This is to certify that the

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A NONLINEAR FINITE ELEMENT
FOR CURVED BEAMS

presented by

Bambang Suhendro

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Civil Engineering (Structures)

Dr. Robert K. Wen
Major professor

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A NONLINEAR FINITE ELEMENT FOR CURVED BEAMS

By

Bambang Suhendro

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Submitted to
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in partial fulfillment of the requirements
for the degree of

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1989

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ABSTRACT

A NONLINEAR FINITE ELEMENT FOR CURVED BEAMS

By

Bambang Suhendro

A procedure for the computation of nonlinear elastic response of curved beam structures is presented. The structure is represented by beam finite elements curved in one plane but deformable in three dimensional space. The curved axis of the element is represented by a second order polynomial in the curvilinear coordinates. Geometric nonlinerities are considered by including the effect of rotations on the longitudinal strains. In deriving the linear stiffness matrix, the displacement functions are approximated by cubic polynomials. However, the incremental (or nonlinear) stiffness matrices are derived by assuming that the longitudinal displacements are interpolated by linear polynomials while the interpolations for the other displacements remain unchanged. The nonlinear terms in the strain expression are averaged over the element length. Differentiation twice of the strain energy expression yields the linear stiffness matrix ,[k], and the first and second order incremental stiffness matrices ,[n1] and [n2], of the element.

Assuming that the system is elastic and conservative, the equilibrium equation is obtained from the first variation of the potential energy. The problem is solved by the Newton-Raphson method using load increments.

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A computer program was prepared for the implementation of the nonlinear equilibrium solution. Numerical results were obtained involving arches with in-plane and out-of-plane behavior. Various types of geometry, loading, and support condition were considered.

Numerical results indicated that the proposed method, which is based on a fixed Lagrangian coordinate system, works very well for "small displacement problems" (2% or less of the arch span) as well as for "intermediate displacement problems" (2-25% of the arch span).

For all of the numerical problems considered, accurate load-deflection curve may be obtained by using at most eight elements to represent the entire arch. For symmetrical problems, only one half of the arch (four elements) need be considered. Many cases required only two elements.

Comparisons of numerical results with those of other methods indicate that the method presented is more accurate and effective than the others.

The solution procedure based on an updated Lagrangian coordinate system is also presented. The procedure is necessary if large displacements (say 25% or more of the arch span) are involved.

In addition to the displacement response, the response of stresses was also investigated. Furthermore, amplification factors for displacement and stresses were studied. The result indicated that the displacement amplification factor was always larger than the stress amplification factor.

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1.1 **GENERAL**

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CHAPTER I

INTRODUCTION

1.1 GENERAL

The concept of basing structural design on ultimate strength has gained increasing acceptance in recent years. In general, the computation of the ultimate strength of a structure would involve load-displacement relationships that are nonlinear. In other words, nonlinear analysis of structure becomes necessary. In the past, such analysis was shunned by engineers because it usually implies a large amount of computations (in addition to theoretical complexities). However, current developments in computers are making such analysis increasingly affordable for engineering practice.

Nonlinear behavior of structures may be due to geometric changes, which represent the effect of distortion of the structure on its response, or to material properties such as a nonlinear stress-strain relation.

In the present study a procedure for the computation of nonlinear elastic response of curved beam members is presented. Only geometric nonlinearity is considered. This study was originated from a search of an efficient method of nonlinear elastic analysis of arches or curved structures in two and three dimensional space.

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This chapter describes the objective and scope of the present work, a literature review of related studies, and the notation used in the subsequent analysis.

1.2 OBJECTIVE AND SCOPE

Many engineering structures have components that may be considered as curved beams. Several examples are the ribs of arch bridges, arch frames, horizontally curved highway bridges, the components of aircraft frames, ship frames, and vessel frames.

Figure 1-1 illustrates a load-displacement curve of a general arch structure (herein the terms "arch" and "curved beam" are used interchangeably) which can be obtained by the solution of the nonlinear equilibrium equations of the system. The curve "OCD" is called the "fundamental path". The point (C) on the fundamental path at which the load is a relative maximum is called a "limit point". Depending on the properties of the arch and loading, a point of "bifurcation" may occur before the limit point (i.e., point A) or after the limit point (i.e., point A'). Immediately beyond the bifurcation point on the fundamental path, the structure is unstable, so that the response could follow the secondary path AB or A'B'. If the bifurcation point occurs before the limit point, the buckling shape would be "antisymmetrical" (sidesway). If the bifurcation point occurs after the limit point, C, the arch would have buckled at C in a "symmetrical" mode (snap through) (48,22)*.

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* Numbers in parantheses refer to entries in the list of references.

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It should be noted that the "classical buckling" theory would assume that, up to the point when buckling takes place , A", the structure would maintain its original undeformed shape. In other words, the prebuckling deformation is neglected. At buckling, it goes into an adjacent equilibrium configuration, B", which would then be unspecified in magnitude.

Considerable amount of work has been done (see literature review) on the development of suitable finite element models for the analysis of curved beams. Most of the previous works have dealt with their linear or stability analyses in the plane of the structure . Past studies that had considered out-of-plane behavior have been limited to buckling analysis (as an eigenvalue problem). Such an analysis represents a short cut procedure to obtaining the ultimate load based on the assumption of no displacement, or a linear relation between displacement and load, prior to buckling. Its application is limited to those situations where the displacement at the incipient buckling is small. For more general cases, i.e., when the latter displacement is not small, it becomes necessary to solve the nonlinear equilibrium problem and obtain the corresponding load-displacement curve from which the ultimate load could be determined.

Nonlinear equilibrium analysis is, in general, difficult to formulate and expensive to carry out the numerical solution. Past studies that dealt with such analysis of curved structures have been limited to behavior in the plane of the arch.

The objective of the present study is to develop a three dimensional nonlinear curved beam finite element which is applicable to

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both linear and nonlinear analyses of arbitrary geometry in two and three dimensional space.

The curved beam element developed herein represents an improvement of the previous model presented by Wen and Lange (45) for buckling analysis. The geometric representation and the displacement functions have been simplified for more convenient application. However, through the use of the "average axial strain", the new model is found to be substantially more effective and accurate than the previous one. The curved axis of the element is represented by a second order polynomial in the curvilinear coordinates. In deriving the linear stiffness matrix, the displacement functions are approximated by cubic polynomials. The incremental (or nonlinear) stiffness matrices are derived by interpolating the transverse and longitudinal displacements respectively by cubic and linear polynomials. A Lagrangian description of the arch displacements is used.

The present study uses the "incremental stiffness matrices" method outlined by Mallett and Marcal (24). The strain energy is written in terms of the displacement variables. Geometrically nonlinear effects are considered by including both the linear and quadratic terms of the displacements in the expression for the generalized strains. Furthermore, following Wen and Rahimzadeh (47), the quadratic (nonlinear) terms are averaged over the element length . By using these functions the expression for the strain energy of an element is derived. This expression consists of three parts : the quadratic , cubic , and quartic terms . Differentiating these expressions twice yields the linear stiffness matrix, [k], and the first and second order incremental stiffness matrices, [n1] and [n2], of the element. The

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linear stiffness matrix need be evaluated numerically by Gauss quadrature method. However, because of the use of lower order interpolation functions for some displacement components, terms of the nonlinear stiffness matrices can be and are evaluated in closed form.

Assuming that the system is elastic and conservative, the equilibrium equation is obtained from the first variation of the potential energy. This represents a set of nonlinear algebraic equations. The equation governing the linear incremental behavior is obtained from the second variation of the potential energy.

The nonlinear equilibrium problem is solved by the Newton-Raphson method for a series of load increments. Possible instability along the solution path is also tested by checking the determinant of the tangent stiffness matrix of the structure at every load increment. In implementing this method, the convergence check is based on the unbalanced force vector.

A computer program was prepared for the implementation of the above described nonlinear equilibrium analysis. Numerical results were obtained involving arches with in-plane and out-of-plane behavior. Various types of loading and support condition were considered. To provide some insight into the effects of variations in the arch profile on its nonlinear response, semi-elliptic, circular, parabolic, and sinusoidal shapes having the same rise to span ratio were considered. The influence of the number of elements on the accuracy of the results was investigated. The amplification factors for stresses and displacement were also studied.

The problems were classified into small, intermediate, and large displacement categories. The small displacement problems are those in which the defl

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which the deflection is less than about 2% of the arch span. Intermediate displacement problems denote those in which the deflection is of the order of 2%-25% of the arch span. Beyond 25% the problems are called large displacement ones.

Comparisons of numerical results with those of other methods indicates that the method presented is very accurate and efficient. The procedure is generally not sensitive to the load step size. For all of the numerical problems considered, accurate load-displacement curve may be obtained by using at most eight elements to represent the entire arch. For symmetrical problems, only one half of the arch (four elements) need be considered. Many cases required only two elements. The method works very well for small and intermediate displacement problems. Most common practical problems would fall into these categories.

For very large displacements, it may be necessary to use the so called "updated Lagrangian coordinates" method of solution as described in Ref. 47 for <u>straight</u> beam elements. Such a procedure for the curved element is outlined in Appendix A.

The nonlinear elastic behavior of structures are often discussed in terms of displacements. The ratio of the displacements obtained from a nonlinear analysis to that obtained from a linear analysis is called a (displacement) "amplification factor". Because of its importance in design application, a look at the maximum stress is taken in this study also. It was found that the amplification factor for stress was always smaller than that of the displacement.

1.3 LITERATURE

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1.3 LITERATURE REVIEW

For a straight beam finite element, it is well known that a cubic polynomial assumed for the transverse displacement and a linear one for the longitudinal displacement yield accurate results. Such is not the case for curved beams.

1.3.1 LINEAR EQUILIBRIUM

Dawe (16) has studied the use of higher order polynomials as shape functions for curved beams. He pointed out that there was a great improvement gained by increasing the order of the assumed displacement components from cubic to quintic. Gallert and Laursen (18) have presented a mixed formulation of finite elements for arches of arbitrary shape. They established the convergence proof for this method. Numerical results indicated that the convergence is rapid. Mebane and Stricklin (25) have pointed out that rigid body motion could be considered to be implicitly included in the polynomial form of shape functions as the number of elements used to represent the structure increases. Ashwell (1) discussed a class of curved finite elements (circular) whose shape functions were derived from independent polynomial expressions for the generalized strains rather than displacements. It was shown that the convergence of the strain element was independent of arch types (shallow, thin moderate, thick moderate, thin deep, and thick deep) and the behavior was superior to other models.

1.3.2 BUCKLING ANALYSIS

For in-plane buckling analysis (as eigenproblems), Austin and Ross (2) have compared the solutions of the in-plane buckling of symmetrically loaded arches between the classical buckling theory and the exact, nonl

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the exact, nonlinear buckling analysis. They found that except for buckling in the symmetric mode (snap-through), the buckling load obtained with the classical theory was very close to the bifurcation load obtained with the exact theory. The conclusion, however, was based on loading types that resulted in relatively small prebuckling deformations.

Ojalvo and Newmann (30) have reported a basic theoretical work on the linear elastic stability of a curved beam in space. Ojalvo, Demuts, and Tokarz (29) followed the preceding work to study the out-of-plane buckling of a member curved in one plane. Tokarz and Sandhu (40) developed the linear differential equations and obtained solutions for the lateral-torsional buckling of a parabolic arch subjected to a uniformly distributed load. Wen and Lange (45) developed a finite element model for a beam initially curved in one plane but deformable in three dimensional space. Geometric nonlinearities have been included in the analysis. Linear as well as nonlinear eigenproblems were formulated by setting the structural incremental stiffness to zero and assuming that the displacement increases linearly with the applied loads. Their curved beam element could be used to calculate the in-plane or out-of-plane buckling loads of arbitrary arch geometry. The curved axis of the element was represented by a fourth order polynomial. The displacement functions in the three dimensional space were each approximated by a cubic polynomial.

1.3.3 NONLINEAR EQUILIBRIUM ANALYSIS

Mallett and Marcal (24) presented the general relationships between the strain energy, the total equilibrium and incremental equilibrium equations in terms of the usual linear stiffness matrix and two nonlinear in

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two nonlinear incremental stiffness matrices. Wen and Rahimzadeh (47) presented a "Finite Element Average" model for a three dimensional nonlinear straight beam element, where the nonlinear part of the axial strain was averaged over the element length. Without such averaging, the element, which was formulated based on an application of the Mallet and Marcal work (24), would generally be excessively stiff. The model, which was based on the Lagrangian coordinate system, worked very well for nonlinear analysis of frames in two or three dimensional space. Solution procedures based on fixed and updated coordinate systems were presented.

For nonlinear equilibrium analysis of arches, Huddleston (20) studied the inplane behavior of two hinged circular arches with any rise to span ratio by formulating the problem as a two point boundary value problem consisting of six nonlinear, first order differential equations and appropriate boundary conditions. The theory was exact in the sense that no restriction were placed on the size of the deflections. The problem was solved by a "shooting method" in which the boundary value problem was converted to an initial value problem, and the Regula-falsi procedure. The formulation was limited to two hinged circular arches subjected to a vertical concentrated load at the crown.

Noor, Green, and Hartley (27) developed a curved beam finite element using the "mixed formulation" for the geometrically nonlinear analysis of deep arches. While the displacement formulation adopted in most works, including this one, and most general computer programs for structural engineering uses only (generalized) displacements at the nodes as degrees of freedom, the mixed formulation employs both displacements

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displacements and forces at nodes as degrees of freedom. The formulation was based on a nonlinear deep arch theory with the effect of transverse shear deformation included. A total Lagrangian description of the arch deformation and Lagrangian interpolation functions were used in the formulation. Newton-Raphson method was used to solve the resulting nonlinear equations. Circular, parabolic, and semi-elliptic arches were analysed to obtain their inplane nonlinear responses. They concluded that their mixed model performance was accurate and less sensitive to variations in the arch geometry than that of the displacement model.

Belytschko and Glaum (5) presented a higher order corotational formulation for the "initially curved beam element" (the shape of a bent beam which was straight before bending) in two dimension. The displacement fields of each element are decomposed into rigid body and deformation displacements. The deformation displacements in the axial and transverse directions are respectively described by linear and cubic shape functions. The nonlinear equilibrium equation was solved by the modified Newton-Raphson method. The model was used to solve several shallow arch problems. It was concluded that the higher order corotational formulation converges to the exact solution more rapidly than the lower order one.

Stolarski and Belytschko (37) pointed out that the preceding curved beam element has the tendency to be too stiff unless the inplane displacement field is of sufficiently high order polynomials. This phenomenon was called "membrane locking". To eliminate this effect, the reduced integration method (i.e., the numerical integration is performed by using only one or two Gauss points) was used and shown to produce reasonab

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produce reasonably good results. They concluded that the use of higher order fields for curved element is not necessary if reduced integration is used.

Calhoun and DaDeppo (8) developed a curved nonlinear finite element to analyze the inplane behavior of circular arches. No restrictions were placed on the magnitude of the rotations. The normal and tangential displacement components were approximated by cubic polynomials. The element had 4 degrees of freedom at each node, one of which was nonessential. The problem was formulated as a system of rate equations that govern the quasi-static deformation of an arch. These equations were integrated using a Runge-Kutta scheme to obtain load-deflection response.

By means of a field transfer matrix method, Fujii and Gong (17) developed a curved beam element of arbitrary geometry for finite displacement analysis of general planar arches. The arbitrary geometry was approximated by "blending functions" of third degree. The total displacements were separated into the rigid-body displacements and the elastic deformations. Sinusoidal, parabolic, and circular arches were analysed. The numerical results indicated that the load-displacement curve of arch problems considered converged to the correct result with 40 elements representing the entire arch.

1.4 NOTATION

The notation shown below has been used in this report :

A = area of cross section;

A, B = end nodes of an element;

b_1, b_2

c_x , c_y

E

G

I_{xx}, I_{yy}

I_{ζ}, I_{η}

$[K]$

$[k]$

$[K_S]$

$[K_T]$

$[k_S]$

$[K_{\epsilon_o}]$

$[k_{\epsilon_o}]$

K_t

$\kappa_x, \kappa_y, \kappa_z$

L

M_{xx} , M_{yy}

$[N1]$

$[N2]$

$[n1]$

$[n2]$

P

$\{P\}$

Δp

Q

$\{Q\}$

$\{q\}$

b_1, b_2 = element geometry coefficients;

c_x , c_y = distance from the y- or x-axis to the extreem fiber;

E = Young's modulus of elasticity;

G = shear modulus;

I_{xx}, I_{yy} = moment of inertia of cross section;

I_ζ, I_η = moment of inertia of cross section;

[K] = structural linear stiffness matrix;

[k] = element linear stiffness matrix;

$[K_S]$ = structural secant stiffness matrix;

$[K_T]$ = structural tangent stiffness matrix;

$[k_S]$ = element secant stiffness matrix;

$[K_{\epsilon_o}]$ = structural initial strain stiffness matrix;

$[k_{\epsilon_o}]$ = element initial strain stiffness matrix;

K_t = torsion constant of cross section;

$\kappa_x,\kappa_y,\kappa_z$ = changes in curvature about x,y,z axes;

L = curved length of element;

M_{xx} , M_{yy} = moment about x- or y-axis;

[N1] = first order structural incremental stiffness matrix;

[N2] = second order structural incremental stiff. matrix;

[n1] = first order element incremental stiffness matrix;

[n2] = second order element incremental stiffness matrix;

P = concentrated load, axial force;

{P} = external load vector;

Δp = load increment;

Q = simbol for exact configuration of the structure;

{Q} = structural generalized displacement vectors;

{q} = generalized coordinates, element gen. disp. vector;

R =

$R1$, $R2$ =

$\{R\}$ =

$\{\Delta R\}_i$ =

$\{r\}$ =

s =

u, v, w

U

U_E

U_ϵ

U_t

U_2, U_3, U_4

X, Y, Z

X_A, Y_A

X_B, Y_B

X_L, Y_L

x, y, z

(x_B, y_B)

α

$\alpha_1, \ldots, \alpha_{16}$

β

$\beta_x, \beta_y, \beta_z$

$\lambda_1, \ldots, \lambda_{16}$

γ

σ

i_{ϵ_o}

ϵ

R = radius of curvature;

R1, R2 = radii of curvature at ends of an element;

$\{R\}$ = resistant force vector;

$\{\Delta R\}_i$ = unbalanced force vector related to ith iteration;

$\{r\}$ = element end force vector;

s = longitudinal axis of curved beam member;

u,v,w = displacements along x,y,z axes, respectively;

U = strain energy of the structure;

U_E = strain energy of an element;

U_ϵ = strain energy due to longitudinal strain;

U_t = strain energy due to torsion;

U_2,U_3,U_4 = quadratic, cubic, and quartic parts of strain energy;

X,Y,Z = structure global coordinate system;

X_A,Y_A = the coordinates of node A in global coord. system;

X_B,Y_B = the coordinates of node B in global coord. system;

X_L,Y_L = relative position of end nodes of an element;

x,y,z = element coordinate system;

(x_B,y_B) = coordinates of node B in element coordinate system;

α = angle of opening of circular arch;

$\alpha_1,..,\alpha_{16}$ = parameters used for definition of shape functions;

β = twist of cross section about z-axis;

β_x,β_y,β_z = rotations about x,y,z axes, respectively;

$\lambda_1,..,\lambda_{16}$ = parameters used for definition of shape functions;

γ = normalized variable;

σ = stress at element end;

${}^i\epsilon_o$ = initial strain at the beginning of ith load incrmnt;

ϵ = longitudinal strain;

ϵ_f =

Φ =

θ =

θ_x, θ_y =

ϕ =

ϕ_A =

ϕ_B =

$\{ \ \}$ =

$[\ ]$ =

$\lfloor \ \rfloor$

ϵ_f	=	unbalanced force vector tolerance;
Φ	=	potential energy of the system;
θ	=	angle of tangent at node B;
θ_x, θ_y	=	rotations about x, y axes, respectively;
ϕ	=	angle of tangent at any point in the element;
ϕ_A	=	angle between global X axis and tangent at node A;
ϕ_B	=	angle between global X axis and tangent at node B;
$\{ \}$	=	column vector;
[ ]	=	rectangular matrix;
$\lfloor \rfloor$	=	row vector.

2.1 GENERAL

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application. Ho

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CHAPTER II

FINITE ELEMENT MODEL FOR A CURVED BEAM

2.1 GENERAL

As mentioned previously, the finite element model discussed in this chapter represents an improvement of the previous model developed by Wen and Lange (45). Although the geometric representation and the displacement functions have been simplified for more convenient application. However, through the use of the "average axial strain", the new model is actually substantially more effective and accurate than the previous one.

In this chapter the strain-displacement relation is first presented. Next, two strain energy expressions based on quartic axial strain function and average axial strain model are described. The geometric representation and the displacement functions of the element are then presented. The strain energy expressions of a typical curved element and the corresponding stiffness matrices (including the linear and nonlinear parts) are derived. Finally, the equilibrium equations and the equations that govern the linear incremental behavior of a structure are developed.

2.2 STRAIN-DISPLACEMENT RELATION

Consider a beam element curved in one plane as shown in Figure 2-1. A right handed coordinate system, x-, y-, and z-axes, represents the local or member coordinates of the element. The displacements corresponding

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corresponding to those axes are denoted by u, v, and w respectively. The rotation about z-axis is denoted by β , as shown in Figure 2-2. The centroidal axis curves in the x-z plane with radius of curvature R, which may vary. The cross-section of the element is taken to be constant and has two axes of symmetry. Assuming that plane sections remain plane after bending deformation, the expression for the longitudinal strain at a point (ζ,η) in a section s , measured along the curved centroidal axis, may be written as :

$$\epsilon \Big|_{s,\zeta,\eta} = \epsilon_z \Big|_o + \eta\,\kappa_x - \zeta\,\kappa_y \qquad \text{.... (2-1)}$$

in which $\epsilon_z \big|_o$ is the longitudinal strain at the centroidal axis, and $\kappa_x = k_x - \bar{k}_x$ and $\kappa_y = k_y - \bar{k}_y$ are the changes in curvature of the centroidal axis (k_x and $\bar{k}_x$ = the current and initial curvatures about the x-axis, respectively; similarly for k_y and $\bar{k}_y$).

For a general case of a beam curved in space, these changes in curvature have been derived by Ojalvo and Newmann (30). For the element considered herein, where the initial curvature about the x-axis is zero , and the initial curvature about y-axis is 1/R , the changes in curvature of the centroidal axis are given in Reference (45) as follows :

$$\kappa_x = \frac{\beta}{R} - \frac{d^2v}{ds^2} \qquad \text{.... (2-2}^a\text{)}$$

$$\kappa_y = \frac{d^2u}{ds^2} + \frac{dw}{ds}\,\frac{1}{R} + w\,\frac{d}{ds}\left(\frac{1}{R}\right) \qquad \text{.... (2-2}^b\text{)}$$

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The longitudinal strain at the centroidal axis may be written as :

$$\epsilon_z \Big|_o = \left(\frac{dw}{ds} - \frac{u}{R} \right) + \frac{1}{2} \left(\frac{du}{ds} + \frac{w}{R} \right)^2 + \frac{1}{2} \left(\frac{dv}{ds} \right)^2 \qquad \ldots (2\text{-}3)$$

in which the quantity in the first parenthesis is the usual linear hoop strain in a curved element, and the next two terms (which are nonlinear), represent the contributions to the strain by the rotations of the centroidal axis about the y- and x-axes, respectively. Substituting Equations (2-2) and (2-3) into Equation (2-1), the expression for the longitudinal strain is obtained :

$$\epsilon = \epsilon_z \Big|_{s,\zeta,\eta} = \left(\frac{dw}{ds} - \frac{u}{R} \right) + \frac{1}{2} \left(\frac{du}{ds} + \frac{w}{R} \right)^2 + \frac{1}{2} \left(\frac{dv}{ds} \right)^2$$

$$+ \eta \left(\frac{\beta}{R} - \frac{d^2 v}{ds^2} \right) \qquad \ldots . (2\text{-}4)$$

$$- \zeta \left[\frac{d^2 u}{ds^2} + \frac{1}{R} \frac{dw}{ds} + w \frac{d}{ds} \left(\frac{1}{R} \right) \right]$$

2.3 STRAIN ENERGY EXPRESSION

The expression for the strain energy of the element, U_E , may be written as :

$$U_E = U_\epsilon + U_t \qquad \ldots . (2\text{-}5)$$

where U_ϵ is the strain energy due to the longitudinal strain and U_t is that due to the shear strain resulting from St. Venant torsion (Note that warping i

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$$U_\epsilon = \int_{vol}$$

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that warping is not considered herein as the cross-section is assumed to be either solid or a closed tubular one). They are given by the following expressions :

$$U_\epsilon = \int_{vol} \frac{E\,\epsilon^2}{2}\, dV = \int_s \int_A \frac{E\,\epsilon^2}{2}\, dA\, ds \qquad \ldots.\ (2\text{-}6^a)$$

$$U_t = \int_s \frac{G\,K_t}{2} \left(\beta_s + \frac{1}{R} v_s \right)^2 ds \qquad \ldots.\ (2\text{-}6^b)$$

in which ϵ is the longitudinal strain, A is the cross-sectional area, E and G are the Young's modulus and shear modulus, and K_t is the torsion constant of the cross section. In the preceding equation, the notation of using a subscript to represent a differentiation has been used, e.g., $\beta_s = d\beta/ds$. This notation will also be used subsequently. The total strain energy of the element becomes :

$$U_E = \int_s \int_A \frac{E\,\epsilon^2}{2}\, dA\, ds + \int_s \frac{G\,K_t}{2} \left(\beta_s + \frac{1}{R} v_s \right)^2 ds$$

$$\ldots.\ (2\text{-}7)$$

2.4 FINITE ELEMENT FORMULATION

In this section, the definition of coordinate systems, the geometric representation, the displacement functions, and the strain energy expression of a typical curved beam element are presented.

2.4.1 **DEFINITI**

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2.4.1 DEFINITION OF COORDINATE SYSTEMS

The global coordinate system used in the analysis (see Figure 2-3) consists of a single set of cartesian axes with the origin located at the crown of the arch. The system is oriented with X-axis horizontal, the Y-axis vertical, and the Z-axis perpendicular to the plane of curvature. The positions of the nodes of the structure are expressed by means of this system. The local or element coordinate system , illustrated in Figure 2-4, consists of one set of cartesian axes with its origin located at node A, with the x-axis in the radial direction, the y-axis normal to the plane of curvature, and the z-axis tangent to the curved centroidal axis forming an angle ϕ_A with the global X-axis. Node B denotes the end node of the element.

2.4.2 ELEMENT GEOMETRY

Referring to the curved element shown in Figure 2-4, the coordinates of the nodes A and B , with respect to the global coordinate system are, (X_A, Y_A) and (X_B, Y_B) respectively. Their relative position is defined by $X_L = X_B - X_A$ and $Y_L = Y_B - Y_A$. The coordinates (x_B, z_B) of node B in the element coordinate system are given by :

$$x_B = - X_L \sin \phi_A + Y_L \cos \phi_A \qquad \ldots. \ (2\text{-}8^a)$$

$$z_B = X_L \cos \phi_A + Y_L \sin \phi_A \qquad \ldots. \ (2\text{-}8^b)$$

Letting ϕ denote the angle between the element z-axis (tangent at the end node A) and the tangent at a given point, Figure 2-5 shows that the angle ϕ is a function of s. It varies from zero at node A to θ at node

B. The radii of

by R1 and R2.

At any poi

$$dz = ds$$

$$dx = ds$$

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$$s = b_1$$

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$$R = \frac{ds}{d\phi}$$

The element le

$$L = b_1$$

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are the follow

(1) x_1

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B. The radii of curvature R at nodes A and B are respectively denoted by R1 and R2.

At any point along the curve, the following relations hold :

$$dz = ds \cos \phi \qquad \text{.... (2-9}^a\text{)}$$

$$dx = ds \sin \phi \qquad \text{.... (2-9}^b\text{)}$$

The curve is approximated by a second order polynomial in ϕ :

$$s = b_1 \phi + b_2 \phi^2 \qquad \text{.... (2-10)}$$

The radius of curvature is obtained by defferentiating Equation (2-10)

$$R = \frac{ds}{d\phi} = b_1 + 2 b_2 \phi \qquad \text{.... (2-11)}$$

The element length is given by

$$L = b_1 \theta + b_2 \theta^2 \qquad \text{.... (2-12)}$$

The boundary conditions used to solve for the coefficients b_1 and b_2 are the following :

$$\text{(1)} \quad x_B = \int_o^{x_B} dx = \int_o^{\theta} R \sin \phi \; d\phi \qquad \text{... (2-13}^a\text{)}$$

$$\text{(2)} \quad z_B = \int_o^{z_B} dz = \int_o^{\theta} R \cos \phi \; d\phi \qquad \text{... (2-13}^b\text{)}$$

The first bounda

$$x_B = \int_o^{\theta}$$

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The first boundary condition gives

$$x_B = \int_0^\theta (b_1 + 2 b_2 \phi) \sin \phi \, d\phi$$

$$= (1 - \cos \theta) b_1 + 2 (\sin \theta - \theta \cos \theta) b_2 \quad \ldots. (2\text{-}14)$$

From the second boundary condition it is found that

$$z_B = \int_0^\theta (b_1 + 2 b_2 \phi) \cos \phi \, d\phi$$

$$= (\sin \theta) b_1 + 2 (\theta \sin \theta + \cos \theta - 1) b_2 \quad \ldots (2\text{-}15)$$

The resulting coefficients b_1 and b_2 obtained from Equations (2-14) and (2-15) are

$$b_2 = \frac{z_B (1 - \cos \theta) - x_B \sin \theta}{2 [(1 - \cos \theta)(\theta \sin \theta + \cos \theta - 1) - \sin \theta (\sin \theta - \theta \cos \theta)]} \quad \ldots (2\text{-}15^a)$$

$$b_1 = \frac{x_B - 2 b_2 (\sin \theta - \theta \cos \theta)}{(1 - \cos \theta)} \quad \ldots (2\text{-}15^b)$$

Thus, the geometry of the finite element as given by Equation (2-10) is completely defined by x_B , z_B and the angle θ. The accuracy of the geometric representation presented in this section , for different numbers of element, is given in Table 2-1.

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$$w = \alpha_9$$
$$\beta = \alpha_{13}$$

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$$u = \lambda_1$$
$$v = \lambda_5$$
$$w = \lambda_9$$
$$\beta = \lambda_1$$

2.4.3 DISPLACEMENT FUNCTIONS

As indicated in Equations (2-4) and (2-7), the strain energy of the curved element considered here depends on four independent displacement functions, i.e., u, v, w, and β , the displacements along the x, y, z axes and the rotation about the z-axis , respectively.

For the finite element, these functions are approximated by polynomials in the variable ϕ. In deriving the linear stiffness matrix, the displacement functions are approximated by **cubic polynomials**

$$
\begin{aligned}
u &= \alpha_1 + \alpha_2 \phi + \alpha_3 \phi^2 + \alpha_4 \phi^3 \\
v &= \alpha_5 + \alpha_6 \phi + \alpha_7 \phi^2 + \alpha_8 \phi^3 \\
w &= \alpha_9 + \alpha_{10} \phi + \alpha_{11} \phi^2 + \alpha_{12} \phi^3 \\
\beta &= \alpha_{13} + \alpha_{14} \phi + \alpha_{15} \phi^2 + \alpha_{16} \phi^3
\end{aligned}
\qquad \text{.... (2-16)}
$$

As shown in Figure 2-5 , ϕ is related to the arch length , s , by Eq.(2-10). For simplicity, the independent variable ϕ in the preceding equations may be normalized by defining $\gamma = \frac{\phi}{\theta}$, and the displacement functions become :

$$
\begin{aligned}
u &= \lambda_1 + \lambda_2 \gamma + \lambda_3 \gamma^2 + \lambda_4 \gamma^3 \\
v &= \lambda_5 + \lambda_6 \gamma + \lambda_7 \gamma^2 + \lambda_8 \gamma^3 \\
w &= \lambda_9 + \lambda_{10} \gamma + \lambda_{11} \gamma^2 + \lambda_{12} \gamma^3 \\
\beta &= \lambda_{13} + \lambda_{14} \gamma + \lambda_{15} \gamma^2 + \lambda_{16} \gamma^3
\end{aligned}
\qquad \text{.... (2-17)}
$$

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$v = \bar{\alpha}_5$

$w = \bar{\alpha}_9$

$\beta = \bar{\alpha}_{11}$

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$u = \bar{\lambda}_1$

$v = \bar{\lambda}_5$

$w = \bar{\lambda}_9$

$\beta = \bar{\lambda}_{1}$

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matrix [k] is

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In deriving the incremental stiffness matrices [n1] and [n2] , however, while the transverse displacements are still approximated by **cubic** polynomials, the longitudinal displacements are approximated by **linear** polynomials as follows :

$$
\begin{aligned}
u &= \bar{\alpha}_1 + \bar{\alpha}_2\,\phi + \bar{\alpha}_3\,\phi^2 + \bar{\alpha}_4\,\phi^3 \\
v &= \bar{\alpha}_5 + \bar{\alpha}_6\,\phi + \bar{\alpha}_7\,\phi^2 + \bar{\alpha}_8\,\phi^3 \\
w &= \bar{\alpha}_9 + \bar{\alpha}_{10}\,\phi \\
\beta &= \bar{\alpha}_{11} + \bar{\alpha}_{12}\,\phi
\end{aligned}
\qquad \ldots.\ (2\text{-}18)
$$

In the normalized variable $\gamma = \phi/\theta$, the above equations can be written as

$$
\begin{aligned}
u &= \bar{\lambda}_1 + \bar{\lambda}_2\,\gamma + \bar{\lambda}_3\,\gamma^2 + \bar{\lambda}_4\,\gamma^3 \\
v &= \bar{\lambda}_5 + \bar{\lambda}_6\,\gamma + \bar{\lambda}_7\,\gamma^2 + \bar{\lambda}_8\,\gamma^3 \\
w &= \bar{\lambda}_9 + \bar{\lambda}_{10}\,\gamma \\
\beta &= \bar{\lambda}_{11} + \bar{\lambda}_{12}\,\gamma
\end{aligned}
\qquad \ldots.\ (2\text{-}19)
$$

The following discussion explains why different approximations were used for w and β in Equations (2-17) and (2-19).

As will be shown in the following sections, the linear stiffness matrix [k] is independent of displacements. It can be obtained directly from the given structural properties and geometry. The shape functions defined in Equ

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defined in Equation (2-17) requires 16 degrees of freedom, 12 of them are essential (for equilibrium) and 4 of them are nonessential. The nonessential degrees of freedom can be condensed out, once for all, for the solution process. The resulting 12 by 12 stiffness matrix is related to the essential degrees of freedom. It can be transformed (into global coordinates) and combined with the stiffness matrices of other finite elements in the usual fashion.

On the other hand, as discussed in later sections, the incremental stiffness matrices, [n1] and [n2] , are respectively linear and quadratic functions of the displacements. If the shape functions Equation (2-17) are used, condensation of the stiffness matrices need be done each there is a displacement change. This would greatly increase the cost of computation. Using Equation (2-19) would avoid such condensations. Of course one could just use Equation (2-19) to derive [k] also. However, the result would be less accurate than that from using Equation (2-17).

2.4.4 ELEMENT STRAIN ENERGY AND STIFFNESS MATRICES

In this section, two strain energy expressions are derived. One of them is based on the **quartic axial strain** function, Equation (2-4), and another one is based on the **average axial strain** model. As will be shown in Chapter IV , the former model, which follows formally the usual potential energy formulation of finite element, results in a nonlinear stiffness that is "excessively high", while the latter model, obtained from a modification of the former, yields more accurate results.

2.4.4.1 QUARTI

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$$\epsilon = (\frac{w}{R}$$

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in which

$$\gamma_s =$$

$$\gamma_{ss} =$$

$$\gamma_{s\gamma} =$$

Using the sam

$$U_\epsilon =$$

$$U_t =$$

2.4.4.1 QUARTIC AXIAL STRAIN MODEL

In terms of the new variable, γ , the longitudinal strain may be rewritten from Equation (2-4) as :

$$\epsilon = \left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right) + \frac{1}{2} \left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 + \frac{1}{2} \left(\frac{v_\gamma}{R\,\theta} \right)^2$$

$$+ \ \eta \left(\frac{\beta}{R} - \frac{v_{\gamma\gamma}}{R^2\theta^2} - v_\gamma\,\gamma_{ss} \right) \qquad \ldots . \ (2\text{-}20)$$

$$- \ \zeta \left(\frac{u_{\gamma\gamma}}{R^2\theta^2} + u_\gamma\,\gamma_{ss} + \frac{w_\gamma}{R^2\theta} + \frac{w}{R}\,\gamma_{s\gamma} \right)$$

in which

$$\gamma_s = \frac{1}{R\,\theta}$$

$$\gamma_{ss} = - \frac{1}{R^3\theta^2}\,R_\gamma \qquad \ldots . \ (2\text{-}21)$$

$$\gamma_{s\gamma} = R\,\theta\,\gamma_{ss}$$

Using the same change of variables for Equation (2-6) ,

$$U_\epsilon = \int_o^1 \int_A \frac{E\,\epsilon^2}{2}\, dA \ \ R\,\theta\,d\gamma$$

$$\ldots . \ (2\text{-}22)$$

$$U_t = \int_o^1 \frac{G\,K_t}{2} \left(\beta_\gamma\,\gamma_s + \frac{1}{R}\,v_\gamma\,\gamma_s \right)^2 R\,\theta\,d\gamma$$

By use of Equat

element may be

$$U_E = U_2$$

in which U_2,

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$$U_2 = \frac{E}{2} \int_0^1$$

$$U_3 = \frac{E\,A}{2}$$

$$U_4 = \frac{E\,A}{8}$$

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By use of Equations (2-5), (2-20), and (2-22), the strain energy of the element may be written as

$$U_E = U_2 + U_3 + U_4 \qquad \text{.... (2-23)}$$

in which U_2, U_3, and U_4 contain respectively the quadratic, cubic, and quartic terms of the displacement field variables :

$$U_2 = \frac{E}{2} \int_o^1 \left[\frac{A}{R\,\theta} (w_\gamma - \theta\, u)^2 + I_\zeta \frac{\theta}{R^3} \left(R\,\beta - \frac{v_{\gamma\gamma}}{\theta^2} - v_\gamma\, \gamma_{ss}\, R^2 \right)^2 \right.$$

$$\left. + \frac{I_\eta}{R^3 \theta^3} (u_{\gamma\gamma} + u_\gamma\, \gamma_{ss}\, R^2 \theta^2 + w_\gamma\, \theta + w\, \gamma_{s\gamma}\, R\, \theta^2)^2 \right] d\gamma$$

$$+ \frac{G\, K_t}{2} \int_o^1 \frac{1}{R^3 \theta} (R\, \beta_\gamma + v_\gamma)^2\, d\gamma \qquad \text{.... (2-24)}$$

$$U_3 = \frac{E\, A}{2} \int_o^1 \frac{1}{R^2 \theta^2} (w_\gamma - \theta\, u) [(u_\gamma + \theta\, w)^2 + v_\gamma^2]\, d\gamma \qquad \text{.. (2-25)}$$

$$U_4 = \frac{E\, A}{8} \int_o^1 \frac{1}{R^3 \theta^3} [(u_\gamma + \theta\, w)^2 + v_\gamma^2]^2\, d\gamma \qquad \text{.... (2-26)}$$

where $I_\zeta = \int_A \eta^2\, dA$ and $I_\eta = \int_A \zeta^2\, dA$ are the moment of inertia of the cross section about ζ- and η-axes respectively.

It is noted that the strain energy is a quartic function of nodal displacements and rotations.

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u_A , u_B

v_A , v_B

w_A , w_B

β_A , β_B

θ_{yA}, θ_{yB}

θ_{xA}, θ_{xB}

and the "none

$$(\frac{dw}{ds})_A$$

These degrees

Thus, U_2 ca

freedom { q }

$$U_2 =$$

2.4.4.1.1 LINEAR STIFFNESS MATRIX

As mentioned previously, in deriving the linear stiffness matrix, the displacement functions are approximated by cubic polynomials given in Equation (2-17). Upon substituting Equation (2-17) into Equation (2-24), U_2 becomes a function of the coefficients λ_i in the displacement functions (Equation 2-17). By use of the boundary conditions at the end nodes, these coefficients may be replaced by the nodal displacement variables. These degrees of freedom are chosen to be :

u_A , u_B = the radial displacements ;

v_A , v_B = the transverse displacements ;

w_A , w_B = the longitudinal displacements ;

β_A , β_B = the twist about the longitudinal axis ;

θ_{yA}, θ_{yB} = $\left(\frac{du}{ds} + \frac{w}{R} \right)$ A, or B = the rotation about y-axis ;

θ_{xA}, θ_{xB} = $\left(- \frac{dv}{ds} \right)$ A, or B = the rotation about x-axis ;

and the "nonessential" degrees of freedom :

$$\left(\frac{dw}{ds} \right)_A , \quad \left(\frac{dw}{ds} \right)_B , \quad \left(\frac{d\beta}{ds} \right)_A , \text{ and } \quad \left(\frac{d\beta}{ds} \right)_B .$$

These degrees of freedom will be denoted collectively by vector { q }. Thus, U_2 can be expressed in terms of the element nodal degrees of freedom { q } :

$$U_2 = \int_o^1 f_2 (\{ q \}) \ d\gamma \qquad \ldots . \ (2\text{-}27)$$

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in which f_2 denotes a quadratic function of the q's .

The linear stiffness matrix , [k] , may be obtained as

$$[\ k\ ] = [\ k_{ij}\ ] = [\ \frac{\partial^2\ U_2}{\partial q_i\ \partial q_j}\ ] = [\ \int_o^1 \frac{\partial^2\ f_2}{\partial q_i\ \partial q_j}\ d\gamma\ ] \quad \ldots (2\text{-}28)$$

The expressions for the integrands in terms of the q's in Equation (2-28) are too lengthy to be presented here. They are given in the subroutine NUMINT of the computer program in Appendix C. It should be noted that the integrals themselves need be evaluated numerically by Gauss quadrature.

The linear stiffness matrix developed in this section has a size of 16 by 16. As mentioned earlier, the nonessential degrees of freedom would then be condensed out. The resulting 12 by 12 linear stiffness matrix is then compatible with the incremental stiffness matrices, [n1] and [n2], developed subsequently.

2.4.4.1.2 INCREMENTAL STIFFNESS MATRICES

The incremental stiffness matrices, [n1] and [n2] , are derived by assuming that the transverse and longitudinal displacements are respectively interpolated by cubic and linear polynomials given in Equations (2-19). As before, by substituting Equations (2-19) into Equations (2-25) and (2-26), U_3 and U_4 become functions of the coefficients $\bar{\lambda}_i$ of the polynomials. The degrees of freedom chosen to replace the coefficients $\bar{\lambda}_i$ are : u_A, u_B, v_A, v_B, w_A, w_B, β_A, β_B, θ_{yA} , θ_{yB} , θ_{xA} , and θ_{xB} . These degrees of freedom will be denoted collectively by vector { $\bar{q}$ }. The relation between { $\bar{q}$ } and { $\bar{\lambda}$ } can be obtained by use of Equations (2-19) and the above definitions. Thus,

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$$U_4 = \int_0$$

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U_3 and U_4 can be expressed in terms of the element nodal degrees of freedom $\{ \bar{q} \}$:

$$U_3 = \int_o^1 f_3 ((\{ \bar{q} \}) \; d\gamma \qquad \ldots. (2\text{-}29)$$

$$U_4 = \int_o^1 f_4 ((\{ \bar{q} \}) \; d\gamma \qquad \ldots. (2\text{-}30)$$

in which f_3 and f_4 are, respectively, cubic and quartic functions of the $\bar{q}$'s.

The incremental stiffness matrices , [n1] and [n2] , may be calculated from :

$$[\ n1\ ] = [\ n1_{ij}\ ] = [\ \frac{\partial^2 U_3}{\partial \bar{q}_i \, \partial \bar{q}_j}\ ] = [\ \int_o^1 \frac{\partial^2 f_3}{\partial \bar{q}_i \, \partial \bar{q}_j} \, d\gamma\ ] \qquad .. (2\text{-}31)$$

$$[\ n2\ ] = [\ n2_{ij}\ ] = [\ \frac{\partial^2 U_4}{\partial \bar{q}_i \, \partial \bar{q}_j}\ ] = [\ \int_o^1 \frac{\partial^2 f_4}{\partial \bar{q}_i \, \partial \bar{q}_j} \, d\gamma\ ] \qquad .. (2\text{-}32)$$

The expressions for the integrands in terms of the $\bar{q}$'s in Equations (2-31) and (2-32) are also too lengthy to be presented here . The integrals need be evaluated numerically by Gauss quadrature. They are given in Appendix D. It should be noted that the elements of the matrices [n1] and [n2] are respectively linear and quadratic functions of the displacements.

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2.4.4.2 AVERAGE AXIAL STRAIN MODEL

An alternative to the strain energy derived in the preceding section is to replace the nonlinear terms in the strain expression by their averages over the element length. Thus, the expression for the axial strain is rewritten from Equation (2-3) as :

$$\epsilon_z \Big|_o = \left(\frac{dw}{ds} - \frac{u}{R} \right) + \frac{1}{2\,L} \int_o^L \left(\frac{du}{ds} + \frac{w}{R} \right)^2 ds$$

$$+ \frac{1}{2\,L} \int_o^L \left(\frac{dv}{ds} \right)^2 ds \qquad \text{.... (2-33)}$$

As before, in terms of γ , the expression for the longitudinal strain can be obtained by substituting Equations (2-2) and (2-33) into Equation (2-1) :

$$\epsilon = \left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right) + \frac{1}{2\,L} \int_o^1 \left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 R\,\theta\; d\gamma$$

$$+ \frac{1}{2\,L} \int_o^1 \left(\frac{v_\gamma}{R\,\theta} \right)^2 R\,\theta\; d\gamma + \eta \left(\frac{\beta}{R} - \frac{v_{\gamma\gamma}}{R^2\theta^2} - v_\gamma\, \gamma_{ss} \right)$$

$$- \zeta \left(\frac{u_{\gamma\gamma}}{R^2\theta^2} + u_\gamma\, \gamma_{ss} + \frac{w_\gamma}{R^2\,\theta} + \frac{w}{R}\, \gamma_{s\gamma} \right) \qquad \text{.... (2-34)}$$

By use of Equations (2-5), (2-22), and (2-34), the strain energy of the element can be written as :

$$U_E = U_2 + U_3 + U_4 \qquad \text{.... (2-35)}$$

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in which U_2 remains the same as given by Equation (2-24), and U_3 and U_4 are given by the following expressions :

$$U_3 = E A \int_o^1 (w_\gamma - \theta u) \mathbf{M} \, d\gamma \qquad \text{.... (2-36)}$$

$$U_4 = \frac{E A}{2} \int_o^1 R \theta (\mathbf{M})^2 \, d \gamma \qquad \text{.... (2-37)}$$

where

$$\mathbf{M} = \frac{1}{2 L} \int_o^1 [(\frac{u_\gamma}{R \theta} + \frac{w}{R})^2 + (\frac{v_\gamma}{R \theta})^2] R \theta \, d\gamma \qquad \text{... (2-38)}$$

It represents the average of the nonlinear part of the axial strain.

2.4.4.2.1 LINEAR STIFFNESS MATRIX

Since the quadratic term of the strain energy U_2 , based on the average axial strain model, remains the same as that of the quartic axial strain model, the resulting linear stiffness matrix is exactly the same as that discussed in Section 2.4.4.1.1. As before, after integrating numerically, the nonessential degrees of freedom can then be condensed out to get a 12 by 12 stiffness matrix.

2.4.4.2.2 INCREMENTAL STIFFNESS MATRICES

By use of Equations (2-36) and (2-37) for U_3 and U_4, respectively, and using exactly the same procedure as described in Section 2.4.4.1.2, the expressions for [n1] and [n2] can be obtained. In this case, the integrals can be evaluated **analytically**. The calculation of [n1] and

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[n2] are very lengthy but straightforward. The intermediate computations are not presented here and expressions for each of those matrices are given in Appendix B in **closed form**. As before, the elements of the matrices [n1] and [n2] are, respectively, linear and quadratic functions of the displacements.

2.4.5 EQUILIBRIUM EQUATIONS

In the preceding sections, the linear stiffness matrix, [k], and the incremental stiffness matrices, [n1] and [n2], for the element in local coordinates have been derived. The structural linear stiffness matrix , [K], and incremental stiffness matrices, [N1] and [N2], can be assembled from the corresponding element matrices in the usual fashion of finite element analysis via the displacement method.

As mentioned previously, the formulation followed in this section is that described by Mallett and Marcal (24). Assuming that the system is elastic and conservative , the potential energy of the system , Φ , may be written as :

$$\Phi = U - \lfloor Q \rfloor \{ P \} \qquad \text{.... (2-39)}$$

in which U is the strain energy, the sum of the strain energy of the constituent elements of the structure ; $\lfloor Q \rfloor$ is the row vector of the degrees of freedom of the structure ; and $\{ P \}$ is the load vector corresponding to $\lfloor Q \rfloor$. The above equation can be written as :

$$\Phi = \lfloor Q \rfloor \left(\frac{1}{2} [K] + \frac{1}{6} [N1] + \frac{1}{12} [N2] \right) \{Q\} - \lfloor Q \rfloor \{P\} \qquad \text{... (2-40)}$$

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The first variation of the potential energy produces the **equilibrium equation** :

$$\left([K] + \frac{1}{2}[N1] + \frac{1}{3}[N2] \right) \{ Q \} = \{ P \} \qquad \text{... (2-41)}$$

This represents a set of nonlinear algebraic equations. The term in the first parenthesis is called the **SECANT STIFFNESS MATRIX**.

The equations governing the linear incremental behavior follow from the second variation of the potential energy and are given by :

$$\left([K] + [N1] + [N2] \right)_{\{\bar{Q}\}} \{ \Delta Q \} = \{ \Delta P \} \qquad \text{... (2-42)}$$

in which $\{\bar{Q}\}$ denotes the displacements at a reference (or the current) equilibrium position , and $\{ \Delta Q \}$ and $\{ \Delta P \}$ are the incremental displacement and load vectors, respectively. The term in the first parenthesis is called the **TANGENT STIFFNESS MATRIX**. The subscript $\{\bar{Q}\}$ denotes the fact that the tangent stiffness matrix is to be evaluated at $\{Q\} = \{\bar{Q}\}$.

Equations (2-41) and (2-42) will be used in the following chapter to develop a procedure for nonlinear equilibrium analysis of curved beam structures.

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CHAPTER III

NONLINEAR EQUILIBRIUM ANALYSIS OF CURVED BEAM STRUCTURES

3.1 GENERAL

In the preceding chapter the equilibrium equation, the equation governing the linear incremental behavior, and the corresponding system stiffness matrices have been developed. The problem of nonlinear equilibrium analysis can be solved by more than one procedure of solution. The most common ones are : (a) direct iteration method; (b) Newton-Raphson method; and (c) straight incremental method (10,44).

The Newton-Raphson method is used herein. It is a second order iterative method using the tangent stiffness. The load is applied as a series of small increments. For each increment an iteration scheme is employed to continuously update the tangent stiffness matrix as improved approximations of the incremental deformations are calculated. The convergence check is based on the unbalanced force vector, which is evaluated by use of the secant stiffness. The method and the computer program implementing the solution procedures are described in this chapter.

3.2 NEWTON-RAPHSON METHOD

3.2.1 CONCEPT

Consider a structure subjected to an external load vector {P}. Let Q represent symbolically the exact deformed configuration of the structure. If we assume an iterative process, and in the ith iteration the approxim

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the approximate configuration Q_i is known, we are interested in improving Q_i in such a way that eventually Q_n $(n > i)$ would get sufficiently close to Q.

The load displacement relation can be written as

$$\{ P \} = \{ f (Q) \} \tag{3-1}$$

Using a first order Taylor series expansion about Q_i we have

$$\{ P \} = \{ f (Q_i) \} + \{ \frac{\partial f}{\partial Q_j} \}_{Q_i} \{ \Delta Q_i \} \tag{3-2}$$

in which, $\{ f (Q_i) \}$ may be interpreted as representing the elastic resistance of the structure corresponding to Q_i , and $\{ \frac{\partial f}{\partial Q_j} \}_{Q_i}$ as the tangent stiffness at Q_i . Then the modification to Q_i is :

$$\{ \Delta Q_i \} = \{ \frac{\partial f}{\partial Q_j} \}_{Q_i}^{-1} \{ P - f (Q_i) \}$$

$$= \{ \frac{\partial f}{\partial Q_j} \}_{Q_i}^{-1} \{ \Delta R_i \} \tag{3-3}$$

in which $\{ \Delta R_i \}$ is the unbalanced force vector at stage Q_i.

The modified displacement is :

$$Q_{i+1} = Q_i + \Delta Q_i \tag{3-4}$$

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The process may be repeated until either ΔQ_{i+k} or ΔR_{i+k} is sufficiently small. For one degree of freedom system, the process is illustrated graphically in Figure 3-1.

The preceding discussion was for the load applied as a single load increment. Greater accuracy in the solution may be obtained by applying the load in increments (e.g., ΔP, $2 \Delta P$, $3 \Delta P$, ..., etc.). For each increment the concept described previously applies.

At the beginning of the increment the geometry of structure may or may not be updated. As will be discussed in chapter IV, the problems of nonlinear analysis of arches are divided into small, intermediate, and large displacement categories. The updated procedure is necessary if large displacements (more than, say, 25% of the arch span) are involved. Since practical designs in civil engineering would result in arches that fall either in the small or intermediate displacement problem type, the updated procedure is not presented in this chapter. However, it is given in Appendix A together with the derivation of the "initial strain stiffness matrix" , $[k_{\epsilon_o}]$, which is needed by the procedure.

3.2.2 NEWTON-RAPHSON METHOD FOR FIXED COORDINATES

In this case the geometry of the structure is not updated. The steps of the calculation are as follows :

1) Given the current state :

displacement $\{Q\} = \{\bar{Q}\}$

resistance $\{R\} = \{\bar{R}\} = [\ K_S\ ]\ \{\bar{Q}\}$

load $\{P\} = \{\bar{P}\}$

2) Check

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2) Check if the intended total load has been applied. If it has, stop. Otherwise, increase load to {P}.

3) Form the structural tangent stiffness matrix, $[K_T]$, as :

$$[\, K_T \,] = [\, K \,] + [\, N1\,(\{\bar{Q}\})\,] + [\, N2\,(\{\bar{Q}\})\,]$$

4) Solve for $\{\, \Delta Q \,\}$ from the linear equations,

$$[\, K_T \,]\,\{\, \Delta Q \,\} = \{\, \Delta R \,\} = \{\, P \,\} - \{\, \bar{R} \,\}$$

5) Add $\{\Delta Q\}$ to the latest $\{\bar{Q}\}$ to obtain a new $\{Q\} = \{\bar{Q}\} + \{\Delta Q\}$.

6) Based on the new {Q} from step 5, evaluate N1({Q}) and N2({Q}).

7) Form the tangent stiffness matrix $[K_T]$, secant stiffness matrix $[K_S]$, and resistant force vector as :

$$[\, K_T \,] = [\, K \,] + [\, N1(\{Q\})\,] + [\, N2(\{Q\})\,]$$

$$[\, K_S \,] = [\, K \,] + \frac{1}{2}\,[\, N1(\{Q\})\,] + \frac{1}{3}\,[\, N2(\{Q\})\,]$$

Resistant force vector $\{\, R \,\} = [\, K_S \,]\,\{\, Q \,\}$

8) Evaluate the unbalanced force vector $\{\, \Delta R \,\}$ as :

$$\{\, \Delta R \,\} = \{\, P \,\} - \{\, R \,\}$$

9) If the unbalanced force vector, $\{\, \Delta R \,\}$, is sufficiently small, return to 2 . Otherwise , set $\{\, \bar{Q} \,\} = \{\, Q \,\}$, and $\{\, \bar{R} \,\} = \{\, R \,\}$, and return to 3.

3.2.3 CONVERGENCE CRITERION

In implementing the above Newton-Raphson method, a convergence criterion based on unbalanced force vector has been used. A tolerance ϵ_f , which has the unit of force or moment, is prescribed for each group of components (i.e., force or moment) of the unbalanced force vector.

After the evaluation of the unbalanced force vector in each iteration the absolute value of each component of the vector is independently compared with the prescribed tolerance. Convergence is considered achieved if, for each component, this absolute value is less than or equal to the tolerance.

