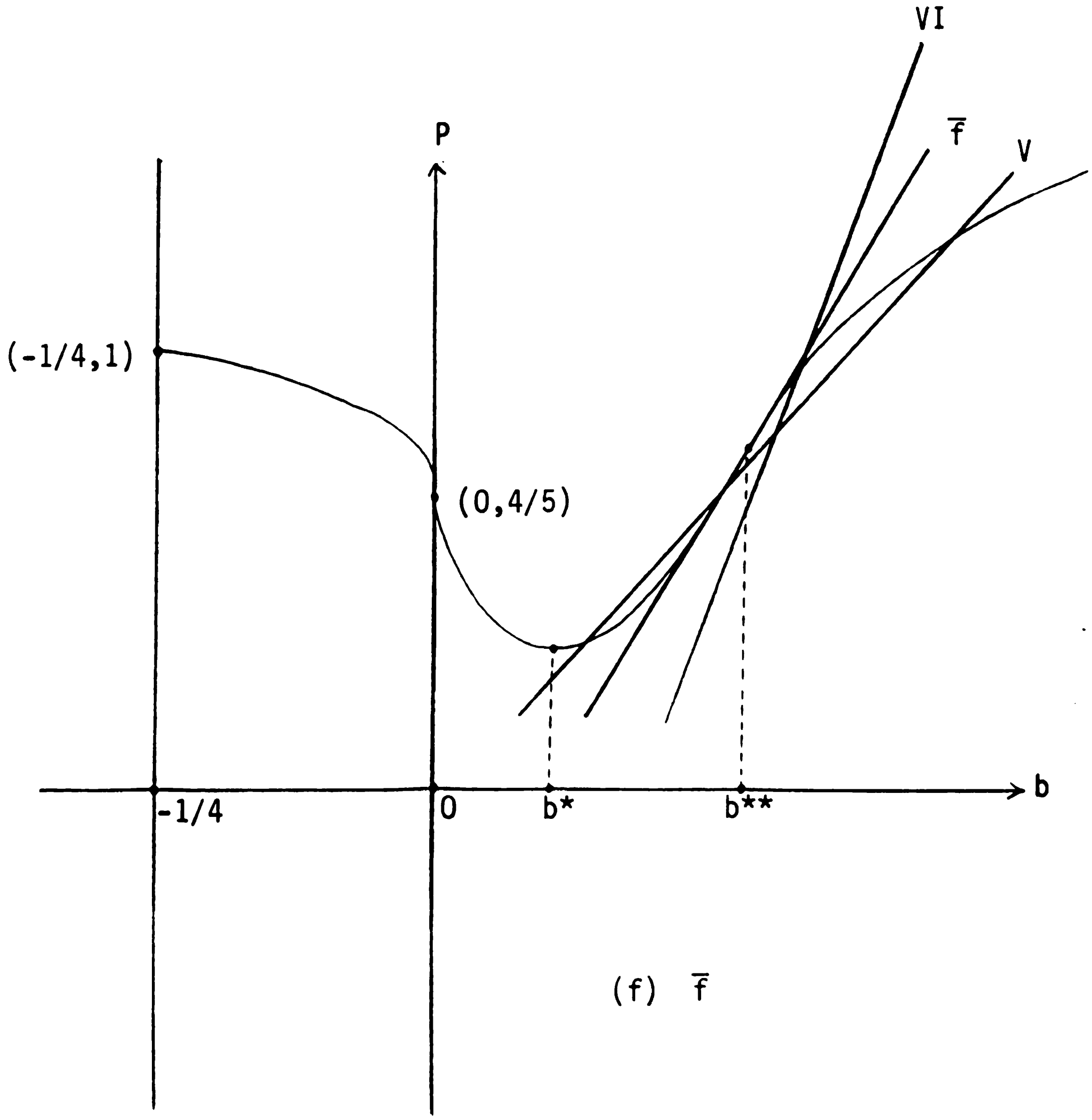
(c) $\bar{c}$



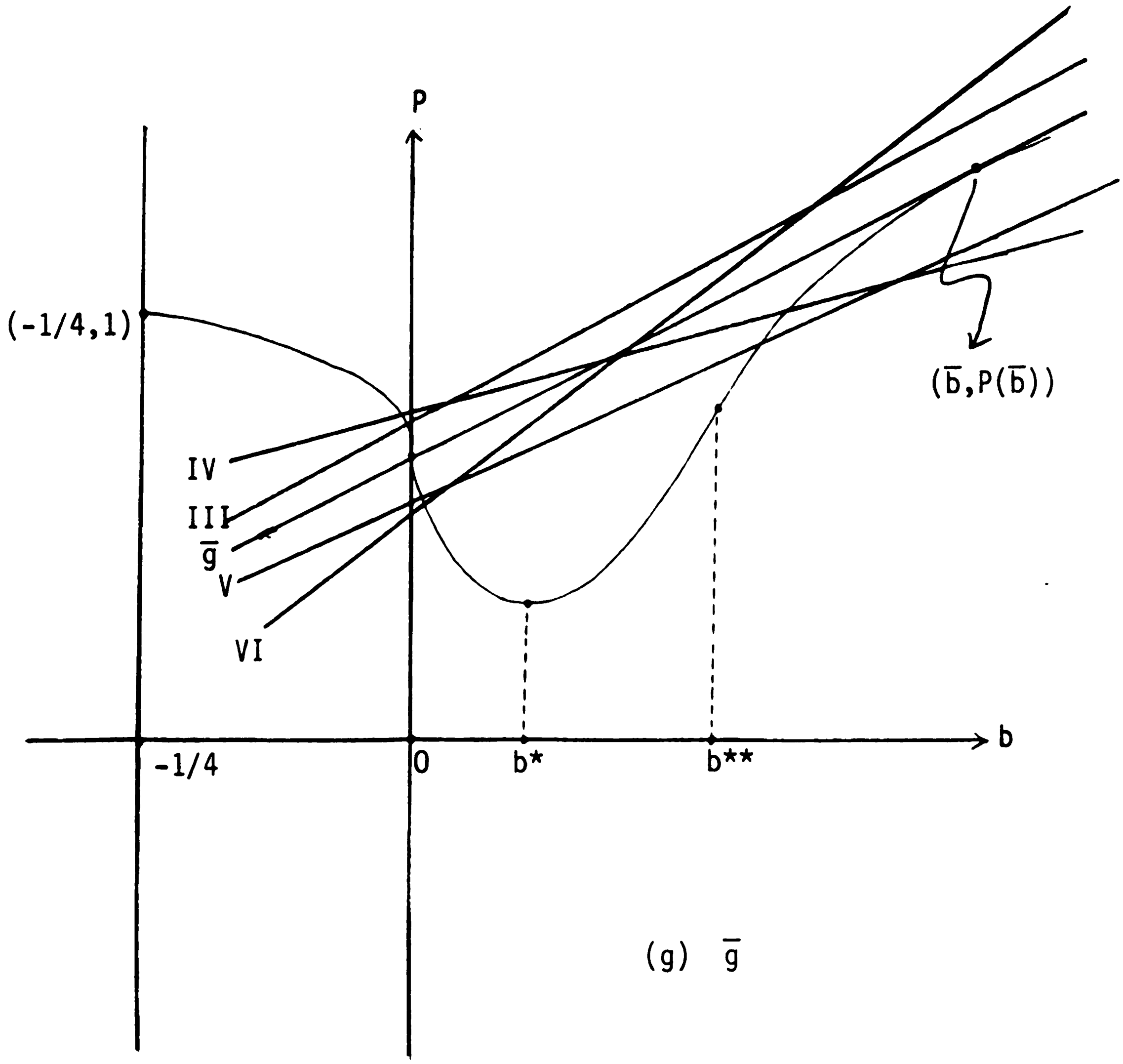
(d) $\bar{d}$



