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by

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ABSTRACT

SERIES REPRESENTATION FOR PROCESSES WITH INFINITE ENERGY AND THEIR PREDICTION

By

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The purpose of this work is to present series representations for stochastic processes $\{X_n, n \in \mathbb{Z}\}$ whose second moments need not exist. In Chapter I, we obtain such a representation for $S\alpha S$ processes in terms of ϵ–invariant exchangeable random variables. For series in ϵ–invariant exchangeable random variables we associate a dispersion distance and study a prediction problem for them in terms of minimizing this distance. In case of series in i.i.d. random variables in the domain of attraction of a stable law our results give those of Cline and Brockwell. In Chapter II we see that the predictors obtained in Chapter I are metric projections. In Chapters III and IV we give nonanticipative series representations in terms of orthogonal random variables. This problem can be looked at as an orthogonal Wold decomposition in certain Banach spaces. The definition of orthogonality is based on the concept of a semi–inner product introduced by Lumer. Under certain geometric conditions the uniqueness of the semi–inner product is proved. If the Banach space is L^p, $p > 1$, our results give the recent work of Cambanis, Hardin and Weron who use James orthogonality.

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CHAPTER 0

INTRODUCTION

Let $\{X_n, n \in \mathbb{Z}\}$ be a second order process with $EX_n = 0$ which is purely nondeterministic. Then X_n has a moving average representation $X_n = \sum_{k=-\infty}^{\infty} a_{n,k}\, \xi_k$. Here the ξ_k's are orthogonal and $< X_n, \xi_k > = 0$, $k > n$. In case $\{X_n, n \in \mathbb{Z}\}$ is a purely non-deterministic Gaussian process $\{\xi_k, k \in \mathbb{Z}\}$ are i.i.d. Gaussian random variables. For non-second order processes with $E|X_n|^p < \infty$ $(1<p<2)$ similar orthogonal representations are studied in [3], [5] and [19]. However even in the symmetric α stable (SαS) case no distributional properties of $\{\xi_i\}$ are obtained. In the non-second order case, it was shown in [18] that there are severe limitations in the use of the spectral theory. However the use of the time domain techniques for processes represented in terms of series of i.i.d. random variables lead to interesting results ([8]). In this approach a suitable dispersion is defined on a sequence space through identification of coefficients of i.i.d. random variables in the series expansion.

In Chapter I we first consider the relationship to the series representation of a SαS process in terms of exchangeable random variables. This allows us to define a dispersion distance on the corresponding sequence space which is shown to be a subspace of ℓ_α. We then introduce an appropriate dispersion distance on the space of series in exchangeable random variables. In case the representing random variables are i.i.d. stable we get that the dispersion distance is the usual ℓ_α distance and if the exchangeable random variables are i.i.d. (not necessarily stable), then the distance is weaker than that given in [8]. Using this distance we study for

ARMA processes with exchangeable input analogues of the prediction results of Cline and Brockwell ([8]). The technique used in the first problem is an adaptation of that due to Dacunha–Castelle and Schreiber ([10]) which also suggests an appropriate dispersion distance. As a by product we also obtain sufficient conditions for the a.s. unconditional convergence for the series in exchangeable random variables. As a consequence we get some results due to Cline ([7]). In some cases the minimum φ–distance predictor obtained coincides with the metric projection. This will be seen in Chapter II.

Let $\{X_n, n \in \mathbb{Z}\}$ be a SαS sequence $(1 < \alpha < 2)$ and $M_n = \overline{sp}\{X_k, k \le n\}$, where closure is taken with respect to the L^p norm $(1 < p < \alpha)$. Cambanis, Hardin and Weron ([5]) have defined concepts of right and left innovations and orthogonal Wold decomposition using James orthogonality. Left innovations (which always exist) and orthogonal Wold decompositions are in terms of $\{\xi_n, n \in \mathbb{Z}\}$ where $\xi_n = X_n - P_{n-1}X_n$ and P_{n-1} is the metric projection on M_{n-1}. Right innovations and Wold decomposition exist in terms of $\{\zeta_n, n \in \mathbb{Z}\}$ if and only if $E[X_n | X_j, j \le n-1] \in M_{n-1}$ and in this case $\zeta_n = X_n - E[X_n | X_j, j \le n-1]$. However, James orthogonality is not enough when we consider general Banach spaces. In Chapter III we define the concept of orthogonality for a Banach space $\mathcal{X}$ using the semi–inner product introduced by Lumer ([17]). If $\mathcal{X} = L^p$, $p > 1$, then Lumer's construction of the semi–inner product ([25]) is the same as that considered by Cambanis and Miller ([4]). The Lumer semi–inner product enables us to extend the definitions of right and left projections as defined by Cambanis and Miamee ([3]) for a general Banach space. It is seen in [3] that if $\{X_n, n \in \mathbb{Z}\}$ is a SαS sequence such that $E[X_n | X_j, j \le n-1] \in M_{n-1}$, then $E[X_n | X_j, j \le n-1]$ is the right projection of

X_n on M_{n-1}. In Chapter III we see that Lumer orthogonality implies James orthogonality. If the Banach space is L^p, $p > 1$, then Lumer orthogonality coincides with James orthogonaltiy ([25]). Let $\underline{x} = \{x_n, n \in \mathbb{Z}\} \subseteq \mathcal{X}$ $M_n(\underline{x}) = \overline{sp}\,\{x_m, m \leq n\}$, P_n denote the metric projection on M_n and r_n denote the right projection on M_n, $n \in \mathbb{Z}$. In Chapter III we see that left innovations always exist if $\mathcal{X}$ is reflexive, rotund and has a rotund dual. Further, left innovations and Wold decompositions are in terms of $\{\xi_n, n \in \mathbb{Z}\}$ where $\xi_n = x_n - P_{n-1}x_n$. In Chapter IV we prove that if $\mathcal{X}$ is reflexive, then the right Wold decomposition and innovations exist if and only if $r_{n-1}(x_n)$ exists for each n. In this case the decomposition is in terms of $\{\zeta_n, n \in \mathbb{Z}\}$ where $\zeta_n = x_n - r_{n-1}(x_n)$.

CHAPTER I

SERIES REPRESENTATION OF STABLE PROCESSES; DISPERSION DISTANCE AND PREDICTION

For a purely non-deterministic Gaussian process $\{X_n,\ n \in \mathbb{Z}\}$ we can choose i.i.d. random variables $\{\xi_n,\ n \in \mathbb{Z}\}$ such that $\{\xi_n,\ n \in \mathbb{Z}\}$ forms a symmetric basis ([16]). In this chapter we first consider the structure of a symmetric stable process $\{X_n, n \in \mathbb{Z}\}$ of index α (in short SαS) for which $M_\alpha(X{:}\infty) = \overline{\text{sp}}^{\alpha}\{X_n, n \in \mathbb{Z}\}$ has a symmetric basis. Here $^{-\alpha}$ denotes the closure with respect to the norm $||\ \ ||_\alpha$ defined by (1.1). This motivates us to study a.s. convergent series in terms of exchangeable random variables. We define a suitable dispersion distance on this space and consider the prediction problem with respect to this dispersion. This extends the work of Cline and Brockwell ([8]).

For a SαS random variable X with characteristic function $Ee^{itX} = \exp(-\gamma|t|^\alpha)$, $\gamma > 0$, define

$$(1.1) \qquad ||X||_\alpha = \begin{cases} \gamma^{1/\alpha} & \text{if} \quad 1 \le \alpha \le 2 \\ \gamma & \text{if} \quad 0 < \alpha < 1 \end{cases}$$

([23]). Then for any $1 \le p < \alpha$, $X \in L^p$ and

$$(1.2) \qquad ||X||_\alpha = c(p,\alpha)\ ||X||_p$$

where $||X||_p$ denotes the L^p norm of X and $c(p,\alpha)$ is a constant which depends on p and α ([4]). Hence all L^p norms are equivalent.

Note that $\| \ \|_\alpha$ gives rise to a metric and if $\alpha > 1$, then $\| \ \|_\alpha$ is a norm ([23]).

We now start with some basic definitions.

Definition 1.3. A basis $\{x_n\}$ of a Banach space is called an unconditional basis if every convergent series of the form $\sum_n a_n x_n$ converges unconditionally. A basis $\{x_n\}$ of a Banach space is said to be a symmetric basis if it is equivalent to the basis $\{x_{\pi(n)}\}$, for any permutation π of the integers.

Note that every symmetric basis is an unconditional basis.

Definition 1.4. Random variables $\{\xi_i, 1 \le i \le n\}$ are said to be exchangeable if their joint distribution function is invariant under permutations of $\{1,\ldots,n\}$. A sequence $\{\xi_n, n \in \mathbb{N}\}$ of random variables is said to be an exchangeable sequence if every finite subset is exchangeable. A sequence $\{\xi_n, n \in \mathbb{N}\}$ of random variables is said to be ϵ–invariant if for every $n \in \mathbb{N}$ and n–tuple $(k_1,\ldots,k_n) \in \mathbb{N}^n$ consisting of distinct elements the 2^n n–dimensional random vectors $(\epsilon_{k_1}\xi_{k_1},\ldots,\epsilon_{k_n}\xi_{k_n})$, $\epsilon_{k_j} = \pm 1$, have the same probability law.

Let $\{X_n, n \in \mathbb{Z}\}$ be a SαS sequence $(\alpha > 1)$. By the Kolmogorov consistency theorem (Theorem 36.1 [1]) we may assume that $\{X_n, n \in \mathbb{Z}\}$ is a sequence on $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}),\mu)$. Since $(\mathbb{R}^{\mathbb{Z}}, \mathcal{B}(\mathbb{R}^{\mathbb{Z}}))$ is a standard Borel space and μ has no atoms, $(\mathbb{R}^{\mathbb{Z}},\mathcal{B}(\mathbb{R}^{\mathbb{Z}}),\mu)$ is Borel isomorphic to $([0,1], \mathcal{B}([0,1]),\lambda)$ where λ denotes Lebesgue measure on $([0,1], \mathcal{B}[0,1])$ ([21] p. 116). Hence we may assume without loss of generality that $\{X_n, n \in \mathbb{Z}\}$ is a SαS sequence on $([0,1], \mathcal{B}([0,1]))$. Define

$$M_p(X:n) = \overline{sp}^p \{X_k, k \leq n\}$$

$$M_p(X:-\infty) = \bigcap_n M_p(X:n)$$

$$M_p(X:\infty) = \overline{sp}^p \{\bigcup_n M_p(X:n)\}$$

where $^{-p}$ denotes closure with respect to the L^p–norm $|| \; ||_p$ if $1 \leq p < \alpha$, $^{-\alpha}$ denotes closure with respect to the norm $|| \; ||_\alpha$ defined by (1.1) and sp denotes linear span.

Let $\{e_n, n \in \mathbb{N}\}$ be a symmetric basis for $M_\alpha(X:\infty)$. Following the proof of Dacunha–Castelle and Schreiber ([10]) we will get for $1 \leq p < \alpha$ a sequence of ϵ–invariant exchangeable random variables $\{\xi_n, n \in \mathbb{N}\}$ in $L^p(\Omega, \mathcal{F}, P)$ such that $\sum_n c_n e_n$ converges in L^p if and only if $\sum_n c_n \xi_n$ converges in $L^p(\Omega, \mathcal{F}, P)$. Since $\{e_n, n \in \mathbb{N}\}$ is a symmetric basis for $M_\alpha(X:\infty)$, we get by (1.2) that $\{e_n, n \in \mathbb{N}\}$ is a symmetric basis for $M_p(X:\infty)$. Propositions 22.2 and 21.4 [24] imply that $\{e_n, n \in \mathbb{N}\}$ is a bounded basis for $M_p(X:\infty)$, $1 \leq p < \alpha$. Let

$$(1.5) \qquad 0 < k_p \leq \inf_{n \in \mathbb{N}} ||e_n||_p \leq \sup_{n \in \mathbb{N}} ||e_n||_p \leq K_p < \infty.$$

Since (1.5) is valid for any $1 \leq p < \alpha$, $\{e_n, n \in \mathbb{Z}\}$ is uniformly integrable.

Let $1 \le p < q < \alpha$. Then $\gamma = q/p > 1$. Further

$$K_q^q \ge ||e_n||_q^q = \int |e_n|^q = \int (|e_n|^p)^\gamma .$$

Hence $\sup_n E(|e_n|^p)^\gamma \le K_q^q < \infty$. Therefore $\{|e_n|^p, n \in \mathbb{N}\}$ is uniformly integrable.

Let $\mu_{e_{k_1},\ldots,e_{k_n}}$ (or $\mu_{k_1,\ldots,k_n}$) denote the probability law of $(e_{k_1},\ldots,e_{k_n})$ for any $n \in \mathbb{N}$ and $k_1,\ldots,k_n \in \mathbb{N}$ distinct. Let S_n denote the group of permutations of $\{1,\ldots,n\}$. If $\sigma \in S_n$ let $\sigma(e_1,\ldots,e_n) = (\sigma(e_1),\ldots,\sigma(e_n)) = (e_{\sigma(1)},\ldots,e_{\sigma(n)})$. Let Γ_n denote the group of multiplication by $(\epsilon_1,\ldots,\epsilon_n)$, $\epsilon_j = \pm 1$, that is ,$\gamma \in \Gamma_n$, $\gamma(e_1,\ldots,e_n) = (\gamma(e_1),\ldots,\gamma(e_n))$ with $\gamma(e_j) = \pm e_j$. Define

$$\mu_n^s = \frac{1}{n!2^n} \sum_\sigma \sum_\gamma \mu_{\sigma\gamma(e_1,\ldots,e_n)}.$$

For $m \le n$ let $\mu_{n,m}^s$ be the marginal of μ_n^s on the first m coordinates. By (1.5), $\{\mu_{n,1}^s, n \ge 1\}$ is tight and hence ([1], p. 331) has a weakly convergent subsequence $\{\mu_{n_k(1),1}^s, n_k(1) \ge 1\}$ converging weakly to $\tilde{\mu}_1$. Using (1.5) again we see that $\{\mu_{n_k(1),2}^s, n_k(1) \ge 2\}$ has a weakly convergent subsequence $\{\mu_{n_k(2),2}^s, n_k(2) \ge 2\}$ converging weakly to $\tilde{\mu}_2$. Continuing in this manner we get for each $m \in \mathbb{N}$, a probability measure $\tilde{\mu}_m$ such that $\{\mu_{n_k(m),m}, n_k(m) \ge m\}$ converges weakly to $\tilde{\mu}_m$ where $\{n_k(m), n_k(m) \ge m\}$ is a subsequence of $\{n_k(m-1), n_k(m-1) \ge m\}$ and $\{n_k(0), n_k(0) \ge 1\} = \{1,2,\ldots.\}$. Since $\mu_{n,m}^s$ is the marginal of $\mu_{n,m+1}^s$, $\tilde{\mu}_m$

defines a sequence $\{\xi_k, k \in \mathbb{N}\}$ of ϵ–invariant exchangeable random variables (by the Kolmogorov consistency theorem) on some probability space $(\Omega,\mathcal{F},P)$. Further, by (1.5), $\{\xi_k\} \subset L^p(\Omega,\mathcal{F},P)$. We will now show that $\sum_n c_n e_n$ converges in L^p if and only if $\sum_n c_n\xi_n$ converges in $L^p(\Omega,\mathcal{F},P)$, for any sequence $\{c_n\}$ of real numbers. But

$$
\begin{aligned}
E\Big|\sum_{k\le m} c_k\xi_k\Big|^p &= \int_{\mathbb{R}^m} |c_1x_1 +\dots+ c_mx_m|^p d\tilde{\mu}_m(x_1,\dots,x_m) \\
\text{(1.6)} \qquad & \\
&= \lim_{k\to\infty} \int_{\mathbb{R}^m} |c_1x_1+\dots+c_mx_m|^p \, d\mu^s_{n_k(m),m}(x_1,\dots,x_m).
\end{aligned}
$$

Since

$$
\begin{aligned}
|c_1\epsilon_{k_1}e_{k_1}+\dots+c_m\epsilon_{k_m}e_{k_m}|^p &\le A(p,m) \sum_{j\le m} |c_j\epsilon_{k_j}e_{k_j}|^p \\
&\le A(p,m)(\sup_{j\le m} |c_j|^p) \sum_{j\le m} |e_{k_j}|^p,
\end{aligned}
$$

where $A(p,m)$ is a constant depending only on m and p, and as $\{|e_n|^p, n \in \mathbb{N}\}$ is uniformly integrable, it follows that the family $\{|c_1\epsilon_{k_1}e_{k_1}+\dots+c_m\epsilon_{k_m}e_{k_m}|^p\colon k_1\ne\cdots\ne k_m, (k_1,\dots,k_m) \in \mathbb{N}^m, \epsilon_j=\pm1, 1\le j\le m\}$ is uniformly integrable. Fix $m \in \mathbb{N}$. For $n \ge m$ let

$$
T_{n,m} = \frac{1}{m!\binom{n}{m}2^m} \sum_{\substack{1\le k_1\ne\cdots\ne k_m\le n \\ \epsilon_{k_j}=\pm 1}} |c_1\epsilon_{k_1}e_{k_1}+\dots+c_m\epsilon_{k_m}e_{k_m}|^p.
$$

The sequence $\{T_{n,m}, n \geq m\}$ is uniformly integrable. We prove this as follows. Suppose not. Then there exists an $\eta > 0$ such that for each $\epsilon > 0$ there exists a set D and an $n \geq m$ such that $\lambda_0(D) < \eta$ and $\int_D T_{n,m} d\lambda_0 > \epsilon$. The definition of $T_{n,m}$ gives

$$\int_D |c_1 \epsilon_{k_1} e_{k_1} + \ldots + c_m \epsilon_{k_m} e_{k_m}|^p d\lambda_0 > \epsilon$$

which contradicts the uniform integrability of $|c_1 \epsilon_{k_1} e_{k_1} + \ldots + c_m \epsilon_{k_m} e_{k_m}|^p$.