3.2.4 STRESS COMPUTATION

Referring to the calculation steps presented in Section 3.2.2 , element end forces and stresses at every load increment can be obtained at the beginning of step 9 as follows :

a) When the unbalanced force vector , $\{\Delta R\}$, is sufficiently small, form the current element displacement vector, $\{q\}$, and the current element secant stiffness matrix, $[k_S]$, as

$$[\, k_S \,] = [\, k \,] + \frac{1}{2} [\, n1(\{q\}) \,] + \frac{1}{3} [\, n2(\{q\}) \,]$$

b) Element end forces, $\{ r \} = [k_S] \{ q \}$

c) Letting P , M_{xx} , M_{yy} , c_x , and c_y denote , respectively, the axial force, the moment about x-axis, the moment about y-axis, the distance from the y-axis to the extreme fiber , and the

distance from the x-axis to the extreme fiber, the stress at any element end, σ, can be computed as

$$\sigma = \frac{P}{A} + \frac{M_{yy}\, c_x}{I_{yy}} + \frac{M_{xx}\, c_y}{I_{xx}}$$

3.3 COMPUTER PROGRAM

In this section, a general description of the computer program developed for this study is presented. The program was named **NANCURVE**, which stands for Nonlinear Analysis of Curved Beam Structures. As discussed previously, the program has a capability of solving nonlinear equilibrium problem of arbitrary curved beam structures either in two or three dimensional space. The program can also be used for solving linear equilibrium problems when the [N1] and [N2] matrices are set equal to zero. The program itself and the corresponding input data example are given in Appendix C. The major steps in the program are described in the same order in which they are executed :

1) The basic information concerning the physical description of the arch is input. This information includes the number of elements, the number of nodal points, and the type of arch.
2) Parameters which specify whether both [N1] and [N2] are to be used, or [N1] only, or neither of them in the solution of linear equilibrium problems, are then input.
3) The global coordinates of the nodes are input with the parameters defining the boundary conditions of the arch.

4) Next, the maximum number of iterations, the unbalanced force tolerance, the applied loads (initial, increment, and total loads) and their orientations are specified.

5) The element data is input. This includes the element number, the node numbers at element ends, the modulus of elasticity, the shear modulus, the cross sectional area, the moments of inertia about the two principal axes, and the torsion constant of the cross section.

6) From the information input in 1 and 3, the slopes of the tangent at the end nodes of each element are calculated. For arbitrary arch type, however, these slopes are to be input (for convenience, these are input in step 3 together with the global coordinates of the nodes). Next, the coefficients b_1 and b_2 for defining the geometry of each curved element are computed. The radius of curvature at each node and the element lengths are then evaluated.

7) The number of Gauss points, which is needed in the numerical integration for evaluating the linear stiffness matrix, [k], is specified. The numbers of Gauss points available in the program are 2, 3, 4, 5, 6, 10, and 15.

8) From the information input in 1 and 3 also, the semi bandwidth of the structural stiffness matrix is computed. The element linear stiffness matrices are then evaluated and assembled into the linear stiffness matrix of the structure. This matrix is assembled in banded format and due to symmetry only the upper semibandwidth is constructed.

9) Based on the initial applied load input in 4, a linear analysis of the arch is performed to obtain the displacements of the nodal points . The displacements so determined are used to compute, for each element, the matrices [n1] and [n2] which are then assembled (also in banded format) into the structure incremental stiffness matrices [N1] and [N2].

10) The rest of the steps , which are given previously in section 3.2.2 , can then be performed to obtain the nonlinear response of the arch . In performing those steps , possible instability along the solution path is tested by checking the determinant of the tangent stiffness matrix, $[K_T]$, at every load increment.

11) Load-displacement relations can then be computed and the critical or limit load of the structure can also be determined.

12) By using the end displacements and the secant stiffness matrix of each element obtained in 10, the element end forces at every load increment can be computed. Finally , the stresses (axial and total) can be evaluated.

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CHAPTER IV

NUMERICAL RESULTS

4.1 GENERAL

In this chapter a number of numerical examples of inplane and out-of-plane behavior of arches are considered. Firstly, a comparison of the finite element solutions of linear equilibrium problems was made with analytical solutions to show that the method presented is also reliable for linear case.

When solving nonlinear load-displacement problems, we divide the problems into "small", "intermediate", and "large" displacement categories. This is a relative classification. What we mean by a "small displacement" problem is the case in which the deflection is less than about 2% of the arch span. "Intermediate displacement" problem means the deflection is of the order of 2 - 25% of the arch span. Beyond 25% the problem is called "large displacement" problem.

Numerical results were obtained involving arches with inplane and out-of-plane behavior. Various types of loading, support condition, arch type, and arch geometry are considered. To provide some insight into the effects of variations in the arch profile on its nonlinear response, semi-elliptic, circular, parabolic, and sinusoidal arches having the same rise to span ratio were considered. The influence of the number of elements on the accuracy of the results was also investigated.

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In addition to the displacement response, the response of stresses in the structure is also investigated. Furthermore, amplification factors for displacement and stresses, which are defined to be the ratios of nonlinear response to linear response, are studied.

For the numerical results presented herein, no units of the data have been given. The dimensions of the various quantities are self-consistent; i.e., if the basic units of length and force are taken to be inches and pounds, then the values of area, moment of inertia, concentrated load, and distributed load given would have units of $in.^2$, $in.^4$, lb., and lb/in., respectively.

4.2 LINEAR EQUILIBRIUM PROBLEMS

Two types of problems were solved. They are linear equilibrium problems for arches subject to a concentrated inplane load and out-of-plane load at the crown.

4.2.1 CONCENTRATED INPLANE LOAD AT CROWN

The solution was obtained for two types of arches, circular and parabolic. In both cases the symmetry of the load and of the structure were used to reduce the number of equations.

The first problem investigated was linear analysis of semi-circular arch subjected to a concentrated inplane load at crown. Figure 4-1 shows the difference between the computed radial displacement at the crown and the analytical solution (Ref. 22), for different numbers of elements. The numerical data are given in Table 4-1.

The data indicated that the differences with the analytical solution decrease rapidly with increase in the number of elements. However, the convergence is seen to be somewhat oscillating within a very small range of error, i.e.: approximately ± 0.25%, when the number of ele

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number of elements is more than five. Such behavior would seem to be the result of the round-off errors accumulated from the increasing amount of computation as the number of elements was increased. For shallower arches, the convergence is better and is illustrated in the following example.

Figure 4-2 and Table 4-2 show similar pattern of results for a parabolic arch subjected to a concentrated inplane load at crown. The convergence here is much faster than before. By using only one element, the error is only about 3%. The errors decrease rapidly from 1.4% for two elements to 0.03% for five elements. When the number of elements is more than five, the convergence is still oscillating but the range of errors is very small, i.e.: ± 0.04%.

4.2.2 CONCENTRATED OUT-OF-PLANE (TRANSVERSE) LOAD AT CROWN

The solution was obtained for the semi-circular arch subjected to a concentrated out-of-plane load at crown. The arch properties remain the same as described in Figure 4-1. Figure 4-3 shows the difference between the computed lateral displacement at the crown and the analytical solution, for different number of elements. By using only 2 elements, the difference is 0.53%. The difference tends to decrease when the number of elements is increased. The numerical data are shown in Table 4-3.

4.3 NONLINEAR LOAD-DISPLACEMENT BEHAVIOR FOR SMALL DISPLACEMENT PROBLEMS

As mentioned previously, the small displacement problem is a class of problems where the deflection is less than about 2% of the arch span. Both inplane and out-of-plane problems were considered. The influence of

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It should be noted that, if only the maximum load carrying capacity is needed, a short-cut procedure may be formulated in terms of an eigenvalue problem. Studies on this subject can be found in References (22) and (45).

The numerical results presented in this section agree very well with those obtained from eigenvalue solutions discussed in References (22) and (45).

4.3.1 INPLANE PROBLEMS

4.3.1.1 A 90^{o}-HINGED CIRCULAR ARCH SUBJECTED TO UNIFORM RADIAL LOAD

The geometry, physical properties and loading condition for this problem are shown in Figure 4-4. It is well known that this type of problem has a buckling mode which is antisymmetry or exhibits sidesway. In order to obtain such mode, a small horizontal perturbing load equal to 1% of P applied at the crown in +X direction has been introduced (P is equal to 5.8905 q).

The resulting load-displacement curves for different number of elements are depicted in Figure 4-4. The results were obtained with a load increment of 20 (equivalent to q = 3.395) and an unbalanced force tolerance ϵ_f= 1% of the load increment. Different load incremenets (i.e.: 10, 40, and 50) were also used to solve the problem. It was observed that the results were not sensitive to the load increment used. However, near the critical load level, where the displacement increases rapidly, smaller load increments are needed to get enough data points for drawing the load-displacement curve. It can be seen from Figure 4-4 that the left quarter point gradually deflects inward until bucklin

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until buckling occurs, while the right quarter point initially deflects inward and when the intensity of the applied load is about 2/3 of the critical load it deflects outward until buckling takes place. Thus the final mode is antisymmetric. As expected, the critical load obtained from nonlinear analysis agrees very well with that obtained from eigenvalue solution reported in References (22) and (47). It is also seen that although the behavior is quite nonlinear, the displacements are small (i.e.: of the order of 0.4% of the arch span).

4.3.1.2 A HINGED PARABOLIC ARCH SUBJECTED TO UNIFORM LOAD ON HORIZONTAL PROJECTION

The problem considered is illustrated in Figure 4-5. As before, in order to obtain an antisymmetrical buckling mode, a small horizontal perturbing load equal to 1% of P (P = 56.25 q) was applied at the crown in +X direction.

The resulting load-displacement curves for different number of elements are shown in Figure 4-5. The results were obtained with a load increment of 200 (equivalent to q = 3.556) and an unbalanced force tolerance ϵ_f= 1% of the load increment. The left quarter point gradually deflects inward until buckling occurs, while the right one initially deflects inward. When the applied load intensity is about 84% of the critical load it deflects outward until buckling takes place. The critical load obtained agrees very well with that of analytical value discussed in Timoshenko's book (39).

4.3.2 OUT-OF-PLANE PROBLEMS

4.3.2.1 A 90^o-HINGED CIRCULAR ARCH SUBJECTED TO UNIFORM RADIAL LOAD

Figure 4-6 shows the load-structural system of the problem. This problem has an out-of-plane buckling mode which is symmetric. To obtain such mode, a

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such mode, a small lateral perturbing load equal to 0.1% of P (P = 5.8905 q) applied at the crown in +Z direction was introduced. Without it the resulting response would be inplane behavior. In this case the rotation degrees of freedom about the x-axis and z-axis at the supports were restrained. Because of the symmetry of the geometry, loading, and buckling mode, only one half of the arch needs be considered.

Figure 4-6 also shows the resulting load-displacement curves for different number of elements. The order of the maximum deflection is approximately 0.02% of the arch span. At this level of displacement, the corresponding load asymptotically approaches the buckling load of the arch obtained numerically by Wen and Lange (45) based on an eigenvalue solution (in the eigenvalue solution, the small initial perturbing load was not needed). The responses were obtained with a load increment of 10 (equivalent to q = 1.698) and an unbalanced force tolerance ϵ_f= 1% of the load increment.

4.3.2.2 A HINGED PARABOLIC ARCH SUBJECTED TO UNIFORM LOAD ON HORIZONTAL PROJECTION

The system considered is identical to that shown previously in Figure 4-5. As in the previous case, a small lateral perturbing load equal to 0.1% of P (P = 56.25 q) applied at the crown in +Z direction was introduced. Because of the symmetry of the geometry, loading, and buckling mode, only one half of the arch was considered. For different number of elements, the resulting load-displacement curves are shown in Figure 4-7. The results were obtained with a load increment of 20 (equivalent to q = 0.3556) and an unbalanced force tolerance ϵ_f= 1.0% of the load increment. As in the previous case, the load-displacement curves indicated ultimate loads very close to the buckling loads obtained by

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4.4 NONLINEAR LOAD-DISPLACEMENT BEHAVIOR FOR INTERMEDIATE DISPLACEMENT PROBLEMS

As defined previously, the intermediate displacement problem is a class of problems where the deflection is of the order of 2 - 25% of the arch span. It should be noted that the common practical proportions of arches fall either in this category or in the previous one, i.e.: the small displacement problem.

The numerical examples presented in this section were chosen because they had been solved by other investigators using various different methods of nonlinear analysis. Thus, a comparison can then be made to examine the accuracy of the proposed method. The example problems chosen also include a range of rise to span ratios covering what may be regarded as "shallow" as well as "deep" arches. The influence of the number of elements on the accuracy of the results was also investigated.

4.4.1 A 28^{o}-CLAMPED CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

The problem, which is illustrated in Figure 4-8, falls into the type of shallow arch (5). Because this arch remains stable, an incremental load procedure described in the proposed method could still be used to determine the entire response.

The load-displacement curves shown in Figure 4-8 were obtained with a load increment of 2000 and ϵ_f= 0.5% of the load increment. The loading was continued until the apex had displaced an amount equal to approximately 1.5 times the initial rise or 9.5% of the arch span. The configuration

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configuration remains symmetric about the apex throughout deformation. Different numbers of element, i.e.: 2, 4, and 8, to represent the one half of the arch were considered.

It can be seen in Figure 4-8 that even when only two elements were used to represent the one half of the arch, the resulting load-displacement curve was close enough to that obtained by Belytschko and Glaum (5) with 10 elements, which is extremely close to an analytical solution which may be considered to be "exact" (5). When 4 or 8 elements were used, the results were of course better, as also shown in the figure.

Figure 4-8 also shows the resulting load-displacement curves obtained by Belytschko and Glaum (5) with 2 and 5 elements. These results, however, are less accurate than that obtained by the proposed method with 2 elements.

4.4.2 A 60°-CLAMPED CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

The problem is illustrated in Figure 4-9. It was solved with a load increment of 50 and ϵ_f= 1% of the load increment. Near the critical load level, where the displacement increases rapidly, the load increment was halved to get more data points for drawing the load displacement curve.

Different number of elements, i.e.: 2, 4, 8, and 16, to represent the one half of the arch were considered. The results are given in Figure 4-9. As shown in the figure, the resulting load-displacement curves for 4, 8, and 16 elements have no significant differences. For 2 elements the curve is somewhat stiffer but is still quite accurate. Those curves agree very well with that obtained by Calhoun and DaDeppo

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(8). The critical load obtained and the corresponding crown displacement also agree very well with those obtained analytically by DaDeppo and Schmidt (11) and Austin and Ross (2) who used 24 elements for their eigensolution. In this problem, the order of the maximum deflection is approximately 5% of the arch span.

4.4.3 A 60°-CLAMPED CIRCULAR ARCH SUBJECTED TO A SKEW CONCENTRATED LOAD AT CROWN

The problem, which is illustrated in Figure 4-10, was solved with the same load increment and tolerance ϵ_f as in the previous example. Because of the present of the horizontal load, the entire arch was considered to obtain the response.

The resulting load-displacement curves for different numbers of element, i.e.: 4, 8, and 16, are shown in Figure 4-10. All of the curves agree very well with that obtained by Calhoun and DaDeppo (8). The resulting critical loads also agree well with that obtained by DaDeppo and Schmidt (11). The maximum deflection is of the order of 4.25% of the arch span. The buckling mode is antisymmetry.

4.4.4 A CLAMPED MULTIPLE RADII CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

This problem demonstrates the extended capability of the proposed method and the computer program for solving an arbitrary arch profile. As discussed in Section 3.3, the only geometric data which have to be input are the nodal coordinates and the end slopes of the elements.

The arch has two different radii, R_1= 200 and R_2= 100. The two radii have a common tangent point at the crown of the arch. A vertical concentrated load is applied at the crown, as shown in Figure 4-11.

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Since the geometry is not symmetry, the entire arch should be considered. Two different numbers of element, i.e.: 4 and 8, were used to solve the problem. A load increment of 50 and a tolerance of 2% of the load increment were used. The resulting load-displacement curves are shown in Figure 4-11. Both curves agree well with that obtained by Calhoun and DaDeppo (8). The resulting buckling mode is symmetric. The maximum deflection is of the order of 6.75% of the arch span.

4.4.5 A HINGED SEMI-CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN.

The problem is illustrated in Figure 4-12. In this example, its symmetrical response was analysed. The problem was solved by the proposed method with a load increment of 1 and $\epsilon_f = 1\%$ of the load increment. The resulting load-displacement curves for 4 and 8 elements representing the one half of the arch are shown in Figure 4-12. As can be seen in the figure, the results agree very well with that obtained by using 16 straight beam elements of Wen and Rahimzadeh (47). The order of the displacement is approximately 25% of the arch span.

4.4.6 A CLAMPED SEMI-CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

The problem is illustrated in Figure 4-13. Two thickness ratios of the arch were considered, namely $h/R = 0.05$ and 0.005. The problem was solved by the proposed method with a load increment of $0.384\ EI/R^2$. The resulting load-displacement curves for 3, 4 and 8 elements (equivalent to 7, 10, and 22 degrees of freedom, respectively) representing the one half of the arch are shown in Figure 4-13. The results agr

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results agree very well with that obtained by using 6 elements (equivalent to 37 degrees of freedom) of Noor et al (27) based on a mixed formulation of finite element. The order of the displacement is approximately 20% of the arch span.

4.4.7 ARCHES WITH DIFFERENT PROFILES

To provide further comparisons of results with existing solutions, arches with semi-elliptic, circular, parabolic, and sinusoidal profiles having the same rise to span ratio, span, and cross sectional properties were considered. All arches are hinged supported at their both ends and are subjected to a concentrated vertical load at crown. The symmetrical buckling of those arches were analyzed by using different numbers of element. All problems were solved with a load increment of 100 and an unbalanced force tolerance $\epsilon_f = 1\%$ of the load increment. The load increment was halved near the critical load level, where the displacement increases rapidly.

The profiles of a "rectangular frame" and a "triangular frame" may be regarded as the limiting cases of the arches mentioned above. The symmetrical responses of these frames were also investigated by using straight beam elemnts of Wen and Rahimzadeh (47). The profiles of these structures are shown in Figure 4-14.

Figure 4-15 shows the load-displacement curves. From the figure it is seen that the stiffness of the sinusoidal arch is higher than that of the parabolic arch; the stiffness of the parabolic arch is higher than that of the circular arch, which is in turn higher than that of the semi-elliptic arch. Furthermore, the stiffness of the "triangular frame" is higher than that of the sinusoidal arch, and the stiffness of

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stiffness of the "rectangular frame" is lower than that of the semi-elliptic arch. Thus it appears that for this type of loading, the shorter the total curved length of the structure, the greater the stiffness. The reason seems to be that shorter length implies a greater proportion of the load being carried by axial force than by bending, and thus greater stiffness.

4.4.7.1 **SEMI-ELLIPTIC ARCH**

The problem has been solved by using 6 elements (38 degrees of freedom) of A.K. Noor et al (27) based on a mixed formulation of finite element. The problem was also solved by using 8 straight beam elements (23 degrees of freedom) of Wen and Rahimzadeh (47) representing the one half of the arch.

As shown in Figure 4-15, the resulting load-displacement curve obtained by proposed method agrees very well with those of other methods. The problem was solved by using 4 elements (11 degrees of freedom). The maximum deflection was of the order of 12.5% of the arch span.

4.4.7.2 **CIRCULAR ARCH**

Using different approaches, the problem had been solved by Huddleston (20) analytically, and Fujii and Gong (17). The latter used 20 elements to represent the one half of the arch.

For different numbers of element, i.e.: 2, 4, and 8, the problem was solved by the proposed method. The resulting load-displacement curves agree very well with those of other methods, as shown in Figure 4-15. Moreover, the curves seem to asymptotically approach the buckling load obtained by Austin and Ross (2). The maximum deflection is of the order of 12.5% of the arch span.

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4.4.7.3 PARABOLIC ARCH

The problem was analysed by the proposed method with 2, 3, 4, and 6 elements representing the one half of the arch. For 2 elements, the behavior seems to be too stiff. For 3, 4, and 6 elements, the curves are very closed to each other, and they agree very well with that obtained by Fujii and Gong (17) using 20 elements. As before, the resulting critical load and the corresponding deflection also agree with that obtained by Austin and Ross (2). The deflection is of the order of 12.5% of the arch span.

4.4.7.4 SINUSOIDAL ARCH

The problem has been solved by Fujii and Gong (17) with 20 elements. The result is plotted in Figure 4-15. The figure also shows the resulting load-displacement curve obtained by the proposed method with 4 elements. The order of the deflection is about 12% of the arch span. As can be seen in Figure 4-15, both curves are quite close to each other.

4.5 STRESSES AND AMPLIFICATION FACTORS

In practice, especially in the preliminary design stage, it is common that a simpler method so called "amplification factor method" is used to estimate nonlinear response of structure from its linear response. Therefore, information regarding the amplification factor, which is defined to be the ratio of the nonlinear response to the linear response, is very useful.

In this section, the response of internal stresses in the structure is first investigated. Two types of stresses, i.e., axial

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stress and total stress (axial stress plus bending stress) are considered. The results are then used to obtain the amplification factor for stresses. Similarly, by using the displacement responses the amplification factor for displacement can also be obtained.

As numerical examples, the resulting stresses and amplification factors of the problems discussed previously in Sections 4.3.1.2 , 4.4.1 , and 4.4.2 are presented.

Figure 4.16 shows the axial and total stresses at the left quarter point of a hinged parabolic arch subjected to uniform load on horizontal projection. Both linear and nonlinear responses are presented. The figure also shows the amplification factors for axial stress, total stress, and displacement. From the figure it is seen that the displacement amplification factor is larger than the total stress amplification factor. Near the critical load level, the magnitude of the displacement and total stress amplification factors are, respectively, 2.3 and 1.3 . The results were obtained by using 8 elements representing the entire arch.

Figure 4-17 shows the axial force at the crown of a 28° clamped shallow circular arch subjected to a vertical concentrated load at crown. In this problem, the axial force initially increases when the load is increased. However, when the intensity of the load is about 27,000. (i.e., when the crown vertical displacement is about the same as the initial rise of the arch), the axial force decreases. The problem was solved by using 8 elements representing the one half of the arch. The result agrees very well with that obtained by Belytschko and Glaum with 10 elements.

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The linear and nonlinear responses of the axial and total stresses are shown in Figure 4-18. The figure also shows the amplification factors for axial stress, total stress, and displacement. The amplification factor for axial stress initially increases when the load is increased. When the load intensity is about 27,000. the amplification factor decreases. When the load intensities are 30,000. and larger than 30,000. the magnitude of the axial force amplification factors are, respectively, equal to one and less than one. From the figure it is seen that the total stress and displacemet amplification factors initially increase when the load is increased. When the intensity of the load is about 30,000. (i.e., when the axial stress amplification factor is equal to one), the amplification factors decrease. As in the previous problem, the displacement amplification factor is larger than the total stress amplification factor. The maximum magnitude of the displacement and total stress amplification factors are, respectively, 3.9 and 2.2 .

Figure 4-19 shows the linear and nonlinear total stress at the crown of a 60^{o} clamped circular arch subjected to a vertical concentrated load at crown. The figure also shows the amplification factors for axial stress, total stress, and displacements. The results were obtained by using 8 elements representing the one half of the arch. As in the previous cases, the displacement amplification factor is larger than that of total stress. Near the critical load level, the magnitude of the displacement and total stress amplification factors are, respectively, 3.8 and 2.0 .

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CHAPTER V

DISCUSSION AND CONCLUSION

5.1 DISCUSSION

In the preceding chapters the development of a three dimensional nonlinear curved beam element and its applications to linear and nonlinear analyses of arches with various geometry in two and three dimensional space have been presented. From the numerical results obtained, the features of the proposed method are discussed in the following sections.

5.1.1 COMPARISON WITH PREVIOUS WORKS

Previous comparisons, as given in Figures 4-8, 4-9, 4-10, 4-11, 4-12 , 4-13 and 4-15 , indicate that the proposed "Averaged Axial Strain" model competes very well with the other models.

a) In Figure 4-8 it is shown that by using only 2 elements with the proposed method , the accuracy of the result is comparable to that obtained by Belytschko and Glaum (5) with 10 elements, the result of which is very close to an analytical solution. The results of Belytschko and Glaum using 2 and 5 elements indicated considerable differences from the correct results.

b) Figure 4-9 indicates that using only 2 elements with the proposed method , the result is as accurate as that obtained by Calhoun and DaDeppo (8) using 8 elements for the converged answe

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c) For the skew loading problem , as shown in Figure 4-10 , the result obtained by the proposed method with 4 elements is comparable to that of Calhoun and DaDeppo (8) with 8 elements . The resulting limit load also agrees very well with the analytical solution.

d) For the multiple radii problem, as shown in Figure 4-11 , the result obtained by the proposed method using 4 elements is in very good agreement with the converged result of Calhoun and DaDeppo (8) using 8 elements.

e) Figure 4-12 indicates that , for the hinged semi-circular arch problem, the result using 4 elements of the proposed method is very close to that of Reference 47 using 16 straight beam elements.

f) Figure 4-13 indicates that , for the clamped semi-circular arch problem , the result using 3 elements (7 degrees of freedom) of the proposed method is in very good agreement with that of Noor et al (27) with 6 elements (37 degrees of freedom).

g) Figure 4-15 indicates that, for parabolic and circular arches, the result using 2 elements of the proposed method is close to that of Fujii and Gong (17) with 20 elements . For sinusoidal arch , 4 elements of the proposed method gives the result comparable to that for 20 elements of Fujii and Gong . For the semi-elliptic arch, 4 elements (11 degrees of freedom) of the proposed method gives the result comparable to that for 6 eleme

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elements (38 degrees of freedom) of Noor et al (27) as well as 8 straight beam elements (23 degrees of freedom) of Ref. 47.

In terms of number of elements needed for accurate results, the preceding comparisons show that the proposed method is more effective than the others. Among the compared results, those related to the clamped semi-circular and elliptic arches of Reference 27 showed the closest competition . However, the results are still in favor of the proposed method . Furthermore , as discussed previously , the formulation in Reference 27 is not as conveniently adaptable for a general structures computer program as the proposed method, and the number of degrees of freedom per end node of the element in that reference is actually twice that in the proposed one.

5.1.2 APPROACHES OF NONLINEAR ELASTIC ANALYSIS

It is generally known that curved beam elements has the tendency to be too stiff unless the in-plane displacement field is represented by sufficiently high order polynomials. This phenomenon is called **membrane locking** (37).

To overcome this problem, four approaches have been suggested :

a) To use higher order polynomials for the displacement fields. This approach was taken by Dawe (16) , using quintic functions for <u>linear analysis</u>.

b) To use a mixed formulation of the finite element , as in the work of Noor et al (27).

c) To use "reduced integration" , as suggested by Stolarski and Belytschko (37).

d) To use the "average axial strain" model, as described herein.

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Using higher order (e.g. quintic) polynomials in a nonlinear analysis would be much more unwieldy than in a linear one. It would involve a large amount of work with no guarantee of success. (Perhaps, that is the reason why it had not been tried thus far.) In approach (b), as mentioned previously, since both nodal displacements and forces are considered as degrees of freedom, matrices of larger size are involved and the formulation is inconvenient for inclusion in a general structures program. In approach (c), the numerical integration is carried out by using only 1 or 2 Gauss points, rather than an accurate evaluation of the integral as defined by the analysis. Thus there is a "mathematical looseness" or computational artificiality involved, which does not appear desirable. Furthermore, the possibility of existence of zero energy mode should be of concern.

The present study indicates that approach (d) overcomes all the above difficulties. The procedure is simpler and more efficient than the other approaches, the integration is carried out as it should be, and the accuracy of the results is generally better.

5.1.3 NATURE OF [n1] AND [n2] MATRICES

The reason why the averaged strain model produces more accurate results than the unaveraged strain model appears to be the fact that the averaging process reduces the strain energy and thus decreases the stiffness to the correct order of magnitude. An analysis is presented in the following. A similar analysis was also given in Reference 47 for a nonlinear **straight** beam element.

Consider a two dimensional curved beam element with the nonlinear strain term , $\frac{1}{2}\left(\frac{u_{\gamma}}{R\,\theta} + \frac{w}{R}\right)^2$, being unaveraged in one case and averaged

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The expressions of U_3 for the unaveraged and averaged strain models may be rewritten respectively from Equations (2-25) and (2-36) as follows :

$$U_{3\,(\text{unaveraged})} = \frac{E\,A}{2} \int_0^1 \left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right) \left[\left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 \right] R\,\theta\, d\gamma \qquad \ldots(5\text{-}1)$$

$$U_{3\,(\text{averaged})} = \frac{E\,A}{2} \int_0^1 \left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right) \left[\frac{1}{L} \int_0^1 \left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 R\,\theta\, d\gamma \right] R\,\theta\, d\gamma \qquad \ldots(5\text{-}2)$$

Consider the quantity $\left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right)$. It represents the linear part of the axial strain in the element. Experience shows that it is slowly varying. (In fact, a linear element based on setting it equal to constant was shown to be very effective (1)). Therefore, the quantity may be taken as a constant. Consequently, $U_{3\,(\text{averaged})}$ (Eq.5-2) may be written as follows (noting $\int_0^1 R\,\theta\, d\gamma = L$) :

$$\frac{E\,A}{2} \int_0^1 \left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right) \left[\frac{1}{L} \int_0^1 \left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 R\,\theta\, d\gamma \right] R\,\theta\, d\gamma =$$

$$\frac{E\,A}{2} \int_0^1 \left(\frac{w_\gamma}{R\,\theta} - \frac{u}{R} \right) \left[\left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 \right] R\,\theta\, d\gamma$$

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The expressions of U_4 for the unaveraged and averaged models can be rewritten from Equations (2-26) and (2-37) as follows :

$$U_{4(\text{unaveraged})} = \frac{E\,A}{8} \int_0^1 \left[\left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 \right]^2 R\,\theta\, d\gamma \qquad \ldots(5\text{-}3)$$

$$U_{4(\text{averaged})} = \frac{E\,A}{8} \int_0^1 \left[\frac{1}{L} \int_0^1 \left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 R\,\theta\, d\gamma \right]^2 R\,\theta\, d\gamma \qquad \ldots(5\text{-}4)$$

Let $\left(\frac{E\,A}{8} \right)^{1/2} \left(\frac{u_\gamma}{R\,\theta} + \frac{w}{R} \right)^2 = f(\gamma)$, and note that $f(\gamma)$ is not constant. Consequently,

$$\left[\frac{1}{L} \int_0^1 f(\gamma)\, R\,\theta\, d\gamma \right]^2 < \frac{1}{L} \int_0^1 [\, f(\gamma)\,]^2\, R\,\theta\, d\gamma \qquad \ldots(5\text{-}5)$$

in which the left term is U_4 for Equation (5-4) and the right term for Equation (5-3). The preceding follows from the fact that, for $f(\gamma) \neq$ constant, the mean (square root of the left side) is always less than the root-mean-square (square root of the right side) (50). It follows from the relationship between the displacement formulation of the finite element method and the principle of potential energy that the nonlinear stiffness as represented by [n2] is lower for the model of Equation (5-4) than that of Equation (5-3). The above observation is illustrated by the following numerical example.

Consider a 2-dimensional curved beam element (circular) having the following properties :

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$A = 2.0$; $L = 3.06960000$

$I = 0.6667$; $\theta = 0.03069600$

It undergoes the following displacements :

$u_A = 0.01803063$; $u_B = 0.08288997$

$w_A = -0.00979038$; $w_B = -0.01899512$

$\theta_{yA} = 0.01269580$; $\theta_{yB} = 0.02964475$

(The above data is taken from the solution of the problem discussed previously in Section 4.4.1 at the applied load level of P = 20,000.)

The resulting entries of [nl] matrix for both averaged and unaveraged models are, respectively, given below :

$$[nl]_{av.} = \begin{bmatrix} -22860.2 & 136863.9 & -5419.8 & 27049.9 & -136097.9 & -8430.3 \\ & -4195.8 & 27966.6 & -136928.2 & -6.4 & -28492.6 \\ & & -27881.8 & 6275.6 & -27787.1 & 6636.3 \\ & \text{symmetry} & & -31239.7 & 136033.6 & 7551.9 \\ & & & & 4183.0 & 28737.9 \\ & & & & & -27870.2 \end{bmatrix}$$

$$[nl]_{un.} = \begin{bmatrix} -23147.7 & 136863.5 & -6895.5 & 27337.4 & -136093.9 & -6450.7 \\ & -4195.7 & 27972.9 & -136928.0 & -6.4 & -28508.0 \\ & & -26706.6 & 7751.2 & -27775.9 & 7022.8 \\ & \text{symmetry} & & -31527.2 & 136029.6 & 5572.2 \\ & & & & 4183.0 & 28730.9 \\ & & & & & -29339.1 \end{bmatrix}$$

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$[n2]_{av.} =$

$[n2]_{un.} =$

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$[\ k\ ] =$

The resulting entries of [n2] matrix for both averaged and unaveraged models are, respectively, given below :

$$[n2]_{av.} = \begin{bmatrix} 4674.60 & -71.745 & 1048.38 & -4674.600 & -71.745 & -135.16 \\ & 1.101 & -16.09 & 71.745 & 1.101 & 2.07 \\ & & 2019.75 & -1048.380 & -16.090 & -597.57 \\ & \text{symmetry} & & 4674.600 & 71.745 & 135.16 \\ & & & & 1.101 & 2.07 \\ & & & & & 2026.16 \end{bmatrix}$$

$$[n2]_{un.} = \begin{bmatrix} 5217.50 & -80.100 & 2280.30 & -5217.500 & -80.100 & -138.66 \\ & 1.229 & -35.00 & 80.077 & 1.229 & 2.12 \\ & & 3655.20 & -2280.300 & -35.000 & -1621.70 \\ & \text{symmetry} & & 5217.500 & 80.100 & 138.66 \\ & & & & 1.229 & 2.12 \\ & & & & & 8037.80 \end{bmatrix}$$

The entries of the linear stiffness matrix, [k], of the element are

$$[\ k\ ] = \begin{bmatrix} 2767012. & 57544. & 4245573. & -2763943. & -142440. & 4244004. \\ & 6515136. & -13998. & 142440. & -6513833. & -116310. \\ & & 8687758. & -4244003. & -116310. & 4343067. \\ & \text{symmetry} & & 2767012. & -57544. & -4245573. \\ & & & & 6515135. & -13998. \\ & & & & & 8687759. \end{bmatrix}$$

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[k] =

[n1] =

It may be observed also that [n1] tends to **decrease** the total stiffness. It represents the effect of membrane flexure coupling. The [n2] matrix tends to **increase** the total stiffness.

For further comparison, the resulting entries of [k], [n1], and [n2] matrices for nonlinear straight beam element (Ref. 47), which has the same cross sectional properties, length, and displacements, are given in the followings :

$$[\,k\,] = \begin{bmatrix} 2766090. & 0. & 4245396. & -2766090. & 0. & 4245396. \\ & 6515507. & 0. & 0. & -6515507. & 0. \\ & & 8687777. & -4245396. & 0. & 4343889. \\ & \text{symmetry} & & 2766090. & 0. & -4245396. \\ & & & & 6515507. & 0. \\ & & & & & 8687777. \end{bmatrix}$$

$$[n1] = \begin{bmatrix} -23445.5 & 137616.8 & -5997.4 & 23445.5 & -137616.8 & -5997.4 \\ & 0.0 & 28166.9 & -137616.8 & 0.0 & -28329.7 \\ & & -24546.0 & 5997.4 & -28166.9 & 6136.5 \\ & \text{symmetry} & & -23445.5 & 137616.8 & 5997.4 \\ & & & & 0.0 & 28329.7 \\ & & & & & -24546.0 \end{bmatrix}$$

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$$[n2] = \begin{bmatrix} 4745.59 & 0.000 & 1065.32 & -4745.590 & 0.000 & -127.96 \\ & 0.000 & 0.00 & 0.000 & 0.000 & 0.00 \\ & & 2047.01 & -1065.320 & 0.000 & -603.78 \\ & \text{symmetry} & & 4745.590 & 0.000 & 127.96 \\ & & & & 0.000 & 0.00 \\ & & & & & 2048.42 \end{bmatrix}$$

It is seen that the differences between $[n1]_{(averaged)}$ and $[n1]_{(unaveraged)}$ are not substantial whereas those between the two versions for [n2] are. A comparison of the stiffness matrices between the curved (averaged model) and straight elements indicates that, for this element with a small value of θ , the matrices are quite similar. This of course can not be expected to be the case when θ is not small. Furthermore, when the magnitude of the radius of curvature of the element is taken to be sufficiently large (and simultaneously the subtending angle is decreased to result in the same element length), all entries of the [k], [n1], and [n2] matrices converge to those for the straight beam element.

5.2 SUMMARY AND CONCLUSION

In this dissertation, a three dimensional nonlinear curved beam element has been developed. It has 12 degrees of freedom in 6 displacements (all "essential") per end node. Thus it can readily be incorporated into a general structural computer program.

The element, which is formulated based on the average axial strain model, is shown to be more accurate, for same number of elements, than all methods compared. Accurate load-displacement curve may be obtained by

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obtained by using at most eight elements to represent the entire arch. For symmetrical problems, only one half of the arch (four elements) need be considered.

The method, which is based on the fixed Lagrangian coordinate system, works very well for small displacement problems (2% or less of the arch span) as well as for intermediate displacement problems (2-25% of the arch span). The solution procedure based on an updated Lagrangian coordinate system is also presented. The procedure is necessary if large displacements (25% or more of the arch span) are involved.

The amplification factor for displacements seems to be always larger than the amplification factor for stresses. This fact and its effects on the amplification factor method, that commonly used in practice, need be studied more thoroughly.

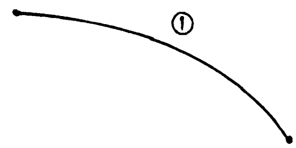
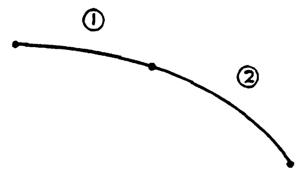
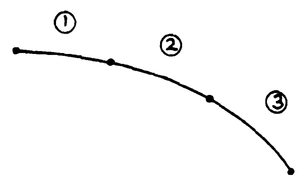
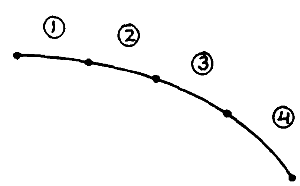
As mentioned previously, the present study is limited to geometric nonlinearity. For many practical problems, when geometric nonlinearity becomes significant, effects of material nonlinearity would become important at the same time. Thus, future studies of nonlinear analysis of curved beam structures should include these effects.

TABLES

TA

NO. OF ELEMENTS
1
2
3
4

TABLE 2-1 ACCURACY OF THE GEOMETRIC REPRESENTATION FOR PARABOLIC ARCH (RISE=9.6", SPAN=48")

NO. OF ELEMENTS	ELEMENT #		APPROX. VALUE	EXACT VALUE	REMARKS
1	1	L	26.3212	26.3575	
		R1	24.1510	30.0000	
		R2	53.8679	63.0067	
2	1	L	12.3115	12.3127	
		R1	28.7201	30.0000	
		R2	35.9910	37.4807	
	2	L	14.0438	14.0447	
		R1	35.2926	37.4807	
		R2	60.1671	63.0067	
3	1	L	8.0937	8.0938	
		R1	29.4501	30.0000	
		R2	32.6650	33.2562	
	2	L	8.6351	8.6352	
		R1	32.5061	33.2562	
		R2	42.7926	43.6711	
	3	L	9.6284	9.6284	
		R1	42.5459	43.6711	
		R2	61.6663	63.0067	
4	1	L	6.0398	6.0398	
		R1	29.6947	30.0000	
		R2	31.4995	31.8178	
	2	L	6.2729	6.2729	
		R1	31.4457	31.8178	
		R2	37.0694	37.4807	
	3	L	6.7153	6.7153	
		R1	36.9778	37.4807	
		R2	47.0094	47.5805	
	4	L	7.3294	7.3294	
		R1	46.9014	47.5805	
		R2	62.2310	63.0067	

TA

NO. OF ELEMENTS
5
6

TABLE 2-1 ACCURACY OF THE GEOMETRIC REPRESENTATION FOR PARABOLIC ARCH (CONTINUED)

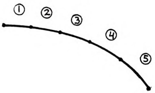
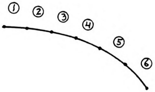
NO. OF ELEMENTS	ELEMENT #		APPROX. VALUE	EXACT VALUE	REMARKS
		L	4.8204	4.8204	
	1	R1	29.8058	30.0000	
		R2	30.9599	31.1593	
		L	4.9410	4.9410	
	2	R1	30.9364	31.1593	
		R2	34.4867	34.7240	
		L	5.1738	5.1738	
5	3	R1	34.4449	34.7240	
		R2	40.6379	40.9440	
		L	5.5048	5.5048	
	4	R1	40.5844	40.9440	
		R2	49.8112	50.2070	
		L	5.9175	5.9175	
	5	R1	49.7525	50.2070	
		R2	62.5036	63.0067	
		L	4.0118	4.0118	
	1	R1	29.8648	30.0000	
		R2	30.6675	30.8035	
		L	4.0820	4.0820	
	2	R1	30.6546	30.8035	
		R2	33.1014	33.2562	
		L	4.2189	4.2189	
	3	R1	33.0797	33.2562	
		R2	37.2916	37.4807	
6		L	4.4163	4.4163	
	4	R1	37.2638	37.4807	
		R2	43.4358	43.6711	
		L	4.6666	4.6666	
	5	R1	43.4046	43.6711	
		R2	51.7889	52.0801	
		L	4.9618	4.9618	
	6	R1	51.7524	52.0801	
		R2	62.6600	63.0067	

TABLE 4-2

NUMBER OF* ELEMENTS
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TABLE 4-2 LINEAR EQUILIBRIUM OF A PARABOLIC ARCH SUBJECTED TO A CONCENTRATED IN-PLANE LOAD AT CROWN

NUMBER OF* ELEMENTS	DISPLACEMENT** AT CROWN (IN. x 10^{-4})	DIFFERENCE*** (%)
1	0.453155953	2.982
2	0.460521696	1.405
3	0.465530466	0.332
4	0.466613128	0.101
5	0.466937636	0.031
6	0.467273494	-0.040
7	0.467232458	-0.031
8	0.467054924	0.006
9	0.467268255	-0.039
10	0.467500213	-0.088

* For one half of the arch

** Analytical solution = 0.467085×10^{-4} in.

*** $\% \text{ Difference} = \dfrac{\text{Analytical - Numerical}}{\text{Analytical}}$

TABLE 4-3 I
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NUMBER OF* ELEMENTS
2
3
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TABLE 4-3 LINEAR EQUILIBRIUM OF A SEMI-CIRCULAR ARCH SUBJECTED TO A CONCENTRATED OUT-OF-PLANE LOAD AT CROWN

NUMBER OF* ELEMENTS	DISPLACEMENT** AT CROWN (IN.)	DIFFERENCE*** (%)
2	2.81090450	0.531
3	2.81890678	0.248
4	2.82110500	0.170
5	2.82034874	0.197
6	2.82473183	0.042
7	2.82618713	-0.009
8	2.82285213	0.108
9	2.81906509	0.242
10	2.82073689	0.183

* For one half of the structure

** Analytical solution = 2.825930 "

*** $\% \text{ Difference} = \dfrac{\text{Analytical - Numerical}}{\text{Analytical}}$

F I G U R E S

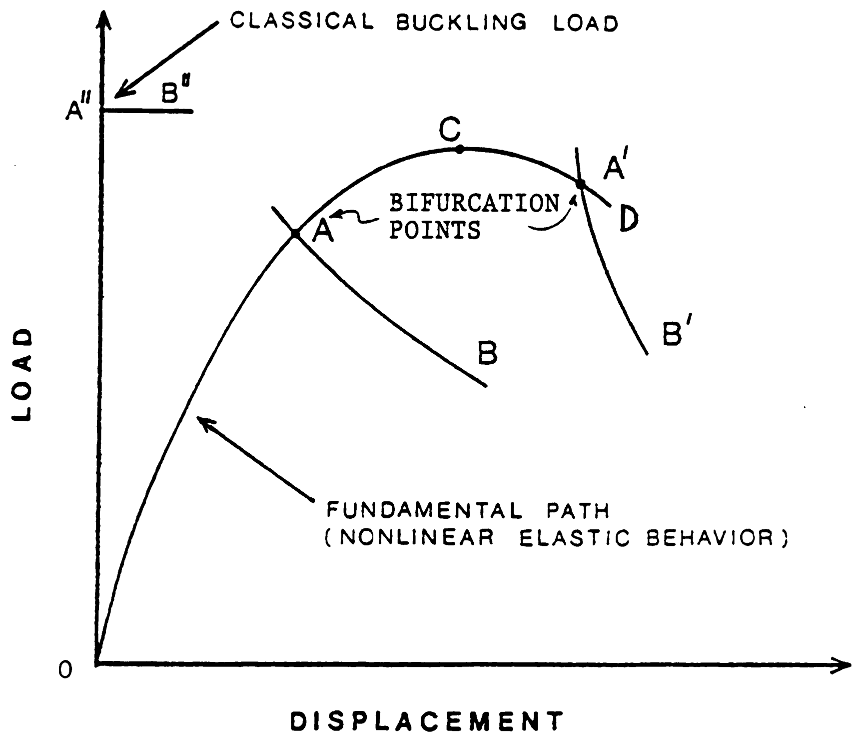

Figure 1-1 : LOAD-DEFLECTION RELATION

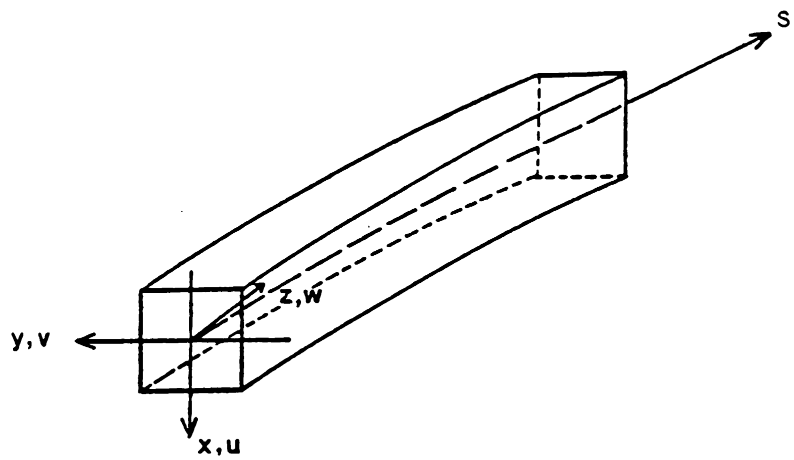

Figure 2-1 : BEAM ELEMENT (Curved In The x-z Plane)

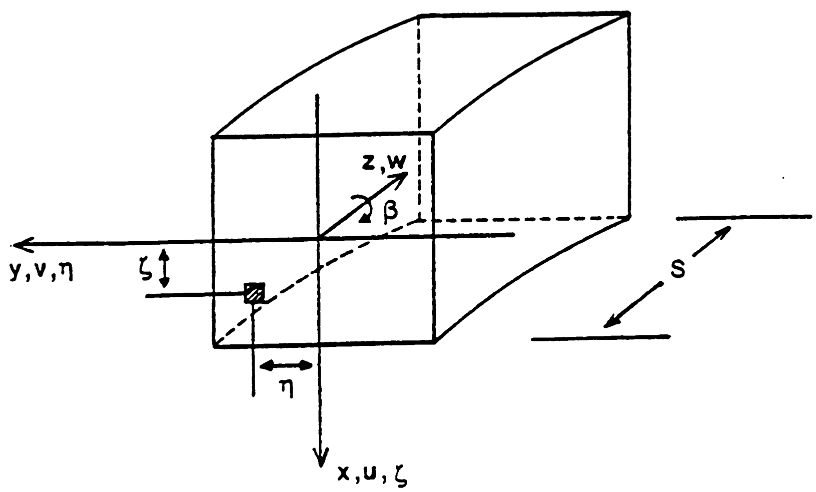

Figure 2-2 : CROSS-SECTION OF PRISMATIC MEMBER

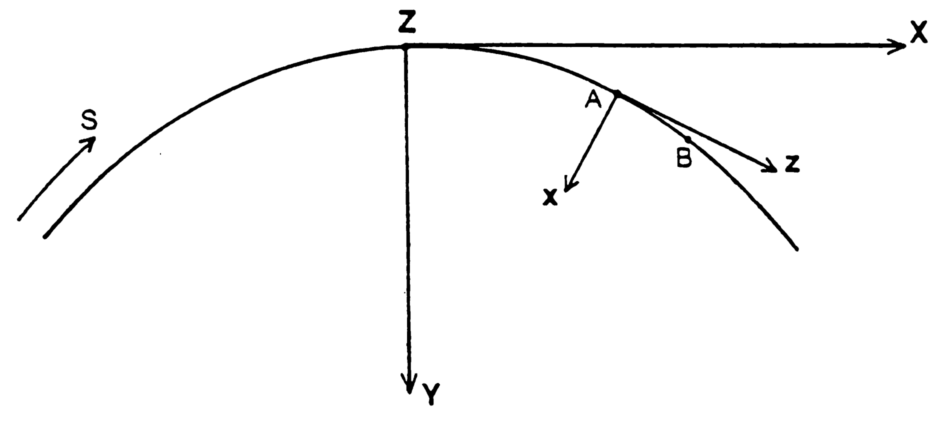

Figure 2-3 : COORDINATE SYSTEMS

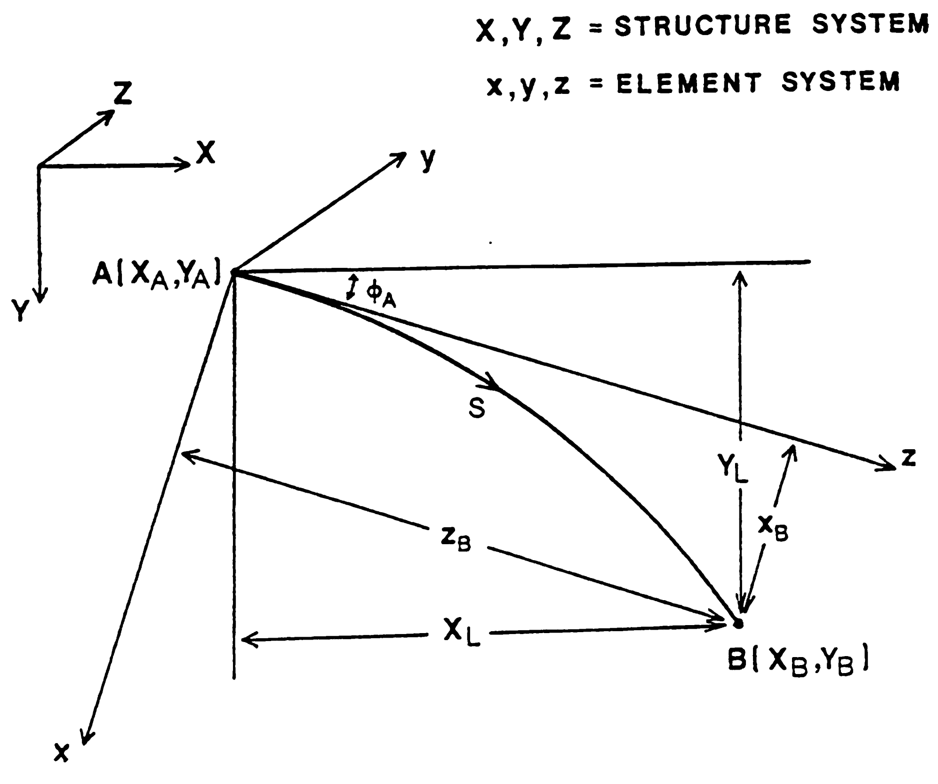

Figure 2-4 : TYPICAL ELEMENT

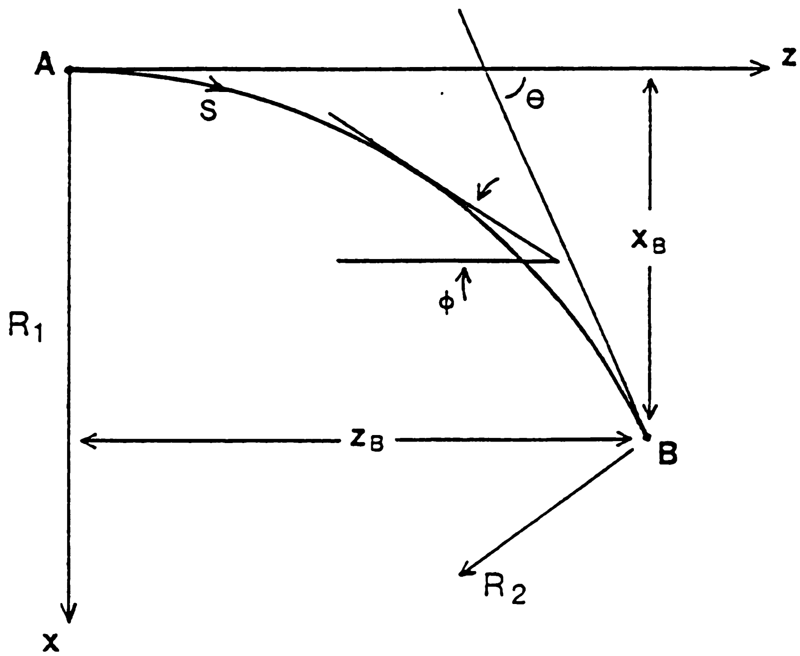

Figure 2-5 : TYPICAL ELEMENT AFTER TRANSFORMATION TO ELEMENT COORDINATE SYSTEM

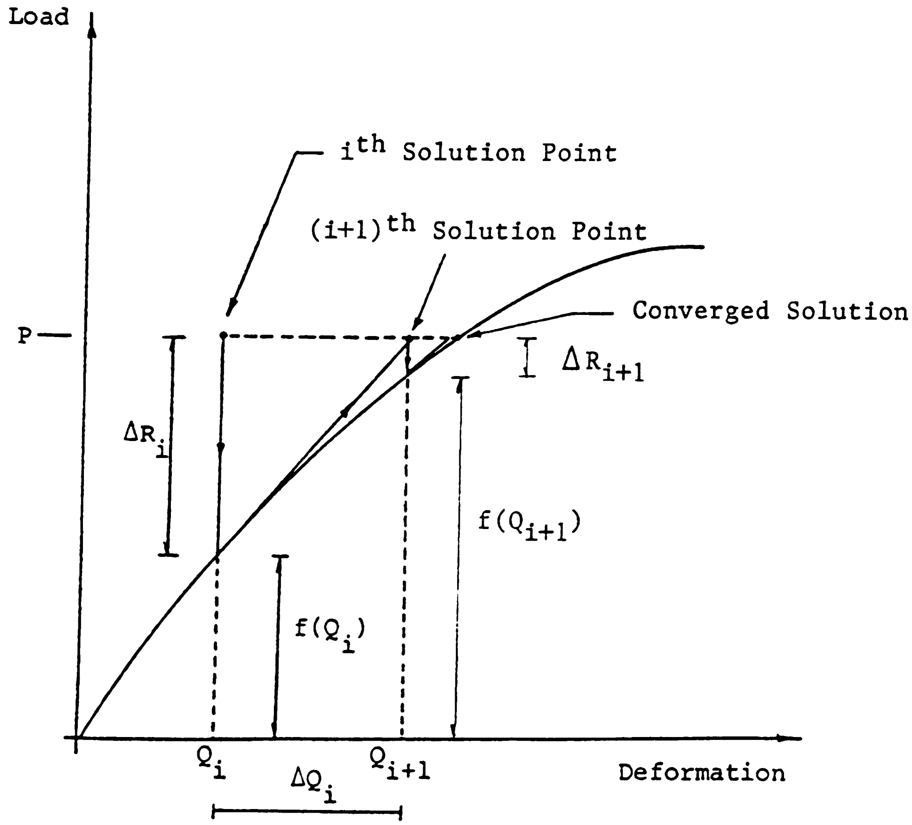

Figure 3-1 : NEWTON-RAPHSON ITERATION


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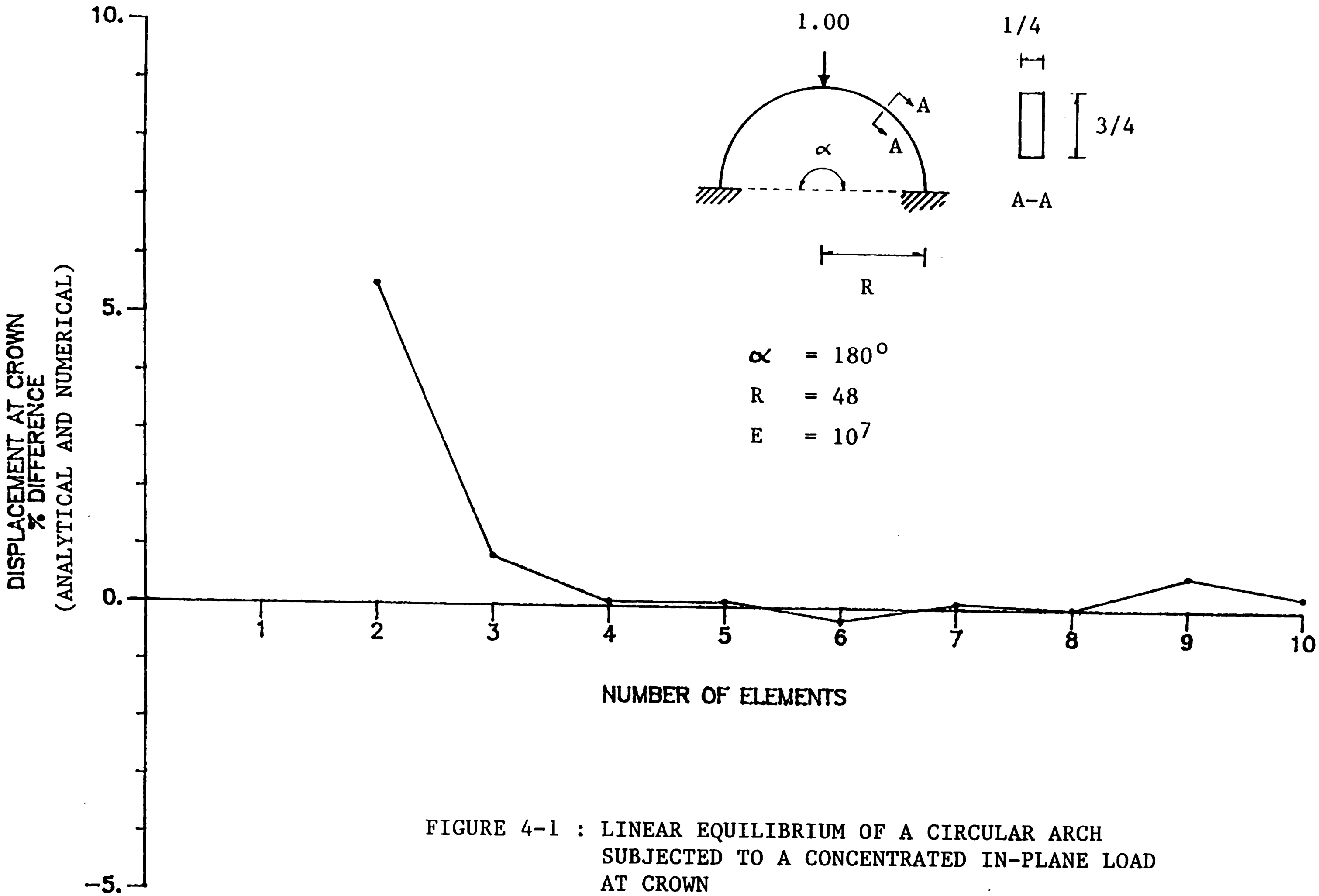


FIGURE 4-1 : LINEAR EQUILIBRIUM OF A CIRCULAR ARCH SUBJECTED TO A CONCENTRATED IN-PLANE LOAD AT CROWN

10.