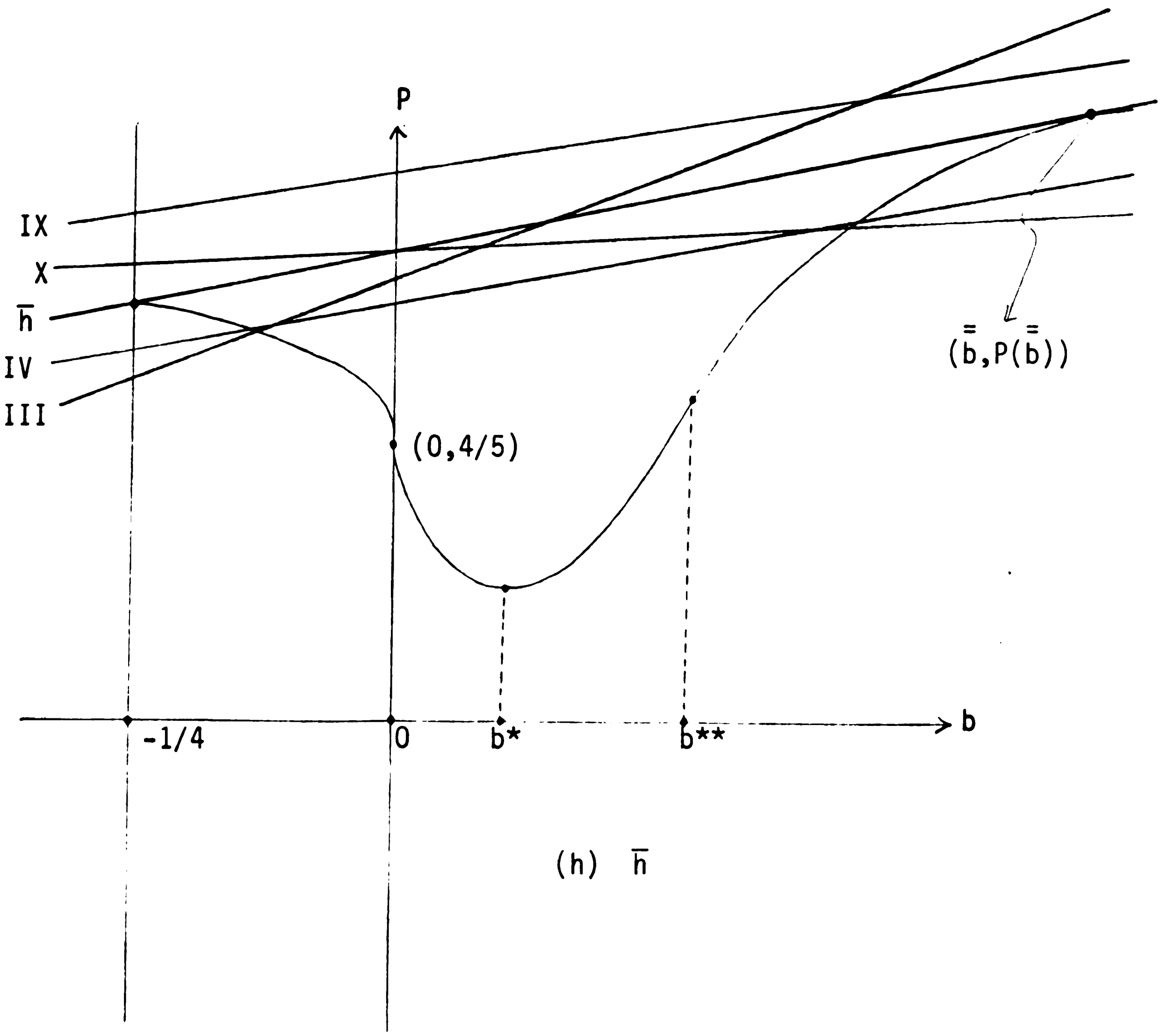
(e) $\bar{e}$



(f) $\bar{f}$



(g) $\overline{g}$



(h) $\bar{h}$

Figure 27: The relative positions of the curve P=P(b) and the straight line P=A(b) (See Figure 26 for $\bar{a}$ to $\bar{h}$ and I to X)

