This is to certify that the

dissertation entitled

QUADRIC REPRESENTATION
AND SUBMANIFOLDS OF FINITE TYPE

presented by

IVKO DIMITRIĆ

has been accepted towards fulfillment
of the requirements for

Ph.D degree in MATHEMATICS

Bang-yen Chen
Major professor

Date July 28, 1989

# QUADRIC REPRESENTATION AND SUBMANIFOLDS OF FINITE TYPE

By

Ivko Dimitrić

A DISSERTATION

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1989

# ABSTRACT

## QUADRIC REPRESENTATION AND SUBMANIFOLDS OF FINITE TYPE

By

Ivko Dimitrić

For an isometric immersion $x : M^n \to E^m$ of a Riemannian manifold into a Euclidean space, one defines the map $\tilde{x} = x \cdot x^t$ ( x regarded as column vector) from M into the set of $m \times m$ symmetric matrices, which we call *quadric representation* of M and propose to study it .

A smooth map $f : M^n \to E^m$ is said to be of finite type (k-type) if it can be decomposed into finitely many ( k, not counting constant vector) eigenvectors of the Laplacian . In particular, a manifold immersed into a Euclidean space is said to be of k-type if the corresponding immersion is of k - type.

We prove some general results about the quadric representation, in particular those related to the condition of $\tilde{x}$ being of finite type. Submanifolds for which $\tilde{x}$ is 1-type map are classified as totally geodesic spherical submanifolds. We show that for minimal submanifold of $E^m$ the quadric representation is of infinite type. Further, we classify compact spherical hypersurfaces which are of 2-type via $\tilde{x}$ as small hyperspheres or standard products $S^p(r_1) \times S^{n-p}(r_2)$ with only three different possibilities for $(r_1, r_2)$. The main result is classification of compact minimal spherical hypersurfaces which are of 3-type and mass-symmetric via $\tilde{x}$ in dimensions $n \leq 5$ . The only such submanifold is the Cartan hypersurface $SO(3)/Z_2 \times Z_2$ . At the end we begin the study of submanifolds of $E^m$ whose mean curvature vector is harmonic. Such submanifolds are shown to be minimal under additional assumptions (e.g. for hypersurfaces having at most two distinct principal curvatures).

To my mother Nadežda ,

brother Radoslav,

and in memory of my father Milan

## ACKNOWLEDGMENTS

I wish to express my deep gratitude to Professor Bang-yen Chen, under whose expert and patient guidance was this work done. He provided not only help on the topic but also valuable insights into Differential geometry as a whole. I would like to express my appreciation to Professors Blair and Ludden for their helpful conversations and sharing their knowledge in preparing this subject. In addition, thanks go to all my teachers at Michigan State University for their excellent teaching. Finally, I especially thank my mother and brother for their love and encouragement.

## TABLE OF CONTENTS


| | | |
|---|---|---|
| INTRODUCTION | | 1 |
| CHAPTER 1. | Preliminaries | 4 |
| | 1. Riemannian geometry and submanifolds | 4 |
| | 2. Homogeneous spaces | 13 |
| | 3. Second standard immersion of a sphere | 17 |
| | 4. Isoparametric spherical hypersurfaces | 21 |
| | 5. Finite type maps and submanifolds | 28 |
| CHAPTER 2. | Quadric representation of a submanifold | 34 |
| CHAPTER 3. | Spherical hypersurfaces which are of low type via the second standard immersion of the sphere | 53 |
| | 1. Spherical hypersurfaces which are of 2 - type via $\tilde{x}$ | 53 |
| | 2. Minimal spherical hypersurfaces which are of 3 - type and mass - symmetric via $\tilde{x}$ | 60 |
| CHAPTER 4. | Submanifolds of $E^m$ with harmonic mean curvature vector | 83 |
| SUMMARY | | 96 |
| BIBLIOGRAPHY | | 100 |

# INTRODUCTION

For an isometric immersion $x : M^n \to E^m$ of a smooth Riemannian manifold $M^n$ into a Euclidean space, one of the first questions one might ask is : " What are the natural maps related to the immersion $x$ ? " Of course, x itself is one such map and we have rich submanifold theory of isometric immersions . Another natural map is the Gauss map which corresponds to each point p of M , the tangent space of M at p , $p \to e_1(p) \wedge \dots \wedge e_n(p)$, and investigation of this map led to many interesting results. Then each vector field $X \in \Gamma(TM)$ on M defines a map $X : M^n \to E^m$ and the Hopf index theory handles one aspect of this map . Also, if we regard x as a column matrix, $x(p) = (x_1(p), \dots, x_m(p))^t$, then one defines a map $\tilde{x}$ from M into the set of $m \times m$ symmetric matrices ( which is also a Euclidean space) by $\tilde{x} = x \cdot x^t$. We call this map the *quadric representation* of M and propose to study it . The map $\tilde{x}$ is not necessarily an isometric immersion but if $\tilde{x}$ is assumed to be isometric (or just conformal ) it follows that M must be a submanifold of a sphere centerted at the origin (Theorems 2.1-2) .