The functions $T_{n,m}$ converge in the $\sigma(L^1, L^\infty)$ topology along $\{n_k(m), n_k(m) \geq m\}$ and consequently $\int T_{n_k(m),m} d\lambda_0$ converge.

$$(1.7) \quad \int (T_{n_k(m),m} \cdot 1) d\lambda_0 = \int_{\mathbb{R}^m} |c_1 x_1 + \ldots + c_m x_m|^p d\mu^s_{n_k(m),m}(x_1, \ldots, x_m) \to E|\sum_{k \leq m} c_k \xi_k|^p.$$

Suppose $\sum_k c_k e_k$ converges in L^p. We now show that $\sum_k c_k \xi_k$ converges in $L^p(\Omega, \mathcal{F}, P)$. Since $\{\xi_n, n \in \mathbb{N}\}$ is a sequence of ϵ–invariant random variables, the random variables $c_n \xi_n$ constitute martingale differences (Remark 2.2.2 [10]) such that $E|\sum_{k \leq n} c_k \xi_k|^p$ is an increasing function of n. By the martingale convergence theorem, $\sum_n c_n \xi_n$ converges in $L^p(\Omega, \mathcal{F}, P)$ if and only if $\lim_n E|\sum_{k \leq n} c_k \xi_k|^p < \infty$. Since $\{e_n, n \in \mathbb{N}\}$ is a symmetric basis for L^p and $\sum_n c_n e_n$ converges in L^p, we see that the set $\{\sum_n c_n \delta_n e_{\pi(n)}\}$, where π runs over all permutations of integers and δ_n are scalars such that $|\delta_n| \leq 1$, is bounded ([16] p.53). Hence there exists a constant K such that

$$(1.8) \qquad E|c_1\epsilon_{k_1}e_{k_1}+\ldots+c_m\epsilon_{k_m}e_{k_m}|^p \le K\ E|c_1e_1+\ldots+c_me_m|^p$$

for every choice of $k_1\neq\cdots\neq k_m$ with $1 \le k_j \le n$ for $j = 1,\ldots,m$. Consequently

$$(1.9) \qquad \int_{\mathbb{R}^m} |c_1x_1+\ldots+c_mx_m|^p\ d\mu^s_{n,m}(x_1,\ldots,x_m)$$

$$\le K\ E|c_1e_1+\ldots+c_me_m|^p.$$

Using (1.7) and (1.9) we see that $\sum_n c_n\ \xi_n$ converges in $L^p(\Omega,\mathcal{F},P)$. Conversely suppose that $\sum_n c_n\xi_n$ converges in $L^p(\Omega,\mathcal{F},P)$. For each $m \in \mathbb{N}$ there exist random variables $e_{k_1},\ldots,e_{k_m}$ such that

$$E|c_1e_{k_1}+\ldots+e_me_{k_m}|^p \le 2E|c_1\xi_1+\ldots+c_m\xi_m|^p.$$

Assume otherwise. Then

$$\frac{1}{m!\binom{n}{m}} \sum_{1\le k_1\neq\cdots\neq k_m\le n} E|c_1e_{k_1}+\ldots+c_me_{k_m}|^p$$

$$> 2E|c_1\xi_1+\ldots+c_m\xi_m|^p$$

which contradicts the fact that

$$E|\sum_{j\leq m} c_j\xi_j|^p =$$

$$\lim_{k\to\infty} \frac{1}{m!\binom{n_k(m)}{m}2^m} \sum_{\substack{1\leq k_1\neq\cdots\neq k_m\leq n_k(m)\\ \epsilon_{k_j}=\pm 1}} E|c_1\epsilon_{k_1}e_{k_1}+\ldots+c_m\epsilon_{k_m}e_{k_m}|^p$$

$$= \lim_{k\to\infty} \frac{1}{m!\binom{n_k(m)}{m}} \sum_{1\leq k_1\neq\cdots\neq k_m\leq n_k(m)} E|c_1e_{k_1}+\ldots+c_me_{k_m}|^p.$$

Since $\sum_n c_n\xi_n$ converges in $L^p(\Omega,\mathcal{F},P)$, $A = \sup_m E|\sum_{k\leq m} c_k\xi_k|^p < \infty$. Therefore $E|c_1e_{k_1}+\ldots+ c_me_{k_m}|^p \leq 2A$. Using this and Theorem 22.1 [24] we see that $||\sum_{k\leq m} c_ke_k||_p^p \leq 2AK_1 < \infty$ for every m. Therefore $\sum_k c_ke_k$ converges in L^p. By Proposition 2.3.8 [10] $\sum_k c_k\xi_k$ converges in L^p if and only if $\sum_k E|c_k\xi_k|^p < \infty$. But

$$\sum_k E|c_k\xi_k|^p = \sum_k |c_k|^p E|\xi_k|^p = E|\xi_1|^p \sum_k |c_k|^p.$$

Hence $\sum_k c_ke_k$ converges in L^p if and only if $\sum_k c_k\xi_k$ converges in $L^p(\Omega,\mathcal{F},P)$ if and only if $\underline{c} = \{c_k\} \in \ell_p$. Therefore $M_\alpha(X:\infty)$ is isomorphic to ℓ_p $(1\leq p<\alpha)$ which can be continuously imbedded in ℓ_α. Using this

imbedding we can define dispersion$(Y_1,Y_2) = \sum |c_n^{(1)}-c_n^{(2)}|^\alpha$ where

$Y_k = \sum c_n^{(k)}e_n \in M_\alpha(X:\infty)$, $k = 1,2$.

Remark 1.10. Since $\{X_n, n \in \mathbb{Z}\}$, is a S$\alpha$S sequence there exist functions $\{f_n, n \in \mathbb{Z}\}$ in $L^\alpha[0,1]$ such that

$$-\log E \exp(i \sum_{j=1}^{n} \lambda_j X_{k_j}) = || \sum_{j=1}^{n} \lambda_j f_{k_j} ||_\alpha^\alpha$$

([15]). Further if $\{Z(s): s \in [0,1]\}$ is an independent increment SαS process with $Ee^{itZ(s)} = \exp(-s|t|^\alpha)$, then the process $\{Y_n, n \in \mathbb{Z}\}$ defined by

$$Y_n = \int_0^1 f_n(s)dZ(s)$$

is stochastically equivalent to $\{X_n, n \in \mathbb{Z}\}$. Hence we may consider $M_\alpha(X:\infty) \subset L^\alpha$. But every symmetric basic sequence for a subspace of L^p, $1 \leq p < 2$, is equivalent to a unit vector sequence in some Orlicz sequence space ([16] p. 149). This therefore gives a geometric condition on $M_\alpha(X:\infty)$ in order that it has a symmetric basis.

We now introduce some definitions and results on Orlicz spaces which will be used throughout the thesis. This material is taken from [14] and [27]. For further information the reader is referred to these books.

Definition 1.11. An Orlicz function φ is a continuous, even, nonnegative function, nondecreasing for positive x such that $\varphi(x) = 0$ if and only if $x = 0$.

Definition 1.12. A measure μ on a measurable space $(\Omega, \mathcal{F})$ is separable if there exists a finite or enumerable subcollection $\mathcal{F}_0$ of measurable sets of finite μ-measure having the property that if E is an arbitrary set of finite μ-measure, there exists to each $\epsilon > 0$ a set $F \in \mathcal{F}_0$ such that $\mu(E \Delta F) < \epsilon$.

Note that if μ = counting measure on ($\mathbb{N}$, power set of $\mathbb{N}$), then μ is separable.

Let μ be a σ-finite separable measure on $(\Omega, \mathcal{F})$. We say that two measurable functions f and g are equivalent if $f = g$ a.e. $[\mu]$. Let $\mathcal{M} = \mathcal{M}(\Omega, \mathcal{F}, \mu)$ be the space of equivalence classes of measurable functions determined by this equivalence relation. For an Orlicz function φ and $f \in \mathcal{M}$ define

$$(1.13) \qquad \rho_\varphi(f) = \int \varphi(f) d\mu$$

and $\mathcal{L}_\varphi = \mathcal{L}_\varphi(\Omega, \mathcal{F}, \mu) = \{f \in \mathcal{M}: \rho_\varphi(f) < \infty\}$. $\mathcal{L}_\varphi$ is not a linear space in general.

Example 1.14. Let $\varphi(x) = e^{|x|} - |x| - 1$. Then φ is a convex Orlicz function such that $\mathcal{L}_\varphi$ is not a linear space ([14]).

Definition 1.15. An Orlicz function φ is said to satisfy the Δ_2-condition if there exists $h > 0$ such that

$$(1.15.1) \qquad \varphi(2x) \leq h\varphi(x) \quad \text{for} \quad x \geq 0.$$

Remark 1.16. If φ satisfies the Δ_2-condition then $\mathcal{L}_\varphi$ is a linear space ([27], p. 81). The Orlicz function in Example 1.14 does not satisfy the Δ_2-condition ([14]).

Examples 1.17. 1) Let $\varphi(x) = |x|^p$. Then φ is an Orlicz function satisfying the Δ_2–condition. Further $\mathscr{L}_\varphi$ is the classical L^p space.
2) $\varphi(x) = (1+|x|)\ln(1+|x|) - |x|$ is a convex Orlicz function which satisfies the Δ_2–condition ([14]). Further $\mathscr{L}_\varphi$ is different from any L^p space ([14]).

Definition 1.18. Two Orlicz functions φ and ψ are said to be complementary if

$$xy \le \varphi(x) + \psi(y) \quad \text{for all} \quad x,y \ge 0. \tag{1.18.1}$$

Let φ be a convex Orlicz function. Then ([14]) φ can be represented in the form $\varphi(x) = \int_0^{|x|} p(t)dt$ where $p(t)$, the right derivative of φ, is a non–decreasing, right continuous, nonnegative function defined for $t \ge 0$. Then the function $q(s) = \sup_{p(t)\le s} t$ is a non–decreasing, right continuous function defined on the nonnegative reals for which $q(t) \ge 0$, $t \ge 0$. If $\psi(x) = \int_0^{|x|} q(s)ds$, then ψ is a convex Orlicz function and ψ is the complementary function of φ.

Example 1.19. 1) Let $\varphi(x) = |x|^p/p$. For $t > 0$, $p(t) = \varphi'(t) = t^{p-1}$. Therefore $q(s) = s^{q-1}$, where $\frac{1}{p} + \frac{1}{q} = 1$. Hence $\psi(x) = \int_0^{|x|} q(s)ds = \frac{|x|^q}{q}$.
(2) If $\varphi(x) = e^{|x|} - |x| - 1$, then the corresponding complementary function is

$$\psi(x) = (1 + |x|) \ln(1 + |x|) - |x|$$

([14]). This is an example where φ does not satisfy the Δ_2–condition but its complementary function ψ satisfies the Δ_2–condition.

Remark 1.20. In many cases it is impossible to find an explicit formula for the complementary function e.g. $\varphi(x) = e^{x^2} - 1$ ([14]).

For $f \in \mathscr{M}(\Omega, \mathscr{F}, \mu)$ define

(1.21) $||f||_\varphi = \inf \{\lambda > 0: \int \varphi(f/\lambda)d\mu \leq 1\}$

and

(1.22) $L_\varphi = L_\varphi(\Omega, \mathscr{F}, \mu) = \{f \in \mathscr{M}: ||f||_\varphi < \infty\}$.

In particular we define

(1.23) $\ell_\varphi = L_\varphi(\mathbb{N}$, power set of $\mathbb{N}$, counting measure).

Then L_φ is a linear space, called an Orlicz space, and $||\cdot||_\varphi$ defines a semi–norm on L_φ. If φ satisfies the Δ_2–condition, then $L_\varphi = \mathscr{L}_\varphi$. If, in addition, φ is convex, then $||\cdot||_\varphi$ is a norm (called the Luxemburg norm) and $(L_\varphi, ||\cdot||_\varphi)$ is a Banach space.

We now assume that the Orlicz function φ is convex and satisfies the Δ_2–condition. Let ψ denote the complementary function of φ. If $f \in L_\varphi$, then $|||f|||_\varphi < \infty$, where

$$|||f|||_{\varphi} = \sup\{|\int fg\,d\mu| : g \in L_{\psi},\ \rho_{\psi}(g) \le 1\}$$

(1.24)

$$= \sup\{\int |fg|\,d\mu : g \in L_{\psi},\ \rho_{\psi}(g) \le 1\}$$

and

(1.25) $$||f||_{\varphi} \le |||f|||_{\varphi} \le 2\,||f||_{\varphi}$$

$|||\cdot|||_{\varphi}$ is also a norm on L_{φ} and is called the Orlicz norm.

Example 1.26. Let $\varphi(x) = |x|^p/p$, $1 < p < \infty$. Then L_{φ} is the classical L^p space with the usual L^p–topology. If $f \in L^p$, then $|||f|||_{\varphi} = q^{1/q}\,||f||_p$ and $||f||_{\varphi} = p^{-1/p}||f||_p$ where $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 1.27. Let φ be a convex Orlicz function satisfying the Δ_2–condition with complementary function ψ. Then, for any $f \in L_{\varphi}$, $g \in L_{\psi}$,

(1.27.1) $$\int fg\,d\mu \le |||f|||_{\varphi}\,\rho_{\psi}(g)$$

Earlier in the chapter we saw that if $\{X_n, n \in \mathbb{Z}\}$ is a SαS sequence $(\alpha > 1)$ and $M_{\alpha}(X:\infty)$ has a symmetric basis, then $M_{\alpha}(X:\infty)$ is isomorphic to any ℓ_p where $1 \le p < \alpha$. Since $\ell_p \subset \ell_{\alpha}$, $1 \le p < \alpha$, we can define a dispersion distance on $M_{\alpha}(X:\infty)$ to be the ℓ_{α} distance.

Motivated by this let us consider the space

(1.28) $$\ell_\xi = \{\{c_n\}: \sum_n c_n\xi_n \text{ converges a.s.}\}$$

where $\{\xi_n, n \in \mathbb{N}\}$ is a sequence of ϵ–invariant exchangeable random variables. Let F denote the distribution function of ξ_1. Assume that F is not concentrated at 0. It will be seen later that this is not a stringent condition. Define

(1.29) $$\varphi(x) = \int_0^\infty (x^2u^2 \wedge 1)dF(u), \quad x \in \mathbb{R},$$

where $x \wedge y = \min(x,y)$ for any $x,y \in \mathbb{R}$. Then φ is an Orlicz function satisfying the Δ_2–condition. Let $\underline{c} = \{c_n\} \in \ell_\varphi$. Then

(1.30) $$\sum_{n=1}^\infty P[|c_n\xi_n|>1] + \sum_{n=1}^\infty E\, c_n^2\xi_n^2[|c_n\xi_n| \le 1] = \sum_{n=1}^\infty E(c_n^2\xi_n^2 \wedge 1) = 2\rho_\varphi(\underline{c}) < \infty.$$

Let $Y_n = c_n\xi_n\,[|c_n\xi_n| \le 1]$. Since $\rho_\varphi(\underline{c}) < \infty$,

(1.30)' $$\sum_{n=1}^\infty P(c_n\xi_n \ne Y_n) = \sum_{n=1}^\infty P(|c_n\xi_n| > 1) < \infty.$$

Hence, by the Borel–Cantelli lemma, $P(c_n\xi_n \ne Y_n \text{ i.o.}) = 0$. Therefore,

$\sum_{n=1}^{\infty} c_n \xi_n$ converges a.s. if and only if $\sum_{n=1}^{\infty} Y_n$ converges a.s.. We shall now prove that $\sum_{n=1}^{\infty} Y_n$ converges a.s.. Let $\mathcal{F}_n = \sigma\{\xi_k, k \le n\}$. Since $E|Y_n| \le 1$, $E^{\mathcal{F}_{n-1}} Y_n$ exists and by the ϵ–invariance property of $\{\xi_n, n \in \mathbb{N}\}$, $\int_A Y_n \, dP = \int_A (-Y_n) \, dP$ for all $A \in \mathcal{F}_{n-1}$. Therefore, $E^{\mathcal{F}_{n-1}} Y_n = 0$ for each $n \in \mathbb{N}$. In view of the fact that $\sum_{n=1}^{\infty} E\, Y_n^2 < \infty$ (by (1.30)) and by Proposition IV.6.1 [20], $\sum_{n=1}^{\infty} Y_n$ converges a.s.. Therefore, $\sum_{n=1}^{\infty} c_n \xi_n$ converges a.s.. Thus we get the following result.

Proposition 1.31. With the notation as in (1.23), (1.28) and (1.29), we have $\ell_\varphi \subset \ell_\xi$.

Remark 1.32. If in addition the random variables $\{\xi_n, n \in \mathbb{N}\}$ are independent, it is known $\ell_\varphi = \ell_\xi$ (see, for example, [2]).