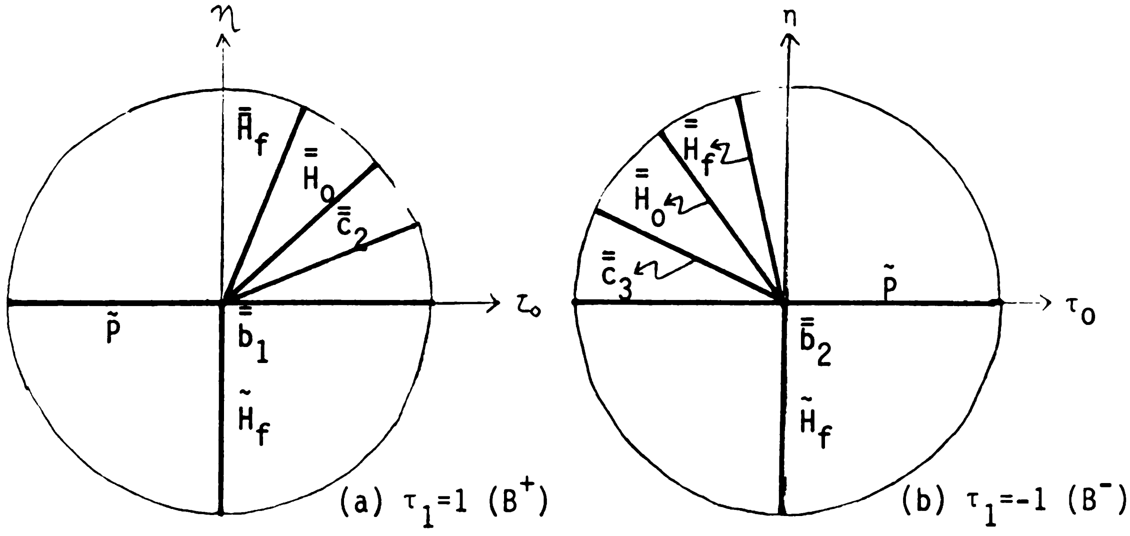


Figure 28

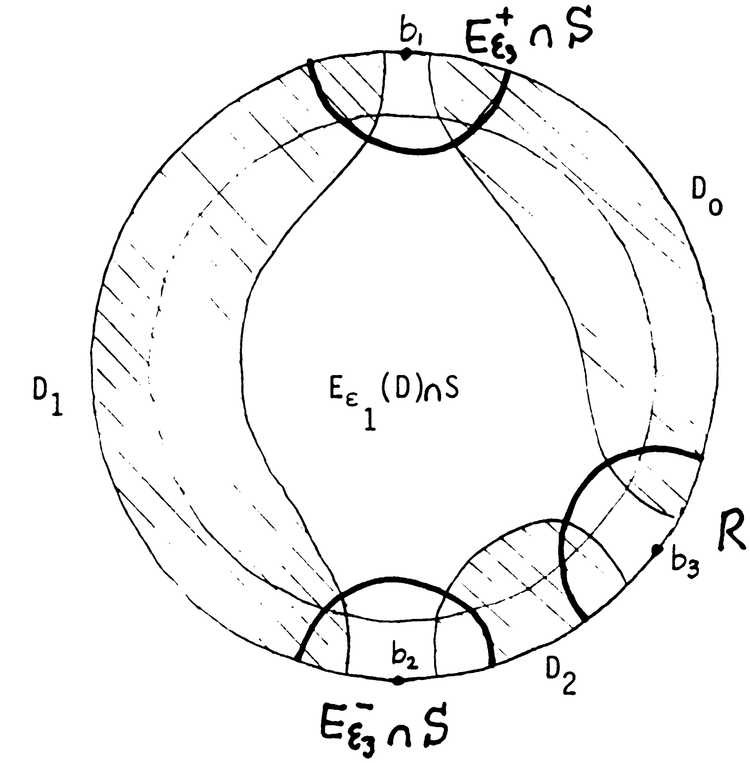


Figure 29

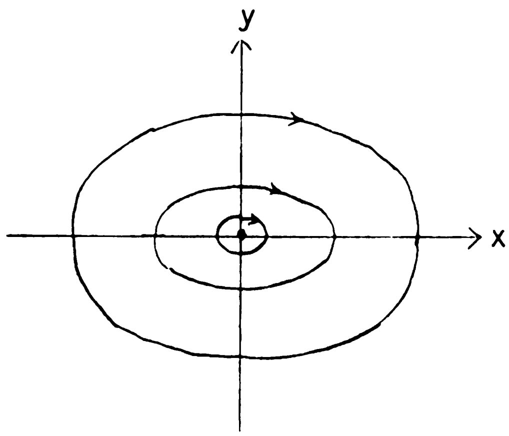
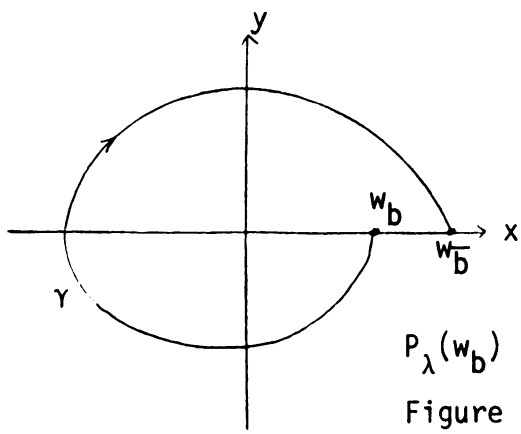