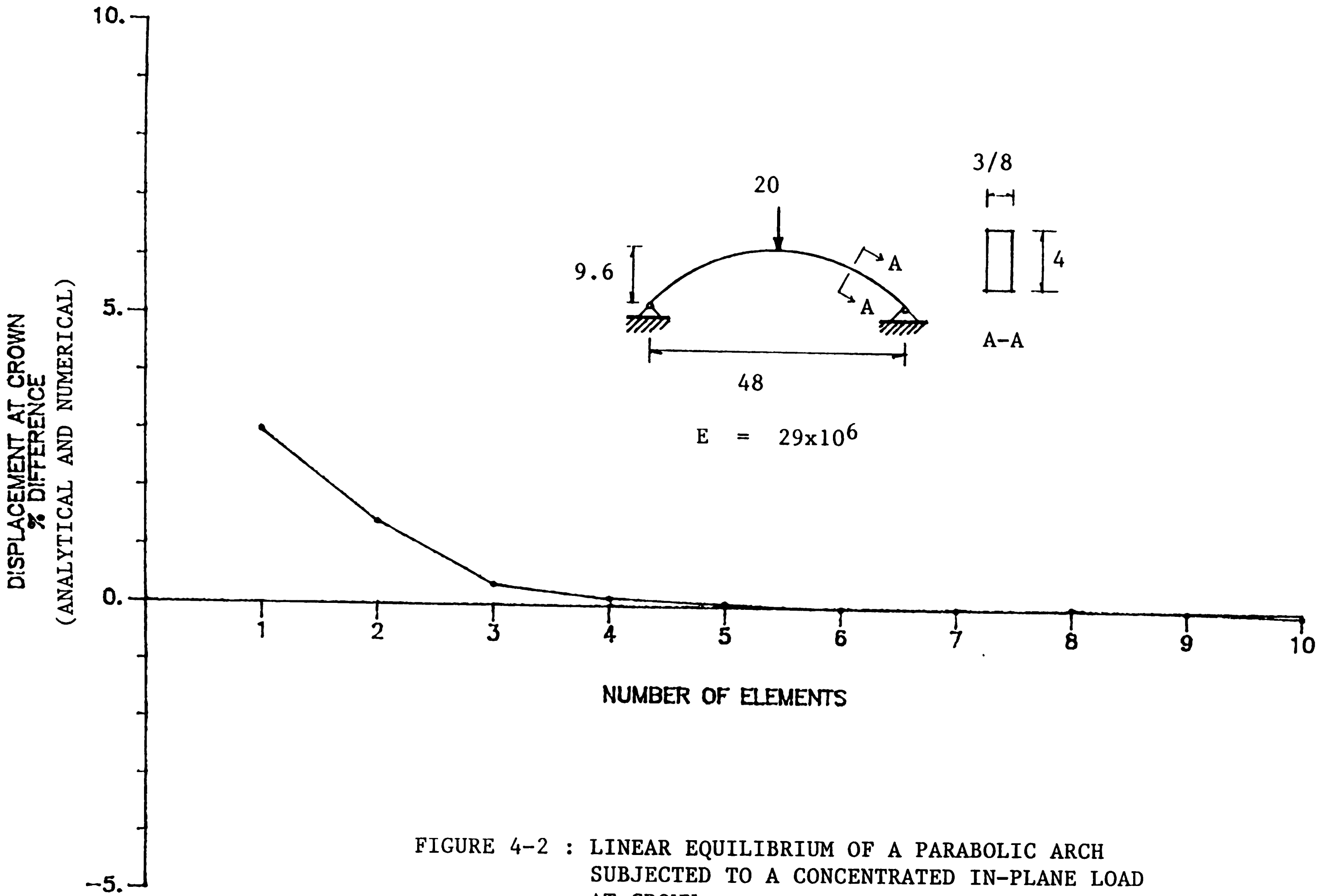


FIGURE 4-2 : LINEAR EQUILIBRIUM OF A PARABOLIC ARCH SUBJECTED TO A CONCENTRATED IN-PLANE LOAD AT CROWN

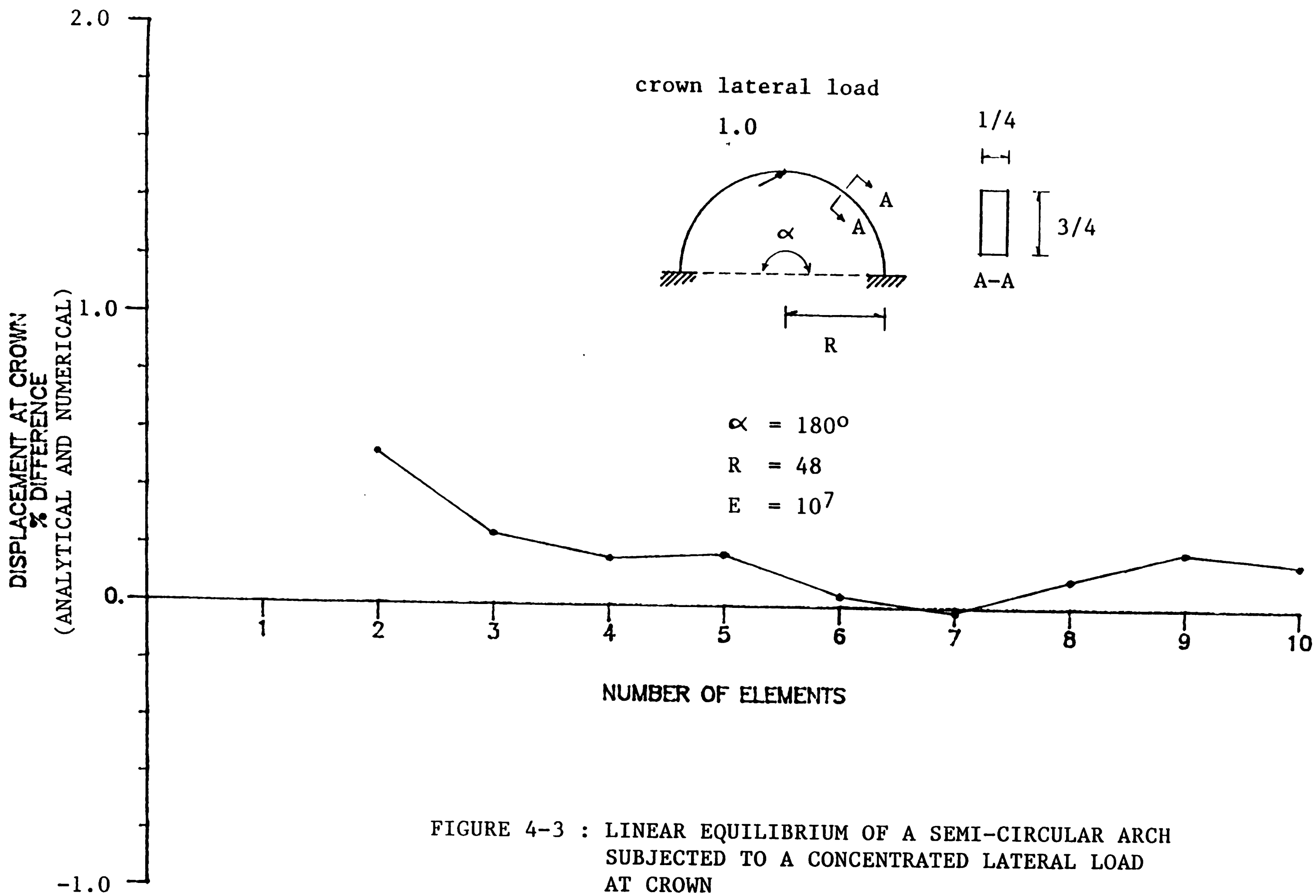
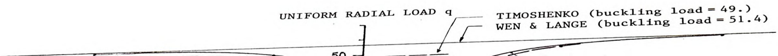


FIGURE 4-3 : LINEAR EQUILIBRIUM OF A SEMI-CIRCULAR ARCH SUBJECTED TO A CONCENTRATED LATERAL LOAD AT CROWN


UNIFORM RADIAL LOAD q
TIMOSHENKO (buckling load = 49.)
WEN & LANGE (buckling load = 51.4)
50

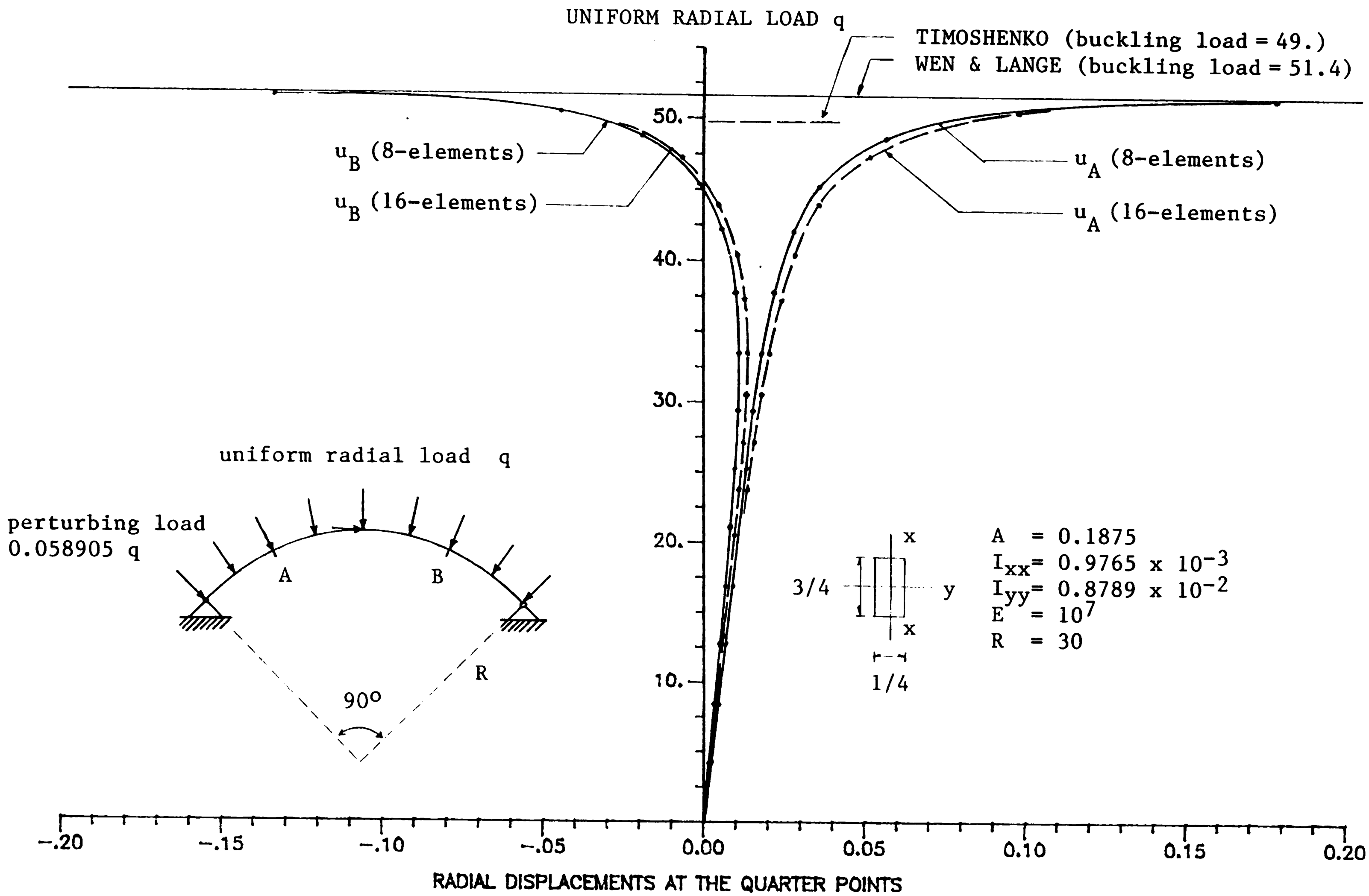
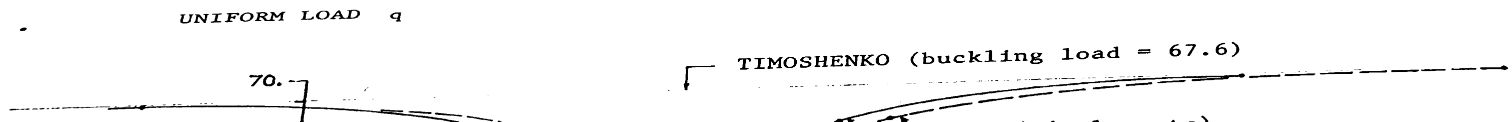


FIGURE 4–4 : A 90 HINGED CIRCULAR ARCH SUBJECTED TO UNIFORM RADIAL LOAD (IN-PLANE BEHAVIOR)



UNIFORM LOAD q
70.
TIMOSHENKO (buckling load = 67.6)

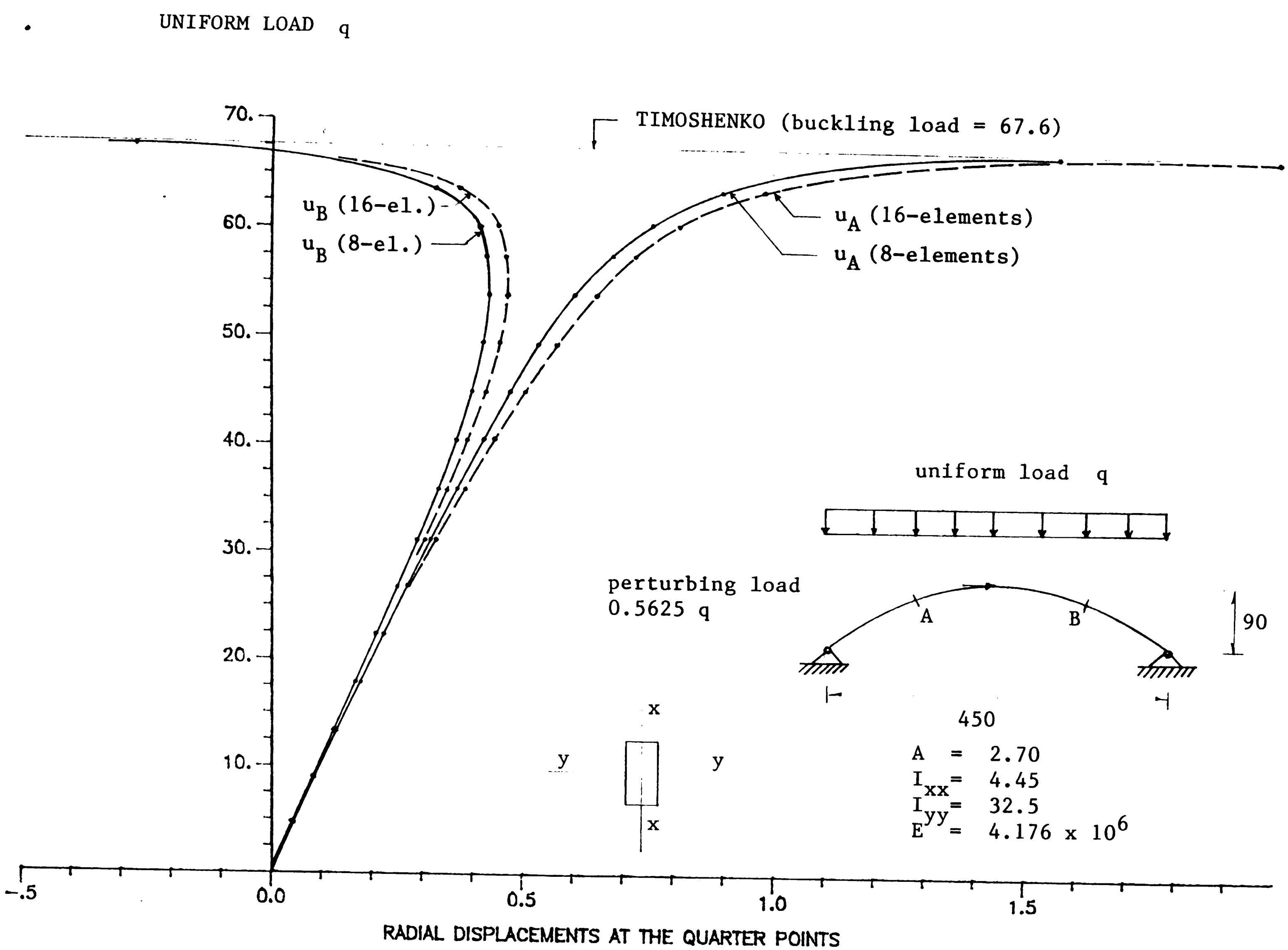
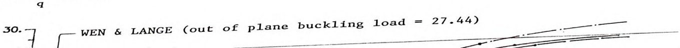


FIGURE 4-6 : A HINGED PARABOLIC ARCH SUBJECTED TO UNIFORM LOAD ON HORIZONTAL PROJECTION (IN-PLANE BEHAVIOR)

UNIFORM RADIAL LOAD
q



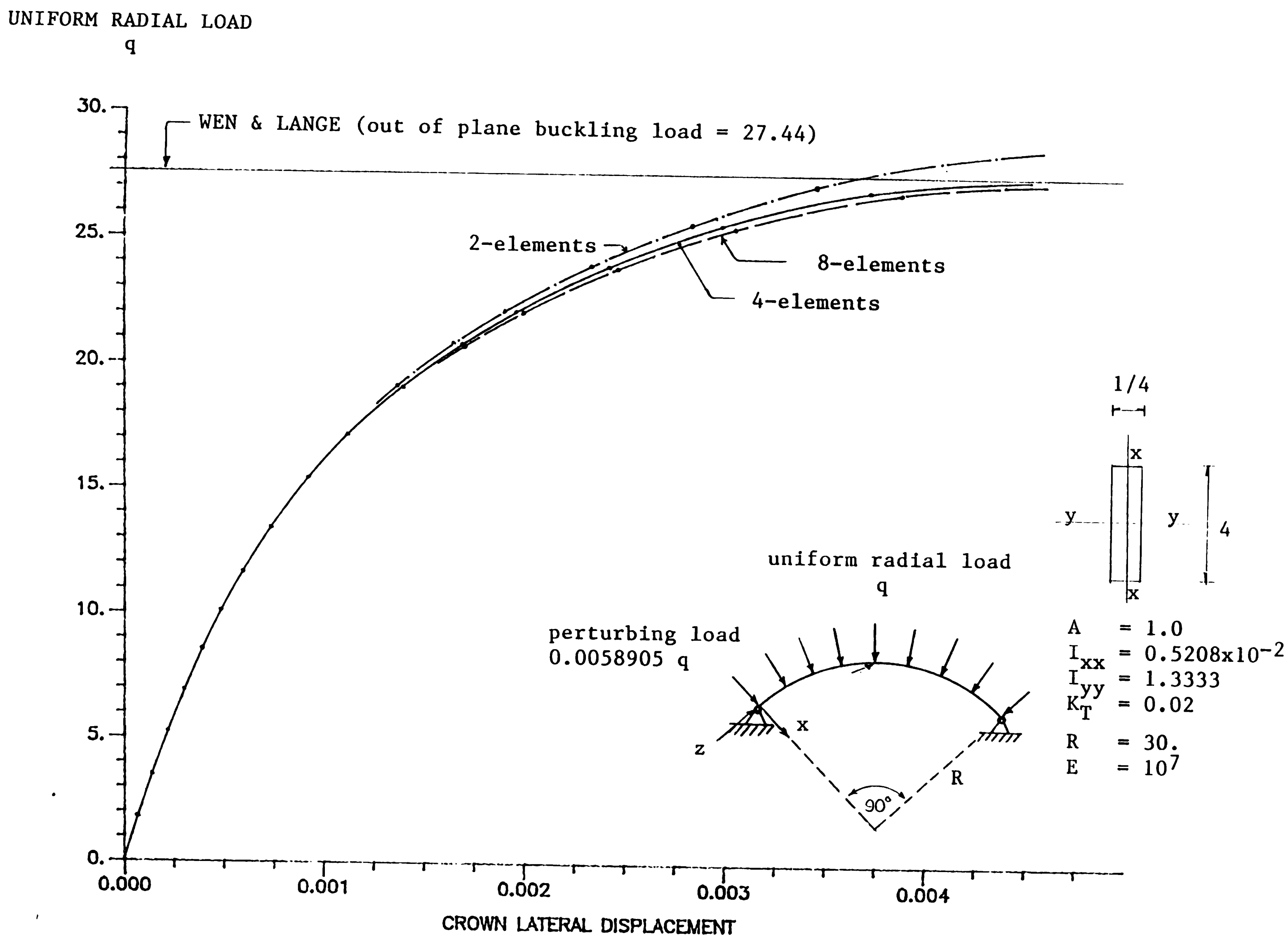
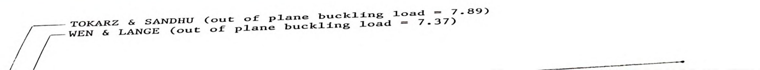


FIGURE 4–6 : A 90° HINGED CIRCULAR ARCH SUBJECTED TO UNIFORM RADIAL LOAD
(OUT-OF-PLANE BEHAVIOR)


TOKARZ & SANDHU (out of plane buckling load = 7.89)
WEN & LANGE (out of plane buckling load = 7.37)

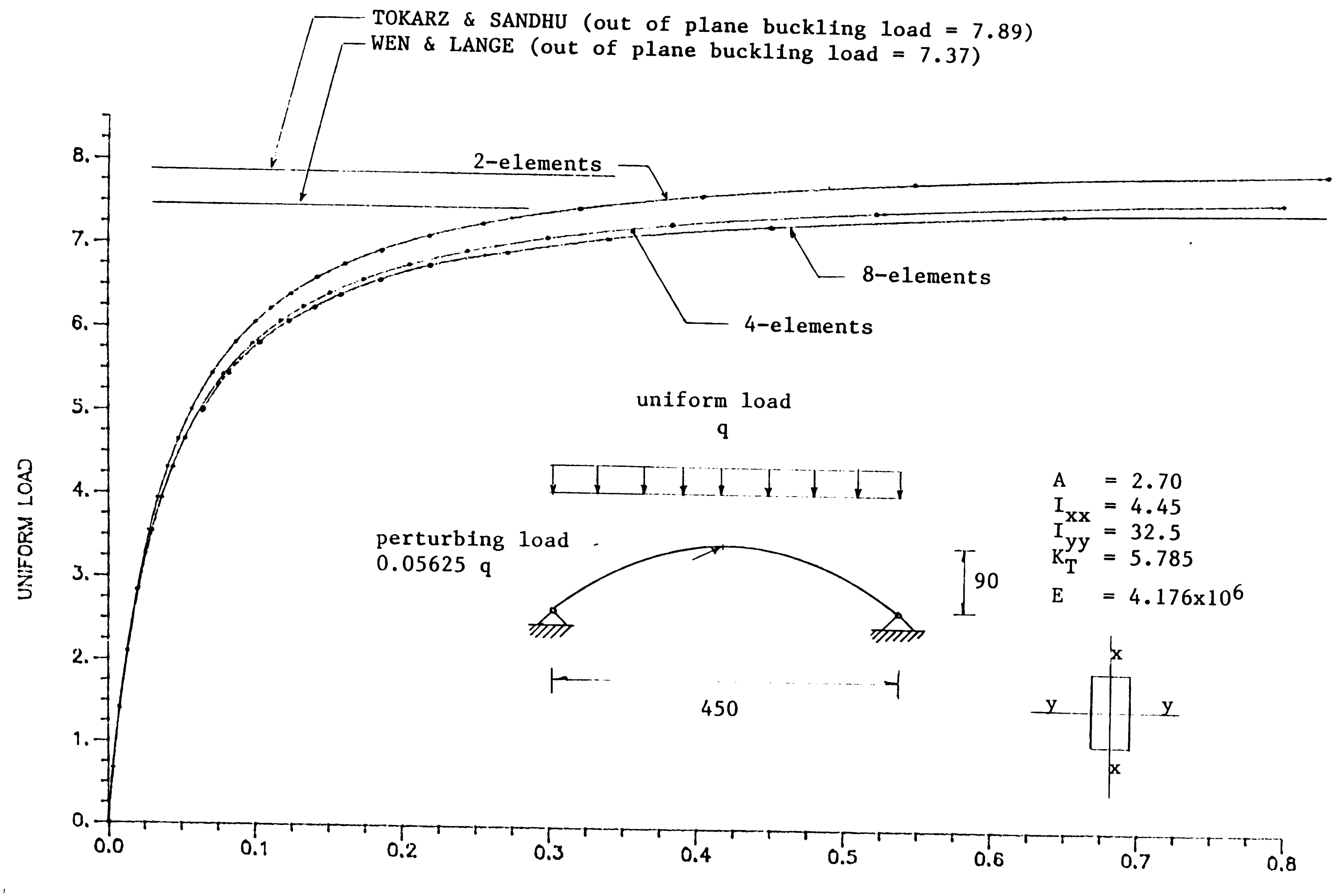
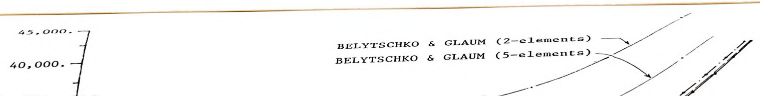


FIGURE 4–7 : A HINGED PARABOLIC ARCH SUBJECTED TO UNIFORM LOAD ON HORIZONTAL PROJECTION (OUT-OF-PLANE BEHAVIOR)



45,000.
40,000.
BELYTSCHKO & GLAUM (2-elements)
BELYTSCHKO & GLAUM (5-elements)

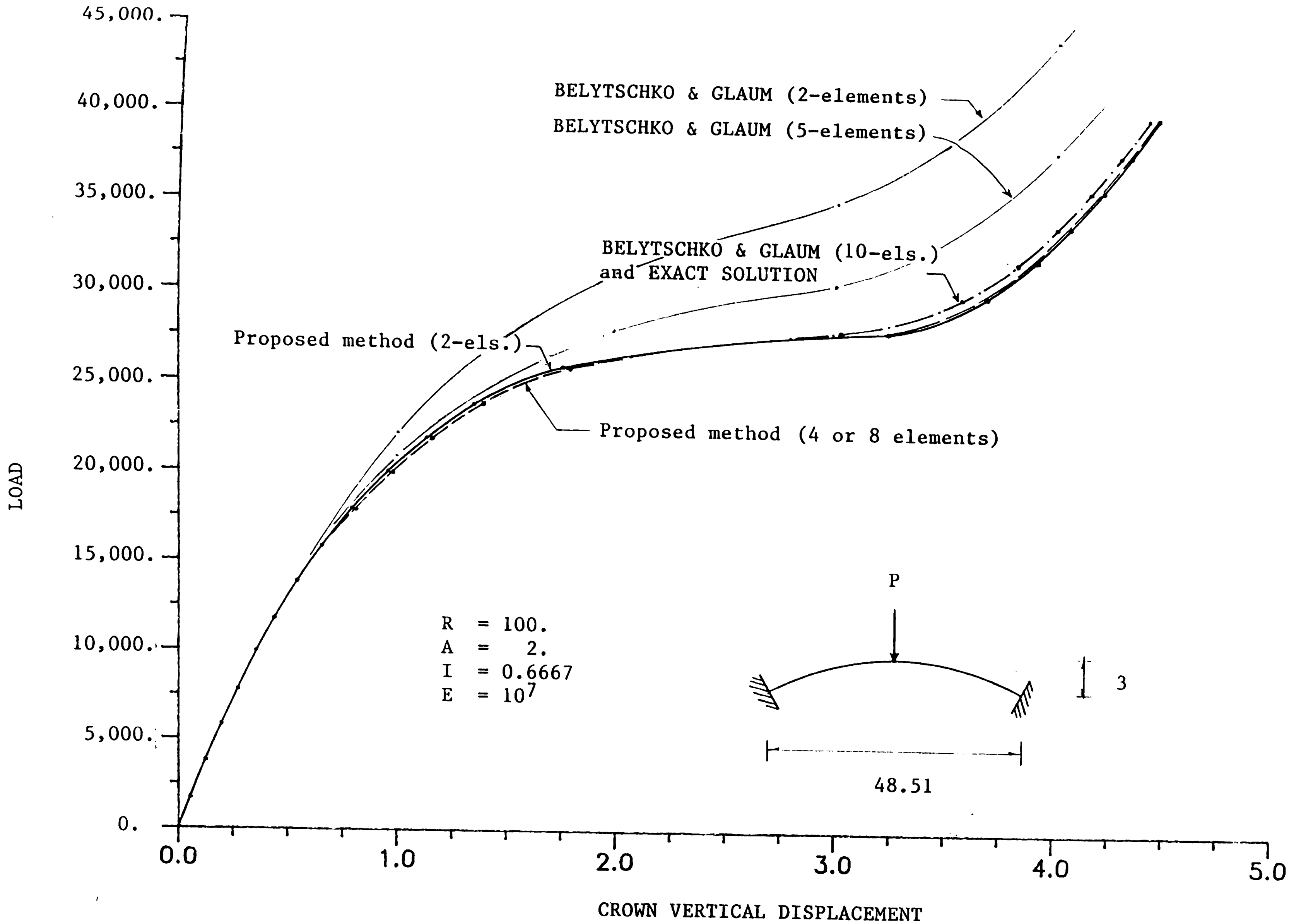
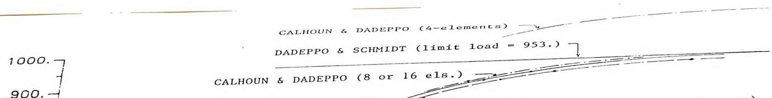


FIGURE 4-8 : A 28° CLAMPED CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN


CALHOUN & DADEPPO (4-elements)
DADEPPO & SCHMIDT (limit load = 953.)
CALHOUN & DADEPPO (8 or 16 els.)
1000.
900.

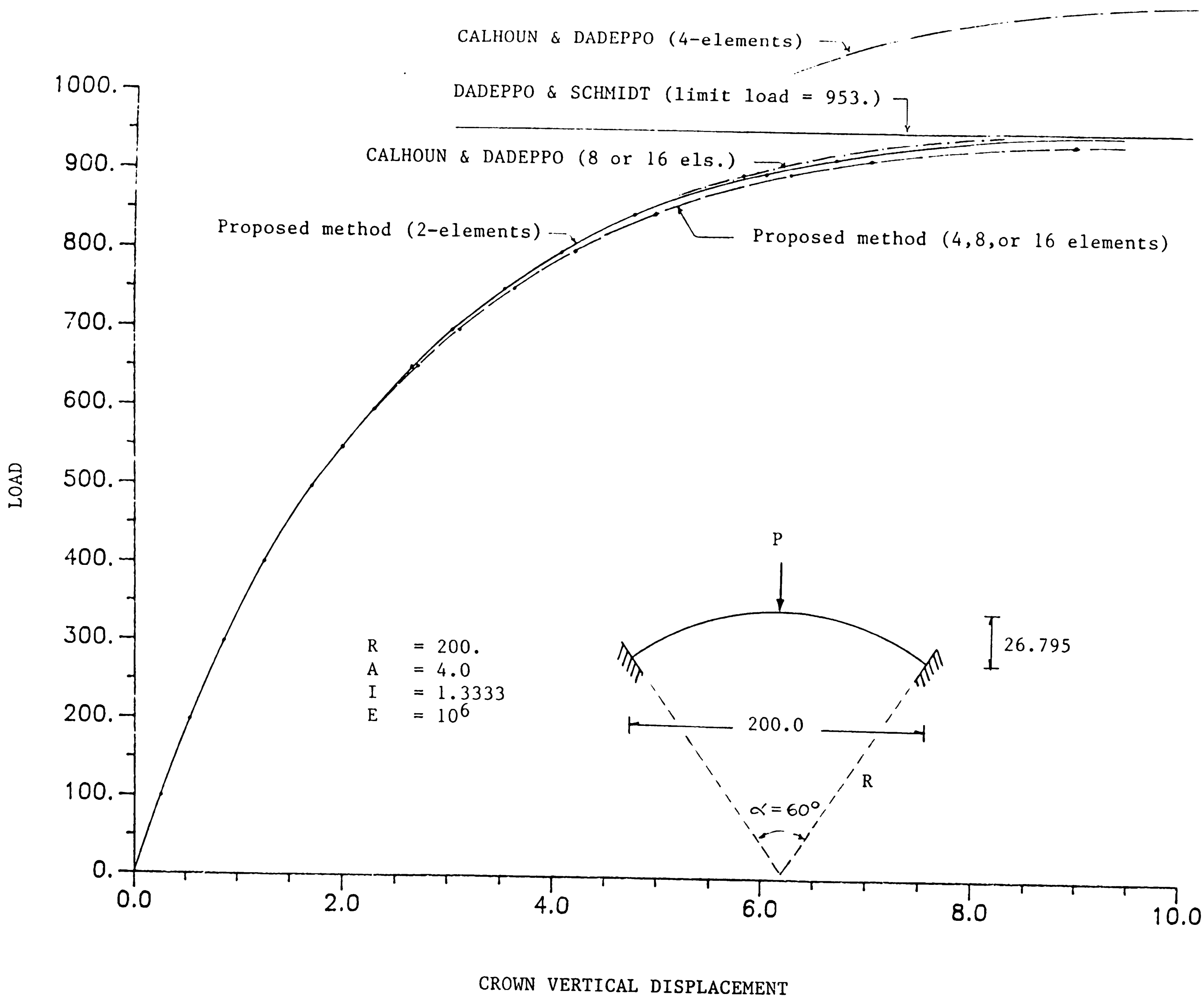
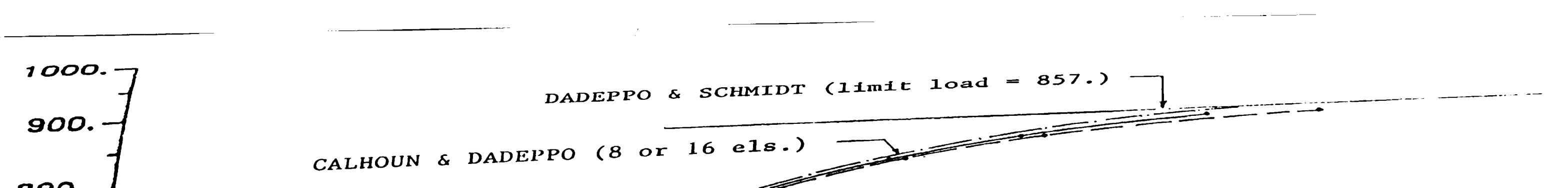


FIGURE 4-9 : A 60° CLAMPED CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN


1000.
900.
DADEPPO & SCHMIDT (limit load = 857.)
CALHOUN & DADEPPO (8 or 16 els.)

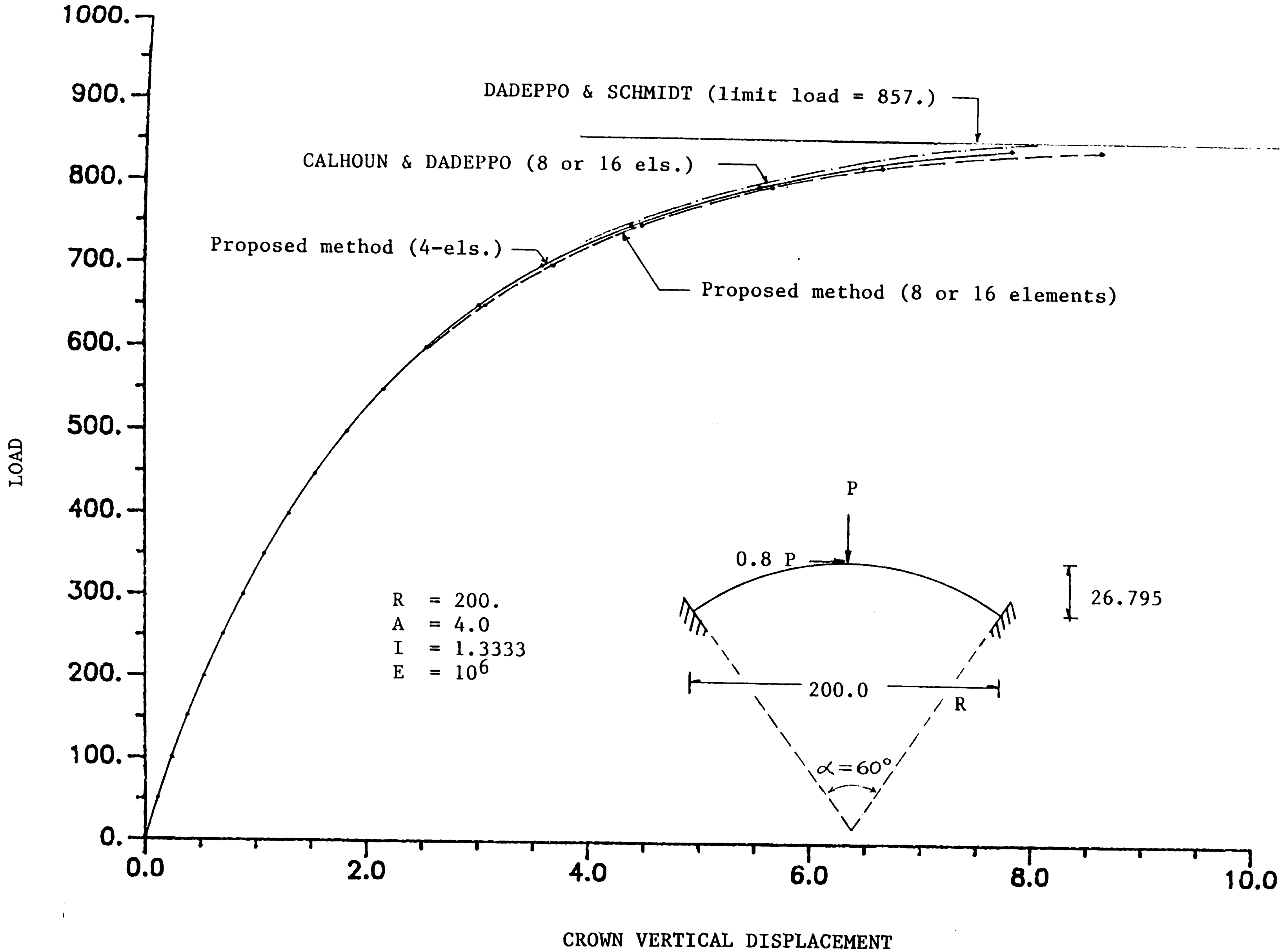
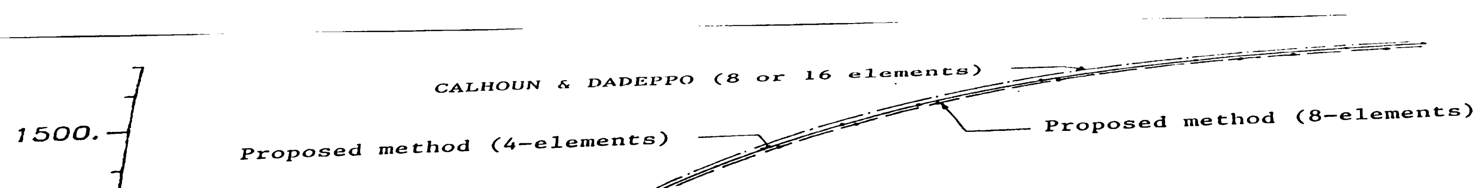


FIGURE 4-10 : A 60 CLAMPED CIRCULAR ARCH SUBJECTED TO A SKEW CONCENTRATED LOAD AT CROWN



CALHOUN & DADEPPO (8 or 16 elements)
1500.
Proposed method (4-elements)
Proposed method (8-elements)

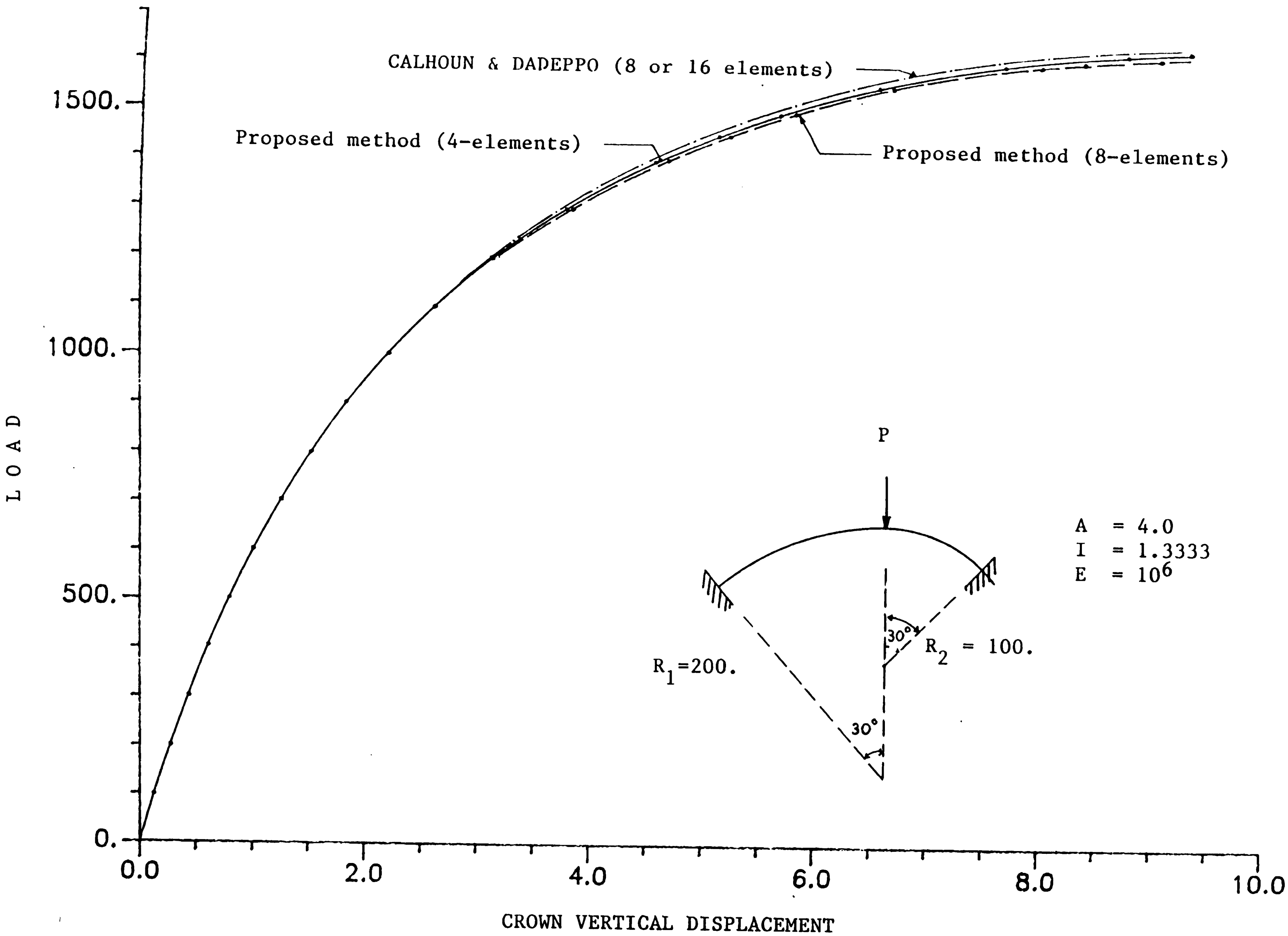


FIGURE 4-11 : A CLAMPED MULTIPLE RADII CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

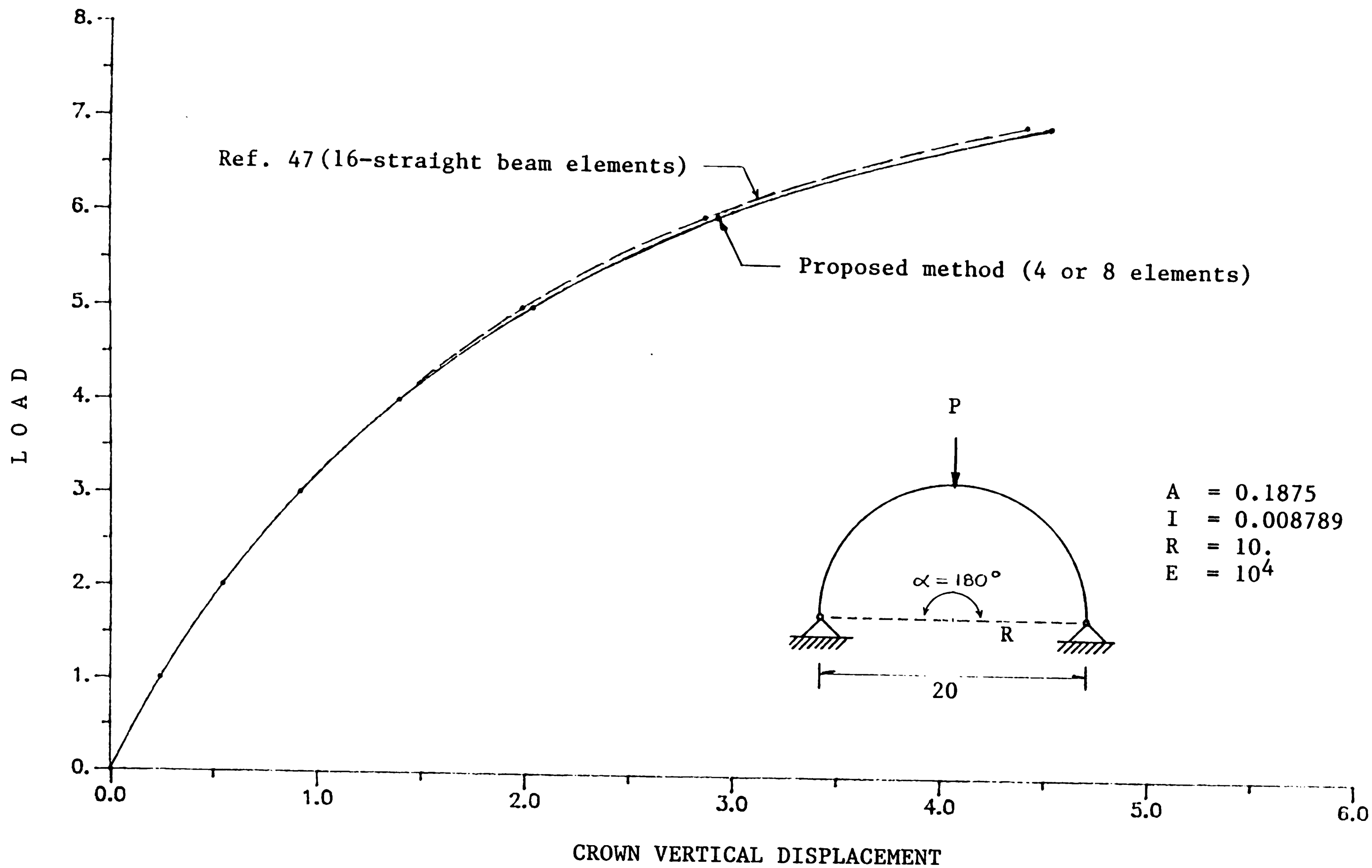


FIGURE 4-12 : A HINGED SEMI-CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

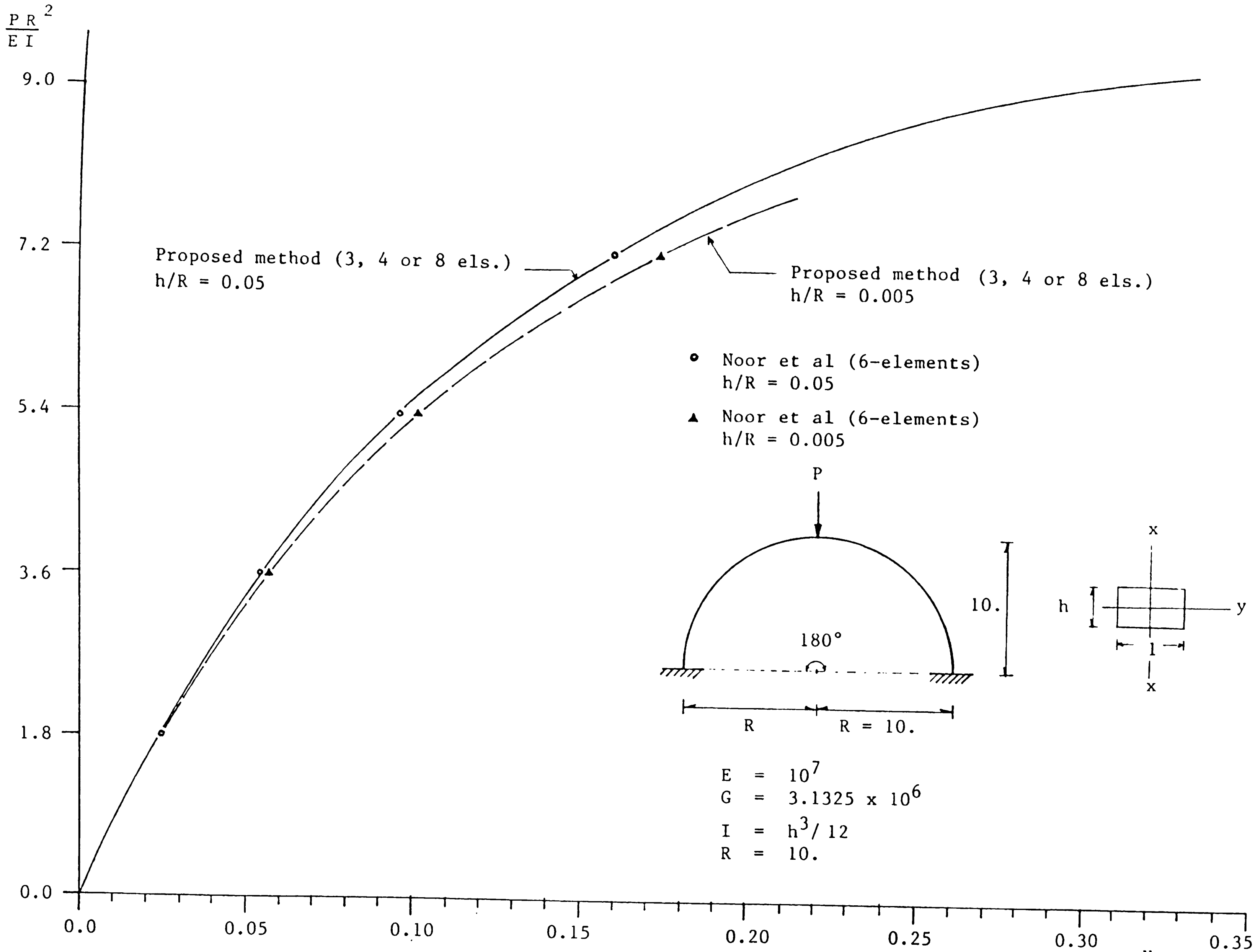


FIGURE 4-13 : A CLAMPED SEMI-CIRCULAR ARCH SUBJECTED TO A VERTICAL CONCENTRATED LOAD AT CROWN

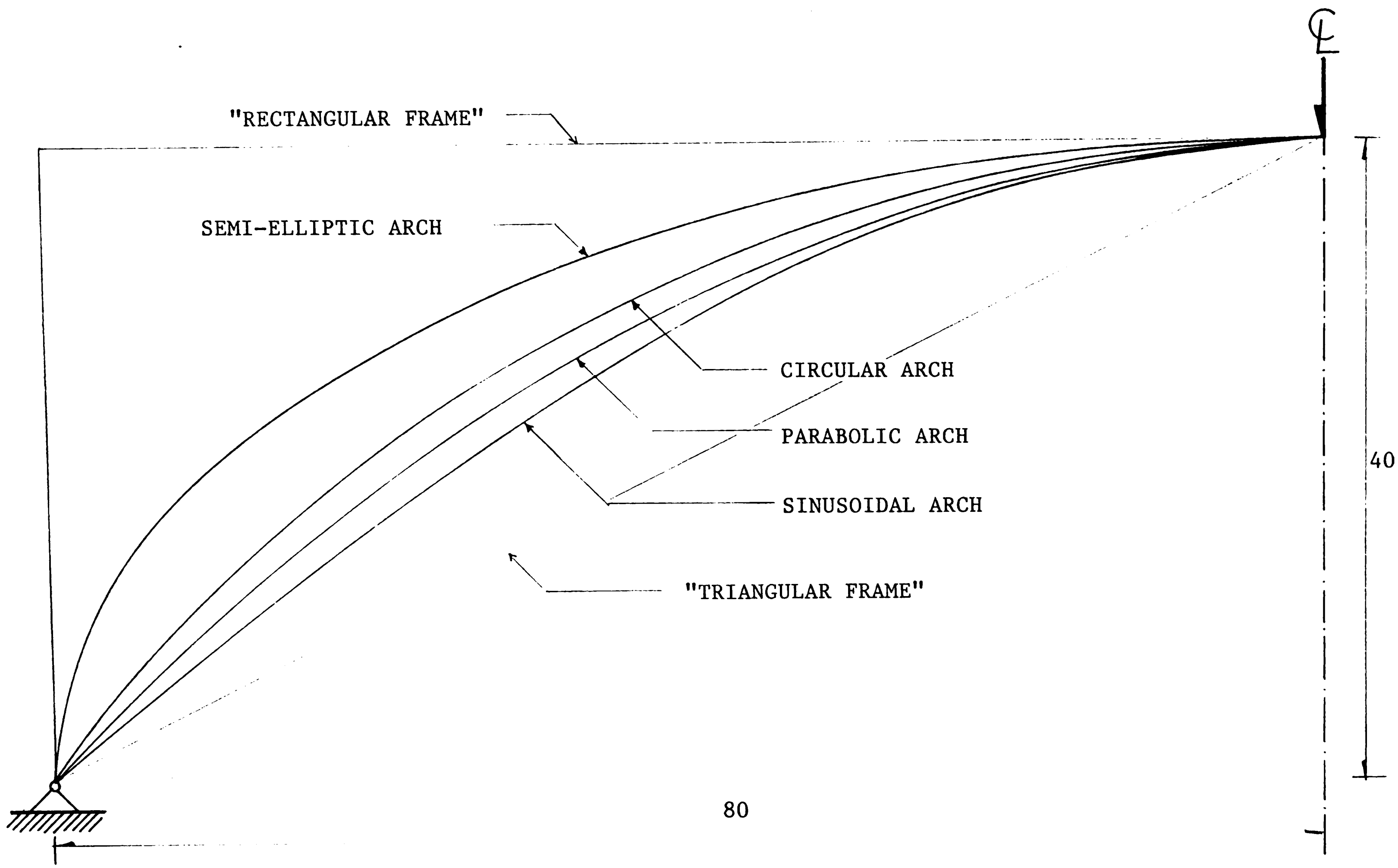
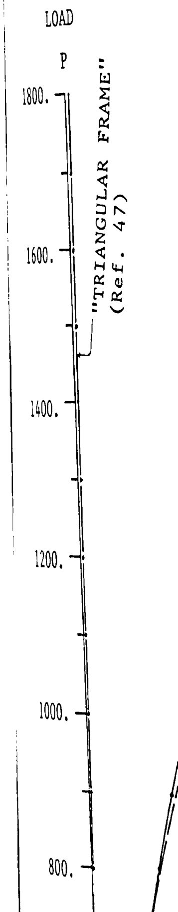