Suppose that $\{\xi_n, n \in \mathbb{N}\}$ are ϵ–invariant, exchangeable random variables . Suppose $\{c_n\}$ is a sequence of real members such that $\sum_n c_n \xi_n$ converges a.s.. Let $X_n = c_n \xi_n$, $X_n^{(1)} = X_n \, [|X_n| \le 1]$, $X_n^{(2)} = X_n - X_n^{(1)}$. Then $\{X_n^{(1)}, X_m^{(2)}, m, n \in \mathbb{N}\}$ is a sequence of ϵ–invariant random variables. Further, as $\{\xi_n, n \in \mathbb{N}\}$ is assumed to be an ϵ–invariant exchangeable sequence

$$(1.33)\quad \mathcal{L}(X_n,\ X_n + X_{n+1},\ldots,\sum_{k=0}^{m} X_{n+k})$$
$$= \mathcal{L}(X_n^{(1)}+X_n^{(2)},X_n^{(1)}+X_n^{(2)}+X_{n+1}^{(1)}+X_{n+1}^{(2)},\ldots,\sum_{k=0}^{m}(X_{n+k}^{(1)}+X_{n+k}^{(2)}))$$
$$= \mathcal{L}(X_n^{(1)}-X_n^{(2)},\ X_n^{(1)}-X_n^{(2)}+X_{n+1}^{(1)}-X_{n+1}^{(2)},\ldots,\sum_{k=0}^{m}(X_{n+k}^{(1)}-X_{n+k}^{(2)}))$$

for each fixed $m \geq 0$, $n \geq 1$. Let

$$S_n = \sum_{k=1}^{n} X_k,\ S_n^{(1)} = \sum_{k=1}^{n} X_k^{(1)},\ S_n^{(2)} = \sum_{k=1}^{n} X_k^{(2)}.$$

Then for any $m \leq n$ and $\epsilon > 0$,

$$P[\max_{m\leq k\leq n} |S_k^{(1)} - S_m^{(1)}| > \epsilon]$$

$$= P[\max_{m\leq k\leq n} |2(S_k^{(1)}-S_m^{(1)})+(S_k^{(2)}-S_m^{(2)})-(S_k^{(2)}-S_m^{(2)})| > 2\epsilon]$$

$$\leq P[\max_{m\leq k\leq n} |(S_k^{(1)}-S_m^{(1)})-(S_k^{(2)}-S_m^{(2)})| > \epsilon]$$

$$+ P[\max_{m\leq k\leq n} |(S_k^{(1)}-S_m^{(1)})+(S_k^{(2)}-S_m^{(2)})| > \epsilon].$$

Using (1.33) we get for any $n \geq m$

$$(1.34)\quad P[\max_{m\leq k\leq n} |S_k^{(1)} - S_m^{(1)}| > \epsilon] \leq 2P[\max_{m\leq k\leq n} |S_k - S_m| > \epsilon].$$

But $[\max_{m\leq k\leq n} |S_k^{(1)}-S_m^{(1)}| > \epsilon] \uparrow [\sup_{k\geq m} |S_k^{(1)}-S_m^{(1)}| > \epsilon]$ and $[\max_{m\leq k\leq n} |S_k-S_m| > \epsilon] \uparrow [\sup_{k\geq m} |S_k-S_m| > \epsilon]$. Letting $n \to \infty$ in (1.34) we get,

$$(1.35) \qquad P[\sup_{k\geq m}|S_k^{(1)}-S_m^{(1)}| > \epsilon] \leq 2P\,[\sup_{k\geq m}|S_k-S_m| > \epsilon].$$

Suppose $\{S_n^{(1)}\}$ diverges with positive probability. Then there exists $\epsilon > 0$ and $\delta > 0$ such that for every m fixed

$$\delta \leq P\,[\sup_{n\geq m}|S_n^{(1)}-S_m^{(1)}| > \epsilon] \leq 2P[\sup_{n\geq m}|S_n-S_m| > \epsilon]$$

so that $\{S_n\}$ diverges with positive probability. This contradicts the fact that $\{S_n\}$ converges a.s.. Therefore $\{S_n^{(1)}\}$ converges a.s. i.e. $\sum_{n=1}^{\infty} X_n^{(1)}$ converges a.s.. Further $\sup_n|X_n^{(1)}| \leq 1$ so that $\sup_n|X_n^{(1)}| \in L^2$. Let $\mathscr{F}_n = \sigma\{\xi_n\colon k \leq n\}$ and $\Omega_0 = \{\sum_n E^{\mathscr{F}_{n-1}}(X_n^{(1)})^2 < \infty\}$. By Proposition IV.6.2 [20], $\Sigma\, X_n^{(1)}$ converges a.s. implies $P(\Omega_0) = 1$. Therefore

$$\begin{aligned}\sum_{n=1}^{\infty} E\, c_n^2\xi_n^2\,[|c_n\xi_n| \leq 1] &= \sum_{n=1}^{\infty} E(X_n^{(1)})^2 \\ &= \sum_{n=1}^{\infty} E\,E^{\mathscr{F}_{n-1}}(X_n^{(1)})^2 = E[\sum_{n=1}^{\infty} E^{\mathscr{F}_{n-1}}(X_n^{(1)})^2]1_{\Omega_0}.\end{aligned}$$

If, in addition, $\{\xi_n, n \in \mathbb{N}\}$ is a sequence of i.i.d. random variables, $\sum_n E\, c_n^2\xi_n^2 \wedge 1 < \infty$ i.e. $\sum_{n=1}^{\infty} \varphi(c_n) < \infty$. If further $\{\xi_n, n \in \mathbb{N}\}$ is a sequence i.i.d. SαS random variables with $\alpha > 1$, then $\sum_{n=1}^{\infty} \varphi(c_n) < \infty$ if and only if $\sum_n |c_n|^{\alpha} < \infty$. Therefore in this case $\ell_\xi = \ell_\varphi = \ell_\alpha$.

Lemma 1.36. Let $\underline{c} = \{c_n\} \in \ell_\varphi$. Then $\sum_{n=1}^{\infty} c_n \xi_n$ converges a.s. unconditionally. Further $\sum_{n=1}^{\infty} c_n \xi_n = \sum_{n=1}^{\infty} c_{\pi(n)} \xi_{\pi(n)}$ a.s. for every rearrangement $\{\pi(n)\}$ of $\{n\}$.

Proof. Let $Y_n = c_n \xi_n \, [|c_n \xi_n| \le 1]$. Let $\sum_{n=1}^{\infty} c_{\pi(n)} \xi_{\pi(n)}$ be a rearrangement of the series $\sum_{n=1}^{\infty} c_n \xi_n$. But

$$\sum_{n=1}^{\infty} P[Y_{\pi(n)} \ne c_{\pi(n)} \xi_{\pi(n)}] = \sum_{n=1}^{\infty} P[|c_{\pi(n)} \xi_{\pi(n)}| > 1]$$
$$= \sum_{n=1}^{\infty} P[|c_n \xi_n| > 1] < \infty \quad \text{(by (1.30)')}.$$

Hence $\sum_{n=1}^{\infty} c_{\pi(n)} \xi_{\pi(n)}$ converges a.s. if and only if $\sum_{n=1}^{\infty} Y_{\pi(n)}$ converges a.s.. Let $\mathcal{F}_n' = \sigma\{\xi_{\pi(k)}, k \le n\}$. Since $E|Y_{\pi(n)}| \le 1$, $E^{\mathcal{F}_{n-1}'} Y_{\pi(n)}$ exists and for all $A \in \mathcal{F}_{n-1}'$, since $\int_A Y_{\pi(n)} \, dP = \int_A (-Y_{\pi(n)}) \, dP$, we get $E^{\mathcal{F}_{n-1}'} Y_{\pi(n)} = 0$. By (1.30) $\sum_{n=1}^{\infty} E\, Y_{\pi(n)}^2 = \sum_{n=1}^{\infty} E\, Y_n^2 < \infty$. Hence, by Proposition IV.6.1 [20], $\sum_{n=1}^{\infty} Y_{\pi(n)}$ converges a.s.. Therefore $\sum_{n=1}^{\infty} c_{\pi(n)} \xi_{\pi(n)}$ converges a.s.. Hence $\sum_{n=1}^{\infty} c_n \xi_n$ converges a.s. unconditionally. Let $\mathcal{F}$ denote the tail σ-algebra of $\{\xi_n, n \in \mathbb{N}\}$. Let $\{\pi(n)\}$ be a rearrangement of $\{n\}$. Let $S_m = \sum_{j=1}^{m} Y_j$, $S_m' = \sum_{j=1}^{m} Y_{\pi(j)}$, $Q_m = \{\pi(1), \dots, \pi(m)\} \Delta \{1, \dots, m\}$. By Corollary 5 in §7.3 [6]

$$\begin{aligned} E(S_m - S'_m)^2 &= \sum_{j\in Q} EY_j^2 + \sum_{\substack{j,k\in Q\\ j\neq k}} EY_jY_k \\ &= \sum_{j\in Q} EY_j^2 + \sum_{\substack{j,k\in Q\\ j\neq k}} E\, E^{\mathcal{X}} Y_jY_k \\ &= \sum_{j\in Q} EY_j^2 + \sum_{\substack{j,k\in Q\\ j\neq k}} E(E^{\mathcal{X}} Y_j)(E^{\mathcal{X}} Y_k) = \sum_{j\in Q} EY_j^2 \end{aligned}$$

If $\{\pi(1),\ldots,\pi(m)\} \supset \{1,\ldots,j\}$, then $E(S_m - S'_m)^2 \le \sum_{k=j+1}^{\infty} E\, Y_j^2 \to 0$ as $j \to \infty$. Hence $S_m - S'_m \to 0$ in probability. But $S_m \to \sum_{n=1}^{\infty} Y_n$ a.s. and $S'_m \to \sum_{n=1}^{\infty} Y_{\pi(n)}$ a.s.. Hence $\sum_{n=1}^{\infty} Y_n = \sum_{n=1}^{\infty} Y_{\pi(n)}$ a.s.. By (1.30)$'$ we get $\sum_{n=1}^{\infty} c_n\xi_n = \sum_{n=1}^{\infty} c_{\pi(n)}\xi_{\pi(n)}$.

<u>Notation 1.37</u>. For $\underline{a} = \{a_n\} \in \ell_\xi$ and $X = \sum_{n=1}^{\infty} a_n\xi_n$, define $\rho_\varphi(X) = \sum_{n=1}^{\infty} \varphi(a_n)$.

The minimization problem considered will be with respect to this translation invariant distance.

Let $\{\xi_n, n \in \mathbb{Z}\}$ be a sequence of ϵ–invariant exchangeable random variables. A special type of process which is a moving average in ϵ–invariant exchangeable random variables is the ExARMA process given by the stochastic difference equation

$$X_n - \alpha_1 X_{n-1} - \ldots - \alpha_p X_{n-p} = \xi_n + \theta_1 \xi_{n-1} + \ldots + \theta_q \xi_{n-q} \tag{1.38}$$

with transfer function $\Theta(z)/A(z)$, where

(1.38.1) $\Theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$

(1.38.2) $A(z) = 1 - \alpha_1 z - \ldots - \alpha_p z^p$

for $z \in \mathbb{D} = \{z \in \mathbb{C}: |z| \leq 1\}$ are such that

(1.38.3) $\Theta(z)A(z) \neq 0$ for all $z \in \mathbb{D}$.

If

(1.38.4) $\frac{\Theta(z)}{A(z)} = \sum_{k=0}^{\infty} \pi_k z^k$, $z \in \mathbb{D}$ (note $\pi_0 = 1$),

then $X_n = \sum_{k=0}^{\infty} \pi_k \xi_{n-k}$, the convergence being a.s.. Note that $\{X_n\}$ defined this way is a stationary sequence. Conversely suppose that $\pi_0 = 1$ and $\{\pi_n\} \in \ell_\varphi$. Then $X_n = \sum_{k=0}^{\infty} \pi_k \xi_{n-k}$, convergence being a.s., is a stationary solution of (1.38) with transfer function $\Theta(z)/A(z)$ satisfying (1.38.4). Note that in view of Remark 1.32 in the special case when ξ_i are i.i.d., $\sum_{j=0}^{\infty} \pi_j \xi_{-j}$ converges a.s. if and only if $\{\pi_k\} \in \ell_\varphi$. As observed earlier one does not necessarily get this condition for non–i.i.d. random variables. We now prove extension of results obtained by Cline and Brockwell ([8]).

Remark 1.39. 1) Suppose $\{\nu_j\}$ is a sequence such that $\sum_{m=1}^{\infty} \varphi(\sum_{j=1}^{m} \nu_j \pi_{m-j}) < \infty$. Then by Lemma 1.36 $\sum_{m=1}^{\infty} (\sum_{j=1}^{m} \nu_j \pi_{m-j}) \xi_{n+1-m}$

converges a.s. unconditionally and $\sum_{m=1}^{\infty} (\sum_{j=1}^{\infty} \nu_j \pi_{m-j}) \xi_{n+1-m} = \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$.

2) Suppose $\sum_{j=n+1}^{\infty} \varphi(\nu_j) < \infty, \sum_{j=-\infty}^{n} \varphi(\nu_j) < \infty$ for some $n \in \mathbb{Z}$. Then $\sum_{j=n+1}^{\infty} \nu_j \xi_j$ and $\sum_{j=-\infty}^{n} \nu_j \xi_j$ converges a.s. unconditionally (Lemma 1.36). Therefore $Y = \sum_{j=\infty}^{\infty} \nu_j \xi_j$ converges a.s.. For Y of this form define $\rho_{\varphi}(Y) = \sum_{j=-\infty}^{\infty} \varphi(\nu_j)$.

Theorem 1.40. Let $\{X_n, n \in \mathbb{Z}\}$ be the process satisfying (1.38) – (1.38.3). Let S_* be the class of random variables of the form

$$Y = \sum_{j=n+1}^{\infty} \delta_j \xi_j + \sum_{j=1}^{\infty} \nu_j X_{n+1-j} \tag{1.40.1}$$

where $\sum_{j=n+1}^{\infty} \varphi(\delta_j) < \infty, \sum_{m=1}^{\infty} \varphi(\sum_{j=1}^{m} \nu_j \pi_{m-j}) < \infty$. For each $Y \in S_*$, define

$$P_{\infty} Y = \{\sum_{j=1}^{\infty} a_j X_{n+1-j} : \sum_{m=1}^{\infty} \varphi(\sum_{j=1}^{m} \nu_j \pi_{m-j}) < \infty \text{ and } \rho_{\varphi}(Y - \sum_{j=1}^{\infty} a_j X_{n+1-j}) \text{ is minimum}\}. \tag{1.40.2}$$

Then $P_{\infty} Y$ consists exactly of one element; namely, $Y^* = \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$. Furthermore the mapping $Y \to Y^*$ is linear on S_*.

Proof. Let $Y \in S_*$ be of the form $Y = \sum_{j=n+1}^{\infty} \delta_j \xi_j + \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$.

Then, by assumptions, Lemma 1.36, and Remark 1.39

$$\begin{aligned} Y - \sum_{j=1}^{\infty} a_j X_{n+1-j} &= \sum_{j=1}^{\infty} \delta_j \xi_j + \sum_{j=1}^{\infty} (\nu_j - a_j)\, X_{n+1-j} \\ &= \sum_{j=1}^{\infty} \delta_j \xi_j + \sum_{m=1}^{\infty} \Big(\sum_{j=1}^{m} (\nu_j - a_j) \pi_{m-j} \Big) \xi_{n+1-m}. \end{aligned}$$

Therefore

$$\begin{aligned} \rho_\varphi\Big(Y - \sum_{j=1}^{\infty} a_j X_{n+1-j}\Big) &= \sum_{j=n+1}^{\infty} \varphi(\delta_j) + \sum_{m=1}^{\infty} \varphi\Big(\sum_{j=1}^{m} (\nu_j - a_j) \pi_{m-j} \Big) \\ &\geq \sum_{j=n+1}^{\infty} \varphi(\delta_j). \end{aligned} \tag{1.40.3}$$

If $\nu_j = a_j$ for each j, then

$$\rho_\varphi\Big(Y - \sum_{j=1}^{\infty} a_j X_{n+1-j}\Big) = \sum_{j=n+1}^{\infty} \varphi(\delta_j).$$

Conversely suppose there is an equality in (1.40.3). Since φ is an Orlicz function,

$$\sum_{j=1}^{m} (\nu_j - a_j) \pi_{m-j} = 0 \quad (m \in \mathbb{N}).$$

If $m = 1$, then $0 = (\nu_1 - a_1)\pi_0 = \nu_1 - a_1$ and hence $\nu_1 = a_1$. Suppose $\nu_m = a_m$ for $m=1,\ldots,k$. Then

$$0 = (\nu_1 - a_1)\, \pi_{k+1-1} + \ldots + (\nu_{k+1} - a_{k+1}) \pi_{(k+1)-(k+1)} = \nu_{k+1} - a_{k+1}$$

so that $\nu_{k+1} = a_{k+1}$. Therefore, by induction on m, $\nu_m = a_m$ for all m.

Hence $\rho_\varphi(Y-\sum_{j=1}^{\infty} a_j X_{n+1-j})$ is minimum if and only if $\nu_m = a_m$ for each m i.e. $P_\infty Y = \{\sum_{j=1}^{\infty} \nu_j X_{n+1-j}\}$. If $Y^* = \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$, then the map $Y \to Y^*$ is linear on S_* (by our assumptions and Lemma 1.36).