Figure 30



$P_\lambda(w_b) = w_{\bar{b}}$, $\gamma = \gamma(b, \lambda_s)$

Figure 31

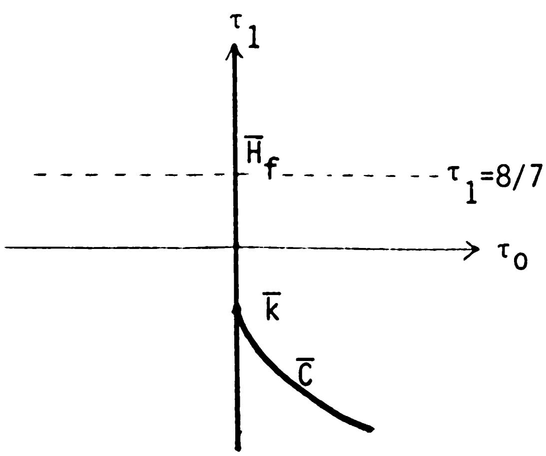


Figure 32

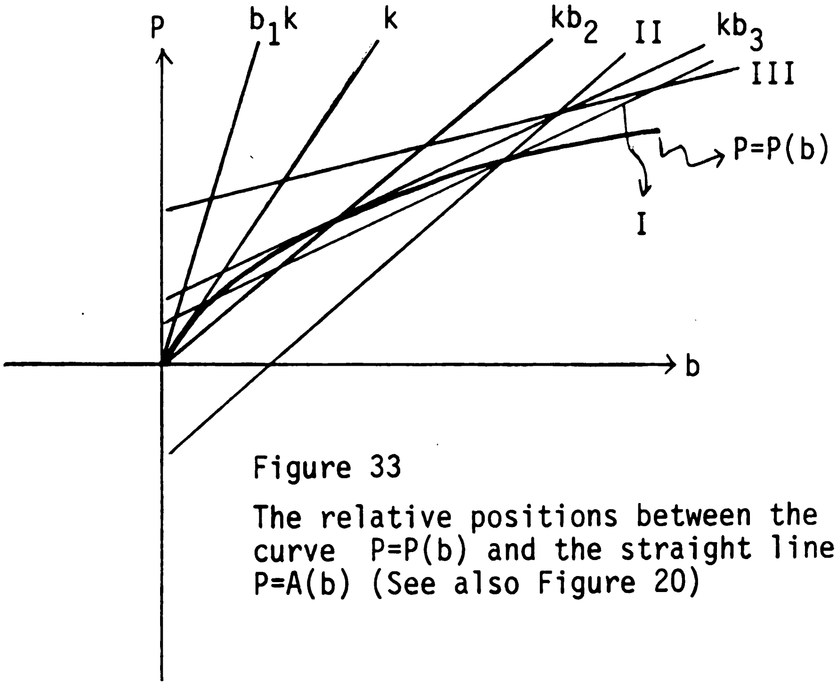


Figure 33

The relative positions between the curve P=P(b) and the straight line P=A(b) (See also Figure 20)