FIGURE 4-14 : ARCH AND FRAME PROFILES


LOAD
P
1800.
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1000.
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600.
400.
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0.0
"TRIANGULAR FRAME"
(Ref. 47)

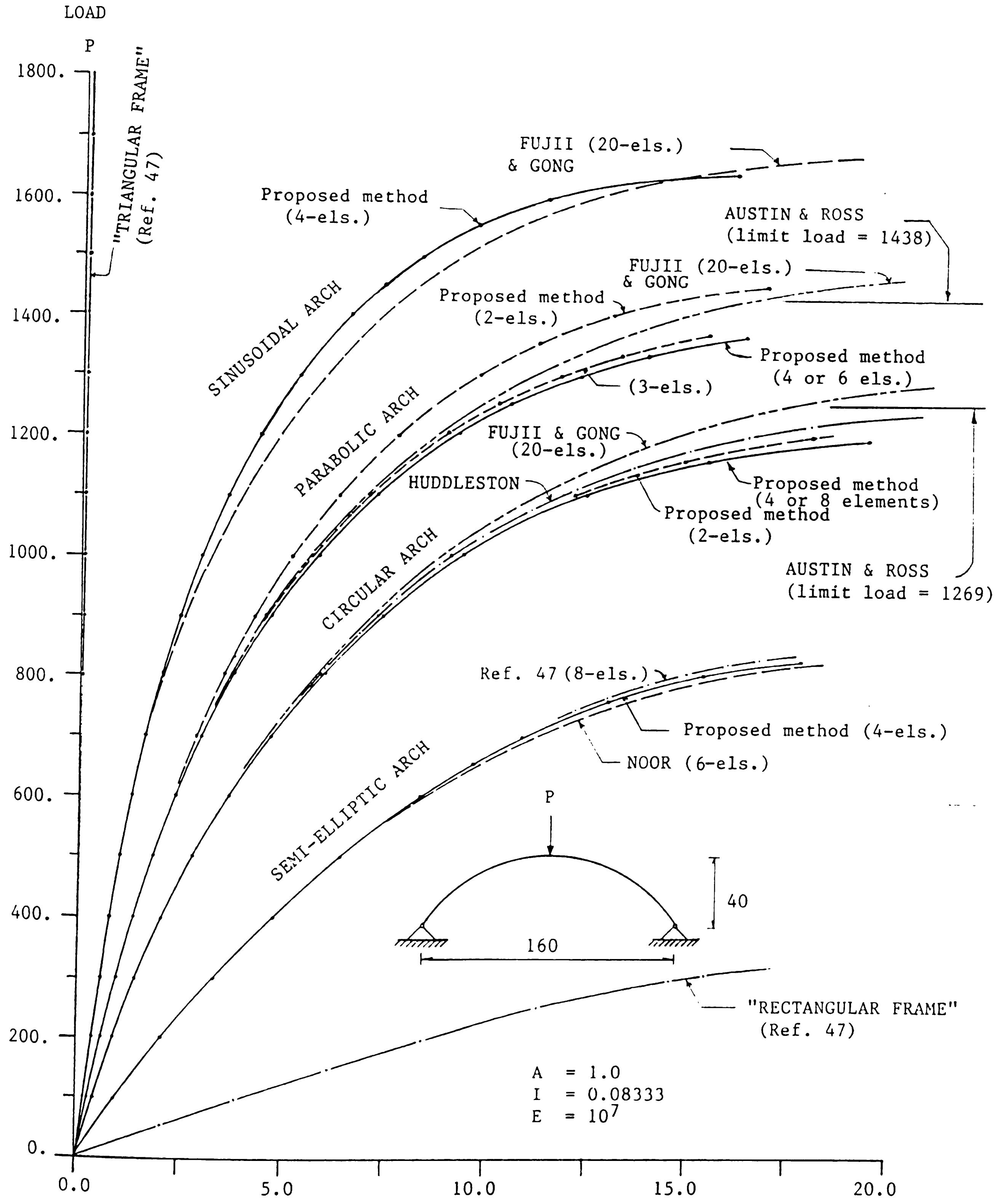
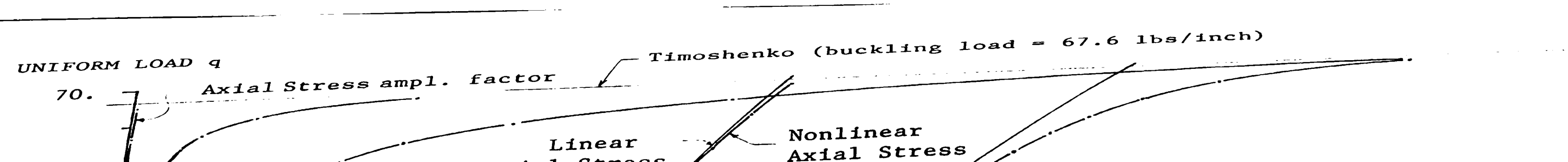


FIGURE 4-15 : SINUSOIDAL, PARABOLIC, CIRCULAR, SEMI-ELLIPTIC ARCHES, TRIANGULAR FRAME, AND RECTANGULAR FRAME SUBJECTED TO A CONCENTRATED LOAD AT THEIR CROWNS



UNIFORM LOAD q
70.
Axial Stress ampl. factor
Timoshenko (buckling load = 67.6 lbs/inch)
Linear
Nonlinear
Axial Stress

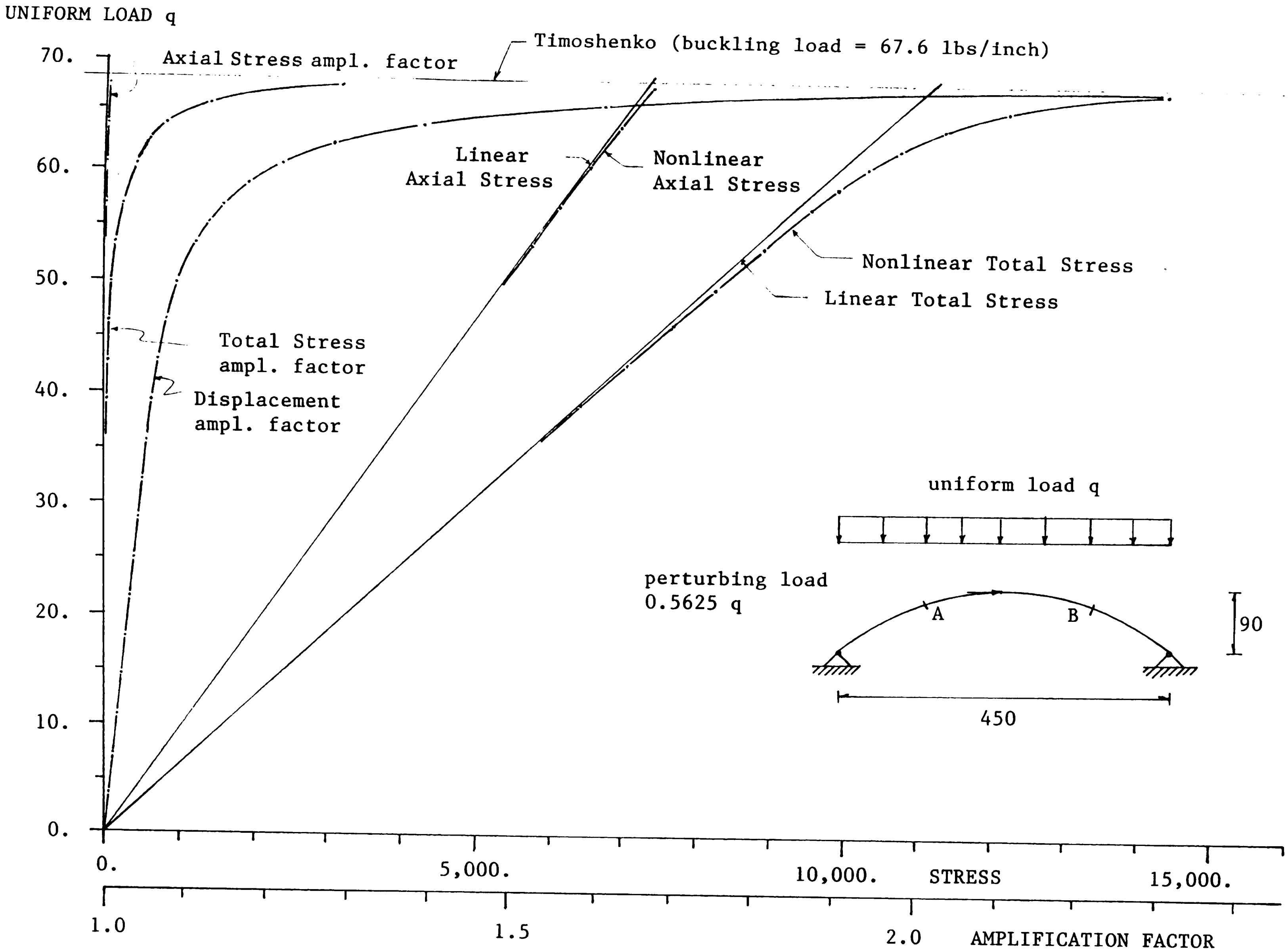


FIGURE 4-16 : STRESSES AND AMPLIFICATION FACTORS AT THE QUARTER POINT A


LOAD P
40,000.

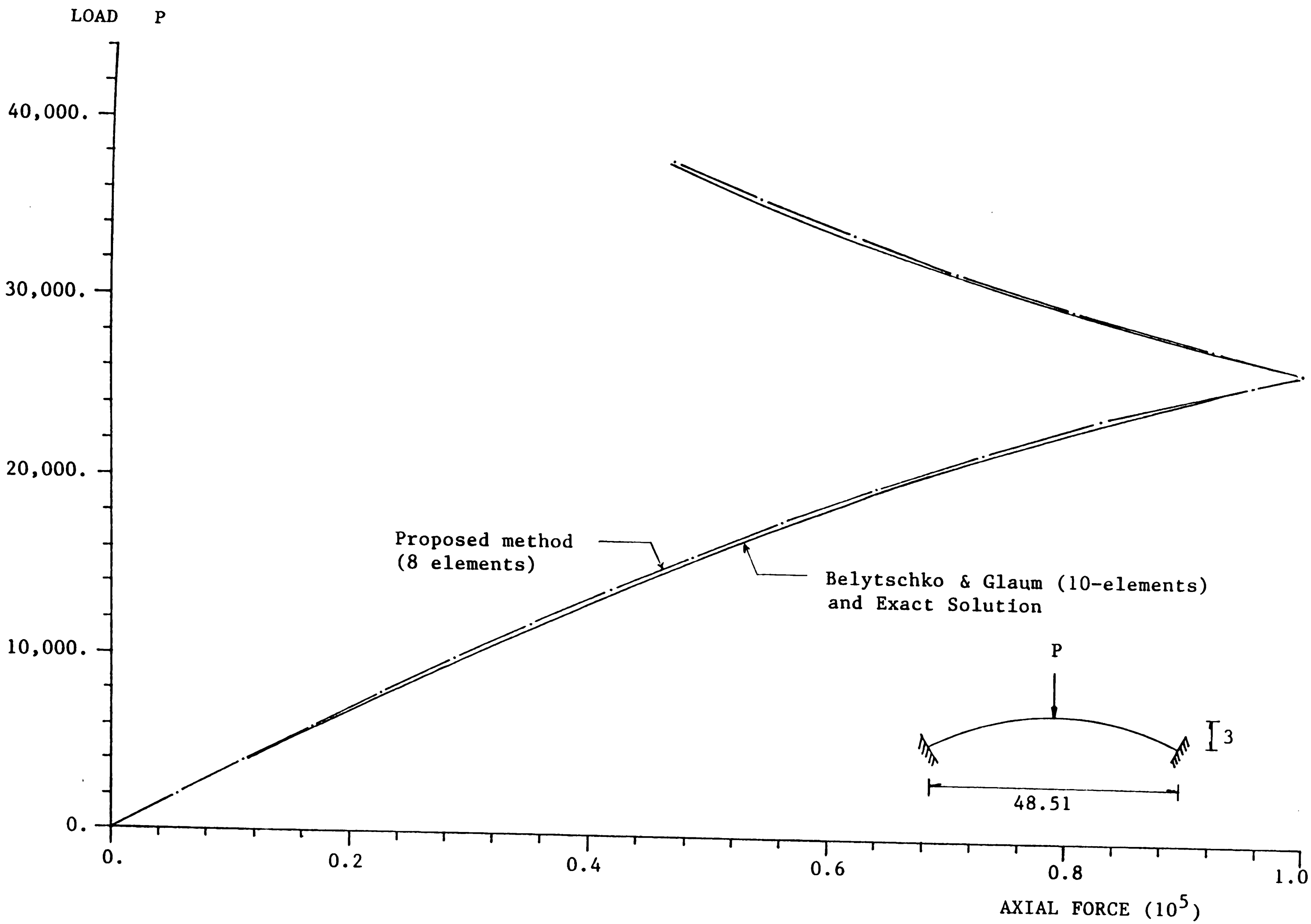


FIGURE 4-17 : AXIAL FORCE AT CROWN


Load P
40,000.
Nonlinear Axial Stress
Linear Axial Stress
Linear Total Stress
Nonlinear

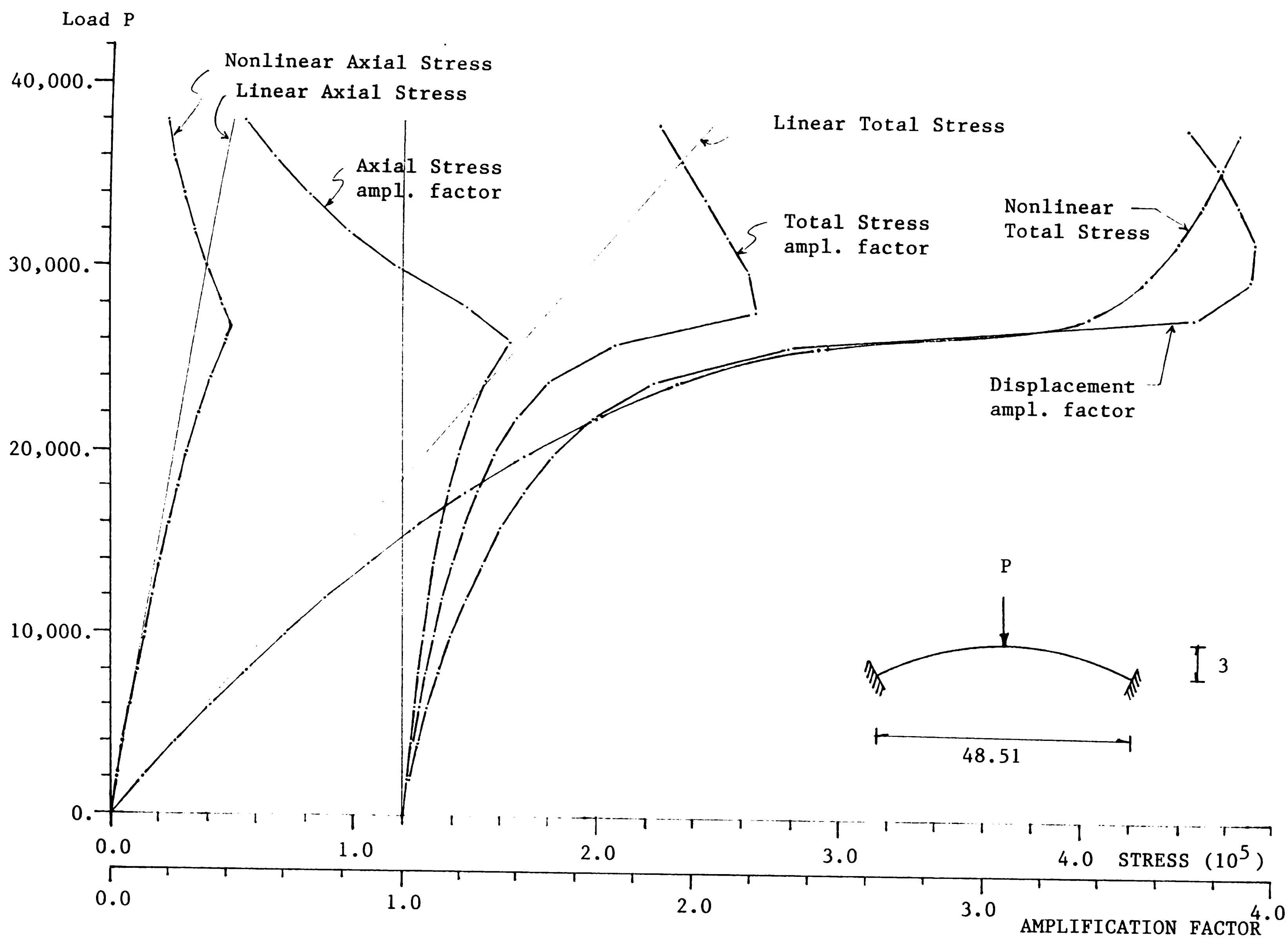
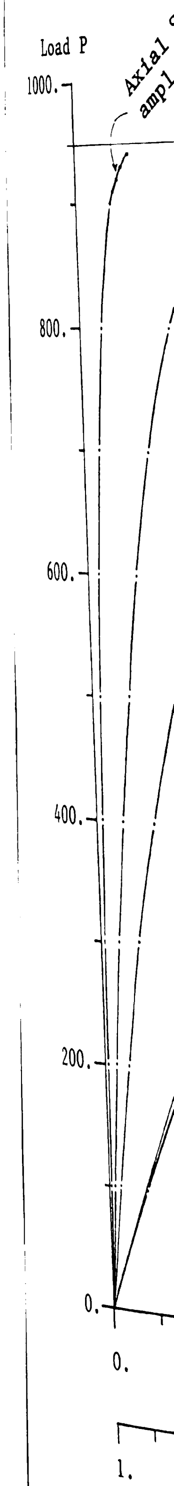


FIGURE 4-18 : STRESSES AND AMPLIFICATION FACTOR AT CROWN


Load P
Axial
1000.
800.
600.
400.
200.
0.
0.
1.

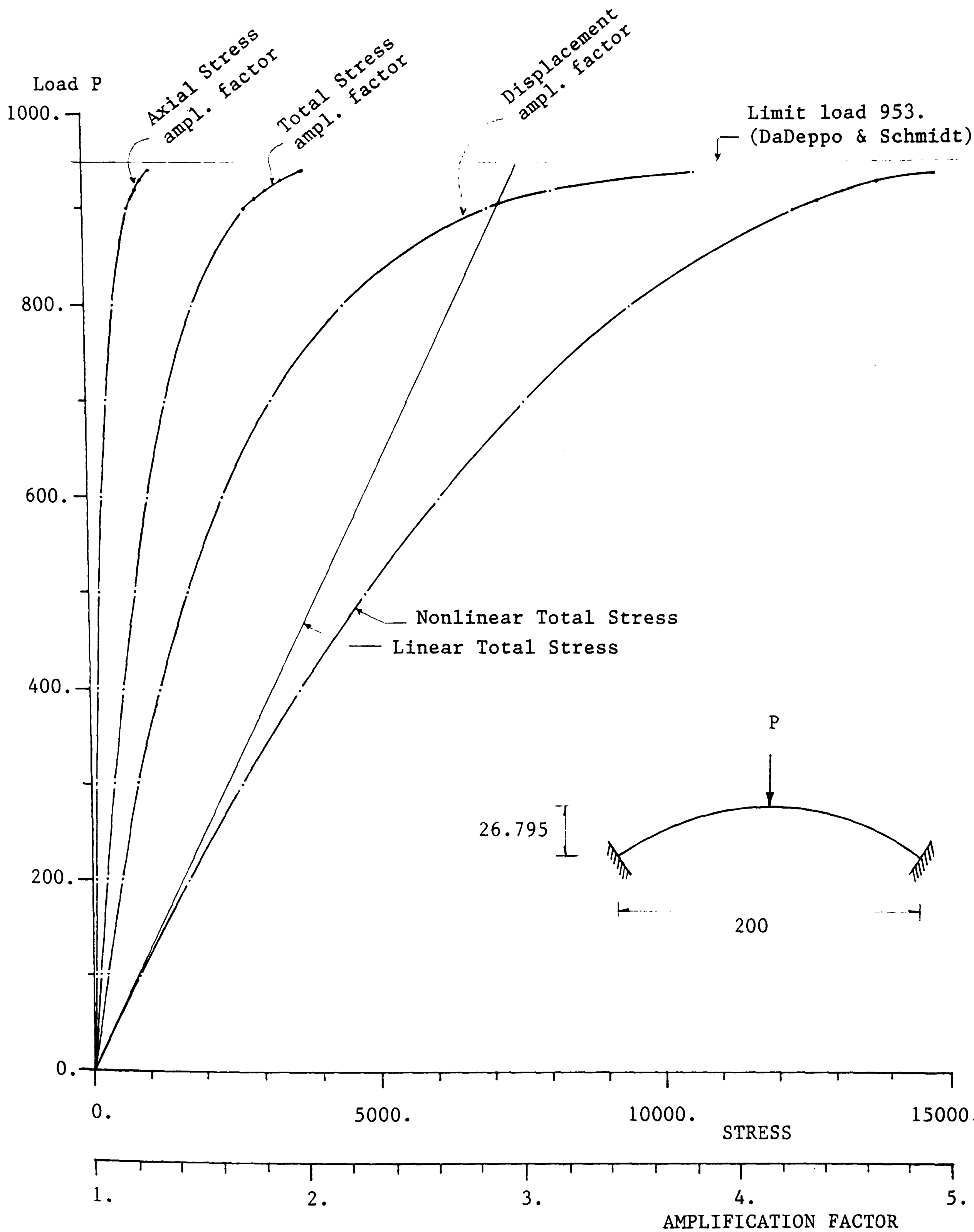


FIGURE 4-19 : STRESSES AND AMPLIFICATION FACTORS AT THE CROWN

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APPENDIX A

NEWTON-RAPHSON METHOD FOR UPDATED COORDINATES

A.1 GENERAL

The formulation discussed in Chapter II, which is refered to Lagrange small rotation formulation, and the corresponding solution procedure described in Chapter III can not be used for large rotation (displacement) problems. For this class of problem, however, we may treat the problem as consisting of a series of increments involving small rotations (displacements). For each increment the concept of Lagrange small rotation formulation applies. At the beginning of each increment the geometry of structure should be updated. For a typical new increment, although the initial displacements in the new coordinates are zero, the strains are not. The strain, which is called the "initial strain", would lead to an "initial strain stiffness matrix". The procedure how to obtain the initial strain stiffness matrix is presented in the next section.

It should be noted that the geometric representation described in Section 2.4.2 makes the updated procedure for curved beam element possible. Without it we would be faced with the problem of defining the radii of curvature at the end nodes of the updated curved beam element, which are required for evaluating the updated stiffness matrices.

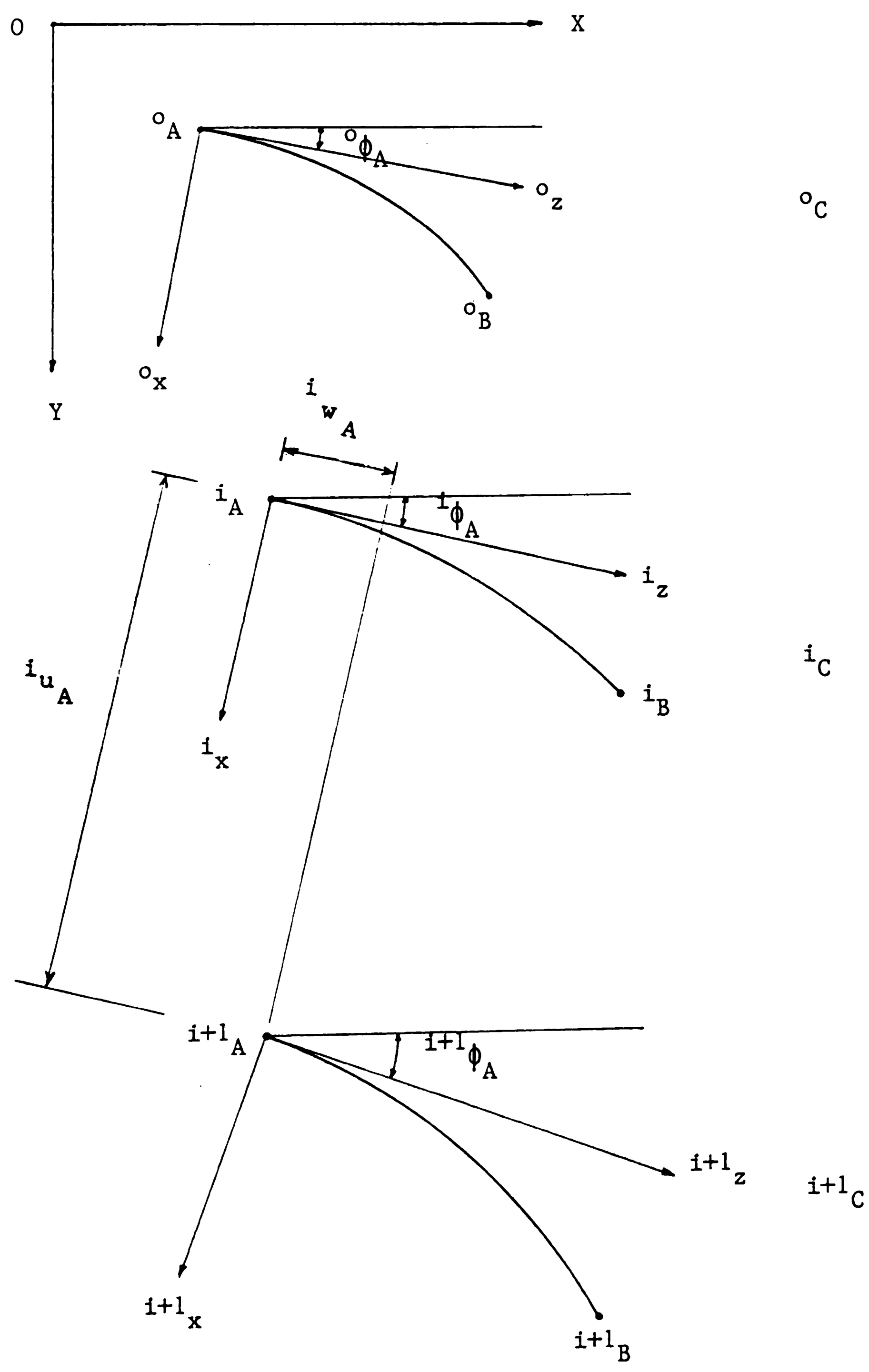

Figure A-1 : CONFIGURATION OF A TWO DIMENSIONAL CURVED BEAM ELEMENT AT SUCCESSIVE LOAD INCREMENTS IN UPDATED LAGRANGE FORMULATION

A.2 INITIAL STRAIN STIFFNESS MATRIX, $[\ k_{\epsilon_o}\ ]$

Consider a two dimensional curved beam element illustrated in Figure A-1. The X and Y axes represent the global coordinate system, the ${}^{i}x$ and ${}^{i}z$ axes denote the member coordinates, and ${}^{i}C$ the member configuration at the beginning of the ith load level.

From the stage from configuration ${}^{i}C$ to ${}^{i+1}C$, the former configuration should be thought as fixed and the latter variable. The displacement components of ${}^{i+1}C$ are measured with respect to the member coordinates of ${}^{i}C$. They are the generalized coordinates at the stage of the analysis. (See the displacement ${}^{i}u_A$ and ${}^{i}w_A$ in Figure A-1.)

Let ${}^{i+1}\epsilon_o$ denote the total strain corresponding to ${}^{i+1}C$, ${}^{i}\epsilon_o$ denote the total strain at ${}^{i}C$, and $\epsilon = {}^{i+1}\epsilon_o - {}^{i}\epsilon_o$.

The strain energy is :

$$U = \int {}^{i+1}\epsilon_o^{\,2} \ \frac{E}{2}\ dV = \int (\ \epsilon + {}^{i}\epsilon_o\ )^2 \ \frac{E}{2}\ dV$$

$$= \int \frac{E}{2} \ (\ \epsilon^2 + 2\ {}^{i}\epsilon_o\ \epsilon + {}^{i}\epsilon_o^{\,2}\ )\ dV \qquad \ldots .\ \text{(A-1)}$$

Since ${}^{i}\epsilon_o$ is independent of the generalized coordinates or displacements, the last term of U may be dropped. The expression for U can then be written as :

$$U = U_\epsilon + {}^{i}U_{\epsilon_o} \qquad \ldots .\ \text{(A-2)}$$

where

$$U_{\epsilon} = \int \frac{E}{2} \epsilon^2 \, dV \qquad \text{.... (A-3)}$$

$${}^{i}U_{\epsilon_o} = \int E \, {}^{i}\epsilon_o \, \epsilon \, dV \qquad \text{.... (A-4)}$$

The tangent stiffness matrix is obtained following the procedure discussed in Section 2.4 . Its m-n entry is equal to :

$$\frac{\partial^2 U}{\partial\, {}^{i}q_m \, \partial\, {}^{i}q_n} = \frac{\partial^2 U_{\epsilon}}{\partial\, {}^{i}q_m \, \partial\, {}^{i}q_n} + \frac{\partial^2 \, {}^{i}U_{\epsilon_o}}{\partial\, {}^{i}q_m \, \partial\, {}^{i}q_n} \qquad \text{(A-5)}$$

The first term is exactly as before, with ${}^{i}q_m$ replacing q_m, resulting in ${}^{i}[k]$, ${}^{i}[n1]$, and ${}^{i}[n2]$. For the second part , we encounter something new.

At the beginning of the ith increment, the initial strain is :

$$\begin{aligned} {}^{i}\epsilon_o(s,\zeta,\eta) = \sum_{j=0}^{i-1} {}^{j}\Big[\Big(\frac{dw}{ds} - \frac{u}{R} \Big) + \frac{1}{2L} \int_o^L \Big(\frac{du}{ds} + \frac{w}{R} \Big)^2 ds \; + \\ \frac{1}{2L} \int_o^L \Big(\frac{dv}{ds} \Big)^2 ds + \eta \Big(\frac{\beta}{R} - \frac{d^2 v}{ds^2} \Big) \; - \\ \zeta \Big\{ \frac{d^2 u}{ds^2} + \frac{1}{R} \frac{dw}{ds} + w \frac{d}{ds} \Big(\frac{1}{R} \Big) \Big\} \Big] \qquad \text{...(A-6)} \end{aligned}$$

in which j denotes the stage of the configuration.

By substituting Equation (A-6) into Equation (A-4) we have the following :

$$
{}^{i}U_{\epsilon_o} = E \int_{vol} \Big[\sum_{j=o}^{i-1} {}^{j}\{ (w_s - \frac{u}{R}) + \frac{1}{2L} \int_o^L (u_s + \frac{w}{R})^2 ds + \frac{1}{2L} \int_o^L (v_s)^2 ds + \eta (\frac{\beta}{R} - v_{ss}) - \zeta (u_{ss} + \frac{1}{R} w_s + w \frac{d}{ds} (\frac{1}{R})) \} \Big] *
$$

$$
{}^{i}[(w_s - \frac{u}{R}) + \frac{1}{2L} \int_o^L (u_s + \frac{w}{R})^2 ds + \frac{1}{2L} \int_o^L (v_s)^2 ds + \eta (\frac{\beta}{R} - v_{ss}) - \zeta (u_{ss} + \frac{1}{R} w_s + w \frac{d}{ds} (\frac{1}{R}))] \, dV \quad \text{...(A-7)}
$$

For two dimensional problem, the previous expression becomes :

$$
{}^{i}U_{\epsilon_o} = E \int_{vol} \Big[\sum_{j=o}^{i-1} {}^{j}\{ (w_s - \frac{u}{R}) + \frac{1}{2L} \int_o^L (u_s + \frac{w}{R})^2 ds - \zeta (u_{ss} + \frac{1}{R} w_s + w \frac{d}{ds} (\frac{1}{R})) \} \Big] *
$$

$$
{}^{i}[(w_s - \frac{u}{R}) + \frac{1}{2L} \int_o^L (u_s + \frac{w}{R})^2 ds - \zeta (u_{ss} + \frac{1}{R} w_s + w \frac{d}{ds} (\frac{1}{R}))] \, dV \quad \text{...(A-8)}
$$

It should be noted that the terms : ${}^{i}(w_s - \frac{u}{R})$, and ${}^{i}(u_{ss} + \frac{1}{R} w_s + w \frac{d}{ds} (\frac{1}{R}))$ have no contribution to the " initial strain stiffness " because they are linear in the generalized coordinates. Therefore :

$$
{}^{i}U_{\epsilon_o} = E \int_{vol} [\sum_{j=o}^{i-1} {}^{j}\{ (w_s - \frac{u}{R}) + \frac{1}{2L} \int_o^L (u_s + \frac{w}{R})^2 ds - \varsigma (u_{ss} + \frac{1}{R} w_s + w \frac{d}{ds} (\frac{1}{R})) \}] * {}^{i}[\frac{1}{2L} \int_o^L (u_s + \frac{w}{R})^2 ds] \; dV
$$

.... (A-9)

In terms of γ , and realizing that $ds = R\, d\phi = R\, \theta\, d\gamma$, the above equation can be written as :

$$
{}^{i}U_{\epsilon_o} = E \int_{vol} [\sum_{j=o}^{i-1} {}^{j}\{ (\frac{dw}{R\, \theta\, d\gamma} - \frac{u}{R}) + \frac{1}{2L} \int_o^1 (\frac{du}{R\, \theta\, d\gamma} + \frac{w}{R})^2 R\theta\, d\gamma - \varsigma (\frac{1}{R^2 \theta^2} \frac{d^2u}{d\gamma^2} + \frac{du}{d\gamma} \frac{d^2\gamma}{ds^2} + \frac{1}{R} \frac{dw}{R\theta\, d\gamma} + \frac{w}{R} \frac{d^2\gamma}{ds\, d\gamma}) \}] * {}^{i}[\frac{1}{2L} \int_o^1 (\frac{du}{R\, \theta\, d\gamma} + \frac{w}{R})^2 R\, \theta\, d\gamma] \; dV
$$

$$ {}^{i}U_{\epsilon_o} = E \int_s \int_A \Big[\sum_{j=o}^{i-1} {}^{j}\Big\{ \frac{1}{R\,\theta} (w_\gamma - \theta\, u) + \frac{1}{2\,L} \int_o^1 \frac{1}{R\,\theta} (u_\gamma + \theta\, w)^2 \, d\gamma $$

$$ - \frac{\varsigma}{R\,\theta} \Big(\frac{1}{R\,\theta} u_{\gamma\gamma} + R\,\theta\, u_\gamma\, \gamma_{ss} + \frac{1}{R} w_\gamma + \theta\, w\, \gamma_{s\gamma} \Big) \Big\}\Big] * $$

$$ {}^{i}\Big[\frac{1}{2\,L} \int_o^1 \frac{1}{R\,\theta} (u_\gamma + \theta\, w)^2 \, d\gamma \Big] \; dA \; ds $$

.... (A-10)

Since the cross section is presumed to have two axes of symmetry , $\int_A \varsigma \, (\; .... \;) \, dA = 0$. Therefore the third part of the first term in Equation (A-10) may be dropped from the equation.

$$ {}^{i}U_{\epsilon_o} = E\,A \int_o^1 \Big[\sum_{j=o}^{i-1} {}^{j}\Big\{ \frac{1}{R\,\theta} (w_\gamma - \theta\, u) + \frac{1}{2\,L} \int_o^1 \frac{1}{R\,\theta} (u_\gamma + \theta\, w)^2 d\gamma \Big\}\Big] * $$

$$ {}^{i}\Big[\frac{1}{2\,L} \int_o^1 \frac{1}{R\,\theta} (u_\gamma + \theta\, w)^2 \, d\gamma \Big] \; R\,\theta\, d\gamma \; . $$

.... (A-11)

The initial strain stiffness matrix ${}^{i}[\, k_{\epsilon_o} \,]$ is equal to the second derivative of ${}^{i}U_{\epsilon_o}$ with respect to the generalized coordinates ${}^{i}q_m$.

A.3 UPDATED COORDINATES PROCEDURE

The loads are applied in increments. At the beginning of each increment the geometry of the structure is updated. In addition to the usual stiffness matrices [k], [n1], and [n2], there is the initial strain stiffness matrix as explained previously. The steps of the calculation are as follows :

1) Set load increment (and check if the intended total load has been applied).
2) Determine the most up-to-date geometry of the structure by using the latest joint displacements and rotations , and update the linear stiffness matrix.
3) Form the tangent stiffness matrix , $[K_T]$, according to the one of the following cases :

   a) For the first load increment :

$$[K_T] = [K] + [N1(\{Q\})] + [N2(\{Q\})]$$

   b) For other load increments :

$$[K_T] = [K] + [K_{\epsilon_o}] + [N1(\{Q\})] + [N2(\{Q\})]$$

   in which $[K_{\epsilon_o}]$ is the structure initial strain stiffness matrix.

4) Solve for $\{\Delta Q\}$ from :

$$\{\Delta Q\} = [K_T]^{-1} \text{(load increment vector)}$$

5) Add $\{\Delta Q\}$ to the latest $\{Q\}$ to obtain a new $\{Q\}$.

6) Based on the new $\{Q\}$, evaluate $[N1(\{Q\})]$ and $[N2(\{Q\})]$.

7) Form tangent and secant stiffness matrices and resistance force vector as

a) For the first load increment :

$$[K_T] = [K] + [N1(\{Q\})] + [N2(\{Q\})]$$

$$[K_S] = [K] + \frac{1}{2}[N1(\{Q\})] + \frac{1}{3}[N2(\{Q\})]$$

b) For other load increments :

$$[K_T] = [K] + [K_{\epsilon_o}] + [N1(\{Q\})] + [N2(\{Q\})]$$

$$[K_S] = [K] + [K_{\epsilon_o}] + \frac{1}{2}[N1(\{Q\})] + \frac{1}{3}[N2(\{Q\})]$$

Resistant force vector $= [K_S]\{Q\}$

8) Evaluate the unbalanced force vector $\{\Delta R\}$ as :

$$\{\Delta R\} = \text{load increment vector} - \text{resistance force vector}$$

9) If the unbalanced force vector , $\{\Delta R\}$, is sufficiently small , return to 1 .

10) Return to 4 but use the unbalanced force vector as the load increment vector.

APPENDIX B

INCREMENTAL STIFFNESS MATRICES, [n1] AND [n2], BASED ON THE AVERAGE AXIAL STRAIN MODEL

B.1 THE FIRST ORDER INCREMENTAL STIFFNESS MATRIX, [n1]

Only nonzero terms are given.

$$n1_{1,1} = \frac{E\ A}{2\ L} [(18\ A_3 - 12\ A_5 + 8\ A_6)\ \lambda_o + \theta\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\}]$$

$$n1_{1,2} = \frac{E\ A\ \theta}{4\ L} [(3\ A_2 - 2\ A_4)\ \lambda_6 + (6\ A_3 - 2\ A_5)\ \lambda_7 + (3\ A_5 - 4\ A_6)\ \lambda_8]$$

$$n1_{1,3} = \frac{E\ A}{2\ L} [\theta\ (-6\ A_3 - 2\ A_4 + 6\ A_5 - 4\ A_6)\ \lambda_o + (1 - \frac{\theta^2}{12})\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\} - \frac{\theta^2}{2}\ \{(1.5\ A_2 - A_4)\ \lambda_2 + (3\ A_3 - A_5)\ \lambda_3 + (1.5\ A_3 - A_4 + 1.5\ A_5 - 2\ A_6)\ \lambda_4 + \theta\ (1.5\ A_2 - 1.5\ A_3)\ \lambda_9 + \theta\ (1.5\ A_3 - 0.5\ A_5)\ \lambda_{10}\}]$$

$$n1_{1,5} = \frac{E\ A}{2\ L}\ R1\ \theta\ [(-3\ A_2 + 12\ A_3 + 2\ A_4 - 7\ A_5 + 4\ A_6)\ \lambda_o + \frac{\theta}{12}\ \{(-12\ A_1 + 15\ A_2 - 8\ A_4)\ \lambda_2 + (-6\ A_2 + 30\ A_3 - 8\ A_5)\ \lambda_3 + (-6\ A_4 + 15\ A_5 - 16\ A_6)\ \lambda_4 + \theta\ (-12\ A_1 + 15\ A_2 - 12\ A_3)\ \lambda_9 + \theta\ (-3\ A_2 + 15\ A_3 - 4\ A_5)\ \lambda_{10}\}]$$

$$n1_{1,6} = \frac{E\ A\ R1\ \theta^2}{4\ L} [(2\ A_1 - 2\ A_2 + A_4)\ \lambda_6 + (A_2 - 4\ A_3 + A_5)\ \lambda_7 + (A_4 - 2\ A_5 + 2\ A_6)\ \lambda_8]$$

$$nl_{1,7} = \frac{E\ A}{2\ L} (- 18\ A_3 + 12\ A_5 - 8\ A_6)\ \lambda_o$$

$$nl_{1,8} = - nl_{1,2}$$

$$nl_{1,9} = \frac{E\ A}{2\ L} [\ \theta (- 9\ A_3 + 6\ A_5 - 4\ A_6)\ \lambda_o +$$

$$(1 - \frac{\theta^2}{12}) \{(2\ A_4 - 3\ A_2)\ \lambda_2 + (2\ A_5 - 6\ A_3)\ \lambda_3 + (4\ A_6 - 3\ A_5)\ \lambda_4$$

$$+ \theta\ (3\ A_3 - 3\ A_2)\ \lambda_9 + \theta\ (A_5 - 3\ A_3)\ \lambda_{10}\} -$$

$$\frac{\theta^2}{2} \{\ (1.5\ A_2 - A_4)\ \lambda_2 + (3\ A_3 - A_5)\ \lambda_3 + (1.5\ A_5 - 2\ A_6)\ \lambda_4 +$$

$$\theta\ (1.5\ A_2 - 1.5\ A_3)\ \lambda_9 + \theta\ (1.5\ A_3 - 0.5\ A_5)\ \lambda_{10}\ \}\ ]$$

$$nl_{1,11} = \frac{E\ A\ R2\ \theta}{2\ L} [\ (6\ A_3 - 5\ A_5 + 4\ A_6)\ \lambda_o + \frac{\theta}{12} \{\ (3\ A_2 - 4\ A_4)\ \lambda_2 +$$

$$(6\ A_3 - 4\ A_5)\ \lambda_3 + (3\ A_5 - 8\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 6\ A_3)\ \lambda_9$$

$$+ \theta\ (3\ A_3 - 2\ A_5)\ \lambda_{10}\ \}\ ]$$

$$nl_{1,12} = \frac{E\ A\ R2\ \theta^2}{4\ L} [\ (A_4 - A_2)\ \lambda_6 + (A_5 - 2\ A_3)\ \lambda_7 + (2\ A_6 - A_5)\ \lambda_8\ ]$$

$$nl_{2,2} = \frac{E\ A}{2\ L} (\ 18\ A_3 - 12\ A_5 + 8\ A_6\ )\ \lambda_o$$

$$nl_{2,3} = \frac{E\ A}{2\ L} (1 - \frac{\theta^2}{12}) [(3\ A_2 - 2\ A_4)\ \lambda_6 + (6\ A_3 - 2\ A_5)\ \lambda_7 + (3\ A_5 - 4\ A_6)\ \lambda_8]$$

$$nl_{2,5} = \frac{E\ A\ R1\ \theta^2}{24\ L} [(3\ A_2 - 2\ A_4)\ \lambda_6 + (6\ A_3 - 2\ A_5)\ \lambda_7 + (3\ A_5 - 4\ A_6)\ \lambda_8]$$

$$nl_{2,6} = \frac{E\ A\ R1\ \theta}{2\ L} (\ 3\ A_2 - 12\ A_3 - 2\ A_4 + 7\ A_5 - 4\ A_6\ )\ \lambda_o$$

$$nl_{2,7} = nl_{1,2}$$

$$nl_{2,8} = - nl_{2,2}$$

$$nl_{2,9} = - nl_{2,3}$$

$$nl_{2,11} = \frac{E\ A\ R2\ \theta^2}{24\ L} [(2\ A_4 - 3\ A_2)\ \lambda_6 + (2\ A_5 - 6\ A_3)\ \lambda_7 + (4\ A_6 - 3\ A_5)\ \lambda_8]$$

$$nl_{2,12} = \frac{E\ A\ R2\ \theta}{2\ L} (- 6\ A_3 + 5\ A_5 - 4\ A_6)\ \lambda_o$$

$$nl_{3,3} = \frac{E\ A}{2\ L} [\ \theta^2 (1.5\ A_3 + 2\ A_4 - 3\ A_5 + 2\ A_6)\ \lambda_o -$$
$$(1 - \frac{\theta^2}{12})\ \theta\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 +$$
$$(3\ A_3 - 2\ A_4 + 3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}$$

$$nl_{3,5} = \frac{E\ A\ R1\ \theta}{2\ L} [\ \theta\ (1.5\ A_2 - 4.5\ A_3 - 2\ A_4 + 3.5\ A_5 - 2\ A_6)\ \lambda_o$$
$$- \frac{\theta^2}{24} \{ (3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 +$$
$$(3\ A_3 - 2\ A_4 + 3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10} \}$$
$$- (1 - \frac{\theta^2}{12}) \{ (2\ A_1 - 2\ A_2 + A_4)\ \lambda_2 + (A_2 - 4\ A_3 + A_5)\ \lambda_3 +$$
$$(A_4 - 2\ A_5 + 2\ A_6)\ \lambda_4 + \theta\ (2\ A_1 - 2\ A_2 + 1.5\ A_3)\ \lambda_9 +$$
$$\theta\ (0.5\ A_2 - 2\ A_3 + 0.5\ A_5)\ \lambda_{10} \}]$$

$$nl_{3,6} = \frac{E\ A\ R1\ \theta}{2\ L} (1 - \frac{\theta^2}{12}) [\ (2\ A_1 - 2\ A_2 + A_4)\ \lambda_6 + (A_2 - 4\ A_3 + A_5)\ \lambda_7$$
$$+ (A_4 - 2\ A_5 + 2\ A_6)\ \lambda_8\ ]$$

$$nl_{3,7} = \frac{E\ A}{2\ L} [\ \theta\ (6\ A_3 + 2\ A_4 - 6\ A_5 + 4\ A_6)\ \lambda_o -$$
$$0.25\ \theta^2 \{ (3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 +$$
$$(3\ A_3 - 2\ A_4 + 3\ A_5 - 4\ A_6)\ \lambda_4 + \theta (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10} \}$$
$$- (1 - \frac{\theta^2}{12}) \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4$$
$$+ \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10} \}]$$

$$nl_{3,8} = - nl_{2,3}$$

$$nl_{3,9} = \frac{E\ A\ \theta}{2\ L} [\ \theta\ (3\ A_3 + A_4 - 3\ A_5 + 2\ A_6)\ \lambda_o + (1 - \frac{\theta^2}{12})\ (1.5\ A_3 - A_4)\ \lambda_4\ ]$$

$$nl_{3,11} = \frac{E\ A\ R2\ \theta}{2\ L} [\ \theta\ (-1.5\ A_3 - A_4 + 2.5\ A_5 - 2\ A_6)\ \lambda_o +$$

$$\frac{\theta^2}{24} \{\ (3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 +$$

$$(3\ A_3 - 2\ A_4 + 3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 +$$

$$\theta\ (\ 3\ A_3 - A_5)\ \lambda_{10}\ \} +$$

$$(1 - \frac{\theta^2}{12})\ \{(A_2 - A_4)\ \lambda_2 + (2\ A_3 - A_5)\ \lambda_3 + (A_5 - 2\ A_6)\ \lambda_4 +$$

$$\theta\ (A_2 - 1.5\ A_3)\ \lambda_9 + \theta\ (A_3 - 0.5\ A_5)\ \lambda_{10}\}\ ]$$

$$nl_{3,12} = \frac{E\ A\ R2\ \theta}{2\ L} (1 - \frac{\theta^2}{12})\ [\ (A_4 - A_2)\ \lambda_6 + (A_5 - 2\ A_3)\ \lambda_7 + (2\ A_6 - A_5)\ \lambda_8\ ]$$

$$nl_{5,5} = \frac{E\ A}{2\ L} (\ R1\ \theta\ )^2\ [\ (2\ A_1 - 4\ A_2 + 8\ A_3 + 2\ A_4 - 4\ A_5 + 2\ A_6)\ \lambda_o -$$

$$\frac{\theta}{6}\ \{(2\ A_1 - 2\ A_2 + A_4)\ \lambda_2 + (A_2 - 4\ A_3 + A_5)\ \lambda_3 +$$

$$(A_4 - 2\ A_5 + 2\ A_6)\ \lambda_4 + \theta\ (2\ A_1 - 2\ A_2 + 1.5\ A_3)\ \lambda_9$$

$$+ \theta\ (0.5\ A_2 - 2\ A_3 + 0.5\ A_5)\ \lambda_{10}\}\ ]$$

$$nl_{5,6} = \frac{E\ A\ R1^2\ \theta^3}{24\ L} [\ (2\ A_1 - 2\ A_2 + A_4)\ \lambda_6 + (A_2 - 4\ A_3 + A_5)\ \lambda_7 +$$

$$(\ A_4 - 2\ A_5 + 2\ A_6\ )\ \lambda_8\ ]$$

$$nl_{5,7} = \frac{E\ A\ R1\ \theta}{2\ L} [\ (3\ A_2 - 12\ A_3 - 2\ A_4 + 7\ A_5 - 4\ A_6)\ \lambda_o -$$

$$\frac{\theta}{12} \{(12\ A_1 - 9\ A_2 + 4\ A_4)\ \lambda_2 + (6\ A_2 - 18\ A_3 + 4\ A_5)\ \lambda_3$$

$$+ (6\ A_4 - 9\ A_5 + 8\ A_6)\ \lambda_4 + \theta\ (12\ A_1 - 9\ A_2 + 6\ A_3)\ \lambda_9$$

$$+ \theta\ (3\ A_2 - 9\ A_3 + 2\ A_5)\ \lambda_{10}\}\ ]$$

$$nl_{5,8} = \frac{E\ A\ R1\ \theta^2}{24\ L} [(2\ A_4 - 3\ A_2)\ \lambda_6 + (2\ A_5 - 6\ A_3)\ \lambda_7 + (4\ A_6 - 3\ A_5)\ \lambda_8]$$

$$nl_{5,9} = \frac{E\ A\ R1\ \theta}{2\ L} \left[\theta\ (1.5\ A_2-6\ A_3-A_4+3.5\ A_5-2\ A_6)\ \lambda_o + \right.$$

$$(1-\frac{\theta^2}{12})\ \{(2\ A_1-2\ A_2+A_4)\ \lambda_2 + (A_2-4\ A_3+A_5)\ \lambda_3 +$$

$$(A_4-2\ A_5+2\ A_6)\ \lambda_4 + \theta\ (2\ A_1-2\ A_2+1.5\ A_3)\ \lambda_9 +$$

$$\theta\ (0.5\ A_2-2\ A_3+0.5\ A_5)\ \lambda_{10}\} -$$

$$\frac{\theta^2}{24}\ \{(3\ A_2-2\ A_4)\ \lambda_2+ (6\ A_3-2\ A_5)\ \lambda_3+(3\ A_5-4\ A_6)\ \lambda_4$$

$$\left. + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\} \right]$$

$$nl_{5,11} = \frac{E\ A}{2\ L}\ R1\ R2\ \theta^2 \left[(-A_2+4\ A_3+A_4-3\ A_5+2\ A_6)\ \lambda_o + \right.$$

$$\frac{\theta}{12}\ \{(2\ A_1-A_2)\ \lambda_2 + (A_2-2\ A_3)\ \lambda_3 + (A_4-A_5)\ \lambda_4 +$$

$$\left. \theta\ (2\ A_1-A_2)\ \lambda_9 + \theta\ (0.5\ A_2-A_3)\ \lambda_{10}\} \right]$$

$$nl_{5,12} = \frac{E\ A\ \theta}{24\ L}\ R1\ R2\ \theta^2\ [\ (A_4-A_2)\ \lambda_6 + (A_5-2\ A_3)\ \lambda_7 + (2\ A_6-A_5)\ \lambda_8]$$

$$nl_{6,6} = \frac{E\ A}{2\ L}\ R1^2\ \theta^2\ (\ 2\ A_1\ -\ 4\ A_2\ +\ 8\ A_3\ +\ 2\ A_4\ -\ 4\ A_5\ +\ 2\ A_6\ )\ \lambda_o$$

$$nl_{6,7} = \frac{E\ A}{4\ L}\ R1\ \theta^2\ [\ (2\ A_1-2\ A_2+A_4)\ \lambda_6 + (A_2-4\ A_3+A_5)\ \lambda_7 +$$

$$(\ A_4\ -\ 2\ A_5\ +\ 2\ A_6\ )\ \lambda_8\ ]$$

$$nl_{6,8} = -\ nl_{2,6}$$

$$nl_{6,9} = \frac{E\ A}{2\ L}\ R1\ \theta\ (1-\frac{\theta^2}{12})\ [\ (-2\ A_1+2\ A_2-A_4)\ \lambda_6 + (-A_2+4\ A_3-A_5)\ \lambda_7 +$$

$$(-A_4+2\ A_5-2\ A_6)\ \lambda_8\ ]$$

$$nl_{6,11} = \frac{E\ A\ \theta}{24\ \ L}\ R1\ R2\ \theta^2\ [\ (-2\ A_1+2\ A_2-A_4)\ \lambda_6 + (-A_2+4\ A_3-A_5)\ \lambda_7 +$$

$$(-A_4+2\ A_5-2\ A_6)\ \lambda_8\ ]$$

$$nl_{6,12} = \frac{E\ A}{2\ L}\ R1\ R2\ \theta^2\ (\ -\ A_2\ +\ 4\ A_3\ +\ A_4\ -\ 3\ A_5\ +\ 2\ A_6\ )\ \lambda_o$$

$$nl_{7,7} = \frac{E\ A}{2\ L} \left[(18\ A_3 - 12\ A_5 + 8\ A_6)\ \lambda_o - \theta\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\} \right]$$

$$nl_{7,8} = -\ nl_{1,2}$$

$$nl_{7,9} = \frac{E\ A}{2\ L} \left[\theta\ (9\ A_3 - 6\ A_5 + 4\ A_6)\ \lambda_o + (1 - \frac{\theta^2}{12})\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\} - \frac{\theta^2}{4}\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\} \right]$$

$$nl_{7,11} = \frac{E\ A\ R2\ \theta}{2\ L} \left[(-\ 6\ A_3 + 5\ A_5 - 4\ A_6)\ \lambda_o + \frac{\theta}{12}\ \{(9\ A_2 - 8\ A_4)\ \lambda_2 + (18\ A_3 - 8\ A_5)\ \lambda_3 + (9\ A_5 - 16\ A_6)\ \lambda_4 + \theta\ (9\ A_2 - 12\ A_3)\ \lambda_9 + \theta\ (9\ A_3 - 4\ A_5)\ \lambda_{10}\} \right]$$

$$nl_{7,12} = \frac{E\ A}{4\ L}\ R2\ \theta^2 \left[(A_4 - A_2)\ \lambda_6 + (A_5 - 2\ A_3)\ \lambda_7 + (2\ A_6 - A_5)\ \lambda_8 \right]$$

$$nl_{8,8} = nl_{2,2}$$

$$nl_{8,9} = nl_{2,3}$$

$$nl_{8,11} = -\ nl_{2,11}$$

$$nl_{8,12} = -\ nl_{2,12}$$

$$nl_{9,9} = \frac{E\ A\ \theta}{2\ L} \left[\frac{\theta}{2}\ (9\ A_3 - 6\ A_5 + 4\ A_6)\ \lambda_0 + (1 - \frac{\theta^2}{12})\ \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 + \theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\} \right]$$

$$nl_{9,11} = \frac{E\ A\ R2\ \theta}{2\ L} \left[\frac{\theta}{2} (- 6\ A_3 + 5\ A_5 - 4\ A_6)\ \lambda_o \right. +$$

$$\frac{\theta^2}{24} \{(3\ A_2 - 2\ A_4)\ \lambda_2 + (6\ A_3 - 2\ A_5)\ \lambda_3 + (3\ A_5 - 4\ A_6)\ \lambda_4 +$$

$$\theta\ (3\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (3\ A_3 - A_5)\ \lambda_{10}\} +$$

$$(1 - \frac{\theta^2}{12})\ \{(A_4 - A_2)\ \lambda_2 + (A_5 - 2\ A_3)\ \lambda_3 + (2\ A_6 - A_5)\ \lambda_4 +$$

$$\left. \theta\ (1.5\ A_3 - A_2)\ \lambda_9 + \theta\ (0.5\ A_5 - A_3)\ \lambda_{10}\} \right]$$

$$nl_{9,12} = \frac{E\ A\ R2\ \theta}{2\ L} (1 - \frac{\theta^2}{12}) \left[(A_2 - A_4)\ \lambda_6 + (2\ A_3 - A_5)\ \lambda_7 + (A_5 - 2\ A_6)\ \lambda_8 \right]$$

$$nl_{11,11} = \frac{E\ A}{2\ L} R2^2\ \theta^2 \left[(2\ A_3 - 2\ A_5 + 2\ A_6)\ \lambda_o \right. -$$

$$\frac{\theta}{12} \{(2\ A_2 - 2\ A_4)\ \lambda_2 + (4\ A_3 - 2\ A_5)\ \lambda_3 + (2\ A_5 - 4\ A_6)\ \lambda_4 +$$

$$\left. \theta\ (2\ A_2 - 3\ A_3)\ \lambda_9 + \theta\ (2\ A_3 - A_5)\ \lambda_{10}\} \right]$$

$$nl_{11,12} = \frac{E\ A\ \theta}{24\ L} R2^2\ \theta^2 \left[(A_2 - A_4)\ \lambda_6 + (2\ A_3 - A_5)\ \lambda_7 + (A_5 - 2\ A_6)\ \lambda_8 \right]$$

$$nl_{12,12} = \frac{E\ A}{2\ L} R2^2\ \theta^2 (2\ A_3 - 2\ A_5 + 2\ A_6)\ \lambda_o$$

in which :

$$A_1 = \frac{1}{2\ b_2\ \theta^2} \log \left(\frac{b_1 + 2\ b_2\ \theta}{b_1} \right)$$

$$A_2 = \frac{4}{\theta} \left[\frac{1}{2\ \theta\ b_2} - \frac{b_1}{(2\ \theta\ b_2)^2} \log \left(\frac{b_1 + 2\ b_2\ \theta}{b_1} \right) \right]$$

$$A_3 = \frac{1}{\theta^2\ b_2} \left[1 - \frac{2\ b_1}{2\ \theta\ b_2} + \frac{2\ b_1^2}{(2\ \theta\ b_2)^2} \log \left(\frac{b_1 + 2\ b_2\ \theta}{b_1} \right) \right]$$

$$A_4 = 1.5\ A_3$$

$$A_5 = \frac{2}{\theta^2 b_2} [1 - \frac{3 b_1}{4 \theta b_2} + \frac{3 b_1^2}{(2 \theta b_2)^2} - \frac{3 b_1^3}{(2 \theta b_2)^3} \log (\frac{b_1+2 b_2 \theta}{b_1})]$$

$$A_6 = \frac{9}{8 \theta^2 b_2} [1 - \frac{4 b_1}{6 \theta b_2} + \frac{2 b_1^2}{(2 \theta b_2)^2} - \frac{4 b_1^3}{(2 \theta b_2)^3} + \frac{4 b_1^4}{(2 \theta b_2)^4} \log (\frac{b_1 + 2 b_2 \theta}{b_1})]$$