<u>Corollary 1.41</u>. Let $\{\xi_n, n \in \mathbb{Z}\}$ be i.i.d. random variables satisfying (1.38) – (1.38.3). The the map $Y \to Y^*$ is continuous at 0 with respect to convergence in probability in both spaces.

<u>Proof</u>. Suppose $Y^{(k)} = \sum_{j=n+1}^{\infty} \delta_j^{(k)} \xi_j + \sum_{j=1}^{\infty} \nu_j^{(k)} X_{n+1-j} \in S_*$ for each k are such that $Y^{(k)} \to 0$ in probability. Then

$$Y^{(k)} = \sum_{j=n+1}^{\infty} \delta_j^{(k)} \xi_j + \sum_{m=1}^{\infty} (\sum_{j=1}^{m} \nu_j^{(k)} \pi_{m-j})\, \xi_{n+1-m}.$$

By Corollary 2.3.5 [2], $\sum_{j=n+1}^{\infty} \varphi(\delta_j^{(k)}) + \sum_{m=1}^{\infty} \varphi(\sum_{j=1}^{m} \nu_j^{(k)} \pi_{m-j}) \to 0$ as $k \to \infty$. In particular $\sum_{m=1}^{\infty} \varphi(\sum_{j=1}^{m} \nu_j^{(k)} \pi_{m-j}) \to 0$ as $k \to \infty$.

Using Corollary 2.3.5 [2] again we get

$$Y^{(k)*} = \sum_{j=1}^{\infty} \nu_j^{(k)} X_{n+1-j} = \sum_{m=1}^{\infty} (\sum_{j=1}^{m} \nu_j^{(k)} \pi_{m-j}) \xi_{n+1-m} \to 0$$

in probability. Therefore the map $Y \to Y^*$ is continuous at 0.

<u>Remark 1.42</u>. Suppose $\{\xi_n, n \in \mathbb{Z}\}$ are i.i.d. random variables which are not necessarily symmetric. Let $\varphi_1(x) = \int_{-\infty}^{\infty} |ux| \wedge 1 dF(u)$ and

$\varphi_2(x) = \int_{-\infty}^{\infty} u^2x^2 \Lambda 1 dF(u)$. Then φ_1 and φ_2 are Orlicz functions satisfying the Δ_2-condition. Let $\underline{b} = \{b_n\}$ be such that $\sum_{n=-\infty}^{\infty} \varphi_1(b_n) < \infty$. Then as $\varphi_2(b_n) \leq \varphi_1(b_n)$ for all n, $\sum_{n=-\infty}^{\infty} \varphi_2(b_n) < \infty$. Further $\sum_{n=-\infty}^{\infty} E|b_n\xi_n|[|b_n\xi_n| \geq 1] \leq \sum_{n=-\infty}^{\infty} \varphi_1(b_n) < \infty$. Hence, by the Kolmogorov three series theorem $\sum_{n=-\infty}^{\infty} |b_n\xi_n|$ converges a.s.. Instead of S_* in Theorem 1.19 consider $\tilde{S}_*$ which is the set of random variables of the form

$$Y = \sum_{j=n+1}^{\infty} \delta_j\xi_j + \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$$

where $\sum_{j=n+1}^{\infty} \varphi_1(\delta_j) < \infty$, $\sum_{m=1}^{\infty} \varphi_1(\sum_{j=1}^{m} \nu_j\pi_{m-j}) < \infty$. Then, as in Theorem 1.40, $P_\infty Y = \{\sum_{j=1}^{\infty} \nu_j X_{n+1-j}\}$ and the map $Y \to \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$ is linear on $\tilde{S}_*$.

The proof of the following theorem is similar to Theorem 2.2 in Cline and Brockwell ([8]) and hence is omitted.

<u>Theorem 1.43</u>. For the ExARMA (p,q) process $\{X_n, n \in \mathbb{Z}\}$ satisfying (1.38)–(1.38.3) there exists a unique minimum φ-dispersion linear prediction X^*_{n+k} for X_{n+k}, $k \geq 1$, based on the infinite past $X_n, X_{n-1},\ldots$. This predictor satisfies the recursive relationship

$$X^*_{n+k} = \sum_{j=1}^{k-1} \psi_j X^*_{n+k-j} + \sum_{j=k}^{\infty} \psi_j X_{n+k-j} \tag{1.43.1}$$

where $1 - \psi_1 z - \psi_2 z^2 - \ldots = A(z)/\Theta(z)$, $z \in \mathbb{D}$.

<u>Remark 1.44</u>. Suppose one has only the data $X_n,\dots,X_1$. For $Y \in S_*$, of the form (1.40.1), define the truncated predictor by $Y^*(n) = \sum_{j=1}^{n} \nu_j X_{n+1-j}$. By the equality in (1.40.3) we have

$$\rho_\varphi(Y-Y^*(n)) - \rho_\varphi(Y-Y^*) = \rho_\varphi(Y-Y^*(n)) - \sum_{j=n+1}^{\infty} \varphi(\delta_j)$$

But $Y-Y^*(n) = \sum_{j=n+1}^{\infty} \delta_j \xi_j + \sum_{m=n+1}^{\infty} \left(\sum_{j=n+1}^{m} \nu_j \pi_{m-j} \right) \xi_{n+1-m}$.

Therefore

$$\rho_\varphi(Y-Y^*(n)) - \rho_\varphi(Y-Y^*)$$

$$= \sum_{j=n+1}^{\infty} \varphi(\delta_j) + \sum_{m=n+1}^{\infty} \varphi\left(\sum_{j=n+1}^{m} \nu_j \pi_{m-j} \right) - \sum_{j=n+1}^{\infty} \varphi(\delta_j)$$

$$= \sum_{m=n+1}^{\infty} \varphi\left(\sum_{j=n+1}^{m} \nu_j \pi_{m-j} \right).$$

In particular, if $Y = X_{n+1}$, then $\rho_\varphi(X_{n+1}-X^*_{n+1}) = \varphi(1)$ and

$$X_{n+1} - X^*_{n+1}(n) = \xi_{n+1} + \sum_{j=n+1}^{\infty} \psi_j X_{n+1-j}$$

$$= \xi_{n+1} + \sum_{m=n+1}^{\infty} \left\{ \sum_{j=n+1}^{m} \psi_j \pi_{m-j} \right\} \xi_{n+1-m}.$$

Therefore

$$\rho_\varphi(X_{n+1}-X^*_{n+1}(n)) = \varphi(1) + \sum_{m=n+1}^{\infty} \varphi\left(\sum_{j=n+1}^{m} \psi_j \pi_{m-j} \right)$$

so for large n the truncation is nearly optimal.

Let X_n satisfy

$$(1.44.1) \qquad X_n - \alpha_1 X_{n-1} - \dots - \alpha_p X_{n-p} = \xi_n$$

and let $n > p$. Then $\psi_j = 0$ if $j \geq p+1$ so that

$$\rho_\varphi(X_{n+1} - X^*_{n+1}(n)) = \varphi(1)$$

and hence the truncated predictor is optimal. Assumption (1.38.3) reduces to $A(z) \neq 0 \;\; \forall \;\; z \in \mathbb{D}$.

We state the following lemma and corollary whose proof is similar to Theorem 1.40 and Theorem 1.43 respectively.

<u>Lemma 1.45</u>. Let $\underline{X}_n = (X_n, X_{n-1}, \dots, X_1)$. Let $S_*(n)$ be the class of random variables of the form $Y = Z + \underline{\nu}' \, \underline{X}_n$ for some $\underline{\nu} \in \mathbb{R}^n$ and $Z = \sum_{j=n+1}^{\infty} \delta_j \xi_j$ such that $\sum_{j=n+1}^{\infty} \varphi(\delta_j) < \infty$. Then, for each $Y \in S_*(n)$, the set $P_n X = \{\underline{a}'\underline{X}_n : \rho_\varphi(Y - \underline{a}'\underline{X}_n)$ is minimum$\}$ consists exactly of one variable. For $Y = Z + \underline{\nu}'\underline{X}_n$, this unique variable is $\hat{Y} = \underline{\nu}'\underline{X}_n$. Furthermore, the mapping $Y \to \hat{Y}$ is linear on $S_*(n)$.

<u>Corollary 1.46</u>. For the process (1.44.1), provided $n > p$, there exists a unique minimum predictor $\hat{X}_{n+k}$ for $X_{n+k} (k \geq 1)$ in terms of $X_1, \dots, X_n$. This predictor satisfies the recursive relationship

$$\hat{X}_{n+k} = \alpha_1 \hat{X}_{n+k-1} + \dots + \alpha_p \hat{X}_{n+k-p}$$

with the initial condition $\hat{X}_j = X_j$, $1 \leq j \leq n$.

Let $\{\xi_n, n \in \mathbb{N}\}$ be i.i.d. random variables with the property that there exists $0 < \alpha < 2$ such that

$$\lim_{t\to\infty} \frac{P(|\xi_1|>tx)}{P(|\xi_1|>t)} = x^{-\alpha} \quad \text{for each} \quad x > 0. \tag{1.47}$$

We now prove a result of Cline ([7]). In order to do so we define regularly varying functions and state a theorem from [12] (p. 275–281).

Definition 1.48. A positive function defined on $(0,\infty)$ varies slowly at infinity if for each $x > 0$,

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1. \tag{1.48.1}$$

A positive function U defined on $(0,\infty)$ varies regularly with exponent ρ if and only if

$$U(x) = x^{\rho}L(x) \tag{1.48.2}$$

where $-\infty < \rho < \infty$ and L is slowly varying.

Theorem 1.49. a) If U varies regularly with exponent γ, then

$$\frac{t^{p+1}U(t)}{U_p(t)} \to p + \gamma + 1, \quad p + \gamma + 1 \geq 0 \tag{1.49.1}$$

where

$$U_p(t) = \int_0^t x^p U(x)dx. \tag{1.49.2}$$

b) If L varies slowly at infinity, then

$$t^{-\epsilon} < L(t) < t^{\epsilon}$$

for any fixed $\epsilon > 0$ and all t sufficiently large.

Suppose $\{\xi_n, n \in \mathbb{N}\}$ is a sequence of ϵ-invariant exchangeable random variables satisfying (1.47) for some $0 < \alpha < 2$. Let φ be defined by (1.29) and let

(1.50) $$L(t) = t^{\alpha}P(|\xi_1| > t), \; U(t) = t^{-\alpha}L(t).$$

By (1.47), L is slowly varying at infinity and U is regularly varying index $-\alpha$. By Theorem 1.49a)

$$\frac{x^2 U(x)}{U_1(x)} \to 2 - \alpha \quad \text{so that}$$

(1.51) $$U_1(x) \sim x^2(2-\alpha)^{-1}\, U(x) = x^2(2-\alpha)P(|\xi_1| > x).$$

Since $2\varphi(a) = Ea^2\xi^2{}_1\Lambda 1$

$$= P(|a\xi_1|>1) + Ea^2\xi^2{}_1[|a\xi_1|\le 1],$$

$$\varphi(a) \ge P(|a\xi_1| > 1) = U(\tfrac{1}{|a|}) \quad \text{for each } a.$$

But $2\varphi(a) = Ea^2\xi_1^2\Lambda 1 = \int_0^{\infty} P(a^2\xi_1^2\Lambda 1>t)dt$

$$= \int_0^1 P(a^2\xi_1^2\Lambda 1>t)dt$$

$$\le \int_0^1 P(a^2\xi_1^2>t)dt$$

$$= 2\int_0^{1/|a|} |a|^2\, sP(|\xi_1| > s)\, ds$$

and hence $\varphi(a) \leq 2|a|^2 \, U_1(\frac{1}{|a|})$ so by (1.51) there exists a constant K such that

$$\varphi(a) \leq KU(\tfrac{1}{|a|}) = KP(|a\xi_1| > 1)$$

for $|a|$ sufficiently close to 0. Therefore

$$(1.52) \qquad P(|a\xi_1|>1) < \varphi(a) < KP(|a\xi_1|>1)$$

for $|a|$ sufficiently close to 0. Theorem 1.49b) gives us

$$(1.53) \qquad (\tfrac{1}{|a|})^{-\epsilon} < |a|^{-\alpha} P(|a\xi_1|>1) < (\tfrac{1}{|a|})^{\epsilon}$$

for $\epsilon > 0$ fixed and all $|a|$ sufficiently small. (1.52) along with (1.53) imply there exists constants $K_1, K_2 > 0$ such that

$$K_1|a|^{\alpha+\epsilon} < \varphi(a) < K_2|a|^{\alpha-\epsilon}$$

for $\epsilon > 0$ fixed and $|a|$ sufficiently small. Thus if $\underline{a} = \{a_n\} \in \ell_p$ for some $p < \alpha$, then $\sum_{n=0}^{\infty} a_n\xi_n$ converges unconditionally and $\sum_{n=0}^{\infty} a_n\xi_n = \sum_{m=0}^{\infty} a_{\pi(m)}\xi_{\pi(m)}$ for any rearrangement $\{\pi(m)\}$ of $\{n\}$ (by Lemma 1.36).

Let $\{\xi_n, n \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables, which are not necessarily symmetric, satisfying (1.47). Let L and U be as defined in (1.50). Let $\varphi_1(a) = E|a\xi|\Lambda 1$. Then

$$\varphi_1(a) = E|a\xi_1|\Lambda 1 = \int_0^\infty P(|a\xi|\Lambda 1>t)dt$$
$$= \int_0^1 P(|a\xi|\Lambda 1>t)dt$$
$$\leq \int_0^1 P(|a\xi|>t)dt$$
$$= \int_0^1 \left(\frac{t}{|a|}\right)^{-\alpha} L(t/|a|)dt.$$

Let $|a|$ be sufficiently close to zero. Hence for any $\epsilon > 0$

$$\varphi_1(a) \leq \int_0^1 \left(\frac{t}{|a|}\right)^{-\alpha} \left(\frac{t}{|a|}\right)^{\epsilon} dt.$$
$$= |a|^{\alpha-\epsilon} \int_0^1 t^{\epsilon-\alpha} ds$$

Let $\epsilon > 0$ be such that $\epsilon + 1 - \alpha > 0$. Then

$$\varphi_1(a) \leq \frac{|a|^{\alpha-\epsilon}}{\epsilon+1-\alpha} = \frac{|a|^{\alpha-\epsilon}}{\epsilon+1-\alpha}.$$

Since $\varphi_2(a) = Ea^2\xi_1^2\Lambda 1 \leq \varphi_1(a)$, we have

$$\varphi_2(a) \leq \varphi_1(a) \leq \frac{|a|^{\alpha-\epsilon}}{\epsilon+1-\alpha}. \tag{1.54}$$

Let $0 < \delta < 1 \Lambda \alpha$ and $\underline{a} = \{a_n\} \in \ell_\delta$. Let $\epsilon = \alpha-\delta$. Then $\epsilon > 0$ and $\epsilon+1-\alpha = \alpha-\delta+1-\alpha = 1-\alpha>0$. By (1.54), $\sum_{n=-\infty}^{\infty} \varphi_2(a_n) < \infty$ and $\sum_{n=-\infty}^{\infty} \varphi_1(a_n) < \infty$. Hence by the Kolmogorov three series theorem $\sum_{n=-\infty}^{\infty} |a_n\xi_n|$ converges. This result is due to Cline ([7]). Prediction problem for $a \in \ell_\delta$, $\delta < 1\Lambda\alpha$ was considered by Cline and Brockwell ([8]).Observe that the dispersion distance used by Cline and Brockwell ([8]) is an appropriate distance.

CHAPTER II

METRIC PROJECTIONS

In Chapter I we considered a minimization problem in Orlicz sequence spaces. In a certain class of Orlicz spaces the minimum φ-dispersion linear predictor is the metric projection considered by Cambanis, Hardin and Weron [5]. In order to define metric projection we need some concepts from the geometry of Banach spaces ([13], p. 342]).

<u>Definition 2.1</u>. A function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is said to be strictly convex if

$$\varphi(\lambda x+(1-\lambda)y) < \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for all $\lambda \in (0,1)$, $x,y \in \mathbb{R}$.

<u>Definition 2.2</u>. A Banach space $(\mathcal{X}||\cdot||)$ is said to be rotund if it satisfies the following property: if $x,y \in \mathcal{X}$ are such that $x \neq y$ and $||x|| = ||y|| = 1$, then $||\frac{1}{2}(x+y)|| < 1$.

<u>Definition 2.3</u>. Let $(\mathcal{X}||\cdot||)$ be a normed linear space, $N \subseteq \mathcal{X}$, $x \in \mathcal{X}$ Define $\mathcal{P}_N(x)$ by

$$\mathcal{P}_N(x) = \{y_0 \in N\colon ||x-y_0|| = \inf_{y\in N} ||x-y||\}.$$

If $\mathcal{P}_N(x)$ consists of exactly one element, denoted by $P_N x$, then $P_N x$ is called the metric projection of x on N.

<u>Remark 2.4</u>. $x \in N$ if and only if $P_N x = x$.