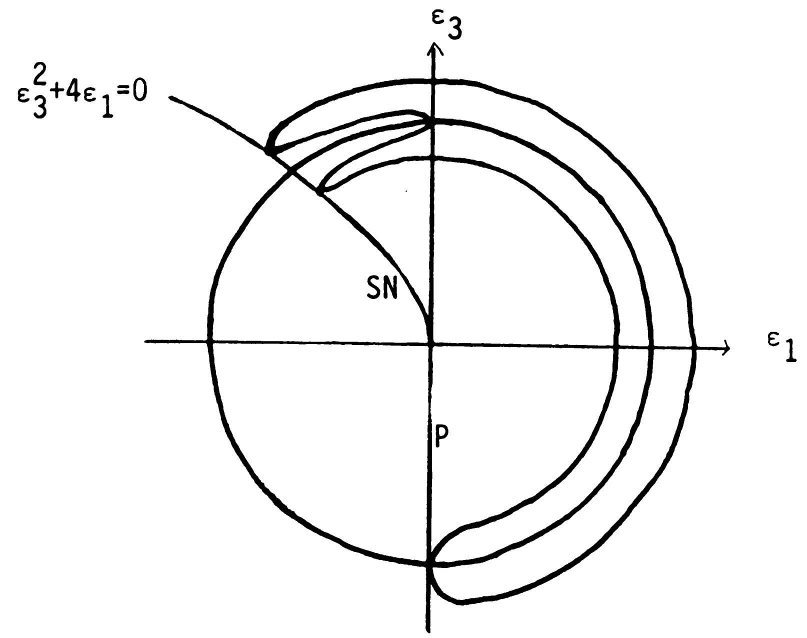


Figure 34

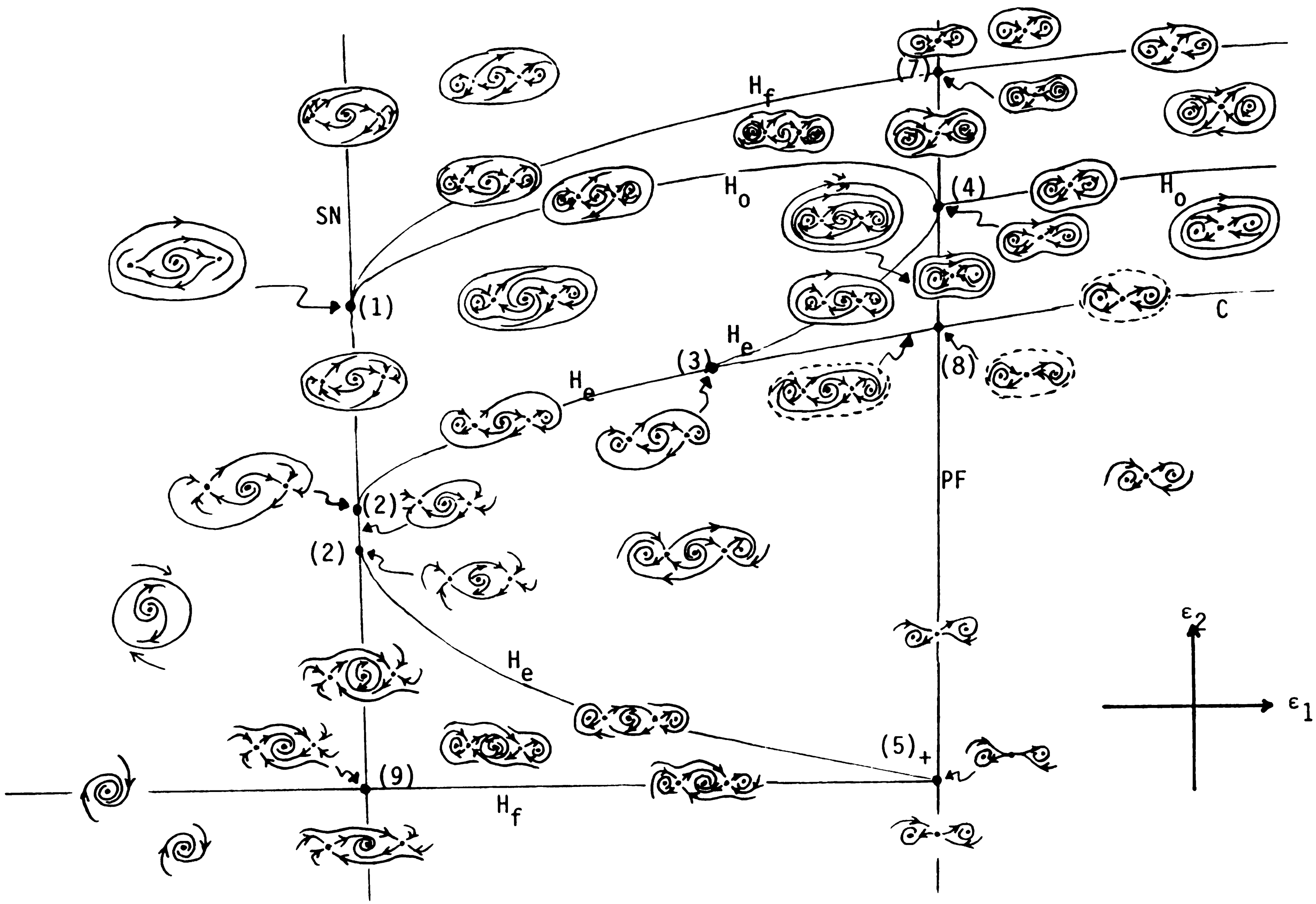


Figure 35 (a) $\varepsilon_3 > 0$

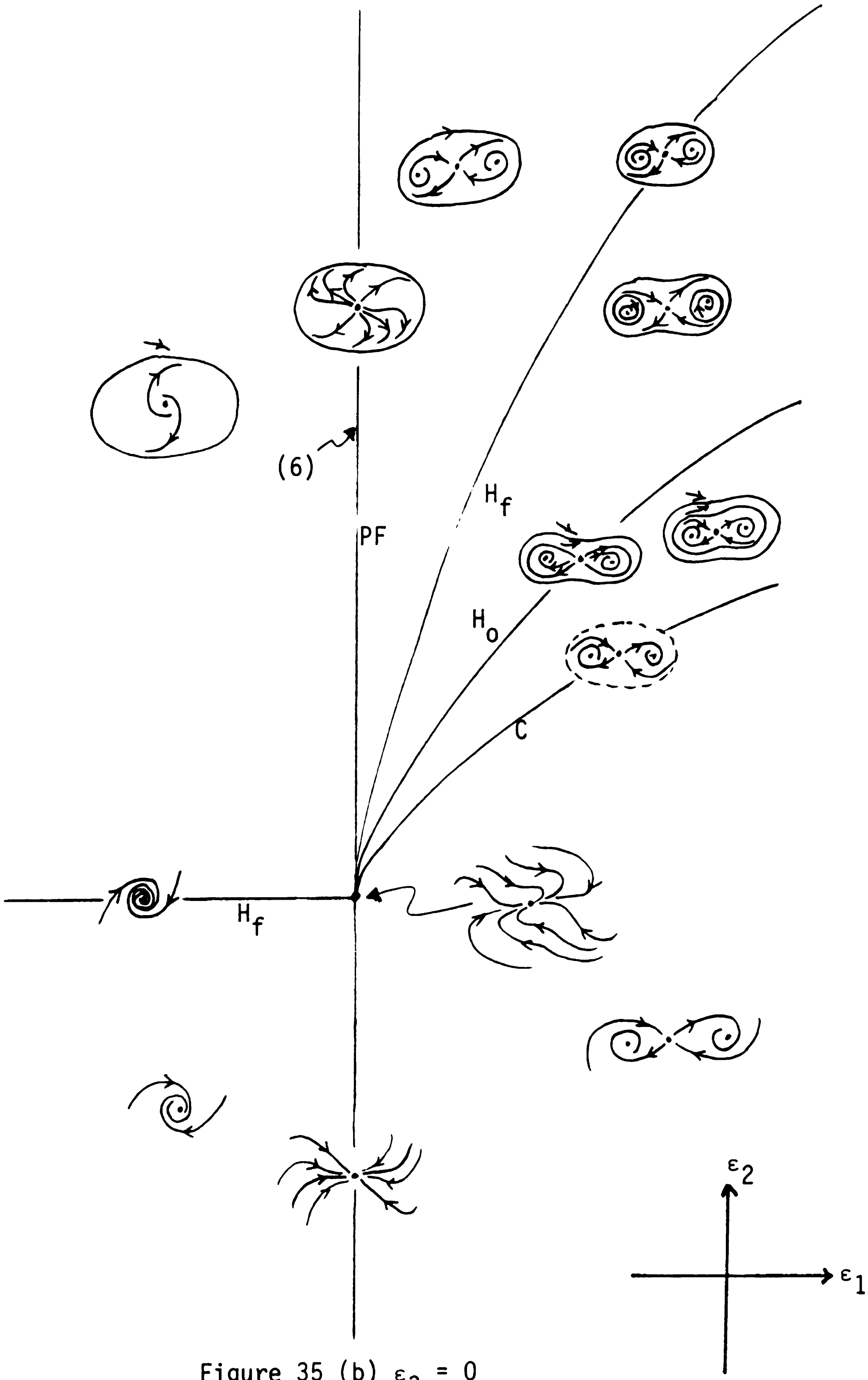


Figure 35 (b) $\varepsilon_3 = 0$