For circular arch, the expressions of A_1, A_2, A_3, A_4, A_5, and A_6 become

$$A_1 = \frac{1}{\theta b_1} \quad ; \quad A_2 = \frac{2}{\theta b_1}$$

$$A_3 = \frac{4}{3 \theta b_1} \quad ; \quad A_4 = 1.5 A_3$$

$$A_5 = \frac{3}{\theta b_1} \quad ; \quad A_6 = \frac{9}{5 \theta b_1}$$

The expressions for λ_o, λ_1, ... λ_{12} are given by :

$$\lambda_o = - \frac{\theta}{2} u_A - (1-\frac{\theta^2}{12}) w_A - \frac{R1 \theta^2}{12} \theta_{yA} - \frac{\theta}{2} u_B + (1-\frac{\theta^2}{12}) w_B + \frac{R2 \theta^2}{12} \theta_{yB}$$

$$\lambda_1 = u_A$$

$$\lambda_2 = - \theta w_A + R1 \theta \theta_{yA}$$

$$\lambda_3 = - 3 u_A + 2 \theta w_A - 2 R1 \theta \theta_{yA} + 3 u_B + \theta w_B - R2 \theta \theta_{yB}$$

$$\lambda_4 = 2 u_A - \theta w_A + R1 \theta \theta_{yA} - 2 u_B - \theta w_B + R2 \theta \theta_{yB}$$

$$\lambda_5 = v_A$$

$$\lambda_6 = - R1 \theta \theta_{xA}$$

$$\lambda_7 = -3\ v_A + 2\ R1\ \theta\ \theta_{xA} + 3\ v_B + R2\ \theta\ \theta_{xB}$$

$$\lambda_8 = 2\ v_A - R1\ \theta\ \theta_{xA} - 2\ v_B - R2\ \theta\ \theta_{xB}$$

$$\lambda_9 = w_A$$

$$\lambda_{10} = -w_A + w_B$$

$$\lambda_{11} = \beta_A$$

$$\lambda_{12} = -\beta_A + \beta_B$$

B.2 THE SECOND ORDER INCREMENTAL STIFFNESS MATRIX, [n2]

Only nonzero terms are given. The expressions for A_1, A_2, ... A_6 and λ_0, λ_1, ... λ_{12} remain the same.

$$n2_{1,1} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ [\ (B_1+B_2+B_3+B_4)\ (18\ A_3-12\ A_5+8\ A_6)\ + \{(2\ A_4-3\ A_2)\ \lambda_2 + (2\ A_5-6\ A_3)\ \lambda_3 + (4\ A_6-3\ A_5)\ \lambda_4 + \theta\ (3\ A_3-3\ A_2)\ \lambda_9 + \theta\ (A_5-3\ A_3)\ \lambda_{10}\}^2\ ]$$

$$n2_{1,2} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ [\ \{\ (2\ A_4-3\ A_2)\ \lambda_6 + (2\ A_5-6\ A_3)\ \lambda_7 + (4\ A_6-3\ A_5)\ \lambda_8\ \}\ \{\ (2\ A_4-3\ A_2)\ \lambda_2 + (2\ A_5-6\ A_3)\ \lambda_3 + (4\ A_6-3\ A_5)\ \lambda_4 + \theta\ (3\ A_3-3\ A_2)\ \lambda_9 + \theta\ (A_5-3\ A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,3} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ \theta\ [\ (B_1+B_2+B_3+B_4)\ (-6\ A_3-2\ A_4+6\ A_5-4\ A_6) + 0.5\ \{(3\ A_2-2\ A_4)\ \lambda_2+(6\ A_3-2\ A_5)\ \lambda_3+(3\ A_3-2\ A_4+3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}\ \{\ (2\ A_4-3\ A_2)\ \lambda_2 + (2\ A_5-6\ A_3)\ \lambda_3 + (4\ A_6-3\ A_5)\ \lambda_4 + \theta\ (3\ A_3-3\ A_2)\ \lambda_9 + \theta\ (A_5-3\ A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,5} = \frac{E\,A\,\theta}{4\,L^2}\ (b_1+\theta\,b_2)\ R1\ \theta\ [\ (B_1+B_2+B_3+B_4)\ (-3\,A_2+12\,A_3+2\,A_4$$
$$-7\,A_5+4\,A_6) + \{(2\,A_1-2\,A_2+A_4)\ \lambda_2 + (A_2-4\,A_3+A_5)\ \lambda_3$$
$$+ (A_4-2\,A_5+2\,A_6)\ \lambda_4 + \theta\ (2\,A_1-2\,A_2+1.5\,A_3)\ \lambda_9 +$$
$$\theta\ (0.5\,A_2-2\,A_3+0.5\,A_5)\ \lambda_{10}\}\ \{(2\,A_4-3\,A_2)\ \lambda_2 +$$
$$(2\,A_5-6\,A_3)\ \lambda_3 + (4\,A_6-3\,A_5)\ \lambda_4 + \theta\ (3\,A_3-3\,A_2)\ \lambda_9 +$$
$$\theta\ (A_5-3\,A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,6} = \frac{E\,A\,\theta}{4\,L^2}\ (b_1+\theta\,b_2)\ R1\ \theta\ \{(-2\,A_1+2\,A_2-A_4)\ \lambda_6 + (-A_2+4\,A_3-A_5)\ \lambda_7$$
$$+ (-A_4+2\,A_5-2\,A_6)\ \lambda_8\}\ \{(2\,A_4-3\,A_2)\ \lambda_2 +$$
$$(2\,A_5-6\,A_3)\ \lambda_3 + (4\,A_6-3\,A_5)\ \lambda_4 + \theta\ (3\,A_3-3\,A_2)\ \lambda_9 +$$
$$\theta\ (A_5-3\,A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,7} = \frac{E\,A\,\theta}{4\,L^2}\ (b_1+\theta\,b_2)\ [\ (B_1+B_2+B_3+B_4)\ (-18\,A_3+12\,A_5-8\,A_6) +$$
$$+ \{(3\,A_2-2\,A_4)\ \lambda_2 + (6\,A_3-2\,A_5)\ \lambda_3 + (3\,A_5-4\,A_6)\ \lambda_4$$
$$+ \theta\ (3\,A_2-3\,A_3)\ \lambda_9 + \theta\ (3\,A_3-A_5)\ \lambda_{10}\}\ \{(2\,A_4-3\,A_2)\ \lambda_2$$
$$+ (2\,A_5-6\,A_3)\ \lambda_3 + (4\,A_6-3\,A_5)\ \lambda_4 + \theta\ (3\,A_3-3\,A_2)\ \lambda_9$$
$$+ \theta\ (A_5-3\,A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,8} = \frac{E\,A\,\theta}{4\,L^2}\ (b_1+\theta\,b_2)\ [\ \{(3\,A_2-2\,A_4)\ \lambda_6 + (6\,A_3-2\,A_5)\ \lambda_7 +$$
$$(3\,A_5-4\,A_6)\ \lambda_8\}\ \{(2\,A_4-3\,A_2)\ \lambda_2 + (2\,A_5-6\,A_3)\ \lambda_3 +$$
$$(4\,A_6-3\,A_5)\ \lambda_4 + \theta\ (3\,A_3-3\,A_2)\ \lambda_9 + \theta\ (A_5-3\,A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,9} = \frac{E\,A\,\theta}{4\,L^2}\ (b_1+\theta\,b_2)\ [\ (B_1+B_2+B_3+B_4)\ \theta\ (-9\,A_3+6\,A_5-4\,A_6) +$$
$$\frac{\theta}{2}\ \{(3\,A_2-2\,A_4)\ \lambda_2 + (6\,A_3-2\,A_5)\ \lambda_3 + (3\,A_5-4\,A_6)\ \lambda_4$$
$$+ \theta\ (3\,A_2-3\,A_3)\ \lambda_9 + \theta\ (3\,A_3-A_5)\ \lambda_{10}\}\ \{(2\,A_4-3\,A_2)\ \lambda_2$$
$$+ (2\,A_5-6\,A_3)\ \lambda_3 + (4\,A_6-3\,A_5)\ \lambda_4 + \theta\ (3\,A_3-3\,A_2)\ \lambda_9$$
$$+ \theta\ (A_5-3\,A_3)\ \lambda_{10}\}\ ]$$

$$n2_{1,11} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R2\,\theta\,[\,(B_1+B_2+B_3+B_4)\,(6\,A_3-5\,A_5+4\,A_6)$$
$$+\{(A_4-A_2)\,\lambda_2 + (A_5-2\,A_3)\,\lambda_3 + (2\,A_6-A_5)\,\lambda_4 +$$
$$\theta\,(1.5\,A_3-A_2)\,\lambda_9 + \theta\,(0.5\,A_5-A_3)\,\lambda_{10}\}\,\{(2\,A_4-3\,A_2)\,\lambda_2$$
$$+ (2\,A_5-6\,A_3)\,\lambda_3 + (4\,A_6-3\,A_5)\,\lambda_4 + \theta\,(3\,A_3-3\,A_2)\,\lambda_9$$
$$+ \theta\,(A_5-3\,A_3)\,\lambda_{10}\}\,]$$

$$n2_{1,12} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R2\,\theta\,\{(A_2-A_4)\,\lambda_6 + (2\,A_3-A_5)\,\lambda_7 +$$
$$(A_5-2\,A_6)\,\lambda_8\}\,\{(2\,A_4-3\,A_2)\,\lambda_2 + (2\,A_5-6\,A_3)\,\lambda_3 +$$
$$(4\,A_6-3\,A_5)\,\lambda_4 + \theta\,(3\,A_3-3\,A_2)\,\lambda_9 + \theta\,(A_5-3\,A_3)\,\lambda_{10}\}$$

$$n2_{2,2} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,[\,(B_1+B_2+B_3+B_4)\,(18\,A_3-12\,A_5+8\,A_6) +$$
$$\{(2\,A_4-3\,A_2)\,\lambda_6 + (2\,A_5-6\,A_3)\,\lambda_7 + (4\,A_6-3\,A_5)\,\lambda_8\}^2\,]$$

$$n2_{2,3} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{\theta}{2}\,\{(3\,A_2-2\,A_4)\,\lambda_2 + (6\,A_3-2\,A_5)\,\lambda_3 +$$
$$(3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4 + \theta\,(3\,A_2-3\,A_3)\,\lambda_9 +$$
$$\theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,\{(2\,A_4-3\,A_2)\,\lambda_6 + (2\,A_5-6\,A_3)\,\lambda_7 +$$
$$(4\,A_6-3\,A_5)\,\lambda_8\}$$

$$n2_{2,5} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,[\,R1\,\theta\,\{(2\,A_1-2\,A_2+A_4)\,\lambda_2 + (A_2-4\,A_3+A_5)\,\lambda_3$$
$$+ (A_4-2\,A_5+2\,A_6)\,\lambda_4 + \theta\,(2\,A_1-2\,A_2+1.5\,A_3)\,\lambda_9 +$$
$$\theta\,(0.5\,A_2-2\,A_3+0.5\,A_5)\,\lambda_{10}\}\,\{(2\,A_4-3\,A_2)\,\lambda_6 +$$
$$(2\,A_5-6\,A_3)\,\lambda_7 + (4\,A_6-3\,A_5)\,\lambda_8\}\,]$$

$$n2_{2,6} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,\theta\,[\,(B_1+B_2+B_3+B_4)\,(3\,A_2-12\,A_3-2\,A_4+7\,A_5$$
$$-4\,A_6) - \{(2\,A_1-2\,A_2+A_4)\,\lambda_6 + (A_2-4\,A_3+A_5)\,\lambda_7 +$$
$$(A_4-2\,A_5+2\,A_6)\,\lambda_8\}\,\{(2\,A_4-3\,A_2)\,\lambda_6 + (2\,A_5-6\,A_3)\,\lambda_7$$
$$+ (4\,A_6-3\,A_5)\,\lambda_8\}\,]$$

$$n2_{2,7} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}\ \{(2\ A_4-3\ A_2)\ \lambda_6 + (2\ A_5-6\ A_3)\ \lambda_7 + (4\ A_6-3\ A_5)\ \lambda_8\}\ ]$$

$$n2_{2,8} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ [\ (B_1+B_2+B_3+B_4)\ (-18\ A_3+12\ A_5-8\ A_6) + \{(3\ A_2-2\ A_4)\ \lambda_6 + (6\ A_3-2\ A_5)\ \lambda_7 + (3\ A_5-4\ A_6)\ \lambda_8\}\ \{(2\ A_4-3\ A_2)\ \lambda_6 + (2\ A_5-6\ A_3)\ \lambda_7 + (4\ A_6-3\ A_5)\ \lambda_8\}\ ]$$

$$n2_{2,9} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ [\ \frac{\theta}{2}\ \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}\ \{(2\ A_4-3\ A_2)\ \lambda_6 + (2\ A_5-6\ A_3)\ \lambda_7 + (4\ A_6-3\ A_5)\ \lambda_8\}\ ]$$

$$n2_{2,11} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ [\ R2\ \theta\ \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4 + \theta\ (1.5\ A_3-A_2)\ \lambda_9 + \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\}\ \{(2\ A_4-3\ A_2)\ \lambda_6 + (2\ A_5-6\ A_3)\ \lambda_7 + (4\ A_6-3\ A_5)\ \lambda_8\}\ ]$$

$$n2_{2,12} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ R2\ \theta\ [\ (B_1+B_2+B_3+B_4)\ (-6\ A_3+5\ A_5-4\ A_6) + \{(A_2-A_4)\ \lambda_6 + (2\ A_3-A_5)\ \lambda_7 + (A_5-2\ A_6)\ \lambda_8\}\ \{(2\ A_4-3A_2)\ \lambda_6 + (2\ A_5-6\ A_3)\ \lambda_7 + (4\ A_6-3\ A_5)\ \lambda_8\}\ ]$$

$$n2_{3,3} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ \frac{\theta^2}{2}\ [\ (B_1+B_2+B_3+B_4)\ (3\ A_3+4\ A_4-6\ A_5+4\ A_6) + 0.5\ \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_3-2\ A_4+3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}^2]$$

$$
\begin{aligned}
n2_{3,5} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{R1\,\theta^2}{2}\,\Big[\,&(B_1+B_2+B_3+B_4)\,(3\,A_2-9\,A_3-4\,A_4+ \\
&7\,A_5-4\,A_6) + \{(2\,A_1-2\,A_2+A_4)\,\lambda_2 + (A_2-4\,A_3+A_5)\,\lambda_3 + \\
&(A_4-2\,A_5+2\,A_6)\,\lambda_4 + \theta\,(2\,A_1-2\,A_2+1.5\,A_3)\,\lambda_9 + \\
&\theta\,(0.5\,A_2-2\,A_3+0.5\,A_5)\,\lambda_{10}\}\,\{(3\,A_2-2\,A_4)\,\lambda_2 + \\
&(6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4 + \\
&\theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,\Big]
\end{aligned}
$$

$$
\begin{aligned}
n2_{3,6} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{R1\,\theta^2}{2}\,&\{(-2\,A_1+2\,A_2-A_4)\,\lambda_6 + (-A_2+4\,A_3- \\
&A_5)\,\lambda_7 + (-A_4+2\,A_5-2\,A_6)\,\lambda_8\}\,\{(3\,A_2-2\,A_4)\,\lambda_2 + \\
&(6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4 + \\
&\theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,]
\end{aligned}
$$

$$
\begin{aligned}
n2_{3,7} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{\theta^2}{2}\,\Big[&(B_1+B_2+B_3+B_4)\,(12\,A_3+4\,A_4-12\,A_5+8\,A_6) \\
&+ \{(3\,A_2-2\,A_4)\,\lambda_2 + (6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_5-4\,A_6)\,\lambda_4 \\
&+ \theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,\{(3\,A_2-2\,A_4)\,\lambda_2 \\
&+ (6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4 \\
&+ \theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,\Big]
\end{aligned}
$$

$$
\begin{aligned}
n2_{3,8} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{\theta}{2}\,&\{(3\,A_2-2\,A_4)\,\lambda_6 + (6\,A_3-2\,A_5)\,\lambda_7 + \\
&(3\,A_5-4\,A_6)\,\lambda_8\}\,\{(3\,A_2-2\,A_4)\,\lambda_2 + (6\,A_3-2\,A_5)\,\lambda_3 + \\
&(3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4 + \theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \\
&\theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,]
\end{aligned}
$$

$$
\begin{aligned}
n2_{3,9} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{\theta^2}{2}\,\Big[\,&(B_1+B_2+B_3+B_4)\,(6\,A_3+2\,A_4-6\,A_5+4\,A_6) \\
&+ 0.5\,\{\,(3\,A_2-2\,A_4)\,\lambda_2 + (6\,A_3-2\,A_5)\,\lambda_3 + \\
&(3\,A_5-4\,A_6)\,\lambda_4 + \theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\} \\
&\{(3\,A_2-2\,A_4)\,\lambda_2 + (6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_3-2\,A_4+3\,A_5- \\
&4\,A_6)\,\lambda_4 + \theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,\Big]
\end{aligned}
$$

$$n2_{3,11} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ \frac{R2\ \theta^2}{2}\ [(B_1+B_2+B_3+B_4)\ (-3\ A_3-2\ A_4+5\ A_5-4\ A_6)$$
$$+ \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4 +$$
$$\theta\ (1.5\ A_3-A_2)\ \lambda_9 + \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\}\ \{(3\ A_2-2\ A_4)\ \lambda_2$$
$$+ (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_3-2\ A_4+3\ A_5-4\ A_6)\ \lambda_4 +$$
$$\theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}\ ]$$

$$n2_{3,12} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ \frac{R2\ \theta^2}{2}\ \{\ (A_2-A_4)\ \lambda_6 + (2\ A_3-A_5)\ \lambda_7 +$$
$$(A_5-2\ A_6)\ \lambda_8\}\ \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 +$$
$$(3\ A_3-2\ A_4+3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 +$$
$$\theta\ (3\ A_3-A_5)\ \lambda_{10}\}$$

$$n2_{5,5} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ R1^2\ \theta^2\ [(B_1+B_2+B_3+B_4)\ (2\ A_1-4\ A_2+8\ A_3+2\ A_4$$
$$-4\ A_5+2\ A_6) + \{(2\ A_1-2\ A_2+A_4)\ \lambda_2 + (A_2-4\ A_3+A_5)\ \lambda_3 +$$
$$(A_4-2\ A_5+2\ A_6)\ \lambda_4 + \theta\ (2\ A_1-2\ A_2+1.5\ A_3)\ \lambda_9 +$$
$$\theta\ (0.5\ A_2-2\ A_3+0.5\ A_5)\ \lambda_{10}\}^2\ ]$$

$$n2_{5,6} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ R1^2\ \theta^2\ \{(2\ A_1-2\ A_2+A_4)\ \lambda_2 + (A_2-4\ A_3+A_5)\ \lambda_3$$
$$+ (A_4-2\ A_5+2\ A_6)\ \lambda_4 + \theta\ (2\ A_1-2\ A_2+1.5\ A_3)\ \lambda_9 +$$
$$\theta\ (0.5\ A_2-2\ A_3+0.5\ A_5)\ \lambda_{10}\}\ \{(-2\ A_1+2\ A_2-A_4)\ \lambda_6 +$$
$$(-A_2+4\ A_3-A_5)\ \lambda_7 + (-\ A_4+2\ A_5-2\ A_6)\ \lambda_8\}$$

$$n2_{5,7} = \frac{E\ A\ \theta}{4\ L^2}\ (b_1+\theta\ b_2)\ R1\ \theta\ [\ (B_1+B_2+B_3+B_4)\ (3\ A_2-12\ A_3-2\ A_4+7\ A_5$$
$$-4\ A_6) + \{(2\ A_1-2\ A_2+A_4)\ \lambda_2 + (A_2-4\ A_3+A_5)\ \lambda_3$$
$$+ (A_4-2\ A_5+2\ A_6)\ \lambda_4 + \theta\ (2\ A_1-2\ A_2+1.5\ A_3)\ \lambda_9 +$$
$$\theta\ (0.5\ A_2-2\ A_3+0.5\ A_5)\ \lambda_{10}\}\ \{(3\ A_2-2\ A_4)\ \lambda_2 +$$
$$(6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_3-2\ A_4+3\ A_5-4\ A_6)\ \lambda_4 +$$
$$\theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}\ ]$$

$$n2_{5,8} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,\theta\,\{(2\,A_1-2\,A_2+A_4)\,\lambda_2+(A_2-4\,A_3+A_5)\,\lambda_3+$$
$$(A_4-2\,A_5+2\,A_6)\,\lambda_4+\theta\,(2\,A_1-2\,A_2+1.5\,A_3)\,\lambda_9+$$
$$\theta\,(0.5\,A_2-2\,A_3+0.5\,A_5)\,\lambda_{10}\}\,\{(3\,A_2-2\,A_4)\,\lambda_6+$$
$$(6\,A_3-2\,A_5)\,\lambda_7+(3\,A_5-4\,A_6)\,\lambda_8\}$$

$$n2_{5,9} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{R1\,\theta^2}{2}\,[\,(B_1+B_2+B_3+B_4)\,(3\,A_2-12\,A_3-2\,A_4+$$
$$7\,A_5-4\,A_6)+\{(2\,A_1-2\,A_2+A_4)\,\lambda_2+(A_2-4\,A_3+A_5)\,\lambda_3+$$
$$(A_4-2\,A_5+2\,A_6)\,\lambda_4+\theta\,(2\,A_1-2\,A_2+1.5\,A_3)\,\lambda_9+$$
$$\theta\,(0.5\,A_2-2\,A_3+0.5\,A_5)\,\lambda_{10}\}\,\{(3\,A_2-2\,A_4)\,\lambda_2+$$
$$(6\,A_3-2\,A_5)\,\lambda_3+(3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4+$$
$$\theta\,(3\,A_2-3\,A_3)\,\lambda_9+\theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,]$$

$$n2_{5,11} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,R2\,\theta^2\,[\,(B_1+B_2+B_3+B_4)\,(-A_2+4\,A_3+A_4-3\,A_5$$
$$+2\,A_6)+\{(2\,A_1-2\,A_2+A_4)\,\lambda_2+(A_2-4\,A_3+A_5)\,\lambda_3+$$
$$(A_4-2\,A_5+2\,A_6)\,\lambda_4+\theta\,(2\,A_1-2\,A_2+1.5\,A_3)\,\lambda_9+$$
$$\theta\,(0.5\,A_2-2\,A_3+0.5\,A_5)\,\lambda_{10}\}\,\{(A_4-A_2)\,\lambda_2+(A_5-2\,A_3)\,\lambda_3$$
$$+(2\,A_6-A_5)\,\lambda_4+\theta\,(1.5\,A_3-A_2)\,\lambda_9+\theta\,(0.5\,A_5-A_3)\,\lambda_{10}\}]$$

$$n2_{5,12} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,R2\,\theta^2\,\{(2\,A_1-2\,A_2+A_4)\,\lambda_2+$$
$$(A_2-4\,A_3+A_5)\,\lambda_3+(A_4-2\,A_5+2\,A_6)\,\lambda_4+\theta\,(2\,A_1-2\,A_2+$$
$$1.5\,A_3)\,\lambda_9+\theta\,(0.5\,A_2-2\,A_3+0.5\,A_5)\,\lambda_{10}\}\,\{(A_2-A_4)\,\lambda_6$$
$$+(2\,A_3-A_5)\,\lambda_7+(A_5-2\,A_6)\,\lambda_8\}$$

$$n2_{6,6} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1^2\,\theta^2\,[\,(B_1+B_2+B_3+B_4)\,(2\,A_1-4\,A_2+8\,A_3+$$
$$2\,A_4-4\,A_5+2\,A_6)+\{(-2\,A_1+2\,A_2-A_4)\,\lambda_6+(-A_2+4\,A_3-$$
$$A_5)\,\lambda_7+(-A_4+2\,A_5-2\,A_6)\,\lambda_8\}^2\,]$$

$$n2_{6,7} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,\theta\,\{(-2\,A_1+2\,A_2-A_4)\,\lambda_6+(-A_2+4\,A_3-A_5)\,\lambda_7 + (-A_4+2\,A_5-2\,A_6)\,\lambda_8\}\,\{(3\,A_2-2\,A_4)\,\lambda_2 + (6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_3-2\,A_4+3\,A_5-4\,A_6)\,\lambda_4 + \theta\,(3\,A_2-3\,A_3)\,\lambda_9 + \theta\,(3\,A_3-A_5)\,\lambda_{10}\}\,]$$

$$n2_{6,8} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,\theta\,[\,(B_1+B_2+B_3+B_4)\,(-3\,A_2+12\,A_3+2\,A_4-7\,A_5+4\,A_6) + \{(-2\,A_1+2\,A_2-A_4)\,\lambda_6+(-A_2+4\,A_3-A_5)\,\lambda_7 + (-A_4+2\,A_5-2\,A_6)\,\lambda_8\}\,\{(3\,A_2-2\,A_4)\,\lambda_6 + (6\,A_3-2\,A_5)\,\lambda_7 + (3\,A_5-4\,A_6)\,\lambda_8\}\,]$$

$$n2_{6,9} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,\frac{R1\,\theta^2}{2}\,\{(-2\,A_1+2\,A_2-A_4)\,\lambda_6+(-A_2+4\,A_3-A_5)\,\lambda_7 +(-A_4+2\,A_5-2\,A_6)\,\lambda_8\}\,\{(3\,A_2-2\,A_4)\,\lambda_2+(6\,A_3-2\,A_5)\,\lambda_3 +(3\,A_5-4\,A_6)\,\lambda_4+\theta\,(3\,A_2-3\,A_3)\,\lambda_9+\theta\,(3\,A_3-A_5)\,\lambda_{10}\}]$$

$$n2_{6,11} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,R2\,\theta^2\,\{(-2\,A_1+2\,A_2-A_4)\,\lambda_6 + (-A_2+4\,A_3-A_5)\,\lambda_7+(-A_4+2\,A_5-2\,A_6)\,\lambda_8\}\,\{(A_4-A_2)\,\lambda_2+(A_5-2\,A_3)\,\lambda_3 +(2\,A_6-A_5)\,\lambda_4+\theta\,(1.5\,A_3-A_2)\,\lambda_9+\theta\,(0.5\,A_5-A_3)\,\lambda_{10}\}]$$

$$n2_{6,12} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,R1\,R2\,\theta^2\,[\,(B_1+B_2+B_3+B_4)\,(-A_2+4\,A_3+A_4-3\,A_5 +2\,A_6) + \{(-2\,A_1+2\,A_2-A_4)\,\lambda_6 + (-A_2+4\,A_3-A_5)\,\lambda_7 + (-A_4+2\,A_5-2\,A_6)\,\lambda_8\}\,\{(A_2-A_4)\,\lambda_6 + (2\,A_3-A_5)\,\lambda_7 + (A_5-2\,A_6)\,\lambda_8\}\,]$$

$$n2_{7,7} = \frac{E\,A\,\theta}{4\,L^2}\,(b_1+\theta\,b_2)\,[\,(B_1+B_2+B_3+B_4)\,(18\,A_3-12\,A_5+8\,A_6) + \{(3\,A_2-2\,A_4)\,\lambda_2+(6\,A_3-2\,A_5)\,\lambda_3 + (3\,A_5-4\,A_6)\,\lambda_4+\theta\,(3\,A_2-3\,A_3)\,\lambda_9+\theta\,(3\,A_3-A_5)\,\lambda_{10}\}^2\,]$$

$$n2_{7,8} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2) \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4+ \theta\ (3\ A_2-3\ A_3)\ \lambda_9+ \theta\ (3\ A_3-A_5)\ \lambda_{10}\} \{(3\ A_2-2\ A_4)\ \lambda_6 + (6\ A_3-2\ A_5)\ \lambda_7 + (3\ A_5-4\ A_6)\ \lambda_8\}$$

$$n2_{7,9} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ \frac{\theta}{2}\ [\ (B_1+B_2+B_3+B_4)\ (18\ A_3-12\ A_5+8\ A_6) + \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4+ \theta\ (3\ A_2-3\ A_3)\ \lambda_9+ \theta\ (3\ A_3-A_5)\ \lambda_{10}\}^2\ ]$$

$$n2_{7,11} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ R2\ \theta\ [\ (B_1+B_2+B_3+B_4)\ (-6\ A_3+5\ A_5-4\ A_6) + \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4+ \theta\ (3\ A_2-3\ A_3)\ \lambda_9+ \theta\ (3\ A_3-A_5)\ \lambda_{10}\} \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4 + \theta\ (1.5\ A_3-A_2)\ \lambda_9 + \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\}\ ]$$

$$n2_{7,12} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ R2\ \theta\ \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4+ \theta\ (3\ A_2-3\ A_3)\ \lambda_9+ \theta\ (3\ A_3-A_5)\ \lambda_{10}\} \{(A_4-A_2)\ \lambda_6 + (2\ A_3-A_5)\ \lambda_7 + (A_5-2\ A_6)\ \lambda_8\}$$

$$n2_{8,8} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ [\ (B_1+B_2+B_3+B_4)\ (18\ A_3-12\ A_5+8\ A_6) + \{(3\ A_2-2\ A_4)\ \lambda_6 + (6\ A_3-2\ A_5)\ \lambda_7 + (3\ A_5-4\ A_6)\ \lambda_8\}^2]$$

$$n2_{8,9} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ \frac{\theta}{2}\ \{\ (3\ A_2-2\ A_4)\ \lambda_6 + (6\ A_3-2\ A_5)\ \lambda_7 + (3\ A_5-4\ A_6)\ \lambda_8\} \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4+ \theta\ (3\ A_2-3\ A_3)\ \lambda_9+ \theta\ (3\ A_3-A_5)\ \lambda_{10}\}$$

$$n2_{8,11} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ R2\ \theta\ \{(3\ A_2-2\ A_4)\ \lambda_6 + (6\ A_3-2\ A_5)\ \lambda_7 + (3\ A_5-4\ A_6)\ \lambda_8\} \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4+ \theta\ (1.5\ A_3-A_2)\ \lambda_9+ \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\}$$

$$n2_{8,12} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ R2\ \theta\ [\ (B_1+B_2+B_3+B_4)\ (6\ A_3-5\ A_5+4\ A_6) + \{(3\ A_2-2\ A_4)\ \lambda_6 + (6\ A_3-2\ A_5)\ \lambda_7 + (3\ A_5-4\ A_6)\ \lambda_8\} \{(A_4-A_2)\ \lambda_6 + (2\ A_3-A_5)\ \lambda_7 + (A_5-2\ A_6)\ \lambda_8\}\ ]$$

$$n2_{9,9} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2) \frac{\theta^2}{4} [\ (B_1+B_2+B_3+B_4)\ (18\ A_3-12\ A_5+8\ A_6) + \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}^2\ ]$$

$$n2_{9,11} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2) \frac{R2\ \theta^2}{2} [\ (B_1+B_2+B_3+B_4)\ (-6\ A_3+5\ A_5-4\ A_6) + \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4 + \theta\ (1.5\ A_3-A_2)\ \lambda_9 + \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\} \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\}\ ]$$

$$n2_{9,12} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2) \frac{R2\ \theta^2}{2} \{(3\ A_2-2\ A_4)\ \lambda_2 + (6\ A_3-2\ A_5)\ \lambda_3 + (3\ A_5-4\ A_6)\ \lambda_4 + \theta\ (3\ A_2-3\ A_3)\ \lambda_9 + \theta\ (3\ A_3-A_5)\ \lambda_{10}\} \{(A_2-A_4)\ \lambda_6 + (2\ A_3-A_5)\ \lambda_7 + (A_5-2\ A_6)\ \lambda_8\}\ ]$$

$$n2_{11,11} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ R2^2\ \theta^2\ [\ (B_1+B_2+B_3+B_4)\ (2\ A_3-2\ A_5+2\ A_6) + \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4 + \theta\ (1.5\ A_3-A_2)\ \lambda_9 + \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\}^2\ ]$$

$$n2_{11,12} = \frac{E\ A\ \theta}{4\ L^2} (b_1+\theta\ b_2)\ R2^2\ \theta^2\ \{(A_4-A_2)\ \lambda_2 + (A_5-2\ A_3)\ \lambda_3 + (2\ A_6-A_5)\ \lambda_4 + \theta\ (1.5\ A_3-A_2)\ \lambda_9 + \theta\ (0.5\ A_5-A_3)\ \lambda_{10}\} \{(A_2-A_4)\ \lambda_6 + (2\ A_3-A_5)\ \lambda_7 + (A_5-2\ A_6)\ \lambda_8\}$$

$$n2_{12,12} = \frac{E\ A\ \theta}{4\ L^2} (b_1 + \theta\ b_2)\ R2^2\ \theta^2\ [\ (B_1+B_2+B_3+B_4)\ (2\ A_3 - 2\ A_5 + 2\ A_6)\ +$$

$$\{(A_2 - A_4)\ \lambda_6 + (2\ A_3 - A_5)\ \lambda_7 + (A_5 - 2\ A_6)\ \lambda_8\}^2\ ]$$

in which :

$$B_1 = A_1\ \lambda_2^2 + A_2\ \lambda_2\ \lambda_3 + A_3\ \lambda_3^2 + A_4\ \lambda_2\ \lambda_4 + A_5\ \lambda_3\ \lambda_4 + A_6\ \lambda_4^2$$

$$B_2 = 2\ \theta\ A_1\ \lambda_2\ \lambda_9 + \theta\ A_2\ \lambda_3\ \lambda_9 + 0.5\ \theta\ A_2\ \lambda_2\ \lambda_{10} +$$

$$1.5\ \theta\ A_3\ \lambda_4\ \lambda_9 + \theta\ A_3\ \lambda_3\ \lambda_{10} + 0.5\ \theta\ A_5\ \lambda_4\ \lambda_{10}$$

$$B_3 = \theta^2\ A_1\ \lambda_9^2 + 0.5\ \theta^2\ A_2\ \lambda_9\ \lambda_{10} + 0.25\ \theta^2\ A_3\ \lambda_{10}^2$$

$$B_4 = A_1\ \lambda_6^2 + A_2\ \lambda_6\ \lambda_7 + A_3\ \lambda_7^2 + A_4\ \lambda_6\ \lambda_8 + A_5\ \lambda_7\ \lambda_8 + A_6\ \lambda_8^2$$

APPENDIX C

COMPUTER PROGRAM

C.1 GENERAL

A general description of the computer program is given in Section 3.3 . A listing of the program, which was named **NANCURVE** (Nonlinear Analysis of Curved Beam Structures), is given at the end of this Appendix. A description of the subroutines used in the program and its corresponding input data examples are presented in the following.

C.2 DESCRIPTION OF SUBROUTINES

The computer program consists of a main program called NANCURVE, fourteen subroutines and one function subprogram. The main program NANCURVE directs the flow of the computations by calling the appropriate subroutines for each step of the solution procedure. The subroutine NODDATA reads data regarding the overall geometry of the arch and the nodal degrees of freedom. It generates the coordinates of the nodes and the equation numbers. The subroutine BAND computes the semi bandwidth that the stiffness matrix of the structure will have. This is done by obtaining the largest difference between the equation numbers of the nodes of any element.

The subroutine ELEMENT calls the appropriate element subroutine. All basic information concerning the curved elements, i.e., material, cross-section, and element properties are read by the subroutine CURVED. The subroutine also directs the computation of the geometric

properties of the curved element, which is accomplished by the subroutine GEOMETRY, the computation of the stiffness matrices of each element, which is performed by the subroutine NUMINT, and the assembly into the structure stiffness matrices, which is carried out by the subroutine ASSEMBLE. The condensation of the element linear stiffness matrix, from 16 by 16 to 12 by 12, is performed by subroutine REOCON. Subroutine STCOND prints out the element and structure stiffness matrices.

The subroutines LINSOLN and GAUSSOL solve the system of linear equations by Gauss elimination. Identification of the displacements obtained from LINSOLN is carried out by subroutine IDENT. The function subprogram DET1 evaluates the determinant of the structural tangent stiffness matrix. Finally subroutine STRESS evaluates the element end forces and stresses.

It should be noted that in addition to the subroutines mentioned previously, there are some more subroutines contained in the program. Those subroutines are necessary for buckling (eigenvalue) analysis, which is not discussed in the present study.

C.2 VARIABLES USED IN THE COMPUTER PROGRAM

The variable names used in the program are listed below in alphabetical order :

A1 = Lower limit of the numerical integration (= 0.0);

A2 = Upper limit of the numerical integration (= 1.0);

A(M) = Area of the cross-section of element M;

DETOPTN = Variable controlling the determinant of $[S_T]$;

If DETOPTN = 0 , no control on the determinant of $[S_T]$.

If DETOPTN = 1 , execution will be terminated if the determinant of $[S_T] \le 0$.

DI = Total number of horizontal intervals in which the span of the parabolic arch is divided into;

DELTA1 = Allowable tolerance for force components of unbalanced force vector;

DELTA2 = Allowable tolerance for moments of unbalanced force vector;

DM = Density of the material (set = 0 in the present study);

EIGVALU = Set equal to 3 in the present study;

E(N) = Modulus of elesticity of element group N;

G(N) = Shear modulus of element group N;

H = Height of parabolic or arbritary arch;

IA(N,I) = Boundary condition code of node N for its Ith degree of freedom. Initially it is defined as follows :

IA(N,I) = 1 if constrained;

IA(N,I) = 0 if free;

After processing,

IA(N,I) = 0 if initially = 1;

IA(N,I) = equation number for the d.o.f if initially=0;

IARCH = Variable that identifies the type of arch being studied

If IARCH = 0 , parabolic arch.

If IARCH = 1 , circular arch.

If IARCH = 2 , arbitrary arch;

IB(N,I) = Additional boundary condition code (in the present study, set = 0);

ICAL1 = Variable controlling print out.

If ICAL1 = 0 , entries of [k], [n1], [n2] are printed.

If ICAL1 = 1 , skip;

ICAL2 = Variable controlling print out.

If ICAL2 = 0, element geometric properties are printed.

If ICAL2 = 1, skip;

ICAL3 = Variable controlling print out.

If ICAL3 = 0, load vector, [K], [N1], [N2] are printed.

If ICAL3 = 1, skip;

ICAL4 = Variable controlling print out.

If ICAL4 = 0 , initial and nodal loads processed into load vector are printed.

If ICAL4 = 1 , skip;

ICAL5 = Variable controlling print out.

If ICAL5 = 0 , print load vector & displacement vector.

If ICAL5 = 1 , skip;

ICAL6 = Variable controlling print out.

If ICAL6 = 0 , print element nodal displacements.

If ICAL6 = 1 , skip.

Set ICAL6 = 2 if ISTRESS = 1 (to get a nice output);

ICAL7 = Variable controlling print out of eigenvalue analysis.

(Set = 1 for the present study);

IDATA = Variable for checking input data.

If IDATA = 0 , execute the program.

If IDATA = 1 , data check only, skip all computations;

IDIRCN = Set equal to 0 for the present study;

IFIX = Set equal to 1 for the present study;

ILOAD - Set equal to 1 for the present study (load is concentrated at the nodes);

IPART - Variable controlling print out.

If IPART=0, intermediate results of the displacement at every iteration process are printed.

If IPART=1, intermediate results of the displacement at every iteration process are not printed;

ISTRESS - Variable controlling the computations of element end forces and stresses.

If ISTRESS = 0 , skip.

If ISTRESS = 1 , compute end nodal forces and stresses.

ITERCHK - Set equal 1 for the present study;

IXX(M) - Moment of inertia about x-axis of the cross-section of element M;

IYY(M) - Moment of inertia about y-axis of the cross-section of element M;

JUSTK - Set equal to 0 for the present study;

KT(M) = Torsion constant of element M;

L = Span of the parabolic or arbitrary arch;

LOADDIR - Set equal to -1 for the present study;

MAXITER - Maximum number of iterations;

MP = Number of Gauss points used in the numerical integration (2,3,4,5,6,10 or 15);

MSUOPTN - Set equal to 1 for the present study;

NE - Total number of elements in the structure;

NODEI(M)- Number of node I of element M;

NODEJ(M)- Number of node J of element M;

NTYPE(N)= Element group N (set equal to 1 in the present study);

NUMEG = Total number of element groups (set = 1 in the present study);

NUMEL(N)= Total number of elements in element group N (set equal to NE in the present study);

NUMNP = Total number of nodal points in the structure;

N1GOPTN = Set equal to 0 in the present study;

N1OPTIN = Variable controlling the use of matrix [N1].
If N1OPTIN = 0 , [N1] is not used in the analysis.
If N1OPTIN = 1 , [N1] is used in the analysis;

N2OPTIN = Variable controlling the use of matrix [N2].
If N2OPTIN = 0 , [N2] is not used in the analysis.
If N2OPTIN = 1 , [N2] is used in the analysis;

PINT(N,DOF) = Initial load applied at node N, in the DOFth direction;

PINC(N,DOF) = Load increment applied at node N, in the DOFth direction;

PTOT(N,DOF) = Total load applied at node N, in the DOFth direction;

PRIOPTN = Variable controlling the print out.
If PRIOPTN = 0 , skip.
If PRIOPTN = 1 , intermediate results at every iteration are printed);

PROTYPE = Set equal to 3 in the present study (fixed Lagrangian);

R = Radius of curvature of the circular arch;

T1,..T8 = Title of problem beeing solved;

T(N) = For circular arch : angle between a node and the center

line of the circular arch (in degrees);

TT(N) = For arbitrary arch : angle between the slope at a node and global X-axis (in radians);

TLDOF = Number of total loads applied to the structure;

TOLER = Set equal to 0 in the present study;

XS = X coordinate of the left most node of parabolic arch;

X(N),Y(N),Z(N) = Global X, Y, and Z coordinates of node N.

C.2.3 INPUT DATA ARRANGEMENT

The input data are arranged in the following order and formats :

DATA CARD	FORMAT
T1,T2,T3,T4,T5,T6,T7,T8	8A10
NE,NUMNP,NUMEG,IDATA,ICAL1,ICAL2,ICAL3,ICAL4,ICAL5,ICAL6,ICAL7	11I5
IARCH,ILOAD,IDIRCN	3I5
PRIOPTN,N2OPTIN,N1OPTIN,ITERCHK,MSUOPTN,N1GOPTN,IFIX,JUSTK, TOLER,DETOPTN	8I5 F10.5,I5
PROTYPE,EIGVALU,ISTRESS,IPART,LOADDIR	5I5
R *	F15.9
N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),T,Z(N) **	I5,12I3,F15.10,F10.6
TLDOF,MAXITER,DELTA1,DELTA2	2I5,2F10.6
N,DOF,PINT(N,DOF),PINC(N,DOF),PTOT(N,DOF)	2I5,3F10.4
NTYPE(N),NUMEL(N)	2I5
E(N),G(N),DM	3E10.2
M,NODEI(M),NODEJ(M),A(M),IXX(M),IYY(M),KT(M)	3I5,4E15.6
A1,A2,MP	2F5.2,I5

* This is for circular arch.

For parabolic arch :	H,L,DI,XS	4F10.5
For arbitrary arch :	H,L	2F10.5

**This is for circular arch.

For parabolic arch :	N,(IA(N,I)),(IB(N,I))	I5,12I3
For arbitrary arch :		
N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),TT(N),X(N),Y(N),Z(N)		I5,12I3, F9.6,3F10.6

```
NONLINEAR ANALYSIS OF BELYTSCHKO'S ARCH (SECTION 4.4.1)
    8    9    1    0    1    1    1    1    1    1    1
    1    1    1    0
    0    1    1    1    1    0    1    0      0.000    0
    3    3    1    0   -1
   100.0
    1  1  1  1  1  1  1  0  0  0  0  0  0 -14.070        0.0
    2  0  1  0  1  0  1  0  0  0  0  0  0 -12.311        0.0
    3  0  1  0  1  0  1  0  0  0  0  0  0 -10.553        0.0
    4  0  1  0  1  0  1  0  0  0  0  0  0  -8.794        0.0
    5  0  1  0  1  0  1  0  0  0  0  0  0  -7.035        0.0
    6  0  1  0  1  0  1  0  0  0  0  0  0  -5.276        0.0
    7  0  1  0  1  0  1  0  0  0  0  0  0  -3.517        0.0
    8  0  1  0  1  0  1  0  0  0  0  0  0  -1.759        0.0
    9  0  1  1  1  1  1  0  0  0  0  0  0     0.0        0.0
    1   90      10.0       10.0
    9    1    1000.0     1000.0    20000.0
    1    8
   1.0E+07    4.0E+06   0.0E+01
    1    1    2      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    2    2    3      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    3    3    4      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    4    4    5      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    5    5    6      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    6    6    7      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    7    7    8      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
    8    8    9      2.0000E+00      0.6667E+00      0.6667E+00   2.25E+00
  0.0  1.0    3
```

```
NONLINEAR ANALYSIS OF DADEPPO'S ARCH (SECTION 4.4.2)
    8     9     1     0     1     1     1     1     1     1     1
    1     1     0
    0     1     1     1     1     0     1     0        0.00     0
    3     3     0     0    -1
     200.0
    1  1  1  1  1  1  1  0  0  0  0  0  0 -30.000          0.0
    2  0  1  0  1  0  1  0  0  0  0  0  0 -26.250          0.0
    3  0  1  0  1  0  1  0  0  0  0  0  0 -22.500          0.0
    4  0  1  0  1  0  1  0  0  0  0  0  0 -18.750          0.0
    5  0  1  0  1  0  1  0  0  0  0  0  0 -15.000          0.0
    6  0  1  0  1  0  1  0  0  0  0  0  0 -11.250          0.0
    7  0  1  0  1  0  1  0  0  0  0  0  0  -7.500          0.0
    8  0  1  0  1  0  1  0  0  0  0  0  0  -3.750          0.0
    9  0  1  1  1  1  1  0  0  0  0  0  0     0.0          0.0
    1    90      0.50       0.50
    9     1      50.0       50.0      500.0
    1     8
   1.0E+06    4.0E+05   0.0E+01
    1     1     2    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    2     2     3    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    3     3     4    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    4     4     5    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    5     5     6    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    6     6     7    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    7     7     8    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
    8     8     9    4.0000E+00     1.3333E+00     1.3333E+00     2.250E+00
  0.0   1.0     3
```