Example 2.5. Suppose that $\{\xi_n, n \in \mathbb{N}\}$ is an symmetric basis for a subspace $\mathscr{X}$ of L^p, $p > 1$. Let $N \subset \mathscr{X}$ be a closed subspace and suppose $N = \overline{sp}\ \{\xi_{n_k}, k \in S\}$ where $S \subseteq \mathbb{N}$ is a finite or a countable subset of $\mathbb{N}$. Let $Y \in \mathscr{X}$ Then there exists a sequence of scalars $\{a_n\}$ such that $Y = \sum_{n=1}^{\infty} a_n\xi_n$, the convergence being with respect to the L^p norm. Let $Y^* = \sum_{k \in S} a_{n_k}\xi_{n_k} \in N$. Then

$$(2.5.1) \qquad ||Y-Y^*||_p \geq \inf_{Z \in N} ||Y-Z||_p.$$

Let $Z = \sum_{k \in S} b_{n_k}\xi_{n_k} \in N$. Let $Y(n) = \sum_{k=1}^{n} a_k\xi_k$, $Z(n) = \sum_{n_k \leq n} b_{n_k}\xi_{n_k}$, $Y^*(n) = \sum_{n_k \leq n} a_{n_k}\xi_{n_k}$, $Y(0) = Z(0) = Y^*(0) = 0$.

Then $Y(n) - Y^*(n) = Y(n) - Z(n)$ if $n < n_1$.

$$(2.5.2) \qquad \text{i.e. } ||Y(n) - Y^*(n)||_p = ||Y(n) - Z(n)||_p \text{ if } n < n_1.$$

Since $\{\xi_n, n \in \mathbb{N}\}$ is a basis ([24] p. 58), there exists a constant $K \geq 1$ such that

$$(2.5.3) \qquad \begin{aligned} K\ ||Y(n_1) - Z(n_1)||_p &\geq ||Y(n_1-1) - Z(n_1-1)||_p \\ &= ||Y(n_1-1) - Y^*(n_1-1)||_p \quad \text{(by (2.5.2))}. \\ &= ||Y(n_1) - Y^*(n_1)||_p \end{aligned}$$

Suppose $n_1 < n < n_2$. Then as $\{\xi_n, n \in \mathbb{N}\}$ is a symmetric basis,

$$(2.5.4) \qquad K\|Y(n) - Z(n)\|_p \geq \|Y(n_1-1)-Z(n_1-1) + \sum_{k=n_1+1}^{n} a_k\xi_k\|_p$$
$$= \|Y(n) - Y^*(n)\|_p.$$

$$(2.5.5) \qquad K\ \|Y(n_2)-Z(n_2)\|_p \geq \|\sum_{\substack{n\neq n_1\\ n\neq n_2}} (a_n-b_n)\ \xi_n\|_p$$
$$= \|Y(n_2-1) - Y^*(n_2-1)\|_p \qquad \text{(by (2.5.4))}$$
$$= \|Y(n_2) - Y^*(n_2)\|_p$$

Continuing in this manner we obtain

$$(2.5.6) \qquad K\ \|Y(n)-Z(n)\|_p \geq \|Y(n) - Y^*(n)\|_p.$$

Letting $n \to \infty$ we get

$$(2.5.7) \qquad K\|Y-Z\|_p \geq \|Y-Y^*\|_p.$$

This is true for any $Z \in N$. Thus

$$\|Y-Y^*\|_p \leq \inf_{Z\in N} K\|Y-Z\|_p \leq K\|Y-Y^*\|.$$

In particular if $K = 1$, then $\|Y-Y^*\|_p = \inf_{Z\in N} \|Y-Z\|_p$. Therefore the metric projection of Y on N exists and is equal to Y^*.

The following proposition gives us conditions when $P_N x$ exists.

Proposition 2.6. Let $\mathcal{X}$ be a Banach space

a) Then $\mathcal{X}$ is reflexive if and only if for every $x \in \mathcal{X}$ and for every closed subspace N of $\mathcal{X}$, $\mathcal{P}_N(x) \neq \phi$.

b) If $\mathscr{X}$ is reflexive and rotund, $P_N x$ exists for each $x \in \mathscr{X}$

For proof we refer the reader to Corollary 2.4 [25] and to §2.6.2 (3) [13].

Remark 2.7. P_N is continuous, bounded and idempotent but not necessarily linear. For example let $\mathscr{X} = L^p[0,1]$, $1<p<2$, $M = \{\alpha f\colon \alpha \in \mathbb{R}\}$ where $f = 1_{(0,1/2)}$, $f_1 = 1_{(0,2/3)} + 1_{(0,1/4)}$, $f_2 = 1_{(0,1/3)} - 1_{(0,1/4)}$, $g = f_1 + f_2 = 1_{(0,2/3)} + 1_{(0,1/3)}$. Let $x^{<p>} = (\operatorname{sgn} x)\,|x|^p$ for any $x \in \mathbb{R}$, $P_M f_k = a_k f_k$, $k=1,2$, and $P_M g = af$. By Theorem 1.11 [25],

$$\int_0^1 f(g-af)^{<p-1>}d\lambda = 0 = \int_0^1 f(f_k-a_k f)^{<p-1>}d\lambda, \quad k=1,2.$$

$$\begin{aligned}
0 &= \int_0^1 f(g-af)^{<p-1>}d\lambda \\
&= \int_0^1 1_{(0,1/2)}(1_{(0,2/3)} + 1_{(0,1/3)} - a1_{(0,1/2)})^{<p-1>}d\lambda \\
&= \int_0^{1/3} (2-a)^{<p-1>}d\lambda + \int_{1/3}^{1/2} (1-a)^{<p-1>}d\lambda \\
&= \tfrac{1}{3}(2-a)^{<p-1>} + \tfrac{1}{6}(1-a)^{<p-1>}
\end{aligned}$$

so that $a = \dfrac{1+2^{\frac{p}{p-1}}}{1+2^{\frac{1}{p-1}}}$.

$$\begin{aligned}
0 &= \int_0^1 f(f_1-a_1 f)^{<p-1>}d\lambda \\
&= \int_0^1 1_{(0,1/2)}(1_{(0,2/3)} + 1_{(0,1/4)} - a_1 1_{(0,1/2)})^{<p-1>}d\lambda \\
&= \int_0^{1/4} (2-a_1)^{<p-1>}d\lambda + \int_{1/4}^{1/2} (1-a_1)^{<p-1>}d\lambda \\
&= \tfrac{1}{4}\,[(2-a_1)^{<p-1>} + (1-a_1)^{<p-1>}]
\end{aligned}$$

so that $a_1 = 3/2$.

$$
\begin{aligned}
0 &= \int_0^1 f(f_2 - a_2 f)^{<p-1>} d\lambda \\
&= \int_0^1 1_{(0,1/2)}\left(1_{(0,1/3)} - 1_{(0,1/4)} - a_2 1_{(0,1/2)}\right)^{<p-1>} d\lambda \\
&= \int_0^{1/4} (-a_2)^{<p-1>} d\lambda + \int_{1/4}^{1/3} (2-a_2)^{<p-1>} d\lambda + \int_{1/3}^{1/2} (-a_2)^{<p-1>} d\lambda \\
&= (-a_2)^{<p-1>}\left(\tfrac{1}{4} + \tfrac{1}{6}\right) + (2-a_2)^{<p-1>} \tfrac{1}{12} \\
&= (-a_2)^{<p-1>} \tfrac{5}{12} + (2-a_2)^{<p-1>} \tfrac{1}{12}
\end{aligned}
$$

so that $a_2 = \dfrac{2}{5^{\frac{1}{p-1}}+1}$ .Therefore

$$
\begin{aligned}
P_M(f_1+f_2) &= \frac{1+2^{\frac{p}{p-1}}}{1+2^{\frac{1}{p-1}}} \, 1_{(0,1/2)} \\
&\neq \tfrac{3}{2} \, 1_{(0,1/2)} + \frac{2}{5^{\frac{1}{p-1}}+1} \, 1_{(0,1/2)} \\
&= P_M f_1 + P_M f_2.
\end{aligned}
$$

However the following properties are true ([5]).

<u>Proposition 2.8</u>. Let $\mathscr{X}$ be a reflexive and rotund Banach space and N a closed subspace of $\mathscr{X}$

a) Then $P_N(\alpha x) = \alpha P_N x$ for all scalars α and for all $x \in \mathscr{X}$
$P_N(x+n) = P_N x + P_N n$ for all $x \in \mathscr{X}$ and all $n \in N$.

b) If further N has codimension one in $\mathscr{X}$, then $P_N\colon \mathscr{X} \to N$ is a linear operator.

<u>Proposition 2.9</u>. Let φ and ψ be complementary convex Orlicz functions satisfying the Δ_2-condition. Assume further that φ is strictly convex. Let N be a closed subspace of L_φ and let $f \in L_\varphi \cap N^c$. Then the metric projection of f on N, namely $P_N f$ exists.

<u>Proof</u>. In view of Proposition 2.6 it suffices to show that L_φ is reflexive and rotund. Since φ and ψ satisfy the Δ_2-condition, L_φ is reflexive ([27 ,p. 154]). Further since φ is strictly convex L_φ is rotund ([22]).

The following proposition shows us that if φ is convex, the minimum φ-dispersion linear predictor obtained in Theorem 1.40 is the metric projection with respect to the distance $||\cdot||_\varphi$ in ℓ_φ.

<u>Proposition 2.10</u>. Let φ be a convex Orlicz function satisfying the Δ_2-condition. Let $N \subset L_\varphi$ and $f \in L_\varphi \cap N^c$ be such that $\tilde{\mathscr{P}}_N f = \{g \in N: \rho_\varphi(f-g)$ is minimum$\}$ consists of exactly one element, denoted by f_φ. Assume further that

(2.10.1) $$\alpha f_\varphi = (\alpha f)_\varphi \quad \text{for any} \quad \alpha \in \mathbb{R}.$$

Then $P_N f$ exists and $P_N f = f_\varphi$.

<u>Proof</u>. For any $g \in L_\varphi$ let

$$T_g = \{\lambda > 0: \rho_\varphi(\lambda^{-1} g) \le 1\}.$$

Let $f_1 \in N$ and $\lambda \in T_{f-f_1}$. By (2.9.1)

$$\rho_\varphi(\lambda^{-1}(f-f_\varphi)) \leq \rho_\varphi(\lambda^{-1}(f-f_1)) \leq 1$$

so that $\lambda \in T_{f-f_\varphi}$. Hence $T_{f-f_1} \subseteq T_{f-f_\varphi}$. Therefore

$$\begin{aligned} ||f-f_\varphi||_\varphi &= \inf \{\rho_\varphi(\lambda^{-1}(f-f_\varphi)) : \lambda \in T_{f-f_\varphi}\} \\ &\leq \inf \{\rho_\varphi(\lambda^{-1}(f-f_1)) : \lambda \in T_{f-f_1}\} = ||f-f_1||_\varphi. \end{aligned}$$

This is true for any $f_1 \in N$. Hence, as $f_\varphi \in N$, we get

$$||f-f_\varphi||_\varphi = \inf_{f_1 \in N} ||f-f_1||_\varphi.$$

Therefore $P_N f$ exists and $P_N f = f_\varphi$.

<u>Corollary 2.11</u>. Suppose that $\{\xi_n, n \in \mathbb{Z}\}$ is a sequence of ϵ–invariant, exchangeable random variables. Suppose the distribution function F of ξ_0 is not concentrated at the origin. We further assume that the Orlicz function φ defined by $\varphi(x) = \int_0^\infty u^2x^2 \Lambda 1 dF(u)$ is a convex function. Assume the conditions of Theorem 1.40 are satisfied. Define

$$\tilde{N} = \{\{b_m\}_{m=-\infty}^{\infty} \in \ell_\varphi : b_k = 0 \text{ if } k \geq n+1 \text{ and}$$
$$b_k = \sum_{j=1}^{n+1-k} a_j \pi_{n+1-j-k} \text{ otherwise}\}.$$

Let $Y = \sum_{j=n+1}^{\infty} \delta_j \xi_j + \sum_{j=1}^{\infty} \nu_j X_{n+1-j} \in S_*$ and let $Y^* = \sum_{j=1}^{\infty} \nu_j X_{n+1-j}$.

Let $\underline{a} = \{a_k\}$ and $\underline{a}^* = \{a_k^*\}$ be defined by

$$a_k = \begin{cases} \delta_k & k \in \{n+1,n+2,....\} \\ \sum_{j=1}^{n+1-k} \nu_j \pi_{n+1-j-k} & \text{otherwise} \end{cases}$$

$$a_k^* = \begin{cases} 0 & k \in \{n+1,n+2,...\} \\ \sum_{j=1}^{n+1-k} \nu_j \pi_{n+1-j-k} & \text{otherwise.} \end{cases}$$

Then $P_{\underline{N}}\underline{a}$ exists and $P_{\underline{N}}\underline{a} = \underline{a}^*$.

Proof. Note that $\underline{a} \in \ell_\varphi$ and $\underline{a}^* \in \tilde{N}$. Further by Proposition 1.40

$$\tilde{\mathcal{P}}_N \underline{a} = \{a^*\} \quad \text{and} \quad (\alpha\underline{a})^* = \alpha\underline{a}^*$$

for any scalar α. Hence by Theorem 2.10, $P_{\underline{N}}\underline{a}$ exists and $P_{\underline{N}}\underline{a} = a^*$.

Remark 2.12. 1) If $\varphi(x) = |x|^\alpha$, $1 \le \alpha < 2$, then the dispersion predictor of Cline and Brockwell ([8]) is the metric projection in ℓ_α.
2) Note that in Example 2.5 we consider $\{\xi_n, n \in \mathbb{N}\}$ be a symmetric basis for a subspace $\mathcal{X}$ of L^p and find conditions for the existence of metric projection with respect to the L^p distance. In Corollary 2.11 we have seen that for a certain subset $S_* \subset \ell_\varphi$, the metric projection exists, the metric here corresponds to the Luxemburg norm in ℓ_φ.

CHAPTER III

THE LEFT WOLD DECOMPOSITION

Let $\{X_n, n \in \mathbb{Z}\}$ be a stationary second order process with $E X_n = 0$ for all n. Then the moving average part of its Wold decomposition is constructed as follows. Let $M_k = \overline{sp}\ \{X_n, n \leq k\}$, and $\xi_k = X_k - P_{M_{k-1}} X_k$, $k \in \mathbb{Z}$, where P_M denotes the projection on M. Let $M_{-\infty} = \bigcap_n M_n = \{0\}$. Then observe that $\{\xi_k, k \in \mathbb{Z}, k \leq 0\}$ is a basis in M_0 and $S_n \xi_k = \xi_{n+k}$ where S_n is the shift operator on $M_\infty = \overline{sp}\{\bigcup_k M_k\}$ given by $S_n X_k = X_{k+n}$. From this we get $X_0 = \sum_{k=0}^{\infty} a_k \xi_{-k}$ and $X_n = \sum_{k=0}^{\infty} a_k \xi_{n-k}$. We note that $\{\xi_k: k \in \mathbb{Z}, k \leq 0\}$ is a symmetric basis for M_0 and in case $\{X_n: n \in \mathbb{Z}\}$ is a Gaussian process $\{\xi_k: k \in \mathbb{Z}, k \leq 0\}$ are i.i.d. random variables.

If $\{X_n\}$ is a symmetric α stable (SαS) process with $\alpha > 1$, Cambanis, Hardin and Weron ([5]) have used the concept of James orthogonality to define left and right Wold decomposition and innovations. In this chapter we extend these concepts to general Banach spaces using a semi–inner product introduced by Lumer ([17]). This gives rise to a definition of orthogonality. We will see later (Proposition 3.14) that Lumer orthogonality implies James orthogonality. In case the Banach space considered is L^p $(p>1)$ the two definitions of orthogonality coincide.

We now introduce the concept of a semi–inner product following Lumer ([17]) to extend the definitions of right and left projections defined in [5]. The Definition 3.1 and Proposition 3.2 are taken from [17]. Here F denotes the field of real or complex numbers.

Definition 3.1. Let $\mathscr{X}$ be a vector space over F. A semi-inner product is said to be defined on $\mathscr{X}$ if for any x, y $\in \mathscr{X}$ there corresponds aelement [x,y] in F with the following properties:

(i) $[x+y,z] = [x,z] + [y,z]$ for all $x,y,z \in \mathscr{X}$

$[\lambda x,y] = \lambda[x,y]$ for all $x, y \in \mathscr{X}, \lambda \in F$.

(ii) $[x,x] > 0$ for $x \neq 0$

(iii) $|[x,y]|^2 \leq [x,x]\,[y,y]$ for all $x, y \in \mathscr{X}$

Proposition 3.2. Let $\mathscr{X}$ be a normed linear space over F and let $\mathscr{X}^*$ denote its dual. For each $x \in \mathscr{X}$ there exists $W_x \in \mathscr{X}^*$ such that $W_x(x) = (x,W_x) = ||x||^2$ and $||W_x|| = ||x||$. For $x, y \in \mathscr{X}$ define $[x,y] = (x,W_y)$. Then $[\cdot,\cdot]$ defines a semi-inner product.