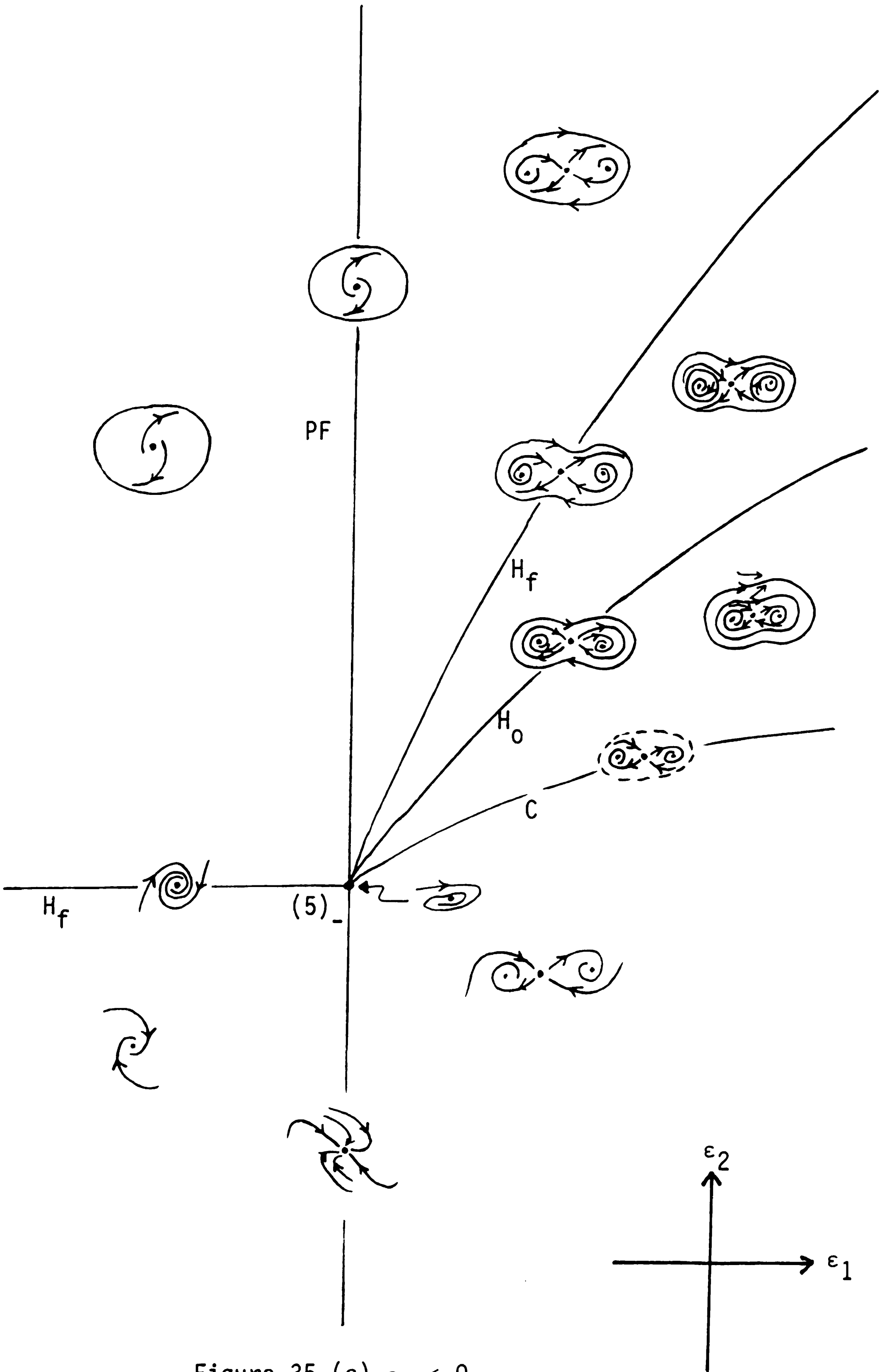


Figure 35 (c) $\varepsilon_3 < 0$

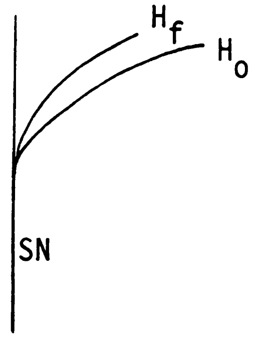


Figure 36

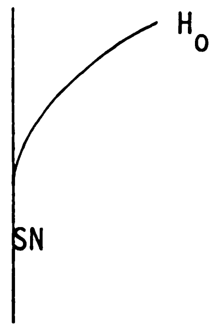


Figure 37

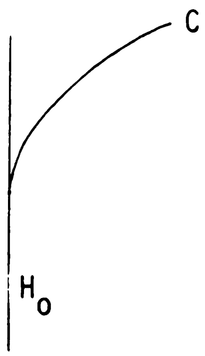


Figure 38

