A COMPARATIVE STUDY OF BOUNDARY ELEMENT SHAPE FUNCTIONS FOR LINEAR ELASTOSTATICS

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ABSTRACT

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Many types of boundary elements are available in the boundary element method. With so many available, however, one needs to know what elements are best suited for different problem types.

This thesis begins to explore this issue. Three simple elements are applied to a select group of problems and compared to exact results. The first element utilizes constant displacement and constant traction approximations, the second employs linear displacements and constant tractions, and the third employs linear approximations for both displacements and tractions. It is found that an element which uses linear displacements and constant tractions yields the most consistent and accurate results.

To the loving memory of my mother, Evelyn

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I wish to thank my academic advisor, Dr. Nicholas Altiero, for his inspiration and invaluable research notes. I also thank my typist, Carrie Coker, for her patience and hard work. Finally, a special thank you to my father, Richard Zwier, for his support and thoughtful guidance.

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CHAPTER 1

INTRODUCTION

Engineers and scientists routinely develop mathematical models to describe the systems they wish to study. These models, however, typically involve differential equations and boundary conditions which are not easily solved. Therefore, in order to obtain a solution, some form of approximate method is employed.

Many such methods exist today. However, the three most powerful are finite differences, finite elements and boundary elements. Although this thesis deals with boundary elements, all three will be discussed briefly because, when viewed historically, they show a trend which, in part, prompted this work.

Before examining the individual methods, it should be pointed out that they possess common characteristics. First, they each require the region of interest to be discretized using a set of points called nodes. Second, through differing mathematical approaches, each method reduces the system of differential equations to a system of algebraic equations, which, when solved, give values for the physical variables at the nodes.

Of the three subject methods, finite differences evolved first (1). It is probably the most natural way to approximate a set of differential equations because it utilizes the basic definition of the derivative. Applying this definition and a finite nodal spacing to the derivatives in the governing equations results in a system of algebraic difference equations.

In theory, finite differences may be applied to any differential equation. In practice, however, it is somewhat limited. This is due to the fact that certain boundary conditions can be cumbersome to incorporate and it is difficult to work with curved or irregular boundaries. Also, since the solution accuracy is primarily dependent on nodal spacing, many equations are necessary to achieve good results. On the other hand, the method works well on problems involving first and second derivatives such as heat transfer, fluid mechanics and two dimensional problems with straight boundaries (2,3).

The finite element method is by far the most popular method in use today. It is, in a sense, a progression from finite differences in that it approximates the physical domain itself with small regions, or elements, which simulate the behavior of the portion of the domain they represent. The differential equations are then applied to each element in turn and rewritten in terms of variational, weighted residual, or virtual work expressions (2).

Finite elements have an advantage over finite differences in that irregular regions pose no significant problems. The method allows for a great deal of flexibility in terms of element types and shapes and, therefore, may be applied to many types of problems and regions. This kind of flexibility is partly responsible for the method's popularity and has prompted the development of many types of elements (2). Another reason for its popularity lies in the fact that it has its roots in the study of differential equations which was well established in the nineteenth century (1) and, therefore, the underlying mathematics are generally understood today by practicing engineers and scientists.

The boundary element method is, in a sense, a progression from the finite element method because it attempts to integrate the differential equations before it discretizes the region. The result is a system of integral equations which involves only the boundary values of the system. Thus, only the region boundary needs to be discretized and the problem dimensionality is reduced by one.

It should be noted that the boundary element method is similar to the finite element method in that they both can be considered weighted residual methods (4,14). However, the boundary element method does have the advantage of only requiring boundary discretization and thus fewer nodes.

In addition, numerical integration is more accurate than numerical differentiation and there is little or no numerical differentiation in the boundary element method (1). Perhaps the biggest disadvantage for the method is that the study of integral equations is more recent and mathematically more difficult than the study of differential equations. This has, until recently, hampered its development.

Integral equations have been studied rigorously since the turn of the century (1,5). However, the contributions which have developed into the boundary element method have come as recently as the 1950's and 1960's (5,15). The nature of the mathematics involved in these works prevented them from being recognized by the majority of engineers and applied scientists. However, they were recognized by some, and in the late 1960's and early 1970's, papers began to be published which applied integral equations to the solution of engineering problems (6,7,8,9,13).

Throughout the 1970's, the number of papers increased and the term "boundary elements" emerged from works by Brebbia and others within the department of civil engineering at the University of Southampton (10). By the end of the 1970's, it became evident that the boundary element method was becoming a legitimate analysis tool and international conferences on the topic were established (11). These conferences are designed to solicit work in the field and promote the method. To that end, they have been successful as evidenced by the increasing number and diversity of the papers presented at each conference (11). Now in the late 1980's, efforts such as the BETECH conferences (12) have begun which are designed to increase the popularity of the method and bridge the gap between industry and academia.

Efforts such as these are increasing the popularity of the method and already, several commercial software packages are available (13). This is not to say that boundary elements are replacing finite elements. It does appear, however, that they will reach equal levels of development and coexist as complementary methods. In fact, since most problems solved by finite elements may also be solved by boundary elements (1), such a coexistence is desirable to make use of the relative merits of both methods and advance the quality of solutions.

Similarly, within each method it is desirable to use different element types in a complementary manner depending on the problem characteristics. By knowing when to apply certain element types, more efficient and accurate solutions may be obtained. This does not mean that each problem should have its own type of element. It does, however, suggest that efforts should be made to study the applicability of basic element types and indeed such works have begun to appear in the literature (17).

The purpose of this thesis then is to examine three types of boundary elements in the direct boundary element method for linear isotropic elasticity. Each element type will be applied to a range of problem types and assessed in terms of accuracy and quality of solution. It is hoped that this work will help establish guidelines for element applicability and show the usefulness of simpler, extrapolated models.

CHAPTER 2

THE BOUNDARY ELEMENT METHOD

There are three natural formulations for the boundary element method. The indirect formulation expresses the integral equations in terms of a unit singular solution of the original differential equations. This solution is distributed over the boundary by way of density functions. The density functions have no physical significance and once they are obtained from the numerical solution of the integral equations, the values of the physical variables may be found by simple integration at any point within the region.

The semi-direct formulation utilizes unknown functions, analogues to airy stress functions, as unknowns in the integral equations. Once these functions are found, the physical variables may be obtained by differentiation at any point within the region. This is the only numerical differentiation required in the boundary element method. As pointed out in Chapter 1, numerical differentiation is less accurate than numerical integration and, therefore, the semi-direct formulation is less useful than the other two formulations.

The direct formulation expresses the integral equations in terms of the actual physical variables. Numerical solution of the integral equations gives the boundary values of the physical variables. Interior values are subsequently found by simple numerical integration. This thesis deals specifically with the direct formulation because it is well suited to problems of plane isotropic elasticity. We will now proceed to develop the basic integral equations and in Chapter 3, we will develop the numerical solution schemes for these equations.

Consider the following planar region:

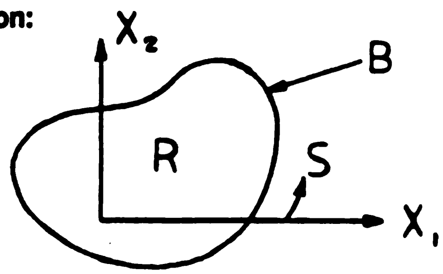


Figure 1. Planar Region

At any point $\underline{x}$ in R, we can write the equilibrium equations as

$$\sigma_{11,1} + \sigma_{12,2} + X_1 = 0$$
$$\sigma_{12,1} + \sigma_{22,2} + X_2 = 0 \qquad [1]$$

where $\underline{\sigma}$ is the stress tensor and $\underline{X}$ is the body force vector. Also, the notation $\sigma_{ij,j}$ indicates differentiation of σ_{ij} with respect to coordinate direction j.

Hooke's law for plane stress is given by:

$$\sigma_{11} = 2G(u_{1,1} + \nu u_{2,2}) / (1 - \nu)$$
$$\sigma_{22} = 2G (u_{2,2} + \nu u_{1,1}) / (1 - \nu) \qquad [2]$$
$$\sigma_{12} = G (u_{1,2} + u_{2,1})$$

where $\underline{u}$ is the displacement vector, G is the shear modulus and ν is Poisson's ratio.

Substitution of eqs. [2] into eqs. [1] results in Navier's equations.

$$(u_{1,11} + u_{2,21}) (1 + \nu) / (1 - \nu) + \nabla^2 u_1 + X_1 / G = 0$$
$$(u_{1,12} + u_{2,22}) (1 + \nu) / (1 - \nu) + \nabla^2 u_2 + X_2 / G = 0 \qquad [3]$$

Shifting attention now from the region to the boundary B, we have the traction vector t given by:

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2$$
$$t_2 = \sigma_{12} n_1 + \sigma_{22} n_2 \qquad [4]$$

for any surface defined by unit normal $\underline{n}$. Inserting eq. [2] into eq. [4] gives:

$$t_1 = G [u_{1,n} + u_{1,1} n_1 + u_{2,1} n_2 + 2\nu (u_{1,1} + u_{2,2}) n_1 / (1 - \nu)]$$
$$t_2 = G [u_{2,n} + u_{1,2} n_1 + u_{2,2} n_2 + 2\nu (u_{1,1} + u_{2,2}) n_2 / (1 - \nu)] \qquad [5]$$

Making use of the coordinate ,s, measured along B, we specify the boundary conditions as

$$u_i = g_i(s) \text{ or } t_i / G = h_i(s) \qquad [6]$$

for $i = 1,2$.

We now have the basic formulation of the elasticity problem with field equations, eq.[3], and boundary conditions, eq. [6]. However, we desire eq. [3] in a more convenient form. To accomplish this, we define a vector function $\underline{\phi}$, such that:

$$u_1 = \nabla^2\phi_1 - (1+\upsilon)(\phi_{1,11} + \phi_{2,12}) / 2$$
$$u_2 = \nabla^2\phi_2 - (1+\upsilon)(\phi_{1,21} + \phi_{2,22}) / 2 \qquad [7]$$

and substitute into eq. [3] to yield the inhomogeneous biharmonic equation

$$\nabla^2(\nabla^2\phi_i) = X_i / G \qquad [8]$$

for $i = 1,2$.

The biharmonic operator is well understood and is known to possess a principal solution which satisfies

$$\nabla^2(\nabla^2\Phi_i) = -\ \delta(\underline{x} - \underline{\xi})\, e_i / G \qquad [9]$$

for $i = 1,2$.. Here $\underline{e}$ is a unit vector and $\delta(\underline{x} - \underline{\xi})$ is a Dirac delta function which represents a point singularity at the source point ξ. For equations such as eq. [9], the solution is a function only of the distance between the source point ξ and the field point $\underline{x}$. This distance is given by:

$$\rho = [(x_1 - \xi_1)^2 + (x_2 - \xi_2)^2]^{0.5}\ .$$

The solution to eq. [9] is then:

$$\Phi_i = -\rho^2 \ln(\rho)\, e_i / 8\pi G. \qquad [10]$$

Substitution of eq. [10] for ϕ_i in eq. [7] gives the displacement field for the principal solution as:

$$U_1 = \{[-(3-\upsilon)\ln\rho + (1+\upsilon)q_1^2]\, e_1 + (1+\upsilon)\, q_1 q_2 e_2\} / 8\pi G$$
$$U_2 = \{[-(3-\upsilon)\ln\rho + (1+\upsilon)q_2^2]\, e2 + (1+\upsilon)\, q_1 q_2 e_1\} / 8\pi G \qquad [11]$$

Here we have defined $q_1$and q_2 as:

$$q_1 = (x_1 - \xi_1)/\rho \qquad \text{and } q_2 = (x_2 - \xi_2)/\rho$$

Similarly, substitution of eq. [11] for u_i into eq. [5] gives the traction field for the principal solution

$$T_1 = \{[2(1+\upsilon)(-n_1 q_1^3 + n_2 q_2^3) - (1-\upsilon)\, n_1 q_1 - (3-\upsilon)\, n_2 q_2]\, e_1$$
$$+ [2(1+\upsilon)(n_2 q_1^3 + n_1 q_2^3) - (3+\upsilon)\, n_2 q_1 - (1+3\upsilon)\, n_1 q_2]\, e_2\} / 4\pi\rho$$
$$T_2 = \{[2(1+\upsilon)(n_2 q_1^3 + n_1 q_2^3) - (1+3\upsilon)\, n_2 q_1 - (3+\upsilon)\, n_1 q_2]\, e_1 \qquad [12]$$
$$+ [2(1+\upsilon)(n_1 q_1^3 - n_2 q_2^3) - 3+\upsilon)\, n1\, q_1 - (1-\upsilon)\, n_2 q_2]\, e_2\} / 4\pi\rho.$$

If we now define the displacement and traction vectors for the principal solution to be:

$$U_i(\underline{x}) = U_{i1}(\underline{x},\underline{\xi})\, e_1(\underline{\xi}) + U_{i2}(\underline{x},\underline{\xi})\, e_2(\underline{\xi})$$
$$T_i(\underline{x}) = T_{i1}(\underline{x},\underline{\xi})\, e_1(\underline{\xi}) + T_{i2}(\underline{x},\underline{\xi})\, e_2(\underline{\xi}), \qquad [13]$$

then we can write

$$U_{11} = [-(3-\nu)\ln\rho + (1+\nu)q_1^2] / 8\pi G$$
$$U_{12} = (1+\nu)\, q_1 q_2 / 8\pi G$$
$$U_{21} = (1+\nu)\, q_1 q_2 / 8\pi G \qquad [14]$$
$$U_{22} = [-(3-\nu)\ln\rho + (1+\nu)q_2^2] / 8\pi G$$

and

$$T_{11} = [2(1+\nu)(-n_1 q_1^3 + n_2 q_2^3) - (1-\nu)\, n_1 q_1 - (3+\nu)\, n_2 q_2] / 4\pi\rho$$
$$T_{12} = [2(1+\nu)(n_2 q_1^3 + n_1 q_2^3) - (3+\nu)\, n_2 q_1 - (1+3\nu)\, n_1 q_2] / 4\pi\rho$$
$$T_{21} = [2(1+\nu)(n_2 q_1^3 + n_1 q_2^3) - (1+3\nu)\, n_2 q_1 - (3+\nu)\, n_1 q_2] / 4\pi\rho \qquad [15]$$
$$T_{22} = [2(1+\nu)(n_1 q_1^3 - n_2 q_2^3) - (3+\nu)\, n_1 q_1 - (1-\nu)\, n_2 q_2] / 4\pi\rho.$$

The quantities given by eq. [14] and eq. [15] represent the effects caused at the field point $\underline{x}$ by a unit force at the point $\underline{\xi}$ in the infinite plane. Thus U_{i1}, $i = 1, 2$, are the displacements and tractions respectively at a point $\underline{x}$ due to a unit force in the x_1- direction applied at $\underline{\xi}$. Similarly, U_{i2} and T_{i2} are generated at $\underline{x}$ by a unit force in the x_2 - direction at $\underline{\xi}$.

We now recall Betti's reciprocal work theorem which states that, if two elastic equilibrium states (X_i, t_i, u_i) and (X_i^*, t_i^*, u_i^*) exist in a region R bounded by B, then the work done by the forces of the first system on the displacements of the second is equal to the work done by the forces of the second on the displacements of the first. Thus, we write:

$$\int_B t_i^*(\underline{x})\, u_i(\underline{x})\, ds(\underline{x}) + \int_R X_i^*(\underline{x})\, u_i(\underline{x})\, da(\underline{x})$$
$$= \int_B t_i(\underline{x})\, u_i^*(\underline{x})\, ds(\underline{x}) + \int_R X_i(\underline{x})\, u_i^*(\underline{x})\, da(\underline{x}) \qquad [16]$$

for $i = 1,2$ and where we have employed Einstein's summation convention, i.e. we infer summation over repeated indices.

Now, if we choose the principal solution to be one of the two elastic states, then

$$u_i^* = U_i = U_{ij}(\underline{x}, \underline{\xi})\, e_j(\underline{\xi})$$

$$t_i^* = T_i = T_{ij}(\underline{x}, \underline{\xi})\, e_j(\underline{\xi})$$

$$X_i^* = \delta(\underline{x} - \underline{\xi})\, e_i(\underline{\xi}).$$

Substituting these into eq. [16] gives

$$e_j(\underline{\xi}) \int_B T_{ij}(\underline{x},\underline{\xi})\, u_i(\underline{x})\, ds(\underline{x}) + e_i(\underline{\xi}) \int_R \delta(\underline{x} - \underline{\xi})\, u_i(\underline{x}) da(\underline{x})$$
$$= e_j(\underline{\xi}) \left[\int_B U_{ij}(\underline{x},\underline{\xi})\, t_i(\underline{x})\, ds(\underline{x}) + \int_R U_{ij}(\underline{x},\underline{\xi})\, X_i(\underline{x})\, da(\underline{x}) \right]. \qquad [17]$$

We can write

$$e_i(\underline{\xi}) \int_R \delta(\underline{x} - \underline{\xi})\, u_i(\underline{x})\, da(\underline{x}) = e_i(\underline{\xi})\, u_i(\underline{\xi}) = e_j(\underline{\xi})\, u_j(\underline{\xi})$$

provided $\underline{\xi}$ is in R. Then, equating coefficients of e_j, we obtain

$$u_j(\underline{\xi}) = -\int_B u_i(\underline{x})\, T_{ij}(\underline{x},\underline{\xi})\, ds(\underline{x}) + \int_B t_i(\underline{x})\, U_{ij}(\underline{x},\underline{\xi})\, ds(\underline{x})$$
$$+ \int_R X_i(\underline{x})\, U_{ij}(\underline{x},\underline{\xi})\, da(\underline{x}). \qquad [18]$$

For convenience, at this point, we recognize that

$$U_{ij}(\underline{\xi},\underline{x}) = U_{ij}(\underline{x},\underline{\xi}) = U_{ji}(\underline{x},\underline{\xi})$$

$$T_{ij}(\underline{\xi},\underline{x}) = -T_{ij}(\underline{x},\underline{\xi})$$

in eq. [14] and eq. [15] provided that we let $\underline{n} = \underline{n}(\underline{\xi})$ in eq. [15]. This allows us to interchange $\underline{x}$ and $\underline{\xi}$ and rewrite eq. [18] as

$$u_i(\underline{x}) = \int_B u_j(\underline{\xi})\, T_{ji}(\underline{x},\underline{\xi})\, ds(\underline{\xi}) + \int_B t_j(\underline{\xi})\, U_{ij}(\underline{x},\underline{\xi})\, ds(\underline{\xi})$$
$$+ \int_R X_j(\underline{\xi})\, U_{ij}(\underline{x},\underline{\xi})\, da(\underline{\xi}) \qquad [19]$$

for $\underline{x}$ in R and $i = 1,2$.

Equation [19] also applies for $\underline{x}$ on B. However, when $\underline{x}$ is on B and $\underline{\xi}$ approaches $\underline{x}$, the integrands in the boundary integrals become singular. Specifically, $T_{ji}(\underline{x},\underline{\xi})$ varies as $1/\rho$ and $U_{ij}(\underline{x},\underline{\xi})$ varies as $\ln \rho$. Since $1/\rho$ is not integrable, we need to integrate around $\underline{x}$ as follows:

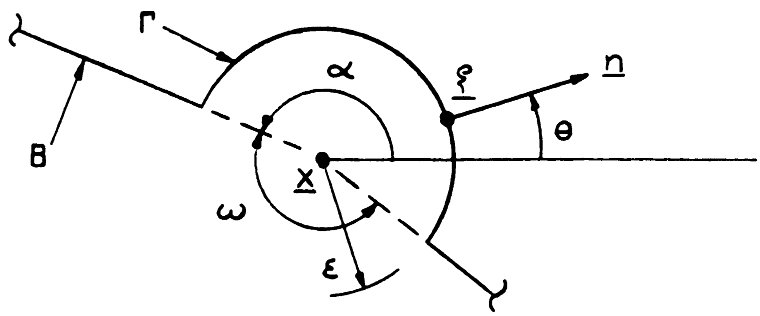


Figure 2. Integration Around Singular Point

and take the limit as ε goes to O. Notice that if the boundary has no corner, $\omega = \pi$

Now on Γ we have

$$q_1 = -\cos\theta \qquad q_2 = -\sin\theta$$

$$n_1 = \cos\theta \qquad n_2 = \sin\theta$$

$$\rho = e \qquad ds = \varepsilon d\theta.$$

If we think of the boundary as being in two parts, i.e., B and B-Γ, then the $1/\rho$ type integral becomes

$$\int_B u_j T_{ji}\, ds = \lim_{\varepsilon \to 0} \left[\int_{B-\Gamma} u_j T_{ji}\, ds + \int_\Gamma u_j T_{ji}\, ds \right].$$

In the limit as ε goes to 0, the B- Γ integral becomes a cauchy principal value integral, and we have

$$\int_B u_j T_{ji}\, ds = ⨍_B u_j T_{ji}\, ds + \lim_{\varepsilon \to 0} \int_\Gamma u_j T_{ji}\, ds.$$

If we define the coefficient β_{ji} such that

$$\lim_{\varepsilon \to 0} \int_\Gamma u_j T_{ji}\, ds = \beta_{ji} u_j$$

and let $\alpha_{ji} = \delta_{ji} - \beta_{ji}$ at $\underline{x}$ on B, we obtain

$$\alpha_{ji} u_j - ⨍_B u_j T_{ji}\, ds = \int_B t_j U_{ij}\, ds + \int_R X_i U_{ij}\, da. \qquad [20]$$

Taking $\underline{X} = 0$ for convenience, we have

$$\alpha_{ji} u_j - ⨍_B u_j T_{ji}\, ds = \int_B t_j U_{ij}\, ds \qquad [21]$$

where U_{ij} and T_{ji} are given by eq. [14] and eq. [15], respectively.

Equation [21] is the desired boundary integral equation. After solving it numerically, we will have the displacement and traction values everywhere on B. Then, to find the displacements at any point within R, we simply integrate, i.e.

$$u_i = \int_B u_j T_{ji}\, ds + \int_B t_j U_{ij}\, ds \qquad [22]$$

Next, employing eq. [2], we have the stresses at any point within R given by

$$\sigma_{ik} = \int_B u_j S_{jki}\, ds + \int_B t_j D_{ikj}\, ds \qquad [23]$$

where S_{jki} and D_{ikj} are given by

$$S_{111} = G(1 + \upsilon)[(8q_1^4 - 4q_1^2 - 1)\, n_1 + 2q_1q_2\,(4q_1^2 - 1)\, n_2] / 2\pi\rho^2$$

$$S_{211} = G(1 + \upsilon)[2q_1q_2(4q_1^2 - 1)\, n_1 + (8q_1^2q_2^2 - 1)\, n_2] / 2\pi\rho^2$$

$$S_{122} = G(1 + \upsilon)[(8q_1^2q_2^2 - 1)\, n_1 + 2q_1q_2\,(4q_2^2 - 1)\, n_2] / 2\pi\rho^2$$

$$S_{222} = G(1 + \upsilon)[2q_1q_2\,(4q_2^2 - 1)\, n_1 + (8q_2^4 - 4q_2^2 - 1)\, n_2] / 2\pi\rho^2 \qquad [24]$$

$$S_{121} = G(1 + \upsilon)[2q_1q_2\,(4q_1^2 - 1)\, n_1 + (8q_1^2q_2^2 - 1)\, n_2] / 2\pi\rho^2$$

$$S_{221} = G(1 + \upsilon)[(8q_1^2q_2^2 - 1)\, n_1 + 2q_1q_2\,(4q_2^2 - 1)\, n_2] / 2\pi\rho^2$$

$$D_{111} = [-2\,(1 + \upsilon)\, q_1^3 - (1 - \upsilon)\, q_1] / 4\pi\rho$$

$$D_{121} = [2(1 + \upsilon)\, q_2^3 - (3 + \upsilon)\, q_2] / 4\pi\rho$$

$$D_{221} = [2(1 + \upsilon)\, q_1^3 - (1 + 3\upsilon)\, q_1] / 4\pi\rho$$

$$D_{112} = [2(1 + \upsilon)\, q_2^3 - (1 + 3\upsilon)\, q_2] / 4\pi\rho \qquad [25]$$

$$D_{122} = [2(1 + \upsilon)\, q_1^3 - (3 + \upsilon)\, q_1] / 4\pi\rho$$

$$D_{222} = [-2(1 + \upsilon)\, q_2^3 - (1 - \upsilon)\, q_2] / 4\pi\rho$$

CHAPTER 3

NUMERICAL TREATMENT

3.1 BOUNDARY APPROXIMATION

To numerically solve the boundary integral equation, eq [21], we will first approximate the boundary B with N straight line segments as shown below.

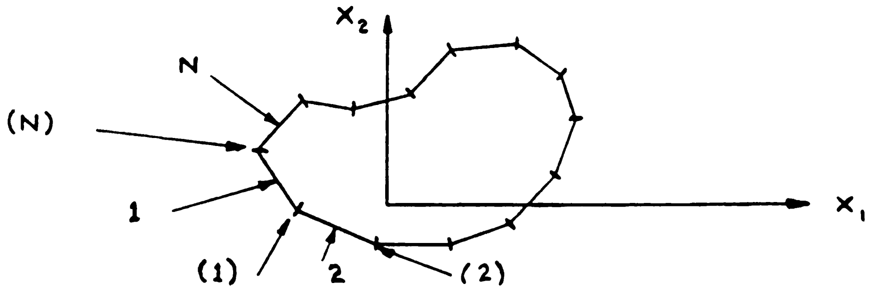


Figure 3. Boundary Approximation

Now for any boundary source point, m , and boundary field point , n , we define the following

$\underline{x}^n$ = Element Field Point

$\underline{x}^{(n)}$ = Nodal Field Point

$\underline{x}^m$ = Element Source Point

$\underline{x}^{(m)}$ = Nodal Source Point

Δs^m = Length of Element m

$= [(x_1^{(m)} - x_1^{(m-1)})^2 + (x_2^{(m)} - x_2^{(m-1)})^2]^{0.5}$

$\underline{n}^m$ = Unit Normal to Element m

$n_1 = (x_2^{(m)} - x_2^{(m-1)}) / \Delta s^m$

$n_2 = (x_1^{(m-1)} - x_1^{(m)}) / \Delta s^m$

ρ = Distance Between Source Point and Field Point

$q_1 = (x_1^{(n)} - x_1^{(m)}) / \rho$ or $(x_1^n - x_1^m) / \rho$

$q_2 = (x_2^{(n)} - x_2^{(m)}) / \rho$ or $(x_2^n - x_2^m) / \rho$

Here the "element points" are at the midpoints of boundary segments and element quantities take their midpoint values.

Now consider a typical straight segment; m ,

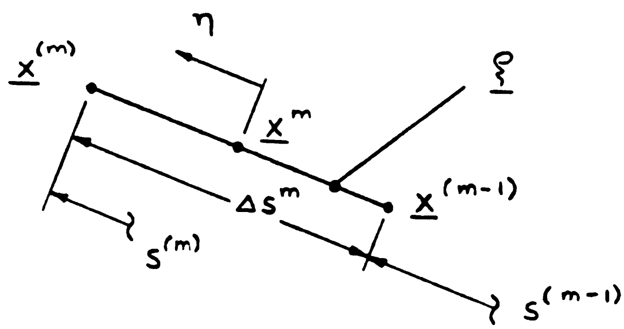


Figure 4. Local Coordinate

with local coordinate, η , where

$$
\begin{aligned}
&-1 \leq \eta \leq 1 \\
&ds = 0.5\,\Delta s^m\, d\eta \qquad\qquad [26] \\
&\xi = 0.5\,\mathbf{x}^{(m-1)}(1-\eta) + 0.5\,\mathbf{x}^{(m)}(1+\eta).
\end{aligned}
$$

We can now rewrite the boundary integral equation

$$\alpha_{ji}\, u_j - \int_B u_j T_{ji}\, ds = \int_B t_j u_{ij}\, ds. \qquad [21]$$

at a given boundary point, $\mathbf{x}$, as

$$
\begin{aligned}
&\alpha_{ji}(\mathbf{x})\, u_j(\mathbf{x}) - 0.5 \sum_{m=1}^{N} [\, \Delta s^m \int_m u_j(\eta)\, T_{ji}(\mathbf{x}, \eta)\, d\eta] \\
&= 0.5 \sum_{m=1}^{N} [\, \Delta s^m \int_m t_j(\eta)\, U_{ij}(\mathbf{x}, \eta)\, d\,\eta.] \qquad [27]
\end{aligned}
$$

3.2 NUMERICAL MODELS

Next, consider three cases on segment m

(a) $u_j = u_j^m$

$t_j = t_j^m$

(b) $u_j = 0.5\, u_j^{(m-1)} (1 - \eta) + 0.5\, u_j^{(m)} (1 + \eta)$

$t_j = t_j^m$

(c) $u_j = 0.5\, u_j^{(m-1)} (1 - \eta) + 0.5\, u_j^{(m)} (1 + \eta)$

$t_j = 0.5\, t_j^{(m-1)} (1 - \eta) + 0.5\, t_{j(m)} (1 + \eta)$

Notice the differences among the cases. Case (a) is the simplest because it assumes both the displacements and tractions to be constant over segment m. Case (b) is slightly more complex in that the displacements are allowed to vary linearly. Case (c) takes the next logical progression and allows the tractions to vary linearly also. Clearly, further progressions are possible such as quadratic and cubic variations, however, for our purposes, these three are sufficient. Also, keep in mind that the constant quantities refer to values at the midpoint of segment m.

We now rewrite eq. [27] for the field point in each of the three cases.

(a) Let $\underline{x} = \underline{x}^n$

$$\hat{\alpha}_{ji}^{n}\, u_j^n - 0.5 \sum_{\substack{m=1 \\ m \neq n}}^{N} \left[\Delta s^m \int T_{ji} (\underline{x}^n, \eta)\, d\eta \right] u_j^m \qquad [28]$$

$$= 0.5 \sum_{m=1}^{n} \left[\Delta s^m \int_m U_{ij} (\underline{x}^n, \eta)\, d\eta \right] t_j^m$$

(b) Let $\underline{x} = \underline{x}^{(n)}$

$$\hat{\alpha}_{ji}^{(n)}\, u_j^{(n)} - 0.25 \sum_{\substack{m=1 \\ m \neq n+1}}^{N} \left[\Delta s^m \int_m (1 - \eta)\, T_{ji} (\underline{x}^{(n)}, \eta)\, d\eta \right] u_j^{(m-1)}$$

$$- 0.25 \sum_{\substack{m=1 \\ m \neq n}}^{N} \left[\Delta s^m \int_m (1 + \eta)\, T_{ji} (\underline{x}^{(n)}, \eta)\, d\eta \right] u_j^{(m)} \qquad [29]$$

$$= 0.5 \sum_{m=1}^{N} \left[\Delta s^m \int_m U_{ij} (\underline{x}^{(n)}, \eta)\, d\eta \right] t_j^m$$

(c) Let $x = x^{(n)}$

$$\hat{\alpha}_{ji}^{(n)} u_j^{(n)} - 0.25 \sum_{\substack{m=1 \\ m \neq n+1}}^{N} [\Delta s^m \int_m (1 - \eta) T_{ji} (\underline{x}^{(n)}, \eta) \, d\eta] \, u_j^{(m-1)}$$

$$- 0.25 \sum_{\substack{m=1 \\ m \neq n}}^{N} [\Delta s^m \int_m (1 + \eta) T_{ji} (\underline{x}^{(n)}, \eta) \, d\eta] \, u_j^{(m)} \qquad [30]$$

$$= 0.25 \sum_{m=1}^{N} [\Delta s^m \int_m (1 - \eta) U_{ij} (\underline{x}^{(n)}, \eta) \, d\eta] \, t_j^{(m-1)}$$

$$+ 0.25 \sum_{m=1}^{N} [\Delta s^m \int_m (1 + \eta) U_{ij} (\underline{x}^{(n)}, \eta) \, d\eta] \, t_j^{(m)}$$

Notice that any integrals multiplying u_j^n or $u_j^{(n)}$ were incorporated into the coefficients α_{ji}^n or $\alpha_{ji}^{(n)}$ in eqs. [28], [29] and [30]. Also notice that the displacements and tractions are now either nodal or midpoint values and are, therefore, taken outside the integrals. This allows us to express the equations in matrix form as

$$\text{Case (a):} \quad [A_a] \{u\}_e = [B_a] \{t\}_e$$

$$\text{Case (b):} \quad [A_b] \{u\}_n = [B_b] \{t\}_e \qquad [31]$$

$$\text{Case (c):} \quad [A_c] \{u\}_n = [B_c] \{t\}_n$$

Essentially, we now have our numerical models. A more convenient form, however, would be to have

$$[F] \{x\} = \{b\}$$

for each case where [F] and {b} are known. We can obtain this desired form by recognizing that {u} and {t} are not known everywhere on the boundary and simply rearrange the equations to have all unknown quantities on the left and known quantities on the right.