Remark 3.3. Suppose $\mathscr{X}^*$, the dual of $\mathscr{X}$ is rotund. Let $x \in \mathscr{X}$ By the Hahn-Banach theorem there exists $W_x \in \mathscr{X}^*$ such that $(x,W_x) = ||x||^2$ and $||W_x|| = ||x||$. Suppose $x_1^* \neq x_2^*$ are in $\mathscr{X}^*$ such that $||x_k^*|| = ||x||$ and $(x,x_k^*) = ||x||^2$, k = 1,2. Since $\mathscr{X}^*$ is rotund $||\frac{1}{2}(x_1^* + x_2^*)|| < ||x||$. But

$$(x, \tfrac{1}{2}(x_1^*+x_2^*)) = \tfrac{1}{2}(x_1^*+x_2^*)(x)$$
$$= \tfrac{1}{2}\, x_1^*(x) + \tfrac{1}{2}\, x_2^*(x) = \tfrac{1}{2}[||x||^2 + ||x||^2] = ||x||^2.$$

This contradicts the fact that $||\frac{1}{2}(x_1^*+x_2^*)|| < ||x||$. Hence for any $x \in \mathscr{X}$ there exists a unique element $W_x \in \mathscr{X}$ such that $||W_x|| = ||x||$ and $(x,W_x) = ||x||^2$. Therefore the semi-inner product is uniquely defined in this case. Note that in a Hilbert space the semi-inner product is the inner product.

Proposition 3.4. Let $\mathscr{X}$ be a reflexive, rotund Banach space such that its dual $\mathscr{X}^*$ is rotund. Let $x \in \mathscr{X}$ and M be a closed subspace of $\mathscr{X}$ Then $P_M x$, the metric projection of x on M, is uniquely determined by

$$[y, x-P_M x] = 0 \quad \text{for all} \quad y \in M.$$

We find the form of W_x for Orlicz function spaces. For this we need extension of [14] (p. 73 and p.88).

Proposition 3.5. Let φ be a convex Orlicz function, let p be its right derivative and ψ be the complementary function φ. Suppose φ and ψ satisfy the Δ_2-condition. If $f \in L_\varphi$ then $p(|f|) \in L_\psi$ and if $|||f|||_\varphi \leq 1$,

$$(3.5.1) \qquad \rho_\psi(p(|f|)) \leq |||f|||_\varphi$$

Proof. Let $\mathscr{F}_1 = \{E \in \mathscr{F}, \mu(E) < \infty\}$. For any $E \in \mathscr{F}_1$, $p(|f|)1_E \in L_\psi$. Further

$$|||f1_E|||_\varphi = \sup \{\int_E |fg|d\mu: \ g \in L_\psi, \ \rho_\psi(g) \leq 1\}$$
$$\leq \sup \{\int |fg|d\mu: \ g \in L_\psi, \ \rho_\psi(g) \leq 1\}$$

That is,

$$(3.5.2) \qquad |||f1_E||| \leq |||f|||_\varphi$$

If $p(|f|) = 0$, then (3.5.1) is trivially true. Now assume that $E_0 = \{x: p(|f|)(x) \neq 0\} \neq \phi$. Let $|||f|||_\varphi \leq 1$. We now show that for each $E \in \mathscr{F}_1$, $\rho_\psi(p(|f|)1_E) \leq |||f|||_\varphi$. Suppose this is not true, i.e., suppose there exists $F \in \mathcal{F}_1$ such that $\rho_\psi(p(|f|)1_F) > |||f|||_\varphi$. Then $F \cap E_0 \neq \phi$. For $x \in F \cap E_0$,

$$\psi(p|f|(x)) = \psi(p(|f|(x))) < \varphi(f(x)) + \psi(p|f|(x)) = |f(x)|p(|f|(x)).$$

Therefore

$$\begin{aligned}\rho_\psi(p(|f|)1_F) &< \int |f|p(|f|)\, 1_F d\mu \\ &\leq |||f1_F|||_\varphi \rho_\psi(p(|f|)1_F) \quad \text{(Proposition 1.27)} \\ &\leq |||f|||_\varphi \rho_\psi(p(|f|)1_F) \qquad \text{(by (3.5.2))}\end{aligned}$$

which contradicts the fact that $|||f|||_\varphi \leq 1$. Therefore for each $E \in \mathcal{F}_1$, $\rho_\psi(p(|f|)1_F) \leq |||f|||_\varphi$. Define $\nu(E) = \rho_\psi(p(|f|)1_E)$ for each $E \in \mathcal{F}$. Then ν is a σ-finite measure. Further $\nu(E) \leq |||f|||_\varphi$ for each $E \in \mathcal{F}_1$. Therefore $\sup_{E \in \mathcal{F}} \nu(E) = \alpha \leq |||f|||_\varphi$. Hence there exists a sequence $\{E_n\}$ in $\mathcal{F}_1$, $E_n \uparrow$ such that $\lim_{n\to\infty} \nu(E_n) = \alpha$. Let $B = \bigcup_n E_n$. Then $\alpha = \nu(B)$. If $E_0 \in \mathcal{F}_1$, we have

$$\rho_\psi(p(|f|)) = \rho_\psi(p|f|1_{E_0}) \leq |||f|||_\varphi;$$

so that (3.5.1) holds in this case. Now suppose that $E_0 \notin \mathcal{F}_1$. Let $E_1 = E_0 \cap B^c$. Let $F \in \mathcal{F}_1$, $F \subset E_1$. Suppose $\nu(F) > 0$. Then

$$\alpha = \nu(B) < \nu(B) + \nu(F) = \nu(B \cup F) = \lim_{n\to\infty} \nu(E_n \cup F) \leq \sup_{E \in \mathcal{F}} \nu(E) = \alpha$$

which is a contradiction. Hence $\nu(F) = 0$. Since ν is σ-finite this implies that $\nu(E_1) = 0$. Thus as $B^c = E_1 \cup (B^c \cap E_0^c)$, we have

$$\nu(B^c) = \nu(E_1) + \nu(B^c \cap E_0^c) = 0.$$

Thus
$$\begin{aligned}\rho_\psi(p(|f|)) &= \rho_\psi(p(|f|)1_B) + \rho_\psi(p(|f|)1_{B^c}) \\ &= \nu(B) + \nu(B^c) = \alpha \leq |||f|||_\varphi\end{aligned}$$

which proves (3.5.1).

Remark 3.6. The following proposition gives another formula for evaluating $|||\cdot|||_{\varphi}$.

Proposition 3.7. Let φ and ψ satisfy the conditions of the previous theorem. Let $f \in L_{\varphi}$ and suppose $k_f^* = k^*$ is a positive number such that

$$\rho_{\psi}(p(k^*|f|)) = 1. \tag{3.7.1}$$

Then

$$\int |f| p(k^*|f|) d\mu = |||f|||_{\varphi}. \tag{3.7.2}$$

Proof. Let k^* be defined as above. Then

$$\int |f| p(k^*|f|) d\mu \leq \sup_{\rho_{\psi}(g)\leq 1} \int |fg| d\mu = |||f|||_{\varphi}.$$

On the other hand, as φ and ψ are complementary functions,

$$\begin{aligned}
|||f|||_{\varphi} &= \frac{1}{k^*} \sup_{\rho_{\psi}(g)\leq 1} \int |f| p(k^*|g|) d\mu \\
&\leq \frac{1}{k^*} \sup_{\rho_{\psi}(g)\leq 1} \left[\int \varphi(k^* f) d\mu + \int \psi(g) d\mu\right] \\
&\leq \frac{1}{k^*} [\rho_{\varphi}(k^* f) + 1] \\
&= \frac{1}{k^*} [\rho_{\varphi}(k^* f) + \rho_{\psi}(p(k^*|f|))] \\
&= \frac{1}{k^*} \int k^* |f| p(k^*|f|) d\mu \\
&= \int |f| p(k^*|f|) d\mu \leq |||f|||_{\varphi}
\end{aligned}$$

which proves the result.

Corollary 3.8. Let φ, ψ be as in Proposition 3.7. Let $[\cdot,\cdot]_\varphi$ be a semi–inner product defined by the norm $|||\cdot|||_\varphi$. Let f, g $\in L_\varphi$ be such that k_g^* exists. Then

$$|||g|||_\varphi = \int |g| p(k_g^*|g|)d\mu = \int g(\operatorname{sgn} g)p(k_g^*|g|)d\mu.$$

Hence

$$[f,g]_\varphi = \int f|||g|||_\varphi(\operatorname{sgn} g)p(k_g^*|g|)d\mu.$$

Example 3.9. This example shows us that the inner product defined by Cambanis and Miller ([4]) is a particular case of the semi–inner product defined here. Let $\varphi(z) = \frac{|x|^\alpha}{\alpha}$ $(1<\alpha<2)$. For $x > 0$, $\varphi'(x) = p(x) = x^{\alpha-1}$. Let $\beta > 0$ be such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Let ψ be the complementary function of φ. Then $\psi(y) = \frac{|y|^\beta}{\beta}$ (Example 1.19.1) so for any $k > 0$

$$\rho_\psi(p(k|g|)) = \frac{1}{\beta}\int |p(k|g|)|^\beta d\lambda = \frac{1}{\beta}\int (k|g|)^{\beta(\alpha-1)}d\lambda$$

$$= \frac{1}{\beta}\int (k|g|)^\alpha d\lambda = \frac{k^\alpha}{\beta}\,||g||_\alpha^\alpha.$$

Hence $\rho_\psi(p(k|g|)) = 1$ implies

$$1 = \frac{k^\alpha}{\beta}\,||g||_\alpha^\alpha; \text{ so that } k = \frac{\beta^{1/\alpha}}{||g||_\alpha}, \text{ i.e. } k_g^* = \frac{\beta^{1/\alpha}}{||g||_\alpha}$$

$$\text{Hence } p(k_g^*|g|) = (k_g^*|g|)^{\alpha-1} = \frac{\beta^{\frac{\alpha-1}{\alpha}}|g|^{\alpha-1}}{||g||_\alpha^{\alpha-1}} = \frac{\beta^{1/\beta}}{||g||_\alpha^{\alpha-1}}\,|g|^{\alpha-1}.$$

By Example 1.26, $|||g|||_\varphi = \beta^{1/\beta}||g||_\alpha$ and hence

$$|||g|||_\varphi \, p(k_g^*|g|) = \beta^{1/\beta} \, ||g||_\alpha^{2-\alpha}|g|^{\alpha-1}.$$

Thus

$$[f,g] = \beta^{1/\beta}||g||_\alpha^{2-\alpha} \int fg^{<\alpha-1>}d\mu,$$

where $g^{<\alpha-1>} = \mathrm{sgn}(g)|g|^{\alpha-1}$. Therefore

$$\frac{[f,g]_\varphi}{|||g|||_\varphi^2} = \frac{\int fg^{<\alpha-1>}d\mu}{\int |g|^\alpha d\mu}.$$

We now assume that $(\mathscr{X}||\cdot||)$ is a Banach space over F with rotund dual space $\mathscr{X}^*$. Let $[\cdot,\cdot]$ be the semi-inner product defined by the norm $||\cdot||$. The following definition extends the concepts of right and left projections as defined by Cambanis and Miamee ([3]).

Definition 3.10. Let $(\mathscr{X}||\cdot||)$ be a Banach space over F and let $[\cdot,\cdot]$ be a semi-inner product defined by the norm $||\cdot||$. Let M be a closed subspace of $\mathscr{X}$ and let $x \in \mathscr{X} \cap M^c$. The right (resp. left) projection of x on N is defined as an element $r(x|M)$(resp.$\ell(x|M)$), of M satisfying

(3.10.1) $\quad [x,y] = [r(x|M),y]$ for all $y \in M$

(resp.

(3.10.2) $\quad [y,x] = [y,\ell(x|M)]$ for all $y \in M$).

Here $[\cdot,\cdot]$ is the Lumer semi-inner product.

Proposition 3.11. Let M be a linear subspace of a Banach space $\mathscr{X}$ and let $x \in \mathscr{X} \cap N^c$. If $r(x|M)$ exists, it is unique.

Proof. Suppose $\gamma_1, \gamma_2 \in M$ are such that $[\gamma_1,y] = [x,y] = [\gamma_2,y]$ for every $y \in M$. By definition 3.1(i) we get $[\gamma_1-\gamma_2,y] = 0$ for every $y \in M$. In particular, as $\gamma_1-\gamma_2 \in M$, $0 = [\gamma_1-\gamma_2, \gamma_1-\gamma_2] = ||\gamma_1-\gamma_2||^2$ so that $\gamma_1 = \gamma_2$ (Definition 3.1).

Definition 3.12. Let $(\mathscr{X}||\cdot||)$ be a normed linear space over F with semi-inner product $[\cdot,\cdot]$. Let $x,y \in \mathscr{X}$ x is James orthogonal to y, denoted by $x \perp_J y$ if

$$||x+\lambda y|| \geq ||x|| \tag{3.12.1}$$

for all $\lambda \in F$. x is said to be orthogonal to y, denoted by $x \perp y$, if

$$[y,x] = 0 \tag{3.12.2}$$

Let $\mathscr{X}_1, \mathscr{X}_2$ be two subspaces of $\mathscr{X}$ $\mathscr{X}_1 \perp \mathscr{X}_2$ (resp. $\mathscr{X}_1 \perp_J \mathscr{X}_2$) if $x_1 \perp x_2$ (resp. $x_1 \perp_J x_2$) for each $x_1 \in \mathscr{X}_1$ and $x_2 \in \mathscr{X}_2$.

Remark 3.13. 1) By the form of the semi inner product in L^p, $1 < p < \infty$, $x \perp y$ if and only if $x \perp_J y$, (Theorem 1.11 and Lemma 1.14 [25]). This is not necessarily true in general. However if $x \perp y$ then $x \perp_J y$ which will be seen in Proposition 3.14.

2) If $\mathscr{X}$ is a Hilbert space then by [25] (p.91), $x \perp y$ if and only if $x \perp_J y$ if and only if $<x, y> = 0$ where $<\cdot,\cdot>$ is the inner product.

3) Note that $x \perp y$ (resp. $x \perp_J x$) does not imply that $y \perp x$ (resp. $y \perp_J x$). For example let $1 < p < 2$, $f_1 = 21_{[0,1/4)} + 1_{[1/4,1/2)} - 31_{[3/4,1]}$ and $f_2 = 1_{[0,1]}$. Then $f_1, f_2 \in L^p$. Let $x^{<p>} = |x|^p \operatorname{sgn} x$.

$$[f_2, f_1] = \int f_1^{<p-1>} f_2$$
$$= \int (21_{[0,1/4]} + 1_{[1/4,1/2)} - 31_{[3/4,1]})^{<p-1>}$$
$$= (1/4)\ (2^{p-1} + 1 - 3^{p-1}) > 0$$

and $[f_1,f_2] = \int f_2^{<p-1>} f_1$

$$= \int (21_{[0,1/4)} + 1_{[1/4,1/2)} - 31_{[3/4,1]})$$
$$= 1/4\ (2+1-3) = 0.$$

Therefore $f_2 \perp f_1$ but f_1 is not orthogonal to f_2. By the previous remark $f_2 \perp_J f_1$ but f_1 is not James orthogonal to f_2.

Proposition 3.14. Let $\mathscr{X}$ be a normed linear space over F and $x,y \in \mathscr{X}$ If $x \perp y$, then $x \perp_J y$.

Proof. Let $\lambda \in F$. Since $x \perp y$, $[y,x] = 0$. Hence

$$[x,x] = [x,x] + \lambda[y,x] = [x+\lambda y,x].$$

So

$$[x,x] = |[x+\lambda y,x]| \leq [x+\lambda y,\ x+\lambda y]^{1/2}\ [x,x]^{1/2}. \tag{3.14.1}$$

If $[x,x] = 0$, then $x = 0$. Hence $||x+\lambda y|| = ||\lambda y|| \geq 0 = ||x||$; i.e. $x \perp_J y$. Let $[x,x] > 0$. Then from (3.14.1) we get

$$[x,x]^{1/2} \le [x+\lambda y,\ x+\lambda y]^{1/2}\ ; \quad \text{i.e.} \quad ||x|| \le ||x+\lambda y||.$$

Therefore $x \perp_J y$.

<u>Proposition 3.15</u>. Let $\mathscr{X}$ be a normed linear space. Suppose $\{x_n\}$ is a sequence in $\mathscr{X}$ converging $x \in \mathscr{X}$ Let $y \in \mathscr{X}$

a) If $y \perp x_n$ for each n, then $y \perp x$.

b) If $x_n \perp y$ for each n, $\mathscr{X}$ is reflexive and $\mathscr{X}^*$ is rotund, then $x \perp y$.