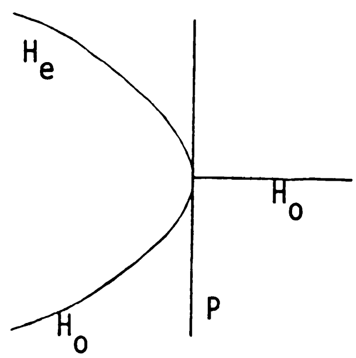
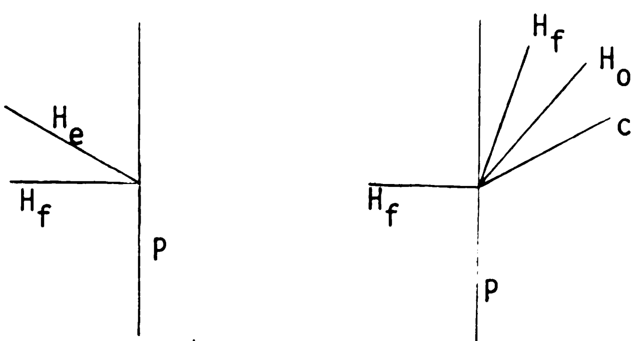


Figure 39



(a) $(5)^+$ (b) $(5)^-$

Figure 40

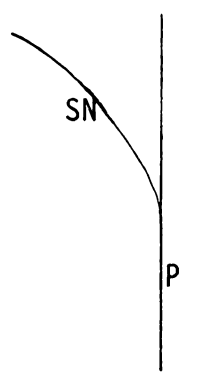


Figure 41

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