C.3 COMPUTER PROGRAM "NANCURVE"

```
      PROGRAM NANCURVE
C     *****************************************************************
C        THIS PROGRAM USES THE FINITE ELEMENT METHOD TO ANALYZE
C        A CURVED ELEMENT IN THREE DIMENSIONS.
C        OTHER ELEMENTS MAY BE ANALYZED BY ADDING A SUBROUTINE
C        FOR EACH NEW TYPE OF ELEMENT BEING USED.
C        NONLINEAR PROPERTIES ARE TAKEN INTO CONSIDERATION IN THE
C        CURVED ELEMENT.
C     *****************************************************************
C
      REAL IXX,IYY,KT,LENGTH,II,JJ,N1STTOT
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,
     +          ICAL3,ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C3/IA(37,8),IB(37,8),X(37),Y(37),Z(37),RAD,AC
      COMMON/C4/SE(16,16)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
     +          KT(36),L(1,36)
      COMMON/C6/A1,A2,MP,B1(36),B2(36),B3(36)
      COMMON/C7/RI(36),RJ(36),PHII(36),PHIJ(36),TETA(36),LENGTH(36),
     +          RIA(36),RJA(36)
      COMMON/C8/PN(37,8),R(296),PINT(37,8)
      COMMON/C9/S(296,16),SP(296,16),IDET
      COMMON/C10/D(296),D10(1184),RC(296),SC(296,16)
      COMMON/C11/DN(16),U(36,12),W(37,8),V(37,8)
      COMMON/C12/ULOC(36,12),RCOL(9),MSUOPTN,N1GOPTIN
      COMMON/C16/PRIOPTN
      COMMON/C17/A7TOT(36),A7OLD(36),BOL(36,5),BTO(36,5),BE(5)
      COMMON/C18/IARCH
      COMMON/C19/TT(36)
C
      DIMENSION DTEMP(296),PTEMP(296),PSTART(296),DTOT(296)
      DIMENSION PACTUAL(296),PSAVE(296),DACTUAL(296),N1STTOT(296,16)
      DIMENSION SOLD(296,16),SRK(296,16),SRN1(296,16),PTOT(37,8)
      DIMENSION REFSTRT(37,8),REFPTMP(37,8),SRN2(37,16),PINC(37,8)
      INTEGER PROTYPE,EIGVALU,PRIOPTN,DETOPTN,DOF,TLDOF
C
      READ(60,1010) T1,T2,T3,T4,T5,T6,T7,T8
      WRITE(61,2020)T1,T2,T3,T4,T5,T6,T7,T8
      READ(60,1015) NE,NUMNP,NUMEG,IDATA,ICAL1,ICAL2,ICAL3,ICAL4,
     +              ICAL5,ICAL6,ICAL7
      WRITE(61,2010)NE,NUMNP,NUMEG,IDATA,ICAL1,ICAL2,ICAL3,ICAL4,
     +              ICAL5,ICAL6,ICAL7
C
C.......READ NODAL POINT DATA
C
      READ(60,1030) IARCH,ILOAD,IDIRCN
      WRITE(61,2030)IARCH,ILOAD,IDIRCN
C
C
      READ(60,6971) PRIOPTN,N2OPTIN,N1OPTIN,ITERCHK,
     +              MSUOPTN,N1GOPTN,IFIX,JUSTK,TOLER,DETOPTN
```

```
6971  FORMAT(8I5,F10.5,I5)
      WRITE(61,6972) PRIOPTN,N2OPTIN,N1OPTIN,ITERCHK,
     +               MSUOPTN,N1GOPTN,IFIX,JUSTK,TOLER,DETOPTN
6972  FORMAT(10X,8HPRIOPTN=,I2/10X,8HN2OPTIN=,I2/
     + 10X,'N1OPTIN=',I2/10X,'ITERCHK=',I2/10X,'MSUOPTN=',I2/10X,
     + 'N1GOPTN=',I2,10X,'IFIX=',I2,10X,'JUSTK=',I2,
     + /10X,'TOLER=',F10.5,10X,'DETOPTN=',I2//)
      READ(60,1) PROTYPE,EIGVALU,ISTRESS,IPART,LOADDIR
1     FORMAT(5I5)
      WRITE(61,8761) PROTYPE,EIGVALU,ISTRESS,IPART,LOADDIR
8761  FORMAT(10X,'PROTYPE=',I2,10X,'EIGVALU=',I2,10X,'ISTRESS=',I2/
     +       10X,'IPART  =',I2,10X,'LOADDIR=',I2//)
C
      DX=0.
      CALL NODDATA (IARCH,DX)
C
      IF(PROTYPE.EQ.2) GO TO 510
C
      IF(ITERCHK.EQ.1) READ(60,1013) TLDOF,MAXITER,DELTA1,DELTA2
      IF(ITERCHK.EQ.1) WRITE(61,1012) TLDOF,MAXITER,DELTA1,DELTA2
1013  FORMAT(2I5,2F10.6)
1012  FORMAT(' ','TLDOF=',I5,5X,8HMAXITER=,I5,5X,7HDELTA1=,F10.6,5X,
     +            7HDELTA2=,F10.6//)
C
      WRITE(61,409)
      DO 407 N=1,NUMNP
      DO 407 I=1,6
      PINT(N,I)=0.0
      PINC(N,I)=0.0
407   PTOT(N,I)=0.0
      I=0
406   CONTINUE
      I=I+1
      READ(60,405) N,DOF,PINT(N,DOF),PINC(N,DOF),PTOT(N,DOF)
      WRITE(61,410) N,DOF,PINT(N,DOF),PINC(N,DOF),PTOT(N,DOF)
      IF(I.LT.TLDOF) GO TO 406
405   FORMAT(2I5,3F10.4)
409   FORMAT(' ',10X,' LOADING CONDITIONS : '//,6X,'NODE',7X,'DOF',16X,
     +                 'PINT',16X,'PINC',16X,'PTOT'//)
410   FORMAT(' ',3X,I5,4X,I5,11X,F10.4,10X,F10.4,10X,F10.4)
      GO TO 513
C
510   CONTINUE
C
C........READ AND STORE INITIAL LOAD DATA
C
      WRITE(61,2015)
      WW=0.
      CALL LOAD (IARCH,ILOAD,IDIRCN,DX,WW)
C
513   CONTINUE
      IF(PROTYPE.NE.3) GO TO 3021
```

```
      SCALE=1.0E+05
      DO 5001 I=1,NEQ
      PSAVE(I)=0.0
      DACTUAL(I)=0.0
5001  DTOT(I)=0.0
C
C     AUTOMATIC GENERATION OF LODPON1 AND CORRESPONDING D.O.F
C
      IF(LOADDIR) 1655,1665,1675
1655  IHORZ=1
      IVERT=0
      ILAT=0
      GO TO 1685
1665  IHORZ=0
      IVERT=1
      ILAT=0
      GO TO 1685
1675  IHORZ=0
      IVERT=0
      ILAT=1
1685  CONTINUE
C
C     IF LOAD WANTED FOR SPECIFIC LOAD LET ITETO=1
C     CHOSE THE APPROPRIATE VALUES OF LODPON1,LNODE1,AND LDOF1
C
      ITETO=0
      IF(ITETO.EQ.0) GO TO 9152
      LODPON1=20
      LNODE1=8
      LDOF1=2
      IF(ITETO.NE.0) GO TO 700
9152  DO 200 N=1,NUMNP
      DO 300 I=1,6
      IF(IA(N,I).EQ.0) GO TO 300
      IF(PINT(N,I).EQ.0) GO TO 300
      IF(I.EQ.1.AND.IHORZ.EQ.0) GO TO 300
      IF(IHORZ.EQ.0) GO TO 909
      LODPON1=IA(N,I)
      LNODE1=N
      LDOF1=I
      GO TO 700
909   IF(IVERT.EQ.0.AND.I.EQ.2) GO TO 300
      IF(IVERT.EQ.0) GO TO 499
      LODPON1=IA(N,I)
      LNODE1=N
      LDOF1=I
      GO TO 700
499   IF(ILAT.EQ.0.AND.I.EQ.3) GO TO 1093
      IF(ILAT.EQ.0) GO TO 1093
      LODPON1=IA(N,I)
      LNODE1=N
      LDOF1=I
```

```
      GO TO 700
1093  PRINT 13
      PRINT 14
13    FORMAT('    PROGRAM CAN NOT CALCULATE THE VALUE OF LODPON1  ')
14    FORMAT('    HELP WANTED, PROGRAM STOPPED AT APPR. LINE 266  ')
      GO TO 900
300   CONTINUE
200   CONTINUE
700   CONTINUE
      WRITE(61,2900) LODPON1,LNODE1,LDOF1
2900  FORMAT(' ',//11X,'THE D.O.F. IN WHICH LOAD HAS BEEN INCREASED=',
     +       I3,//10X,'AT NODE=',I3,5X,'WITH D.O.F.=',I3//)
C
      DO 3010 I=1,NUMNP
      DO 3010 J=1,6
3010  U(I,J)=0.0
      ICHECK=1
1001  DO 3020 I=1,NUMNP
      IF(IFIX.EQ.0)  X(I)=X(I)+U(I,1)
      IF(IFIX.EQ.0)  Y(I)=Y(I)+U(I,2)
      IF(IFIX.EQ.0)  Z(I)=Z(I)+U(I,3)
3020  CONTINUE
      IF(PRIOPTN.EQ.0)   GO TO 4994
      WRITE(61,4995)
4995  FORMAT(/,10X,'NODE',10X,'X(I)',10X,'Y(I)',10X,'Z(I)',/)
      DO 4996  I=1,NUMNP
      WRITE(61,4997)  I,X(I),Y(I),Z(I)
4997  FORMAT(/,10X,I5,3F15.8)
4996  CONTINUE
4994  CONTINUE
3021  CONTINUE
C        READ AND STORE ELEMENT DATA
C     ************************************************************
C
      IPAR=1
      NUMITER=1
C
      IF(ICHECK.NE.1) GO TO 5928
      DO 100 N=1,NUMEG
      READ(60,1020) NTYPE(N),NUMEL(N)
      CALL ELEMENT(N,IDATA,IARCH)
100   CONTINUE
5928  CONTINUE
C
      IF(PROTYPE.NE.3) GO TO 3335
      IF(ICHECK.NE.1)  GO TO 3333
3335  CONTINUE
C        COMPUTE SEMIBANDWIDTH OF STRUCTURE STIFFNESS MATRIX
C     ************************************************************
      CALL BAND
      IF(JUSTK.EQ.1)     GO TO 2110
      IF(PROTYPE.NE.3)   GO TO 2110
```

```
      DO 5745  NN=1,NUMEG
      IF(NUMEL(NN).EQ.0)   GO TO 5745
      NAME=NUMEL(NN)
      DO 5341  K=1,NAME
      M=L(NN,K)
      IF(MSUOPTN.EQ.1)  A7OLD(M)=0.0
      DO 5341 I=1,5
      IF(MSUOPTN.EQ.2)  BOL(M,I)=0.0
5341  CONTINUE
5745  CONTINUE
2110  CONTINUE
C
C        ASSEMBLE INITIAL LOADS AND NODAL LOADS INTO LOAD VECTOR
C        SET ARRAYS -S- AND -R- EQUAL TO ZERO
C     ************************************************************
C
      CALL ASEMBLE (M)
3333  CONTINUE
      IF(PROTYPE.NE.3)  GO TO 3336
      IF(ICHECK.EQ.1)   GO TO 2111
      GO TO 2112
2111  DO 5003  I=1,NEQ
5003  PSTART(I)=0.0
      J=0
      DO 415 N=1,NUMNP
      DO 420 I=1,6
      IF(IA(N,I).EQ.0) GO TO 420
      J=J+1
      PSTART(J)=PINT(N,I)
420   CONTINUE
415   CONTINUE
      IF(PRIOPTN.EQ.0) GO TO 423
      WRITE(61,421)
421   FORMAT(' ',10X,'PSTART :')
      WRITE(61,422) (PSTART(I),I=1,NEQ)
422   FORMAT(' ',8X,F10.5)
423   CONTINUE
C
2112  IF(JUSTK.EQ.1)  ICHECK=2
      IF(JUSTK.EQ.1)  GO TO 3336
      IF(ICHECK.EQ.1) GO TO 3336
C     IPAR=2
C     DO 2113  I=1,NSIZE
C     DO 2113  J=1,MBAND
C2113  S(I,J)=0.0
C     DO 111 N=1,NUMEG
C     CALL ELEMENT(N,IDATA,IARCH)
C111   CONTINUE
C     IF(ICAL3.EQ.0) CALL STCONDN
C
C1901  IF(ITERCHK.NE.0) CALL INVTRNS
1901  IF(ICHECK.EQ.3)  GO TO 4988
```

```
      IF(N1OPTIN.EQ.0) GO TO 4987
      DO 1801 I=1,NSIZE
      DO 1801 J=1,MBAND
1801  S(I,J)=0.0
      N=1
      IPAR=3
      CALL CURVED(N,IDATA,IARCH)
      IF(ICAL3.EQ.0) CALL STCONDN
4987  CONTINUE
      IF(ICHECK.EQ.3)  GO TO 4988
      IF(N2OPTIN.EQ.0) GO TO 4988
      DO 7691 I=1,NSIZE
      DO 7691 J=1,MBAND
7691  S(I,J)=0.0
      IPAR=4
      CALL CURVED(N,IDATA,IARCH)
      IF(ICAL3.EQ.0) CALL STCONDN
4988  CONTINUE
      IF(ICHECK.EQ.2.OR.PSAVE(LODPON1).EQ.0.) GO TO 5010
C     DO 9436 I=1,NSIZE
C     DO 9436 J=1,MBAND
C9436  S(I,J)=0.0
C     IPAR=7
C     DO  9437 N=1,NUMEG
C     IF(NUMEL(N).EQ.0) GO TO 9437
C     CALL KEPSI01(N)
C9437  CONTINUE
      IF(ICHECK.EQ.2)   GO TO 5010
      DO 3071 I=1,NEQ
      DO 3081 J=1,MBAND
      READ(4,10) RK
      SRK(I,J)=RK
C     READ(16,10) RN1STAR
C     SRN1(I,J)=RN1STAR
C     N1STTOT(I,J)=RN1STAR
      IF(IFIX.EQ.1)  S(I,J)=RK
      IF(IFIX.EQ.1)  SOLD(I,J)=RK
C     IF(IFIX.EQ.0)  S(I,J)=RK+N1STTOT(I,J)
C     IF(IFIX.EQ.0)  SOLD(I,J)=RK+N1STTOT(I,J)
3081  CONTINUE
3071  CONTINUE
      REWIND 4
C     REWIND 16
      IF(PRIOPTN.EQ.0) GO TO 7233
C     WRITE(61,8005)
C8005  FORMAT(///,10X,'KEPSIO MATRIX',/)
C     WRITE(61,8002)  ((SRN1(I,J),J=1,MBAND),I=1,NEQ)
      WRITE(61,8008)
8008  FORMAT(///,10X,'K LINEAR STIFFNESS MATRIX',/)
      WRITE(61,8002)  ((SRK(I,J),J=1,MBAND),I=1,NEQ)
      WRITE(61,8009)
8009  FORMAT(///,10X,'S(I,J)   MATRIX',/)
```

```
      WRITE(61,8002)  ((S(I,J),J=1,MBAND),I=1,NEQ)
 7233 CONTINUE
      GO TO 5011
 5010 DO 4071 I=1,NEQ
      DO 4081 J=1,MBAND
      IF(N1OPTIN.EQ.1)READ(5,10) RN1
      IF(PSAVE(LODPON1).EQ.0.)   READ(4,10)  RK
      IF(N2OPTIN.EQ.1) READ(16,10) RN2
      IF(N1OPTIN.EQ.1)SRN1(I,J)=RN1
      IF(PSAVE(LODPON1).EQ.0..AND.N2OPTIN.EQ.1)SP(I,J)=RK+.5*RN1+RN2/3.
      IF(PSAVE(LODPON1).NE.0..AND.N2OPTIN.EQ.1)
     +SP(I,J)=SOLD(I,J)+.5*RN1+RN2/3.
      IF(PSAVE(LODPON1).EQ.0..AND.N1OPTIN.EQ.0)SP(I,J)=RK
      IF(PSAVE(LODPON1).EQ.0..AND.N1OPTIN.EQ.0)S(I,J)=RK
      IF(PSAVE(LODPON1).EQ.0..AND.N1OPTIN.EQ.1.AND.N2OPTIN.EQ.0)SP(I,J)=
     +RK+.5*RN1
      IF(PSAVE(LODPON1).NE.0..AND.N1OPTIN.EQ.0)SP(I,J)=SOLD(I,J)
      IF(PSAVE(LODPON1).NE.0..AND.N1OPTIN.EQ.0)S(I,J)=SOLD(I,J)
      IF(PSAVE(LODPON1).NE.0..AND.N1OPTIN.EQ.1.AND.N2OPTIN.EQ.0)SP(I,J)=
     +SOLD(I,J)+.5*RN1
      IF(PSAVE(LODPON1).EQ.0..AND.N2OPTIN.EQ.1)S(I,J)=RK+RN1+RN2
      IF(PSAVE(LODPON1).NE.0..AND.N2OPTIN.EQ.1)S(I,J)=SOLD(I,J)+RN1+RN2
      IF(PSAVE(LODPON1).EQ.0..AND.N1OPTIN.EQ.1.AND.N2OPTIN.EQ.0)
     +S(I,J)=RK+RN1
      IF(PSAVE(LODPON1).NE.0..AND.N1OPTIN.EQ.1.AND.N2OPTIN.EQ.0)
     +S(I,J)=SOLD(I,J)+RN1
 4081 CONTINUE
 4071 CONTINUE
      IF(N1OPTIN.EQ.1) REWIND 5
      IF(N2OPTIN.EQ.1) REWIND 16
      IF(PSAVE(LODPON1).EQ.0.)  REWIND 4
      IF(PRIOPTN.EQ.0) GO TO 5011
      IF(N2OPTIN.EQ.0) GO TO 4989
      WRITE(61,7693)
 7693 FORMAT(///,10X,11HN2   MATRIX,/)
      DO 7694 I=1,NEQ
      DO 7695 J=1,MBAND
      READ(16,10) RN2
      SRN2(I,J)=RN2
 7695 CONTINUE
 7694 CONTINUE
 4989 CONTINUE
      IF(N2OPTIN.EQ.1) REWIND 16
      IF(N2OPTIN.EQ.1) WRITE(61,8002)((SRN2(I,J),J=1,MBAND),I=1,NEQ)
      IF(N1OPTIN.EQ.1) WRITE(61,8004)
 8004 FORMAT(//,10X,'N1 NONLINEAR STIFFNESS MATRIX',/)
      IF(N1OPTIN.EQ.1) WRITE(61,8002)  ((SRN1(I,J),J=1,MBAND),I=1,NEQ)
      IF(PSAVE(LODPON1).NE.0.) WRITE(61,8010)
 8010 FORMAT(///,10X,'SOLD(I,J)  MATRIX',/)
      IF(PSAVE(LODPON1).NE.0.) WRITE(61,8002)
     +((SOLD(I,J),J=1,MBAND),I=1,NEQ)
      WRITE(61,8018)
```

```
8018  FORMAT(///,10X,'SP(I,J) MATRIX',/)
      WRITE(61,8002)  ((SP(I,J),J=1,MBAND),I=1,NEQ)
      WRITE(61,8009)
      WRITE(61,8002)  ((S(I,J),J=1,MBAND),I=1,NEQ)
5011  IF(ICHECK.NE.3)  GO TO 7001
      GO TO 6001
7001  ICHECK=3
      GO TO 3339
3336  CONTINUE
C
C        COMPUTE ELEMENT LINEAR STIFFNESS AND ASSEMBLE INTO STRUCTURE
C        LINEAR STIFFNESS
C     *****************************************************************
C
C
C
      DO 2114  I=1,NSIZE
      DO 2114  J=1,MBAND
2114  S(I,J)=0.0
      IPAR=2
      DO 110 N=1,NUMEG
      CALL ELEMENT (N,IDATA,IARCH)
110   CONTINUE
      IF(ICAL3.EQ.0) CALL STCONDN
C
      IF(PROTYPE.NE.3)  GO TO 3337
      DO 1071 I=1,NEQ
      DO 1081 J=1,MBAND
      READ(4,10) RK
      S(I,J)=RK
      SP(I,J)=RK
1081  CONTINUE
1071  CONTINUE
      REWIND 4
      IF(NCOND.EQ.0) GO TO 1809
      CALL STCONDN
1809  CONTINUE
      IF(PRIOPTN.EQ.0) GO TO 9431
      IF(ICHECK.EQ.1) WRITE(61,8008)
      IF(ICHECK.EQ.1) WRITE(61,8002)((S(I,J),J=1,MBAND),I=1,NEQ)
9431  CONTINUE
6001  IDET=1
8002  FORMAT(1X,6(2X,E19.13),/)
      CALL LINSOLN
      DETRMNT=DET1(SCALE)
      DO 5005 I=1,NEQ
      DTOT(I)=DTOT(I)+D(I)
      DACTUAL(I)=DACTUAL(I)+D(I)
      IF(IFIX.EQ.0)  D(I)=DTOT(I)
      IF(IFIX.EQ.1)  D(I)=DACTUAL(I)
5005  CONTINUE
C     IF(NCOND.NE.0)  CALL RECOVER
```

```
      CALL IDENT
      IF(JUSTK.EQ.1)  GO TO 3339
C     IF(ITERCHK.EQ.0) CALL INVTRNS
      IF(ITERCHK.NE.0) GO TO 8537
      DO 5763  NN=1,NUMEG
      IF(NUMEL(NN).EQ.0.)  GO TO 5763
      NAME=NUMEL(NN)
      DO 5002  K=1,NAME
      M=L(NN,K)
 5002 A7TOT(M)=A7OLD(M)+U(M,7)-U(M,1)
 5763 CONTINUE
 8537 CONTINUE
      ICHECK=2
      IF(ITERCHK.NE.0) GO TO 2120
      DO 2121 I=1,NEQ
      R(I)=0.0
 2121 PACTUAL(I)=PSAVE(I)+PSTART(I)
      GO TO 3342
 2120 CONTINUE
      GO TO 1901
 3339 DO 2001 I=1,NEQ
      PTEMP(I)=0.0
      IM=I+1
      IF(IM.GT.NEQ) GO TO 2001
      DO 3901 J=2,MBAND
      IF(SP(I,J).EQ.0.)  GO TO 1804
      PTEMP(I)=PTEMP(I)+SP(I,J)*D(IM)
 1804 IM=IM+1
      IF(IM.GT.NEQ)  GO TO 2001
 3901 CONTINUE
 2001 CONTINUE
      DO 2301 I=1,NEQ
      IM=I
      JM=1
 2108 IF(SP(IM,JM).EQ.0.) GO TO 2201
      PTEMP(I)=PTEMP(I)+SP(IM,JM)*D(IM)
 2201 IM=IM-1
      JM=JM+1
      IF(IM.EQ.0)       GO TO 2301
      IF(JM.GT.MBAND)   GO TO 2301
      GO TO 2108
 2301 CONTINUE
      DO 5006 I=1,NEQ
      IF(IFIX.EQ.0)  PACTUAL(I)=PTEMP(I)+PSAVE(I)
      IF(IFIX.EQ.1)  PACTUAL(I)=PTEMP(I)
 5006 CONTINUE
      IF(ITERCHK.EQ.0) GO TO 6975
      IF(PRIOPTN.EQ.1) WRITE(61,8011)
 8011 FORMAT(15X,'I',5X,'PACTUAL(I)',10X,'PTEMP(I)',10X,'PSAVE(I)',//)
      IF(PRIOPTN.EQ.0) GO TO 4990
      DO 8012 I=1,NEQ
      WRITE(61,8013)  I,PACTUAL(I),PTEMP(I),PSAVE(I)
```

```
8012  CONTINUE
4990  CONTINUE
8013  FORMAT(10X,I5,5X,E21.15,10X,E21.15,10X,E21.15,/)
      WRITE(61,8547)
8547  FORMAT(/,10X,'DTOT(I)',/)
      IF(IPART.EQ.0) GO TO 1003
      DO 8541 MM=1,NEQ
8541  WRITE(61,8542) DACTUAL(MM)
8542  FORMAT(10X,E21.15)
1003  CONTINUE
      IF(IHORZ.EQ.1) LODPON2=LODPON1+1
      IF(IHORZ.EQ.1) LODPON3=LODPON1+2
      IF(IVERT.EQ.1) LODPON1=LODPON1-1
      IF(IVERT.EQ.1) LODPON2=LODPON1+1
      IF(IVERT.EQ.1) LODPON3=LODPON1+2
      IF(ILAT.EQ.1)  LODPON1=LODPON1-2
      IF(ILAT.EQ.1)  LODPON2=LODPON1+1
      IF(ILAT.EQ.1)  LODPON3=LODPON1+1
      WRITE(61,9001)PACTUAL(LODPON1),PACTUAL(LODPON2),PACTUAL(LODPON3),
     +              DACTUAL(LODPON1),DACTUAL(LODPON2),DACTUAL(LODPON3)
9001  FORMAT(' ',10X,11HPACTUAL(X)=,E21.15,10X,11HPACTUAL(Y)=,E21.15,
     +         10X,11HPACTUAL(Z)=,E21.15//16HDISPLACEMENT(X)=,E21.15,
     + 10X,16HDISPLACEMENT(Y)=,E21.15,10X,16HDISPLACEMENT(Z)=,E21.15//)
      IF(IVERT.EQ.1) LODPON1=LODPON1+1
      IF(ILAT.EQ.1)  LODPON1=LODPON1+2
6975  CONTINUE
      DO 2115 I=1,NEQ
2115  R(I)=PSTART(I)-PTEMP(I)
      IF(PRIOPTN.EQ.0) GO TO 6976
      WRITE(61,9731)
9731  FORMAT(//,20X,'R(I)',15X,'PSTART(I)',15X,'PTEMP(I)',/)
      DO 9732  IMM=1,3
9732  WRITE(61,9733)  R(IMM),PSTART(IMM),PTEMP(IMM)
9733  FORMAT(//,10X,E21.15,10X,E21.15,10X,E21.15,/)
6976  CONTINUE
      IF(ITERCHK.EQ.0)    GO TO 3342
      DO 2451  NN=1,NUMEG
      IF(NUMEL(NN).EQ.0)  GO TO 2451
      NAME=NUMEL(NN)
      DO 2351  K=1,NAME
      M=L(NN,K)
      NI=NODEI(M)
      NJ=NODEJ(M)
      DO 2351  K1=1,2
      IF(K1.EQ.1)  NP=NI
      IF(K1.EQ.2)  NP=NJ
      DO 2251  I=1,6
      IF(IA(NP,I))  1651,1551,1571
1571  NL=IA(NP,I)
      REFSTRT(NP,I)=PSTART(NL)
      REFPTMP(NP,I)=PTEMP(NL)
      GO TO 2251
```

```
1551  REFSTRT(NP,I)=0.0
      REFPTMP(NP,I)=0.0
      GO TO 2251
1651  IF(IB(NP,I).LT.0)  GO TO 1751
      NM=IB(NP,I)
      GO TO 1851
1751  NL=-IB(NP,I)+NEQ
      REFSTRT(NP,I)=PSTART(NL)
      REFPTMP(NP,I)=PTEMP(NL)
      GO TO 2251
1851  IF(IA(NM,I)) 1951,2051,2151
1951  NL=-IB(NM,I)+NEQ
      REFSTRT(NP,I)=PSTART(NL)
      REFPTMP(NP,I)=PTEMP(NL)
      GO TO 2251
2051  REFSTRT(NP,I)=0.0
      REFPTMP(NP,I)=0.0
      GO TO 2251
2151  NL=IA(NM,I)
      REFSTRT(NP,I)=PSTART(NL)
      REFPTMP(NP,I)=PTEMP(NL)
2251  CONTINUE
2351  CONTINUE
2451  CONTINUE
      DO 6949 NP=1,NUMNP
      DO 6949 J=1,3
      KJJ=J+3
      PART1=ABS(REFSTRT(NP,J)-REFPTMP(NP,J))
      PART2=ABS(REFSTRT(NP,KJJ)-REFPTMP(NP,KJJ))
      IF(PART1.GT.DELTA1.OR.PART2.GT.DELTA2)  GO TO 6950
      WRITE(61,2116)  PART1,PART2
6949  CONTINUE
      GO TO 3342
6950  WRITE(61,2116)  PART1,PART2
2116  FORMAT(10X,6HPART1=,E21.15,10X,6HPART2=,E21.15)
      NUMITER=NUMITER+1
      IF(NUMITER.LE.MAXITER)  GO TO  2117
      GO TO 900
2117  GO TO 6001
3342  CONTINUE
      IF(ITERCHK.NE.1)  GO TO 8945
      IF(MSUOPTN.EQ.2)  GO TO 8945
      DO 8538  NN=1,NUMEG
      IF(NUMEL(NN).EQ.0)  GO TO 8538
      NAME=NUMEL(NN)
      DO 8539 K=1,NAME
      M=L(NN,K)
      TO=(U(M,8)-U(M,2))/LENGTH(M)
      SIO=(U(M,3)-U(M,9))/LENGTH(M)
      TA=U(M,6)-TO
      TB=U(M,12)-TO
      SIA=U(M,5)-SIO
```

```
      SIB=U(M,11)-SIO
8539  A7TOT(M)=A7OLD(M)+U(M,7)-U(M,1)
     ++.5*(TO**2+SIO**2)*LENGTH(M)
     ++LENGTH(M)*(2.*TA**2-TA*TB+2.*TB**2)/30.
     ++LENGTH(M)*(2.*SIA**2-SIA*SIB+2.*SIB**2)/30.
8538  CONTINUE
8945  IF(ITERCHK.NE.1)  GO TO 4993
      IF(MSUOPTN.EQ.1)  GO TO 4993
      DO 4992  NN=1,NUMEG
      IF(NUMEL(NN).EQ.0)   GO TO 4992
      NAME=NUMEL(NN)
      DO 5344  K=1,NAME
      M=L(NN,K)
      ALFA1=U(M,6)
      ALFA2=2.*(-3.*U(M,2)-2.*U(M,6)*LENGTH(M)+3.*U(M,8)-
     +U(M,12)*LENGTH(M))/LENGTH(M)
      ALFA3=3.*(2.*U(M,2)+U(M,6)*LENGTH(M)-2.*U(M,8)+U(M,12)
     +*LENGTH(M))/LENGTH(M)
      BETA1=-U(M,5)
      BETA2=2.*(-3.*U(M,3)+2.*U(M,5)*LENGTH(M)+3.*U(M,9)
     ++U(M,11)*LENGTH(M))/LENGTH(M)
      BETA3=3.*(2.*U(M,3)-U(M,5)*LENGTH(M)-2.*U(M,9)-
     +U(M,11)*LENGTH(M))/LENGTH(M)
      BE(1)=(-U(M,1)+U(M,7))/LENGTH(M)+(ALFA1**2+BETA1**2)/2.
      BE(2)=ALFA1*ALFA2+BETA1*BETA2
      BE(3)=(ALFA2**2+BETA2**2)/2.+ALFA1*ALFA3+BETA1*BETA3
      BE(4)=ALFA2*ALFA3+BETA2*BETA3
      BE(5)=(ALFA3**2+BETA3**2)/2.
      DO 5343  I=1,5
5343  BTO(M,I)=BOL(M,I)+BE(I)
5344  CONTINUE
4992  CONTINUE
4993  CONTINUE
      WRITE(61,8649)
8649  FORMAT(/,10X,'DACTUAL(I)',/)
      DO 8653  I=1,NEQ
8653  WRITE(61,8654)   DACTUAL(I)
8654  FORMAT(10X,E21.15)
      WRITE(61,399) PACTUAL(LODPON1),DACTUAL(LODPON1),
     +                    DETRMNT,NUMITER
      IF(DETRMNT.LE.0..AND.DETOPTN.EQ.1)        GO TO 900
      IF(ABS(PACTUAL(LODPON1)).GE.ABS(PTOT(LNODE1,LDOF1))) GO TO 900
      DO 5007 I=1,NEQ
      PSAVE(I)=PACTUAL(I)
      DTOT(I)=0.0
5007  CONTINUE
      IF(JUSTK.EQ.1)  GO TO 2118
      DO 5281  NN=1,NUMEG
      IF(NUMEL(NN).EQ.0)   GO TO 5281
      NAME=NUMEL(NN)
      DO 5342  K=1,NAME
      M=L(NN,K)
```

```
      IF(MSUOPTN.EQ.1)   A7OLD(M)=A7TOT(M)
      DO 5342  I=1,5
      IF(MSUOPTN.EQ.2)   BOL(M,I)=BTO(M,I)
5342  CONTINUE
5281  CONTINUE
2118  CONTINUE
      J=0
      DO 450 N=1,NUMNP
      DO 451 I=1,6
      IF(IA(N,I).EQ.0) GO TO 451
      J=J+1
      R(J)=R(J)+PINC(N,I)
451   CONTINUE
450   CONTINUE
      DO 2119 I=1,NEQ
      IF(IFIX.EQ.0)  PSTART(I)=R(I)
      IF(IFIX.EQ.1)  PSTART(I)=R(I)+PSAVE(I)
2119  CONTINUE
      IF(PRIOPTN.EQ.0) GO TO 6977
      WRITE(61,9735)
9735  FORMAT(///,10X,'R(I)',/)
      DO 9736  IMM=1,NEQ
9736  WRITE(61,9737)  R(IMM)
9737  FORMAT(//,10X,E21.15,/)
6977  CONTINUE
      ICHECK=3
      GO TO 1001
3337  CONTINUE
C
C        CONDENSE LINEAR STIFFNESS AND LOAD VECTOR OF STRUCTURE
C     *******************************************************************
      IF(NCOND.EQ.0) GO TO 801
      CALL STCONDN
801   CONTINUE
C
C
C        SOLVE SYSTEM OF LINEAR EQUATIONS S*D=R
C     *******************************************************************
      IF(PROTYPE.NE.1) GO TO 601
      SCALE=1.0E+05
299   J=0
      DO 4666 N=1,NUMNP
      DO 4888 I=1,6
      IF(IA(N,I).EQ.0) GO TO 4888
      J=J+1
      R(J)=R(J)+PINC(N,I)
4888  CONTINUE
4666  CONTINUE
      IDET=1
601   CALL LINSOLN
      IF(PROTYPE.EQ.2)  GO TO 1778
      IF(R(LODPON1).EQ.PINT(LNODE1,LDOF1)) CALL IDENT
```

```
C     IF(R(LODPON1).EQ.PINT(LNODE1,LDOF1)) CALL INVTRNS
C     *****************************************************************
C        IN CASE WHICH WE WANT THE END FORCES DUE TO
C        THE LINEAR SOLUTION SUBROUTINE ENDFORC MAY BE CALLED
C        AT THIS STAGE(THE FIRST ITERATION OF THE FIRST
C        LOAD INCREMENT)
C     *****************************************************************
      IF(R(LODPON1).EQ.PINT(LNODE1,LDOF1).AND.ISTRESS.EQ.1) CALL STRESS
      IF(PROTYPE.NE.1) GO TO 1778
      DETER=DET1(SCALE)
      WRITE(61,399) PINT(LNODE1,LDOF1),D(LODPON1),DETER,NUMITER
      GO TO 709
C
777   NUMITER=0
C        RECOVER INTERNAL D.O.F."S OF STRUCTURE
C     *****************************************************************
C
      DO 1555 I=1,NEQ
1555  DTEMP(I)=D(I)
      IF(PROTYPE.NE.1) GO TO 715
778   NUMITER=NUMITER+1
715   CONTINUE
      NN=MAXITER+1
      IF(NUMITER.EQ.NN)  GO TO 9999
1778  IF(NCOND.NE.0)  CALL RECOVER
C
C        IDENTIFY DISPLACEMENTS FOUND FROM SOLUTION OF S*D=R AND FROM
C        THE RECOVERY PROCESS
C     *****************************************************************
      CALL IDENT
C        TO HAVE NODAL DEGREES OF FREEDOM IN LOCAL COORDINATES
C     *****************************************************************
C     CALL INVTRNS
C
C
C
      DO 180 I=1,NSIZE
      DO 180 J=1,MBAND
180   S(I,J)=0.0
      N=1
      IPAR=3
      CALL CURVED(N,IDATA,IARCH)
      IF(ICAL3.EQ.0) CALL STCONDN
      IF(NCOND.EQ.0) GO TO 802
      CALL STCONDN
802   CONTINUE
      IF(N2OPTIN.EQ.0)  GO TO 4991
      DO 190 I=1,NSIZE
      DO 190 J=1,MBAND
190   S(I,J)=0.0
      IPAR=4
      CALL CURVED(N,IDATA,IARCH)
```

```
      IF(ICAL3.EQ.0) CALL STCONDN
4991  CONTINUE
      IF(NCOND.EQ.0) GO TO 899
      CALL STCONDN
899   CONTINUE
      IF(PROTYPE.EQ.1) GO TO 222
      IF(PROTYPE.EQ.2.AND.EIGVALU.EQ.1)   CALL EIGENVL(EIGEN,IDATA)
      IF(EIGVALU.EQ.2) GO TO 444
      GO TO 900
222   CONTINUE
      DO 107 I=1,NEQ
      DO 108 J=1,MBAND
      READ(4,10) RK
      READ(5,10) RN1
      IF(N2OPTIN.EQ.1) READ(16,10) RN2
      IF(N2OPTIN.EQ.0) S(I,J)=RK+.5*RN1
      IF(N2OPTIN.EQ.1) S(I,J)=RK+.5*RN1+RN2/3.
      IF(N2OPTIN.EQ.0) SP(I,J)=RK+RN1
      IF(N2OPTIN.EQ.1) SP(I,J)=RK+RN1+RN2
108   CONTINUE
107   CONTINUE
      REWIND 4
      REWIND 5
      IF(N2OPTIN.EQ.1) REWIND 16
      IF(NUMITER.EQ.1) GO TO 701
      GO TO 702
9999  J=0
      DO 16 N=1,NUMNP
      DO 17 I=1,6
      IF(IA(N,I).EQ.0) GO TO 17
      J=J+1
      R(J)=R(J)+0.5*PINC(N,I)
17    CONTINUE
16    CONTINUE
      DO 333 I=1,NEQ
333   D(I)=DTEMP(I)
      WRITE(61,1899)  PINT(LNONE1,LDOF1),PINC(LNODE1,LDOF1)
1899  FORMAT(15X,23HLOADINCREMENT IS HALVED/
     +15X,6HPLOAD=,F10.5/15X,5HPINC=,F10.5)
      GO TO 1777
701   UOLD=D(LODPON1)
      GO TO 778
702   IF(ABS((UOLD-D(LODPON1))/D(LODPON1)).LE.TOLER)  GO TO 708
      UOLD=D(LODPON1)
      GO TO 778
708   DO 2555 I=1,NEQ
2555  DTEMP(I)=D(I)
      IDET=3
      WRITE(61,1399) PINT(LNODE1,LDOF1),D(LODPON1),NUMITER
1399  FORMAT(//,10X,5HLOAD=,F10.5/10X,7HDEFLEC=,F15.10/
     +10X,11HITERATIONS=,I5)
      DETER=DET1(SCALE)
```

```
      WRITE(61,399)  PINT(LNODE1,LDOF1),D(LODPON1),DETER,NUMITER
      IF(DETER.LE.0..AND.DETOPTN.EQ.1)    GO TO 900
709   J=0
      DO 161 N=1,NUMNP
      DO 171 I=1,6
      IF(IA(N,I).EQ.0) GO TO 171
      J=J+1
      R(J)=R(J)+PINC(N,I)
171   CONTINUE
161   CONTINUE
1777  IF(ABS(PINT(LNODE1,LDOF1)).GT.ABS(PTOT(LNODE1,LDOF1))) GO TO 900
      GO TO 299
C
444   CONTINUE
C     ******************************************************************
C        TO HAVE EIGENVALUE SOLUTION USING DETERMINANT SEARCH METHOD
C        INTHE CASE OF IEIGEN=1  (N1) STIFFNESS MATRIX WOULD
C        BE CONSIDERED IN SUBROUTINE NLEIGNP FOR NONLINEAR
C        EIGENVALUE PROBLEM.
C        FOR IEIGEN=2  (N1+K)  WOULD BE CONSIDERED
C     ******************************************************************
      IEIGEN=1
      SCALE=1.0E+05
      CALL NLEIGNP(SCALE)
900   CONTINUE
C
CCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCCC
10    FORMAT(E21.6)
399   FORMAT(///10X,5HLOAD=,F15.9/10X,'D(LODPON1)=',F15.10/
     + 10X,12HDETERMINANT=,E25.15/10X,11HITERATIONS=,I5/)
699   FORMAT(6F10.6,I5)
1010  FORMAT(A10,A10,A10,A10,A10,A10,A10,A10)
1015  FORMAT(11I5)
1020  FORMAT(2I5)
1030  FORMAT(3I5)
2010  FORMAT(////7X,6HNE    =,I3//7X,6HNUMNP=,I3//7X,
     +        6HNUMEG=,I3//7X,6HIDATA=,I3//7X,6HICAL1=,I3//7X,6HICAL2=,
     +        I3//7X,6HICAL3=,I3//7X,6HICAL4=,I3//7X,6HICAL5=,I3//
     +        7X,6HICAL6=,I3//7X,6HICAL7=,I3)
2015  FORMAT('1',15H  INITIAL LOADS//7H   NODE,27X,14HLOAD DIRECTION//
     +        7H NUMBER,9X,1HU,9X,1HV,9X,1HW,9X,1HB,8X,2HTY,8X,2HTX//)
2020  FORMAT('1',A10,A10,A10,A10,A10,A10,A10,A10)
2030  FORMAT(////7X,8HIARCH   =,I3//7X,8HILOAD   =,I3//7X,8HIDIRCN =,I3)
C
      END
C
      SUBROUTINE NODDATA (IARCH,DX)
C
C     ******************************************************************
C        TO READ AND PRINT NODAL POINT DATA
C        TO CALCULATE EQUATION NUMBERS AND CONDENSATION NUMBERS AND
C        STORE THEM IN ARRAYS  -IA- AND -IB- RESPECTIVELY
```

```
C     ******************************************************************
C
      REAL L
      COMMON/C1/NE,NUMNP,D1(15)
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C3/IA(37,8),IB(37,8),X(37),Y(37),Z(37),R,A
      COMMON/C19/TT(36)
C
C........READ NODAL POINT DATA
C
      WRITE(61,2000)
      WRITE(61,2010)
      WRITE(61,2015)
      IF (IARCH.EQ.0) GO TO 101
      IF (IARCH.EQ.2) GO TO 104
      READ(60,1010) R
100   READ(60,1000) N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),T,Z(N)
      PI=4.*ATAN(1.)
      T=T*PI/180.
      X(N)=SIN(T)*R
      Y(N)=R*(1.-COS(T))
      WRITE(61,2020) N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),X(N),Y(N),Z(N)
      IF (N.NE.NUMNP) GO TO 100
      GO TO 103
101   CONTINUE
      READ(60,1020) H,L,DI,XS
      A=H/(XS*XS)
      DX=L/DI
      DL=0.
102   READ(60,1000) N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),T,Z(N)
      X(N)=XS+DL
      Y(N)=A*X(N)*X(N)
      WRITE(61,2020) N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),X(N),Y(N),Z(N)
      DL=DL+DX
      IF (N.NE.NUMNP) GO TO 102
      GO TO 103
104   CONTINUE
      READ(60,1021) H,L
106   READ(60,1001) N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),TT(N),X(N),Y(N)
     +               ,Z(N)
      WRITE(61,2021)N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),TT(N),X(N),Y(N)
     +               ,Z(N)
      IF(N.NE.NUMNP) GO TO 106
C
C........PROCESS ARRAYS -IA- AND -IB- TO FIND EQUATION NUMBERS AND
C        CONDENSATION NUMBERS. STORE NEQ"S AND NCOND"S IN ARRAYS IA AND
C        IB RESPECTIVELY.
C
103   NEQ=0
      NCOND=0
      DO 125 N=1,NUMNP
      DO 120 I=1,6
```

```
      IF(IA(N,I).NE.1) GO TO 105
      IA(N,I)=0
      GO TO 120
105   IA(N,I)=-1
      IF(IB(N,I)) 110,115,120
110   NCOND=NCOND+1
      IB(N,I)=-NCOND
      GO TO 120
115   NEQ=NEQ+1
      IA(N,I)=NEQ
120   CONTINUE
125   CONTINUE
      NSIZE=NEQ+NCOND
C
C........WRITE GENERATED NODAL POINT DATA
C
      WRITE(61,2030)
      WRITE(61,2040)
      WRITE(61,2050) (N,(IA(N,I),I=1,6),(IB(N,I),I=1,6),N=1,NUMNP)
      WRITE(61,2060) NSIZE,NEQ,NCOND
      RETURN
C
1000  FORMAT(I5,12I3,F15.10,F10.6)
1001  FORMAT(I5,12I3,F9.6,3F10.6)
1010  FORMAT(F15.9)
1020  FORMAT(4F10.5)
1021  FORMAT(2F10.5)
2000  FORMAT('1',33H N O D A L   P O I N T   D A T A  //)
2010  FORMAT(18H INPUT NODAL DATA //)
2015  FORMAT(7H   NODE,26X,36HNODAL POINT BOUNDARY CONDITION CODES,33X,
     +       23HNODAL POINT COORDINATES/7H NUMBER,21X,7HIA(N,I),33X,
     +       7HIB(N,I)/11X,2(4X,1HU,4X,1HV,4X,1HW,4X,1HB,3X,2HTY,3X,
     +       2HTX,10X),9X,4HX(N),8X,4HY(N),8X,4HZ(N))
2020  FORMAT(I5,6X,12I5,4X,3F12.3)
2021  FORMAT(I5,6X,12I5,4X,F9.6,3F12.3)
2030  FORMAT(///22H GENERATED NODAL DATA //)
2040  FORMAT(7H   NODE,16X,16HEQUATION NUMBERS,22X,
     +       20HCONDENSATION NUMBERS/7H NUMBER,21X,7HIA(N,I),33X,
     +       7HIB(N,I)/11X,2(4X,1HU,4X,1HV,4X,1HW,4X,1HB,3X,2HTY,3X,
     +       2HTX,10X))
2050  FORMAT(I5,6X,12I5)
2060  FORMAT('-',6HNSIZE=,I3,3X,4HNEQ=,I3,3X,6HNCOND=,I3)
C
      END
C
      SUBROUTINE LOAD (IARCH,ILOAD,IDIRCN,DX,WW)
C
C     ******************************************************************
C        TO READ AND STORE INITIAL LOAD DATA
C     ******************************************************************
C
      REAL LENGTH
```

```
      COMMON/C1/NE,NUMNP,D1(15)
      COMMON/C3/IA(37,8),IB(37,8),X(37),Y(37),Z(37),RAD,AC
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),D5(180)
      COMMON/C7/RI(36),RJ(36),PHII(36),PHIJ(36),TETA(36),LENGTH(36),
     +          RIA(36),RJA(36)
      COMMON/C8/PN(37,8),R(296),PINT(37,8)
C
C........CHECK TYPES OF LOADS TO BE READ
C          ILOAD.EQ.0 , LOAD IS UNIFORMLY DISTRIBUTED
C          ILOAD.EQ.1 , LOADS ARE CONCENTRATED
C
      IF (ILOAD.EQ.0) GO TO 200
100   READ(60,1023) MN,(PN(MN,I),I=1,6)
      WRITE(61,2020) MN,(PN(MN,I),I=1,6)
      IF(MN.NE.NUMNP) GO TO 100
      RETURN
200   CONTINUE
      READ(60,1020) WW
      IF (IDIRCN.EQ.0) WRITE(61,2030) WW
      IF (IDIRCN.EQ.1) WRITE(61,2040) WW
      DO 300 NM=1,NUMNP
      DO 300 I=1,6
300   PN(NM,I)=0.
      DO 400 M=1,NE
      NI=NODEI(M)
      NJ=NODEJ(M)
C
C........CHECK IF DISTRIBUTED LOAD IS VERTICAL OR HORIZONTAL
C        THEN CONCENTRATE IT AT THE NODES IN COMPONENTS
C          IDIRCN.EQ.0 , VERTICAL
C          IDIRCN.EQ.1 , HORIZONTAL
C
      IF (IDIRCN.EQ.1) GO TO 350
      IF (IARCH.EQ.1) DX=ABS(X(NJ)-X(NI))
      PN(NI,1)=PN(NI,1)+DX/2.*W*COS(PHII(M))
      PN(NI,3)=PN(NI,3)+DX/2.*W*SIN(PHII(M))
      PN(NJ,1)=PN(NJ,1)+DX/2.*W*COS(PHIJ(M))
      PN(NJ,3)=PN(NJ,3)+DX/2.*W*SIN(PHIJ(M))
      GO TO 400
350   CONTINUE
      PN(NI,2)=PN(NI,2)+W*LENGTH(M)/2.
      PN(NJ,2)=PN(NJ,2)+W*LENGTH(M)/2.
400   CONTINUE
      WRITE(61,2020) (N,(PN(N,I),I=1,6),N=1,NUMNP)
      RETURN
C
1023  FORMAT(I5,6F7.1)
1020  FORMAT(F10.5)
2020  FORMAT(I5,6X,6F10.3)
2030  FORMAT('-',38HUNIFORMLY DISTRIBUTED VERTICAL LOAD W=,F10.5//)
2040  FORMAT('-',40HUNIFORMLY DISTRIBUTED HORIZONTAL LOAD W=,F10.5//)
C
```

```
      END
C
```

```
      SUBROUTINE ELEMENT (N,IDATA,IARCH)
C
C     *****************************************************************
C        TO CALL THE APPROPRIATE ELEMENT SUBROUTINE
C     *****************************************************************
C
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),D12(11)
C
      IF (NTYPE(N).GT.1) GO TO 200
      CALL CURVED (N,IDATA,IARCH)
      RETURN
200   RETURN
C
      END
C
      SUBROUTINE BAND
C
C     *****************************************************************
C        TO COMPUTE SEMIBANDWIDTH OF STRUCTURE STIFFNESS MATRIX
C        DONE BY FINDING THE MAXIMUMN DIFFERENCE BETWEEN THE
C        EQUATION NUMBERS ASSOTIATED WITH THE NODES OF A
C        PARTICULAR ELEMENT
C     *****************************************************************
C
      COMMON/C1/NE,NUMNP,D1(15)
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C3/IA(37,8),D3(409)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),D5(180)
C
      MBAND=0
      DO 900 M=1,NE
      NI=NODEI(M)
      NJ=NODEJ(M)
      DO 800 I=1,6
      IF (IA(NI,I).LE.0) GO TO 800
      N1=IA(NI,I)
      DO 700 J=1,6
      IF (IA(NJ,J).LE.0) GO TO 700
      N2=IA(NJ,J)
      MB=N2-N1
      IF (MB.LT.0) MB=-MB+1
      IF (MB.GT.0) MB=MB+1
      IF (MB.GT.MBAND) MBAND=MB
700   CONTINUE
800   CONTINUE
900   CONTINUE
      WRITE(61,2000) MBAND
      RETURN
C
2000  FORMAT('1',20HSEMIBANDWIDTH MBAND=,I3)
C
      END
```

```
C
      SUBROUTINE CURVED (N,IDATA,IARCH)
C     *****************************************************************
C        CURVED ELEMENT SUBROUTINE
C     *****************************************************************
C
      REAL IXX,IYY,KT,II,JJ,LENGTH
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,
     +         ICAL3,ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C4/SE(16,16)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
     +         KT(36),L(1,36)
      COMMON/C6/A1,A2,MP,B1(36),B2(36),B3(36)
      COMMON/C7/RI(36),RJ(36),PHII(36),PHIJ(36),TETA(36),LENGTH(36),
     +          RIA(36),RJA(36)
      COMMON/C8/PN(37,8),R(296),PINT(37,8)
      COMMON/C9/S(296,16),SP(296,16),IDET
      COMMON/C10/D(296),D10(1184),RC(296),SC(296,16)
      COMMON/C11/DN(16),U(36,12),W(37,8),V(37,8)
      COMMON/C12/ULOC(36,12),RCOL(9),MSUOPTN,N1GOPTIN
      COMMON/C19/TT(36)
C
      GO TO (100,200,300,400),IPAR
C
C........READ MATERIAL INFORMATION
C
100   WRITE(61,2000) NTYPE(N)
      READ(60,1010) E(N),G(N),DM
      WRITE(61,2020) NUMEL(N),E(N),G(N),DM
C
C........READ ELEMENT AND CROSS SECTION INFORMATION
C
      WRITE(61,2021)
      K=0
105   READ(60,1020) M,NODEI(M),NODEJ(M),A(M),IXX(M),IYY(M),KT(M)
      WRITE(61,2022) M,A(M),IXX(M),IYY(M),KT(M)
      K=K+1
      L(N,K)=M
      IF(K.NE.NUMEL(N)) GO TO 105
C
C........READ AND CALCULATE ELEMENT GEOMETRIC PROPERTIES
C
      IF (ICAL2.EQ.0) WRITE(61,2025)
      DO 110 KK=1,K
      M=L(N,KK)
      CALL GEOMTRY (M,IARCH)
110   CONTINUE
      RETURN
C
200   CONTINUE
C
```

```
C........CALCULATE LINEAR STIFFNESS MATRIX OF EACH ELEMENT AND STORE IN
C        ARRAY SE(M,I,J). READ LIMITS OF INTEGRATION AND THE MP-POINTS
C        OF INTEGRATION TO BE USED IN THE GAUSS-LEGENDRE QUADRATURE
C
      IF (IDATA.EQ.0) READ(60,1030) A1,A2,MP
      IF (ICAL1.EQ.0) WRITE(61,2030) A1,A2,MP
C
C........OBTAIN LINEAR STIFFNESS FOR EACH ELEMENT. INTEGRATE NUMERICALLY
C
      INUMEL=NUMEL(N)
      DO 220 K=1,INUMEL
      M=L(N,K)
      CALL NUMINT (N,M)
      IF(ICAL1.NE.0) GO TO 215
      WRITE(61,2032) M
      WRITE(61,2034) ((SE(I,J),J=1,6),I=1,6)
      WRITE(61,2036)
      WRITE(61,2034) ((SE(I,J),J=7,12),I=1,6)
      WRITE(61,2038)
      WRITE(61,2034) ((SE(I,J),J=1,6),I=7,12)
      WRITE(61,2040)
      WRITE(61,2034) ((SE(I,J),J=7,12),I=7,12)
215   WRITE(1,10) ((SE(I,J),J=1,12),I=1,12)
220   CONTINUE
      REWIND 1
C
C........ASSEMBLE LINEAR STIFFNESS OF EACH ELEMENT
C        INTO LINEAR STIFFNESS OF STRUCTURE
C
      DO 230 K=1,INUMEL
      M=L(N,K)
      READ(1,10) ((SE(I,J),J=1,12),I=1,12)
      CALL ASEMBLE (M)
230   CONTINUE
      REWIND 1
      REWIND 9
      REWIND 8
      WRITE(4,10) ((S(I,J),J=1,MBAND),I=1,NSIZE)
      REWIND 4
      RETURN
C
300   CONTINUE
C
C........OBTAIN NONLINEAR STIFFNESS -SE1- FOR EACH ELEMENT.
C        INTEGRATE NUMERICALLY.
C
      DO 320 K=1,INUMEL
      M=L(N,K)
      NI=NODEI(M)
      NJ=NODEJ(M)
      DO 305 ID=1,6
      DN(ID)=U(NI,ID)
```

```
305   DN(ID+6)=U(NJ,ID)
      CALL NUMINT (N,M)
      IF(ICAL1.NE.0) GO TO 315
      WRITE(61,2050) M
      WRITE(61,2034) ((SE(I,J),J=1,6),I=1,6)
      WRITE(61,2036)
      WRITE(61,2034) ((SE(I,J),J=7,12),I=1,6)
      WRITE(61,2038)
      WRITE(61,2034) ((SE(I,J),J=1,6),I=7,12)
      WRITE(61,2040)
      WRITE(61,2034) ((SE(I,J),J=7,12),I=7,12)
315   WRITE(2,10) ((SE(I,J),J=1,12),I=1,12)
320   CONTINUE
      REWIND 2
C
C........ASSEMBLE NONLINEAR STIFFNESS SE1 OF EACH ELEMENT
C         INTO NONLINEAR STIFFNESS OF STRUCTURE, S1.
C         STORE MATRIX S1
C
      DO 330 K=1,INUMEL
      M=L(N,K)
      READ(2,10) ((SE(I,J),J=1,12),I=1,12)
      CALL ASEMBLE (M)
330   CONTINUE
      REWIND 2
      WRITE(5,10) ((S(I,J),J=1,MBAND),I=1,NSIZE)
      REWIND 5
      RETURN
C
400   CONTINUE
C
C........OBTAIN NONLINEAR STIFFNESS -SE2- FOR EACH ELEMENT.
C         INTEGRATE NUMERICALLY.
C
      DO 420 K=1,INUMEL
      M=L(N,K)
      NI=NODEI(M)
      NJ=NODEJ(M)
      DO 405 ID=1,6
      DN(ID)=U(NI,ID)
405   DN(ID+6)=U(NJ,ID)
      CALL NUMINT (N,M)
      IF(ICAL1.NE.0) GO TO 415
      WRITE(61,2055) M
      WRITE(61,2034) ((SE(I,J),J=1,6),I=1,6)
      WRITE(61,2036)
      WRITE(61,2034) ((SE(I,J),J=7,12),I=1,6)
      WRITE(61,2038)
      WRITE(61,2034) ((SE(I,J),J=1,6),I=7,12)
      WRITE(61,2040)
      WRITE(61,2034) ((SE(I,J),J=7,12),I=7,12)
415   WRITE(3,10) ((SE(I,J),J=1,12),I=1,12)
```

```
420    CONTINUE
       REWIND 3
C
C........ASSEMBLE NONLINEAR STIFFNESS SE2 OF EACH ELEMENT
C           INTO NONLINEAR STIFFNESS OF STRUCTURE, S2.
C           STORE MATRIX S2
C
       DO 430 K=1,INUMEL
       M=L(N,K)
       READ(3,10) ((SE(I,J),J=1,12),I=1,12)
       CALL ASEMBLE (M)
430    CONTINUE
       REWIND 3
       WRITE(16,10) ((S(I,J),J=1,MBAND),I=1,NSIZE)
       REWIND 16
       RETURN
C
10     FORMAT(E21.6)
1010   FORMAT(3E10.2)
1020   FORMAT(3I5,4E15.6)
1030   FORMAT(2F5.2,I5)
2000   FORMAT('1',23HG R O U P   N U M B E R ,I2//2X,6HNUMBER,6X,7HMODULUS
     +        ,11X,5HSHEAR,8X,7HDENSITY/4X,2HOF,11X,2HOF,12X,7HMODULUS/
     +        1X,8HELEMENTS,4X,10HELASTICITY)
2020   FORMAT(I6,3E17.6)
2021   FORMAT(//8H ELEMENT,9X,4HA(M),10X,6HIXX(M);9X,6HIYY(M),9X,5HKT(M))
2022   FORMAT(I6,5X,4E15.6)
2025   FORMAT(//8H ELEMENT,3X,8HNODEI(M),3X,8HNODEJ(M),9X,
     +        19HRADIUS OF CURVATURE,18X,17HNODAL SLOPE ANGLE/8H  NUMBER/
     +        39X,5HRI(M),10X,5HRJ(M),15X,7HPHII(M),8X,7HPHIJ(M),7X,
     +        7HTETA(M))
2030   FORMAT('1',21HLIMITS OF INTEGRATION,3X,3HA1=,F3.1,3X,3HA2=,F3.1//
     +        26H QUADRATURE FORMULA POINTS,3X,3HMP=,I2)
2032   FORMAT('1',44HUNCONDENSED LINEAR STIFFNESS (SE) OF ELEMENT,I3///
     +        9H BLOCK II)
2034   FORMAT(//1X,6F15.6)
2036   FORMAT(///9H BLOCK IJ)
2038   FORMAT('1'//9H BLOCK JI)
2040   FORMAT(///9H BLOCK JJ)
2050   FORMAT('1',48HUNCONDENSED NONLINEAR STIFFNESS (SE1) OF ELEMENT,I3
     +        ///9H BLOCK II)
2055   FORMAT('1',48HUNCONDENSED NONLINEAR STIFFNESS (SE2) OF ELEMENT,I3
     +        ///9H BLOCK II)
C
       END
C
       SUBROUTINE GEOMTRY (M,IARCH)
C
C      *****************************************************************
C         TO CALCULATE AND STORE GEOMETRIC PROPERTIES OF CURVED ELEMENTS.
C      *****************************************************************
C
```

```
      REAL IXX,IYY,KT,II,JJ,LENGTH
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,
     +         ICAL3,ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C3/IA(37,8),IB(37,8),X(37),Y(37),Z(37),R,AC
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
     +         KT(36),L(1,36)
      COMMON/C6/A1,A2,MP,B1(36),B2(36),B3(36)
      COMMON/C7/RI(36),RJ(36),PHII(36),PHIJ(36),TETA(36),LENGTH(36),
     +          RIA(36),RJA(36)
      COMMON/C19/TT(36)
C
      XI=X(NODEI(M))
      YI=Y(NODEI(M))
      XJ=X(NODEJ(M))
      YJ=Y(NODEJ(M))
C
C........READ ELEMENT GEOMETRIC PROPERTIES
C
      IF (IARCH.EQ.0) GO TO 100
      IF (IARCH.EQ.2) GO TO 190
      RI(M)=R
      RJ(M)=R
      DY=XI/SQRT(R*R-XI*XI)
      PHII(M)=ATAN(DY)
      DY=XJ/SQRT(R*R-XJ*XJ)
      PHIJ(M)=ATAN(DY)
      GO TO 200
100   CONTINUE
      D2Y=2.*AC
      DY=2.*AC*XI
      RI(M)=(1.+DY*DY)**1.5/D2Y
      PHII(M)=ATAN(DY)
      DY=2.*AC*XJ
      RJ(M)=(1.+DY*DY)**1.5/D2Y
      PHIJ(M)=ATAN(DY)
      GO TO 200
190   CONTINUE
      PHII(M)=TT(NODEI(M))
      PHIJ(M)=TT(NODEJ(M))
200   CONTINUE
      XL=XJ-XI
      YL=YJ-YI
      TETA(M)=ABS(PHII(M)-PHIJ(M))
      T=TETA(M)
C
C........CALCULATE NODAL LOCAL COORDINATES AFTER ROTATION
C
      ZR=XL*COS(PHII(M))+YL*SIN(PHII(M))
      XR=-XL*SIN(PHII(M))+YL*COS(PHII(M))
C
C........CHECK DATA GENERATION
C
```

```
      IF (ICAL2.EQ.0) WRITE(61,2010) M,NODEI(M),NODEJ(M),RI(M),RJ(M),
     +                               PHII(M),PHIJ(M),T
C
C........SOLVE SYSTEM OF EQUATIONS IN CLOSED FORM ,
C        OBTAIN VARIABLES B1, B2, RIA(M), RJA(M), AND LENGTH(M),
C        FIRST CALCULATE COEFFICIENTS OF THE VARIABLES
C
      AA11=1.-COS(T)
      AA12=2.*(SIN(T)-T*COS(T))
      AA21=SIN(T)
      AA22=2.*(T*SIN(T)+COS(T)-1.)
C
C........CALCULATE B1,B2,RIA(M),RJA(M),LENGTH(M)
C
      B2(M)=(AA11*ZR-AA21*XR)/(AA11*AA22-AA12*AA21)
      B1(M)=XR/AA11-AA12*B2(M)/AA11
      LENGTH(M)=B1(M)*T+B2(M)*T*T
      RIA(M)=B1(M)
      RJA(M)=B1(M)+2.*B2(M)*T
      IF (IARCH.EQ.2) RI(M)=RIA(M)
      IF (IARCH.EQ.2) RJ(M)=RJA(M)
C
C........CHECK DATA GENERATION
C
      IF(ICAL2.EQ.0) WRITE(61,2020) XR,ZR,B1(M),B2(M),LENGTH(M),
     +                 RIA(M),RJA(M)
      RETURN
C
2010  FORMAT(//I6,5X,I5,6X,I5,6X,2F15.6,6X,3F15.6)
2020  FORMAT(/10X,3HXR=,F15.10,3X,3HZR=,F15.10//10X,6HB1(M)=,E15.9,3X,
     +        6HB2(M)=,E15.9,3X,10HLENGTH(M)=,F13.6//10X,11HRI(APPROX)=
     +        ,F15.9,10X,11HRJ(APPROX)=,F15.9//)
C
      END
      SUBROUTINE NUMINT (N,M)
      DOUBLE PRECISION A01,A02,A3,A4,A5,A6,BTG,BLG
C     *****************************************************************
C         TO INTEGRATE NUMERICALLY THE TERMS OF THE CURVED ELEMENT
C         STIFFNESS MATRICES SE, SE1, SE2, IT USES THE GAUSS-LEGENDRE
C         QUADRATURE FORMULA.
C         THE ROUTINE NUMINT USES THE MP-POINT GAUSS-LEGENDRE QUADRATURE
C         FORMULA TO COMPUTE THE INTEGRAL OF FUNCTN(GM)*DGM BETWEEN
C         INTEGRATION LIMITS A1 AND A2. THE ROOTS OF SEVEN LEGENDRE
C         POLYNOMIALS AND THE WEIGHT FACTORS FOR CORRESPONDING
C         QUADRATURES ARE STORED IN THE Z AND WEIGHT ARRAYS RESPECTIVELY.
C         MP MAY ASSUME VALUES 2, 3, 4, 5, 6, 10, AND 15 ONLY. THE
C         APPROPRIATE VALUES FOR THE MP-POINT FORMULA ARE LOCATED IN
C         ELEMENTS Z(KEY(I))...Z(KEY(I+1)-1) AND WEIGHT(KEY(I))...
C         WEIGHT(KEY(I+1)-1) WHERE THE PROPER VALUE FOR I IS DETERMINED
C         BY FINDING THE SUBSCRIPT OF THE ELEMENT OF THE ARRAY NPOINT
C         WHICH HAS THE VALUE MP. IF AN INVALID VALUE OF MP IS USED, A
C         TRUE ZERO IS RETURNED AS THE VALUE OF GAUSS.
```

```
C     ******************************************************************
C
      REAL IXX,IYY,KT,II,JJ,LENGTH,L1,L2,K,KK,LL,MM,NN,MS
      DIMENSION NPOINT(7),KEY(8),Z(24),WEIGHT(24),K(16,16)
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,ICAL3,
     +          ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C4/SE(16,16)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
     +          KT(36),L(1,36)
      COMMON/C6/A1,A2,MP,B1(36),B2(36),B3(36)
      COMMON/C7/RI(36),RJ(36),PHII(36),PHIJ(36),TETA(36),LENGTH(36),
     +           RIA(36),RJA(36)
      COMMON/C11/DN(16),U(36,12),W(37,8),V(37,8)
      COMMON/C18/IARCH
      DATA NPOINT/ 2, 3, 4, 5, 6, 10, 15/
      DATA KEY/ 1, 2, 4, 6, 9, 12, 17, 25/
      DATA Z              / 0.577350269,0.0        ,0.774596669,
     1          0.339981044,0.861136312,0.0        ,0.538469310,
     2          0.906179846,0.238619186,0.661209387,0.932469514,
     3          0.148874339,0.433395394,0.679409568,0.865063367,
     4          0.973906529,0.0        ,0.201194094,0.394151347,
     5          0.570972173,0.724417731,0.848206583,0.937273392,
     6          0.987992518 /
      DATA WEIGHT         / 1.0        ,0.888888889,0.555555556,
     1          0.652145155,0.347854845,0.568888889,0.478628671,
     2          0.236926885,0.467913935,0.360761573,0.171324493,
     3          0.295524225,0.269266719,0.219086363,0.149451349,
     4          0.066671344,0.202578242,0.198431485,0.186161000,
     5          0.166269206,0.139570678,0.107159221,0.070366047,
     6          0.030753242 /
C
      T=TETA(M)
      R1=RI(M)
      R2=RJ(M)
      L1=R1*T
      L2=R2*T
C
C........FIND SUBSCRIPT OF FIRST Z AND WEIGHT VALUE
C
      DO 100 I=1,7
      IF(MP.EQ.NPOINT(I)) GO TO 200
100   CONTINUE
C
C........INVALID MP USED
C
      GAUSS=0.0
      WRITE(61,2000) GAUSS
      RETURN
C
C........SET UP INITIAL PARAMETERS
C
200   JFIRST=KEY(I)
```

```
      JLAST=KEY(I+1)-1
      C=(A2-A1)/2.
      D=(A2+A1)/2.
C
C........ACCUMULATE THE SUM IN THE MP-POINT FORMULA
C
CCCC
      IF (IPAR.GE.3) GO TO 543
      DO 249 I=1,16
      DO 249 J=1,16
249   K(I,J)=0.0



      GO TO 248
543   CONTINUE
CCCC
      DO 250 I=1,12
      DO 250 J=1,12
250   K(I,J)=0.0



248   CONTINUE
                              IF(IPAR.GE.3) GO TO 390
      DO 500 J=JFIRST,JLAST
      I=0
      IF (Z(J).EQ.0.) GO TO 350
300   I=I+1
      IF (I.EQ.1) GM=Z(J)*C+D
      IF (I.EQ.2) GM=-Z(J)*C+D
      GO TO 360
350   GM=D
360   AA=6.*GM**2-6.*GM
      BB=3.*GM**2-4.*GM+1.
      CC=3.*GM**2-2.*GM
      DD=12.*GM-6.
      EE=6.*GM-4.
      FF=6.*GM-2.
      GG=2.*GM**3-3.*GM**2+1.
      HH=GM**3-2.*GM**2+GM
      II=-2.*GM**3+3.*GM**2
      JJ=GM**3-GM**2
      KK=1.-GM
      R=B1(M)+2.*B2(M)*T*GM
      GMSS=(-1./(R**3*T))*(2.*B2(M))
      GMSG=R*T*GMSS
C
C........CHECK WHICH PART OF THE STIFFNESS MATRIX IS BEING COMPUTED
C                  IPAR=2, COMPUTE ARRAY SE
C                  IPAR=3, COMPUTE ARRAY SE1
C                  IPAR=4, COMPUTE ARRAY SE2
```

```
C
C
C
C
C
C........INTEGRANDS OF CURVED ELEMENT LINEAR STIFFNESS (SYMMETRIC)
C
      C1=-E(N)*A(M)*GG/R
      C2=(E(N)*IYY(M)/(R**3*T**3))*(DD+AA*GMSS*R**2*T**2)
      C3=(E(N)*IXX(M)*T/R**3)*(-DD/T**2-GMSS*R**2*AA)
      C4=G(N)*KT(M)*AA/(R**3*T)
      C5=(E(N)*A(M)/(R*T))*(AA+T**2*HH)
      C6=(E(N)*IYY(M)/(R**3*T**3))*(-T*EE-GMSS*R**2*T**3*BB+T*AA+
     +   GMSG*R*T**2*GG)
      C7=E(N)*IXX(M)*T*GG/R**2
      C8=G(N)*KT(M)*AA/(R**2*T)
      C9=-E(N)*A(M)*L1*HH/R
      C10=(E(N)*IYY(M)/(R**3*T**3))*(L1*EE+GMSS*R**2*T**2*L1*BB)
      C11=(E(N)*IXX(M)*T/R**3)*(L1*EE/T**2+GMSS*R**2*L1*BB)
      C12=-G(N)*KT(M)*L1*BB/(R**3*T)
      C13=E(N)*A(M)*R1*BB/(R*T)
      C14=(E(N)*IYY(M)/(R**3*T**3))*(T*R1*BB+GMSG*R*T**2*R1*HH)
      C15=E(N)*IXX(M)*T*L1*HH/R**2
      C16=G(N)*KT(M)*L1*BB/(R**2*T)
      C17=-E(N)*A(M)*II/R
      C18=(E(N)*IYY(M)/(R**3*T**3))*(-DD-GMSS*R**2*T**2*AA)
      C19=-C3
      C20=-C4
      C21=(E(N)*A(M)/(R*T))*(-AA+T**2*JJ)
      C22=(E(N)*IYY(M)/(R**3*T**3))*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +    GMSG*R*T**2*II)
      C23=E(N)*IXX(M)*T*II/R**2
      C24=-C8
      C25=(-E(N)*A(M)*L2*JJ)/R
      C26=(E(N)*IYY(M)/(R**3*T**3))*(L2*FF+GMSS*R**2*T**2*L2*CC)
      C27=(E(N)*IXX(M)*T/R**3)*(L2*FF/T**2+GMSS*R**2*L2*CC)
      C28=-G(N)*KT(M)*L2*CC/(R**3*T)
      C29=E(N)*A(M)*R2*CC/(R*T)
      C30=(E(N)*IYY(M)/(R**3*T**3))*(T*R2*CC+GMSG*R*T**2*R2*JJ)