<u>Proof</u>. a) If $y = 0$, then $y \perp x$. Now assume $y \neq 0$ so that $||y|| > 0$. Let $\epsilon > 0$. Since $x_n \to x$, there exists $n_0 \in \mathbb{N}$ such that $n \ge n_0$ implies $||x_n - x|| < \sqrt{\epsilon}/||y||$. But $y \perp x_n$; so $[x_n,y] = 0$. Hence $[x,y] = [x-x_n,\ y]$. Therefore $|[x,y]|^2 \le [x-x_n,\ x-x_n][y,y] < \epsilon$ for $n \ge n_0$; i.e. $|[x,y]| < \epsilon$. This is true for any $\epsilon > 0$. Hence $[x,y] = 0$.

b) Let W_{x_n}, W_x be elements of $\mathscr{X}^*$ corresponding to x_n, $x \in \mathscr{X}$ (cf. Proposition 3.2 and Remark 3.3). Since $x_n \to x$, there exists $M > 0$ such that

$$(3.15.1) \qquad ||W_{x_n}|| = ||x_n|| \le M \quad \text{for each } n.$$

Further $||W_{x_n}|| = ||x_n|| \to ||x|| = ||W_x||$.Since $\mathscr{X}$ is reflexive, $\mathscr{X}^*$ is reflexive ([19], p. 135). Hence by (3.15.1) $\{W_{x_n}\}$ has a weakly convergent subsequence $\{W_{x_{n_k}}\}$. Without loss of generality assume that W_{x_n} converges weakly to an element $x^* \in \mathscr{X}^*$. Then

$$(3.15.2) \qquad |x^*(x)| = \lim_{n\to\infty} |W_{x_n}(x)| \le \lim_{n\to\infty} ||W_{x_n}||\ ||x|| = ||x||^2$$

But $||x_n||^2 = W_{x_n}(x_n) = W_{x_n}(x_n-x) + W_{x_n}(x)$.Since

$$|W_{x_n}(x_n-x)| \le ||W_{x_n}||\ ||x_n-x|| \le M\ ||x_n-x|| \to 0,$$

$W_{x_n}(x) \to x^*(x)$ and $||x_n||^2 \to ||x||^2$. We get

$$(3.15.3) \qquad ||x||^2 = x^*(x).$$

(3.15.2) and (3.15.3) together imply that $||x^*|| = ||x||$. By Remark 3.3, $x^* = W_x$. Since $x_n \perp y$, $0 = [y,x_n] = W_{x_n}(y)$. Hence $[y,x] = W_x(y) = \lim_{n\to\infty} W_{x_n}(y) = 0$; i.e. $x \perp y$.

Let $\mathcal{X}$ be a Banach space over F with rotund dual space $\mathcal{X}^*$ and $\mathcal{X}_1, \mathcal{X}_2, \ldots$ be closed linear subspaces of $\mathcal{X}$ We now define a concept of an orthogonal ($\perp$) decomposition for general Banach spaces. For certain class of Banach spaces orthogonal decompositions where considered in ([5]).

<u>Definition 3.16</u>. The symbol $\mathcal{X}_1+\ldots+\mathcal{X}_n$ (or $\sum_{j=1}^{n} \mathcal{X}_j$) denotes the subspace $\{x_1+\ldots+x_n\colon\ x_j\in\mathcal{X}_j,\ 1\le k\le n\}$. $\mathcal{X}_1+\mathcal{X}_2+\ldots$ (or $\sum_{j=1}^{\infty} \mathcal{X}_j$) denotes the subspace $\overline{\mathrm{sp}} \bigcup_n (\sum_{j=1}^{n} \mathcal{X}_j)$. $\mathcal{X} = \mathcal{X}_1 \underset{\rightarrow}{\oplus} \ldots \underset{\rightarrow}{\oplus} \mathcal{X}_n$ (or $\mathcal{X} = \sum_{j=1}^{n} \underset{\rightarrow}{\oplus} \mathcal{X}_j$) means $\mathcal{X} = \mathcal{X}_1+\ldots+\mathcal{X}_n$ and

$$(3.16.1) \qquad \mathcal{X}_1+\ldots+\mathcal{X}_k \perp \mathcal{X}_{k+1}+\ldots+\mathcal{X}_n \text{ for all } 1 \le k < n.$$

Writing $\mathscr{X} = \mathscr{X}_1 \underset{\leftarrow}{\oplus} \ldots \underset{\leftarrow}{\oplus} \mathscr{X}_n$ means that $\mathscr{X} = \mathscr{X}_1 + \ldots + \mathscr{X}_n$ and

(3.16.2) $\qquad \mathscr{X}_n + \ldots + \mathscr{X}_{k+1} \perp \mathscr{X}_k + \ldots + \mathscr{X}_1$ for all $1 \leq k < n$.

$\mathscr{X} = \sum_{j=1}^{\infty} \underset{\rightarrow}{\oplus} \mathscr{X}_j$ (resp. $\mathscr{X} = \sum_{j=1}^{\infty} \underset{\leftarrow}{\oplus} \mathscr{X}_j$) means $\mathscr{X} = \sum_{j=1}^{\infty} \mathscr{X}_j$ and (3.16.1) (resp. (3.16.2)) holds for all n.

Remark 3.17. 1) Note that $\mathscr{X}_1 \underset{\rightarrow}{\oplus} \mathscr{X}_2 = \mathscr{X}_2 \underset{\leftarrow}{\oplus} \mathscr{X}_1$. Also, as noted in Remark 3.13.1, the statements $\mathscr{X} = \mathscr{X}_1 \underset{\rightarrow}{\oplus} \mathscr{X}_2$ and $\mathscr{X} = \mathscr{X}_1 \underset{\leftarrow}{\oplus} \mathscr{X}_2$ are, in general, distinct.

2) Let $\mathscr{X} = \sum_{j=1}^{\infty} \underset{\rightarrow}{\oplus} \mathscr{X}_j$. Let $0 \neq x_j \in \mathscr{X}_j$. For any $m \leq n$ and $\beta_1, \beta_2, \ldots \in F$ we have by the definition and Lemma 3.14 that

$$\left|\left| \sum_{j=1}^{m} \beta_j x_j \right|\right| \leq \left|\left| \sum_{j=1}^{n} \beta_j x_j \right|\right|.$$

Hence ([24], p. 54) $\{x_j\}$ forms a basis for its closed linear span, i.e. each $x \in \overline{sp}\,\{x_j: j = 1,2,\ldots\}$ has a unique norm convergent expansion $x = \sum_{j=1}^{\infty} \lambda_j x_j$ for some $\lambda_1, \lambda_2, \ldots \in F$. Note that the same argument cannot be made when $\mathscr{X} = \sum_{j=1}^{\infty} \underset{\leftarrow}{\oplus} \mathscr{X}_j$.

We now consider Wold decomposition under definition of orthogonality. For this we need the following proposition .

Proposition 3.18. Let $\mathcal{X}$ be a reflexive Banach space such that $\mathcal{X}^*$ is rotund. Suppose there exist subspaces $\mathcal{X}_n$ and L_n of $\mathcal{X}$ such that

(3.18.1) $\qquad \mathcal{X} = \mathcal{X}_n \underset{\leftarrow}{\oplus} L_n \underset{\leftarrow}{\oplus} \dots \underset{\leftarrow}{\oplus} L_1$ for each $n \geq 1$,

Then

(3.18.2) $\qquad \mathcal{X} = \sum_{n=1}^{\infty} (\underset{\rightarrow}{\oplus} L_n) \underset{\rightarrow}{\oplus} (\bigcap_n \mathcal{X}_n)$

Proof. Let $\mathcal{X}_{-\infty} = \bigcap_n \mathcal{X}_n$, $K_n = L_n \underset{\leftarrow}{\oplus} \dots \underset{\leftarrow}{\oplus} L_1$ and $K_\infty = \overline{sp}\,(\bigcup_n K_n)$. By the definition of $\underset{\leftarrow}{\oplus}$, $K_n \perp \mathcal{X}_n$ for each n. Since $\mathcal{X}_{-\infty} \subseteq \mathcal{X}_n$, for each n, $K_n \perp \mathcal{X}_{-\infty}$ for each n. Let $k \in \bigcup_n K_n$. Then $\exists\, n \in \mathbb{N}$ such that $k \in K_n$. Hence, as $K_n \perp \mathcal{X}_{-\infty}$, $[x,k] = 0$ for any $x \in \mathcal{X}_{-\infty}$. But $k \in K_n$ was arbitrary. Therefore $\bigcup_n K_n \perp \mathcal{X}_{-\infty}$. Let $k \in K_\infty$. Then there exists a sequence $\{k_n\}$ in $\bigcup_n K_n$ such that $||k_n - k|| \to 0$ as $n \to \infty$. Let $x \in \mathcal{X}_{-\infty}$. Since $\{k_n\} \subseteq \bigcup_m K_n$, $[x,k_n] = 0$ for each n. By Proposition 3.15 b), $[x,k] = 0$. This is true for any $x \in \mathcal{X}_{-\infty}$ and $k \in K_\infty$. Hence $K_\infty \perp \mathcal{X}_{-\infty}$. Thus in order to prove (3.18.2) we need to show that $\mathcal{X} = K_\infty + \mathcal{X}_{-\infty}$. Let $x \in \mathcal{X}$ Then $x = x_n + k_n$ where $x_n \in \mathcal{X}_n$, $k_n \in K_n$. Since $k_n \perp x_n$, $k_n \perp_J x_n$ (Proposition 3.14). Therefore

$$||k_n|| \leq ||k_n + x_n|| = ||x||;$$

so that $||x_n|| = ||x - k_n|| \leq 2||x||$. Hence the sequences $\{x_n\}$ and $\{k_n\}$ are norm bounded. Since $\mathcal{X}$ is reflexive, they have simultaneously weakly convergent subsequences $\{x_{n_j}\}$ and $\{k_{n_j}\}$ with weak limits $x_{-\infty}$ and k_∞ respectively. Hence $x = x_{-\infty} + k_\infty$. Since $\mathscr{X}_{-\infty}$ is a closed subspace of $\mathscr{X}$, it is convex. Hence $\mathscr{X}_{-\infty}$ is a weakly closed subspace of $\mathscr{X}$ Therefore $x_{-\infty} \in \mathscr{X}_{-\infty}$. Further there exists a subsequence $\{y_{n_j}\}$ such that $y_{n_j} \in \mathrm{co}\{k_{n_1},\dots,k_{n_j}\}$ = convex hull $\{k_{n_1},\dots,k_{n_j}\}$ and $\|y_{n_j} - k_\infty\| \to 0$. Hence $k_\infty \in K_\infty$. Therefore $\mathscr{X} = K_\infty \underset{\rightarrow}{\oplus} \mathscr{X}_{-\infty}$.

<u>Remark 3.19</u>. From Remark 3.17.2, we recall that any $k_\infty \in K_\infty$ has a unique norm convergent expansion $k = \sum_{n=1}^{\infty} k_n$, $k_n \in L_n$ for each n.

Let us observe that ([25], p. 111) if $\mathscr{X}$ is a reflexive, rotund Banach space and M is a closed subspace of $\mathscr{X}$, then $P_M x$, the projection of x on M, exists for each $x \in \mathscr{X}$ and satisfies

$$(3.20) \qquad \|x - P_M x\| = \inf_{y \in M} \|x-y\|.$$

We now show that if in addition $\mathscr{X}^*$ is rotund, then $(x-P_M x) \perp M$. Note that by the Hahn–Banach theorem ([25] p. 18) there exists $x^* \in \mathscr{X}^*$ such that $\|x^*\| = \|x-P_M x\|$, $x^*(y) = 0$ for every $y \in M$ and $x^*(x-P_M x) = \|x-P_M x\|^2$. In view of Remark 3.3, $x^* = W_{x-P_M x}$. Thus $P_M x$ is uniquely determined by the equation

$$(3.21) \qquad [y,\, x-P_M x] = 0 \quad \text{for any } y \in M.$$

<u>Remark 3.22</u>. Notice that $x = P_M x + (x - P_M x)$ and $(x - P_M x) \perp P_M x$. We want to show that this is a unique representation. Suppose there exists $x_1 \in M$ and $y_1 \in \mathcal{X}$ such that

$$x = P_M x + (I - P_M)x = x_1 + y_1 \quad \text{and} \quad y_1 \perp M.$$

Since $y_1 \perp M$, we have by Proposition 3.14 that $y_1 \perp_J M$ so that for any $y \in M$,

$$||x-y|| = ||y_1 + (x_1 - y)|| \geq ||y_1|| = ||x - x_1||;$$

$$\text{i.e.} \quad ||x - x_1|| = \inf_{y \in M} ||x-y||.$$

Since $P_M x$ is unique, $x_1 = P_M x$. Thus $x = P_M x + (I - P_M)x$ is a unique representation of x as a sum of an element of M and an element of $\mathcal{X}$ orthogonal to M. In particular $x \perp M$ if and only if $P_M x = 0$.

2) Let $Q: \mathcal{X} \to M$ be an operator (not necessarily linear). Suppose $(I-Q)\mathcal{X} \perp M$. Let $x \in \mathcal{X}$ Then $x = Qx + (I-Q)x = P_M x + (I - P_M)x$. But $Qx \in M$ and $(I-Q)x \perp M$. Hence by Remark 3.21 $P_M x = Qx$. But $x \in \mathcal{X}$ was arbitrary. Hence $P_M = Q$. Conversely suppose $P_M = Q$. Then by (3.21), $(I - P_M)x \perp M$. Thus $Q = P_M$ if and only if $(I - P_M)\mathcal{X} \perp M$.

3) If the Banach space considered is a Hilbert space H with inner product $\langle \cdot, \cdot \rangle$, then $x \perp y$ if and only if $\langle x, y \rangle = 0$ $(x, y \in H)$. In this case P_M is linear for every closed subspace M of H and satisfies (3.21) for any $x \in H$ ([25] p. 57). Then by the above remarks $H = M \underset{\leftarrow}{\oplus} M^{\perp}$ for any closed subspace M of H where

$$M^{\perp} = \{y \in H: \langle x, y \rangle = 0 \text{ for all } x \in M\}.$$

Following [5] for $\underline{x} = \{x_n, n \in \mathbb{Z}\} \subseteq \mathcal{X}$ a Banach space we now give the left Wold decomposition.

Define

(3.23) $M_n = M(\underline{x}:n) = \overline{sp}\ \{x_k: k \leq n\}$ (past and present of $\{x_n\}$)

(3.24) $M_\infty = M(\underline{x}:\infty) = \overline{sp}\ \{\bigcup_n M(\underline{x}:n)\}$ (time domain of $\{x_n\}$)

(3.25) $M_{-\infty} = M(\underline{x}:-\infty) = \bigcap_n M(\underline{x}:n)$. (remote past of $\{x_n\}$)

$\underline{x} = \{x_n\}$ is said to have left innovations if for each n there exists a subspace $N_n(x) = N_n$ so that

$$M(\underline{x}:n) = M(\underline{x}:n-1) \underset{\leftarrow}{\oplus} N_n(\underline{x}). \tag{3.26}$$

Notice that $N_n(x)$ is necessarily one or zero dimensional. $\underline{x} = \{x_n\}$ is said to have a left Wold decomposition if there are subspaces $N_n = N_n(\underline{x})$, $-\infty < n < \infty$, such that

$$M(x:n) = \Big(\sum_{k=0}^{\infty} \underset{\rightarrow}{\oplus} N_{n-k}(x)\Big) \underset{\rightarrow}{\oplus} M(\underline{x}:-\infty). \tag{3.27}$$

<u>Proposition 3.28</u>. Let $\mathscr{X}$ be a reflexive, rotund Banach space with a rotund dual space $\mathscr{X}^*$. Then $\{x_n\}$ has left innovations.

<u>Proof</u>. For the sake of convenience let P_n denote the metric projection onto $M(\underline{x}:n)$. Since the codimension of $M(\underline{x}:n-1)$ on $\mathscr{X}$ is one, P_{n-1}: $M(\underline{x}:n) \rightarrow M(x:n-1)$ is linear. Hence $N_n(\underline{x}) = (I-P_{n-1})M(x:n)$ is a

linear subspace. In view of (3.21), $N_n(\underline{x}) \perp M(\underline{x}{:}n-1)$. Thus in order to complete the proof of this proposition we need to prove that
$M(\underline{x}{:}n) = M(\underline{x}{:}n-1) + N_n(\underline{x})$. Let $y \in M(\underline{x}{:}n)$. Then
$P_{n-1}y \in M(\underline{x}{:}n-1)$ and $y - P_{n-1}y = (I-P_{n-1})\, y \in N_n(\underline{x})$. Further
$y = P_{n-1}y + (y-P_{n-1}y)$. Therefore $M(\underline{x}{:}n) = M(\underline{x}{:}n-1) \underset{\leftarrow}{\oplus} N_n(\underline{x})$.

Notation 3.29. For the sake of convenience let P_n denote the metric projection operator onto $M(\underline{x}{:}n)$.

Theorem 3.30. Let $\mathscr{X}$ be a reflexive, rotund Banach space with rotund dual space $\mathscr{X}^*$. The following are equivalent:

(i) $\{x_n\}$ has a left Wold decomposition.

(ii) $P_n\colon M_\infty \to M_n$ are linear.

(iii) The operators $P_n\colon M_\infty \to M_n$ commute.

(iv) If $P_{n,m}$ denotes the restriction of P_n on M_m, then for all $k \geq 1$, $P_{n,n+1}P_{n+1,n+2}\cdots P_{n+k-1,n+k} = P_{n,n+k}$.

Proof. We will show that (iv) $\to$ (ii) $\to$ (i) $\to$ (iv) and (ii) $\leftrightarrow$ (iii).

(iv) $\to$ (ii) Assume (iv) holds. By Proposition 2.7 b) each $P_{n+\ell,n+\ell+1}$ is linear. Hence by (iv) $P_{n,n+k}$ is linear for each $k \geq 1$ so that P_n is linear on each M_{n+k}. Since P_n is continuous, $P_n\colon M_\infty \to M_n$ is linear.