For convenience, we will make one further adjustment to the models. Since we will be comparing results for each case on a number of example problems, we desire the input data to be consistent. As it stands, our models use both element and nodal quantities. We may change this if we let

$$\{u\}_e = [\Gamma]^T \{u\}_n$$

$$\{t\}_e = [\Gamma]^T \{t\}_n$$

so that

$$[A_a]\{u\}_e = [B_a]\{t\}_e$$

$$[A_b][\Gamma]^{-T}\{u\}_e = [B_b]\{t\}_e \qquad [32]$$

$$[A_c][\Gamma]^{-T}\{u\}_e = [B_c][\Gamma]^{-T}\{t\}_e$$

where the number of nodes is odd and

$$[\Gamma]^{-T} = \begin{bmatrix} I & I & -I & I & \cdot & \cdot & \cdot & -I \\ -I & I & I & -I & \cdot & \cdot & \cdot & I \\ I & -I & I & I & \cdot & \cdot & \cdot & -I \\ -I & I & I & I & \cdot & \cdot & \cdot & I \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & & \cdot \\ I & -I & I & -I & \cdot & \cdot & \cdot & I \end{bmatrix} \qquad [\Gamma]^T = \begin{bmatrix} I & 0 & 0 & 0 & \cdot & I \\ I & I & 0 & 0 & \cdot & 0 \\ 0 & I & I & 0 & \cdot & 0 \\ 0 & 0 & I & I & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & I \end{bmatrix} \qquad [33]$$

3.3 MATRIX EVALUATION

The coefficients in matrices [A] and [B] for each case may be calculated using simple Gauss quadrature. This is a well-known technique and will not be discussed here, however, details of the procedure may be found in (1).

To apply Gauss quadrature, we must have T_{ji} and U_{ij} in proper form. This requires ρ, q_1, q_2, n_1 and n_2 in terms of η. Employing eq. [26], we have

$$x^{(n)} - \xi = [x^{(n)} - x^m] - [x^{(m)} - x^m]\,\eta$$

$$= \delta^{(nm)} - \delta^{mm}\,\eta$$

and

$$x^n - \delta = [x^n - x^m] - [x^{(m)} - x^m]\,\eta$$

$$= \delta^{nm} - \delta^{mm}\,\eta$$

where we have defined

$$\delta^{(nm)} = x^{(n)} - x^m$$

$$\delta^{nm} = x^n - x^m \qquad [34]$$

Then for case (a)

$$\rho^2 = [\underset{\sim}{\delta}^{nm} \cdot \underset{\sim}{\delta}^{nm}] - 2\,[\underset{\sim}{\delta}^{nm} \cdot \underset{\sim}{\delta}^{mm}]\,\eta + [\,\underset{\sim}{\delta}^{mm} \cdot \underset{\sim}{\delta}^{mm}]\,\eta^2$$

$$q_1 = (\delta_1^{nm} - \delta_1^{mm}\,\eta) / \rho$$

$$q_2 = (\delta_2^{nm} - \delta_2^{mm}\,\eta) / \rho \qquad [35]$$

$$n_1 = 2\,\delta_2^{mm} / \Delta s^m$$

$$n_2 = -2\,\delta_1^{mm} / \Delta s^m$$

and for cases (b) and (c)

$$\rho^2 = [\,\underset{\sim}{\delta}^{(nm)} \cdot \underset{\sim}{\delta}^{(nm)}] - 2[\,\underset{\sim}{\delta}^{(nm)} \cdot \underset{\sim}{\delta}^{mm}]\,\eta + [\underset{\sim}{\delta}^{mm} \cdot \underset{\sim}{\delta}^{mm}]\,\eta^2$$

$$q_1 = (\delta_1^{(nm)} - \delta_1^{mm}\,\eta) / \rho$$

$$q_2 = (\delta_2^{(nm)} - \delta_2^{mm}\,\eta) / \rho \qquad [36]$$

$$n_1 = 2\,\delta_2^{mm} / \Delta s^m$$

$$n_2 = -2\,\delta_1^{mm} / \Delta s^m$$

Equations [28], [29], [30], [35] and [36], together with the matrix forms given in [31] and [32], provide the basic numerical models for each of the three cases. However, further discussion is required because the integrals become singular when the source point and field point coincide. The coefficients given by the singular integrals will be referred to as "special" coefficients and are:

case (a): $\Delta s^m \left(\int_m T_{ji}\, d\eta \right) / 2 \qquad m = n$

$\Delta s^m \left(\int_m U_{ij}\, d\eta \right) / 2 \qquad m = n$

case (b): $\Delta s^m \left[\int_m (1 - \eta)\, T_{ji}\, d\eta \right] / 4 \qquad m = n, n + 1$

$\Delta s^m \left[\int_m (1 + \eta)\, T_{ji}\, d\eta \right] / 4 \qquad m = n, n + 1$

$\Delta s^m \left(\int_m U_{ij}\, d\eta \right) / 2 \qquad m = n, n + 1$

case (c): $\Delta s^m [\int_m (1-\eta)\, T_{ji}\, d\eta\,] / 4 \qquad m = n, n+1$

$$\Delta s^m [\int_m (1-\eta)\, T_{ji}\, d\eta] \;/ 4 \qquad m = n, n+1$$

$$\Delta s^m [\int_m (1-\eta)\, U_{ij}\, d\eta] \;/ 4 \qquad m = n, n+1$$

$$\Delta s^m [\int_m (1+\eta)\, U_{ij}\, d\eta\,] / 4 \qquad m = n, n+1.$$

Altogether then, sixteen special coefficients need to be dispositioned. Recall, however, that five of these are incorporated into the α_{ji} coefficients as shown in eqs. [28], [29] and [30]. Let's look at these coefficients more closely.

Suppose $\underset{\sim}{x}$ is a corner point on B and define the following angles:

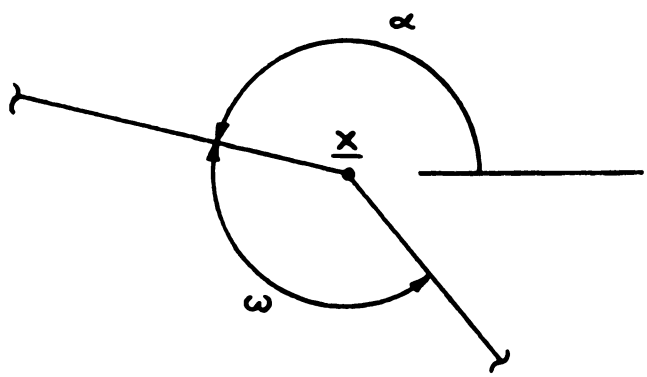


Figure 5. Free Term Coefficient

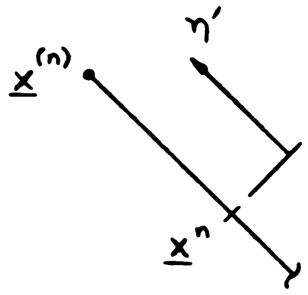
Then, proceeding as we did in Chapter 2, we get

$$\alpha_{ji} = \begin{cases} \omega/2\pi \pm (1+\nu)\,[\sin 2(\alpha + w) - \sin 2\,\alpha] \,/\, 8\,\pi & j = i \\ \\ (1+\nu)\,[\sin^2(\alpha + w) - \sin^2 \alpha] \,/\, 4\,\pi & j \neq i \end{cases}$$

so that when $\omega = \pi$, $\alpha_{ji} = \delta_{ji} / 2$. It is clear , then, that these coefficients represent only rigid-body displacements and may, therefore, be ignored.

Now we are left with eleven special coefficients. However, those involving T_{ji} in cases (b) and (c) are identical so we are actually left with nine. To evaluate these, we need the following local coordinates:

On m = n:



$$0 \le \eta \le 1$$

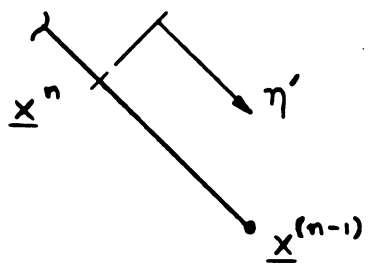
$$\eta = \eta'$$

$$d\eta = d\eta'$$

$$\rho = \Delta s^m \eta' / 2 \qquad [37]$$

$$q_1 = n_2$$

$$q_2 = - n_1$$



$$-1 \le \eta \le 0$$

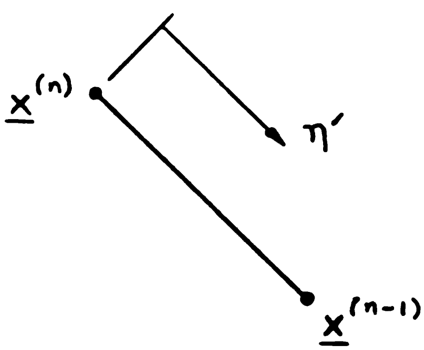
$$\eta = -\eta'$$

$$d\eta = -d\eta'$$

$$\rho = \Delta s^m \eta' / 2 \qquad [38]$$

$$q_1 = -n_2$$

$$q_2 = n_1$$



$$-1 \le \eta \le 1$$

$$\eta = 1 - 2\eta'$$

$$d\eta = -2\, d\eta'$$

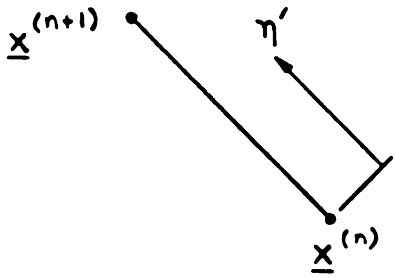
$$\rho = \eta' \Delta s^m \qquad [39]$$

$$q_1 = - n_2$$

$$q_2 = n_1$$

Figure 6. Local Coordinates On m = n

On $\quad m = n + 1$:



$$-1 \le \eta \le 1$$
$$\eta = 2\eta' - 1$$
$$d\eta = 2\, d\eta'$$
$$\rho = \eta' \Delta s^m \qquad [40]$$
$$q_1 = n_2$$
$$q_2 = -n_1$$

Figure 7. Local Coordinates On m = n + 1

Now consider case (a), which has the following special coefficient:

$$\Delta s^m \Big(\int_m U_{ij}\, d\eta \Big) / 2 \qquad m = 1.$$

Using equations [37] and [38], this becomes

$$\int_{-1}^{1} U_{ij}\,(x^n, \eta)\, d\eta = -\int_{1}^{0} U_{ij}\,(\eta')\, d\eta' + \int_{0}^{1} U_{ij}\,(\eta')\, d\eta'. \qquad [41]$$

Equations [37] and [38] both yield identical results when substituted into equations [14] and we have

$$\begin{aligned}
U_{11} &= [-(3-\nu) \ln (\Delta s^m \eta'/2) + (1+\nu)\, n_2^2] / 8\pi G \\
U_{12} &= -(1+\nu)\, n_1 n_2 / 8\pi G \\
U_{21} &= -(1+\nu)\, n_1 n_2 / 8\pi G \\
U_{22} &= [-(3-\nu) \ln (\Delta s^m \eta'/2) + (1+\nu)\, n_1^2] / 8\pi G.
\end{aligned} \qquad [42]$$

This result allows us to write

$$-\int_{1}^{0} U_{ij}\,(\eta')\, d\eta' + \int_{0}^{1} U_{ij}\,(\eta')\, d\eta' = 2\int_{0}^{1} U_{ij}\,(\eta')\, d\eta'. \qquad [43]$$

Using equations [42] and [43], we now write the special coefficients for case (a)

$$\Delta s^n \left(\int_n U_{11}\, d\eta \right) / 2 = (\Delta s^n / 8\pi G)\, [(3-\upsilon) - (3-\upsilon) \ln (\Delta s^m / 2) + (1+\upsilon)\, n_2^2]$$

$$\Delta s^n \left(\int_n U_{12}\, d\eta \right) / 2 = -(\Delta s^n / 8\pi G)\, (1+\upsilon)\, n_1 n_2$$

$$\Delta s^n \left(\int_n U_{21}\, d\eta \right) / 2 = -(\Delta s^n / 8\pi G)\, (1+\upsilon)\, n_1 n_2 \qquad [44]$$

$$\Delta s^n \left(\int_n U_{22}\, d\eta \right) / 2 = (\Delta s^n / 8\pi G)\, [(3-\upsilon) - (3-\upsilon) \ln (\Delta s^m / 2) + (1+\upsilon)\, n_1^2]$$

Both case (b) and case (c) have the following special coefficients:

$$\Delta s^m \left[\int_m (1-\eta)\, T_{ji}\, d\eta \right] / 4 \qquad m = n$$

$$\Delta s^m \left[\int_m (1+\eta)\, T_{ji}\, d\eta \right] / 4 \qquad m = n+1$$

The singularity in T_{ji}, however, is of the form $1/\rho$, and ρ goes to zero at the same rate as $(1 \pm \eta)$. This means that the integrals are well-behaved and may be evaluated numerically.

We now turn our attention to the U_{ij} - type coefficients of cases (b) and (c). From equations [39] and [40], we have

m = n:

$$\text{(b)} \quad \Delta s^n \left[\int_{-1}^{1} U_{ij} (\underline{x}^{(n)}, \eta)\, d\eta \right] / 2 = \Delta s^n \int_{0}^{1} U_{ij} (\eta')\, d\eta'$$

$$\text{(c)} \quad \Delta s^n \left[\int_{-1}^{1} (1-\eta)\, U_{ij} (\underline{x}^{(n)}, \eta)\, d\eta \right] / 4 = \Delta s^n \int_{0}^{1} \eta 1\, U_{ij} (\eta')\, d\eta' \qquad [45]$$

$$\Delta s^n \left[\int_{-1}^{1} (1+\eta)\, U_{ij} (\underline{x}^{(n)}, \eta)\, d\eta \right] / 4 = \Delta s^n \int_{0}^{1} (1-\eta')\, U_{ij} (\eta')\, d\eta'$$

$m = n + 1$:

$$\text{(b)}\quad \Delta s^{n+1}\left[\int_{-1}^{1} U_{ij}(\underline{x}^{(n+1)}, \eta)\, d\eta\right] / 2 = \Delta s^{n+1}\int_{0}^{1} U_{ij}(\eta')\, d\eta'$$

$$\text{(c)}\quad \Delta s^{n+1}\left[\int_{-1}^{1} (1-\eta)\, U_{ij}(\underline{x}^{(n+1)}, \eta)\, d\eta\right] / 4 = \Delta s^{n+1}\int_{0}^{1} (1-\eta')\, U_{ij}(\eta')\, d\eta' \qquad [46]$$

$$\Delta s^{n+1}\left[\int_{-1}^{1} (1+\eta)\, U_{ij}(\underline{x}^{(n+1)}, \eta)\, d\eta\right] / 4 = \Delta s^{n+1}\int_{0}^{1} \eta'\, U_{ij}(\eta')\, d\eta'$$

Just as equations [45] and [46] are identical in form, substitution of equations [39] and [40] into equations [15] shows that U_{ij} is identical in form on $m + n$ and $m = n + 1$.

$$\begin{aligned} U_{11} &= [-(3-\upsilon)\ln(\Delta s^{m}\eta') + (1+\upsilon)\, n_2^2] / 8\pi G \\ U_{12} &= -(1+\upsilon)\, n_1 n_2 / 8\pi G \\ U_{21} &= -(1+\upsilon)\, n_1 n_2 / 8\pi G \\ U_{22} &= [-(3-\upsilon)\ln(\Delta s^{m}\eta') + (1+\upsilon)\, n_1^2] / 8\pi G \end{aligned} \qquad [47]$$

After making the substitutions and carrying out the integrations, we get the following results:

case (b):

$$\begin{aligned} \Delta s^{m}\int_{0}^{1} U_{11}\, d\eta' &= \Delta s^{m}[-(3-\upsilon)\ln\Delta s^{m} + (3-\upsilon) + (1+\upsilon)\, n_2^2] / 8\pi G \\ \Delta s^{m}\int_{0}^{1} U_{12}\, d\eta' &= -\Delta s^{m}(1+\upsilon)\, n_1 n_2 / 8\pi G \\ \Delta s^{m}\int_{0}^{1} U_{21}\, d\eta' &= -\Delta s^{m}(1+\upsilon)\, n_1 n_2 / 8\pi G \\ \Delta s^{m}\int_{0}^{1} U_{22}\, d\eta' &= \Delta s^{m}[-(3-\upsilon)\ln\Delta s^{m} + (3-\upsilon) + (1+\upsilon)\, n_1^2] / 8\pi G \end{aligned} \qquad [48]$$

case (c):

$$\Delta s^m \int_0^1 \eta^1 U_{11}\, d\eta' = \Delta s^m [-0.5 (3 - \upsilon) \ln \Delta s^m + 0.25 (3 - \upsilon) + 0.5 (1 + \upsilon) n_2^2] / 8\pi G$$

$$\Delta s^m \int_0^1 \eta' U_{12}\, d\eta' = \Delta s^m (0.5) (1 + \upsilon) n_1 n_2 / 8\pi G$$

[49]

$$\Delta s^m \int_0^1 \eta' U_{21}\, d\eta' = -\Delta s^m (0.5) (1 + \upsilon) n_1 n_2 / 8\pi G$$

$$\Delta s^m \int_0^1 \eta' U_{22}\, d\eta' = \Delta s^m [-0.5 (3 - \upsilon) \ln \Delta s^m + 0.25 (3 - \upsilon) + 0.5 (1 + \upsilon) n_1^2] / 8\pi G$$

$$\Delta s^m \int_0^1 (1 - \eta') U_{11}\, d\eta' = \Delta s^m [(3 - \upsilon) (0.75 - 0.5 \ln \Delta s^m) + 0.5(1 + \upsilon) n_2^2] / 8\pi G$$

$$\Delta s^m \int_0^1 (1 - \eta') U_{12}\, d\eta' = - \Delta s^m (0.5) (1 + \upsilon) n_1 n_2 / 8\pi G$$

[50]

$$\Delta s_m \int_0^1 (1 - \eta') U_{21}\, d\eta' = - \Delta s^m (0.5) (1 + \upsilon) n_1 n_2 / 8\pi G$$

$$\Delta s^m \int_0^1 (1 - \eta') U_{22}\, d\eta' = \Delta s^m [(3 - \upsilon) (0.75 - 0.5 \ln \Delta s^m) + 0.5 (1 + \upsilon) n_1^2] / 8\pi G$$

where $m = n, n + 1$.

It is important to note that in case (c), the $(1 - \eta')$ - type integrals add together for $m = n$ and $m = n + 1$.

Recall that the coefficients in [A] , which multiplied U_j^n, for all three cases, were incorporated into the α_{ji} coefficients. These were then found to represent rigid - body displacements and are therefore ignored. In doing this, however, we have created a diagonal void in the [A] matrix. We may fill this void by determining the diagonal entries in the following manner.

Consider a rigid - body displacement applied to the system. This will produce no stress and, therefore,

$$[A] \{u\} = [B] \{0\} = \{0\}.$$

In other words,

$$A_{11}\,u_1 + A_{12}\,u_2 + A_{13}\,u_3 + \ldots + A_{12n}\,U_{2n} = 0.$$

So we have that the diagonal entries of [A] are the negative sums of the off-diagonal entries. Of course, the diagonal entries referred to here are actually 2 x 2 block matrices and are given by

$$
\begin{aligned}
A_{(2n-1)(2n-1)} &= -\sum_{m=1}^{N} A_{(2n-1)(2m-1)} && m \neq n \\
A_{(2n-1)(2n)} &= -\sum_{m=1}^{N} A_{(2n-1)(2m)} && m \neq n \\
A_{(2n)(2n-1)} &= -\sum_{m=1}^{N} A_{(2n)(2m-1)} && m \neq n \\
A_{(2n)(2n)} &= -\sum_{m=1}^{N} A_{(2n)(2m)} && m \neq n
\end{aligned}
\qquad [51]
$$

In summary, the following equations are used to calculate the matrices:

[A] Matrix:	case (a)	[15], [35], [51]
	case (b)	[15], [36], [51]
	case (c)	[15], [36], [51]
[B] Matrix:	case (a)	[14], [35], [44]
	case (b)	[14], [36], [48]
	case (c)	[14], [36], [49], [50]

3.4 INTERIOR FIELD POINTS

Following the same reasoning used in the previous section, we may rewrite equations [22] and [23] for the three cases.

case (a):

$$u_i = \sum_{m=1}^{N} [(\Delta s^m/2) \int_m T_{ji} (\underline{x}, \eta)\, d\eta]\, u_j^m + \sum_{m=1}^{N} [(\Delta s^m/2) \int_m U_{ij} (\underline{x},\eta)\, d\eta]\, t_j^m$$

$$\sigma_{ik} = \sum_{m=1}^{N} [(\Delta s^m/2) \int_m S_{jki} (\underline{x},\eta)\, d\eta]\, u_j^m + \sum_{m=1}^{N} [(\Delta s^m/2) \int_m D_{ikj} (\underline{x},\eta)\, d\eta]\, t_j^m \qquad [52]$$

case (b):

$$u_i = \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1-\eta)\, T_{ji} (\underline{x},\eta)\, d\eta]\, u_j^{(m-1)} + \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1+\eta)\, T_{ji} (\underline{x},\eta)\, d\eta]\, u_j^{(m)} + \sum_{m=1}^{N} [(\Delta s^m/2) \int_m U_{ij} (\underline{x},\eta)\, d\eta]\, t_j^m$$

$$\sigma_{ik} = \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1-\eta)\, S_{jki} (\underline{x},\eta)\, d\eta]\, u_j^{(m-1)} + \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1+\eta)\, S_{jki} (\underline{x},\eta)\, d\eta]\, u_j^{(m)} + \sum_{m=1}^{N} [(\Delta s^m/2) \int_m D_{ikj} (\underline{x},\eta)\, d\eta]\, t_j^m \qquad [53]$$

case (c):

$$u_i = \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1-\eta)\, T_{ji} (\underline{x},\eta)\, d\eta]\, u_j^{(m-1)} + \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1+\eta)\, T_{ji} (\underline{x},\eta)\, d\eta]\, u_j^{(m)} + \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1-\eta)\, U_{ij} (\underline{x},\eta)\, d\eta]\, t_j^{(m-1)} + \sum_{m=1}^{N} [(\Delta s^m/4) \int_m (1+\eta)\, U_{ij} (\underline{x},\eta)\, d\eta]\, t_j^{(m)}$$

[54]

$$\sigma_{ik} = \sum_{m=1}^{N} \left[(\Delta s^{m}/4) \int_{m} (1 - \eta)\, S_{jki}\, (\underline{x},\eta)\, d\eta \right] u_j^{(m-1)}$$

$$+ \sum_{m=1}^{N} \left[(\Delta s^{m}/4) \int_{m} (1 + \eta)\, S_{jki}\, (\underline{x},\eta)\, d\eta \right] u_j^{(m)}$$

$$+ \sum_{m=1}^{N} \left[(\Delta s^{m}/4) \int_{m} (1 - \eta)\, D_{ikj}\, (\underline{x},\eta)\, d\eta \right] t_j^{(m-1)}$$

$$+ \sum_{m=1}^{N} \left[(\Delta s^{m}/4) \int_{m} (1 + \eta)\, D_{ikj}\, (\underline{x},\eta)\, d\eta \right] t_j^{(m)}$$

Applying equations [24] and [25] and Gauss quadrature, these equations may be used to obtain displacements and stresses at any point in the body.

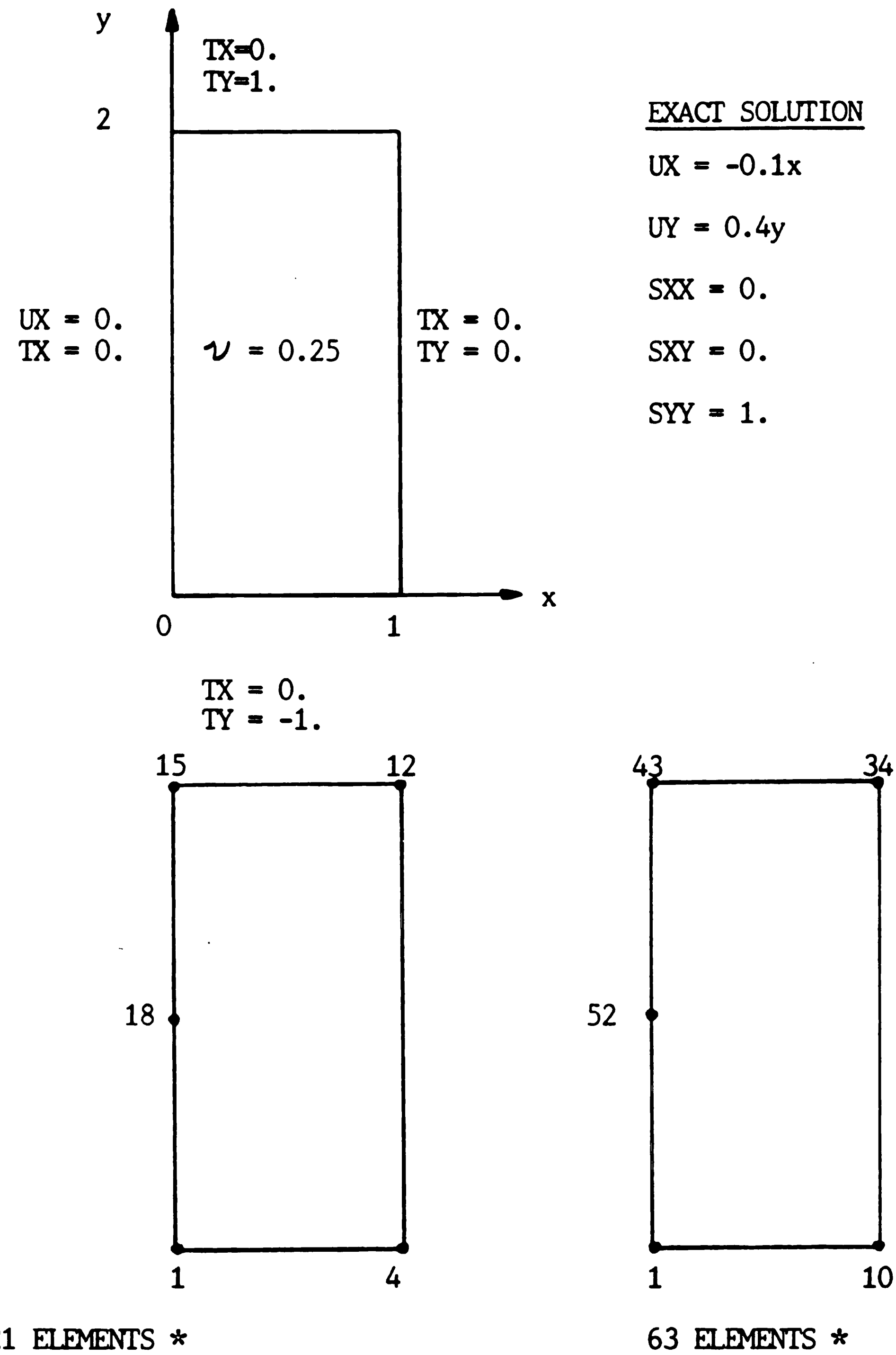
CHAPTER 4

EXAMPLES AND RESULTS

4.1 PROBLEM DESCRIPTIONS

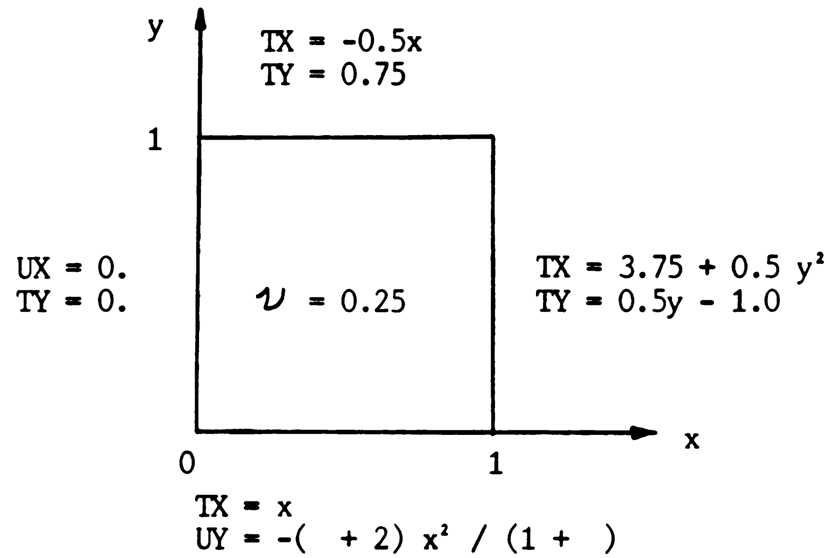
Our objective is to assess the performance of each of the three numerical models in solving linear isotropic elasticity problems. Three example problems were chosen for this purpose. They represent a range of geometries and boundary conditions intended to highlight the relative strengths and weaknesses of each model.

The example problems are described in the following figures along with boundary conditions, exact solutions and boundary meshes. These figures are followed by tables containing the output data and percent differences for all three models in each problem.



* Between each node number, elements are of equal size.

Figure 8. Problem One - Pure Extension



<u>EXACT SOLUTION (18)</u>

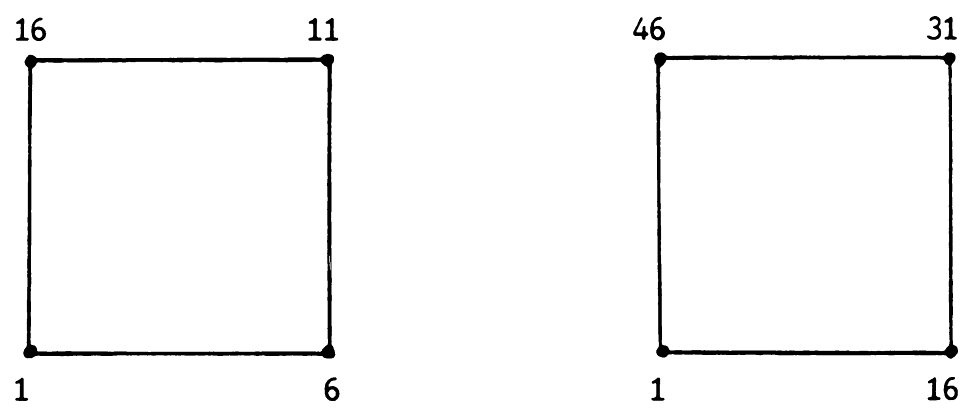
$UX = 0.5[4x - x^3/12 + y^2/2 - \quad x(y - y^2/4)] / (1 + \quad)$

$UY = 0.5[y^2/2 - y^3/12 - \quad (4y - x^2y/4 + y^3/6) - (1 + \quad)x^2 + \quad x^2/2]/(1 + \quad)$

$SXX = 4 - x^2/4 + y^2/2$

$SXY = xy/2 - x$

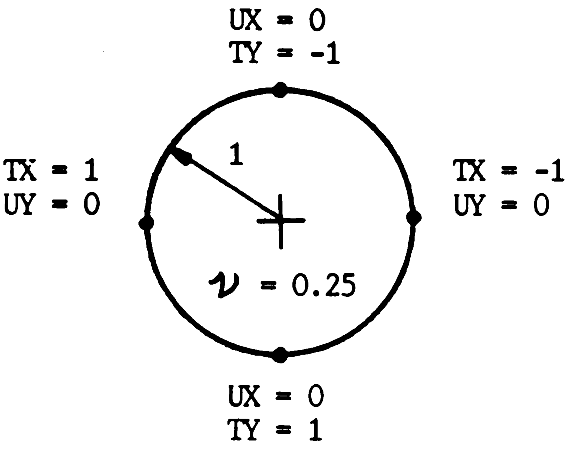
$SYY = y - y^2/4$



21 ELEMENTS * 63 ELEMENTS *

* Between each node number, elements are of equal size

Figure 9. Problem Two - Square With Variable Boundary Conditions



Otherwise: TX = - nx
TY = - ny

<u>EXACT SOLUTION</u>

UX = -0.3x

UY = -0.3y

SXX = -1

SXY = 0

SYY = -1

19 ELEMENTS *

35 ELEMENTS *

* Between node numbers shown, elements are of equal size.