      C31=E(N)*IXX(M)*T*L2*JJ/R**2
      C32=G(N)*KT(M)*L2*CC/(R**2*T)
C
        SE(1,1)=C1*T*(-GG)+C2*(DD+AA*GMSS*R**2*T**2)
        SE(1,2)=0.0
        SE(1,4)=0.0
        SE(1,6)=0.0
        SE(1,8)=0.0
        SE(1,10)=0.0
        SE(1,12)=0.0
        SE(1,14)=0.0
        SE(1,16)=0.0
        SE(1,3)=C1*(AA+T**2*HH)+C2*(-T*EE-GMSS*R**2*T**3*BB+T*AA+
```

```
     +          GMSG*R*T**2*GG)
      SE(1,5)=C1*T*(-L1)*HH+C2*L1*(EE+BB*GMSS*R**2*T**2)
      SE(1,7)=C1*R1*BB+C2*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(1,9)=C1*(-T*II)-C2*(DD+AA*GMSS*R**2*T**2)
      SE(1,11)=C1*(-AA+T**2*JJ)+C2*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +           GMSG*R*T**2*II)
      SE(1,13)=C1*(-T*L2*JJ)+C2*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(1,15)=C1*(R2*CC)+C2*(T*R2*CC+GMSG*R*T**2*JJ)
C
      SE(2,2)=C3*(-DD/T**2-GMSS*R**2*AA)+C4*AA
      SE(2,3)=0.0
      SE(2,5)=0.0
      SE(2,7)=0.0
      SE(2,9)=0.0
      SE(2,11)=0.0
      SE(2,13)=0.0
      SE(2,15)=0.0
      SE(2,4)=C3*R*GG+C4*R*AA
      SE(2,6)=C3*(L1*EE/T**2+GMSS*R**2*L1*BB)-C4*L1*BB
      SE(2,8)=C3*R*L1*HH+C4*R*L1*BB
      SE(2,10)=C3*(DD/T**2+GMSS*R**2*AA)-C4*AA
      SE(2,12)=C3*R*II-C4*R*AA
      SE(2,14)=C3*(L2*FF/T**2+GMSS*R**2*L2*CC)-C4*L2*CC
      SE(2,16)=C3*R*L2*JJ+C4*R*L2*CC
C
      SE(3,3)=C5*(AA+T**2*HH)+C6*(-T*EE-GMSS*R**2*T**3*BB+T*AA+
     +          GMSG*R*T**2*GG)
      SE(3,4)=0.0
      SE(3,6)=0.0
      SE(3,8)=0.0
      SE(3,10)=0.0
      SE(3,12)=0.0
      SE(3,14)=0.0
      SE(3,16)=0.0
      SE(3,5)=C5*(-T*L1*HH)+C6*(L1*EE+GMSS*R**2*T**2*L1*BB)
      SE(3,7)=C5*R1*BB+C6*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(3,9)=C5*(-T*II)+C6*(-DD-GMSS*R**2*T**2*AA)
      SE(3,11)=C5*(-AA+T**2*JJ)+C6*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +           GMSG*R*T**2*II)
      SE(3,13)=C5*(-T*L2*JJ)+C6*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(3,15)=C5*R2*CC+C6*(T*R2*CC+GMSG*R*T**2*JJ)
C
      SE(4,4)=C7*R*GG+C8*R*AA
      SE(4,5)=0.0
      SE(4,7)=0.0
      SE(4,9)=0.0
      SE(4,11)=0.0
      SE(4,13)=0.0
      SE(4,15)=0.0
      SE(4,6)=C7*(L1*EE/T**2+GMSS*R**2*L1*BB)+C8*(-L1*BB)
      SE(4,8)=C7*R*L1*HH+C8*R*L1*BB
      SE(4,10)=C7*(DD/T**2+GMSS*R**2*AA)+C8*(-AA)
```

```
      SE(4,12)=C7*R*II+C8*(-R*AA)
      SE(4,14)=C7*(L2*FF/T**2+GMSS*R**2*L2*CC)+C8*(-L2*CC)
      SE(4,16)=C7*R*L2*JJ+C8*R*L2*CC
C
      SE(5,5)=C9*(-T*L1*HH)+C10*(L1*EE+GMSS*R**2*T**2*L1*BB)
      SE(5,6)=0.0
      SE(5,8)=0.0
      SE(5,10)=0.0
      SE(5,12)=0.0
      SE(5,14)=0.0
      SE(5,16)=0.0
      SE(5,7)=C9*R1*BB+C10*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(5,9)=C9*(-T*II)+C10*(-DD-GMSS*R**2*T**2*AA)
      SE(5,11)=C9*(-AA+T**2*JJ)+C10*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +         GMSG*R*T**2*II)
      SE(5,13)=C9*(-T*L2*JJ)+C10*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(5,15)=C9*R2*CC+C10*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(6,6)=C11*(L1*EE/T**2+GMSS*R**2*L1*BB)+C12*(-L1*BB)
      SE(6,7)=0.0
      SE(6,9)=0.0
      SE(6,11)=0.0
      SE(6,13)=0.0
      SE(6,15)=0.0
      SE(6,8)=C11*R*L1*HH+C12*R*L1*BB
      SE(6,10)=C11*(DD/T**2+GMSS*R**2*AA)+C12*(-AA)
      SE(6,12)=C11*R*II+C12*(-R*AA)
      SE(6,14)=C11*(L2*FF/T**2+GMSS*R**2*L2*CC)+C12*(-L2*CC)
      SE(6,16)=C11*R*L2*JJ+C12*R*L2*CC
C
      SE(7,7)=C13*R1*BB+C14*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(7,8)=0.0
      SE(7,10)=0.0
      SE(7,12)=0.0
      SE(7,14)=0.0
      SE(7,16)=0.0
      SE(7,9)=C13*(-T*II)+C14*(-DD-GMSS*R**2*T**2*AA)
      SE(7,11)=C13*(-AA+T**2*JJ)+C14*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +         GMSG*R*T**2*II)
      SE(7,13)=C13*(-T*L2*JJ)+C14*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(7,15)=C13*R2*CC+C14*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(8,8)=C15*R*L1*HH+C16*R*L1*BB
      SE(8,9)=0.0
      SE(8,11)=0.0
      SE(8,13)=0.0
      SE(8,15)=0.0
      SE(8,10)=C15*(DD/T**2+GMSS*R**2*AA)+C16*(-AA)
      SE(8,12)=C15*R*II+C16*(-R*AA)
      SE(8,14)=C15*(L2*FF/T**2+GMSS*R**2*L2*CC)+C16*(-L2*CC)
      SE(8,16)=C15*R*L2*JJ+C16*R*L2*CC
C
```

```
      SE(9,9)=C17*T*(-II)+C18*(-DD-GMSS*R**2*T**2*AA)
      SE(9,10)=0.0
      SE(9,12)=0.0
      SE(9,14)=0.0
      SE(9,16)=0.0
      SE(9,11)=C17*(-AA+T**2*JJ)+C18*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +         GMSG*R*T**2*II)
      SE(9,13)=C17*(-T*L2*JJ)+C18*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(9,15)=C17*R2*CC+C18*(T*R2*CC+GMSG*R*T**2*JJ)
C
      SE(10,10)=C19*(DD/T**2+GMSS*R**2*AA)+C20*(-AA)
      SE(10,11)=0.0
      SE(10,13)=0.0
      SE(10,15)=0.0
      SE(10,12)=C19*R*II+C20*(-R*AA)
      SE(10,14)=C19*(L2*FF/T**2+GMSS*R**2*L2*CC)+C20*(-L2*CC)
      SE(10,16)=C19*R*L2*JJ+C20*R*L2*CC
C
      SE(11,11)=C21*(-AA+T**2*JJ)+C22*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +          GMSG*R*T**2*II)
      SE(11,12)=0.0
      SE(11,14)=0.0
      SE(11,16)=0.0
      SE(11,13)=C21*(-T*L2*JJ)+C22*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(11,15)=C21*R2*CC+C22*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(12,12)=C23*R*II+C24*(-R*AA)
      SE(12,13)=0.0
      SE(12,15)=0.0
      SE(12,14)=C23*(L2*FF/T**2+GMSS*R**2*L2*CC)+C24*(-L2*CC)
      SE(12,16)=C23*R*L2*JJ+C24*R*L2*CC
C
      SE(13,13)=C25*(-T*L2*JJ)+C26*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(13,14)=0.0
      SE(13,16)=0.0
      SE(13,15)=C25*R2*CC+C26*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(14,14)=C27*(L2*FF/T**2+GMSS*R**2*L2*CC)+C28*(-L2*CC)
      SE(14,15)=0.0
      SE(14,16)=C27*R*L2*JJ+C28*R*L2*CC
C
      SE(15,15)=C29*R2*CC+C30*(T*R2*CC+GMSG*R*T**2*R2*JJ)
      SE(15,16)=0.0
C
      SE(16,16)=C31*R*L2*JJ+C32*R*L2*CC
C
    DO 380 IE=1,16
    DO 380 JE=IE,16
380 K(IE,JE)=K(IE,JE)+WEIGHT(J)*SE(IE,JE)
    IF (I.EQ.1) GO TO 300
    GO TO 500
C
```

```
390   CONTINUE
C
C........ENTRIES OF CURVED ELEMENT NONLINEAR STIFFNESS SE1
C
C     A01,A02,A3,A4,A5,A6, ARE IN CLOSED FORM SOLUTIONS
C
C
      IF (IARCH.EQ.1) GO TO 105
      BTG = 2.*T*B2(M)
      BLG = LOG((B1(M)+BTG)/B1(M))
      A01 = BLG/(T*BTG)
      A02 = 4.*(1./BTG - B1(M)*BLG/(BTG*BTG))/T
      A3  = (1.-2.*B1(M)/BTG+2.*B1(M)*B1(M)*BLG/(BTG*BTG))/(T*T*B2(M))
      A4  = 1.5*A3
      A5  = 2.*(1.-1.5*B1(M)/BTG+3.*B1(M)*B1(M)/(BTG*BTG)
     +          -3.*B1(M)**3*BLG/(BTG**3))/(T*T*B2(M))
      A6  = 1.125*(1.-4.*B1(M)/(3.*BTG)+2.*B1(M)*B1(M)/(BTG*BTG)
     +             -4.*B1(M)**3/(BTG**3)+4.*B1(M)**4*BLG/(BTG**4))
     +            /(T*T*B2(M))
      GO TO 106
105   CONTINUE
      A01=1./(T*(B1(M)))
      A02=2./(T*(B1(M)))
      A3=4./(3.*T*(B1(M)))
      A4=1.5*A3
      A5=3./(T*(B1(M)))
      A6=9./(5.*T*(B1(M)))
106   CONTINUE
C
C
C........LAMDA0 TO LAMDA12 ARE WRITTEN AS XLD0 TO XLD12
C
      XLD0=-0.5*T*DN(1)-(1.-T*T/12.)*DN(3)-R1*T*T*DN(5)/12.-0.5*T*DN(7)
     +     +(1.-T*T/12.)*DN(9)+R2*T*T*DN(11)/12.
      XLD1=DN(1)
      XLD2=-T*DN(3)+R1*T*DN(5)
      XLD3=-3.*DN(1)+2.*T*DN(3)-2.*R1*T*DN(5)+3.*DN(7)+T*DN(9)-
     +       R2*T*DN(11)
      XLD4=2.*DN(1)-T*DN(3)+R1*T*DN(5)-2.*DN(7)-T*DN(9)+R2*T*DN(11)
      XLD5=DN(2)
      XLD6=-R1*T*DN(6)
      XLD7=-3.*DN(2)+2.*R1*T*DN(6)+3.*DN(8)+R2*T*DN(12)
      XLD8=2.*DN(2)-R1*T*DN(6)-2.*DN(8)-R2*T*DN(12)
      XLD9=DN(3)
      XLD10=DN(9)-DN(3)
      XLD13=DN(4)
      XLD14=DN(10)-DN(4)
C
      C1=E(N)*A(M)/(2.*LENGTH(M))
      C2=3.*A02-2.*A4
      C3=6.*A3-2.*A5
      C4=3.*A5-4.*A6
```

```
      C5=3.*(A02-A3)
      C6=3.*A3-A5
      C7=3.*A3-2.*A4+3.*A5-4.*A6
C
      IF(IPAR.EQ.4) GO TO 410
C
      SE(1,1)=C1*((18.*A3-12.*A5+8.*A6)*XLD0+T*(C2*XLD2+C3*XLD3+C4*XLD4
     +             )+T**2*(C5*XLD9+C6*XLD10))
      SE(1,2)=C1*0.5*T*(C2*XLD6+C3*XLD7+C4*XLD8)
      SE(1,3)=C1*(T*(-6.*A3-2.*A4+6.*A5-4.*A6)*XLD0+
     +            (1-T**2/12.)*(C2*XLD2+C3*XLD3+C4*XLD4+T*(C5*XLD9+
     +                          C6*XLD10))-
     +            0.25*T**2*(C2*XLD2+C3*XLD3+(3.*A3-2.*A4+3.*A5-4.*A6)*
     +                        XLD4+T*(C5*XLD9+C6*XLD10)))
      SE(1,4)=0.0
      SE(1,5)=C1*R1*T*((-3.*A02+12.*A3+2.*A4-7.*A5+4.*A6)*XLD0+
     +                 T/12.*((-12.*A01+15.*A02-8.*A4)*XLD2+(-6.*A02+
     +                 30.*A3-8.*A5)*XLD3+(-6.*A4+15.*A5-16.*A6)*XLD4+
     +                 T*((-12.*A01+15.*A02-12.*A3)*XLD9+
     +                 (-3.*A02+15.*A3-4.*A5)*XLD10)))
      SE(1,6)=C1*0.5*R1*T**2*((2.*A01-2.*A02+A4)*XLD6+(A02-4.*A3+A5)*
     +                 XLD7+(A4-2.*A5+2.*A6)*XLD8)
      SE(1,7)=C1*(-18.*A3+12.*A5-8.*A6)*XLD0
      SE(1,8)=-SE(1,2)
      SE(1,9)=C1*(T*(-9.*A3+6.*A5-4.*A6)*XLD0 - (1.-T**2/12.)*
     +           (C2*XLD2+C3*XLD3+C4*XLD4+T*(C5*XLD9+C6*XLD10))
     +          -0.25*T**2*(C2*XLD2+C3*XLD3+C4*XLD4+
     +                      T*(C5*XLD9+0.5*C3*XLD10)))
      SE(1,10)=0.0
      SE(1,11)=C1*R2*T*((6.*A3-5.*A5+4.*A6)*XLD0+T/12.*((3.*A02-4.*A4)
     +                  *XLD2+(6.*A3-4.*A5)*XLD3+(3.*A5-8.*A6)*XLD4 +
     +                  T*((3.*A02-6.*A3)*XLD9+(3.*A3-2.*A5)*XLD10)))
      SE(1,12)=C1*0.5*R2*T**2*((A4-A02)*XLD6+(A5-2.*A3)*XLD7+
     +                          (2.*A6-A5)*XLD8)
C
      SE(2,2)=C1*(18.*A3-12.*A5+8.*A6)*XLD0
      SE(2,3)=C1*(1.-T**2/12.)*(C2*XLD6+C3*XLD7+C4*XLD8)
      SE(2,4)=0.0
      SE(2,5)=C1*R1*T**2/12.*(C2*XLD6+C3*XLD7+C4*XLD8)
      SE(2,6)=C1*R1*T*(3.*A02-12.*A3-2.*A4+7.*A5-4.*A6)*XLD0
      SE(2,7)=C1*T*0.5*(C2*XLD6+C3*XLD7+C4*XLD8)
      SE(2,8)=-SE(2,2)
      SE(2,9)=-SE(2,3)
      SE(2,10)=0.0
      SE(2,11)=C1*R2*T**2/12.*(-C2*XLD6-C3*XLD7-C4*XLD8)
      SE(2,12)=C1*R2*T*(-6.*A3+5.*A5-4.*A6)*XLD0
C
      SE(3,3)=C1*(T**2*(1.5*A3+2.*A4-3.*A5+2*A6)*XLD0
     +            -(1.-T**2/12.)*T*(C2*XLD2+C3*XLD3+C7*XLD4+
     +              T*(C5*XLD9+C6*XLD10)))
      SE(3,4)=0.0
      SE(3,5)=C1*R1*T*(T*(1.5*A02-4.5*A3-2.*A4+3.5*A5-2.*A6)*XLD0
```

```
+         -T**2/24.*(C2*XLD2+C3*XLD3+C7*XLD4+T*(C5*XLD9+C6*XLD10))
+       -(1.-T**2/12.)*((2.*A01-2.*A02+A4)*XLD2+(A02-4.*A3+A5)*XLD3
+         +(A4-2.*A5+2.*A6)*XLD4
+   +T*((2.*A01-2.*A02+1.5*A3)*XLD9+(0.5*A02-2.*A3+0.5*A5)*XLD10)))
  SE(3,6)=C1*R1*T*(1.-T**2/12.)*((2.*A01-2.*A02+A4)*XLD6+
+        (A02-4.*A3+A5)*XLD7+(A4-2.*A5+2.*A6)*XLD8)
  SE(3,7)=C1*(T*(6.*A3+2.*A4-6.*A5+4.*A6)*XLD0 - T**2/4.*
+             (C2*XLD2+C3*XLD3+C7*XLD4+T*(C5*XLD9+C6*XLD10))
+             -(1.-T**2/12.)*(C2*XLD2+C3*XLD3+C4*XLD4+
+                            T*(C5*XLD9+C6*XLD10)))
  SE(3,8)=C1*(1.-T**2/12.)*(-C2*XLD6-C3*XLD7-C4*XLD8)
  SE(3,9)=C1*T*(T*(3.*A3+A4-3.*A5+2.*A6)*XLD0 +
+               (1.-T**2/12.)*0.5*(3.*A3-2.*A4)*XLD4)
  SE(3,10)=0.0
  SE(3,11)=C1*R2*T*(T*(-1.5*A3-A4+2.5*A5-2.*A6)*XLD0 +
+          T**2/24.*(C2*XLD2+C3*XLD3+C7*XLD4+T*(C5*XLD9+C6*XLD10))
+         +(1.-T**2/12.)*((A02-A4)*XLD2+(2.*A3-A5)*XLD3+
+        (A5-2.*A6)*XLD4+T*((A02-1.5*A3)*XLD9+(A3-0.5*A5)*XLD10)))
  SE(3,12)=C1*R2*T*(1.-T**2/12.)*((A4-A02)*XLD6+(A5-2.*A3)*XLD7+
+                                (2.*A6-A5)*XLD8)
C
  SE(4,4)=0.0
  SE(4,5)=0.0
  SE(4,6)=0.0
  SE(4,7)=0.0
  SE(4,8)=0.0
  SE(4,9)=0.0
  SE(4,10)=0.0
  SE(4,11)=0.0
  SE(4,12)=0.0
C
  C16=2.*A01-2.*A02+A4
  C17=A02-4.*A3+A5
  C18=A4-2.*A5+2.*A6
  C19=2.*A01-2.*A02+1.5*A3
  C20=0.5*A02-2.*A3+0.5*A5
  C21=0.5*A3-2./3.*A5+2./3.*A6
  C22=A5/6.-8./9.*A6+3.*A7
C
  SE(5,5)=C1*((R1*T)**2)*((2.*A01-4.*A02+8.*A3+2.*A4-4.*A5+2.*A6)*
+         XLD0-T/6.*(C16*XLD2+C17*XLD3+C18*XLD4+T*(C19*XLD9+
+         C20*XLD10)))
  SE(5,6)=C1*R1**2*T**3/12.*(C16*XLD6+C17*XLD7+C18*XLD8)
  SE(5,7)=C1*R1*T*((3*A02-12.*A3-2.*A4+7*A5-4.*A6)*XLD0-T/12.*
+        ((12.*A01-9.*A02+4.*A4)*XLD2+(6.*A02-18.*A3+4.*A5)*XLD3+
+       (6.*A4-9.*A5+8.*A6)*XLD4+T*((12.*A01-9.*A02+6.*A3)*XLD9+
+       (3.*A02-9.*A3+2.*A5)*XLD10)))
  SE(5,8)=-SE(2,5)
  SE(5,9)=C1*R1*T*(T*(1.5*A02-6.*A3-A4+3.5*A5-2.*A6)*XLD0
+        +(1.-T**2/12.)*(C16*XLD2+C17*XLD3+C18*XLD4+T*(C19*XLD9+
+        C20*XLD10))-T**2/24.*(C2*XLD2+C3*XLD3+C4*XLD4+
+        T*(C5*XLD9+C6*XLD10)))
```

```
      SE(5,10)=0.0
      SE(5,11)=C1*R1*R2*T**2*((-A02+4.*A3+A4-3.*A5+2.*A6)*XLD0+T/12.*
     +           ((2.*A01-A02)*XLD2+(A02-2.*A3)*XLD3+(A4-A5)*XLD4 +
     +           T*((2.*A01-A02)*XLD9+(0.5*A02-A3)*XLD10)))
      SE(5,12)=C1*R1*R2*T**3/12.*((A4-A02)*XLD6+(A5-2.*A3)*XLD7+
     +                       (2.*A6-A5)*XLD8)
C
      SE(6,6)=C1*R1**2*T**2*(2.*A01-4.*A02+8.*A3+2.*A4-4.*A5+2.*A6)*XLD0
      SE(6,7)=C1*R1*T**2/2.*(C16*XLD6+C17*XLD7+C18*XLD8)
      SE(6,8)=-SE(2,6)
      SE(6,9)=-SE(1,6)*(1.-T**2/12.)*2./T
      SE(6,10)=0.0
      SE(6,11)=-SE(1,6)*R2*T/6.
      SE(6,12)=C1*R1*R2*T**2*(-A02+4.*A3+A4-3.*A5+2.*A6)*XLD0
C
      SE(7,7)=C1*((18.*A3-12.*A5+8.*A6)*XLD0-T*(C2*XLD2+C3*XLD3+
     +        C4*XLD4+T*(C5*XLD9+C6*XLD10+C7/3.*XLD11+C8/3.*XLD12)))
      SE(7,8)=-SE(2,7)
      SE(7,9)=C1*(T*(9.*A3-6.*A5+4.*A6)*XLD0+(1.-T**2/12.)
     +         *(C2*XLD2+C3*XLD3+C4*XLD4+T*(C5*XLD9+C6*XLD10))
     +        -T**2/4.*(C2*XLD2+C3*XLD3+C4*XLD4+T*(C5*XLD9+C6*XLD10)))
      SE(7,10)=0.0
      SE(7,11)=C1*R2*T*((-6.*A3+5.*A5-4.*A6)*XLD0+T/12.*((9.*A02-8.*A4)
     +                *XLD2+(18.*A3-8.*A5)*XLD3+(9.*A5-16.*A6)*XLD4+
     +                T*((9.*A02-12.*A3)*XLD9+(9.*A3-4.*A5)*XLD10)))
      SE(7,12)=C1*R2*T**2/2.*((A4-A02)*XLD6+(A5-2.*A3)*XLD7+
     +                     (2.*A6-A5)*XLD8)
C
      SE(8,8)=SE(2,2)
      SE(8,9)=-SE(2,9)
      SE(8,10)=0.0
      SE(8,11)=-SE(2,11)
      SE(8,12)=-SE(2,12)
C
      SE(9,9)=C1*T*(T*(4.5*A3-3.*A5+2.*A6)*XLD0
     +            +(1.-T**2/12.)*(C2*XLD2+C3*XLD3+C4*XLD4+
     +             T*(C5*XLD9+C6*XLD10)))
      SE(9,10)=0.0
      SE(9,11)=C1*R2*T*(T*(-3.*A3+2.5*A5-2.*A6)*XLD0 + T**2/24.*
     +           (C2*XLD2+C3*XLD3+C4*XLD4+T*(C5*XLD9+C6*XLD10)) +
     +           (1.-T**2/12.)*((A4-A02)*XLD2+
     +           (A5-2.*A3)*XLD3+(2.*A6-A5)*XLD4+T*((1.5*A3-A02)*XLD9
     +           +(0.5*A5-A3)*XLD10)))
      SE(9,12)=C1*R2*T*(1.-T**2/12.)*((A02-A4)*XLD6+(2.*A3-A5)*XLD7+
     +                               (A5-2.*A6)*XLD8)
C
      SE(10,10)=0.0
      SE(10,11)=0.0
      SE(10,12)=0.0
C
      SE(11,11)=C1*R2**2*T**2*((2.*A3-2.*A5+2.*A6)*XLD0+T/6.*
     +         ((A4-A02)*XLD2+(A5-2.*A3)*XLD3+(2.*A6-A5)*XLD4+
```

```
     +          T*((1.5*A3-A02)*XLD9+(0.5*A5-A3)*XLD10)))
      SE(11,12)=-SE(1,12)*R2*T/6.
C
      SE(12,12)=C1*R2**2*T**2*(2.*A3-2.*A5+2.*A6)*XLD0
C
      DO 400 IE=1,12
      DO 400 JE=IE,12
400   K(IE,JE)=SE(IE,JE)
C
      GO TO 500
C
410   CONTINUE
C
C........INTEGRANDS ON CURVED ELEMENT NONLINEAR STIFFNESS SE2
C
      AS=A01*XLD2**2+A02*XLD2*XLD3+A3*XLD3**2+A4*XLD2*XLD4
     +                             +A5*XLD3*XLD4+A6*XLD4**2
      BS=T*(2.*A01*XLD2*XLD9+A02*XLD3*XLD9+0.5*A02*XLD2*XLD10+
     +      1.5*A3*XLD4*XLD9+A3*XLD3*XLD10+0.5*A5*XLD4*XLD10)
      CS=T**2*(A01*XLD9**2+0.5*A02*XLD9*XLD10+0.25*A3*XLD10**2)
      DS=A01*XLD6**2+A02*XLD6*XLD7+A3*XLD7**2+A4*XLD6*XLD8
     +                             +A5*XLD7*XLD8+A6*XLD8**2
      MS=AS+BS+CS+DS
C
      DQ1=(-C2*XLD2-C3*XLD3-C4*XLD4-T*(C5*XLD9+C6*XLD10))
      DQ2=(-C2*XLD6-C3*XLD7-C4*XLD8)
      DQ3=0.5*T*(C2*XLD2+C3*XLD3+C7*XLD4+T*(C5*XLD9+C6*XLD10))
      DQ4=0.0
      DQ5=R1*T*((2.*A01-2.*A02+A4)*XLD2+(A02-4.*A3+A5)*XLD3+
     +          (A4-2.*A5+2.*A6)*XLD4+T*((2.*A01-2.*A02+1.5*A3)*XLD9+
     +          (0.5*A02-2.*A3+0.5*A5)*XLD10))
      DQ6=R1*T*((-2.*A01+2.*A02-A4)*XLD6+(-A02+4.*A3-A5)*XLD7+
     +          (-A4+2.*A5-2.*A6)*XLD8)
      DQ7=-DQ1
      DQ8=-DQ2
      DQ9=0.5*T*(C2*XLD2+C3*XLD3+C4*XLD4+T*(C5*XLD9+C6*XLD10))
      DQ10=0.0
      DQ11=R2*T*((A4-A02)*XLD2+(A5-2.*A3)*XLD3+(2.*A6-A5)*XLD4+
     +          T*((1.5*A3-A02)*XLD9+(0.5*A5-A3)*XLD10))
      DQ12=R2*T*((A02-A4)*XLD6+(2.*A3-A5)*XLD7+(A5-2.*A6)*XLD8)
C
      C01=C1*T*(B1(M)+B2(M)*T)/(2.*LENGTH(M))
C
      SE(1,1)=C01*(MS*(18.*A3-12.*A5+8.*A6)+DQ1**2)
      SE(1,2)=C01*DQ1*DQ2
      SE(1,3)=C01*(T*MS*(-6.*A3-2.*A4+6.*A5-4.*A6)+DQ1*DQ3)
      SE(1,4)=0.0
      SE(1,5)=C01*(R1*T*MS*(-3.*A02+12.*A3+2.*A4-7.*A5+4.*A6)+DQ1*DQ5)
      SE(1,6)=C01*DQ1*DQ6
      SE(1,7)=-SE(1,1)
      SE(1,8)=-SE(1,2)
      SE(1,9)=C01*(MS*T*(-9.*A3+6.*A5-4.*A6)+DQ1*DQ9)
```

```
      SE(1,10)=0.0
      SE(1,11)=C01*(R2*T*MS*(6.*A3-5.*A5+4.*A6)+DQ1*DQ11)
      SE(1,12)=C01*DQ1*DQ12
C
      SE(2,2)=C01*(MS*(18.*A3-12.*A5+8.*A6)+DQ2**2)
      SE(2,3)=C01*DQ2*DQ3
      SE(2,4)=0.0
      SE(2,5)=C01*DQ2*DQ5
      SE(2,6)=C01*(R1*T*MS*(3.*A02-12.*A3-2.*A4+7.*A5-4.*A6)+DQ2*DQ6)
      SE(2,7)=-SE(1,2)
      SE(2,8)=-SE(2,2)
      SE(2,9)=C01*DQ2*DQ9
      SE(2,10)=0.0
      SE(2,11)=C01*DQ2*DQ11
      SE(2,12)=C01*(R2*T*MS*(-6.*A3+5.*A5-4.*A6)+DQ2*DQ12)
C
      SE(3,3)=C01*(T*T*MS*(1.5*A3+2.*A4-3.*A5+2.*A6)+DQ3**2)
      SE(3,4)=0.0
      SE(3,5)=C01*(R1*T*T*0.5*MS*(3.*A02-9.*A3-4.*A4+7.*A5-4.*A6)+
     +              DQ3*DQ5)
      SE(3,6)=C01*DQ3*DQ6
      SE(3,7)=-SE(1,3)
      SE(3,8)=-SE(2,3)
      SE(3,9)=C01*(T*T*MS*(3.*A3+A4-3.*A5+2.*A6)+DQ3*DQ9)
      SE(3,10)=0.0
      SE(3,11)=C01*(R2*T*T*MS*(-1.5*A3-A4+2.5*A5-2.*A6)+DQ3*DQ11)
      SE(3,12)=C01*DQ3*DQ12
C
      SE(4,4)=0.0
      SE(4,5)=0.0
      SE(4,6)=0.0
      SE(4,7)=0.0
      SE(4,8)=0.0
      SE(4,9)=0.0
      SE(4,10)=0.0
      SE(4,11)=0.0
      SE(4,12)=0.0
C
      SE(5,5)=C01*((R1*T)**2*MS*(2.*A01-4.*A02+8.*A3+2.*A4-4.*A5+2.*A6)
     +              +DQ5**2)
      SE(5,6)=C01*DQ5*DQ6
      SE(5,7)=C01*(R1*T*MS*(3.*A02-12.*A3-2.*A4+7.*A5-4.*A6)+DQ5*DQ7)
      SE(5,8)=C01*DQ5*DQ8
      SE(5,9)=C01*(R1*T*T*MS*(1.5*A02-6.*A3-A4+3.5*A5-2.*A6)+DQ5*DQ9)
      SE(5,10)=0.0
      SE(5,11)=C01*(R1*R2*T*T*MS*(-A02+4.*A3+A4-3.*A5+2.*A6)+DQ5*DQ11)
      SE(5,12)=C01*DQ5*DQ12
C
      SE(6,6)=C01*((R1*T)**2*MS*(2.*A01-4.*A02+8.*A3+2.*A4-4.*A5+2.*A6)
     +              +DQ6**2)
      SE(6,7)=C01*DQ6*DQ7
      SE(6,8)=-SE(2,6)
```

```
      SE(6,9)=C01*DQ6*DQ9
      SE(6,10)=0.0
      SE(6,11)=C01*DQ6*DQ11
      SE(6,12)=C01*(R1*R2*T*T*MS*(-A02+4.*A3+A4-3.*A5+2.*A6)+DQ6*DQ12)
C
      SE(7,7)=SE(1,1)
      SE(7,8)=SE(1,2)
      SE(7,9)=-SE(1,9)
      SE(7,10)=0.0
      SE(7,11)=-SE(1,11)
      SE(7,12)=-SE(1,12)
C
      SE(8,8)=SE(2,2)
      SE(8,9)=-SE(2,9)
      SE(8,10)=0.0
      SE(8,11)=-SE(2,11)
      SE(8,12)=-SE(2,12)
C
      SE(9,9)=C01*(T*T*MS*(4.5*A3-3.*A5+2.*A6)+DQ9**2)
      SE(9,10)=0.0
      SE(9,11)=C01*(R2*T*T*MS*(-3.*A3+2.5*A5-2.*A6)+DQ9*DQ11)
      SE(9,12)=C01*DQ9*DQ12
C
      SE(10,10)=0.0
      SE(10,11)=0.0
      SE(10,12)=0.0
C
      SE(11,11)=C01*((R2*T)**2*MS*(2.*A3-2.*A5+2.*A6)+DQ11**2)
      SE(11,12)=C01*DQ11*DQ12
C
      SE(12,12)=C01*((R2*T)**2*MS*(2.*A3-2.*A5+2.*A6)+DQ12**2)
C
      DO 420 IE=1,12
      DO 420 JE=IE,12
420   K(IE,JE)=SE(IE,JE)
C
500   CONTINUE
C
C........MAKE INTERVAL CORRECTION AND RETURN
CCCC
      IF (IPAR.GE.3) GO TO 1
      IQ=16
      GO TO 2
1     IQ=12
      C=1.0
2     CONTINUE
CCCC
      DO 550 I=1,IQ
      DO 550 J=I,IQ
      SE(I,J)=C*K(I,J)
550   SE(J,I)=SE(I,J)
CCCC
```

```
C
      IF (IPAR.EQ.1) CALL REOCON
      IF (IPAR.EQ.2) CALL REOCON
C
CCCC
      RETURN
C
2000  FORMAT('1',15HINVALID MP USED///7H GAUSS=,F4.1)
C
      END
C
```

```
      SUBROUTINE ASEMBLE (M)
C     ******************************************************************
C        TO PROCESS AND ASSEMBLE ELEMENT STIFFNESS MATRICES AND NODAL
C        LOAD VECTORS INTO THEIR CORRESPONDING STRUCTURE ARRAYS.
C     ******************************************************************
C
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,ICAL3,
     +         ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C3/IA(37,8),IB(37,8),D31(113)
      COMMON/C4/SE(16,16)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),D5(180)
      COMMON/C8/PN(37,8),R(296),PINT(37,8)
      COMMON/C9/S(296,16),SP(296,16),IDET
C
      IF (IPAR.NE.1) GO TO 90
C
C........PROCESSING OF INITIAL LOADS AND NODAL LOADS INTO LOAD VECTOR
C
      IF(ICAL4.EQ.0) WRITE(61,2000)
      DO 80 N=1,NUMNP
      DO 70 I=1,6
      IF(IA(N,I)) 20,70,10
10    II=IA(N,I)
      GO TO 60
20    IF (IB(N,I).LT.0) GO TO 30
      NN=IB(N,I)
      GO TO 35
30    II=-IB(N,I)+NEQ
      GO TO 60
35    IF (IA(NN,I)) 40,70,50
40    II=-IB(NN,I)+NEQ
      GO TO 60
50    II=IA(NN,I)
60    R(II)=PN(N,I)
      IF(ICAL4.EQ.0) WRITE(61,2010) II,N,I,R(II)
70    CONTINUE
80    CONTINUE
      RETURN
C
C........ASSEMBLE ELEMENT STIFFNESS INTO STRUCTURE STIFFNESS
C
90    NI=NODEI(M)
      NJ=NODEJ(M)
      DO 165 K1=1,2
      IF(K1.EQ.1) NP=NI
      IF(K1.EQ.2) NP=NJ
      DO 160 I=1,6
      IF(IA(NP,I)) 105,160,100
100   II=IA(NP,I)
      GO TO 115
105   IF (IB(NP,I).LT.0) GO TO 110
```

```
      NN=IB(NP,I)
      GO TO 111
110   II=-IB(NP,I)+NEQ
      GO TO 115
111   IF (IA(NN,I)) 112,160,113
112   II=-IB(NN,I)+NEQ
      GO TO 115
113   II=IA(NN,I)
115   CONTINUE
      DO 155 K2=1,2
      IF(K2.EQ.1) ND=NI
      IF(K2.EQ.2) ND=NJ
      DO 150 J=1,6
      IF(IA(ND,J)) 125,150,120
120   JJ=IA(ND,J)
      GO TO 145
125   IF (IB(ND,J).LT.0) GO TO 130
      NN=IB(ND,J)
      GO TO 132
130   JJ=-IB(ND,J)+NEQ
      GO TO 145
132   IF (IA(NN,J)) 135,150,140
135   JJ=-IB(NN,J)+NEQ
      GO TO 145
140   JJ=IA(NN,I)
145   CONTINUE
C
C........FILL-IN STRUCTURE STIFFNESS MATRIX IN BANDED FORMAT
C        ONLY UPPER SEMIBANDWIDTH INCLUDING DIAGONAL
C
      IF (JJ.LT.II) GO TO 150
      IF(K1.EQ.1) IE=I
      IF(K1.EQ.2) IE=I+6
      IF(K2.EQ.1) JE=J
      IF(K2.EQ.2) JE=J+6
C
C........CHANGE -JJ- SUBSCRIPT OF FULL MATRIX TO -JJ- SUBSCRIPT
C        OF BANDED FORMAT. LOOP OVER TERMS OUTSIDE OF BAND
C
      JJ=JJ-II+1
      S(II,JJ)=S(II,JJ)+SE(IE,JE)
150   CONTINUE
155   CONTINUE
160   CONTINUE
165   CONTINUE
      RETURN
C
2000  FORMAT('1',43HINITIAL AND NODAL LOADS PROCESSED INTO LOAD,
     +       12H VECTOR R(I)//)
2010  FORMAT('0',2HR(,I3,4H)=P(,I2,1H,,I2,2H)=,F16.6)
C
      END
```

```
      SUBROUTINE STCONDN
C     ******************************************************************
C        TO WRITE THE UNCONDENSED STRUCTURE STIFFNESS ACCORDING
C        TO THE VALUES OF IPAR=2,3,4 FOR S, S1, S2 RESPECTIVELY
C     ******************************************************************
C
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,ICAL3,
     +         ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C8/PN(37,8),R(296),PINT(37,8)
      COMMON/C9/S(296,16),SP(296,16),IDET
C
      IF (IPAR.NE.2) GO TO 40
      IF (ICAL3.EQ.0) WRITE(61,2050)
      IF (ICAL3.EQ.0) WRITE(61,2060) (I,R(I),I=1,NSIZE)
C
C........WRITE UNCONDENSED STRUCTURE LINEAR STIFFNESS -S- OR UNCONDENSED
C        NONLINEAR STIFFNESS S1 OR S2 DEPENDING ON VALUE OF IPAR
C
40    IF (ICAL3.NE.0) GO TO 90
      IF(IPAR.EQ.2) WRITE(61,2030)
      IF (IPAR.EQ.3) WRITE(61,2040)
      IF (IPAR.EQ.4) WRITE(61,2045)
      K1=1
      K2=8
      K3=MBAND-K1
      IF (K3.LE.7) GO TO 60
50    WRITE(61,2015) K1,K2
      WRITE(61,2020) ((S(I,J),J=K1,K2),I=1,NSIZE)
      K1=K1+8
      K2=K2+8
      K3=MBAND-K1
      IF(K3.LE.7) GO TO 60
      GO TO 50
60    WRITE(61,2015) K1,MBAND
      IF (K3.EQ.0) WRITE(61,2027) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.1) WRITE(61,2021) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.2) WRITE(61,2022) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.3) WRITE(61,2023) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.4) WRITE(61,2024) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.5) WRITE(61,2025) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.6) WRITE(61,2026) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
      IF (K3.EQ.7) WRITE(61,2020) ((S(I,J),J=K1,MBAND),I=1,NSIZE)
90    CONTINUE
      REWIND 4
      REWIND 5
      REWIND 16
      RETURN
C
2015  FORMAT('-',7HCOLUMNS,I4,7HTHROUGH,I4)
2020  FORMAT('0',8E16.5)
2021  FORMAT('0',2E16.5)
```

```
2022  FORMAT('0',3E16.5)
2023  FORMAT('0',4E16.5)
2024  FORMAT('0',5E16.5)
2025  FORMAT('0',6E16.5)
2026  FORMAT('0',7E16.5)
2027  FORMAT('0',E16.5)
2030  FORMAT('1',45HUNCONDENSED LINEAR STIFFNESS OF STRUCTURE (S))
2040  FORMAT('1',49HUNCONDENSED NONLINEAR STIFFNESS OF STRUCTURE (S1))
2045  FORMAT('1',49HUNCONDENSED NONLINEAR STIFFNESS OF STRUCTURE (S2))
2050  FORMAT('1',28HUNCONDENSED LOAD VECTOR R(I)//)
2060  FORMAT(' ',2HR(,I3,2H)=,F16.6)
C
      END
C
      SUBROUTINE LINSOLN
C     ******************************************************************
C        TO SOLVE SYSTEM OF LINEAR EQUATIONS S*D=R BY CALLING THE
C        APPROPRIATE SUBROUTINE
C            S= STUCTURE"S LINEAR STIFFNESS
C            D= VECTOR OF D.O.F."S
C            R= LOAD VECTOR
C     ******************************************************************
C
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,ICAL3,
     +         ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C8/PN(37,8),R(296),PINT(37,8)
      COMMON/C9/S(296,16),SP(296,16),IDET
      COMMON/C10/D(296),D10(1184),RC(296),SC(296,16)
C
C........FILL-IN ARRAY D(I) WITH VALUES OF LOAD VECTOR R(I)
C        AFTER SOLUTION D(I) WILL CONTAIN THE DISPLACEMENT VALUES
C
      DO 110 I=1,NEQ
110   D(I)=R(I)
C
C........CHECK DATA GENERATION FOR SOLUTION OF EQUATIONS
C
      IF(ICAL5.EQ.0) WRITE(61,2020)
      IF (ICAL5.EQ.0) WRITE(61,2010) (I,D(I),I=1,NEQ)
C
C........SOLVE SYSTEM OF -NEQ- LINEAR EQUATIONS
C
      CALL GAUSSOL
C
C........CHECK DATA GENERATION
C
      IF (ICAL5.NE.0) GO TO 140
      WRITE(61,2000)
      WRITE(61,2010) (I,D(I),I=1,NEQ)
140   RETURN
C
```

```
2000  FORMAT('1',34HDISPLACEMENTS FROM LINEAR SOLUTION//)
2010  FORMAT(' ',2HD(,I3,2H)=,E25.15)
2020  FORMAT('1',31HLOAD VECTOR FOR LINEAR SOLUTION//)
C
      END
      SUBROUTINE GAUSSOL
C     ******************************************************************
C        GAUSS ELIMINATION EQUATION SOLVER, BANDED FORMAT
C        FROM BOOK BY ROBERT D. COOK, FIG. 2.8.1., PAGE 45
C        CONCEPTS AND APPLICATIONS OF FINITE ELEMENT ANALYSIS
C     ******************************************************************
C
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C9/S(296,16),SP(296,16),IDET
      COMMON/C10/D(296),D10(1184),RC(296),SC(296,16)
C
C........FORWARD REDUCTION OF MATRIX (GAUSS ELIMINATION)
C
      DO 790 N=1,NEQ
      DO 780 L=2,MBAND
      IF (S(N,L).EQ.0.) GO TO 780
      I=N+L-1
      C=S(N,L)/S(N,1)
      J=0
      DO 750 K=L,MBAND
      J=J+1
750   S(I,J)=S(I,J)-C*S(N,K)
      S(N,L)=C
780   CONTINUE
790   CONTINUE
C
C........FORWARD REDUCTION OF CONSTANTS (GAUSS ELIMINATION)
C
      DO 830 N=1,NEQ
      DO 820 L=2,MBAND
      IF (S(N,L).EQ.0.) GO TO 820
      I=N+L-1
      D(I)=D(I)-S(N,L)*D(N)
820   CONTINUE
830   D(N)=D(N)/S(N,1)
C
C........SOLVE FOR UNKNOWNS BY BACK SUBSTITUTION
C
      DO 860 M=2,NEQ
      N=NEQ+1-M
      DO 850 L=2,MBAND
      IF (S(N,L).EQ.0.)GO TO 850
      K=N+L-1
      D(N)=D(N)-S(N,L)*D(K)
850   CONTINUE
860   CONTINUE
      RETURN
```

```
C
      END
C
      SUBROUTINE IDENT
C     *****************************************************************
C        TO IDENTFY THE DISPLACEMENTS FOUND IN THE SOLUTION OF
C        EQUATIONS S*D=R AND THE ONES FOUND IN THE RECOVERY PROCESS
C     *****************************************************************
C
      COMMON/C1/NL,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,ICAL3,
     +         ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C3/IA(37,8),IB(37,8),D31(113)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
     +         KT(36),L(1,36)
      COMMON/C10/D(296),D10(1184),RC(296),SC(296,16)
      COMMON/C11/DN(16),U(36,12),W(37,8),V(37,8)
C
C........IDENTIFICATION OF DISPLACEMENTS
C
      IF (ICAL6.EQ.0) WRITE(61,2000)
      DO 230 NN=1,NUMEG
      INUMEL=NUMEL(NN)
      DO 230 K=1,INUMEL
      M=L(NN,K)
      IF (ICAL6.EQ.0) WRITE(61,2010) M
      NI=NODEI(M)
      NJ=NODEJ(M)
      DO 230 K1=1,2
      IF(K1.EQ.1) NP=NI
      IF(K1.EQ.2) NP=NJ
      DO 220 I=1,6
      IF(IA(NP,I)) 160,155,150
150   NE=IA(NP,I)
      U(NP,I)=D(NE)
      IF (ICAL6.EQ.0) WRITE(61,2020) NE,NP,I,U(NP,I)
      GO TO 220
155   U(NP,I)=0.
      IF (ICAL6.EQ.0) WRITE(61,2020) NE,NP,I,U(NP,I)
      GO TO 220
160   IF (IB(NP,I).LT.0) GO TO 170
      NM=IB(NP,I)
      GO TO 180
170   NE=-IB(NP,I)+NEQ
      U(NP,I)=D(NE)
      IF (ICAL6.EQ.0) WRITE(61,2020) NE,NP,I,U(NP,I)
      GO TO 220
180   IF (IA(NM,I)) 190,200,210
190   NE=-IB(NM,I)+NEQ
      U(NP,I)=D(NE)
      IF (ICAL6.EQ.0) WRITE(61,2020) NE,NP,I,U(NP,I)
      GO TO 220
```

```
 200    U(NP,I)=0.
        IF (ICAL6.EQ.0) WRITE(61,2020) NE,NP,I,U(NP,I)
        GO TO 220
 210    NE=IA(NM,I)
        U(NP,I)=D(NE)
        IF (ICAL6.EQ.0) WRITE(61,2020) NE,NP,I,U(NP,I)
 220    CONTINUE
 230    CONTINUE
        RETURN
 C
 2000   FORMAT('1',35HNODAL DISPLACEMENTS ON EACH ELEMENT)
 2010   FORMAT('-',7HELEMENT,I3//)
 2020   FORMAT(' ',2HD(,I3,1H),5X,2HU(,I2,1H,,I1,2H)=,E25.15)
 C
        END
 C
        SUBROUTINE STRESS
 C      ****************************************************************
 C         TO COMPUTE NODAL FORCES AND STRESSES IN THE STRUCTURE
 C      ****************************************************************
 C
        DIMENSION D(16)
        COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),D11(8)
        COMMON/C4/SE(16,16)
        COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
       +         KT(36),L(1,36)
        COMMON/C8/PN(37,8),R(296),PINT(37,8)
        COMMON/C11/DN(16),U(36,12),W(37,8),V(37,8)
 C
        WRITE(61,2000)
        DO 100 N=1,NUMNP
        DO 100 I=1,6
 100    PN(N,I)=0.0
 C
 C........PROCESS EVERY ELEMENT OF EACH ELEMENT GROUP
 C
        DO 200 K=1,NUMEG
        INUMEL=NUMEL(K)
        DO 190 KK=1,INUMEL
        M=L(K,KK)
        NI=NODEI(M)
        NJ=NODEJ(M)
        READ(1,10) ((SE(I,J),J=1,12),I=1,12)
 C
 C........IDENTIFY NODAL DISPLACEMENTS ON EACH ELEMENT
 C
        N=NI
        DO 110 I=1,6
 110    D(I)=U(N,I)
        N=NJ
        DO 120 I=7,12
 120    D(I)=U(NI,I-6)
```

```
C
C........OBTAIN RESULTANT LOADS
C
      N=NI
      DO 145 I=1,6
      DN(I)=0.
      DO 140 J=1,12
140   DN(I)=DN(I)+SE(I,J)*D(J)
145   PN(N,I)=PN(N,I)+DN(I)
      N=NJ
      DO 160 I=7,12
      DN(I)=0.
      DO 155 J=1,12
155   DN(I)=DN(I)+SE(I,J)*D(J)
160   PN(N,I-6)=PN(N,I-6)+DN(I)
C
C........WRITE RESULTANT LOADS OF THE NODES OF EACH ELEMENT
C
      WRITE(61,2010) M,(DN(I),I=1,6),(DN(I),I=7,12)
190   CONTINUE
200   CONTINUE
C
C........WRITE NODAL RESULTANTS OF STRUCTURE
C
      WRITE(61,2020)
      WRITE(61,2030) (N,(PN(N,I),I=1,6),N=1,NUMNP)
      REWIND 8
      RETURN
C
10    FORMAT(E21.6)
2000  FORMAT('1',31HRESULTANT LOADS ON EACH ELEMENT///)
2010  FORMAT('-',8H ELEMENT,I3//4X,6HNODE-I,6E15.9//
     +             4X,6HNODE-J,6E15.9)
2020  FORMAT('1',48HSTRUCTURE RESULTANTS DUE TO LINEAR DISPLACEMENTS///
     +        1X,4HNODE,15X,3HPN1,15X,3HPN2,15X,3HPN3,15X,3HPN4,
     +        15X,3HPN5,15X,3HPN6//)
2030  FORMAT('0',I5,4X,6E15.9)
C
      END
C
      SUBROUTINE EIGENVL (EIGEN,IDATA)
C     ****************************************************************
C        TO SOLVE EIGENVALUE PROBLEM S*X=-(LAMBDA)*S1*X
C        WILL OBTAIN ONLY THE LOWEST EIGENVALUE AND CORRESPONDING
C        EIGENVECTOR. USES INVERSE VECTOR ITERATION WITH THE
C        RAYLEIGH QUOTIENT.
C     ****************************************************************
C
      COMMON/C1/DUMMY(16),ICAL7
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C9/S(296,16),SP(296,16),IDET
      COMMON/C10/XB(296),YB(296),X(296),Y(296),EIGNVTR(296)
```

```
C
C........ASSUME STARTING SHIFT, STARTING VECTOR, AND
C        MAXIMUN NUMBER OF ITERATIONS ALLOWED.
C
      WRITE(61,2010)
      READ(60,1000) MAX,EPSI,RHO
      WRITE(61,2000)MAX,EPSI,RHO
      DO 100 I=1,NEQ
100   X(I)=1.
C
C........OBTAIN VECTOR Y(I) FROM Y(I)=S1(I,J)*X(I)
C        FIRST CHANGE SIGN OF MATRIX S1
C
      READ(5,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
      REWIND 5
      DO 107 I=1,NEQ
      DO 105 J=1,MBAND
      S(I,J)=-S(I,J)
105   CONTINUE
107   CONTINUE
      WRITE(5,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
      REWIND 5
C
C........HORIZONTAL SWEEP OF S1(I,J)*X(I), DIAGONAL NOT INCLUDED
C
      DO 130 I=1,NEQ
      Y(I)=0.
      II=I+1
      IF (II.GT.NEQ) GO TO 130
      DO 120 J=2,MBAND
      IF (S(I,J).EQ.0.) GO TO 110
      Y(I)=Y(I)+S(I,J)*X(II)
110   II=II+1
      IF (II.GT.NEQ) GO TO 130
120   CONTINUE
130   CONTINUE
C
C........DIAGONAL SWEEP OF S1(I,J)*X(I)
C
      DO 160 I=1,NEQ
      II=I
      JJ=1
140   IF (S(II,JJ).EQ.0.) GO TO 150
      Y(I)=Y(I)+S(II,JJ)*X(II)
150   II=II-1
      JJ=JJ+1
      IF (II.EQ.0) GO TO 160
      IF (JJ.GT.MBAND) GO TO 160
      GO TO 140
160   CONTINUE
C
C........START ITERATION PROCEDURE BY STABLISHING THE SYSTEM OF
```

```
C          EQUATIONS S(I,J)*XB(I)=Y(I) AND SOLVING FOR XB(I).
C          STORE VALUES OF Y(I) INTO XB(I) FOR GAUSS SOLUTION.
C
      DO 300 K = 1,MAX
      DO 165 I=1,NEQ
165   XB(I)=Y(I)
      IF (IDATA.EQ.0) READ(4,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
      IF (IDATA.EQ.1) READ(7,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
      REWIND 4
      REWIND 7
      IF (ICAL7.NE.0) GO TO 176
C
C........PRINT DATA SENT TO SUBROUTINE GAUSSOL
C
      WRITE(61,2100) K
      K1=1
      K2=6
      K3=MBAND-K1
      IF (K3.LE.7) GO TO 174
172   WRITE(61,2110) K1,K2
      WRITE(61,2115) ((S(I,J),J=K1,K2),I=1,NEQ)
      K1=K1+6
      K2=K2+6
      K3=MBAND-K1
      IF (K3.LE.7) GO TO 174
      GO TO 172
174   WRITE(61,2110) K1,MBAND
      IF (K3.EQ.0) WRITE(61,2120) ((S(I,J),J=K1,MBAND),I=1,NEQ)
      IF (K3.EQ.1) WRITE(61,2121) ((S(I,J),J=K1,MBAND),I=1,NEQ)
      IF (K3.EQ.2) WRITE(61,2122) ((S(I,J),J=K1,MBAND),I=1,NEQ)
      IF (K3.EQ.3) WRITE(61,2123) ((S(I,J),J=K1,MBAND),I=1,NEQ)
      IF (K3.EQ.4) WRITE(61,2124) ((S(I,J),J=K1,MBAND),I=1,NEQ)
      IF (K3.EQ.5) WRITE(61,2125) ((S(I,J),J=K1,MBAND),I=1,NEQ)
C     IF (K3.EQ.6) WRITE(61,2126) ((S(I,J),J=K1,MBAND),I=1,NEQ)
C     IF (K3.EQ.7) WRITE(61,2115) ((S(I,J),J=K1,MBAND),I=1,NEQ)
      WRITE(61,2130)
      WRITE(61,2135) (XB(I),I=1,NEQ)
176   CONTINUE
C
C........SOLVE SYSTEM OF EQUATIONS S(I,J)*XB(I)=Y(I)
C
      CALL GAUSSOL
      IF (ICAL7.EQ.0) WRITE(61,2030)
C
C........OBTAIN VECTOR YB(I) FROM YB(I)=S1(I,J)*XB(I)
C
      READ(5,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
      REWIND 5
C
C........HORIZONTAL SWEEP OF S1(I,J)*XB(I), DIAGONAL NOT INCLUDED
C
      DO 200 I=1,NEQ
```

```
      YB(I)=0.
      II=I+1
      IF (II.GT.NEQ) GO TO 200
      DO 190 J=2,MBAND
      IF (S(I,J).EQ.0.) GO TO 180
      YB(I)=YB(I)+S(I,J)*XB(II)
180   II=II+1
      IF (II.GT.NEQ) GO TO 200
190   CONTINUE
200   CONTINUE
C
C........DIAGONAL SWEEP OF S1(I,J)*XB(I)
C
      DO 230 I=1,NEQ
      II=I
      JJ=1
210   IF (S(II,JJ).EQ.0.) GO TO 220
      YB(I)=YB(I)+S(II,JJ)*XB(II)
220   II=II-1
      JJ=JJ+1
      IF (II.EQ.0) GO TO 230
      IF (JJ.GT.MBAND) GO TO 230
      GO TO 210
230   CONTINUE
C
C........COMPUTE RAYLEIGH QUOTIENT
C
      RQ=RHO
      Q1=0.0
      Q2=0.0
      DO 240 I=1,NEQ
      Q1=Q1+XB(I)*Y(I)
240   Q2=Q2+XB(I)*YB(I)
      RHO=Q1/Q2
      DO 250 I=1,NEQ
250   Y(I)=YB(I)/(Q2**.5)
C
C........CHECK CONVERGENCE TO DESIRED EIGENVALUE
C
      CHECK=ABS(RHO-RQ)/RHO
      IF (CHECK.LE.EPSI) GO TO 310
      EIGEN=RHO
      DO 260 I=1,NEQ
260   EIGNVTR(I)=XB(I)/(Q2**.5)
      IF (ICAL7.NE.0) WRITE(61,2035) K,EIGEN
      IF (ICAL7.NE.0) GO TO 300
      WRITE(61,2040) K,RHO,CHECK,EIGEN
      WRITE(61,2050) (XB(I),YB(I),Y(I),EIGNVTR(I),I=1,NEQ)
300   CONTINUE
C
C........OBTAIN EIGENVALUE AND CORRESPONDING EIGENVECTOR
C
```

```
310    EIGEN=RHO
       DO 320 I=1,NEQ
320    EIGNVTR(I)=XB(I)/(Q2**.5)
       ILAST=K
       WRITE(61,2070) ILAST
       WRITE(61,2080) EIGEN
       WRITE(61,2090) (EIGNVTR(I),I=1,NEQ)
       RETURN
C
10     FORMAT(E21.6)
1000   FORMAT(I5,2F10.6)
2000   FORMAT('-',4HMAX=,I3///6H EPSI=,F10.6///6H  RHO=,F10.6)
2010   FORMAT('1',45HLINEAR EIGENVALUE PROBLEM (INVERSE ITERATION)//)
2030   FORMAT('1',38HINVERSE VECTOR ITERATION WITH SHIFTING///2X,1HK,9X,
     +        2HXB,16X,2HYB,16X,3HRHO,14X,5HCHECK,15X,1HY,15X,5HEIGEN,
     +        12X,7HEIGNVTR///)
2035   FORMAT('-',2HK=,I3,5X,6HEIGEN=,E15.9)
2040   FORMAT('-',I3,39X,E15.9,3X,E15.9,21X,E15.9)
2050   FORMAT(' ',6X,E15.9,3X,E15.9,39X,E15.9,21X,E15.9)
2100   FORMAT('1',34HDATA FOR GAUSSOL S(I,J) AND XB(I)//1X,2HK=,I3//)
2110   FORMAT('-',7HCOLUMNS,I4,10H   THROUGH,I4)
2115   FORMAT('0',6E16.8)
2120   FORMAT('0',E16.8)
2121   FORMAT('0',2E16.8)
2122   FORMAT('0',3E16.8)
2123   FORMAT('0',4E16.8)
2124   FORMAT('0',5E16.8)
2125   FORMAT('0',6E16.8)
C 2126   FORMAT('0',7E16.8)
2130   FORMAT('-',37HVECTOR Y(I), SENT TO GAUSSOL AS XB(I)//)
2135   FORMAT('0',10X,E15.9)
2070   FORMAT('1',5X,10HEIGENVALUE,9X,11HEIGENVECTOR,5X,6HILAST=,I3)
2080   FORMAT(' ',E15.9)
2090   FORMAT(' ',20X,E15.9)
C
       END
C
       SUBROUTINE NLEIGNP (SCALE)
C      ************************************************************
C         THIS ROUTINE WILL COMPUTE THE EIGENVALUE OF THE
C         QUADRATIC EIGENVALUE PROBLEM (K+L*N1+L*L*N2)*X=0
C         IT USES THE MODIFIED REGULA FALSI METHOD
C      ************************************************************
C
       EXTERNAL DET
       REAL L
C
       READ(60,1000) XTOL,FTOL,NTOL,DINCR
       WRITE(61,2010) XTOL,FTOL,NTOL,DINCR
       WRITE(61,2030)
       A=0.
100    FA=DET(A,SCALE)
```

```
      WRITE(61,2020) A,FA
      IF (FA.LT.0.) GO TO 110
      A=A+DINCR
      GO TO 100
 110  CONTINUE
      B=A
      A=A-DINCR
      CALL MRGFLS (DET,A,B,XTOL,FTOL,NTOL,IFLAG,SCALE)
      IF (IFLAG.GT.2) GO TO 500
      L=(A+B)/2.
      ERROR=ABS(B-A)/2.
      FL=DET(L,SCALE)
      WRITE(61,2000) L,ERROR,FL
 500  CONTINUE
      RETURN
C
1000  FORMAT(2E10.2,I10,F10.5)
2000  FORMAT(////14H  THE ROOT IS ,E25.15,10X,12H PLUS/MINUS ,E25.15//
     +        15H  DETERMINANT =,E25.15)
2010  FORMAT('1',28HQUADRATIC EIGENVALUE PROBLEM///6H XTOL=,E10.2///
     +        6H FTOL=,E10.2///6H NTOL=,I3///7H DINCR=,F10.5)
2020  FORMAT('-',E25.15,5X,E25.15//)
2030  FORMAT(////13X,6HLAMBDA,17X,11HDETERMINANT//)
C
      END
C
      SUBROUTINE MRGFLS (F,A,B,XTOL,FTOL,NTOL,IFLAG,SCALE)
C     ************************************************************
C        ITERATES TO A SUFFICIENTLY SMALL VALUE OF THE DETERMINANT
C        OR TO A SUFFICIENTLY SMALL INTERVAL WHERE THE ROOT MAY
C        BE FOUND
C     ************************************************************
C
      IFLAG=0
      FA=F(A,SCALE)
      SIGNFA=FA/ABS(FA)
      FB=F(B,SCALE)
C
C........CHECK FOR SIGN CHANGE
C
      IF (SIGNFA*FB.LE.0.) GO TO 100
      IFLAG=3
      WRITE(61,2010) A,B
      RETURN
C
100   W=A
      FW=FA
      DO 400 N=1,NTOL
C
C........CHECK FOR SUFFICIENTLY SMALL INTERVAL
C
      IF (ABS(B-A)/2..LE.XTOL) RETURN
```

```
C
C........CHECK FOR SUFFICIENTLY SMALL DETERMINANT VALUE
C
      IF (ABS(FW).GT.FTOL) GO TO 200
      A=W
      B=W
      IFLAG=1
      RETURN
200   W=(FA*B-FB*A)/(FA-FB)
      PREVFW=FW/ABS(FW)
      FW=F(W,SCALE)
C
C........TEMPORARY PRINT OUT
C
      NM1=N-1
      WRITE(61,2020) NM1,A,W,B,FA,FW,FB
C
C........CHANGE TO NEW INTERVAL
C
      IF (SIGNFA*FW.LT.0.) GO TO 300
      A=W
      FA=FW
      IF (FW*PREVFW.GT.0.) FB=FB/2.
      GO TO 400
300   B=W
      FB=FW
      IF (FW*PREVFW.GT.0.) FA=FA/2.
400   CONTINUE
      IFLAG=2
      WRITE(61,2030) NTOL
      RETURN
C
2010  FORMAT(////43H F(X) IS OF SAME SIGN AT THE TWO ENDPOINTS ,
     +       2E25.15)
2020  FORMAT('-',I3,9H L-VALUES,3E25.15//4X,9H F-VALUES,3E25.15//)
2030  FORMAT(////19H  NO CONVERGENCE IN,I5,11H ITERATIONS)
C
      END
C
      FUNCTION DET (L,SCALE)
C     ************************************************************
C        THIS FUNCTION COMPUTES THE VALUE OF THE DETERMINANT
C        OF THE MATRIX S = K + L * N1 + L * L * N2
C            K=LINEAR STIFFNESS OF STRUCTURE
C            N1=NONLINEAR STIFFNESS OF STRUCTURE (CUBIC TERMS)
C            N2=NONLINEAR STIFFNESS OF STRUCTURE (QUARTIC TERMS)
C            L=VALUE OF LAMBDA FOR WHICH S IS COMPUTED
C     ************************************************************
C
      REAL K,N1,N2,L
      COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
      COMMON/C9/S(296,16),SP(296,16),IDET
```

```
C
C........COMPUTE MATRIX S=K+L*N1+L*L*N2 (BANDED FORMAT)
C
      IF (L.EQ.0.) GO TO 220
      DO 210 I=1,NEQ
      DO 200 J=1,MBAND
      READ(4,10) K
      IF (IEIGEN.EQ.1) READ(5,10) N1
      IF (IEIGEN.EQ.2) READ(7,10) N1
      READ(16,10) N2
      S(I,J)=K+L*N1+L*L*N2
200   CONTINUE
210   CONTINUE
      GO TO 230
220   READ(4,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
230   REWIND 4
      REWIND 5
      REWIND 16
      REWIND 7
C
C........FORWARD REDUCTION OF MATRIX (GAUSS ELIMINATION)
C
      DO 390 N=1,NEQ
      DO 380 LL=2,MBAND
      IF (S(N,LL).EQ.0.) GO TO 380
      I=N+LL-1
      C=S(N,LL)/S(N,1)
      J=0
      DO 350 KK=LL,MBAND
      J=J+1
350   S(I,J)=S(I,J)-C*S(N,KK)
      S(N,LL)=C
380   CONTINUE
390   CONTINUE
C
C........COMPUTE DETERMINANT OF MATRIX S
C          SCALE DOWN "DET" BY A "SCALE" VALUE AFTER EACH STEP
C
      DT=1.
      DO 400 I=1,NEQ
      DT=DT*S(I,1)/SCALE
400   CONTINUE
      DET=DT
      RETURN
C
10    FORMAT(E21.6)
C
      END
C
      SUBROUTINE TILTED (IDATA,SCALE,DX,W)
C     ****************************************************************
C        TO COMPUTE THE BUCKLING LOAD OF A DECK BRIDGE OR
```

```
C          A THROUGH BRIDGE DUE TO TILTED LOADS
C       **************************************************************
C
        COMMON/C1/NE,NUMNP,D1(15)
        COMMON/C2/NSIZE,NEQ,NCOND,MBAND,IEIGEN
        COMMON/C3/IA(37,8),IB(37,8),X(37),Y(37),H(37),RAD,AC
        COMMON/C9/S(296,16),SP(296,16),IDET
C
C........CHECK TYPE OF BRIDGE BEING CONSIDERED
C            IDECK.EQ.0 , THROUGH BRIDGE
C            IDECK.EQ.1 , DECK BRIDGE
C            IDECK.EQ.2 , HALF-THROUGH BRIDGE
C
        READ(60,1010) IDECK,HD
        WRITE(61,2000)
        IF (IDECK.EQ.0) WRITE(61,2020) IDECK,HD
        IF (IDECK.EQ.1) WRITE(61,2010) IDECK,HD
        IF (IDECK.EQ.2) WRITE(61,2025) IDECK,HD
C
C........MODIFY MATRIX S1 BY PARAMETER P/H ACCORDING TO
C          EQUATION NUMBERS IN ARRAY IA(N,I). STORE MATRIX S1
C
        READ(5,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
        REWIND 5
        P=W*DX
        WRITE(61,2030) P
        DO 110 N=1,NUMNP
        IF (IA(N,2).LE.0) GO TO 110
        I=IA(N,2)
        H(N)=Y(N)-HD
        WRITE(61,2040) N,H(N)
        IF (H(N).EQ.0.) GO TO 100
        S(I,1)=S(I,1)-P/H(N)
100     CONTINUE
110     CONTINUE
        IF (IEIGEN.EQ.0) WRITE(5,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
        IF (IEIGEN.EQ.1) WRITE(5,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
        IF (IEIGEN.EQ.2) WRITE(7,10) ((S(I,J),J=1,MBAND),I=1,NEQ)
        REWIND 5
        REWIND 7
C
C........SOLVE EIGENVALUE PROBLEM
C          IF IEIGEN.EQ.0 , SOLVE LINEAR CASE
C          IF IEIGEN.EQ.1 , SOLVE QUADRATIC CASE
C          IF IEIGEN.EQ.2 , SOLVE BOTH
C
        EIGEN=0.
        IF (IEIGEN.EQ.0) CALL EIGENVL (EIGEN,IDATA)
        IF (IEIGEN.EQ.1) CALL NLEIGNP (SCALE)
        IF (IEIGEN.EQ.2) GO TO 120
        RETURN
120     CALL NLEIGNP (SCALE)
```

```
      CALL EIGENVL (EIGEN,IDATA)
      RETURN
C
10    FORMAT(E21.6)
1010  FORMAT(I5,F10.5)
2000  FORMAT('1',18H TILTED LOAD CASE //)
2010  FORMAT(////13H  DECK BRIDGE///9H  IDECK =,I2///6H  HD =,F10.5)
2020  FORMAT(////16H  THROUGH BRIDGE///9H  IDECK =,I2///6H  HD =,F10.5)
2025  FORMAT(////21H  HALF-THROUGH BRIDGE///9H  IDECK =,I2///
     +        6H  HD =,F10.5)
2030  FORMAT(////25H  LOAD ON EACH COLUMN IS ,F10.5////
     +       16H  COLUMN LENGTHS//)
2040  FORMAT('0',7H     H(,I2,3H) =,F7.3)
C
      END
C
CCCC
      SUBROUTINE REOCON
C
      COMMON/C4/SE(16,16)
      DIMENSION ID(4),XA(14)
C
C     CONDENSE FROM 16 BY 16 TO 14 BY 14
C
      N=16
      NC=2
      DO 230 K=1,NC
      LL=N-K
      KK=LL+1
      DO 230 L=1,LL
      DUM=SE(KK,L)/SE(KK,KK)
      DO 220 M=1,L
220   SE(L,M)=SE(L,M)-SE(KK,M)*DUM
230   CONTINUE
      LL=N-NC
      DO 250 I=1,LL
      DO 240 J=I,LL
      SE(I,J)=SE(J,I)
240   CONTINUE
250   CONTINUE
C
      ID(1)=7
      ID(2)=8
C
C     REORDER COLUMNS AND ROWS
C
      N=14
      NR=2
      NC=2
      DO 200 K=1,NR
      DO 100 J=1,N
      XA(J)=SE(J,ID(K))
```

```
100   CONTINUE
      DO 120 L=ID(K),N-1
      DO 110 J=1,N
      SE(J,L)=SE(J,L+1)
110   CONTINUE
120   CONTINUE
      DO 130 J=1,N
      SE(J,N)=XA(J)
130   CONTINUE
      DO 140 J=1,N
      XA(J)=SE(ID(K),J)
140   CONTINUE
      DO 160 L=ID(K),N-1
      DO 150 J=1,N
      SE(L,J)=SE(L+1,J)
150   CONTINUE
160   CONTINUE
      DO 170 J=1,N
      SE(N,J)=XA(J)
170   CONTINUE
      KK=K+1
      DO 180 M=KK,NR
      ID(M)=ID(M)-1
180   CONTINUE
200   CONTINUE
C
C     DO CONDENSATION
C
      DO 231 K=1,NC
      LL=N-K
      KK=LL+1
      DO 231 L=1,LL
      DUM=SE(KK,L)/SE(KK,KK)
      DO 221 M=1,L
221   SE(L,M)=SE(L,M)-SE(KK,M)*DUM
231   CONTINUE
      LL=N-NC
      DO 251 I=1,LL
      DO 241 J=I,LL
      SE(I,J)=SE(J,I)
241   CONTINUE
251   CONTINUE
CC
CCCC
      RETURN
C
      END
C
      FUNCTION DET1(SCALE)
C
C     *****************************************************************
C        THIS FUNCTION COMPUTES THE VALUE OF THE DETERMINANT OF
```

```
C          THE MATRIX S=K+N1+N2
C       *****************************************************************
C
        COMMON/C2/ NSIZE,NEQ,NCOND,MBAND,IEIGEN
        COMMON/C9/ S(296,16),SP(296,16),IDET
C
        IF(IDET.EQ.1) GO TO 250
        IF(IDET.EQ.2) GO TO 450
        DO 490 I=1,NEQ
        DO 490 J=1,MBAND
490     S(I,J)=SP(I,J)
C          FORWARD REDUCTION OF MATRIX(GAUSS ELEMINATION)
C       *****************************************************************
450     DO 390 LN=1,NEQ
        DO 380 LL=2,MBAND
        IF(S(LN,LL).EQ.0.) GO TO 380
        I=LN+LL-1
        C=S(LN,LL)/S(LN,1)
        J=0
        DO 350 KK=LL,MBAND
        J=J+1
350     S(I,J)=S(I,J)-C*S(LN,KK)
        S(LN,LL)=C
380     CONTINUE
390     CONTINUE
250     CONTINUE
C
C          COMPUTE DETERMINANT OF MATRIX S
C          SCALE DOWN"DET1" BY A "SCALE" VALUE AFTER EACH STEP
C       *****************************************************************
        DT=1.
        DO 400 I=1,NEQ
        DT=DT*S(I,1)/SCALE
400     CONTINUE
        DET1=DT
        RETURN
C
        END
```