(ii) $\to$ (i) Assume each $P_n\colon M_\infty \to M_n$ is linear. Define $N_n = (I-P_{n-1})M_n$. Let $z_n \in M_n$. By (3.21) $z_n \perp M_{n-1}$ and thus $z_n \perp M_{n-\ell}$ for $\ell \geq 1$. Then $P_{n-\ell}\, z_n = 0$ (by Remark 3.22.1). Since P_n is linear, we have

$P_{n-k}(z_n + z_{n-1}+\dots+z_{n-k+1}) = 0$. Hence using Remark 3.22, $N_n +\dots+N_{n-k+1} \perp M_{n-k}$. Therefore $M_n = (\sum_{\ell=0}^{k-1} \vec{\oplus} N_{n-\ell}) \vec{\oplus} M_{n-k}$. By Proposition 3.18 we get (i).

(i) $\to$ (iv) Suppose $\{x_n\}$ has a left Wold decomposition. Then for all n and $\ell \geq 1$, $M_{n+\ell} = N_{n+\ell} \vec{\oplus} N_{n+\ell-1} \vec{\oplus}\dots\vec{\oplus} N_{n+1} \vec{\oplus} M_n$ so any $y \in M_{n+\ell}$ is uniquely expressed as $y = z_{n+\ell} + z_{n+\ell-1}+\dots+z_{n+1} + y_n$ where $z_j \in N_j$ and $y_n \in M_n$. Further $P_{n,n+\ell}\, y = y_n$. But

$$\begin{aligned} &P_{n,n+1}\dots P_{n+\ell-1,n+\ell}(y) \\ &\quad = P_{n,n+1}\dots P_{n+\ell-1,n+\ell}(z_{n+\ell}+\dots+z_{n+1}+y_n) \\ &\quad = P_{n,n+1}\dots P_{n+\ell-2,n+\ell-1}(z_{n+\ell-1}+\dots z_{n+1}+y_n) \\ &\quad = P_{n,n+1}(z_{n+1}+y_n)=y_n=P_{n,n+\ell}(y) \end{aligned}$$

and this proves (iv).

(ii) $\to$ (iii) Assume (ii) holds. Let $x \in M_\infty$ and $m \leq n$. Then

$$\begin{aligned} P_mP_n(x) &= P_m\{x-(x-P_nx)\} \\ &= P_mx - P_m(x-P_nx) \quad (\text{ as } P_m \text{ is linear }) \\ &= P_mx \qquad (\text{Remark 3.22}). \end{aligned}$$

But $m \leq n$ implies $M_m \subseteq M_n$ so that $P_mx \in M_n$. Hence $P_mx = P_nP_mx$. Therefore $P_nP_mx = P_mP_nx$. Thus $P_n\colon M_\infty \to M_n$ commute.

(iii) $\to$ (ii). Suppose (iii) holds. Since each P_n is continuous, it suffices to show that each P_n is linear on each M_{n+k}. In view of Proposition 2.7 a) we need to show that each P_n is additive on each M_{n+k}. By Proposition 2.7 a), P_n is additive on M_n. Assume P_n is additive on M_{n+k-1}. Let $x_1, x_2 \in M_{n+k}$ be arbitrary. Let $y_j = P_{n+k-1}x_j$, $z_j = (I-P_{n+k-1})x_j$,

$j = 1,2$. Note that $z_j \perp M_{n+k-1}$. By Proposition 2.7, P_{n+k-1} is a linear operator on M_{n+k}. Thus

$$
\begin{aligned}
P_n(x_1+x_2) &= P_n(y_1+y_2+z_1+z_2) \\
&= P_{n+k-1}P_n\,(y_1+y_2+z_1+z_2) \\
&= P_n\,P_{n+k-1}(y_1+y_2+z_1+z_2) \qquad \text{(by (iii))} \\
&= P_n\,(y_1+y_2) \\
&= P_n y_1 + P_n y_2 \qquad \text{(induction assumption)} \\
&= P_n\,P_{n+k-1}(x_1) + P_n\,P_{n+k-1}(x_2) \\
&= P_{n+k-1}\,P_n x_1 + P_{n+k-1}P_n x_2 \qquad \text{(by (iii))} \\
&= P_n\,x_1 + P_n x_2
\end{aligned}
$$

and this proves the result.

Remark 3.31. The above Wold decomposition was proved in [5] for the case L^p, $p > 1$.

CHAPTER IV

THE RIGHT WOLD DECOMPOSITION

In this chapter we will discuss an extension of the right Wold decomposition introduced by Cambanis, Hardin and Weron ([5]) by using the right projection (c.f. Definition 3.10) and Definition 3.16. Throughout this section we will assume that $\mathscr{X}$ is a Banach space over F with rotund dual space $\mathscr{X}^*$. For this we need the following proposition.

<u>Proposition 4.1</u>. Let $\mathscr{X}$ be a reflexive Banach space over F. Suppose there exist closed subspaces $\mathscr{X}_n$ and L_n of $\mathscr{X}$ with $\mathscr{X} = \mathscr{X}_n \underset{\rightarrow}{\oplus} L_n \underset{\rightarrow}{\oplus} \dots \underset{\rightarrow}{\oplus} L_1$ for each $n \geq 1$. Then $\mathscr{X} = (\sum_{j=1}^{\infty} \underset{\leftarrow}{\oplus} L_n) \underset{\leftarrow}{\oplus} (\bigcap_n \mathscr{X}_n)$ and each $k \in \sum_{j=1}^{\infty} \underset{\leftarrow}{\oplus} L_n$ has a unique norm convergent expansion $k = \sum_{\ell=1}^{\infty} \ell_n$, $\ell_n \in L_n$.

<u>Proof</u>. Let $\mathscr{X}_{-\infty} = \bigcap_n \mathscr{X}_n$, $K_n = L_n \underset{\rightarrow}{\oplus} \dots \underset{\rightarrow}{\oplus} L_1$ and $K_\infty = \overline{sp}\,\{\bigcup_n K_n\}$. Then, by a proof similar to the one in Proposition 3.18 (we use Proposition 3.15 a) instead of Proposition 3.15 b)) $\mathscr{X} = \mathscr{X}_{-\infty} \underset{\rightarrow}{\oplus} K_\infty$. So in order to complete the proof of the theorem it remains to show that each $k \in K_\infty$ has a unique norm convergent expansion $k = \sum_{n=1}^{\infty} \ell_n$, $\ell_n \in L_n$. For each n we can write $k = x_n + k_n$ uniquely (follows from the definition of orthogonality) with $x_n \in \mathscr{X}_n$ and $k_n \in K_n$. In turn we may write $k_n = \ell_1 + \dots + \ell_n$ uniquely with $\ell_j \in L_j$. Define Q_n: $K_\infty \rightarrow K_n$ by $Q_n k = k_n$. Then $Q_n\, Q_\ell = Q_{n \wedge \ell}$. Also by Proposition 3.14.

$$\begin{aligned} \|Q_n k\| &= \|k - Q_n k + k\| \\ &\leq \|k - Q_n k\| + \|k\| = \|x_n\| + \|k\| \\ &\leq \|x_n + k_n\| + \|k\| \quad \text{(by orthogonality)} \\ &\leq 2\,\|k\|. \end{aligned}$$

Therefore, by the uniform boundedness principle, $\{Q_n\}$ is a bounded sequence ([9], p. 98). Let $k \in \bigcup_n K_n$. Then there exists $n_0 \in \mathbb{N}$ such that $k \in K_{n_0}$ so that $k \in K_n$ for each $n \geq n_0$. Hence $Q_n k = k$ for each $n \geq n_0$. Therefore $\text{s-lim}_{n\to\infty} Q_n k = k$ for any $k \in \bigcup_n K_n$. Let $k \in K_\infty$ and $\epsilon>0$. Let $\sup_n \|Q_n\| \leq M$. Then there exists $k_\epsilon \in \bigcup_n K_n$ such that $\|k-k_\epsilon\| < \epsilon/(M+1)$. Further there exists $n_\epsilon \in \mathbb{N}$ such that $k_\epsilon \in K_n$ for each $n \geq n_\epsilon$ so that $Q_n k_\epsilon = k_\epsilon$ for each $n \geq n_\epsilon$. But $n \geq n_\epsilon$ implies

$$\begin{aligned} \|k - Q_n k\| &= \|k - k_\epsilon - (Q_n k - Q_n k_\epsilon)\| \\ &\leq \|k - k_\epsilon\| + \|Q_n\|\,\|k - k_\epsilon\| < \epsilon. \end{aligned}$$

$$k = \text{s-lim}_{n\to\infty} Q_n k = \text{s-lim}_{n\to\infty} \sum_{j=1}^{n} \ell_j = \sum_{j=1}^{\infty} \ell_j.$$

<u>Proposition 4.2</u>. Let $\mathscr{X}$ be a Banach space and M a closed subspace of $\mathscr{X}$ Let ${}^\perp M = \{y \in \mathscr{X}\colon [y,x] = 0 \text{ for all } x \in M\}$. Then ${}^\perp M$ is a closed subspace of $\mathscr{X}$ Suppose further that for each $x \in \mathscr{X}$, the right projection of x on M, $r(x|M)$, exists. Then $\mathscr{X} = M \underset{\rightarrow}{\oplus} {}^\perp M$.

<u>Proof</u>. Let $y_1, y_2 \in {}^\perp M$, $\alpha_1, \alpha_2 \in F$. Then for any $x \in M$, $[\alpha_1 y_1 + \alpha_2 y_2, x] = \alpha_1[y_1,x] + \alpha_2[y_2,x] = 0$ so that $\alpha_1 y_1 + \alpha_2 Y_2 \in {}^\perp M$. Hence ${}^\perp M$ is a subspace of $\mathscr{X}$ By Proposition 3.15, ${}^\perp M$ is closed.

Let $\mathscr{X}$ be a Banach space such that $r(x|M) \in M$ exists for all $x \in \mathscr{X}$ Then by (3.10.1)

$$[r(x|M), y] = [x,y] \quad \text{for each} \quad y \in M,$$

$$\text{i.e. } [x-r(x|M), y] = 0 \quad \text{for each} \quad y \in M.$$

Therefore $x - r(x|M) \in {}^{\perp}M$. Further as $x = r(x|M) + [x - r(x|M)]$ and as $r(x|M) \in M$, $\mathscr{X} = M \underset{\rightarrow}{\oplus} {}^{\perp}M$.

For any sequence $\underline{x} = \{x_n\}$ in a Banach space $\mathscr{X}$ let $M(\underline{x}:n)$, $M(\underline{x}:\infty)$ and $M(\underline{x}:-\infty)$ be as defined in (3.23), (3.24) and (3.25) respectively. Following [5], we say that $\{x_n\}$ has right innovations if for each n there is a subspace $N_n = N_n(\underline{x})$ such that $M(\underline{x}:n) = M(\underline{x}:n-1) \underset{\rightarrow}{\oplus} N_n(\underline{x})$. Note that $N_n(\underline{x})$ is necessarily one or zero dimensional. $\underline{x} = \{x_n\}$ is said to have a right Wold decomposition if there all subspaces $N_n(\underline{x})$, $-\infty < n < \infty$, such that

$$M(\underline{x}:n) = \sum_{k=0}^{\infty} \underset{\leftarrow}{\oplus} N_{n-k}(\underline{x}) \underset{\leftarrow}{\oplus} M(\underline{x}:-\infty), \; M(\underline{x}:n) \perp N_m(\underline{x}) \quad \text{for each}$$

$m > n$ and further each $z \in \sum_{k=0}^{\infty} \underset{\leftarrow}{\oplus} N_{n-k}(\underline{x})$ has a unique norm convergent expansion $z = \sum_{k=0}^{\infty} w_{n-k}$, $w_j \in N_j(\underline{x})$.

<u>Theorem 4.3</u>. Let $\underline{x} = \{x_n\}$ be a sequence in a reflexive Banach space. The following are equivalent.

(i) $\underline{x}$ has right Wold decomposition

(ii) $\underline{x}$ has right innovations

(iii) $r_n(y) = r(y|M_{n-1})$ exists for each n and for each $y \in \bigcup_n M(\underline{x}:n)$.

<u>Proof</u>. We will show that (i) $\rightarrow$ (ii), (ii) $\leftrightarrow$ (iii) and (ii), (iii) $\rightarrow$ (i).

(i) $\rightarrow$ (ii). This follows from the definition of right Wold decomposition and right innovations.

(ii) $\rightarrow$ (iii). Suppose $\underline{x}$ has right innovations. Let $y \in \bigcup_n M(\underline{x}{:}n)$. Then there exists $n \in \mathbb{Z}$ such that $y \in M(x{:}n)$. The definition of a right projection and Proposition 3.11 imply that $y = r(y|M(\underline{x}{:}m))$ for all $m \geq n$. Let $n < m$. Then there exists $y_n \in M(\underline{x}{:}n)$ and $z_j \in N_j(\underline{x})$, $n + 1 \leq j \leq m$, such that $y = y_n + z_{n+1} + z_{n+2} + \ldots + z_m$. Note that each $z_j \perp M(\underline{x}{:}n)$, $n + 1 \leq j \leq m$. Hence, for any $z \in M(\underline{x}{:}n)$,

$$[y,z] = [y_n + z_{n+1} + \ldots + z_m, z] = [y_n, z] + \sum_{j=n+1}^{m} [z_j, z] = [y_n, z].$$

Therefore $r(y|M(\underline{x}{:}n))$ exists and is equal to y_n.

(iii) $\rightarrow$ (ii). Suppose (iii) holds. Let $L_{n-1} = \{r_{n-1}(y)\colon y \in M_n\}$. Let $y_1, y_2 \in M_n$, $\alpha_1, \alpha_2 \in F$. Then, for any $z \in M_{n-1}$,

$$\begin{aligned} &[\alpha_1 r_{n-1}(y_1) + \alpha_2 r_{n-1}(y_2), z] \\ &\qquad = \alpha_1[r_{n-1}(y_1), z] + \alpha_2[r_{n-1}(y_2), z] \\ &\qquad = \alpha_1[y_1, z] + \alpha_2[y_2, z] \\ &\qquad = [(\alpha_1 y_1 + \alpha_2 y_2), z] \\ &\qquad = [r_{n-1}(\alpha_1 y_1 + \alpha_2 y_2), z]. \end{aligned}$$

Since $\alpha_1 r_{n-1}(y_1) + \alpha_2 r_{n-1}(y_2) \in M_{n-1}$, by the definition of a right projection and Proposition 3.11, we have that

$\alpha_1 r_{n-1}(y_1) + \alpha_2 r_{n-2}(y_2) \in M_{n-1}$. Hence L_{n-1} is a subspace of M_{n-1}. Conversely suppose $y \in M_{n-1}$. Then $y = r_{n-1}(y)$. Thus $M_{n-1} \subseteq L_{n-1}$. Therefore $M_{n-1} = \{r_{n-1}(y)\colon y \in M_n\}$. Let

$N_n = N_n(\underline{x}) = \{y - r_{n-1}(y)\colon y \in M_n\}$. By Proposition 4.2, N_n is a subspace of M_n and $M_n = M_{n+1} \underset{\rightarrow}{\oplus} N_n$. Therefore $\underline{x}$ has right innovations.

(ii), (iii) $\rightarrow$ (i). Now assume (ii) and (iii). Then for each n,

$$M_{n-1} = \{r_{n-1}(y)\colon y \in M_n\},$$
$$N_n = \{y - r_{n-1}(y)\colon y \in M_n\} \text{ and}$$
$$M_n = M_{n-1} \underset{\rightarrow}{\oplus} N_n = (M_{n-2} \underset{\rightarrow}{\oplus} N_{n-1}) \underset{\rightarrow}{\oplus} N_n.$$

We now show that

$$M_n = (M_{n-2} \underset{\rightarrow}{\oplus} N_{n-1}) \underset{\rightarrow}{\oplus} N_n$$
$$= M_{n-2} \underset{\rightarrow}{\oplus} (N_{n-1} \underset{\rightarrow}{\oplus} N_n) = M_{n-2} \underset{\rightarrow}{\oplus} N_n$$

Let $y_1 \in M_{n-2}$, $y_2 \in N_{n-1}$, $y_3 \in N_n$. Then there exist $z_1, z_2, z_3 \in M_n$ such that $y_1 = r_{n-2}(r_{n-1}(z_1))$, $y_2 = r_{n-2}(r_{n-1}(z_2)) - r_{n-1}(z_2)$, $y_3 = z_3 - r_{n-1}(z_3)$. Since $N_{n-1} \subseteq M_{n-1}$ and $M_{n-1} \perp N_n$, $N_{n-1} \perp N_n$. Further $M_{n-2} \perp N_{n-1}$. But

$$[y_2+y_3, y_1] = [y_2, y_1] + [y_3, y_1]$$
$$= [r_{n-2}(r_{n-1}(z_2)), y_1] - [r_{n-1}(z), y_1] + [z_3, y_1] - [r_{n-1}(z_3), y_1]$$
$$= 0$$

(definition of a right projection and as $y_1 \in M_{n-2} \subseteq M_{n-1}$).

Therefore $M_{n-2} \perp N_{n-1} \underset{\rightarrow}{\oplus} N_n$. Hence $M_n = M_{n-2} \underset{\rightarrow}{\oplus} N_{n-1} \underset{\rightarrow}{\oplus} N_n$.

Continuing in this manner we get $M_n = M_{n-k} \underset{\rightarrow}{\oplus} N_{n-k+1} \underset{\rightarrow}{\oplus} \dots \underset{\rightarrow}{\oplus} N_n$. Using this and Proposition 4.1 we see that $\underline{x} = \{x_n\}$ has a right Wold decomposition.

<u>Remark</u> 4.4. In view of Theorem 3.30 and 4.3 this extends the work of [5].

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