Figure 10. Problem Three - Circular Disk Under Uniform Pressure

4.2 TABLES OF RESULTS

The following tables summarize the numerical solutions of the three example problems. Each problem was solved twice using an increasing number of nodes.

The tables should be self-explanatory, however, two small clarifications are necessary. First, since percent difference is meaningless for elements where the quantity is either specified or equal to zero, these elements are omitted from the percent difference table. Second, all displacement values are multiplied by the shear modulus.

4.3 DISCUSSION

The problems described in the previous section represent a range of geometries and boundary conditions. These problems were chosen to include smooth boundaries and boundaries with corners as well as constant and variable boundary conditions. The performance of each model in these basic problems will then provide information on their applicability in general.

Problem one has a straight boundary with corners and linear displacements with piecewise constant tractions. The straight boundaries allow an exact boundary representation and the variable behavior is identical to that assumed in case (b). Therefore, one would expect case (b) to solve this problem exactly and, indeed, it does. The other cases differ in their assumed variable behavior and, therefore, yield approximate results. In case (a), the assumed traction behavior is consistent with the problem, however, the displacements are piecewise constant. Conversely, in case (c), the assumed displacement behavior is consistent, but the tractions are piecewise continuous. Therefore, in cases (a) and (c), we see increasing inaccuracy near corner points where there is a change in variable behavior. These inaccuracies in the boundary solution carry over into the interior solution as well. However, interior solutions are, in general, more accurate except for points near the boundary.

Problem two also has straight boundaries with corners, however, the boundary conditions are more complex. One would still expect to see inaccuracy near corners, however, it is not clear what effect the assumed variable behavior would have. Since the piecewise constant behavior is more forgiving at corners, it would appear cases (a) and (b) might prevail.

Problem three has a circular boundary which introduces boundary approximation errors in all three cases. The variable behavior does not immediately favor any case so we expect, and find, similar results in each.

Table 1. Pure Extension - 21 Nodes: X - Displacements

X-DISPLACEMENTS

NO	X	Y	UXA	UXB	UXC	EXACT
1	0.00000	.12500	0.00000	0.00000	0.00000	0.00000
2	.16667	0.00000	-.02301	-.01667	-.02247	-.01667
3	.50000	0.00000	-.05480	-.05000	-.04920	-.05000
4	.83333	0.00000	-.08671	-.08333	.00321	-.08333
5	1.00000	.12500	-.11030	-.10000	-.03224	-.10000
6	1.00000	.37500	-.11029	-.10000	-.11801	-.10000
7	1.00000	0.62500	-.10990	-.10000	-.10915	-.10000
8	1.00000	.87500	-.10980	-.10000	-.10597	-.10000
9	1.00000	1.12500	-.11000	-.10000	-.10615	-.10000
10	1.00000	1.37500	-.11060	-.10000	-.10578	-.10000
11	1.00000	1.62500	-.11162	-.10000	-.09857	-.10000
12	1.00000	1.87500	-.11216	-.10000	-.02072	-.10000
13	.83333	2.00000	-.08878	-.08333	.01120	-.08333
14	.50000	2.00000	-.05730	-.05000	-.01054	-.05000
15	.16667	2.00000	-.02706	-.01667	.02797	-.01667
16	0.00000	1.83333	0.00000	0.00000	0.00000	0.00000
17	0.00000	1.50000	0.00000	0.00000	0.00000	0.00000
18	0.00000	1.16667	0.00000	0.00000	0.00000	0.00000
19	0.00000	.87500	0.00000	0.00000	0.00000	0.00000
20	0.00000	.62500	0.00000	0.00000	0.00000	0.00000
21	0.00000	.37500	0.00000	0.00000	0.00000	0.00000

Table 2. Pure Extension - 21 Nodes: Y - Displacements

Y-DISPLACEMENTS

NO	X	Y	UYA	UYB	UYC	EXACT
1	0.00000	.12500	.05323	.05000	.12917	.05000
2	.16667	0.00000	0.00000	0.00000	0.00000	0.00000
3	.50000	0.00000	0.00000	0.00000	0.00000	0.00000
4	.83333	0.00000	0.00000	0.00000	0.00000	0.00000
5	1.00000	.12500	.05362	.05000	.16087	.05000
6	1.00000	.37500	.15182	.15000	.14766	.15000
7	1.00000	.62500	.25156	.25000	.22970	.25000
8	1.00000	.87500	.35152	.35000	.33063	.35000
9	1.00000	1.12500	.45156	.45000	.42866	.45000
10	1.00000	1.37500	.55155	.55000	.51974	.55000
11	1.00000	1.62500	.65119	.65000	.59108	.65000
12	1.00000	1.87500	.74901	.75000	.61680	.75000
13	.83333	2.00000	.80226	.80000	.73813	.80000
14	.50000	2.00000	.80312	.80000	.76959	.80000
15	.16667	2.00000	.79939	.80000	.87782	.80000
16	0.00000	1.83333	.73029	.73333	.78077	.73333
17	0.00000	1.50000	.59941	.60000	.63285	.60000
18	0.00000	1.16667	.46688	.46667	.49113	.46667
19	0.00000	.87500	.35067	.35000	.35124	.35000
20	0.00000	.62500	.25100	.25000	.20905	.25000
21	0.00000	.37500	.15141	.15000	.15842	.15000

Table 3. Pure Extension - 21 Nodes: X - Tractions

X-TRACTIONS

NO	X	Y	TXA	TXB	TXC	EXACT
1	0.00000	.12500	-.00825	.00000	-.11642	0.00000
2	.16667	0.00000	0.00000	0.00000	0.00000	0.00000
3	.50000	0.00000	0.00000	0.00000	0.00000	0.00000
4	.83333	0.00000	0.00000	0.00000	0.00000	0.00000
5	1.00000	.12500	0.00000	0.00000	0.00000	0.00000
6	1.00000	.37500	0.00000	0.00000	0.00000	0.00000
7	1.00000	.62500	0.00000	0.00000	0.00000	0.00000
8	1.00000	.87500	0.00000	0.00000	0.00000	0.00000
9	1.00000	1.12500	0.00000	0.00000	0.00000	0.00000
10	1.00000	1.37500	0.00000	0.00000	0.00000	0.00000
11	1.00000	1.62500	0.00000	0.00000	0.00000	0.00000
12	1.00000	1.87500	0.00000	0.00000	0.00000	0.00000
13	.83333	2.00000	0.00000	0.00000	0.00000	0.00000
14	.50000	2.00000	0.00000	0.00000	0.00000	0.00000
15	.16667	2.00000	0.00000	0.00000	0.00000	0.00000
16	0.00000	1.83333	-.00885	.00000	-.32744	0.00000
17	0.00000	1.50000	-.00473	.00000	-.01711	0.00000
18	0.00000	1.16667	-.02091	.00000	-.00401	0.00000
19	0.00000	.87500	.02341	.00000	.05992	0.00000
20	0.00000	.62500	.00012	.00000	.04106	0.00000
21	0.00000	.37500	-.00363	.00000	-.12700	0.00000

Table 4. Pure Extension - 21 Nodes: Y - Tractions

Y-TRACTIONS

NO	X	Y	TYA	TYB	TYC	EXACT
1	0.00000	.12500	0.00000	0.00000	0.00000	0.00000
2	.16667	0.00000	-1.00351	-1.00000	-.86657	-1.00000
3	.50000	0.00000	-.99128	-1.00000	-.97328	-1.00000
4	.83333	0.00000	-1.00616	-1.00000	-1.45664	-1.00000
5	1.00000	.12500	0.00000	0.00000	0.00000	0.00000
6	1.00000	.37500	0.00000	0.00000	0.00000	0.00000
7	1.00000	.62500	0.00000	0.00000	0.00000	0.00000
8	1.00000	.87500	0.00000	0.00000	0.00000	0.00000
9	1.00000	1.12500	0.00000	0.00000	0.00000	0.00000
10	1.00000	1.37500	0.00000	0.00000	0.00000	0.00000
11	1.00000	1.62500	0.00000	0.00000	0.00000	0.00000
12	1.00000	1.87500	0.00000	0.00000	0.00000	0.00000
13	.83333	2.00000	1.00000	1.00000	1.00000	1.00000
14	.50000	2.00000	1.00000	1.00000	1.00000	1.00000
15	.16667	2.00000	1.00000	1.00000	1.00000	1.00000
16	0.00000	1.83333	0.00000	0.00000	0.00000	0.00000
17	0.00000	1.50000	0.00000	0.00000	0.00000	0.00000
18	0.00000	1.16667	0.00000	0.00000	0.00000	0.00000
19	0.00000	.87500	0.00000	0.00000	0.00000	0.00000
20	0.00000	.62500	0.00000	0.00000	0.00000	0.00000
21	0.00000	.37500	0.00000	0.00000	0.00000	0.00000

Table 5. Pure Extension - 21 Nodes: Interior Values

INTERIOR X-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	.25000	.50000	-.02964	-.02500	-.02002	-.02500
2	.25000	1.00000	-.03032	-.02500	-.03363	-.02500
3	.25000	1.50000	-.02984	-.02500	-.00236	-.02500
4	.50000	.50000	-.05507	-.05000	-.04395	-.05000
5	.50000	1.00000	-.05523	-.05000	-.05881	-.05000
6	.50000	1.50000	-.05591	-.05000	-.04264	-.05000
7	.75000	.50000	-.08047	-.07500	-.08055	-.07500
8	.75000	1.00000	-.08036	-.07500	-.08084	-.07500
9	.75000	1.50000	-.08145	-.07500	-.07538	-.07500

INTERIOR Y-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	.25000	.50000	.20142	.20000	.20889	.20000
2	.25000	1.00000	.40116	.40000	.40849	.40000
3	.25000	1.50000	.59992	.60000	.61530	.60000
4	.50000	.50000	.20143	.20000	.19088	.20000
5	.50000	1.00000	.40114	.40000	.39594	.40000
6	.50000	1.50000	.60077	.60000	.59726	.60000
7	.75000	.50000	.20165	.20000	.18026	.20000
8	.75000	1.00000	.40132	.40000	.38623	.40000
9	.75000	1.50000	.60101	.60000	.58238	.60000

INTERIOR SXX

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	.00035	.00000	.05022	0.00000
2	.25000	1.00000	.00967	.00000	-.00209	0.00000
3	.25000	1.50000	-.02427	.00000	1.31848	0.00000
4	.50000	.50000	-.00587	.00000	-.04520	0.00000
5	.50000	1.00000	-.00348	.00000	.02039	0.00000
6	.50000	1.50000	-.00754	.00000	-.06053	0.00000
7	.75000	.50000	.00081	.00000	-.06660	0.00000
8	.75000	1.00000	.00281	.00000	.06332	0.00000
9	.75000	1.50000	-.00018	.00000	-.02828	0.0000

INTERIOR SXY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	-.00061	.00000	-.24064	0.00000
2	.25000	1.00000	.00162	.00000	-.23927	0.00000
3	.25000	1.50000	.00385	.00000	-.13439	0.00000
4	.50000	.50000	.00010	.00000	-.15109	0.00000
5	.50000	1.00000	.00056	.00000	-.08109	0.00000
6	.50000	1.50000	.00015	.00000	.00618	0.00000
7	.75000	.50000	.00186	.00000	.04726	0.00000
8	.75000	1.00000	.00019	.00000	-.01461	0.00000
9	.75000	1.50000	-.00320	.00000	-.03663	0.00000

Table 5 (cont'd.)

INTERIOR SKY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	.99402	1.00000	.86865	1.00000
2	.25000	1.00000	.98745	1.00000	1.06814	1.00000
3	.25000	1.50000	1.00367	1.00000	1.03900	1.00000
4	.50000	.50000	.99949	1.00000	1.03050	1.00000
5	.50000	1.00000	.99635	1.00000	1.00854	1.00000
6	.50000	1.50000	.99900	1.00000	1.03846	1.00000
7	.75000	.50000	.99370	1.00000	1.07657	1.00000
8	.75000	1.00000	.99351	1.00000	.95636	1.00000
9	.75000	1.50000	.99346	1.00000	.98414	1.00000

Table 6. Pure Extension - 21 Nodes: Percent Differences

X-DISPLACEMENTS

NO	A	B	C
2	-38.07	.00	-34.83
3	-9.61	.00	1.60
4	-4.06	.00	103.86
5	-10.30	.00	67.76
6	-10.29	.00	-18.01
7	-9.90	.00	-9.15
8	-9.80	.00	-5.97
9	-10.00	.00	-6.15
10	-10.60	.00	-5.78
11	-11.62	.00	1.43
12	-12.16	.00	79.28
13	-6.53	.00	113.44
14	-14.61	.00	78.92
15	-62.34	.00	267.81

Y-DISPLACEMENTS

NO	A	B	C
1	-6.46	.00	-158.34
5	-7.23	.00	-221.73
6	-1.22	.00	1.56
7	-.62	.00	8.12
8	-.43	.00	5.54
9	-.35	.00	4.74
10	-.28	.00	5.50
11	-.18	.00	9.07
12	.13	.00	17.76
13	-.28	.00	7.73
14	-.39	.00	3.80
15	.08	.00	-9.73
16	.41	.00	-6.47
17	.10	.00	-5.47
18	-.05	.00	-5.24

Table 6 (cont'd.)

19	-.19	.00	-.35
20	-.40	.00	16.38
21	-.94	.00	-5.61

Y-TRACTIONS

NO	A	B	C
2	-.35	.00	13.34
3	.87	.00	2.67
4	-.62	.00	-45.66

INTERIOR X-DISP

NO	A	B	C
1	-18.57	.00	19.91
2	-21.28	.00	-34.52
3	-19.38	.00	90.57
4	-10.14	.00	12.10
5	-10.46	.00	-17.62
6	-11.83	.00	14.72
7	-7.29	.00	-7.40
8	-7.15	.00	-7.79
9	-8.60	.00	-.51

INTERIOR Y-DISP

NO	A	B	C
1	-.71	.00	-4.44
2	-.29	.00	-2.12
3	.01	.00	-2.55
4	-.72	.00	4.56
5	-.28	.00	1.02
6	-.13	.00	.46
7	-.82	.00	9.87
8	-.33	.00	3.44
9	-.17	.00	2.94

INTERIOR SYY

NO	A	B	C
1	.60	.00	13.13
2	1.25	.00	-6.81
3	-.37	.00	-3.90
4	.05	.00	-3.05
5	.37	.00	-.85
6	.10	.00	-3.85
7	.63	.00	-7.66
8	.65	.00	4.36
9	.65	.00	1.59

Table 7. Pure Extension - 63 Nodes: X - Displacements

X-DISPLACEMENTS

NO	X	Y	UXA	UXB	UXC	EXACT
1	0.00000	.04167	0.00000	0.00000	0.00000	0.00000
2	.05556	0.00000	-.00785	-.00556	-.01587	-.00556
3	.16667	0.00000	-.01849	-.01667	-.02705	-.01667
4	.27778	0.00000	-.02950	-.02778	-.03015	-.02778
5	.38889	0.00000	-.04055	-.03889	-.03953	-.03889
6	.50000	0.00000	-.05161	-.05000	-.04983	-.05000
7	.61111	0.00000	-.06268	-.06111	-.06018	-.06111
8	.72222	0.00000	-.07374	-.07222	-.06975	-.07222
9	.83333	0.00000	-.08478	-.08333	-.07539	-.08333
10	.94444	0.00000	-.09545	-.09444	-.05542	-.09444
11	1.00000	.04167	-.10337	-.10000	-.07145	-.10000
12	1.00000	.12500	-.10349	-.10000	-.10864	-.10000
13	1.00000	.20833	-.10343	-.10000	-.10299	-.10000
14	1.00000	.29167	-.10338	-.10000	-.09963	-.10000
15	1.00000	.37500	-.10335	-.10000	-.09837	-.10000
16	1.00000	.45833	-.10332	-.10000	-.09787	-.10000
17	1.00000	.54167	-.10330	-.10000	-.09769	-.10000
18	1.00000	.62500	-.10329	-.10000	-.09767	-.10000
19	1.00000	.70833	-.10329	-.10000	-.09773	-.10000
20	1.00000	.79167	-.10330	-.10000	-.09783	-.10000
21	1.00000	.87500	-.10331	-.10000	-.09796	-.10000
22	1.00000	.95833	-.10334	-.10000	-.09809	-.10000
23	1.00000	1.04167	-.10338	-.10000	-.09821	-.10000
24	1.00000	1.12500	-.10342	-.10000	-.09831	-.10000
25	1.00000	1.20833	-.10348	-.10000	-.09841	-.10000
26	1.00000	1.29167	-.10355	-.10000	-.09849	-.10000
27	1.00000	1.37500	-.10362	-.10000	-.09858	-.10000
28	1.00000	1.45833	-.10370	-.10000	-.09869	-.10000
29	1.00000	1.54167	-.10379	-.10000	-.09884	-.10000
30	1.00000	1.62500	-.10388	-.10000	-.09911	-.10000
31	1.00000	1.70833	-.10395	-.10000	-.09971	-.10000
32	1.00000	1.79167	-.10400	-.10000	-.10104	-.10000
33	1.00000	1.87500	-.10398	-.10000	-.10075	-.10000
34	1.00000	1.95833	-.10367	-.10000	-.07178	-.10000
35	.94444	2.00000	-.09569	-.09445	-.06085	-.09444
36	.83333	2.00000	-.08509	-.08333	-.07232	-.08333
37	.72222	2.00000	-.07421	-.07222	-.06603	-.07222
38	.61111	2.00000	-.06326	-.06111	-.05711	-.06111
39	.50000	2.00000	-.05230	-.05000	-.04746	-.05000
40	.38889	2.00000	-.04134	-.03889	-.03783	-.03889
41	.27778	2.00000	-.03040	-.02778	-.02868	-.02778
42	.16667	2.00000	-.01952	-.01667	-.01662	-.01667
43	.05556	2.00000	-.00927	-.00556	.00876	-.00556
44	0.00000	1.94444	0.00000	0.00000	0.00000	0.00000
45	0.00000	1.83333	0.00000	0.00000	0.00000	0.00000
46	0.00000	1.72222	0.00000	0.00000	0.00000	0.00000
47	0.00000	1.61111	0.00000	0.00000	0.00000	0.00000
48	0.00000	1.50000	0.00000	0.00000	0.00000	0.00000
49	0.00000	1.38889	0.00000	0.00000	0.00000	0.00000

Table 7 (cont'd.)

50	0.00000	1.27778	0.00000	0.00000	0.00000	0.00000
51	0.00000	1.16667	0.00000	0.00000	0.00000	0.00000
52	0.00000	1.05556	0.00000	0.00000	0.00000	0.00000
53	0.00000	.95833	0.00000	0.00000	0.00000	0.00000
54	0.00000	.87500	0.00000	0.00000	0.00000	0.00000
55	0.00000	.79167	0.00000	0.00000	0.00000	0.00000
56	0.00000	.70833	0.00000	0.00000	0.00000	0.00000
57	0.00000	.62500	0.00000	0.00000	0.00000	0.00000
58	0.00000	.54167	0.00000	0.00000	0.00000	0.00000
59	0.00000	.45833	0.00000	0.00000	0.00000	0.00000
60	0.00000	.37500	0.00000	0.00000	0.00000	0.00000
61	0.00000	.29167	0.00000	0.00000	0.00000	0.00000
62	0.00000	.20833	0.00000	0.00000	0.00000	0.00000
63	0.00000	.12500	0.00000	0.00000	0.00000	0.00000

Table 8. Pure Extension - 63 Nodes: Y - Displacements

Y-DISPLACEMENTS

NO	X	Y	UYA	UYB	UYC	EXACT
1	0.00000	.04167	.01789	.01667	.06515	.01667
2	.05556	0.00000	0.00000	0.00000	0.00000	0.00000
3	.16667	0.00000	0.00000	0.00000	0.00000	0.00000
4	.27778	0.00000	0.00000	0.00000	0.00000	0.00000
5	.38889	0.00000	0.00000	0.00000	0.00000	0.00000
6	.50000	0.00000	0.00000	0.00000	0.00000	0.00000
7	.61111	0.00000	0.00000	0.00000	0.00000	0.00000
8	.72222	0.00000	0.00000	0.00000	0.00000	0.00000
9	.83333	0.00000	0.00000	0.00000	0.00000	0.00000
10	.94444	0.00000	0.00000	0.00000	0.00000	0.00000
11	1.00000	.04167	.01797	.01667	.07223	.01667
12	1.00000	.12500	.05074	.05000	.03992	.05000
13	1.00000	.20833	.08398	.08333	.06209	.08333
14	1.00000	.29167	.11727	.11667	.09577	.11667
15	1.00000	.37500	.15058	.15000	.13026	.15000
16	1.00000	.45833	.18390	.18333	.16454	.18333
17	1.00000	.54167	.21723	.21667	.19858	.21667
18	1.00000	.62500	.25056	.25000	.23244	.25000
19	1.00000	.70833	.28389	.28333	.26615	.28333
20	1.00000	.79167	.31723	.31667	.29973	.31667
21	1.00000	.87500	.35057	.35000	.33320	.35000
22	1.00000	.95833	.38390	.38333	.36657	.38333
23	1.00000	1.04167	.41724	.41667	.39985	.41667
24	1.00000	1.12500	.45057	.45000	.43305	.45000
25	1.00000	1.20833	.48390	.48333	.46618	.48333
26	1.00000	1.29167	.51722	.51667	.49923	.51667
27	1.00000	1.37500	.55053	.55000	.53220	.55000
28	1.00000	1.45833	.58383	.58333	.58505	.58333
29	1.00000	1.54167	.61710	.61667	.59777	.61667
30	1.00000	1.62500	.65035	.65000	.63026	.65000
31	1.00000	1.70833	.68355	.68333	.66227	.68333

Table 8 (cont'd.)

32	1.00000	1.79167	.71669	.71667	.69246	.71667
33	1.00000	1.87500	.74970	.75000	.71451	.75000
34	1.00000	1.95833	.78223	.78333	.71602	.78333
35	.94444	2.00000	.80016	.80000	.76016	.80000
36	.83333	2.00000	.80096	.80000	.76540	.80000
37	.72222	2.00000	.80115	.80000	.76775	.80000
38	.61111	2.00000	.80120	.80000	.76837	.80000
39	.50000	2.00000	.80116	.80000	.76879	.80000
40	.38889	2.00000	.80106	.80000	.76899	.80000
41	.27778	2.00000	.80088	.80000	.76822	.80000
42	.16667	2.00000	.80054	.80000	.76479	.80000
43	.05556	2.00000	.79916	.80000	.81089	.80000
44	0.00000	1.94444	.77607	.77778	.77810	.77778
45	0.00000	1.83333	.73259	.73334	.72948	.73333
46	0.00000	1.72222	.68852	.68889	.67976	.68889
47	0.00000	1.61111	.64429	.64445	.63330	.64444
48	0.00000	1.50000	.59998	.60000	.58801	.60000
49	0.00000	1.38889	.55563	.55556	.54319	.55556
50	0.00000	1.27778	.51126	.51111	.49872	.51111
51	0.00000	1.16667	.46687	.46667	.45473	.46667
52	0.00000	1.05556	.42247	.42222	.41294	.42222
53	0.00000	.95833	.38362	.38333	.36846	.38333
54	0.00000	.87500	.35031	.35000	.34443	.35000
55	0.00000	.79167	.31701	.31667	.30231	.31667
56	0.00000	.70833	.28370	.28333	.26954	.28333
57	0.00000	.62500	.25040	.25000	.23668	.25000
58	0.00000	.54167	.21709	.21667	.20383	.21667
59	0.00000	.45833	.18379	.18333	.17105	.18333
60	0.00000	.37500	.15049	.15000	.13840	.15000
61	0.00000	.29167	.11719	.11667	.10594	.11667
62	0.00000	.20833	.08391	.08333	.04068	.08333
63	0.00000	.12500	.05067	.05000	.04905	.05000

Table 9. Pure Extension - 63 Nodes: X - Tractions

X-TRACTIONS

NO	X	Y	TXA	TXB	TXC	EXACT
1	0.00000	.04167	-.00253	.00000	-.09736	0.00000
2	.05556	0.00000	0.00000	0.00000	0.00000	0.00000
3	.16667	0.00000	0.00000	0.00000	0.00000	0.00000
4	.27778	0.00000	0.00000	0.00000	0.00000	0.00000
5	.38889	0.00000	0.00000	0.00000	0.00000	0.00000
6	.50000	0.00000	0.00000	0.00000	0.00000	0.00000
7	.61111	0.00000	0.00000	0.00000	0.00000	0.00000
8	.72222	0.00000	0.00000	0.00000	0.00000	0.00000
9	.83333	0.00000	0.00000	0.00000	0.00000	0.00000
10	.94444	0.00000	0.00000	0.00000	0.00000	0.00000
11	1.00000	.04167	0.00000	0.00000	0.00000	0.00000
12	1.00000	.12500	0.00000	0.00000	0.00000	0.00000
13	1.00000	.20833	0.00000	0.00000	0.00000	0.00000

Table 9 (cont'd.)

14	1.00000	.29167	0.00000	0.00000	0.00000	0.00000
15	1.00000	.37500	0.00000	0.00000	0.00000	0.00000
16	1.00000	.45833	0.00000	0.00000	0.00000	0.00000
17	1.00000	.54167	0.00000	0.00000	0.00000	0.00000
18	1.00000	.62500	0.00000	0.00000	0.00000	0.00000
19	1.00000	.70833	0.00000	0.00000	0.00000	0.00000
20	1.00000	.79167	0.00000	0.00000	0.00000	0.00000
21	1.00000	.87500	0.00000	0.00000	0.00000	0.00000
22	1.00000	.95833	0.00000	0.00000	0.00000	0.00000
23	1.00000	1.04167	0.00000	0.00000	0.00000	0.00000
24	1.00000	1.12500	0.00000	0.00000	0.00000	0.00000
25	1.00000	1.20833	0.00000	0.00000	0.00000	0.00000
26	1.00000	1.29167	0.00000	0.00000	0.00000	0.00000
27	1.00000	1.37500	0.00000	0.00000	0.00000	0.00000
28	1.00000	1.45833	0.00000	0.00000	0.00000	0.00000
29	1.00000	1.54167	0.00000	0.00000	0.00000	0.00000
30	1.00000	1.62500	0.00000	0.00000	0.00000	0.00000
31	1.00000	1.70833	0.00000	0.00000	0.00000	0.00000
32	1.00000	1.79167	0.00000	0.00000	0.00000	0.00000
33	1.00000	1.87500	0.00000	0.00000	0.00000	0.00000
34	1.00000	1.95833	0.00000	0.00000	0.00000	0.00000
35	.94444	2.00000	0.00000	0.00000	0.00000	0.00000
36	.83333	2.00000	0.00000	0.00000	0.00000	0.00000
37	.72222	2.00000	0.00000	0.00000	0.00000	0.00000
38	.61111	2.00000	0.00000	0.00000	0.00000	0.00000
39	.50000	2.00000	0.00000	0.00000	0.00000	0.00000
40	.38889	2.00000	0.00000	0.00000	0.00000	0.00000
41	.27778	2.00000	0.00000	0.00000	0.00000	0.00000
42	.16667	2.00000	0.00000	0.00000	0.00000	0.00000
43	.05556	2.00000	0.00000	0.00000	0.00000	0.00000
44	0.00000	1.94444	-.00055	.00000	-.32752	0.00000
45	0.00000	1.83333	-.00387	.00000	-.00196	0.00000
46	0.00000	1.72222	-.00280	-.00001	.06990	0.00000
47	0.00000	1.61111	-.00227	.00002	.03961	0.00000
48	0.00000	1.50000	-.00194	-.00001	.00949	0.00000
49	0.00000	1.38889	-.00188	.00001	-.00142	0.00000
50	0.00000	1.27778	-.00220	.00000	-.00271	0.00000
51	0.00000	1.16667	-.00227	.00000	.00460	0.00000
52	0.00000	1.05556	-.01993	.00000	-.02681	0.00000
53	0.00000	.95833	.02456	.00000	-.04560	0.00000
54	0.00000	.87500	.00163	.00000	-.03089	0.00000
55	0.00000	.79167	.00148	.00002	.03702	0.00000
56	0.00000	.70833	.00080	-.00002	.01685	0.00000
57	0.00000	.62500	.00041	.00002	-.00200	0.00000
58	0.00000	.54167	.00009	-.00002	-.00897	0.00000
59	0.00000	.45833	-.00020	.00002	-.00738	0.00000
60	0.00000	.37500	-.00052	-.00001	-.02542	0.00000
61	0.00000	.29167	-.00091	.00001	.10233	0.00000
62	0.00000	.20833	-.00136	-.00001	.09651	0.00000
63	0.00000	.12500	-.00420	.00000	-.22024	0.00000

Table 10. Pure Extension - 63 Nodes: Y - Tractions

Y-TRACTIONS

NO	X	Y	TYA	TYB	TYC	EXACT
1	0.00000	.04167	0.00000	0.00000	0.00000	0.00000
2	.05556	0.00000	-1.00852	-.99999	-.92313	-1.00000
3	.16667	0.00000	-.99576	-1.00002	-1.05889	-1.00000
4	.27778	0.00000	-.99784	-.99997	-1.05179	-1.00000
5	.38889	0.00000	-.99804	-1.00003	-1.02182	-1.00000
6	.50000	0.00000	-.99820	-.99999	-1.00505	-1.00000
7	.61111	0.00000	-.99823	-1.00000	-.99108	-1.00000
8	.72222	0.00000	-.99818	-.99999	-.98759	-1.00000
9	.83333	0.00000	-.99608	-1.00001	-.72985	-1.00000
10	.94444	0.00000	-1.01001	-1.00000	-1.51632	-1.00000
11	1.00000	.04167	0.00000	0.00000	0.00000	0.00000
12	1.00000	.12500	0.00000	0.00000	0.00000	0.00000
13	1.00000	.20833	0.00000	0.00000	0.00000	0.00000
14	1.00000	.29167	0.00000	0.00000	0.00000	0.00000
15	1.00000	.37500	0.00000	0.00000	0.00000	0.00000
16	1.00000	.45833	0.00000	0.00000	0.00000	0.00000
17	1.00000	.54167	0.00000	0.00000	0.00000	0.00000
18	1.00000	.62500	0.00000	0.00000	0.00000	0.00000
19	1.00000	.70833	0.00000	0.00000	0.00000	0.00000
20	1.00000	.79167	0.00000	0.00000	0.00000	0.00000
21	1.00000	.87500	0.00000	0.00000	0.00000	0.00000
22	1.00000	.95833	0.00000	0.00000	0.00000	0.00000
23	1.00000	1.04167	0.00000	0.00000	0.00000	0.00000
24	1.00000	1.12500	0.00000	0.00000	0.00000	0.00000
25	1.00000	1.20833	0.00000	0.00000	0.00000	0.00000
26	1.00000	1.29167	0.00000	0.00000	0.00000	0.00000
27	1.00000	1.37500	0.00000	0.00000	0.00000	0.00000
28	1.00000	1.45833	0.00000	0.00000	0.00000	0.00000
29	1.00000	1.54167	0.00000	0.00000	0.00000	0.00000
30	1.00000	1.62500	0.00000	0.00000	0.00000	0.00000
31	1.00000	1.70833	0.00000	0.00000	0.00000	0.00000
32	1.00000	1.79167	0.00000	0.00000	0.00000	0.00000
33	1.00000	1.87500	0.00000	0.00000	0.00000	0.00000
34	1.00000	1.95833	0.00000	0.00000	0.00000	0.00000
35	.94444	2.00000	1.00000	1.00000	1.00000	1.00000
36	.83333	2.00000	1.00000	1.00000	1.00000	1.00000
37	.72222	2.00000	1.00000	1.00000	1.00000	1.00000
38	.61111	2.00000	1.00000	1.00000	1.00000	1.00000
39	.50000	2.00000	1.00000	1.00000	1.00000	1.00000
40	.38889	2.00000	1.00000	1.00000	1.00000	1.00000
41	.27778	2.00000	1.00000	1.00000	1.00000	1.00000
42	.16667	2.00000	1.00000	1.00000	1.00000	1.00000
43	.05556	2.00000	1.00000	1.00000	1.00000	1.00000
44	0.00000	1.94444	0.00000	0.00000	0.00000	0.00000
45	0.00000	1.83333	0.00000	0.00000	0.00000	0.00000
46	0.00000	1.72222	0.00000	0.00000	0.00000	0.00000
47	0.00000	1.61111	0.00000	0.00000	0.00000	0.00000
48	0.00000	1.50000	0.00000	0.00000	0.00000	0.00000
49	0.00000	1.38889	0.00000	0.00000	0.00000	0.00000

Table 10 (cont'd.)