APPENDIX D

INCREMENTAL STIFFNESS MATRICES, [n1], AND [n2]

BASED ON THE QUARTIC AXIAL STRAIN MODEL

The following subroutine contains the entries of the [n1] and [n2] matrices based on the quartic axial strain model.

```
      SUBROUTINE NUMINT (N,M)
C     IMPLICIT DOUBLE PRECISION (A-H,O-Z)
C     *****************************************************************
C        TO INTEGRATE NUMERICALLY THE TERMS OF THE CURVED ELEMENT
C        STIFFNESS MATRICES SE, SE1, SE2, IT USES THE GAUSS-LEGENDRE
C        QUADRATURE FORMULA.
C        THE ROUTINE NUMINT USES THE MP-POINT GAUSS-LEGENDRE QUADRATURE
C        FORMULA TO COMPUTE THE INTEGRAL OF FUNCTN(GM)*DGM BETWEEN
C        INTEGRATION LIMITS A1 AND A2. THE ROOTS OF SEVEN LEGENDRE
C        POLYNOMIALS AND THE WEIGHT FACTORS FOR CORRESPONDING
C        QUADRATURES ARE STORED IN THE Z AND WEIGHT ARRAYS RESPECTIVELY.
C        MP MAY ASSUME VALUES 2, 3, 4, 5, 6, 10, AND 15 ONLY. THE
C        APPROPRIATE VALUES FOR THE MP-POINT FORMULA ARE LOCATED IN
C        ELEMENTS Z(KEY(I))...Z(KEY(I+1)-1) AND WEIGHT(KEY(I))...
C        WEIGHT(KEY(I+1)-1) WHERE THE PROPER VALUE FOR I IS DETERMINED
C        BY FINDING THE SUBSCRIPT OF THE ELEMENT OF THE ARRAY NPOINT
C        WHICH HAS THE VALUE MP. IF AN INVALID VALUE OF MP IS USED, A
C        TRUE ZERO IS RETURNED AS THE VALUE OF GAUSS.
C     *****************************************************************
C
      REAL IXX,IYY,KT,II,JJ,LENGTH,L1,L2,K,KK
      DIMENSION NPOINT(7),KEY(8),Z(24),WEIGHT(24),K(16,16)
      COMMON/C1/NE,NUMNP,NUMEG,NTYPE(3),NUMEL(3),IPAR,ICAL1,ICAL2,ICAL3,
     +          ICAL4,ICAL5,ICAL6,ICAL7
      COMMON/C4/SE(16,16)
      COMMON/C5/E(3),G(3),NODEI(36),NODEJ(36),A(36),IXX(36),IYY(36),
     +          KT(36),L(1,36)
      COMMON/C6/A1,A2,MP,B1(36),B2(36),B3(36)
      COMMON/C7/RI(36),RJ(36),PHII(36),PHIJ(36),TETA(36),LENGTH(36),
     +           RIA(36),RJA(36)
      COMMON/C11/DN(16),U(36,12),W(37,8),V(37,8)
      DATA NPOINT/ 2, 3, 4, 5, 6, 10, 15/
      DATA KEY/ 1, 2, 4, 6, 9, 12, 17, 25/
```

```
      DATA Z              / 0.577350269,0.0         ,0.774596669,
     1        0.339981044,0.861136312,0.0         ,0.538469310,
     2        0.906179846,0.238619186,0.661209387,0.932469514,
     3        0.148874339,0.433395394,0.679409568,0.865063367,
     4        0.973906529,0.0         ,0.201194094,0.394151347,
     5        0.570972173,0.724417731,0.848206583,0.937273392,
     6        0.987992518 /
      DATA WEIGHT         / 1.0         ,0.888888889,0.555555556,
     1        0.652145155,0.347854845,0.568888889,0.478628671,
     2        0.236926885,0.467913935,0.360761573,0.171324493,
     3        0.295524225,0.269266719,0.219086363,0.149451349,
     4        0.066671344,0.202578242,0.198431485,0.186161000,
     5        0.166269206,0.139570678,0.107159221,0.070366047,
     6        0.030753242 /
C
      T=TETA(M)
      R1=RI(M)
      R2=RJ(M)
      L1=R1*T
      L2=R2*T
C
C........FIND SUBSCRIPT OF FIRST Z AND WEIGHT VALUE
C
      DO 100 I=1,7
      IF(MP.EQ.NPOINT(I)) GO TO 200
100   CONTINUE
C
C........INVALID MP USED
C
      GAUSS=0.0
      WRITE(61,2000) GAUSS
      RETURN
C
C........SET UP INITIAL PARAMETERS
C
200   JFIRST=KEY(I)
      JLAST=KEY(I+1)-1
      C=(A2-A1)/2.
      D=(A2+A1)/2.
C
C........ACCUMULATE THE SUM IN THE MP-POINT FORMULA
C
CCCC
      IF (IPAR.GE.3) GO TO 543
      DO 249 I=1,16
      DO 249 J=1,16
249   K(I,J)=0.0
      GO TO 248
```

```
543   CONTINUE
CCCC
      DO 250 I=1,12
      DO 250 J=1,12
250   K(I,J)=0.0
248   CONTINUE
      DO 500 J=JFIRST,JLAST
      I=0
      IF (Z(J).EQ.0.) GO TO 350
300   I=I+1
      IF (I.EQ.1) GM=Z(J)*C+D
      IF (I.EQ.2) GM=-Z(J)*C+D
      GO TO 360
350   GM=D
360   AA=6.*GM**2-6.*GM
      BB=3.*GM**2-4.*GM+1.
      CC=3.*GM**2-2.*GM
      DD=12.*GM-6.
      EE=6.*GM-4.
      FF=6.*GM-2.
      GG=2.*GM**3-3.*GM**2+1.
      HH=GM**3-2.*GM**2+GM
      II=-2.*GM**3+3.*GM**2
      JJ=GM**3-GM**2
      KK=1.-GM
      R=B1(M)+2.*B2(M)*T*GM
      GMSS=(-1./(R**3*T))*(2.*B2(M))
      GMSG=R*T*GMSS
C
C........CHECK WHICH PART OF THE STIFFNESS MATRIX IS BEING COMPUTED
C                 IPAR=2, COMPUTE ARRAY SE
C                 IPAR=3, COMPUTE ARRAY SE1
C                 IPAR=4, COMPUTE ARRAY SE2
C
      GO TO (370,370,390,410),IPAR
C
370   CONTINUE
C
C........INTEGRANDS OF CURVED ELEMENT LINEAR STIFFNESS (SYMMETRIC)
C
      C1=-E(N)*A(M)*GG/R
      C2=(E(N)*IYY(M)/(R**3*T**3))*(DD+AA*GMSS*R**2*T**2)
      C3=(E(N)*IXX(M)*T/R**3)*(-DD/T**2-GMSS*R**2*AA)
      C4=G(N)*KT(M)*AA/(R**3*T)
      C5=(E(N)*A(M)/(R*T))*(AA+T**2*HH)
      C6=(E(N)*IYY(M)/(R**3*T**3))*(-T*EE-GMSS*R**2*T**3*BB+T*AA+
     +   GMSG*R*T**2*GG)
      C7=E(N)*IXX(M)*T*GG/R**2
```

```
      C8=G(N)*KT(M)*AA/(R**2*T)
      C9=-E(N)*A(M)*L1*HH/R
      C10=(E(N)*IYY(M)/(R**3*T**3))*(L1*EE+GMSS*R**2*T**2*L1*BB)
      C11=(E(N)*IXX(M)*T/R**3)*(L1*EE/T**2+GMSS*R**2*L1*BB)
      C12=-G(N)*KT(M)*L1*BB/(R**3*T)
      C13=E(N)*A(M)*R1*BB/(R*T)
      C14=(E(N)*IYY(M)/(R**3*T**3))*(T*R1*BB+GMSG*R*T**2*R1*HH)
      C15=E(N)*IXX(M)*T*L1*HH/R**2
      C16=G(N)*KT(M)*L1*BB/(R**2*T)
      C17=-E(N)*A(M)*II/R
      C18=(E(N)*IYY(M)/(R**3*T**3))*(-DD-GMSS*R**2*T**2*AA)
      C19=-C3
      C20=-C4
      C21=(E(N)*A(M)/(R*T))*(-AA+T**2*JJ)
      C22=(E(N)*IYY(M)/(R**3*T**3))*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +    GMSG*R*T**2*II)
      C23=E(N)*IXX(M)*T*II/R**2
      C24=-C8
      C25=(-E(N)*A(M)*L2*JJ)/R
      C26=(E(N)*IYY(M)/(R**3*T**3))*(L2*FF+GMSS*R**2*T**2*L2*CC)
      C27=(E(N)*IXX(M)*T/R**3)*(L2*FF/T**2+GMSS*R**2*L2*CC)
      C28=-G(N)*KT(M)*L2*CC/(R**3*T)
      C29=E(N)*A(M)*R2*CC/(R*T)
      C30=(E(N)*IYY(M)/(R**3*T**3))*(T*R2*CC+GMSG*R*T**2*R2*JJ)
      C31=E(N)*IXX(M)*T*L2*JJ/R**2
      C32=G(N)*KT(M)*L2*CC/(R**2*T)
C
        SE(1,1)=C1*T*(-GG)+C2*(DD+AA*GMSS*R**2*T**2)
        SE(1,2)=0.0
        SE(1,4)=0.0
        SE(1,6)=0.0
        SE(1,8)=0.0
        SE(1,10)=0.0
        SE(1,12)=0.0
        SE(1,14)=0.0
        SE(1,16)=0.0
        SE(1,3)=C1*(AA+T**2*HH)+C2*(-T*EE-GMSS*R**2*T**3*BB+T*AA+
     +          GMSG*R*T**2*GG)
        SE(1,5)=C1*T*(-L1)*HH+C2*L1*(EE+BB*GMSS*R**2*T**2)
        SE(1,7)=C1*R1*BB+C2*(T*R1*BB+GMSG*R*T**2*R1*HH)
        SE(1,9)=C1*(-T*II)-C2*(DD+AA*GMSS*R**2*T**2)
        SE(1,11)=C1*(-AA+T**2*JJ)+C2*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +           GMSG*R*T**2*II)
        SE(1,13)=C1*(-T*L2*JJ)+C2*(L2*FF+GMSS*R**2*T**2*L2*CC)
        SE(1,15)=C1*(R2*CC)+C2*(T*R2*CC+GMSG*R*T**2*JJ)
C
        SE(2,2)=C3*(-DD/T**2-GMSS*R**2*AA)+C4*AA
        SE(2,3)=0.0
```

```
      SE(2,5)=0.0
      SE(2,7)=0.0
      SE(2,9)=0.0
      SE(2,11)=0.0
      SE(2,13)=0.0
      SE(2,15)=0.0
      SE(2,4)=C3*R*GG+C4*R*AA
      SE(2,6)=C3*(L1*EE/T**2+GMSS*R**2*L1*BB)-C4*L1*BB
      SE(2,8)=C3*R*L1*HH+C4*R*L1*BB
      SE(2,10)=C3*(DD/T**2+GMSS*R**2*AA)-C4*AA
      SE(2,12)=C3*R*II-C4*R*AA
      SE(2,14)=C3*(L2*FF/T**2+GMSS*R**2*L2*CC)-C4*L2*CC
      SE(2,16)=C3*R*L2*JJ+C4*R*L2*CC
C
      SE(3,3)=C5*(AA+T**2*HH)+C6*(-T*EE-GMSS*R**2*T**3*BB+T*AA+
     +        GMSG*R*T**2*GG)
      SE(3,4)=0.0
      SE(3,6)=0.0
      SE(3,8)=0.0
      SE(3,10)=0.0
      SE(3,12)=0.0
      SE(3,14)=0.0
      SE(3,16)=0.0
      SE(3,5)=C5*(-T*L1*HH)+C6*(L1*EE+GMSS*R**2*T**2*L1*BB)
      SE(3,7)=C5*R1*BB+C6*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(3,9)=C5*(-T*II)+C6*(-DD-GMSS*R**2*T**2*AA)
      SE(3,11)=C5*(-AA+T**2*JJ)+C6*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +         GMSG*R*T**2*II)
      SE(3,13)=C5*(-T*L2*JJ)+C6*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(3,15)=C5*R2*CC+C6*(T*R2*CC+GMSG*R*T**2*JJ)
C
      SE(4,4)=C7*R*GG+C8*R*AA
      SE(4,5)=0.0
      SE(4,7)=0.0
      SE(4,9)=0.0
      SE(4,11)=0.0
      SE(4,13)=0.0
      SE(4,15)=0.0
      SE(4,6)=C7*(L1*EE/T**2+GMSS*R**2*L1*BB)+C8*(-L1*BB)
      SE(4,8)=C7*R*L1*HH+C8*R*L1*BB
      SE(4,10)=C7*(DD/T**2+GMSS*R**2*AA)+C8*(-AA)
      SE(4,12)=C7*R*II+C8*(-R*AA)
      SE(4,14)=C7*(L2*FF/T**2+GMSS*R**2*L2*CC)+C8*(-L2*CC)
      SE(4,16)=C7*R*L2*JJ+C8*R*L2*CC
C
      SE(5,5)=C9*(-T*L1*HH)+C10*(L1*EE+GMSS*R**2*T**2*L1*BB)
      SE(5,6)=0.0
      SE(5,8)=0.0
```

```
      SE(5,10)=0.0
      SE(5,12)=0.0
      SE(5,14)=0.0
      SE(5,16)=0.0
      SE(5,7)=C9*R1*BB+C10*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(5,9)=C9*(-T*II)+C10*(-DD-GMSS*R**2*T**2*AA)
      SE(5,11)=C9*(-AA+T**2*JJ)+C10*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +         GMSG*R*T**2*II)
      SE(5,13)=C9*(-T*L2*JJ)+C10*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(5,15)=C9*R2*CC+C10*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(6,6)=C11*(L1*EE/T**2+GMSS*R**2*L1*BB)+C12*(-L1*BB)
      SE(6,7)=0.0
      SE(6,9)=0.0
      SE(6,11)=0.0
      SE(6,13)=0.0
      SE(6,15)=0.0
      SE(6,8)=C11*R*L1*HH+C12*R*L1*BB
      SE(6,10)=C11*(DD/T**2+GMSS*R**2*AA)+C12*(-AA)
      SE(6,12)=C11*R*II+C12*(-R*AA)
      SE(6,14)=C11*(L2*FF/T**2+GMSS*R**2*L2*CC)+C12*(-L2*CC)
      SE(6,16)=C11*R*L2*JJ+C12*R*L2*CC
C
      SE(7,7)=C13*R1*BB+C14*(T*R1*BB+GMSG*R*T**2*R1*HH)
      SE(7,8)=0.0
      SE(7,10)=0.0
      SE(7,12)=0.0
      SE(7,14)=0.0
      SE(7,16)=0.0
      SE(7,9)=C13*(-T*II)+C14*(-DD-GMSS*R**2*T**2*AA)
      SE(7,11)=C13*(-AA+T**2*JJ)+C14*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +         GMSG*R*T**2*II)
      SE(7,13)=C13*(-T*L2*JJ)+C14*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(7,15)=C13*R2*CC+C14*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(8,8)=C15*R*L1*HH+C16*R*L1*BB
      SE(8,9)=0.0
      SE(8,11)=0.0
      SE(8,13)=0.0
      SE(8,15)=0.0
      SE(8,10)=C15*(DD/T**2+GMSS*R**2*AA)+C16*(-AA)
      SE(8,12)=C15*R*II+C16*(-R*AA)
      SE(8,14)=C15*(L2*FF/T**2+GMSS*R**2*L2*CC)+C16*(-L2*CC)
      SE(8,16)=C15*R*L2*JJ+C16*R*L2*CC
C
      SE(9,9)=C17*T*(-II)+C18*(-DD-GMSS*R**2*T**2*AA)
      SE(9,10)=0.0
      SE(9,12)=0.0
```

```
      SE(9,14)=0.0
      SE(9,16)=0.0
      SE(9,11)=C17*(-AA+T**2*JJ)+C18*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +        GMSG*R*T**2*II)
      SE(9,13)=C17*(-T*L2*JJ)+C18*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(9,15)=C17*R2*CC+C18*(T*R2*CC+GMSG*R*T**2*JJ)
C
      SE(10,10)=C19*(DD/T**2+GMSS*R**2*AA)+C20*(-AA)
      SE(10,11)=0.0
      SE(10,13)=0.0
      SE(10,15)=0.0
      SE(10,12)=C19*R*II+C20*(-R*AA)
      SE(10,14)=C19*(L2*FF/T**2+GMSS*R**2*L2*CC)+C20*(-L2*CC)
      SE(10,16)=C19*R*L2*JJ+C20*R*L2*CC
C
      SE(11,11)=C21*(-AA+T**2*JJ)+C22*(-T*FF-GMSS*R**2*T**3*CC-T*AA+
     +        GMSG*R*T**2*II)
      SE(11,12)=0.0
      SE(11,14)=0.0
      SE(11,16)=0.0
      SE(11,13)=C21*(-T*L2*JJ)+C22*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(11,15)=C21*R2*CC+C22*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(12,12)=C23*R*II+C24*(-R*AA)
      SE(12,13)=0.0
      SE(12,15)=0.0
      SE(12,14)=C23*(L2*FF/T**2+GMSS*R**2*L2*CC)+C24*(-L2*CC)
      SE(12,16)=C23*R*L2*JJ+C24*R*L2*CC
C
      SE(13,13)=C25*(-T*L2*JJ)+C26*(L2*FF+GMSS*R**2*T**2*L2*CC)
      SE(13,14)=0.0
      SE(13,16)=0.0
      SE(13,15)=C25*R2*CC+C26*(T*R2*CC+GMSG*R*T**2*R2*JJ)
C
      SE(14,14)=C27*(L2*FF/T**2+GMSS*R**2*L2*CC)+C28*(-L2*CC)
      SE(14,15)=0.0
      SE(14,16)=C27*R*L2*JJ+C28*R*L2*CC
C
      SE(15,15)=C29*R2*CC+C30*(T*R2*CC+GMSG*R*T**2*R2*JJ)
      SE(15,16)=0.0
C
      SE(16,16)=C31*R*L2*JJ+C32*R*L2*CC
C
      DO 380 IE=1,16
      DO 380 JE=IE,16
380   K(IE,JE)=K(IE,JE)+WEIGHT(J)*SE(IE,JE)
      IF (I.EQ.1) GO TO 300
      GO TO 500
```

```
C
C
390   CONTINUE
C
C........INTEGRANDS OF CURVED ELEMENT NONLINEAR STIFFNESS SE1
C
C
C
      UD=GG*DN(1)+HH*(L1*DN(5)-T*DN(3))+II*DN(7)+JJ*(L2*DN(11)
     +   -T*DN(9))
      VD=GG*DN(2)-HH*L1*DN(6)+II*DN(8)-JJ*L2*DN(12)
      WD=KK*DN(3)+GM*DN(9)
      BD=KK*DN(4)+GM*DN(10)
      UG=AA*(DN(1)-DN(7))+BB*(L1*DN(5)-T*DN(3))+CC*(L2*DN(11)
     +   -T*DN(9))
      VG=AA*(DN(2)-DN(8))-BB*L1*DN(6)-CC*L2*DN(12)
      WG=-DN(3)+DN(9)
      BG=-DN(4)+DN(10)
C
      C1=UG+T*WD
      C2=WG-T*UD
      C3=E(N)*A(M)/(R**2*T)
C
      SE(1,1)=C3*(-2.*AA*GG*C1+(AA**2*C2)/T)
      SE(1,2)=C3*(-GG)*VG*AA
      SE(1,3)=C3*(C1*(-GG*T*(-BB+KK)+(AA*(-1.+HH*T**2))/T)
     +         +AA*C2*(-BB+KK))
      SE(1,5)=C3*(C1*L1*(-GG*BB-AA*HH)+(AA*C2*BB*L1)/T)
      SE(1,6)=C3*GG*VG*BB*L1
      SE(1,7)=C3*(C1*AA*(GG-II)-(AA**2*C2)/T)
      SE(1,8)=C3*GG*AA*VG
      SE(1,9)=C3*(C1*(-GG*T*(-CC+GM)+(AA*(1.+JJ*T**2))/T)
     +          +AA*C2*(-CC+GM))
      SE(1,11)=C3*(C1*L2*(-GG*CC-AA*JJ)+(AA*C2*CC*L2)/T)
      SE(1,12)=C3*CC*GG*L2*VG
C
      C4=C3/T
C
      SE(2,2)=C4*AA**2*C2
      SE(2,3)=C4*VG*AA*(-1.+HH*T**2)
      SE(2,5)=C4*VG*AA*(-HH*L1*T)
      SE(2,6)=C4*C2*AA*(-BB*L1)
      SE(2,7)=C4*VG*AA*(-II*T)
      SE(2,8)=-C4*AA**2*C2
      SE(2,9)=C4*AA*VG*(1.+JJ*T**2)
      SE(2,11)=C4*VG*AA*(-JJ*L2*T)
      SE(2,12)=C4*AA*C2*(-CC*L2)
C
```

```
      C5=C4*(-1.+HH*T**2)
      C6=C4*C2*T*(-BB+KK)
C
      SE(3,3)=C5*2.*C1*T*(-BB+KK)+C6*T*(-BB+KK)
      SE(3,5)=C5*C1*BB*L1+C4*C1*T**2*(-HH*L1)*(-BB+KK)+C6*BB*L1
      SE(3,6)=C5*VG*(-BB*L1)
      SE(3,7)=C5*C1*(-AA)+C4*C1*T**2*II*(BB-KK)-C6*AA
      SE(3,8)=C5*VG*(-AA)
      SE(3,9)=C5*C1*T*(-CC+GM)+C4*C1*T*(-BB+KK)*(1.+JJ*T**2)
     +          +C6*T*(-CC+GM)
      SE(3,11)=C5*C1*CC*L2+C4*C1*T**2*(-BB+KK)*(-JJ*L2)
     +          +C6*CC*L2
      SE(3,12)=C5*VG*(-CC*L2)
C
      C7=C4*(-HH*L1*T)
      C8=C4*BB*L1*C2
C
      SE(5,5)=C7*C1*2.*BB*L1+C8*BB*L1
      SE(5,6)=C7*VG*(-BB*L1)
      SE(5,7)=C7*C1*(-AA)+C4*C1*BB*L1*(-II*T)+C8*(-AA)
      SE(5,8)=C7*VG*(-AA)
      SE(5,9)=C7*C1*T*(-CC+GM)+C4*C1*BB*L1*(1.+JJ*T**2)
     +          +C8*T*(-CC+GM)
      SE(5,11)=C7*C1*CC*L2+C4*BB*L1*(-JJ*L2*T)*C1+C8*CC*L2
      SE(5,12)=C7*VG*(-CC*L2)
C
      C9=C4*(-BB*L1)
C
      SE(6,6)=C9*(-BB*L1)*C2
      SE(6,7)=C9*VG*(-II*T)
      SE(6,8)=C9*C2*(-AA)
      SE(6,9)=C9*VG*(1.+JJ*T**2)
      SE(6,11)=C9*VG*(-JJ*L2*T)
      SE(6,12)=C9*C2*(-CC*L2)
C
      SE(7,7)=2.*C4*AA*II*T*C1+C4*AA**2*C2
      SE(7,8)=C4*II*T*VG*AA
      SE(7,9)=C4*(-II*T)*C1*T*(-CC+GM)-C4*AA*C1*(1.+JJ*T**2)
     +          -C4*C2*T*(-CC+GM)*AA
      SE(7,11)=C4*(-II*T)*C1*CC*L2+C4*AA*JJ*L2*T*C1-C4*C2*CC*L2*AA
      SE(7,12)=C4*II*T*VG*CC*L2
C
      SE(8,8)=C4*AA**2*C2
      SE(8,9)=C4*(-AA*VG)*(1.+JJ*T**2)
      SE(8,11)=C4*AA*VG*JJ*L2*T
      SE(8,12)=C4*C2*AA*CC*L2
C
      C10=C4*(1.+JJ*T**2)
```

```
      C11=C4*T*(-CC+GM)*C2
C
      SE(9,9)=C10*2.*C1*T*(-CC+GM)+C11*T*(-CC+GM)
      SE(9,11)=C10*C1*CC*L2+C4*C1*T*(-CC+GM)*(-JJ*L2*T)+C11*CC*L2
      SE(9,12)=C10*VG*(-CC*L2)
C
      SE(11,11)=C4*(-2.*JJ*L2*L2*T*CC*C1+CC*CC*L2*L2*C2)
      SE(11,12)=C4*JJ*L2*T*VG*CC*L2
C
      SE(12,12)=C4*C2*CC*L2*CC*L2
C
      DO 400 IE=1,12
      DO 400 JE=IE,12
400   K(IE,JE)=K(IE,JE)+WEIGHT(J)*SE(IE,JE)
      IF (I.EQ.1) GO TO 300
      GO TO 500
C
C
410   CONTINUE
C
C........INTEGRANDS ON CURVED ELEMENT NONLINEAR STIFFNESS SE2
C
C
      WD=KK*DN(3)+GM*DN(9)
      UG=AA*(DN(1)-DN(7))+BB*(L1*DN(5)-T*DN(3))+CC*(L2*DN(11)-T*DN(9))
      VG=AA*(DN(2)-DN(8))-L1*BB*DN(6)-L2*CC*DN(12)
C
      C1=E(N)*A(M)*AA/(2.*R**3*T**3)
      C2=3.*(UG+T*WD)**2+VG**2
      C3=2.*VG*(UG+T*WD)
C
      SE(1,1)=C1*C2*AA
      SE(1,2)=C1*C3*AA
      SE(1,3)=C1*C2*(-T*BB+T*KK)
      SE(1,4)=0.0
      SE(1,5)=C1*C2*L1*BB
      SE(1,6)=C1*C3*(-L1*BB)
      SE(1,7)=-SE(1,1)
      SE(1,8)=-SE(1,2)
      SE(1,9)=C1*C2*(-T*CC+T*GM)
      SE(1,10)=0.0
      SE(1,11)=C1*C2*L2*CC
      SE(1,12)=C1*C3*(-L2*CC)
C
      C4=3.*VG**2+(UG+T*WD)**2
C
      SE(2,2)=C1*C4*AA
      SE(2,3)=C1*C3*(-T*BB+T*KK)
```

```
      SE(2,4)=0.0
      SE(2,5)=-SE(1,6)
      SE(2,6)=C1*C4*(-L1*BB)
      SE(2,7)=-SE(1,2)
      SE(2,8)=-SE(2,2)
      SE(2,9)=C1*C3*(-T*CC+T*GM)
      SE(2,10)=0.0
      SE(2,11)=-SE(1,12)
      SE(2,12)=C1*C4*(-L2*CC)
C
      C5=E(N)*A(M)*T*(-BB+KK)/(2.*R**3*T**3)
C
      SE(3,3)=C5*C2*T*(-BB+KK)
      SE(3,4)=0.0
      SE(3,5)=C5*C2*L1*BB
      SE(3,6)=C5*C3*(-L1*BB)
      SE(3,7)=-SE(1,3)
      SE(3,8)=-SE(2,3)
      SE(3,9)=C5*C2*T*(-CC+GM)
      SE(3,10)=0.0
      SE(3,11)=C5*C2*L2*CC
      SE(3,12)=C5*C3*(-L2*CC)
C
      SE(4,4)=0.0
      SE(4,5)=0.0
      SE(4,6)=0.0
      SE(4,7)=0.0
      SE(4,8)=0.0
      SE(4,9)=0.0
      SE(4,10)=0.0
      SE(4,11)=0.0
      SE(4,12)=0.0
C
      C6=E(N)*A(M)*L1*BB/(2.*R**3*T**3)
C
      SE(5,5)=C6*C2*L1*BB
      SE(5,6)=C6*C3*(-L1*BB)
      SE(5,7)=C6*C2*(-AA)
      SE(5,8)=C6*C3*(-AA)
      SE(5,9)=C6*C2*(-T*CC+T*GM)
      SE(5,10)=0.0
      SE(5,11)=C6*C2*L2*CC
      SE(5,12)=C6*C3*(-L2*CC)
C
      SE(6,6)=C6*C4*L1*BB
      SE(6,7)=-SE(5,8)
      SE(6,8)=-SE(2,6)
      SE(6,9)=-C6*C3*(-T*CC+T*GM)
```

[illegible] 132
[illegible] 132
[illegible] 132
[illegible]

```
      SE(6,10)=0.0
      SE(6,11)=SE(5,12)
      SE(6,12)=C6*C4*L2*CC
C
      SE(7,7)=SE(1,1)
      SE(7,8)=SE(1,2)
      SE(7,9)=-SE(1,9)
      SE(7,10)=0.0
      SE(7,11)=-SE(1,11)
      SE(7,12)=-SE(1,12)
C
      SE(8,8)=SE(2,2)
      SE(8,9)=-SE(2,9)
      SE(8,10)=0.0
      SE(8,11)=-SE(2,11)
      SE(8,12)=-SE(2,12)
C
      C8=E(N)*A(M)*(-T*CC+T*GM)/(2.*R**3*T**3)
C
      SE(9,9)=C8*C2*(-T*CC+T*GM)
      SE(9,10)=0.0
      SE(9,11)=C8*C2*L2*CC
      SE(9,12)=C8*C3*(-L2*CC)
C
      SE(10,10)=0.0
      SE(10,11)=0.0
      SE(10,12)=0.0
C
      C9=E(N)*A(M)*L2*CC/(2.*R**3*T**3)
C
      SE(11,11)=C9*C2*L2*CC
      SE(11,12)=C9*C3*(-L2*CC)
C
      SE(12,12)=C9*C4*L2*CC
C
      DO 420 IE=1,12
      DO 420 JE=IE,12
420   K(IE,JE)=K(IE,JE)+WEIGHT(J)*SE(IE,JE)
      IF (I.EQ.1) GO TO 300
500   CONTINUE
C
C........MAKE INTERVAL CORRECTION AND RETURN
CCCC
      IF (IPAR.GE.3) GO TO 1
      IQ=16
      GO TO 2
1     IQ=12
2     CONTINUE
```

```
CCCC
      DO 550 I=1,IQ
      DO 550 J=I,IQ
      SE(I,J)=C*K(I,J)
550   SE(J,I)=SE(I,J)
CCCC
C
      IF (IPAR.EQ.1) CALL REOCON
      IF (IPAR.EQ.2) CALL REOCON
C
CCCC
      RETURN
C
2000  FORMAT('1',15HINVALID MP USED///7H GAUSS=,F4.1)
C
      END
```