50	0.00000	1.27778	0.00000	0.00000	0.00000	0.00000
51	0.00000	1.16667	0.00000	0.00000	0.00000	0.00000
52	0.00000	1.05556	0.00000	0.00000	0.00000	0.00000
53	0.00000	.95833	0.00000	0.00000	0.00000	0.00000
54	0.00000	.87500	0.00000	0.00000	0.00000	0.00000
55	0.00000	.79167	0.00000	0.00000	0.00000	0.00000
56	0.00000	.70833	0.00000	0.00000	0.00000	0.00000
57	0.00000	.62500	0.00000	0.00000	0.00000	0.00000
58	0.00000	.54167	0.00000	0.00000	0.00000	0.00000
59	0.00000	.45833	0.00000	0.00000	0.00000	0.00000
60	0.00000	.37500	0.00000	0.00000	0.00000	0.00000
61	0.00000	.29167	0.00000	0.00000	0.00000	0.00000
62	0.00000	.20833	0.00000	0.00000	0.00000	0.00000
63	0.00000	.12500	0.00000	0.00000	0.00000	0.00000

Table 11. Pure Extension - 63 Nodes: Interior Values

INTERIOR X-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	.25000	.50000	-.02651	-.02500	-.02447	-.02500
2	.25000	1.00000	-.02668	-.02500	-.02551	-.02500
3	.25000	1.50000	-.02686	-.02500	-.02491	-.02500
4	.50000	.50000	-.05167	-.05000	-.04911	-.05000
5	.50000	1.00000	-.05179	-.05000	-.04945	-.05000
6	.50000	1.50000	-.05204	-.05000	-.04929	-.05000
7	.75000	.50000	-.07682	-.07500	-.07360	-.07500
8	.75000	1.00000	-.07689	-.07500	-.07376	-.07500
9	.75000	1.50000	-.07726	-.07500	-.07422	-.07500

INTERIOR Y-DISP

NO	X	Y	UYA	UYB	UYC	EXACT
1	.25000	.50000	.20044	.20000	.18762	.20000
2	.25000	1.00000	.40041	.40000	.38628	.40000
3	.25000	1.50000	.60018	.60000	.58542	.60000
4	.50000	.50000	.20046	.20000	.18645	.20000
5	.50000	1.00000	.40043	.40000	.38477	.40000
6	.50000	1.50000	.60041	.60000	.58321	.60000
7	.75000	.50000	.20052	.20000	.18472	.20000
8	.75000	1.00000	.40049	.40000	.38359	.40000
9	.75000	1.50000	.60043	.60000	.58149	.60000

INTERIOR SXX

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	-.00166	.00000	.00296	0.00000
2	.25000	1.00000	-.00128	.00000	.01982	0.00000
3	.25000	1.50000	-.00140	.00000	.01597	0.00000
4	.50000	.50000	-.00161	.00000	.00258	0.00000
5	.50000	1.00000	-.00127	.00000	.00792	0.00000

Table 11 (cont'd.)

6	.50000	1.50000	-.00214	.00000	.00068	0.00000
7	.75000	.50000	-.00145	.00000	.00737	0.00000
8	.75000	1.00000	-.00107	.00000	.00631	0.00000
9	.75000	1.50000	-.00193	.00000	.00033	0.00000

INTERIOR SXY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	-.00011	.00000	-.00569	0.00000
2	.25000	1.00000	-.00044	.00000	-.01226	0.00000
3	.25000	1.50000	.00114	.00000	-.00895	0.00000
4	.50000	.50000	.00009	.00000	-.00481	0.00000
5	.50000	1.00000	-.00025	.00000	-.00663	0.00000
6	.50000	1.50000	.00001	.00000	-.00678	0.00000
7	.75000	.50000	.00039	.00000	-.00607	0.00000
8	.75000	1.00000	-.00017	.00000	-.00480	0.00000
9	.75000	1.50000	-.00110	.00000	-.00648	0.00000

INTERIOR SYY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	.99946	1.00000	.98993	1.00000
2	.25000	1.00000	.99902	1.00000	.99072	1.00000
3	.25000	1.50000	.99873	1.00000	.98517	1.00000
4	.50000	.50000	.99978	1.00000	.99343	1.00000
5	.50000	1.00000	.99928	1.00000	.99285	1.00000
6	.50000	1.50000	1.00025	1.00000	.99240	1.00000
7	.75000	.50000	.99960	1.00000	.99394	1.00000
8	.75000	1.00000	.99942	1.00000	.99454	1.00000
9	.75000	1.50000	.99949	1.00000	.98585	1.00000

Table 12. Pure Extension - 63 Nodes: Percent Differences

X-DISPLACEMENTS

NO	A	B	C
2	-41.21	.00	-185.61
3	-10.93	.00	-62.30
4	-6.20	.00	-8.53
5	-4.26	.00	-1.64
6	-3.22	.00	.34
7	-2.56	.00	1.53
8	-2.11	.00	3.42
9	-1.74	.00	9.54
10	-1.06	.00	41.32
11	-3.37	.00	28.55
12	-3.49	.00	-8.64
13	-3.43	.00	-2.99
14	-3.38	.00	.37
15	-3.35	.00	1.63
16	-3.32	.00	2.13

Table 12 (cont'd.)

17	-3.30	.00	2.31
18	-3.29	.00	2.33
19	-3.29	.00	2.27
20	-3.30	.00	2.17
21	-3.31	.00	2.04
22	-3.34	.00	1.91
23	-3.38	.00	1.79
24	-3.42	.00	1.69
25	-3.48	.00	1.59
26	-3.55	.00	1.51
27	-3.62	.00	1.42
28	-3.70	.00	1.31
29	-3.79	.00	1.16
30	-3.88	.00	.89
31	-3.95	.00	.29
32	-4.00	.00	-1.04
33	-3.98	.00	-.75
34	-3.67	.00	28.22
35	-1.32	.00	35.57
36	-2.11	.00	13.22
37	-2.75	.00	8.58
38	-3.52	.00	6.55
39	-4.60	.00	5.08
40	-6.30	.00	2.73
41	-9.43	.00	-3.23
42	-17.12	.00	.26
43	-66.85	.00	257.77

Y-DISPLACEMENTS

NO	A	B	C
1	-7.37	.00	-290.90
11	-7.80	.00	-333.39
12	-1.48	.00	20.16
13	-.77	.00	25.49
14	-.51	.00	17.91
15	-.38	.00	13.16
16	-.31	.00	10.25
17	-.26	.00	8.35
18	-.22	.00	7.02
19	-.20	.00	6.06
20	-.18	.00	5.35
21	-.16	.00	4.80
22	-.15	.00	4.37
23	-.14	.00	4.04
24	-.13	.00	3.77
25	-.12	.00	3.55
26	-.11	.00	3.37
27	-.10	.00	3.24
28	-.08	.00	3.13
29	-.07	.00	3.06
30	-.05	.00	3.04

Table 12 (cont'd.)

31	-.03	.00	3.08
32	.00	.00	3.38
33	.04	.00	4.73
34	.14	.00	8.59
35	-.02	.00	4.98
36	-.12	.00	4.33
37	-.14	.00	4.03
38	-.15	.00	3.95
39	-.15	.00	3.90
40	-.13	.00	3.88
41	-.11	.00	3.97
42	-.07	.00	4.40
43	.10	.00	-1.36
44	.22	.00	-.04
45	.10	.00	.53
46	.05	.00	1.32
47	.02	.00	1.73
48	.00	.00	2.00
49	-.01	.00	2.23
50	-.03	.00	2.42
51	-.04	.00	2.56
52	-.06	.00	2.20
53	-.07	.00	3.88
54	-.09	.00	1.59
55	-.11	.00	4.53
56	-.13	.00	4.87
57	-.16	.00	5.33
58	-.20	.00	5.92
59	-.25	.00	6.70
60	-.32	.00	7.73
61	-.45	.00	9.19
62	-.69	.00	51.19
63	-1.34	.00	1.90

Y-TRACTIONS

NO	A	B	C
2	-.85	.00	7.69
3	.42	.00	-5.89
4	.22	.00	-5.18
5	.20	.00	-2.18
6	.18	.00	-.51
7	.18	.00	.89
8	.18	.00	1.24
9	.39	.00	27.02
10	-1.00	.00	-51.63

INTERIOR X-DISP

NO	A	B	C
1	-6.04	.00	2.11
2	-6.74	.00	-2.02

Table 12 (cont'd.)

3	-7.46	.00	.36
4	-3.33	.00	1.78
5	-3.58	.00	1.11
6	-4.08	.00	1.42
7	-2.42	.00	1.86
8	-2.53	.00	1.66
9	-3.01	.00	1.04

INTERIOR Y-DISP

NO	A	B	C
1	-.22	.00	6.19
2	-.10	.00	3.43
3	-.03	.00	2.43
4	-.23	.00	6.78
5	-.11	.00	3.81
6	-.07	.00	2.80
7	-.26	.00	7.64
8	-.12	.00	4.10
9	-.07	.00	3.09

INTERIOR SYY

NO	A	B	C
1	.05	.00	1.01
2	.10	.00	.93
3	.13	.00	1.48
4	.02	.00	.66
5	.07	.00	.72
6	-.03	.00	.76
7	.04	.00	.61
8	.06	.00	.55
9	.05	.00	1.42

Table 13. Square - 21 Nodes: X - Displacements

X-DISPLACEMENTS

NO	X	Y	UXA	UXB	UXC	EXACT
1	0.00000	.08333	0.00000	0.00000	0.00000	0.00000
2	.10000	0.00000	.16661	.15998	.12491	.15997
3	.30000	0.00000	.48854	.47872	.42453	.47910
4	.50000	0.00000	.81024	.79528	.75824	.79583
5	.70000	0.00000	1.12711	1.10769	1.10029	1.10857
6	.90000	0.00000	1.43617	1.41354	1.47046	1.41570
7	1.00000	.10000	1.57028	1.55964	1.65303	1.55892
8	1.00000	.30000	1.56760	1.55867	1.42869	1.55692
9	1.00000	.50000	1.57974	1.57500	1.44284	1.57292
10	1.00000	.70000	1.60466	1.60855	1.43724	1.60692
11	1.00000	.90000	1.62674	1.65723	1.37350	1.65892
12	.90000	1.00000	1.47094	1.52260	1.05197	1.52820

Table 13 (cont'd.)

13	.70000	1.00000	1.16171	1.19346	1.02668	1.19607
14	.50000	1.00000	.83687	.85689	.77929	.85833
15	.30000	1.00000	.50457	.51575	.49220	.51660
16	.10000	1.00000	.17078	.17233	.16281	.17247
17	0.00000	.91667	0.00000	0.00000	0.00000	0.00000
18	0.00000	.75000	0.00000	0.00000	0.00000	0.00000
19	0.00000	.58333	0.00000	0.00000	0.00000	0.00000
20	0.00000	.41667	0.00000	0.00000	0.00000	0.00000
21	0.00000	.25000	0.00000	0.00000	0.00000	0.00000

Table 14. Square - 21 Nodes: Y - Displacements

Y-DISPLACEMENTS

NO	X	Y	UYA	UYB	UYC	EXACT
1	0.00000	.08333	-.06238	-.02709	-.03212	-.03197
2	.10000	0.00000	-.00450	-.00450	-.00450	-.00450
3	.30000	0.00000	-.04050	-.04050	-.04050	-.04050
4	.50000	0.00000	-.11250	-.11250	-.11250	-.11250
5	.70000	0.00000	-.22050	-.22050	-.22050	-.22050
6	.90000	0.00000	-.36450	-.36450	-.36450	-.36450
7	1.00000	.10000	-.49160	-.48032	-.43473	-.48555
8	1.00000	.30000	-.55198	-.54201	-.46370	-.54585
9	1.00000	.50000	-.59429	-.59006	-.50192	-.59375
10	1.00000	.70000	-.62192	-.62775	-.48306	-.63165
11	1.00000	.90000	-.62797	-.65726	-.40117	-.66195
12	.90000	1.00000	-.61124	-.59490	-.40115	-.59425
13	.70000	1.00000	-.49212	-.46004	-.50421	-.45825
14	.50000	1.00000	-.39729	-.35799	-.41121	-.35625
15	.30000	1.00000	-.33432	-.29000	-.34075	-.28825
16	.10000	1.00000	-.30638	-.25592	-.36235	-.25425
17	0.00000	.91667	-.26109	-.23466	-.36142	-.23712
18	0.00000	.75000	-.23952	-.20654	-.24584	-.20859
19	0.00000	.58333	-.20439	-.17272	-.21656	-.17520
20	0.00000	.41667	-.16231	-.13256	-.16971	-.13556
21	0.00000	.25000	-.11245	-.08471	-.13878	-.08828

Table 15. Square - 21 Nodes: X - Tractions

X-TRACTIONS

NO	X	Y	TXA	TXB	TXC	EXACT
1	0.00000	.08333	-4.13240	-4.01886	-3.30061	-4.00347
2	.10000	0.00000	.10000	.10000	.10000	.10000
3	.30000	0.00000	.30000	.30000	.30000	.30000
4	.50000	0.00000	.50000	.50000	.50000	.50000
5	.70000	0.00000	.70000	.70000	.70000	.70000
6	.90000	0.00000	.90000	.90000	.90000	.90000
7	1.00000	.10000	3.75500	3.75500	3.75500	3.75500
8	1.00000	.30000	3.79500	3.79500	3.79500	3.79500

Table 15 (cont'd.)

9	1.00000	.50000	3.87500	3.87500	3.87500	3.87500
10	1.00000	.70000	3.99500	3.99500	3.99500	3.99500
11	1.00000	.90000	4.15500	4.15500	4.15500	4.15500
12	.90000	1.00000	-.45000	-.45000	-.45000	-.45000
13	.70000	1.00000	-.35000	-.35000	-.35000	-.35000
14	.50000	1.00000	-.25000	-.25000	-.25000	-.25000
15	.30000	1.00000	-.15000	-.15000	-.15000	-.15000
16	.10000	1.00000	-.05000	-.05000	-.05000	-.05000
17	0.00000	.91667	-4.47629	-4.44779	-1.86205	-4.42014
18	0.00000	.75000	-4.18303	-4.24155	-5.08293	-4.28125
19	0.00000	.58333	-4.13571	-4.18825	-4.50472	-4.17014
20	0.00000	.41667	-4.07330	-4.07308	-3.96936	-4.08681
21	0.00000	.25000	-3.99275	-4.02877	-5.90831	-4.03125

Table 16. Square - 21 Nodes: Y - Tractions

Y-TRACTIONS

NO	X	Y	TYA	TYB	TYC	EXACT
1	0.00000	.08333	0.00000	0.00000	0.00000	0.00000
2	.10000	0.00000	.00965	.01257	-.40149	0.00000
3	.30000	0.00000	.03890	.00244	.28765	0.00000
4	.50000	0.00000	.02188	.01382	.27392	0.00000
5	.70000	0.00000	-.00711	-.00094	.11283	0.00000
6	.90000	0.00000	-.12226	.00752	-.43576	0.00000
7	1.00000	.10000	-.95000	-.95000	-.95000	-.95000
8	1.00000	.30000	-.85000	-.85000	-.85000	-.85000
9	1.00000	.50000	-.75000	-.75000	-.75000	-.75000
10	1.00000	.70000	-.65000	-.65000	-.65000	-.65000
11	1.00000	.90000	-.55000	-.55000	-.55000	-.55000
12	.90000	1.00000	.75000	.75000	.75000	.75000
13	.70000	1.00000	.75000	.75000	.75000	.75000
14	.50000	1.00000	.75000	.75000	.75000	.75000
15	.30000	1.00000	.75000	.75000	.75000	.75000
16	.10000	1.00000	.75000	.75000	.75000	.75000
17	0.00000	.91667	0.00000	0.00000	0.00000	0.00000
18	0.00000	.75000	0.00000	0.00000	0.00000	0.00000
19	0.00000	.58333	0.00000	0.00000	0.00000	0.00000
20	0.00000	.41667	0.00000	0.00000	0.00000	0.00000
21	0.00000	.25000	0.00000	0.00000	0.00000	0.00000

Table 17. Square - 21 Nodes: Interior Values

INTERIOR X-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	.25000	.50000	.40295	.40096	.46337	.40104
2	.50000	.50000	.80191	.79890	.74203	.79896
3	.75000	.50000	1.19548	1.19084	1.02113	1.19062

Table 17 (cont'd.)

4	.50000	.75000	.81498	.82104	.76488	.82161
5	.50000	.25000	.79826	.79032	.72923	.79036

INTERIOR Y-DISP

NO	X	Y	UYA	UYB	UYC	EXACT
1	.25000	.50000	-.21033	-.18155	-.23262	-.18359
2	.50000	.50000	-.28775	-.26358	-.30050	-.26563
3	.75000	.50000	-.41491	-.39997	-.39314	-.40234
4	.50000	.75000	-.34319	-.31498	-.35766	-.31641
5	.50000	.25000	-.21685	-.19660	-.22326	-.19922

INTERIOR SXX

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	4.10227	4.10770	3.91982	4.10938
2	.50000	.50000	4.06440	4.06207	4.04759	4.06250
3	.75000	.50000	4.00304	3.98706	4.05478	3.98438
4	.50000	.75000	4.17205	4.21496	3.72216	4.21875
5	.50000	.25000	3.98660	3.96764	4.00292	3.96875

INTERIOR SXY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	-.19181	-.18875	-.15594	-.18750
2	.50000	.50000	-.37550	-.37543	-.34196	-.37500
3	.75000	.50000	-.54214	-.56115	-.44944	-.56250
4	.50000	.75000	-.33247	-.31565	-.28413	-.31250
5	.50000	.25000	-.43381	-.43622	-.59197	-.43750

INTERIOR SYY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	.37973	.43102	.41555	.43750
2	.50000	.50000	.38910	.43100	.26430	.43750
3	.75000	.50000	.42416	.43216	.34010	.43750
4	.50000	.75000	.55615	.60311	.68714	.60938
5	.50000	.25000	.19261	.22925	.31365	.23438

Table 18. Square - 21 Nodes: Percent Differences

X-DISPLACEMENTS

NO	A	B	C
2	-4.15	-.01	21.92
3	-1.99	.08	11.39
4	-1.81	.07	4.72
5	-1.67	.08	.75
6	-1.45	.15	-3.87
7	-.73	-.05	-6.04
8	-.69	-.11	8.24

Table 18 (cont'd.)

9	-.43	-.13	8.27
10	.14	-.10	10.56
11	1.94	.10	17.21
12	3.75	.37	31.16
13	2.87	.22	14.16
14	2.50	.17	9.21
15	2.33	.16	4.72
16	.98	.08	5.60

Y-DISPLACEMENTS

NO	A	B	C
1	-95.10	15.27	-.45
7	-1.25	1.08	10.47
8	-1.12	.70	15.05
9	-.09	.62	15.47
10	1.54	.62	23.52
11	5.13	.71	39.40
12	-2.86	-.11	32.50
13	-7.39	-.39	-10.03
14	-11.52	-.49	-15.43
15	-15.98	-.61	-18.21
16	-20.50	-.66	-42.52
17	-10.11	1.04	-52.42
18	-14.83	.98	-17.85
19	-16.66	1.42	-23.61
20	-19.73	2.21	-25.19
21	-27.38	4.04	-57.20

X-TRACTIONS

NO	A	B	C
1	-3.22	-.38	17.56
17	-1.27	-.63	57.87
18	2.29	.93	-18.73
19	.83	-.43	-8.02
20	.33	.34	2.87
21	.95	.06	-46.56

INTERIOR X-DISP

NO	A	B	C
1	-.48	.02	-15.54
2	-.37	.01	7.13
3	-.41	-.02	14.24
4	.81	.07	6.90
5	-1.00	.01	7.73

INTERIOR Y-DISP

NO	A	B	C
1	-14.56	1.11	-26.70

Table 18 (cont'd.)

2	-8.33	.77	-13.13
3	-3.12	.59	2.29
4	-8.47	.45	-13.04
5	-8.85	1.31	-12.07

INTERIOR SXX

NO	A	B	C
1	.17	.04	4.61
2	-.05	.01	.37
3	-.47	-.07	-1.77
4	1.11	.09	11.77
5	-.45	.03	-.86

INTERIOR SXY

NO	A	B	C
1	-2.30	-.67	16.83
2	-.13	-.11	8.81
3	3.62	.24	20.10
4	-6.39	-1.01	9.08
5	.84	.29	-35.31

INTERIOR SYY

NO	A	B	C
1	13.21	1.48	5.02
2	11.06	1.49	39.59
3	3.05	1.22	22.26
4	8.73	1.03	-12.76
5	17.82	2.19	-33.82

Table 19. Square - 63 Nodes: X - Displacements

X-DISPLACEMENTS

NO	X	Y	UXA	UXB	UXC	EXACT
1	0.00000	.02778	0.00000	0.00000	0.00000	0.00000
2	.03333	0.00000	.05511	.05332	.00493	.05333
3	.10000	0.00000	.16156	.15993	.10622	.15997
4	.16667	0.00000	.26859	.26648	.21864	.26651
5	.23333	0.00000	.37552	.37287	.32572	.37291
6	.30000	0.00000	.48225	.47906	.43269	.47910
7	.36667	0.00000	.58869	.58497	.53930	.58502
8	.43333	0.00000	.69480	.69056	.64551	.69062
9	.50000	0.00000	.80049	.79577	.75136	.79583
10	.56667	0.00000	.90570	.90052	.85691	.90060
11	.63333	0.00000	1.01036	1.00477	.96229	1.00487
12	.70000	0.00000	1.11439	1.10846	1.06778	1.10857
13	.76667	0.00000	1.21772	1.21151	1.17407	1.21165
14	.83333	0.00000	1.32029	1.31387	1.28300	1.31404

Table 19 (cont'd.)

15	.90000	0.00000	1.42203	1.41548	1.40058	1.41570
16	.96667	0.00000	1.52258	1.51619	1.53791	1.51656
17	1.00000	.03333	1.56699	1.56354	1.59452	1.56358
18	1.00000	.10000	1.56300	1.55899	1.47722	1.55892
19	1.00000	.16667	1.56058	1.55640	1.47243	1.55625
20	1.00000	.23333	1.55991	1.55577	1.47058	1.55558
21	1.00000	.30000	1.56106	1.55713	1.47129	1.55692
22	1.00000	.36667	1.56409	1.56048	1.47415	1.56025
23	1.00000	.43333	1.56901	1.56582	1.47882	1.56558
24	1.00000	.50000	1.57580	1.57316	1.48506	1.57292
25	1.00000	.56667	1.58445	1.58248	1.49268	1.58225
26	1.00000	.63333	1.59491	1.59381	1.50141	1.59358
27	1.00000	.70000	1.60711	1.60712	1.51085	1.60692
28	1.00000	.76667	1.62093	1.62243	1.52022	1.62225
29	1.00000	.83333	1.63609	1.63971	1.52695	1.63958
30	1.00000	.90000	1.65191	1.65893	1.51862	1.65892
31	1.00000	.96667	1.66326	1.67983	1.45536	1.68025
32	.96667	1.00000	1.61185	1.63651	1.32756	1.63739
33	.90000	1.00000	1.50966	1.52768	1.38275	1.52820
34	.83333	1.00000	1.40348	1.41785	1.31287	1.41821
35	.76667	1.00000	1.29534	1.30720	1.21982	1.30748
36	.70000	1.00000	1.18590	1.19585	1.12010	1.19607
37	.63333	1.00000	1.07546	1.08386	1.01687	1.08403
38	.56667	1.00000	.96419	.97129	.91144	.97143
39	.50000	1.00000	.85224	.85821	.80462	.85833
40	.43333	1.00000	.73971	.74469	.69699	.74479
41	.36667	1.00000	.62670	.63078	.58906	.63086
42	.30000	1.00000	.51328	.51653	.48153	.51660
43	.23333	1.00000	.39953	.40202	.37541	.40208
44	.16667	1.00000	.28553	.28730	.27219	.28735
45	.10000	1.00000	.17138	.17243	.16830	.17247
46	.03333	1.00000	.05782	.05749	.03847	.05750
47	0.00000	.97222	0.00000	0.00000	0.00000	0.00000
48	0.00000	.91667	0.00000	0.00000	0.00000	0.00000
49	0.00000	.86111	0.00000	0.00000	0.00000	0.00000
50	0.00000	.80556	0.00000	0.00000	0.00000	0.00000
51	0.00000	.75000	0.00000	0.00000	0.00000	0.00000
52	0.00000	.69444	0.00000	0.00000	0.00000	0.00000
53	0.00000	.63889	0.00000	0.00000	0.00000	0.00000
54	0.00000	.58333	0.00000	0.00000	0.00000	0.00000
55	0.00000	.52778	0.00000	0.00000	0.00000	0.00000
56	0.00000	.47222	0.00000	0.00000	0.00000	0.00000
57	0.00000	.41667	0.00000	0.00000	0.00000	0.00000
58	0.00000	.36111	0.00000	0.00000	0.00000	0.00000
59	0.00000	.30556	0.00000	0.00000	0.00000	0.00000
60	0.00000	.25000	0.00000	0.00000	0.00000	0.00000
61	0.00000	.19444	0.00000	0.00000	0.00000	0.00000
62	0.00000	.13889	0.00000	0.00000	0.00000	0.00000
63	0.00000	.08333	0.00000	0.00000	0.00000	0.00000

Table 20. Square - 63 Nodes: Y - Displacements

Y-DISPLACEMENTS

NO	X	Y	UYA	UYB	UYC	EXACT
1	0.00000	.02778	-.02059	-.01039	.00194	-.01096
2	.03333	0.00000	-.00050	-.00050	-.00050	-.00050
3	.10000	0.00000	-.00450	-.00450	-.00450	-.00450
4	.16667	0.00000	-.01250	-.01250	-.01250	-.01250
5	.23333	0.00000	-.02450	-.02450	-.02450	-.02450
6	.30000	0.00000	-.04050	-.04050	-.04050	-.04050
7	.36667	0.00000	-.06050	-.06050	-.06050	-.06050
8	.43333	0.00000	-.08450	-.08450	-.08450	-.08450
9	.50000	0.00000	-.11250	-.11250	-.11250	-.11250
10	.56667	0.00000	-.14450	-.14450	-.14450	-.14450
11	.63333	0.00000	-.18050	-.18050	-.18050	-.18050
12	.70000	0.00000	-.22050	-.22050	-.22050	-.22050
13	.76667	0.00000	-.26450	-.26450	-.26450	-.26450
14	.83333	0.00000	-.31250	-.31250	-.31250	-.31250
15	.90000	0.00000	-.36450	-.36450	-.36450	-.36450
16	.96667	0.00000	-.42050	-.42050	-.42050	-.42050
17	1.00000	.03333	-.46505	-.46169	-.43345	-.46228
18	1.00000	.10000	-.48979	-.48510	-.44758	-.48555
19	1.00000	.16667	-.51138	-.50674	-.47872	-.50718
20	1.00000	.23333	-.53101	-.52683	-.50141	-.52725
21	1.00000	.30000	-.54904	-.54544	-.52122	-.54585
22	1.00000	.36667	-.56563	-.56268	-.53864	-.56308
23	1.00000	.43333	-.58085	-.57863	-.55390	-.57901
24	1.00000	.50000	-.59479	-.59337	-.56710	-.59375
25	1.00000	.56667	-.60751	-.60700	-.57820	-.60738
26	1.00000	.63333	-.61906	-.61961	-.58699	-.61998
27	1.00000	.70000	-.62947	-.63128	-.59291	-.63165
28	1.00000	.76667	-.63871	-.64210	-.59453	-.64248
29	1.00000	.83333	-.64665	-.65214	-.58783	-.65255
30	1.00000	.90000	-.65275	-.66148	-.56221	-.66195
31	1.00000	.96667	-.65355	-.67016	-.50541	-.67078
32	.96667	1.00000	-.64727	-.64714	-.52250	-.64714
33	.90000	1.00000	-.60051	-.59444	-.59797	-.59425
34	.83333	1.00000	-.55417	-.54536	-.55297	-.54514
35	.76667	1.00000	-.51049	-.50003	-.51040	-.49981
36	.70000	1.00000	-.47006	-.45847	-.47120	-.45825
37	.63333	1.00000	-.43311	-.42068	-.43535	-.42047
38	.56667	1.00000	-.39975	-.38668	-.40309	-.38647
39	.50000	1.00000	-.37006	-.35645	-.37460	-.35625
40	.43333	1.00000	-.34407	-.33000	-.34998	-.32981
41	.36667	1.00000	-.32181	-.30733	-.32931	-.30714
42	.30000	1.00000	-.30332	-.28844	-.31259	-.28825
43	.23333	1.00000	-.28861	-.27333	-.29958	-.27314
44	.16667	1.00000	-.27774	-.26200	-.28882	-.26181
45	.10000	1.00000	-.27085	-.25446	-.27733	-.25425
46	.03333	1.00000	-.26840	-.25068	-.31549	-.25047
47	0.00000	.97222	-.25372	-.24555	-.32345	-.24579
48	0.00000	.91667	-.24767	-.23695	-.25774	-.23712
49	0.00000	.86111	-.23845	-.22788	-.24586	-.22807

Table 20 (cont'd.)

50	0.00000	.80556	-.22876	-.21837	-.23541	-.21858
51	0.00000	.75000	-.21858	-.20837	-.22473	-.20859
52	0.00000	.69444	-.20785	-.19783	-.21357	-.19807
53	0.00000	.63889	-.19650	-.18670	-.20180	-.18696
54	0.00000	.58333	-.18450	-.17492	-.18935	-.17520
55	0.00000	.52778	-.17178	-.16245	-.17617	-.16275
56	0.00000	.47222	-.15820	-.14924	-.16222	-.14956
57	0.00000	.41667	-.14401	-.13522	-.14747	-.13556
58	0.00000	.36111	-.12887	-.12036	-.13192	-.12072
59	0.00000	.30556	-.11282	-.10460	-.11562	-.10498
60	0.00000	.25000	-.09583	-.08788	-.09867	-.08828
61	0.00000	.19444	-.07785	-.07017	-.08134	-.07058
62	0.00000	.13889	-.05885	-.05139	-.06268	-.05183
63	0.00000	.08333	-.03886	-.03151	-.04767	-.03197

Table 21. Square - 63 Nodes: X - Tractions

X-TRACTIONS

NO	X	Y	TXA	TXB	TXC	EXACT
1	0.00000	.02778	-4.11647	-4.00678	-3.24222	-4.00039
2	.03333	0.00000	.03333	.03333	.03333	.03333
3	.10000	0.00000	.10000	.10000	.10000	.10000
4	.16667	0.00000	.16667	.16667	.16667	.16667
5	.23333	0.00000	.23333	.23333	.23333	.23333
6	.30000	0.00000	.30000	.30000	.30000	.30000
7	.36667	0.00000	.36667	.36667	.36667	.36667
8	.43333	0.00000	.43333	.43333	.43333	.43333
9	.50000	0.00000	.50000	.50000	.50000	.50000
10	.56667	0.00000	.56667	.56667	.56667	.56667
11	.63333	0.00000	.63333	.63333	.63333	.63333
12	.70000	0.00000	.70000	.70000	.70000	.70000
13	.76667	0.00000	.76667	.76667	.76667	.76667
14	.83333	0.00000	.83333	.83333	.83333	.83333
15	.90000	0.00000	.90000	.90000	.90000	.90000
16	.96667	0.00000	.96667	.96667	.96667	.96667
17	1.00000	.03333	3.75056	3.75056	3.75056	3.75056
18	1.00000	.10000	3.75500	3.75500	3.75500	3.75500
19	1.00000	.16667	3.76389	3.76389	3.76389	3.76389
20	1.00000	.23333	3.77722	3.77722	3.77722	3.77722
21	1.00000	.30000	3.79500	3.79500	3.79500	3.79500
22	1.00000	.36667	3.81722	3.81722	3.81722	3.81722
23	1.00000	.43333	3.84389	3.84389	3.84389	3.84389
24	1.00000	.50000	3.87500	3.87500	3.87500	3.87500
25	1.00000	.56667	3.91056	3.91056	3.91056	3.91056
26	1.00000	.63333	3.95056	3.95056	3.95056	3.95056
27	1.00000	.70000	3.99500	3.99500	3.99500	3.99500
28	1.00000	.76667	4.04389	4.04389	4.04389	4.04389
29	1.00000	.83333	4.09722	4.09722	4.09722	4.09722
30	1.00000	.90000	4.15500	4.15500	4.15500	4.15500
31	1.00000	.96667	4.21722	4.21722	4.21722	4.21722

Table 21 (cont'd.)

32	.96667	1.00000	-.48333	-.48333	-.48333	-.48333
33	.90000	1.00000	-.45000	-.45000	-.45000	-.45000
34	.83333	1.00000	-.41667	-.41667	-.41667	-.41667
35	.76667	1.00000	-.38333	-.38333	-.38333	-.38333
36	.70000	1.00000	-.35000	-.35000	-.35000	-.35000
37	.63333	1.00000	-.31667	-.31667	-.31667	-.31667
38	.56667	1.00000	-.28333	-.28333	-.28333	-.28333
39	.50000	1.00000	-.25000	-.25000	-.25000	-.25000
40	.43333	1.00000	-.21667	-.21667	-.21667	-.21667
41	.36667	1.00000	-.18333	-.18333	-.18333	-.18333
42	.30000	1.00000	-.15000	-.15000	-.15000	-.15000
43	.23333	1.00000	-.11667	-.11667	-.11667	-.11667
44	.16667	1.00000	-.08333	-.08333	-.08333	-.08333
45	.10000	1.00000	-.05000	-.05000	-.05000	-.05000
46	.03333	1.00000	-.01667	-.01667	-.01667	-.01667
47	0.00000	.97222	-4.58134	-4.48431	-.98029	-4.47261
48	0.00000	.91667	-4.35249	-4.40426	-5.11678	-4.42014
49	0.00000	.86111	-4.34655	-4.38360	-5.53915	-4.37076
50	0.00000	.80556	-4.30382	-4.31159	-4.94881	-4.32446
51	0.00000	.75000	-4.26452	-4.29182	-4.42764	-4.28125
52	0.00000	.69444	-4.22762	-4.23081	-4.18993	-4.24113
53	0.00000	.63889	-4.19348	-4.21241	-4.12636	-4.20409
54	0.00000	.58333	-4.16213	-4.16225	-4.11612	-4.17014
55	0.00000	.52778	-4.13353	-4.14524	-4.10435	-4.13927
56	0.00000	.47222	-4.10769	-4.10613	-4.08172	-4.11150
57	0.00000	.41667	-4.08454	-4.09044	-4.05309	-4.08681
58	0.00000	.36111	-4.06408	-4.06224	-4.02181	-4.06520
59	0.00000	.30556	-4.04622	-4.04791	-3.98148	-4.04668
60	0.00000	.25000	-4.03071	-4.03077	-3.93725	-4.03125
61	0.00000	.19444	-4.01726	-4.01768	-3.62106	-4.01890
62	0.00000	.13889	-4.00656	-4.01160	-3.54151	-4.00964
63	0.00000	.08333	-3.96093	-3.99992	-6.35215	-4.00347

Table 22. Square - 63 Nodes: Y - Tractions

Y-TRACTIONS

NO	X	Y	TYA	TYB	TYC	EXACT
1	0.00000	.02778	0.00000	0.00000	0.00000	0.00000
2	.03333	0.00000	-.01197	.00116	-.87809	0.00000
3	.10000	0.00000	.01260	.00062	.02971	0.00000
4	.16667	0.00000	.01300	.00181	.30095	0.00000
5	.23333	0.00000	.01249	.00016	.24658	0.00000
6	.30000	0.00000	.01155	.00194	.15212	0.00000
7	.36667	0.00000	.01020	-.00017	.09823	0.00000
8	.43333	0.00000	.00839	.00206	.07835	0.00000
9	.50000	0.00000	.00611	-.00044	.07126	0.00000
10	.56667	0.00000	.00339	.00214	.06529	0.00000
11	.63333	0.00000	.00029	-.00072	.05820	0.00000
12	.70000	0.00000	-.00308	.00226	.05251	0.00000
13	.76667	0.00000	-.000650	-.00107	.05223	0.00000

Table 22 (cont'd.)

14	.83333	0.00000	-.00967	.00247	.08417	0.00000
15	.90000	0.00000	-.00986	-.00127	.06669	0.00000
16	.96667	0.00000	-.07304	.00171	-.57466	0.00000
17	1.00000	.03333	-.98333	-.98333	-.98333	-.98333
18	1.00000	.10000	-.95000	-.95000	-.95000	-.95000
19	1.00000	.16667	-.91667	-.91667	-.91667	-.91667
20	1.00000	.23333	-.88333	-.88333	-.88333	-.88333
21	1.00000	.30000	-.85000	-.85000	-.85000	-.85000
22	1.00000	.36667	-.81667	-.81667	-.81667	-.81667
23	1.00000	.43333	-.78333	-.78333	-.78333	-.78333
24	1.00000	.50000	-.75000	-.75000	-.75000	-.75000
25	1.00000	.56667	-.71667	-.71667	-.71667	-.71667
26	1.00000	.63333	-.68333	-.68333	-.68333	-.68333
27	1.00000	.70000	-.65000	-.65000	-.65000	-.65000
28	1.00000	.76667	-.61667	-.61667	-.61667	-.61667
29	1.00000	.83333	-.58333	-.58333	-.58333	-.58333
30	1.00000	.90000	-.55000	-.55000	-.55000	-.55000
31	1.00000	.96667	-.51667	-.51667	-.51667	-.51667
32	.96667	1.00000	.75000	.75000	.75000	.75000
33	.90000	1.00000	.75000	.75000	.75000	.75000
34	.83333	1.00000	.75000	.75000	.75000	.75000
35	.76667	1.00000	.75000	.75000	.75000	.75000
36	.70000	1.00000	.75000	.75000	.75000	.75000
37	.63333	1.00000	.75000	.75000	.75000	.75000
38	.56667	1.00000	.75000	.75000	.75000	.75000
39	.50000	1.00000	.75000	.75000	.75000	.75000
40	.43333	1.00000	.75000	.75000	.75000	.75000
41	.36667	1.00000	.75000	.75000	.75000	.75000
42	.30000	1.00000	.75000	.75000	.75000	.75000
43	.23333	1.00000	.75000	.75000	.75000	.75000
44	.16667	1.00000	.75000	.75000	.75000	.75000
45	.10000	1.00000	.75000	.75000	.75000	.75000
46	.03333	1.00000	.75000	.75000	.75000	.75000
47	0.00000	.97222	0.00000	0.00000	0.00000	0.00000
48	0.00000	.91667	0.00000	0.00000	0.00000	0.00000
49	0.00000	.86111	0.00000	0.00000	0.00000	0.00000
50	0.00000	.80556	0.00000	0.00000	0.00000	0.00000
51	0.00000	.75000	0.00000	0.00000	0.00000	0.00000
52	0.00000	.69444	0.00000	0.00000	0.00000	0.00000
53	0.00000	.63889	0.00000	0.00000	0.00000	0.00000
54	0.00000	.58333	0.00000	0.00000	0.00000	0.00000
55	0.00000	.52778	0.00000	0.00000	0.00000	0.00000
56	0.00000	.47222	0.00000	0.00000	0.00000	0.00000
57	0.00000	.41667	0.00000	0.00000	0.00000	0.00000
58	0.00000	.36111	0.00000	0.00000	0.00000	0.00000
59	0.00000	.30556	0.00000	0.00000	0.00000	0.00000
60	0.00000	.25000	0.00000	0.00000	0.00000	0.00000
61	0.00000	.19444	0.00000	0.00000	0.00000	0.00000
62	0.00000	.13889	0.00000	0.00000	0.00000	0.00000
63	0.00000	.08333	0.00000	0.00000	0.00000	0.00000

Table 23. Square - 63 Nodes: Interior Values

INTERIOR X-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	.25000	.50000	.40186	.40104	.65922	.40104
2	.50000	.50000	.80031	.79896	.76110	.79896
3	.75000	.50000	1.19277	1.19065	.86015	1.19062
4	.50000	.75000	.82009	.82157	.82222	.82161
5	.50000	.25000	.79309	.79036	.70008	.79036

INTERIOR Y-DISP

NO	X	Y	UYA	UYB	UYC	EXACT
1	.25000	.50000	-.19207	-.18337	-.19715	-.18359
2	.50000	.50000	-.27281	-.26540	-.27561	-.26563
3	.75000	.50000	-.40690	-.40210	-.40024	-.40234
4	.50000	.75000	-.32509	-.31625	-.33046	-.31641
5	.50000	.25000	-.20494	-.19893	-.20476	-.19922

INTERIOR SXX

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	4.10915	4.10917	4.04201	4.10938
2	.50000	.50000	4.06554	4.06247	4.00402	4.06250
3	.75000	.50000	3.99143	3.98465	3.93337	3.98438
4	.50000	.75000	4.20651	4.21840	4.07419	4.21875
5	.50000	.25000	3.97444	3.96861	3.95307	3.96875

INTERIOR SXY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	-.18875	-.18758	-.19350	-.18750
2	.50000	.50000	-.37560	-.37504	-.37101	-.37500
3	.75000	.50000	-.55765	-.56239	-.51642	-.56250
4	.50000	.75000	-.31989	-.31281	-.32148	-.31250
5	.50000	.25000	-.4654	-.4735	-.43464	-.43750

INTERIOR SYY

NO	X	Y	A	B	C	EXACT
1	.25000	.50000	.41995	.43683	.37092	.43750
2	.50000	.50000	.42242	.43678	.37659	.43750
3	.75000	.50000	.43477	.43686	.41777	.43750
4	.50000	.75000	.59417	.60871	.55431	.60938
5	.50000	.25000	.22260	.23379	.18115	.23438

Table 24. Square - 63 Nodes: Percent Differences

X-DISPLACEMENTS

NO	A	B	C
2	-3.34	.02	90.75

Table 24 (cont'd.)

3	-1.00	.02	33.60
4	-.78	.01	17.96
5	-.70	.01	12.65
6	-.66	.01	9.69
7	-.63	.01	7.82
8	-.60	.01	6.53
9	-.59	.01	5.59
10	-.57	.01	4.85
11	-.55	.01	4.24
12	-.53	.01	3.68
13	-.50	.01	3.10
14	-.48	.01	2.36
15	-.45	.02	1.07
16	-.40	.02	-1.41
17	-.22	.00	-1.98
18	-.26	.00	5.24
19	-.28	-.01	5.39
20	-.28	-.01	5.46
21	-.27	-.01	5.50
22	-.25	-.01	5.52
23	-.22	-.02	5.54
24	-.18	-.02	5.59
25	-.14	-.01	5.66
26	-.08	-.01	5.78
27	-.01	-.01	5.98
28	.08	-.01	6.29
29	.21	-.01	6.87
30	.42	.00	8.46
31	1.01	.02	13.38
32	1.56	.05	18.92
33	1.21	.03	9.52
34	1.04	.03	7.43
35	.93	.02	6.70
36	.85	.02	6.35
37	.79	.02	6.20
38	.75	.01	6.18
39	.71	.01	6.26
40	.68	.01	6.42
41	.66	.01	6.62
42	.64	.01	6.79
43	.63	.01	6.63
44	.63	.02	5.27
45	.63	.02	2.41
46	-.55	.02	33.10

Y-DISPLACEMENTS

NO	A	B	C
1	-87.88	5.15	117.67
17	-.60	.13	6.24
18	-.87	.09	7.82
19	-.83	.09	5.61

Table 24 (cont'd.)

20	-.71	.08	4.90
21	-.59	.07	4.51
22	-.45	.07	4.34
23	-.32	.07	4.34
24	-.18	.06	4.49
25	-.02	.06	4.80
26	.15	.06	5.32
27	.35	.06	6.13
28	.59	.06	7.46
29	.90	.06	9.92
30	1.39	.07	15.07
31	2.57	.09	24.65
32	-.02	.00	19.26
33	-1.05	-.03	-.63
34	-1.66	-.04	-1.44
35	-2.14	-.04	-2.12
36	-2.58	-.05	-2.83
37	-3.01	-.05	-3.54
38	-3.44	-.05	-4.30
39	-3.88	-.06	-5.15
40	-4.32	-.06	-6.12
41	-4.78	-.06	-7.22
42	-5.23	-.07	-8.45
43	-5.66	-.07	-9.68
44	-6.09	-.08	-10.32
45	-6.53	-.08	-9.08
46	-7.16	-.08	-25.96
47	-3.22	.10	-31.59
48	-4.45	.07	-8.70
49	-4.55	.08	-7.80
50	-4.66	.09	-7.70
51	-4.79	.11	-7.74
52	-4.93	.12	-7.82
53	-5.10	.14	-7.94
54	-5.30	.16	-8.08
55	-5.55	.18	-8.25
56	-5.85	.21	-8.47
57	-6.23	.25	-8.78
58	-6.75	.29	-9.28
59	-7.47	.36	-10.14
60	-8.55	.45	-11.77
61	-10.30	.59	-15.24
62	-13.55	.85	-20.93
63	-21.55	1.44	-49.08

X-TRACTIONS

NO	A	B	C
1	-2.90	-.16	18.95
47	-2.43	-.26	78.08
48	1.53	.36	-15.76
49	.55	-.29	-26.73

Table 24 (cont'd.)

50	.48	.30	-14.44
51	.39	-.25	-3.42
52	.32	.24	1.21
53	.25	-.20	1.85
54	.19	.19	1.30
55	.14	-.14	.84
56	.09	.13	.72
57	.06	-.09	.83
58	.03	.07	1.07
59	.01	-.03	1.61
60	.01	.01	2.33
61	.04	.03	9.90
62	.08	-.05	11.68
63	1.06	.09	-58.67

INTERIOR X-DISP

NO	A	B	C
1	-.20	.00	-64.38
2	-.17	.00	4.74
3	-.18	.00	27.76
4	.19	.01	-.07
5	-.35	.00	11.42

INTERIOR Y-DISP

NO	A	B	C
1	-4.62	.12	-7.39
2	-2.70	.08	-3.76
3	-1.13	.06	.52
4	-2.74	.05	-4.44
5	-2.87	.15	-2.78

INTERIOR SXX

NO	A	B	C
1	.01	.01	1.64
2	-.07	.00	1.44
3	-.18	-.01	1.28
4	.29	.01	3.43
5	-.14	.00	.40

INTERIOR SXY

NO	A	B	C
1	-.66	-.04	-3.20
2	-.16	-.01	1.07
3	.86	.02	8.19
4	-2.37	-.10	-2.87
5	.22	.03	.65

Table 24 (cont'd.)

INTERIOR SYY

NO	A	B	C
1	4.01	.15	15.22
2	3.45	.17	13.92
3	.62	.15	4.51
4	2.50	.11	9.04
5	5.02	.25	22.71

Table 25. Disk - 19 Nodes: X - Dixplacements

X-DISPLACEMENTS

NO	X	Y	UXA	UXB	UXC	EXACT
1	-.41620	-.89254	.09604	.09747	.09219	.12486
2	-.09671	-.98189	0.00000	0.00000	0.00000	0.00000
3	.22404	-.96087	-.08836	-.09208	-.10161	-.06721
4	.52101	-.83786	-.17181	-.17603	-.17864	-.15630
5	.76269	-.62592	-.24019	-.24563	-.24145	-.22881
6	.92341	-.34755	-.28573	-.29242	-.28136	-.27702
7	.98612	-.03228	-.30439	-.31165	-.29958	-.29583
8	.94416	.28641	-.29276	-.29985	-.28646	-.28325
9	.80199	.57470	-.25069	-.25682	-.24143	-.24060
10	.57470	.80199	-.18337	-.18732	-.17379	-.17241
11	.28641	.94416	-.09732	-.09753	-.08950	-.08592
12	-.03228	.98612	0.00000	0.00000	0.00000	0.00000
13	-.34754	.92341	.08575	.09300	.08594	.10426
14	-.62592	.76269	.16330	.16836	.15901	.18778
15	-.83786	.52102	.22274	.22891	.21336	.25136
16	-.96087	.22404	.25702	.26424	.24247	.28826
17	-.98189	-.09671	.26296	.27063	.24123	.29457
18	-.89254	-.41620	.23594	.24417	.20644	.26776
19	-.69636	-.69636	.17842	.18455	.15161	.20891

Table 26. Disk - 19 Nodes: Y - Displacements

Y-DISPLACEMENTS

NO	X	Y	UYA	UYB	UYC	EXACT
1	-.41620	-.89254	.23594	.24417	.22012	.26776
2	-.09671	-.98189	.26296	.27063	.25817	.29457
3	.22404	-.96087	.25702	.26424	.23543	.28826
4	.52101	-.83786	.22274	.22891	.19908	.25136
5	.76269	-.62592	.16330	.16836	.14050	.18778
6	.92341	-.34755	.08575	.09030	.07150	.10426
7	.98612	-.03228	0.00000	0.00000	0.00000	0.00000
8	.94416	.28641	-.09732	-.09753	-.11315	-.08592
9	.80199	.57470	-.18337	-.18732	-.19350	-.17241
10	.57470	.80199	-.25069	-.25682	-.25599	-.24060
11	.28641	.94416	-.29276	-.29985	-.29518	-.28325

Table 26 (cont'd.)

12	-.03228	.98612	-.30439	-.31165	-.30243	-.29583
13	-.34754	.92341	-.28573	-.29242	-.28109	-.27702
14	-.62592	.76269	-.24019	-.24563	-.23471	-.22881
15	-.83786	.52102	-.17181	-.17603	-.16457	-.15630
16	-.96087	.22404	-.08837	-.09208	-.08488	-.06721
17	-.98189	-.09671	0.00000	0.00000	0.00000	0.00000
18	-.89254	-.41620	.09604	.09747	.07830	.12486
19	-.69636	-.69636	.17842	.18455	.13606	.20891

Table 27. Disk - 19 Nodes: X - Tractions

X-TRACTIONS

NO	X	Y	TXA	TXB	TXC	EXACT
1	-.41620	-.89254	.41620	.41620	.41620	.41620
2	-.09671	-.98189	.05623	.04803	.09908	.09671
3	.22404	-.96087	-.22404	-.22404	-.22404	-.22404
4	.52101	-.83786	-.52101	-.52101	-.52101	-.52101
5	.76269	-.62592	-.76269	-.76269	-.76269	-.76269
6	.92341	-.34755	-.92341	-.92341	-.92341	-.92341
7	.98612	-.03228	-.98612	-.98612	-.98612	-.98612
8	.94416	.28641	-.94416	-.94416	-.94416	-.94416
9	.80199	.57470	-.80199	-.80199	-.80199	-.80199
10	.57470	.80199	-.57470	-.57470	-.57470	-.57470
11	.28641	.94416	-.28641	-.28641	-.28641	-.28641
12	-.03228	.98612	.08018	.08478	.04518	.03228
13	-.34754	.92341	.34754	.34754	.34754	.34754
14	-.62592	.76269	.62592	.62592	.62592	.62592
15	-.83786	.52102	.83786	.83786	.83786	.83786
16	-.96087	.22404	.96087	.96087	.96087	.96087
17	-.98189	-.09671	.98189	.98189	.98189	.98189
18	-.89254	-.41620	.89254	.89254	.89254	.89254
19	-.69636	-.69636	.69636	.69636	.69636	.69636

Table 28. Disk - 19 Nodes: Y - Tractions

Y-TRACTIONS

NO	X	Y	TYA	TYB	TYC	EXACT
1	-.41620	-.89254	.89254	.89254	.89254	.89254
2	-.09671	-.98189	.98189	.98189	.98189	.98189
3	.22404	-.96087	.96087	.96087	.96087	.96087
4	.52101	-.83786	.83786	.83786	.83786	.83786
5	.76269	-.62592	.62592	.62592	.62592	.62592
6	.92341	-.34755	.34755	.34755	.34755	.34755
7	.98612	-.03228	.08019	.08475	.18549	.03228
8	.94416	.28641	-.28641	-.28641	-.28641	-.28641
9	.80199	.57470	-.57470	-.57470	-.57470	-.57470
10	.57470	.80199	-.80199	-.80199	-.80199	-.80199
11	.28641	.94416	-.94416	-.94416	-.94416	-.94166

Table 28 (cont'd.)

12	-.03228	.98612	-.98612	-.98612	-.98612	-.98612
13	-.34754	.92341	-.92341	-.92341	-.92341	-.92341
14	-.62592	.76269	-.76269	-.76269	-.76269	-.76269
15	-.83786	.52102	-.52102	-.52102	-.52102	-.52102
16	-.96087	.22404	-.22404	-.22404	-.22404	-.22404
17	-.98189	-.09671	.05623	.04803	.04756	.09671
18	-.89254	-.41620	.41620	.41620	.41620	.41620
19	-.69636	-.69636	.69636	.69636	.69636	.69636

Table 29. Disk - 19 Nodes: Interior Values

INTERIOR X-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	-.75000	0.00000	.19790	.20154	.17640	.22500
2	-.50000	0.00000	.12514	.12773	.10661	.15000
3	0.00000	0.00000	-.02036	-.02006	-.02803	0.00000
4	.50000	0.00000	-.16586	-.16801	-.16305	-.15000
5	.75000	0.00000	-.23854	-.24203	-.23132	-.22500
6	0.00000	.75000	-.01614	-.01534	-.01275	0.00000
7	0.00000	.50000	-.01862	-.01835	-.02011	0.00000
8	0.00000	-.50000	-.02141	-.02120	-.03115	0.00000
9	0.00000	-.75000	-.02309	-.02359	-.03178	0.00000
10	-.50000	.50000	.12698	.12932	.11774	.15000
11	.50000	.50000	-.16492	-.16710	-.15513	-.15000

INTERIOR Y-DISP

NO	X	Y	UYA	UYB	UYC	EXACT
1	-.75000	0.00000	-.02309	-.02359	-.03124	0.00000
2	-.50000	0.00000	-.02141	-.02120	-.03198	0.00000
3	0.00000	0.00000	-.02036	-.02006	-.02984	0.00000
4	.50000	0.00000	-.01862	-.01835	-.02799	0.00000
5	.75000	0.00000	-.01614	-.01534	-.02288	0.00000
6	0.00000	.75000	-.23854	-.24203	-.24701	-.22500
7	0.00000	.50000	-.16586	-.16801	-.17536	-.15000
8	0.00000	-.50000	.12514	.12773	.11918	.15000
9	0.00000	-.75000	.19790	.20154	.19559	.22500
10	-.50000	.50000	-.16602	-.16821	-.17369	-.15000
11	.50000	.50000	-.16492	-.16710	-.17396	-.15000

INTERIOR SXX

NO	X	Y	A	B	C	EXACT
1	-.75000	0.00000	-.97027	-.98360	-.99003	-1.00000
2	-.50000	0.00000	-.96936	-.98436	-.96725	-1.00000
3	0.00000	0.00000	-.96990	-.98602	-.95970	-1.00000
4	.50000	0.00000	-.97086	-.98694	-.97295	-1.00000
5	.75000	0.00000	-.96497	-.98748	-.97296	-1.00000
6	0.00000	.75000	-.97838	-.98944	-.98242	-1.00000
7	0.00000	.50000	-.97191	-.98757	-.97390	-1.00000

Table 29 (cont'd.)

8	0.00000	-.50000	-.96750	-.98468	-.95030	-1.00000
9	0.00000	-.75000	-.96429	-.98053	-.97728	-1.00000
10	-.50000	.50000	-.96944	-.97989	-.99931	-1.00000
11	.50000	.50000	-.97850	-.99558	-.95242	-1.00000

INTERIOR SXY

NO	X	Y	A	B	C	EXACT
1	-.75000	0.00000	.01565	.01243	-.01252	0.00000
2	-.50000	0.00000	.00521	.00561	.01081	0.00000
3	0.00000	0.00000	.00293	.00179	.00686	0.00000
4	.50000	0.00000	.00575	.00640	-.00157	0.00000
5	.75000	0.00000	.00570	.01358	.02050	0.00000
6	0.00000	.75000	.00570	.01358	.05688	0.00000
7	0.00000	.50000	.00575	.00640	.01720	0.00000
8	0.00000	-.50000	.00521	.00561	-.00616	0.00000
9	0.00000	-.75000	.01565	.01243	-.01394	0.00000
10	-.50000	.50000	.01015	.01084	.02086	0.00000
11	.50000	.50000	.00738	.01195	.04680	0.00000

INTERIOR SYY

NO	X	Y	A	B	C	EXACT
1	-.75000	0.00000	-.96429	-.98053	-.93358	-1.00000
2	-.50000	0.00000	-.96750	-.98468	-.94655	-1.00000
3	0.00000	0.00000	-.96990	-.98602	-.97341	-1.00000
4	.50000	0.00000	-.97191	-.98757	-.97297	-1.00000
5	.75000	0.00000	-.97838	-.98944	-.98667	-1.00000
6	0.00000	.75000	-.96497	-.98748	-.95095	-1.00000
7	0.00000	.50000	-.97086	-.98694	-.96699	-1.00000
8	0.00000	-.50000	-.96936	-.98436	-.99498	-1.00000
9	0.00000	-.75000	-.97027	-.98360	-1.01433	-1.00000
10	-.50000	.50000	-.95983	-.97670	-.92942	-1.00000
11	.50000	.50000	-.97850	-.99558	-.99591	-1.00000

Table 30. Disk - 19 Nodes: Percent Differences

X-DISPLACEMENTS

NO	A	B	C
1	23.08	21.94	26.17
3	-31.47	-37.00	-51.17
4	-9.92	-12.62	-14.29
5	-.498	-.735	-5.52
6	-3.14	-5.56	-1.56
7	-2.89	-5.35	-1.27
8	-3.36	-5.86	-1.13
9	-4.19	-6.74	-.35
10	-6.36	-8.65	-.80
11	-13.26	-13.51	-4.16
13	17.76	13.40	17.57

Table 30 (cont'd.)

14	13.03	10.34	15.32
15	11.39	8.93	15.12
16	10.84	8.33	15.89
17	10.73	8.13	18.11
18	11.88	8.81	22.90
19	14.59	11.66	27.43

Y-DISPLACEMENTS

NO	A	B	C
1	11.88	8.81	17.79
2	10.73	8.13	12.36
3	10.84	8.33	18.33
4	11.39	8.93	20.80
5	13.03	10.34	25.18
6	17.76	13.40	31.42
8	-13.26	-13.51	-31.69
9	-6.36	-8.65	-12.23
10	-4.19	-6.74	-6.40
11	-3.36	-5.86	-4.21
12	-2.89	-5.35	-2.23
13	-3.14	-5.56	-1.47
14	-4.98	-7.35	-2.58
15	-9.92	-12.62	-5.29
16	-31.47	-37.00	-26.28
17	100.00	100.00	100.00
18	23.08	21.94	37.29
19	14.59	11.66	34.87

INTERIOR X-DISP

NO	A	B	C
1	12.04	10.43	21.60
2	16.57	14.85	28.93
4	-10.57	-12.01	-8.70
5	-6.02	-7.57	-2.81
10	15.35	13.79	21.51
11	-9.95	-11.40	-3.42

INTERIOR Y-DISP

NO	A	B	C
6	-6.02	-7.57	-9.78
7	-10.57	-12.01	-16.90
8	16.57	14.85	20.55
9	12.04	10.43	13.07
10	-10.68	-12.14	-15.80
11	-9.95	-11.40	-15.97

Table 30 (cont'd.)

INTERIOR SXX

NO	A	B	C
1	2.97	1.64	1.00
2	3.06	1.56	3.28
3	3.01	1.40	4.03
4	2.91	1.31	2.71
5	3.50	1.25	2.70
6	2.16	1.06	1.76
7	2.81	1.24	2.61
8	3.25	1.53	4.97
9	3.57	1.95	2.27
10	3.06	2.01	.07
11	2.15	.44	4.76

INTERIOR SYY

NO	A	B	C
1	3.57	1.95	6.64
2	3.25	1.53	5.34
3	3.01	1.40	2.66
4	2.81	1.24	2.70
5	2.16	1.06	1.33
6	3.50	1.25	4.91
7	2.91	1.31	3.30
8	3.06	1.56	.50
9	2.97	1.64	-1.43
10	4.02	2.33	7.06
11	2.15	.44	.41

Table 31. Disk - 35 Nodes: X - Displacements

X-DISPLACEMENTS

NO	X	Y	UXA	UXB	UXC	EXACT
1	-.18499	-.97768	.05388	.05511	.05228	.05550
2	.00000	-.99619	0.00000	0.00000	0.00000	0.00000
3	.17299	-.98106	-.05166	-.05177	-.05979	-.05190
4	.34072	-.93612	-.10143	-.10203	-.10351	-.10222
5	.49810	-.86273	-.14803	-.14912	-.14748	-.14943
6	.64034	-.76313	-.19011	-.19167	-.18669	-.19210
7	.76313	-.64034	-.22642	-.22840	-.21993	-.22894
8	.86273	-.49810	-.25587	-.25819	-.24673	-.25882
9	.93612	-.34072	-.27756	-.2801	-.26555	-.28084
10	.98106	-.17299	-.29086	-.29359	-.27728	-.29432
11	.99619	.00000	-.29550	-.29816	-.28271	-.29886
12	.98106	.17299	-.29119	-.29369	-.27870	-.29432
13	.93612	.34072	-.27791	-.28025	-.26437	-.28084
14	.86273	.49810	-.25618	-.25829	-.24259	-.25882
15	.76313	.64034	-.22670	-.22849	-.21387	-.22894
16	.64034	.76313	-.19034	-.19175	-.17840	-.19210

Table 31 (cont'd.)

17	.49810	.86273	-.14820	-.14918	-.13809	-.14943
18	.34072	.93612	-.10154	-.10207	-.09378	-.10222
19	.17299	.98106	-.05169	-.05181	-.04715	-.05190
20	0.00000	.99619	0.00000	0.00000	0.00000	0.00000
21	-.17299	.98106	.05051	.05164	.05022	.05190
22	-.34072	.93612	.09975	.10167	.09623	.10222
23	-.49810	.86273	.14607	.14866	.13998	.14943
24	-.64034	.76313	.18796	.19115	.17875	.19210
25	-.76313	.64034	.22412	.22782	.21213	.22894
26	-.86273	.49810	.25345	.25755	.23865	.25882
27	-.93612	.34072	.27503	.27944	.25771	.28084
28	-.98106	.17299	.28820	.29280	.26888	.29432
29	-.99619	.00000	.29247	.29716	.27142	.29886
30	-.97768	-.18499	.28548	.29144	.26604	.29331
31	-.92164	-.37504	.26901	.27461	.24728	.27649
32	-.82906	-.55023	.24187	.24696	.21789	.24872
33	-.70359	-.70359	.20515	.20955	.17904	.21108
34	-.55023	-.82906	.16030	.16384	.13310	.16507
35	-.37504	-.92164	.10914	.11166	.09247	.11251

Table 32. Disk - 35 Nodes: Y - Displacements

Y-DISPLACEMENTS

NO	X	Y	UYA	UYB	UYC	EXACT
1	-.18499	-.97768	.28548	.29144	.26595	.29331
2	.00000	-.99619	.29247	.29716	.28391	.29886
3	.17299	-.98106	.28820	.29280	.26343	.29432
4	.34072	-.93612	.27503	.27945	.25045	.28084
5	.49810	-.86273	.25345	.25755	.29910	.25882
6	.64034	-.76313	.22412	.22782	.20108	.22894
7	.76313	-.64034	.18796	.19115	.16642	.19210
8	.86273	-.49810	.14607	.14866	.12742	.14943
9	.93612	-.34072	.09975	.10167	.08480	.10222
10	.98106	-.17299	.05051	.05164	.04168	.05190
11	.99619	.00000	0.00000	0.00000	0.00000	0.00000
12	.98106	.17299	-.05169	-.05181	-.05944	-.05190
13	.93612	.34072	-.10154	-.10207	-.10503	-.10222
14	.86273	.49810	-.14820	-.14918	-.15006	-.14943
15	.76313	.64034	-.19034	-.19175	-.19019	-.19210
16	.64034	.76313	-.22670	-.22849	-.22400	-.22894
17	.49810	.86273	-.25618	-.25829	-.25137	-.25882
18	.34072	.93612	-.27791	-.28025	-.27059	-.28084
19	.17299	.98106	-.29119	-.29368	-.28230	-.29432
20	.00000	.99619	-.29550	-.29816	-.28550	-.29886
21	-.17299	.98106	-.29086	-.29359	-.28035	-.29432
22	-.34072	.93612	-.27756	-.28013	-.26637	-.28084
23	-.49810	.86273	-.25587	-.25819	-.24460	-.25882
24	-.64034	.76313	-.22642	-.22840	-.21581	-.22894
25	0.76313	.64034	-.19011	-.19167	-.18035	-.19210
26	-.86273	.49810	-.14803	-.14912	-.14011	-.14943

Table 32 (cont'd.)

27	-.93612	.34072	-.10143	-.10203	-.09540	-.10222
28	-.98106	.17299	-.05166	-.05177	-.04847	-.05190
29	-.99619	.00000	0.00000	0.00000	0.00000	0.00000
30	-.97768	-.18499	.05388	.05511	.05124	.05550
31	-.92164	-.37504	.10914	.11166	.10496	.11251
32	-.82906	-.55023	.16030	.16384	.15235	.16507
33	-.70359	-.70359	.20515	.20955	.19281	.21108
34	-.5502	-.82906	.24187	.24697	.20781	.24872
35	-.37504	-.92164	.26901	.27461	.21729	.27649

Table 33. Disk - 35 Nodes: X - Tractions

X-TRACTIONS

NO	X	Y	TXA	TXB	TXC	EXACT
1	-.18499	-.97768	.18499	.18499	.18499	.18499
2	.00000	-.99619	.00552	.00317	.06044	.00000
3	.17299	-.98106	-.17299	-.17299	-.17299	-.17299
4	.34072	-.93612	-.34072	-.34072	-.34072	-.34072
5	.49810	-.86273	-.49810	-.49810	-.49810	-.49810
6	.64034	-.76313	-.64034	-.64034	-.64034	-.64034
7	.76313	-.64034	-.76313	-.76313	-.76313	-.76313
8	.86273	-.49810	-.86273	-.86273	-.86273	-.86273
9	.93612	-.34072	-.93612	-.93612	-.93612	-.93612
10	.98106	-.17299	-.98106	-.98106	-.98106	-.98106
11	.99619	.00000	-.99619	-.99619	-.99619	-.99619
12	.98106	.17299	-.98106	-.98106	-.98106	-.98106
13	.93612	.34072	-.93612	-.93612	-.93612	-.93612
14	.86273	.49810	-.86273	-.86273	-.86273	-.86273
15	.76313	.64034	-.76313	-.76313	-.76313	-.76313
16	.64034	.7631	-.64034	-.64034	-.64034	-.64034
17	.49810	.86273	-.49810	-.49810	-.49810	-.49810
18	.34072	.93612	-.34072	-.34072	-.34072	-.34072
19	.17299	.98106	-.17299	-.17299	-.17299	-.17299
20	.00000	.99619	.00960	.00276	-.01894	.00000
21	-.17299	.98106	.17299	.17299	.17299	.17299
22	-.34072	.93612	.34072	.34072	.34072	.34072
23	-.49810	.86273	.49810	.49810	.49810	.49810
24	-.64034	.76313	.64034	.64034	.64034	.64034
25	-.76313	.64034	.76313	.76313	.76313	.76313
26	-.86273	.49810	.86273	.86273	.86273	.86273
27	-.93612	.34072	.93612	.93612	.93612	.93612
28	-.98106	.17299	.98106	.98106	.98106	.98106
29	-.99619	.00000	.99619	.99619	.99619	.99619
30	-.97768	-.18499	.97768	.97768	.97768	.97768
31	-.92164	-.37504	.92164	.92164	.92614	.92614
32	-.82906	-.55023	.82906	.82906	.82906	.82906
33	-.70359	-.70359	.70359	.70359	.70359	.70359
34	-.55023	-.82906	.55023	.55023	.55023	.55023
35	-.37504	-.92164	.37504	.37504	.37504	.37504

Table 34. Disk - 35 Nodes: Y - Tractions

Y-TRACTIONS

NO	X	Y	TYA	TYB	TYC	EXACT
1	-.18499	-.97768	.97768	.97768	.97768	.97768
2	.00000	-.99619	.99619	.99619	.99619	.99619
3	.17299	-.98106	.98106	.98106	.98106	.98106
4	.34072	-.93612	.93612	.93612	.93612	.93612
5	.49810	-.86273	.86273	.86273	.86273	.86273
6	.64034	-.76313	.76313	.76313	.76313	.76313
7	.76313	-.64034	.64034	.64034	.64034	.64034
8	.86273	-.49810	.49810	.49810	.49810	.49810
9	.93612	-.34072	.34072	.34072	.34072	.34072
10	.98106	-.17299	.17299	.17299	.17299	.17299
11	.99619	.00000	.00960	.00275	.11302	.00000
12	.98106	.17299	-.17299	-.17299	-.17299	-.17299
13	.93612	.34072	-.34072	-.34072	-.34072	-.34072
14	.86273	.49810	-.49810	-.49810	-.49810	-.49810
15	.76313	.64034	-.64034	-.64034	-.64034	-.64034
16	.64034	.76313	-.76313	-.76313	-.76313	-.76313
17	.49810	.86273	-.86273	-.86273	-.86273	-.86273
18	.34072	.93612	-.93612	-.93612	-.93612	-.93612
19	.17299	.98106	-.98106	-.98106	-.98106	-.98106
20	.00000	.99619	-.99619	-.99619	-.99619	-.99619
21	-.17299	.98106	-.98106	-.98106	-.98106	-.98106
22	-.34072	.93612	-.93612	-.93612	-.93612	-.93612
23	-.49810	.86273	-.86273	-.86273	-.86273	-.86273
24	-.64034	.76313	-.76313	-.76313	-.76313	-.76313
25	-.76313	.64034	-.64034	-.64034	-.64034	-.64034
26	-.86273	.49810	-.49810	-.49810	-.49810	-.49810
27	-.98612	.34072	-.34072	-.34072	-.34072	-.34072
28	-.98106	.17299	-.17299	-.17299	-.17299	-.17299
29	-.99619	.00000	.00551	.00316	.00457	.00000
30	-.97768	-.18499	.18499	.18499	.18499	.18499
31	-.92164	-.37504	.37504	.37504	.37504	.37504
32	-.82906	-.55023	.55023	.55023	.55023	.55023
33	-.70359	-.70359	.70359	.70359	.70359	.70359
34	-.55023	-.82906	.82906	.82906	.82906	.82906
35	-.37504	-.92164	.92164	.92164	.92164	.92164

Table 35. Disk - 35 Nodes: Interior Values

INTERIOR X-DISP

NO	X	Y	UXA	UXB	UXC	EXACT
1	-.75000	0.00000	.22094	.2259	.13353	.22500
2	-.50000	0.00000	.14664	.14888	.08700	.15000
3	0.00000	0.00000	-.00179	-.00053	-.00619	0.00000
4	.50000	0.00000	-.15009	-.14991	-.09844	-.15000
5	.75000	0.00000	-.22426	-.22460	-.14488	-.22500
6	0.00000	.75000	-.00105	-.00027	-.00856	0.00000
7	0.00000	.50000	-.00139	-.00039	-.00798	0.00000

Table 35 (cont'd.)

8	0.00000	-.50000	-.00178	-.00052	-.00396	0.00000
9	0.00000	-.75000	-.00137	-.00040	-.00044	0.00000
10	-.50000	.50000	.14728	.14912	-.08554	.15000
11	.50000	.50000	-.14999	-.14986	-.10125	-.15000

INTERIOR Y-DISP

NO	X	Y	UYA	UYB	UYC	EXACT
1	-.75000	0.00000	-.00137	-.00039	-.00522	0.00000
2	-.50000	0.00000	-.00178	-.00052	-.00773	0.00000
3	0.00000	0.00000	-.00179	-.00053	-.00943	0.00000
4	.50000	0.00000	-.00139	-.00039	-.00890	0.00000
5	.75000	0.00000	-.00105	-.00027	-.00767	0.00000
6	0.00000	.75000	-.22426	-.22460	-.22755	-.22500
7	0.00000	.50000	-.15009	-.14991	-.15467	-.15000
8	0.00000	-.50000	.14664	.14888	.13703	.15000
9	0.00000	-.75000	.22094	.22359	.21340	.22500
10	-.50000	.50000	-.14983	-.14982	-.15251	-.15000
11	.50000	.50000	-.14999	-.14986	-.15515	-.15000

INTERIOR SXX

NO	X	Y	A	B	C	EXACT
1	-.75000	0.00000	-.99078	-.99593	-.97552	-1.00000
2	-.50000	0.00000	-.98928	-.99593	-.97486	-1.00000
3	0.00000	0.00000	-.98899	-.99594	-.97109	-1.00000
4	.50000	0.00000	-.98917	-.99601	-.97167	-1.00000
5	.75000	0.00000	-.98932	-.99610	-.97701	-1.00000
6	0.00000	.75000	-.99070	-.99651	-.97460	-1.00000
7	0.00000	.50000	-.99044	-.99644	-.97591	-1.00000
8	0.00000	-.50000	-.98722	-.99539	-.95302	-1.00000
9	0.00000	-.75000	-.98676	-.99531	-.94871	-1.00000
10	-.50000	.50000	-.99036	-.99644	-.97754	-1.00000
11	.50000	.50000	-.99073	-.99652	-.97310	-1.00000

INTERIOR SXY

NO	X	Y	A	B	C	EXACT
1	-.75000	0.00000	.00060	.00004	-.00594	0.00000
2	-.50000	0.00000	.00099	.00027	-.00388	0.00000
3	0.00000	0.00000	.00107	.00036	.00439	0.00000
4	.50000	0.00000	.00071	.00029	.00040	0.00000
5	.75000	0.00000	.00079	.00033	-.00450	0.00000
6	0.00000	.75000	.00079	.00033	.00919	0.00000
7	0.00000	.50000	.00071	.00029	.00319	0.00000
8	0.00000	-.50000	.00099	.00027	.00997	0.00000
9	0.00000	-.75000	.00060	.00004	-.00834	0.00000
10	-.50000	.50000	.00013	.00010	.00082	0.00000
11	.50000	.50000	.00147	.00053	.00677	0.00000

Table 35 (cont'd.)

INTERIOR SYY

NO	X	Y	A	B	C	EXACT
1	-.75000	0.00000	-.98676	-.99531	-.96770	-1.00000
2	-.50000	0.00000	-.98722	-.99539	-.96397	-1.00000
3	0.00000	0.00000	-.98899	-.99594	-.96913	-1.00000
4	.50000	0.00000	-.99044	-.99644	-.97358	-1.00000
5	.75000	0.00000	-.99070	-.99651	-.97341	-1.00000
6	0.00000	.75000	-.98932	-.99610	-.97538	-1.00000
7	0.00000	.50000	-.98917	-.99601	-.97121	-1.00000
8	0.00000	-.50000	-.98928	-.99593	-.98397	-1.00000
9	0.00000	-.75000	-.99078	-.99593	-1.02409	-1.00000
10	-.50000	.50000	-.98864	-.99578	-.96878	-1.00000
11	.50000	.50000	-.99073	-.99652	-.98002	-1.00000

Table 36. Disk - 35 Nodes: Percent Differences

X-DISPLACEMENTS

NO	A	B	C
1	2.92	.70	5.80
3	.45	.24	-15.21
4	.77	.19	-1.26
5	.94	.21	1.30
6	1.04	.22	2.82
7	1.10	.24	3.94
8	1.14	.24	4.67
9	1.17	.25	5.44
10	1.17	.25	5.79
11	1.12	.23	5.40
12	1.06	.22	5.31
13	1.04	.21	5.86
14	1.02	.20	6.27
15	.98	.19	6.58
16	.92	.18	7.13
17	.82	.16	7.59
18	.66	.14	8.25
19	.39	.18	9.14
21	2.68	.50	3.22
22	2.41	.54	5.85
23	2.25	.51	6.32
24	2.16	.50	6.95
25	2.10	.49	7.34
26	2.08	.49	7.79
27	2.07	.50	8.23
28	2.08	.52	8.64
29	2.14	.57	9.18
30	2.67	.64	9.30
31	2.71	.68	10.57
32	2.75	.70	12.29
33	2.81	.72	15.18

Table 36 (cont'd.)

34	2.89	.74	19.37
35	3.00	.76	17.82

Y-DISPLACEMENTS

NO	A	B	C
1	2.67	.64	9.33
2	2.14	.57	5.00
3	2.08	.52	10.50
4	2.07	.49	10.82
5	2.08	.49	11.48
6	2.10	.49	12.17
7	2.16	.50	13.37
8	2.25	.51	14.73
9	2.41	.54	17.04
10	2.68	.50	19.68
12	.39	.18	-14.54
13	.66	.14	-2.75
14	.82	.16	-.42
15	.92	.18	1.00
16	.98	.20	2.16
17	1.02	.20	2.88
18	1.04	.21	3.65
19	1.06	.22	4.08
20	1.12	.23	4.47
21	1.17	.25	4.75
22	1.17	.25	5.15
23	1.14	.24	5.50
24	1.10	.24	5.74
25	1.04	.22	6.12
26	.94	.21	6.24
27	.77	.19	6.67
28	.45	.25	6.60
30	2.92	.70	7.67
31	3.00	.76	6.72
32	2.89	.74	7.70
33	2.81	.72	8.65
34	2.75	.70	16.45
35	2.71	.68	21.41

INTERIOR X-DISP

NO	A	B	C
1	1.81	.63	40.65
2	2.24	.75	42.00
4	-.06	.06	34.37
5	.33	.18	35.61
10	1.82	.59	42.97
11	.01	.09	32.50

Table 36 (cont'd.)

INTERIOR Y-DISP

NO	A	B	C
6	.33	.18	-1.13
7	-.06	.06	-3.11
8	2.24	.75	8.65
9	1.81	.63	5.16
10	.11	.12	-1.67
11	.01	.09	-3.43

INTERIOR SXX

NO	A	B	C
1	.92	.41	2.45
2	1.07	.41	2.51
3	1.10	.41	2.89
4	1.08	.40	2.83
5	1.07	.39	2.30
6	.93	.35	2.54
7	.96	.36	2.41
8	1.28	.46	4.70
9	1.32	.47	5.13
10	.96	.36	2.25
11	.93	.35	2.69

INTERIOR SYY

NO	A	B	C
1	1.32	.47	3.23
2	1.28	.46	3.60
3	1.10	.41	3.09
4	.96	.36	2.64
5	.93	.35	2.66
6	1.07	.39	2.46
7	1.08	.40	2.88
8	1.07	.41	1.60
9	.92	.41	-2.41
10	1.14	.42	3.12
11	.93	.35	2.00

CHAPTER 5

CONCLUSIONS

The results from each problem favor case (b) as the best model. It is a closer approximation for linear elasticity since displacements are linear, but tractions are not, in general, piecewise continuous. Case (a), with its simple assumptions, is the next best approximation. It appears that an inaccurate assumption in displacement behavior is less damaging than an inaccurate traction assumption. The corner effect in case (c) is too pronounced to favor it over the other two cases.

It is possible to devise schemes for avoiding or reducing the corner effect. However, this will only add to the complexity of the input data or the model itself and, before taking that step, one should examine the benefits. This brings us to the ideas bought out in Chapter 1. Since the time of the first approximation methods, we have progressed to ever-increasing levels of sophistication. As we keep progressing, however, we should learn to use what we have more efficiently. That is what prompted this work and the results indicate that much can be learned about the applicability of our models through simple comparative studies.

APPENDIX

APPENDIX

COMPUTER CODE

```
      program bem
c
      parameter(mn=75)
      parameter(mnn=2*mn)
      parameter(mnn1-mnn+1)
      parameter(mmm=50)
c
      dimension x(mn),y(mn),xm(mn),ym(mn)
      dimension rnx(mn),rny(mn),s(mn)
      dimension r(10),w(10),w1(10),w2(10)
      dimension ua(mnn),ub(mnn),uc(mnn),ue(mnn)
      dimension ta(mnn),tb(mnn),tc(mnn),te(mnn)
      dimension uxa(mmm),uxb(mmm),uxc(mmm)
      dimension uya(mmm),uyb(mmm),uyc(mmm)
      dimension uxe(mmm),uye(mmm)
      dimension sxxa(mmm),sxxb(mmm),sxxc(mmm)
      dimension sxya(mmm),sxyb(mmm),sxyc(mmm)
      dimension syya(mmm),syyb(mmm),syyc(mmm)
      dimension sxxe(mmm),syye(mmm),sxye(mmm)
      dimension ibc(mn,2),bc(mnn),bcon(mnn),rhs(mnn)
      dimension a(mnn,mnn),b(mnn,mnn),ucc(mnn,mnn)
      dimension xf(mmm),yf(mmm)
c
      character*70 title
c
      open(unit=3,file='sol')
c
c OBTAIN INPUT DATA
c
      write(*,*)'THIS PROGRAM EMPLOYS THE DIRECT BOUNDARY ELEMENT'
      write(*,*)'METHOD TO SOLVE PLANE PROBLEMS OF LINEAR
      write(*,*) ELASTICITY'
      write(*,*)'          '
c
      write(*,*)'DO YOU WANT TO FIND VALUES AT INTERIOR POINTS?'
      write(*,*)'          '
      write(*,*)'    1 = YES'
      write(*,*)'    2 = NO'
      write(*,*)'          '
      write(*,*)'ENTER THE NUMBER OF YOUR CHOICE'
      read(*,*)ifield
```

```
      write(*,*)'          '
c
      write(*,*)'ENTER A DESCRIPTION OF THE PROBLEM ON ONE LINE'
      write(*,*)'COLUMNS 1-70.  (ENCLOSE IN QUOTES)'
      read(*,*)title
      write(*,*)'                 '
c
      write(*,*)'ENTER THE APPROPRIATE NUMBER'
      write(*,*)'    1 = PLANE STRESS'
      write(*,*)'    2 = PLANE STRAIN'
      read(*,*)iplane
      write(*,*)'                 '
c
      write(*,*)'ENTER THE NUMBER OF NODES'
      read(*,*)n
      write(*,*)'             '
c
      write(*,*)'ENTER POISSON'S RATIO'
      read(*,*)pr
      if(iplane.ne.1)then
      pr=pr/(1.-pr)
      else
      endif
      write(*,*)'          '
c
      call bndry (n,x,y,xm,ym,rmx,rmy,s)
c
1     write(*,*)'HOW MANY INTERIOR POINTS?'
      read(*,*)mm
      write(*,*)'                '
      write(*,*)'ENTER XF,YF ONE SET PER LINE'
      do 5 i=1,mm
      read(*,*)xf(i),yf(i)
5     continue
c
      write(*,*)'DO YOU WANT TO CHANGE THESE?'
      write(*,*)'    1 = YES'
      write(*,*)'    2 = NO'
      read(*,*)ichange
      write(*,*)'          '
      if(ichange.eq.1)goto 1
c
      nn=2*n
c
      do 20 i=1,n
      do 20 l=1,2
      ibc(i,l)=0
20    continue
c
      do 21 i=1,nn
      bc(i)=0.
      bcon(i)=0.
21    continue
c
```

```
C BOUNDARY CONDITION PROGRAM
C
      call bdc(n,mm,pr,xm,ym,xf,yf,rnx,rny,ue,te,bc,ibc,bcon,uxe,
&     uye,sxxe,sxye,syye)
C
      write(*,*)'ALL INPUT DATA RECEIVED'
      write(*,*)'          '
C
C INTEGRATION POINTS
C
      call ingpts(r,w,w1,w2)
C
C COMPUTE COEFFICIENT MATRICES A AND B
C
      do 30 l=1,3
C
C INITIALIZE MATRICES A AND B
C
      do 31 i=1,nn
      bc(i)=bcon(i)
      do 31 j=1,nn
      a(j,i)=0.
      b(j,i)=0.
31    continue
C
C ELEMENT SUBROUTINES
C
      if(l.eq.1)then
      write(*,*)'COMPUTING FOR CASE A'
      write(*,*)'          '
      call casea(n,pr,x,y,xm,ym,rnx,rny,s,r,w,a,b)
      else if(l.eq.2)then
      write(*,*)'COMPUTING FOR CASE B'
      write(*,*)'              '
      call caseb(n,pr,x,y,s,r,w,w1,w2,a,b)
      else
      write(*,*)'COMPUTING FOR CASE C'
      write(*,*)'            '
      call casec(n,pr,x,y,s,r,w,w1,w2,a,b)
      endif
C
C POST-MULTIPLY BY GAMMA INVERSE TRANSPOSE
C
      if(l.eq.2)then
      do 32 i=1,n
      ii2=2*i
      ii1=ii2-1
      aa=-1.
      temp1=a(ii1,1)
      temp2=a(ii2,1)
      temp3=a(ii1,2)
      temp4=a(ii2,2)
C
      do 33 k=2,n
```

```
      kk2=2*k
      kk1=kk2-1
c
      a(ii1,1)=a(ii1,1)+a(ii1,kk1)*aa
      a(ii2,1)=a(ii2,1)+a(ii2,kk1)*aa
      a(ii1,2)=a(ii1,2)+a(ii1,kk2)*aa
      a(ii2,2)=a(ii2,2)+a(ii2,kk2)*aa
      aa=-aa
33    continue
c
      do 34 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      temp5=a(ii1,kk1)
      temp6=a(ii2,kk1)
      temp7=a(ii1,kk2)
      temp8=8(ii2,kk2)
c
      a(ii1,kk1)=-a(ii1,kk1-2)+2.*temp1
      a(ii2,kk1)=-a(ii2,kk1-2)+2.*temp2
      a(ii1,kk2)=-a(ii1,kk2-2)+2.*temp3
      a(ii2,kk2)=-a(ii2,kk2-2)+2.*temp4
c
      temp1=temp5
      temp2=temp6
      temp3=temp7
      temp4=temp8
34    continue
c
32    continue
c
      else if(l.eq.3)then
      do 35 i=1,n
      ii2=2*i
      ii1=ii2-1
      aa=-1.
      temp1=a(ii1,1)
      temp2=a(ii2,1)
      temp3=a(ii1,2)
      temp4=a(ii2,2)
c
      temp11=b(ii1,1)
      temp12=b(ii2,1)
      temp13=b(ii1,2)
      temp14=b(ii2,2)
c
      do 36 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      a(ii1,1)=a(ii1,1)+a(ii1,kk1)*aa
      a(ii2,1)=a(ii2,1)+a(ii2,kk1)*aa
      a(ii1,2)=a(ii1,2)+a(ii1,kk2)*aa
```

```
      a(ii2,2)=a(ii2,2)+a(ii2,kk2)*aa
c
      b(ii1,1)=b(ii1,1)+b(ii1,kk1)*aa
      b(ii2,1)=b(ii2,1)+b(ii2,kk1)*aa
      b(ii1,2)=b(ii1,2)+b(ii1,kk2)*aa
      b(ii2,2)=b(ii2,2)+b(ii2,kks)*aa
      aa=-aa
36    continue
c
      do 37 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      temp5=a(ii1,kk1)
      temp6=a(ii2,kk1)
      temp7=a(ii1,kk2)
      temp8=a(ii2,kk2)
c
      temp15=b(ii1,kk1)
      temp16=b(ii2,kk1)
      temp17=b(ii1,kk2)
      temp18=b(ii2,kk2)
c
      a(ii1,kk1)=-a(ii1,kk1-2)+2.*temp1
      a(ii2,kk1)=-a(ii2,kk1-2)+2.*temp2
      a(ii1,kk2)=-a(ii1,kk2-2)+2.*temp3
      a(ii2,kk2)=-a(ii2,kk2-2)+2.*temp4
c
      b(ii1,kk1)=-b(ii1,kk1-2)+2.*temp11
      b(ii2,kk1)=-b(ii2,kk1-2)+2.*temp12
      b(ii1,kk2)=-b(ii1,kk2-2)+2.*temp13
      b(ii2,kk2)=-b(ii2,kk2-2)+2.*temp14
c
      temp1=temp5
      temp2=temp6
      temp3=temp7
      temp4=temp8
c
      temp11=temp15
      temp12=temp16
      temp13=temp17
      temp14=temp18
37    continue
c
35    continue
c
      else
      endif
c
c RE-ORDER SYSTEM BASED ON KNOWN BOUNDARY CONDITIONS
c
      do 38 i=1,n
      do 38 k=1,2
      ii=2*(i-1)+k
```

```
      if(ibc(i,k).eq.0)goto 38
      do 39 j=1,nn
      temp=b<j,ii)
      b(j,ii)=-a(j,ii)
      a(j,ii)=-temp
39    continue
38    continue
C
C DETERMINE THE KNOWN RIGHT HAND SIDE
C
      do 40 i=1,nn
      rhs(i)=0.
40    continue
C
      do 41 i=1,nn
      do 41 j=1,nn
      rhs(i)=rhs(i)+b(i,j)*bc(j)
41    continue
C
C SOLVE FOR UNKNOWN BOUNDARY VALUES
C
      do 42 i=1,nn
      do 42 j=1,nn
      ucc(i,j)=a(i,j)
42    continue
C
      call gauss(ucc,rhs,nn)
C
      do 43 i=1,n
      do 43 k=1,2
      ii=2*(i-1)+k
      if(ibc(i,k).ne.1) goto 4
      temp=bc(ii)
      bc(ii)=rhs(ii)
      rhs(ii)=temp
43    continue
C
      do 44 i=1,nn
      if(l.eq.1)then
      ua(i)=rhs(i)
      ta(i)=bc(i)
      else if(l.eq.2)then
      ub(i)=rhs(i)
      tb(i)=bc(i)
      else
      uc(i)=rhs(i)
      tc(i)=bc(i)
      endif
44    continue
C
C COMPUTE INTERIOR VALUES
C
      if(ifield.eq.1)then
C
```

```
      if(l.eq.1)then
      call flda(n,mm,pr,x,y,r,w,w1,w2,ua,ta,xf,yf,uxa,uya,sxxa,
&     sxya,syya)
c
      else if(l.eq.2)then
      call fldb(n,mm,pr,x,y,r,w,w1,w2,ub,tb,xf,yf,uxb,uyb,sxxb,
&     sxyb,syyb)
c
      else
      call fldc(n,mm,pr,x,y,r,w,w1,w2,uc,tc,xf,yf,uxc,uyc,sxxc,
&     sxyc,syyc)
      endif
c
      else
      endif
c
30    continue
c
c OUTPUT
c
c TITLE AND INPUT DATA
c
      write(3,100)
      write(3,105)
      write(3,*)title
      write(3,100)
      write(3,110)
      write(3,115)
c
      write(3,120)n
      write(3,125)pr
c
      if(iplane,eq.1)then
      write(3,130)
      else
      write(3,135)
      endif
c
c NODAL COORDINATES AND BOUNDARY CONDITIONS
c
      write(3,200)
      write(3,205)
      do 50 i=1,n
50    write(3,210)i,x,(i),y(i),xm(i),ym(i),s(i),rnx(i),rny(i)
c
      write(3,215)
c
      do 52 j=1,n
      i1=2*(j-1)+1
      i2=2*(j-1)+2
      bc1=bcon(i1)
      bc2=bcon(i2)
      k=1
      if(ibc(j,k).eq.1)then
```

```
      iprin1=0
      else
      iprin1=1
      endif
      k=2
      if(ibc(j,k).eq.1)then
      iprin2=1
      else
      iprin2=3
      endif
      iwrite=iprin1+iprin2
C
      if(iwrite.eq.1)then
      write(3,220)j,bc1,bc2
      else if(iwrite.eq.2)then
      write(3,225)j,bc1,bc2
      else if(iwrite.eq.3)then
      write(3,230)j,bc1,bc2
      else
      write(3,235)j,bc1,bc2
      endif
52    continue
      write(3,100)
C
C CALCULATED VALUES
C
      write(3,315)
C
      write(3,320)
      do 53 i=1,n
      i1=2*(i-1)+1
      write(3,322)i,xm(i),ym(i),ua(i1),ub(i1),uc(i1),ue(i1)
53    continue
      write(3,330)
      write(3,335)
      do 54 i=1,n
      i2=2*(i-1)+2
      write(3,322)i,xm(i),ym(i),ua(i2),ub(i2),uc(i2),ue(i2)
54    continue
      write(3,340)
      write(3,345)
      do55 i=1,n
      i=2*(i-1)+1
      write(3,322)i,xm(i),ym(i),ta(i1),tb(i1),tc(i1),te(i1)
55    continue
      write(3,350)
      write(3,355)
      do 56 i=,n
      i2=2*(i-1)+2
      write(3,322)i,xm(i),ym(i),ta(i2),tb(i2),tc(i2),te(i2)
56    continue
      write(3,100)
      write(3,360)
      write(3,365)
```

```
      write(3,370)
      do 57 i=1,mm
      write(3,322)i,xf(i),yf(i),uxa(i),uxb(i),uxc(i),uxe(i)
57    continue
      write(3,375)
      write(3,380)
      do 58 i=1,mm
      write(3,322)i,xf(i),yf(i),uya(i),uyb(i),uyc(i),uye(i)
58    continue
      write(3,385)
      write(3,390)
      do 59 i=1,mm
      write(3,322)i,xf(i),yf(i),sxxa(i),sxxb(i),sxxc(i),sxxe(i)
59    continue
      write(3,395)
      write(3,390)
      do 60 i=1,mm
       write(3,322)i,xf(i),yf(i),sxya(i),sxyb(i),sxyc(i),sxye(i)
60    continue
      write(3,400)
      write(3,390)
      do 61 i=1,mm
       write(3,322)i,xf(i),yf(i),syya(i),syyb(i),syyc(i),syye(i)
61    continue
      write(3,100)
C
C PERCENT DIFFERENCES
C
      write(3,405)
      write(3,315)
      write(3,392)
      do 70 i=1,n
      i1=2*(i-1)+1
      if(ue(i1).eq.0.)then
      write(3,410)i
      else
      da=((ue(i1)-ua(i1))/ue(i1))*100.
      db=((ue(i1)-ub(i1))/ue(i1))*100.
      dc=((ue(i1)-uc(i1))/ue(i1))*100.
      write(3,325)i,da,db,dc
      endif
70    continue
C
      write(3,330)
      write(3,392)
      do 71 i=1,n
      i2=2*(i-1)+2
      if(ue(i2).eq.0.)then
      write(3,410)i
      else
      da=((ue(i2)-ua(i2))/ue(i2))*100.
      db=((ue(i2)-ub(i2))/ue(i2))*100.
      dc=((ue(i2)-uc(i2))/ue(i2))*100.
      write(3,325)i,da,db,dc
```

```
      endif
71    continue
c
      write(3,340)
      write(3,392)
      do 72 i=1,n
      i1=2*(i-1)+1
      if(te(i1).eq.0.)then
      write(3,410)i
      else
      da=((te(i1)-ta(i1))/te(i1))*100.
      db=((te(i1)-tb(i1))/te(i1))*100.
      dc=((te(i1)-tc(i1))/te(i1))*100.
      write(3,325)i,da,db,dc
      endif
72    continue
c
      write(3,350)
      write(3,392)
      do 73 i=1,n
      i2=2*(i-1)+2
      if(te(i2).eq.0.)then
      write(3,410)i
      else
      da=((te(i2)-ta(i2))/te(i2))*100.
      db=((te(i2)-tb(i2))/te(i2))*100.
      dc=((te(i2)-tc(i2))/te(i2))*100.
      write(3,325)i,da,db,dc
      endif
73    continue
c
      write(3,365)
      write(3,392)
      do 74 i=1,mm
      if(uxe(i).eq.0.)then
      write(3,410)i
      else
      da=((uxe(i)-uxa(i))/uxe(i))*100.
      db=((uxe(i)-uxb(i))/uxe(i))*100.
      dc=((uxe(i)-uxc(i))/uxe(i))*100.
      write(3,325)i,da,db,dc
      endif
74    continue
c     write(3,375)
      write(3,392)
      do 75 i=1,mm
      if(uye(i).eq.0.)then
      write(3,410)i
      else
      da=((uye(i)-uya(i))/uye(i))*100.
      db=((uye(i)-uyb(i))/uye(i))*100.
      dc=((uye(i)-uyc(i))/uye(i))*100.
      write(3,325)i,da,db,dc
      endif
```

```
75   continue
c
     write(3,385)
     write(3,392)
     do 76 i=1,mm
     if(sxxe(i).eq.0.)then
     write(3,410)i
     else
     da=((sxxe(i)-sxxa(i))/sxxe(i))*100.
     db=((sxxe(i)-sxxb(i))/sxxe(i))*100.
     dc=((sxxe(i)-sxxc(i))/sxxe(i))*100.
     write(3,325)i,da,db,dc
     endif
76   continue
c
     write(3,395)
     write(3,392)
     do 77 i=1,mm
     if(sxye(i).eq.0.)then
     write(3,410)i
     else
     da=((sxye(i)-sxya(i))/sxye(i))*100.
     db=((sxye(i)-sxyb(i))/sxye(i))*100.
     dc=((sxye(i)-sxyc(i))/sxye(i))*100.
     write(3,325)i,da,db,dc
     endif
77   continue
c
     write(3,400)
     write(3,392)
     do 78 i=1,mm
     if(syye(i).eq.0.)then
     write(3,410)i
     else
     da=((syye(i)-syya(i))/syye(i))*100.
     db=((syye(i)-syyb(i))/syye(i))*100.
     dc=((syye(i)-syyc(i))/syye(i))*100.
     write(3,325)i,da,db,dc
     endif
78   continue
c
c FORMATS
c
100  format(/,'----------------------------------------')
105  format( '                              ')
110  format(//,30x,'INPUT DATA')
115  format( 28x,'____________')
120  format(/,'NUMBER OF NODES =',i3)
125  format(/,"POISSON''S RATIO =',f7.4)
130  format(1,'PLANE STRESS')
135  format(/,'PLANE STRAIN')
200  format(//,'NODE AND ELEMENT INFORMATION')
205   format(/,'NO',4x,'X',9x,'Y',8x,'XM',7x,'YM',6x,'S',5x,'RNX',
&    5x,'RNY')
```

```
210  format( i2,f10.5,f10.5,f8.5,1x,f8.5,1x,f6.3,1x,f6.3,1x,f6.3)
215  format(/,"BOUNDARY CONDITIONS')
220  format( i3,1x,' U1=',f8.4,' U2=',f8.4)
225  format( i3,1x,' T1=',f8.4,' U2=',f8.4)
230  format( i3,1x,' U1=',f8.4,' T2=',f8.4)
235  format( i3,1x,' T1=',f8.4,' T2=',f8.4)
299  format(/,'***** CALCULATED RESULTS *****')
315  format(//,'X-DISPLACEMENTS')
320   format(/,'NO',4x,'X',9x,'Y',8x,'UXA',7x,'UXB',7x,'UXC',6x,
&    'EXACT')
322  format( i2,f10.5,f10.5,f10.5,f10.5,f10.5,f10.5)
325  format( i2,f7.2,2x,f7.2,2x,f7.2)
330  format(//,'Y-DISPLACEMENTS')
340  format(//,'X-TRACTIONS')
345   format(/,'NO',4x,'X',9x,'Y',8x,'TXA',7x,'TXB',7x,'TXC',6x,
&    'EXACT')
350  format(//,'Y-TRACTIONS')
355   format(/,'NO',4x,'X',9x,'Y',8x,'TYA',7x,'TYB',7x,'TYC',6x,
&    'EXACT')
360  format(/,'INTERIOR VALUES')
365  format(//,'INTERIOR X-DISP')
370   format(/,'NO',4x,'X',9x,'Y',8x,'UXA',7x,'UXB',7x,'UXC',6x,
&    'EXACT')
375  format(//,'INTERIOR Y-DISP')
380   format(/,'NO',4x,'X',9x,'Y',8x,'UYA',7x,'UYB',7x,'UYC',6x,
&    'EXACT')
385  format(//,'INTERIOR SXX')
390   format(/,'NO',4x,'X',9x,'Y',9x,'A',9x,'B',9x,'C',9x,'EXACT')
392  format(/,'NO',4x,'A',7x,'B',6x,'C')
395  format(//,'INTERIOR SXY')
400  format(//,'INTERIOR SYY')
405  format(//,'PERCENT DIFFERENCE')
410  format( i2,1x,'******  ******  ******')
c
999  stop
     end
     subroutine bndry(n,x,y,xm,ym,rnx,rny,s)
c
     parameter (mn=75)
c
     dimension xm(mn),ym(mn),rnx(mn),rny(mn),s(mn)
     dimension x(mn),y(mn)
c
     istop=0
c
1    write(*,*)'ENTER THE TOTAL NUMBER OF BOUNDARY SEGMENTS.'
     read(*,*)nb
     nel=0.   do 10 i=1,nb
     ichange=0
99   write(*,90)i
90   format(/,'IS SEGMENT',i2,'?'
     write(*,*)'               '
     write(*,*)'  1. STRAIGHT'
     write(*,*)'  2. CIRCULAR'
```

```
      write(*,*)'  3. ELLIPTICAL'
      write(*,*)'          '
      read(*,*)ntype
C
      write(*,*)'DO YOU WANT TO CHANGE THIS?'
      write(*,*)'      '
      write(*,*)'  1 = YES'
      write(*,*)'  2 = NO'
      read(*,*)ichange
      write(*,*)'     '
      if(ichange.eq.1)goto 99
C
      if(ntype.eq.1)then
98    write(*,101)i
101   format(/,'ENTER THE FOLLOWING INFORMATION FOR SEGMENT ',i2)
      write(*,102)
102   format(/,' BEG. NODE NO.,X(I),Y(I),X(J),Y(J),NO. OF
&     ELEMENTS')
      write(*,*)'                    '
      read(*,*)nbeg,xi,yi,xj,yj,ne
C
      write(*,*)'DO YOU WANT TO CHANGE THIS?'
      write(*,*)'      '
      write(*,*)'   1 = YES'
      write(*,*)'   2 = NO'
      read(*,*)ichange
      write(*,*)'         '
      if(ichange.eq.1)goto 98
C
      x(nbeg)=xi
      y(nbeg)=yi
      dx=xj-xi
      dy=yj-yi
      sx=dx/ne
      sy=dy/ne
      ne1=ne-1
      do 20 j=1,ne1
      k=nbeg+j
      k1=k-1
      x(k)=x(k1)+sx
      y(k)-y(k1)+sy
20    continue
      else if(ntype.eq.2)then
97    write(*,101)i
      write(*,103)
103   format(/,' NBEG,XI,YI,XC,YC,R,THETA,NO.OF ELEMENTS')
      write(*,104)
104   format(/,' (THETA=ARC IN DEGREES)')
      write(*,*)'          '
      read(*,*)nbeg,xi,yi,xc,yc,r,beta,ne
C
      write(*,*)'DO YOU WANT TO CHANGE THIS?'
      write(*,*)'      '
      write(*,*)'   1 = YES'
```

```
      write(*,*)'   2 = NO'
      read(*,*)ichange
      write(*,*)'       '
      if(ichange.eq.1)goto 97
c
      write(*,*)'                 '
      pi=4.*atan(1.)
      theta=(beta*pi)/180.
      phi=theta/ne
      delx=(xi-xc)/r
      dely=(yi-yc)/r
      if(dely.ge.0.)then
      alpha=acos(delx)
      else
      alpha=2.*pi-acos(delx)
      endif
      x(nbeg)=xi
      y(nbeg)=yi
      ne1=ne-1
      do 30 j=1,ne1
      k=nbeg+j
      k1=K-1
      kp1=k+1
      alpha1=alpha+2.*phi
      alpha=alpha+phi
      x(k)=xc+r*cos(alpha)
      y(k)=yc+r*sin(alpha)
      x(kp1)-xc+r*cos(alpha1)
      y(kp1)=yc+r*sin(alpha1)
30    continue
      nbeg1=nbeg+1
      else if(ntype.eq.3)then
96    write(*,101)i
      write(*,105)
105   format(/,' NBEG,X(I),Y(I),ALPHA,A,B,THETA,NO. OF ELEMENTS')
      WRITE(*,106)
106   format(/,'(CENTER AT ORIGIN ; THETA = ARC IN DEGREES)')
      write(*,*)'(ALPHA = ANGLE TO FIRST POINT)'
      read(*,*)nbeg,xi,yi,alpha,a,b,beta,ne
c
      write(*,*)'DO YOU WANT TO CHANGE THIS?'
      write(*,*)'       '
      write(*,*)'   1 = YES'
      write(*,*)'   2 = NO'
      read(*,*)ichange
      write(*,*)'        '
      if(ichange.eq.1)goto 96
c
      write(*,*)'                 '
      pi=4.*atan(1.)
      theta=(beta*pi)/180.
      phi=theta/ne
      x(nbeg)=xi
      y(nbeg)=yi
```

```
      ne1=ne-1
      a2=a*a
      b2=b*b
      top=a2*b2
      do 40 j=1,ne1
      k=nbeg+j
      k1=k-1
      kp1=k+1
      alpha=alpha+phi
      bot=a2*sin(alpha)*sin(alpha)+b2*cos(alpha)*cos(alpha)
      r2=top/bot
      r=sqrt(r2)
      x(k)=r*cos(alpha)
      y(k)=r*sin(alpha)
40    continue
      nbeg1=nbeg+1
      else
      write(*,*)'AT THIS TIME I CANNOT GENERATE ELEMENTS FOR THIS'
      write(*,*)'TYPE OF BOUNDARY SEGMENT.'
      goto 99
      endif
      nel=nel+ne
10    continue
      write(*,*)'THIS IS THE MESH'
      write(*,*)'NODE      X                Y          '
      do 50 m=1,nel
      write(*,503)m,x(m),y(m)
503   format(1x,i3,3x,f10.6,2x,f10.6)
50    continue
      write(*,*)'          '
      do 60 i=1,n
      if(i.eq.1)then
      im1=n
      else
      im1=i-1
      endif
      xm(i)=(x(i)+x(im1))/2.
      ym(i)=(y(i)+y(im1))/2.
      dx=x(i)-x(im1)
      dy=y(i)-y(im1)
      s(i)=sqrt(dx*dx+dy*dy)
      if(ntype.eq.2)then
      rnx(i)=xm(i)
      rny(i)=ym(i)
      else
      rnx(i)=dy/s(i)
      rny(i)=-dx/s(i)
      endif
60    continue
c
70    return
      end

      subroutine bdc(n,mm,pr,xm,ym,xf,yf,rnx,rny,ue,te,bc,ibc,
```

```
&     bcon,uxe,uye,sxxe,sxye,syye)
c
      parameter (mn=75)
      parameter (mnn=2*mn)
      parameter (mmm=50)
c
      dimension xm(mn),ym(mn),xf(mn),yf(mn),rnx(mn),rny(mn)
      dimension bc(mnn),ibc(mn,2),bcon(mnn)
      dimension ue(mnn),te(mnn)
      dimension uxe(mmm),uye(mmm),sxxe(mmm),sxye(mmm),syye(mmm)
c
      nn=2*n
c
100   write(*,*)'ENTER PROBLEM TYPE'
      write(*,*)'   1 = PURE EXTENSION'
      write(*,*)'   2 = RECTANGLE'
      write(*,*)'   3 = CIRCLE'
      read(*,*)iprob
      write(*,*)'         '
c
      if(iprob.eq.1)then
      do 10 i=1,n
      i1=2*(i-1)+1
      i2=2*(i-1)+2
      xm2=xm(i)*xm(i)
      ym2=ym(I)*ym(i)
      ym3=xm2*xm(i)
      ym3=ym2*ym(i)
c
      if(ym(i).eq.0.)then
      bc(i1)=0.
      bc(i2)=0.
      ibc(i,2)=1
      bcon(i1)=0.
      bcon(i2)=0.
      ue(i1)=-0.1*xm(i)
      ue(i2)=0.
      te(i1)=0.
      te(i1)=-1.
c
      else if(xm(i).eq.1.)then
      bc(i1)=0.
      bc(i2)=0.
      bcon(i1)=0.
      bcon(i2)=0.
      ue(i1)=-0.1
      ue(i2)=0.4*ym(i)
      te(i1)=0.
      te(i2)=0.
c
      else if(ym(i).eq.2.)then
      bc(i1)=0.
      bc(i2)=1.
      bcon(i1)=0.
```

```
      bcon(i2)=1.
      ue(i1)=-0.1*xm(i)
      ue(i2)=0.4*ym(i)
      te(i1)=0.
      te(i2)=1.
c
      else
      bc(i1)=0.
      bc(i2)=0.
      ibc(i,1)=1
      bcon(i1)=0.
      bcon(i2)=0.
      ue(i1)=0.
      ue(i2)=0.4*ym(i)
      te(i1)=0.
      te(i2)=0.
      endif
c
10    continue
c
      do 15 i=1,mm
c
      sxxe(i)=0.
      syye(i)=1.
      sxye(i)=0.
      uxe(i)=-0.1*xf(i)
      uye(i)=0.4*yf(i)
15    continue
c
      else if(iprob.eq.2)then
      do 20 i=1,n
      i1=2*(i-1)+1
      i2=2*(i-1)+2
      xm2=xm(i)*xm(i)
      ym2=ym(i)*ym(i)
      xm3=xm2*xm(i)
      ym3=ym2*ym(i)
c
      if(ym(i).eq.0.)then
      bc(i1)=xm(i)
      bc(i2)=-0.45*xm2
      ibc(i,2)=1
      bcon(i1)=bc(i1)
      bcon(i2)=bc(i2)
      ue(i1)=(4.*xm(i)-xm3/12.)/2.5
      ue(i2)=(-1.125*xm2)/2.5
      te(i1)=xm(i)
      te(i2)=0.
c
      else if(xm(i).eq.1.)then
      bc(i1)=3.75+ym2/2.
      bc(i2)=0.5*ym(i)-1.
      bcon(i1)=bc(i1)
      bcon(i2)=bc(i2)
```

```
      ue(i1)=(47./12.+0.5625*ym2-ym(i)/4.)/2.5
      ue(i2)=(ym2/2.-ym3/8.-15.*ym(i)/16.-1.125)/2.5
      te(i1)=ym2/2.+3.75
      te(i2)=ym(i)/2.-1.
c
      else if(ym(i).eq.1.)then
      bc(i1)=-0.5*xm(i)
      bc(i2)=0.75
      bcon(i1)=bc(i1)
      bcon(i2)=bc(i2)
      ue(i1)=(69.*xm(i)/16.-xm3/12.)/2.5
      ue(i2)=(-5./8.-17.*xm2/16.)/2.5
      te(i1)=-0.5*xm(i)
      te(i2)=0.75
c
      else
      bc(i1)=0.
      bc(i2)=0.
      ibc(i,1)=1
      bcon(i1)=0.
      bcon(i2)=0.
      ue(i1)=0.
      ue(i2)=(ym2/2.-ym3/8.-ym(i))/2.5
      te(i1)=-ym2/2.-4.
      te(i2)=0.
      endif
c
20    continue
c
      do 25 i=1,mm
      xf2=xf(i)*xf(i)
      yf2=yf(i)*yf(i)
      xf3=xf2*xf(i)
      yf3=yf2*yf(i)
c
      sxxe(i)=4.-xf2/4.+yf2/2.
      syye(i)=yf(i)-yf2/4.
      sxye(i)=xf(i)*yf(i)/2.-xf(i)
      uxe(i)=(xf(i)*(4.-xf2/12.tyf2/2.-(yf(i)-yf2/4.)/4.))/2.5
      uye(i)=(yf2/2.-yf3/12.-yf(i)*(4.-xf2/4.+yf2/6.)/4.-1.125*
&     xf2)/2.5
25    continue
c
      else if(iprob.eq.3)then
c
      write(*,*)'ENTER ELEMENTS WHERE UX=0'
      read(*,*)ix1,ix2
      write(*,*)'            '
      write(*,*)'ENTER ELEMENTS WHERE UY=0'
      read(*,*)iy1,iy2
      write(*,*)'      '
c
      do 30 i=1,n
      i1=2*(i-1)+1
```

```
      i2=2*(I-1)+2
c
      if(i.eq.ix1.or.i.eq.ix2)then
      bc(i1)=0.
      bc(i2)=-rny(i)
      ibc(i,1)=1
      bcon(i1)=bc(i1)
      bcon(i2)=bc(i2)
c
      else if(i.eq.iy1.or.i.eq.iy2)then
      bc(i1)=-rnx(i)
      bc(i2)=0.
      ibc(i,2)=1
      bcon(i1)=bc(i1)
      bcon(i2)=bc(i2)
c
      else
      bc(i1)=-rnx(i)
      bc(i2)=1rny(i)
      bcon(i1)=bc(i1)
      bcon(i2)=bc(i2)
      endif
c
      ue(i1)=-0.3*rnx(i)
      ue(i2)=-0.3*rny(i)
      te(i1)=-rnx(i)
      te(i2)=-rny(i)
c
30    continue
c
      do 35 i=1,mm
      uxe(i)=-0.3*xf(i)
      uye(i)=-0.3*yf(i)
      sxxe(i)=-1.
      syye(i)=-1.
      sxye(i)=0.
35    continue
c
      else
      goto 100
      endif
c
      write(*,*)'BOUNDARY CONDITION VECTORS'
      write(*,*)'       '
      write(*,115)
115   format( 'NO',4x,'BC(I1)',4x,"BC(I2)',4x,'IBC(I,1)',4x,'IBC
&     (I,2)')
      do 40 i=1,n
      i1=2*(i-1)+1
      i2=2*(i-1)+2
      write(*,120)i,bc(i1),bc(i2),ibc(i,1),ibc(i,2)
120   format( i2,4x,f6.3,4x,f6.3,7x,i1,11x,i1)
40    continue
c
```

```
      write(*,*)'        '
      write(*,*)'ENTER ANY NUMBER TO CONTINUE'
      read(*,*)igo
      write(*,*)'        '
c
      return
      end

      subroutine ingpts(r,w,w1,w2)
c
      dimension r(10,w(10),w1(10),w2(10)
c
c POINTS AND WEIGHTS FOR NUMERICAL INTEGRATION
c
      r(1)=-.97390652851717
      r(2)=-.86506336668898
      r(3)=-.67940956829902
      r(4)=-.43339539412925
      r(5)=-.14887433898163
      r(6)=-r(5)
      r(7)=-r(4)
      r(8)=-r(3)
      r(9)=-r(2)
      r(10=-r(1)
c
      w(1)=.066671344308688
      w(2)=.14945134915058
      w(3)=.21908636251598
      w(4)=.26926671931000
      w(5)=.29552422471475
      w(6)=w(5)
      w(7)=w(4)
      w(8)=w(3)
      w(9)=w(2)
      w(10=w(1)
c
      do 10 l=1,10
      w1(l)=1.-r(l)
      w2(l)=1.r(l)
10    continue
c
      return
      end

      subroutine casea(n,pr,x,y,xm,ym,rmx,rmy,s,r,w,a,b)
c
      parameter (mn=75)
      parameter (mmn=2*mn)
c
      dimension x(mn),y(mn)
      dimension xm(mn),ym(mn),rmx(mn),rmy(mn),s(mn)
      dimension a(mmn,mmn),b(mmn,mmn),r(10),w(10)
c
c COMPUTE COEFFICIENT MATRICES
```

```
c
      nn=2*n
      pi=4.*atan(1.)
      c1=(1.+pr)
      c2=(1.-pr)
      c3=(3.+pr)
      c4=(3.-pr)
      c5=(1.+3.*pr)
c
      do 2 j=1,n
      jj2=2*j
      jj1=jj2-1
c
      if(j.eq.1)then
      jm1=n
      jj2m2=nn
      else
      jm1=j-1
      jj2m2=jj2-2
      endif
      jj1m2=jj2m2-1
c
      if(j.eq.n)then
      jp1=1
      jj2p2=2
      else
      jp1=j+1
      jj2p2=jj2+2
      endif
      jj1p2=jj2p2-1
c
      rn1=rnx(j)
      rn2=rny(j)
      ds=s(j)
      rn12=rn1*rn1
      rn22=rn2*rn2
c
      b(jj1,jj1)=(c1*rn22-c4*(log(0.5*ds)-1.))*ds/2.
      b(jj1,jj2)=1c1*rn1*rn2*ds/2.
      b(jj2,jj1)=b(jj1,jj2)
      b(jj2,jj2)=(c1*rn12-c4*(log(ds*0.5)-1.))*ds/2.
c
      dmm1=x(j)-xm(j)
      dmm2=y(j)-ym(j)
      c=dmm1*dmm1+dmm2*dmm2
c
      do 2 i=1,n
      if(i.eq.j)goto2
      ii2=2*i
      ii1=ii2-1
c
      drm1=xm(i)-xm(j)
      drm2=ym(i)-ym(j)
      aa=drm1*drm1+drm2*drm2
```

```
      bb=-2.*(drm1*dmm1+drm2*dmm2)
c
      do 3 l=1,10
      rho=sqrt(aa+bb*r(l)+c*r(l)*r(l))
      qx=(drm1-dmm1*r(l))/rho
      qy=(drm2-dmm2*r(l))/rho
      qx2=qx*qx
      qy2=qy*qy
      qx3=qx2*qx
      qy3=qy2*qy
      c12=2.*c1
c

      t11=(c12*(dmm2*qx3+dmm1*qy3)+c2*dmm2*qx-c3*dmm1*qy)/rho
      t12=(c12*(dmm1*qx3-dmm2*qy3)-c3*dmm1*qx+c5*dmm2*qy)/rho
      t21=(c12*(dmm1*qx3-dmm2*qy3)-c5*dmm1*qx+c3*dmm2*qy)/rho
      t22=(-c12*(dmm2*qx3+dmm1*qy3)+c3*dmm2*qx-c2*dmm1*qy)/rho
c
      u11=(c1*qx2-c4*log(rho))*ds/4.
      u12=c1*qx*qy*ds/4.
      u22=(c1*qy2-c4*log(rho))*ds/4.
c
      a(ii1,jj1)=a(ii1,jj1)+w(l)*t11
      a(ii1,jj2)=a(ii1,jj2)+w(l)*t21
      a(ii2,jj1)=a(ii2,jj1)+w(l)*t12
      a(ii2,jj2)=a(ii2,jj2)+w(l)*t22
c
      b(ii1,jj1)=b(ii1,jj1)+w(l)*u11
      b(ii1,jj2)=b(ii1,jj2)+w(l)*u12
      b(ii2,jj1)=b(ii2,jj1)+w(l)*u12
      b(ii2,jj2)=b(ii2,jj2)+w(l)*u22
c
3     continue
2     continue
c
      do 4 i=1,n
      ii2=2*i
      ii1=ii2-1
      do 4 k=1,n
      if(k.eq.i)goto 4
      kk2=2*k
      kk1=kk2-1
c
      a(ii1,ii1)=a(ii1,ii1)-a(ii1,kk1)
      a(ii1,ii2)=a(ii1,ii2)-a(ii1,kk2)
      a(ii2,ii1)=a(ii2,ii1)-a(ii2,kk1)
      a(ii2,ii2)=a(ii2,ii2)-a(ii2,kk2)
4     continue
c
      write(*,*)'CONSTANT MATRICES CALCULATED'
      write(*,*)'          '
c
      return
      end
```

```
      subroutine caseb(n,pr,x,y,s,r,w,w1,w2,a,b)
c
      parameter (mn=75)
      parameter (mnn=2*mn)
c
      dimension x(mn),y(mn),s(mn)
      dimension r(10),w(10),w1(10),w2(10)
      dimension a(mnn,mnn),b(mnn,mnn)
c
c COMPUTE COEFFICIENT MATRICES
c
      nn=2*n
      pi=4.*atan(1.)
      c1=(1.+pr)
      c12=c1*2.
      c2=(1.-pr)
      c3=(3.+pr)
      c4=(3.-pr)
      c5=(1.+3.*pr)
c
      do 2 j=1,n
      jj2=2*j
      jj1=jj2-1
c
      if(j.eq.1)then
      jm1=n
      jj2m2=nn
      else
      jm1=j-1
      jj2m2=jj2-2
      endif
      jj1m2=jj2m2-1
c
      if(j.eq.n)then
      jp1=1
      jj2p2=2
      else
      jp1=j+1
      jj2p2=jj2+2
      endif
      jj1p2=jj2p2-1
c
      dx=x(j)-x(jm1)
      dy=y(j)-y(jm1)
      ds=sqrt(dx*dx+dy*dy)
      rnx=dy/ds
      rny=-dx/ds
c
      b(jj1,jj1)=(c4*(1.-log(ds))+c1*rny(rny)*ds/2.
      b(jj1,jj2)=1c1*rnx*rny*ds/2.
      b(jj2,jj1)=b(jj1,jj2)
      b(jj2,jj2)=(c4*(1.-log(ds))+c1*rnx*rnx)*ds/2.
c
```

```
      dx=x(jp1)-x(j)
      dy=y(jp1)-y(j)
      dss=sqrt(dx*dx+dy*dy)
      rnx=dy/dss
      rny=-dx/dss
c
      b(jj1,jj1p2)=(c4*(1.-log(dss))+c1*rny*rny)*dss/2.
      b(jj1,jj2p2)=-c1*rnx*rny*dss/2.
      b(jj2,jj1p2)=b(jj1,jj2)
      b(jj2,jj2p2)=(c4*(1.-log(dss))+c1*rnx*rnx)*dss/2.
c
      xm=(x(j)+x(jm1))/2.
      ym=(y(j)+y(jm1))/2.
      dmm1=x(j)-xm
      dmm2=y(j)-ym
      c=dmm1*dmm1+dmm2*dmm2
c
      do 2 i=1,n
      ii2=2*i
      ii1=ii2-1
c
      drm1=x(i)-xm
      drm2=y(i)-ym
      aa=drm1*drm1+drm2*drm2
      bb=-2.*(drm1*dmm1+drm2*dmm2)
c
      do 3 l=1,10
      rho=sqrt(aa+bb*r(l)+c*r(l)*r(l))
      qx=(drm1-dmm1*r(l))/rho
      qy=(drm2-dmm2*r(l))/rho
      qx2=qx*qx
      qy2=qy*qy
      qx3=qx2*qx
      qy3=qy2*qy
c
      t11=(c12*(dmm2*qx3+dmm1*qy3)+c2*dmm2*qx-c3*dmm1*qy)/(2.*rho)
      t12=(c12*(dmm1*qx3-dmm2*qy3)-c3*dmm1*qx+c5*dmm2*qy)/(2.*rho)
      t21=(c12*(dmm1*qx3-dmm2*qy3)-c5*dmm1*qx+c3*dmm2*qy)/(2.*rho)
      t22=(-c12*(dmm2*qx3+dmm1*qy3)+c3*dmm2*qx-c2*dmm1*qy)/
     &     (2.*rho)
c
      if(i.eq.j)then
c
      a(ii1,jj1m2)=a(ii1,jj1m2)+w(l)*w1(l)*t11
      a(ii1,jj2m2)=a(ii1,jj2m2)+w(l)*w1(l)*t21
      a(ii2,jj1m2)=a(ii2,jj1m2)+w(l)*w1(l)*t12
      a(ii2,jj2m2)=a(ii2,jj2m2)+w(l)*w1(l)*t22
c
      else if(i.eq.jm1)then
c
      a(ii1,jj1)=a(ii1,jj1)+w(l)*w2(l)*t11
      a(ii1,jj2)=a(ii1,jj2)+w(l)*w2(l)*t21
      a(ii2,jj1)=a(ii2,jj1)+w(l)*w2(l)*t12
      a(ii2,jj2)=a(ii2,jj2)+w(l)*w2(l)*t22
```

```
c
      else
c
      a(ii1,jj1m2)=a(ii1,jj1m2)+w(l)*w1(l)*t11
      a(ii1,jj2m2)=1(ii1,jj2m2)+w(l)*w1(l)*t21
      a(ii2,jj1m2)=a(ii2,jj1m2)+w(l)*w1(l)*t12
      a(i22,jj2m2)=a(ii2,jj2m2)+w(l)*w1(l)*t22
c
      a(ii1,jj1)=a(ii1,jj1)+w(l)*w2(l)*t11
      a(ii1,jj2)=a(ii1,jj2)+w(l)*w2(l)*t21
      a(ii2,jj1)=a(ii2,jj1)+w(l)*w2(l)*t12
      a(ii2,jj2)=a(ii2,jj2)+w(l)*w2(l)*t22
c
      endif
c
      if(i.eq.j.or.l.eq.jm1)goto 3
c
      u11=(c1*qx2-c4*log(rho))*ds/4.
      u12=c1*qx*qy*ds/4.
      u22=(c1*qy2-c4*log(rho))*ds/4.
c
      b(ii1,jj1)=b(ii1,jj1)+w(l)*u11
      b(ii1,jj2)=b(ii1,jj2)+w(l)*u12
      b(ii2,jj1)=b(ii2,jj1)+w(l)*u12
      b(ii2,jj2)=b(ii2,jj2)+w(l)*u22
c
3     continue
2     continue
c
c COMPUTE DIAGONAL ENTRIES OF [A]
c
      do 4 i=1,n
      ii2=2*i
      ii1=ii2-1
      do 4 k=1,n
      if(i.eq.i)goto4
      kk2=2*k
      kk1=kk2-1
c
      a(ii1,ii1)=a(ii1,ii1)-a(ii1,kk1)
      a(ii1,ii2)=a(ii1,ii2)-a(ii1,kk2)
      a(ii2,ii1)=a(ii2,ii1)-a(ii2,kk1)
      a(ii2,ii2)=a(ii2,ii2)-a(ii2,kk2)
4     continue
c
      write(*,*)'NONISOPARAMETRIC MATRICES CALCULATED'
      write(*,*)'        '
c
      return
      end

      subroutine casec(n,pr,x,y,s,r,w,w1,w2,a,b)
c
      parameter (mn=75)
```

```
      parameter (mmn=2*mn)
c
      dimension x(mn),y(mn),s(mn)
      dimension r(10),w(10),w1(10),w2(10)
      dimension a(mmn,mmn),b(mmn,mmn)
c
c COMPUTE COEFFICIENT MATRICES
c
      nn=2*n
      pi=4,*atan(1.)
      c1=(1.+pr)
      c12=c1*2.
      c2=(1.-pr)
      c3=(3.+pr)
      c4=(3.-pr)
      c5=(1.+3.*pr)
c
      do 2 j=1,n
      jj2=2*j
      jj1=jj2-1
c
      if(j.eq.1)then
      jm1=n
      jj2m2=nn
      else
      jm1=j-1
      jj2m2=jj2-2
      endif
      jj1m2=jj2m2-1
c
      if(j.eq.n)then
      jp1=1
      jj2p2=2
      else
      jp1=j+1
      jj2p2=jj2+2
      endif
      jj1p2=jj2p2-1
c
      dx=x(j)-x(jm1)
      dy=y(j)-y(jm1)
      ds=sqrt(dx*dx+dy*dy)
      rnx=dy/ds
      rny=-dx/ds
c
      b(jj1,jj1m2)=(c4*(0.5-log(ds))+c1*rny*rny)*ds/4.
      b(jj1,jj2m2)=1c1*rnx*rny(ds/4.
      b(jj2,jj1m2)=b(jj1,jj2)
      b(jj2,jj2m2)=(c4*(0.5-log(ds))+c1*rnx*rnx)*ds/4.
c
      b(jj1,jj1)=(c4*(1.5-log(ds))+c1*rny*rny)*ds/4.
      b(jj1,jj2)=-c1*rnx*rny*ds/4.
      b(jj2,jj1)=b(jj1,jj2)
      b(jj2,jj2)=(c4*(1.5-log(ds))+c1*rnx*rnx)*ds/4.
```

```
c
      dx=x(jp1)-x(j)
      dy=y(jp1)-y(j)
      dss=sqrt(dx*dx+dy*dy)
      rnx_dy/dss
      rny=-dx/dss
c
      b(jj1,jj1)=b(jj1,jj1)+(c4*(1.5-log(dss))+c1*rny*rny)*dss/4.
      b(jj1,jj2)=b(jj1,jj2)-c1*rnx*rny(dss/4.
      b(jj2,jj1)=b(jj1,jj2)
      b(jj2,jj2)=b(jj2,jj2)+(c4*(1.5-log(dss))+c1*rnx*rnx)*dss/4.
c
      b(jj1,jj1p2)=(c4*(0.5-log(dss))+c1*rny*rny)*dss/4.
      b(jj1,jj2p2)=-c1*rnx*rny(dss/4.
      b(jj2,jj1p2)=b(jj1,jj2)
      b(jj2,jj2p2)=(c4*(0.5-log(dss))+c1*rnx*rnx)*dss/4.
c
      xm=(x(j)+x(jm1))/2.
      ym=(y(j)+y(jm1))/2.
      dmm1=x(j)-xm
      dmm2=y(j)-ym
      c=dmm1*dmm1+dmm2*dmm2
c
      do 2 i=1,n
      ii2=2*i
      ii1=ii2-1
c
      drm1=x(i)-xm
      drm2=y(i)-ym
      aa=drm1*drm1+drm2*drm2
      bb=-2.*(drm1*dmm1+drm2*dmm2)
c
      do 3 l=1,10
      rho=sqrt(aa+bb*r(l)+c*r(l)*r(l))
      qx=(drm1-dmm1*r(l))/rho
      qy=(drm2-dmm2*r(l))/rho
      qx2=qx*qx
      qy2=qy*qy
      qx3=qx2*qx
      qy3=qy2*qy
c
      t11=(c12*(dmm2*qx3+dmm1*qy3)+c2*dmm2*qx-c3*dmm1*qy)/(2.*rho)
      t12=(c12*(dmm1*qx3-dmm2*qy3)-c3*dmm1*qx+c5*dmm2*qy)/(2.*rho)
      t21=(c12*(dmm1*qx3-dmm2*qy3)-c5*dmm1*qx+c3*dmm2*qy)/(2.*rho)
      t22=(-c12*(dmm2*qx3+dmm1*qy3)+c3*dmm2*qx-c2*dmm1*qy)/
     &      (2.*rho)
c
      if(i.eq.j)then
c
      a(ii1,jj1m2)=a(ii1,jj1m2)+w(l)*w1(l)*t11
      a(ii1,jj2m2)=a(ii1,jj2m2)+w(l)*w1(l)*t21
      a(ii2,jj1m2)=a(ii2,jj1m2)+w(l)*w1(l)*t12
      a(ii2,jj2m2)=a(ii2,jj2m2)+w(l)*w1(l)*t22
c
```

```
      else if(i.eq.jm1)then
c
      a(ii1,jj1)=a(ii1,jj1)+w(l)*w2(l)*t11
      a(ii1,jj2)=a(ii1,jj2)+w(l)*w2(l)*t21
      a(ii2,jj1)=a(ii2,jj1)+w(l)*w2(l)*t12
      a(ii2,jj2)=a(ii2,jj2)+w(l)*w2(l)*t22
c
      else
c
      a(ii1,jj1m2)=a(ii1,jj1m2)+w(l)*w1(l)*t11
      a(ii1,jj2m2)=a(ii1,jj2m2)+w(l)*w1(l)*t21
      a(ii2,jj1m2)=a(ii2,jj1m2)+w(l)*w1(l)*t12
      a(ii2,jj2m2)=a(ii2,jj2m2)+w(l)*w1(l)*t22
c
      a(ii1,jj1)=a(ii1,jj1)+w(l)*w2(l)*t11
      a(ii1,jj2)=a(ii1,jj2)+w(l)*w2(l)*t21
      a(ii2,jj1)=a(ii2,jj1)+w(l)*w2(l)*t12
      a(ii2,jj2)=a(ii2,jj2)+w(l)*w2(l)*t22
c
      endif
c
      if(i.eq.j.or.i.eq.jm1)goto 3
c
      u11=(c1*qx2-c4*log(rho))*ds/8.
      U12=c1*qx*qy(ds/8.
      u22=(c1*qy2-c4*log(rho))*ds/8.
c
      b(ii1,jj1m2)=b(ii1,jj1m2)+w(l)*w1(l)*u11
      b(ii1,jj2m2)=b(ii1,jj2m2)+w(l)*w1(l)*u12
      b(ii2,jj1m2)=b(ii2,jj1m2)+w(l)*w1(l)*u12
      b(ii2,jj2m2)=b(ii2,jj2m2)+w(l)*w1(l)*u22
c
      b(ii1,jj1)=b(ii1,jj1)+w(l)*w2(l)*u11
      b(ii1,jj2)=b(ii1,jj2)+w(l)*w2(l)*u12
      b(ii2,jj1)=b(ii2,jj1)+w(l)*w2(l)*u12
      b(ii2,jj2)=b(ii2,jj2)+w(l)*w2(l)*u22
c
3     continue
2     continue
c
c COMPUTE DIAGONAL ENTRIES OF [A]
c
      do 4 i=1,n
      ii2=2*i
      ii1=ii2-1
      do 4 k=1,n
      if(k.eq.i)goto 4
      kk2=2*k
      kk1=kk2-1
c
      a(ii1,ii1)=a(ii1,ii1)-a(ii1,kk1)
      a(ii1,ii2)=a(ii1,ii2)-a(ii1,kk2)
      a(ii2,ii1)=a(ii2,ii1)-a(ii2,kk1)
      a(ii2,ii2)=a(ii2,ii2)-a(ii2,kk2)
```

```
4     continue
C
      write(*,*)'ISOPARAMETRIC MATRICES CALCULATED'
      write(*,*)'                '
C
      return
      end

      subroutine gauss(ucc,rhs,nn)
C
      parameter (mn=75)
      parameter (mnn=2*mn)
      parameter (mnn1=mnn+1)
C
      dimension ucc(mnn,mnn),rhs(mnn)
      dimension aug(mnn,mnn1)
C
      nm1=nn-1
      np1=nn+1
C
C SET UP THE AUTMENTED MATRIX FOR AX=B
C
      do 2 i=1,nn
      do 1 j=1,nn
      aug(i,j)=ucc(i,j)
1     continue
      aug(i,np1)=rhs(i)
2     continue
C
C THE OUTER LOOP USES ELEMENTARY ROW OPERATIONS TO TRANSFORM THE
C AUGMENTED MATRIX INTO ECHELON FORM.
C
      do 8 i=1,nm1
C
C SEARCH FOR THE LARGEST ENTRY IN COLUMN I,ROWS I THROUGH N.
C IPIVOT IS THE ROW INDEX OF THE LARGEST ENTRY.
C
      pivot=0.
      do 3 j=1,nn
      temp=abs(aug(j,i))
      if(pivot.ge.temp) go to 3
      pivot=temp
      ipivot=j
3     continue
      if(pivot.eq.0.) goto 13
      if(ipivot,eq.i) goto 5
C
C INTERCHANGE ROW I AND ROW IPIVOT.
C
      do 4 k=1,np1
      temp=aug(i,k)
      aug(i,k)=aug(ipivot,k)
      aug(ipivot,k)=temp
4     continue
```

```
C
C ZERO ENTRIES (I+1,I),(I+2,I),...,(N,I) IN THE AUGMENTED MATRIX.
C
5     ip1=i+1
      do 7 k=ip1,nn
      q=-qug(k,i)/aug(i,i)
      aug(k,i)=0.
      do 6 j=ip1,np1
      qug(k,j)=a*aug(i,j)+aug(k,j)
6     continue
7     continue
8     continue
      if(aug(nn,nn).eq.0.) goto 13
C
C BACKSOLVE TO OBTAIN SOLUTION TO AX=B.
C
      rhs(nn)=aug(nn,np1)/aug(nn,nn)
      do 10 k=1,nm1
      q=0.
      do 9 j=1,k
      q=q+aug(nn-k,np1-j)*rhs(np1-j)
9     continue
      rhs(nn-k)=(aug(nn-k,np1)-q)/aug(nn-k,nn-k)
10    continue
C
C CALCULATE THE NORM OF THE RESIDUAL VECTOR,B-AX, SET IERROR=1
C AND RETURN.
C
      rsq=0.
      do 12 i=1,nn
      q=0.
      do 11 j=1,nn
      q=q+ucc(i,j)*rhs(j)
11    continue
      resi=rhs(i)-q
      rmag=abs(resi)
      rsq=rsq+rmag*rmag
12    continue
      rnorm=sqrt(rsq)
      ierror=1
C
      write(*,*)'SYSTEM OF EQUATIONS SOLVED'
      write(*,*)'
C
      return
C
C ABNORMAL RETURN---REDUCTION TO ECHELON FORM PRODUCES A ZERO ON
C THE DIAGONAL.  THE MATRIX A MAY BE SINGULAR.
C
13    ierror=2
C
      write(*,*)'WARNING--MATRIX MAY BE SINGULAR!'
      write(*,*)'      '
      write(*,*)'ENTER ANY NUMBER TO CONTINUE'
```

```
      read(*,*)igoes
      write(*,*)'       '
c
      return
      end

      subroutine flda(n,mm,pr,x,y,r,w,w1,w2,ua,ta,xf,yf,uxa,uya,
&     sxxa,sxya,syya)
c
      parameter(mn=75)
      parameter(mnn=2*mn)
      parameter(mnn1=mnn+1)
      parameter(mmm=5)
c
      dimension x(mn),y(mn)
      dimension r(10,w(10,w1(10,w2(10)
      dimension ua(mnn),ta(mnn),fa(mnn)
      dimension xf(mmm),yf(mmm)
      dimension ul(5,mnn),ur(5,mnn)
      dimension uxa(mmm),uya(mmm)
      dimension sxxa(mmm),sxya(mmm),syya(mmm)
c
      nn=2*n
c
      pl=4,*atan(1.)
c
      cl=1.+pr
      c2=1.-pr
      c3=3.+pr
      c4=3.-pr
      c5=1.+3.*pr
c
      do 135 ii=1,mm
c
      do 136 i=1,nn
      do 136 k=1,5
      ul(k,i)=0.
      ur(k,i)=0.
136   continue
c
      do 137 j=1,n
      if(j.eq.1)then
      jml=n
      else
      jml=j-1
      endif
      jj2=2*j
      jj1=jj2-1
c
      if(j.eq.1)then
      jj2m2=nn
      else
      jj2m2=jj2-2
      endif
```

```
      jj1m2=jj2m2-1
c
      dx=x(j)-x(jm1)
      dy=y(j)-y(jm1)
      s=sqrt(dx*dx+dy(dy)
      fa(jj1)=ta(jj1)*s
      fa(jj2)=ta(jj2)*s
c
      xmi=(x(j)+x(jm1))/2.
      ymi=(y(j)+y(jm1))/2.
      dmm1=x(j)-xmi
      dmm2=y(j)-ymi
      c=dmm1*dmm1+dmm2*dmm2
c
      drm1=xf(ii)-xmi
      drm2=yf(ii)-ymi
      aa=drm1*drm1+drm2*drm2
      bb=-2.*(drm1*dmm1+drm2*dmm2)
c
      do 137 l=1,10
      rho=sqrt(aa+bb*r(l)+c*r(l)*r(l))
      rho2=rho*rho
      qx=(drm1-dmm1*r(l))/rho
      qy=(drm2-dmm2*r(l))/rho
      qx2=qx*qx
      qy2=qy*qy
      qx3=qx2*qx
      qy3=qy2*qy
      qx4=qx2*qx2
      qy4=qy2*qy2
c
      t11=2.*(2.*c1*(dmm2*qx3+dmm1*qy3)+c2*dmm2*qx-c3*dmm1*qy)/rho
      t21=2.*(2.*c1*(dmm1*qx3-dmm2*qy3)-c5*dmm1*qx+c3*dmm2*qy)/rho
      t12=2.*(2.*c1*(dmm1*qx3-dmm2*qy3)-c3*dmm1*qx+c5*dmm2*qy)/rho
      t22=2.*(2.*c1*(-dmm2*qx3-dmm1*qy3)+c3*dmm2*qx-c2*dmm1*qy)
     &      /rho
c
      ul(1,jj1)=ul(1,jj1)+w(l)*t11
      ul(1,jj2)=ul(1,jj2)+w(l)*t21
      ul(2,jj1)=ul(2,jj1)+w(l)*t12
      ul(2,jj2)=ul(2,jj2)+w(l)*t22
c
      u11=(-c4*log(rho)+c1*qx*qx)/2.
      u12=c1*qx*qy/2.
      u22=(-c4*log(rho)+c1*qy*qy)/2.
c
      ur(1,jj1)=ur(1,jj1)+w(l)*u11
      ur(1,jj2)=ur(1,jj2)+w(l)*u12
      ur(2,jj1)=ur(2,jj1)+w(l)*u12
      ur(2,jj2)=ur(2,jj2)+w(l)*u22
c
      s111=4.*((8.*qx4-4.*qx2-1.)*dmm2-2.*qx*qy*(4.*qx2-1.)*dmm1)
     &      /rho2
      s211=4.*(2.*qx*qy*(4.*qx2-1.)*dmm2-(8,*qx2*qy2-1.)*dmm1)
```

```
&     /rho2
      s221=4.*((8.*qx2*qy2-1.)*dmm2-2.*qx*qy((4.*qy2-1.)*dmm1)
&     /rho2
      s222=4.*(2.*qx*qy()4.*qy2-1.)*dmm2-(8,*qy4-4.*qy2-1.)*dmm1/
&     /rho2
c
      ul(3,jj1)=ul(3,jj1)+w(l)*s111
      ul(3,jj2)=ul(3,jj2)+w(l)*s211
      ul(r,jj1)=ul(4,jj1)+w(l)*s211
      ul(4,jj2)=ul(4,jj2)+w(l)*s221
      ul(5,jj1)=ul(5,jj1)+w(l)*s221
      ul(5,jj2)=ul(5,jj2)+w(l)*s222
c
      d111=(-2.*c1*qx3-c2*qx)/rho
      d112=(8.*c1*qy3-c5*qy)/rho
      d121=(2.*c1*qy3-c3*qy)/rho
      d122=(2.*c1*qx3-c3*qx)/rho
      d221=(2.*c1*qx3-c5*qx)/rho
      d222=(-2.*c1*qy3-c2*qy)/rho
c
      ur(3,jj1)=ru(3,jj1)+w(l)*d111
      ur(3,jj2)=ur(3,jj2)+w(l)*d112
      ur(4,jj1)=ur(4,jj1)+w(l)*d121
      ur(4,jj2)=ur(4,jj2)+w(l)*d122
      ur(5,jj1)=ur(5,jj1)+w(l)*d221
      ur(5,jj2)=ur(5,jj2)+w(l)*d222
c
137   continue
c
      uux=0.
      uuy=0.
      ssxx=0.
      ssxy=0.
      ssyy=0.
c
      do 142 i=1,nn
      uux=uux-ul(1,i)*ua(i)+ur(1,i)*fa(i)
      uuy=uuy-ul(2,1)*us(i)+ur(2,i)*fa(i)
      ssxx=ssxx+c1*ul(3,i)*ua(i)+ur(3,i)*fa(i)
      ssxy=ssxy+c1*ul(4,i)*ua(i)+ur(4,i)*fa(i)
      ssyy=ssyy+c1*ul(5,i)*ua(i)+ur(5,i)*fa(i)
142   continue
c
      uxa(ii)=uux/(8.*pi)
      uya(ii)=uuy/(8.*pi)
      sxxa(ii)=ssxx/(8.*pi)
      sxya(ii)=ssxy/(8.*pi)
      syya(ii)=ssyy/(8.*pi)
c
135   continue
c
      return
      end
```

```
      subroutine fldb(n,mm,pr,x,y,r,w,w1,w2,ub,tb,sf,yf,uxb,uyb,
     &     sxxb,sxyb,syyb)
c
      parameter(mn=75)
      parameter(mnn=2*mn)
      parameter(mnn1=mnn+1)
      parameter(mmm=50
c
      dimension x(mn),y(mn)
      dimension r(10),w(10),w1(10,w2(10)
      dimension ub(mnn),tb(mnn),fb(mnn)
      dimension xf(mmm),yf(mmm)
      dimension ul(5,mnn),ur(5,mnn)
      dimension uxb(mmm),uyb(mmm)
      dimension sxxb(mmm),sxyb(mmm),syyb(mmm)
      dimension temp1(5),temp2(5),temp3(5),temp4(5)
c
      nn=2*n
c
      pi=4.*atan(1.)
c
      c1=1.+pr
      c2=1.-pr
      c3=3.+pr
      c4=3.-pr
      c5=1.+3.*pr
c
      do 135 ii=1,mm
c
      do 136 i=1,nn
      do 136 k=1,5
      ul(k,i)=0.
      ur(k,i)=0.
136   continue
c
      do 137 j=1,n
      if(j.eq.1)then
      jm1=n
      else
      jm1=j-1
      endif
      jj2=2*j
      jj1=jj2-1
c
      if(j.eq.1)then
      jj2m2=nn
      else
      jj2m2=jj2-2
      endif
      jj1m2=jj2m2-1
c
      dx=x(j)-x(jm1)
      dy=y(j)-y(jm1)
      s=sqrt(dx*dx+dy*dy)
```

```
      fb(jj1)=tb(jj1)*s
      fb(jj2)=tb(jj2)*s
c
      xmi=(x(j)+x(jm1))/2.
      ymi=(y(j)+y(jm1))/2.
      dmm1=x(j)-xmi
      dmm2=y(j)-ymi
      c=dmm1+dmm1+dmm2+dmm2
c
      drm1=xf(ii)-xmi
      drm2=yf(ii)-ymi
      aa=drm1*drm1+drm2*drm2
      bb=-2.*(drm1*dmm1+drm2*dmm2)
c
      do 137 l=1,10
      rho=sqrt(aa+bb*r(l)+c*r(l)*r(l))
      rho2=rho*rho
      qx=(drm1-dmm1*r(l))/rho
      qy=(drm2-dmm2*r(l))/rho
      qx2=qx*qx
      qy2=qy*qy
      qx3=qx2*qx
      qy3=qy2*qy
      qx4=qx2*qx2
      qy4=qy2*qy2
c
      t11=(2.*c1*(dmm2*qx3+dmm1*qy3)+c2*dmm2*qx-c3*dmm1*qy)/rho
      t21=(2.*c1*(dmm1*qx3-dmm2*qy3)-c5*dmm1*qx+c3*dmm2*qy)/rho
      t12=(2.*c1*(dmm1*qx3-dmm2*qy3)-c3*dmm1*qx+c5*dmm2*qy)/rho
      t22=(2.*c1*(-dmm2*qx3-dmm1*qy3)+c3*dmm2*qx-c2*dmm1*qy)/rho
c
      ul(1,jj1m2)=ul(1,jj1m2)+w(l)*w1(l)*t11
      ul(1,jj2m2)=ul(1,jj2m2)+w(l)*w1(l)*t21
      ul(2,jj1m2)=ul(2,jj1m2)+w(l)*w1(l)*t12
      ul(2,jj2m2)=ul(2,jj2m2)+w(l)*w1(l)*t22
c
      ul(1,jj1)=ul(1,jj1)+w(l)*w2(l)*t11
      ul(1,jj2)+ul(1,jj2)+w(l)*w2(l)*t21
      ul(2,jj1)=ul(2,jj1)+w(l)*w2(l)*t12
      ul(2,jj2)+ul(2,jj2)+w(l)*w2(l)*t22
c
      u11=(-c4*log(rho)+c1*qx*qx)/2.
      u12=c1*qx*qy/2.
      u22=(-c4*log(rho)+c1*qy(qy)/2.
c
      ur(1,jj1)=ur(1,jj1)+w(l)*u11
      ur(1,jj2)=ur(1,jj2)+w(l)*u12
      ur(2,jj1)=ur(2,jj1)+2(l)*u12
      ur(2,jj2)=ur(2,jj2)+w(l)*u22
c
      s111=2.*((8.*qx4-4.*qx2-1.)*dmm2-2.*qx*qy*(4.*qx2-2.)*dmm1)
     &     /rho2
      s211=2.*(2.*qx*qy*(4.*qx2-1.)*dmm2-(8.*qx2*qy2-1.)*dmm1)
     &     /rho2
```

```
      s221=2.*((8.*qx2*qy2-1.)*dmm2-2.*qx*qy*(4.*qy2-1.)*dmm1)
     &     /rho2
      s222=2.*(2.*qx*qy*(4.*qy2-1.)*dmm2-(8.*qy4-4.*qy2-1.)*dmm1)
     &     /rho2
c
      ul(3,jj1m2)=ul(3,jj1m2)+w(l)*w1(l)*s111
      ul(3,jj2m2)=ul(3,jj2m2)+w(l)*w1(l)*s211
      ul(4,jj1m2)=ul(4,jj1m2)+w(l)*w1(l)*s211
      ul(4,jj2m2)=ul(4,jj2m2)+w(l)*w1(l)*s221
      ul(5,jj1m2)=ul(5,jj1m2)+w(l)*w1(l)*s221
      ul(5,jj2m2)=ul(5,jj2m2)+w(l)*w1(l)*s222
c
      ul(3,jj1)+ul(3,jj1)+w(l)*w2(l)*s111
      ul(3,jj2)+ul(3,jj2)+w(l)*w2(l)*s211
      ul(4,jj1)+ul(4,jj1)+w(l)*w2(l)*s211
      ul(4,jj2)+ul(4,jj2)+w(l)*w2(l)*s221
      ul(5,jj1)+ul(5,jj1)+w(l)*w2(l)*s221
      ul(5,jj2)+ul(5,jj2)+w(l)*w2(l)*s222
c
      d111=(-2.*c1*qx3-c2*qx)/rho
      d112=(2.*c1*qy3-c5*qy)/rho
      d121=(2.*c1*qy3-c3*qy)/rho
      d122=(2.*c1*qx3-c3*qx)/rho
      d221=(2.*c1*qx3-c5*qx)/rho
      d222=(-2.*c1*qy3-c2*qy)/rho
c
      ur(3,jj1)=ur(3,jj1)+w(l)*d111
      ur(3,jj2)=ur(3,jj2)+w(l)*d112
      ur(4,jj1)=ur(4,jj1)+w(l)*d121
      ur(4,jj2)=ur(4,jj2)+w(l)*d122
      ur(5,jj1)=ur(5,jj1)+w(l)*d221
      ur(5,jj2)=ur(5,jj2)+w(l)*d222
c
137   continue
c
c POST-MULTIPLY UL BY GAMMA TRANSPOSE INVERSE
c
      aa=-1
      do 138 l=1,5
      temp1(l)=ul(l,1)
      temp2(l)=ul(l,2)
138   continue
c
      do 139 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      do 140 l=1,5
      ul(l,1)=ul(l,1)+ul(l,kk1)*aa
      ul(l,2)=ul(l,2)+ul(l,kk2)*aa
140   continue
      aa=-aa
139   continue
c
```

```
      do 141 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      do 141 l=1,5
      temp3(l)=ul(l,kk1)
      temp4(l)=ul(l,kk2)
      ul(l,kk1)=-ul(l,kk1-2)+2.*temp1(l)
      ul(l,kk2)=-ul(l,kk2-2)+2.*temp2(l)
      temp1(l)=temp3(l)
      temp2(l)=temp4(l)
141   continue
c
      uux=0.
      uuy=0.
      ssxx=0.
      ssxy=0.
      ssyy=0.
c
      do 142 i=1,nn
      uux=uux-ul(1,i)*ub(i)+ur(1,i)*fb(i)
      uuy=uuy-ul(2,i)*ub(i)+ur(2,i)*fb(i)
      ssxx=ssxx+c1*ul(3,i)*ub(i)+ur(3,i)*fb(i)
      ssxy=ssxy+c1*ul(4,i)*ub(i)+ur(4,i)*fb(i)
      ssyy=ssyy+c1*ul(5,i)*ub(i)+ur(5,i)*fb(i)
142   continue
c
      uxb(ii)=uux/(8.*pi)
      uyb(ii)=uuy/(8.*pi)
      sxxb(ii)=ssxx/(8.*pi)
      sxyb(ii)=ssxy/(8.*pi)
      syyb(ii)=ssyy/(8.*pi)
c
135   continue
c
      return
      end

      subroutine fldc(n,mm,pr,x,y,r,w,w1,w2,uc,tc,xf,yf,uxc,uyc,
&     sxxc,sxyc,syyc)
c
      parameter(mn=75)
      parameter(mnn=2*mn)
      parameter(mnn1=mnn+1)
      parameter(mmm=50)
c
      dimension x(mn),y(mn)
      dimension r(10),w(10),w1(10),w2(10)
      dimension uc(mnn),tc(mnn),fc(mnn)
      dimension xf(mmm),yf(mmm)
      dimension ul(5,mnn),ur(5,mnn)
      dimension uxc(mmm),uyc(mmm)
      dimension sxxc(mmm),sxyc(mmm),syyc(mmm)
      dimension temp1(5),temp2(5),temp3(5),temp4(5)
```

```
      dimension temp11(5),temp12(5),temp13(5),temp14(5)
c
      nn=2*n
c
      pi=4.*atan(1.)
c
      c1=1.+pr
      c2=1.-pr
      c3=3.+pr
      c4=3.-pr
      c5=1.+3.*pr
c
      do 135 ii=1,nn
c
      do 136 i=1,nn
      do 136 k=1,5
      ul(k,i)=0.
      ur(k,i)=0.
136   continue
c
      do 137 j=1,n
      if(j.eq.1)then
      jm1=n
      else
      jm1=j-1
      endif
      jj2=2*j
      jj1=jj2-1
c
      if(j.eq.1)then
      jj2m2=nn
      else
      jj2m2=jj2-2
      endif
      jj1m2=jj2m2-1
c
      dx=x(j)-x(jm1)
      dy-y(j)-y(jm1)
      s+sqrt(dx*dx+dy*dy)
      fc(jj1)=tc(jj1)*s
      fc(jj2)=tc(jj2)*s
c
      xmi=(x(j)+x(jm1))/2.
      ymi=(y(j)+y(jm1))/2.
      dmm1=x(j)-xmi
      dmm2=y(j)-ymi
      c=dmm1*dmm1+dmm2*dmm2
c
      dnm1=xf(ii)-xmi
      dnm2=yf(ii)-ymi
      aa=dnm1*dnm1+dnm2*dnm2
      bb=-2.*(dnm1*dmm1+dnm2*dmm2)
c
      do 137 l=1,10
```

```
      rho=sqrt(aa+bb*r(l)+c*r(l)*r(l))
      rho2=rho*rho
      qx=(drm1-dmm1*r(l))/rho
      qy=(drm2-dmm2*r(l))/rho
      qx2=qx*qx
      qy2=qy*qy
      qx3=qx2*qx
      qy3=qy2*qy
      qx4=qx2*qx2
      qy4=qy2*qy2
c
      t11=(2.*c1*(dmm2*qx3+dmm1*ay3)+c2*dmm2*qx-c3*dmm1*qy)/rho
      t21=(2.*c1*(dmm1*qx3-dmm2*qy3)-c5*dmm1*qx+c3*dmm2*qy)/rho
      t12=(2.*c1*(dmm1*qx3-dmm2*qy3)-c3*dmm1*qx+c5*dmm2*qy)/rho
      t22=(2.*c1*(-dmm2*qx3-dmm1*qy3)+c3*dmm2*qx-c2*dmm1*qy)/rho
c
      ul(1,jj1m2)=ul(1,jj1m2)+w(l)*w1(l)*t11
      ul(1,jj2m2)=ul(1,jj2m2)+w(l)*w1(l)*t21
      ul(2,jj1m2)=ul(2,jj1m2)+w(l)*w1(l)*t12
      ul(2,jj2m2)=ul(2,jj2m2)+w(l)*w1(l)*t22
c
      ul(1,jj1)=ul(1,jj1)+w(l)*w2(10*t11
      ul(1,jj2)=ul(1,jj2)+w(l)*w2(l)*t21
      ul(2,jj1)=ul(2,jj1)+w(l)*w2(l)*t12
      ul(2,jj2)=ul(2,jj2)+w(l)*w2(l)*t22
c
      u11=(-c4*log(rho)+c1*qx*qx)/4.
      u12=c1*qx*qy/4.
      u22=(-c4*log(rho)+c1*qy*qy)/4.
c
      ur(1,jj1m2)=ur(1,jj1m2)+w(l)*w1(l)*u11
      ur(1,jj2m2)=ur(1,jj2m2)+w(l)*w1(l)*u12
      ur(2,jj1m2)=ur(2,jj1m2)+w(l)*w1(l)*u12
      ur(2,jj2m2)=ur(2,jj2m2)+w(l)*w1(l)*u22
c
      ur(1,jj1)=ur(1,jj1)+w(l)*w2(i)*u11
      ur(1,jj2)=ur(1,jj2)+w(l)*w2(l)*u12
      ur(2,jj1)=ur(2,jj1)+w(l)*w2(l)*u12
      ur(2,jj2)=ur(2,jj2)+w(l)*w2(l)*u22
c
      s111=2.*((8.*qx4-4.*qx2-1.)*dmm2-2.*qx*qy*(4.*qx2-1.)*dmm1)
     &     /rho2
      s211=2.*(2.*qx*qy*(4.*qx2-1.)*dmm2-(8.*qx2*qy2-1.)*dmm1)
     &     /rho2
      s221=2.*((8.*qx2*qy2-1.)*dmm2-2.*qx*qy*(4.*qy2-1.)*dmm1)
     &     /rho2
      s222=2.*(2.*qx*qy*(4.*qy2-1.)*dmm2-(8.*qy4-4.*qy2-1.)*dmm1)
     &     /rho2
c
      ul(3,jj1m2)=ul(3,jj1m2)+w(l)*w1(l)*s111
      ul(3,jj2m2)=ul(3,jj2m2)+w(l)*w1(l)*s211
      ul(4,jj1m2)=ul(4,jj1m2)+w(l)*w1(l)*s211
      ul(4,jj2m2)=ul(4,jj2m2)+w(l)*w1(l)*s221
      ul(5,jj1m2)=ul(5,jj1m2)+w(l)*w1(l)*s221
```

```
      ul(5,jj2m2)=ul(5,jj2m2)+w(l)*w1(l)*s222
c
      ul(3,jj1)=ul(3,jj1)+w(l)*w2(l)*s111
      ul(3,jj2)=ul(3,jj2)+w(l)*w2(l)*s211
      ul(4,jj1)=ul(4,jj1)+w(l)*w2(l)*s211
      ul(4,jj2)=ul(4,jj2)+w(l)*w2(l)*s221
      ul(5,jj1)=ul(5,jj1)+w(l)*w2(l)*s221
      ul(5,jj2)=ul(5,jj2)+w(l)*w2(l)*s222
c
      d111=(-2.*c1*qx3-c2*qx)/(rho*2.)
      d112=(2.*c1*qy3-c5*qy)/(rho*2.)
      d121=(2.*c1*qy3-c3*qy)/(rho*2.)
      d122=(2.*c1*qx3-c3*qx)/(rho*2.)
      d221=(2.*c1*qx3-c5*qx)/(rho*2.)
      d222=(-2.*c1*qy3-c2*qy)/(rho*2.)
c
      ur(3,jj1m2)=ur(3,jj1m2)+w(l)*w1(l)*d111
      ur(3,jj2m2)=ur(3,jj2m2)+w(l)*w1(l)*d112
      ur(4,jj1m2)=ur(4,jj1m2)+w(l)*w1(l)*d121
      ur(4,jj2m2)=ur(4,jj2m2)+w(l)*w1(l)*d122
      ur(5,jj1m2)=ur(5,jj1m2)+w(l)*w1(l)*d221
      ur(5,jj2m2)=ur(5,jj2m2)+w(l)*w1(l)*d222
c
      ur(3,jj1)=ur(3,jj1)+w(l)*w2(l)*d111
      ur(3,jj2)=ur(3,jj2)+w(l)*w2(l)*d112
      ur(4,jj1)=ur(4,jj1)+w(l)*w2(l)*d121
      ur(4,jj2)=ur(4,jj2)+w(l)*w2(l)*d122
      ur(5,jj1)=ur(5,jj1)+w(l)*w2(l)*d221
      ur(5,jj2)=ur(5,jj2)+w(l)*w2(l)*d222
c
137   continue
c
c POST-MULTIPLY UL AND UR BY GAMMA TRANSPOSE INVERSE
c
      aa=-1
      do 138 l=1,5
      temp1(l)=ul(l,1)
      temp2(l)=ul(l,2)
      temp11(l)=ur(l,1)
      temp12(l)=ur(l,2)
38    continue
c
      do 139 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      do 140 l=1,5
      ul(l,1)=ul(l,1)+ul(l,kk1)*aa
      ul(l,2)=ul(l,2)+ul(l,kk2)*aa
      ur(l,1)=ur(l,1)+ur(l,kk1)*aa
      ur(l,2)=ur(l,2)+ur(l,kk2)*aa
140   continue
      aa=-aa
139   continue
```

```
c
      do 141 k=2,n
      kk2=2*k
      kk1=kk2-1
c
      do 141 l=1,5
      temp3(l)=ul(l,kk1)
      temp4(l)=ul(l,kk2)
      temp13(l)=ur(l,kk1)
      temp14(l)=ur(l,kk2)
      ul(l,kk1)=-ul(l,kk1-2)+2.*temp1(l)
      ul(l,kk2)=-ul(l,kk2-2)+2.*temp2(l)
      ur(l,kk1)=-ur(l,kk1-2)+2.*temp11(l)
      ur(l,kk2)=-ur(l,kk2-2)+2.*temp12(l)
      temp1(l)=temp3(l)
      temp2(l)=temp4(l)
      temp11(l)=temp13(l)
      temp12(l)=temp(14(l)
141   continue
c
      uux=0.
      uuy=0.
      ssxx=0.
      ssxy=0.
      ssyy=0.
c
      do 142 i=1,nn
      uux=uux-ul(1,i)*uc(i)+ur(1,i)*fc(i)
      uuy=uuy-ul(2,i)*uc(i)+ur(2,i)*fc(i)
      ssxx=ssxx+c1*ul(3,i)*uc(i)+ur(3,i)*fc(i)
      ssxy=ssxy+c1*ul(4,i)*uc(i)+ur(4,i)*fc(i)
      ssyy=ssyy+c1*ul(5,i)*uc(i)+ur(5,i)*fc(i)
142   continue
c
      usc(ii)=uux/(8.*pi)
      uyc(ii)=uuy/(8.*pi)
      sxxc(ii)=ssxx/(8.*pi)
      sxyc(ii)=ssxy/(8.*pi)
      syyc(ii)=ssyy/(8.*pi)
c
135   continue
c
      return
      end
```

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