There are several important results about integrals of geometric quantities on a compact Riemannian manifold M . The classical theorem of Gauss - Bonnet states that $\int_M K dV = 2\pi\, \chi(M)$ , i.e. the integral of the Gauss curvature is a topological invariant - the Euler characteristic. Also the celebrated inequality of Chern and Lashof gives a universal lower bound (topological) for so called total absolute curvature: $TA(x) \geq b(M)$ , where $b(M)$ is the total Betti number of M [Ch-L]. Up to late 1970's there were some indications that one could find estimates for the total mean curvature in terms of the Riemannian structure of M. Finally, in 1979, B.Y. Chen gave the following best possible estimate for the total mean curvature [C 2]

$$\frac{\lambda_p}{n}\ \mathrm{vol}(M) \ \le\ \int_M |H|^2 dV \ \le\ \frac{\lambda_q}{n}\ \mathrm{vol}(M) \ ,$$

where $\lambda_p$ and $\lambda_q$ are two eigenvalues of the Laplacian uniquely determined by the spectral behavior of the immersion x . Thus we get an invariant [p, q] associated with M where p is an integer $\ge 1$ and q is either an integer $\ge p$ or $\infty$ (in latter case right hand side of the inequality is $\infty$ ). A submanifold M (or an immersion x ) is said to be of finite type if q is finite. Equivalently, M is of finite type if the immersion x decomposes into finitely many eigenvectors of the Laplacian ,

$$x = x_0 + x_p + \dots + x_q \ , \quad \text{where } x_0 = \text{const} \quad \text{and} \quad \Delta x_t = \lambda_t x_t \ \text{ for all } p \le t \le q \ .$$

If M is compact, the constant vector $x_0$ is the center of mass of M . A submanifold M is of k - type if there are exactly k nonzero vectors $x_t$ $(t > 0)$ in the decomposition above. The same definition can be adopted if we do not assume M compact, and also if x is not necessarily an isometric immersion but simply an arbitrary smooth map from M into $E^m$ . Since its inception, the theory of finite type submanifolds has become an area of active research [C 4]. According to the well known theorem of Takahashi [Ta 1], compact 1 - type submanifolds of $E^m$ are characterized as being minimal in hypersphere and one can expect that 2 - type and higher type submanifolds are more general. Indeed, the classification of even 2 - type spherical submanifolds is virtually impossible, but finite type submanifolds are still "nice" examples of submanifolds.

In Chapter 2 we classify submanifolds $x : M^n \to E^m$ for which the quadric representation $\tilde{x}$ is of 1 - type as totally geodesic submanifolds of hypersphere of $E^m$. While it is relatively easy to construct nonspherical submanifolds for which $\tilde{x}$ is of finite type, we show that if M is minimal in $E^m$ than its quadric representation is of infinite type

(Theorem 2.4) . Next, in Chapter 3 we study spherical hypersurfaces which are of low type via quadric representation. Studying submanifolds $x : M^n \to E^m$ whose quadric representation is of finite type amounts to studying spectral behavior of products of coordinate functions $x_i \cdot x_j$ . We classify spherical hypersurfaces which are of 2 - type via $\tilde{x}$ as products of two spheres with three different possibilities for the radii, thus generalizing a result of M. Barros and B.Y. Chen [B-C] . Investigation of 3 - type spherical submanifold is much more complicated because of the computation of iterated Laplacians involved. The only known result about spherical submanifolds being of 3 - type via $\tilde{x}$ is classification of minimal surfaces (n = 2) in sphere which are of 3 - type by M. Barros and F. Urbano [B-U] (See also [U]). In Chapter 3 we undertake study of minimal hypersurfaces of sphere which are mass - symmetric and of 3 - type via $\tilde{x}$ . The only such submanifold in dimensions $n \leq 5$ is the Cartan hypersurface $SO(3)/Z_2 \times Z_2$ (Theorem 3.2.2). Actually, all minimal isoparametric spherical hypersurfaces with three distinct principal curvatures are also mass - symmetric and of 3 - type via $\tilde{x}$ (Lemma 3.2.3).

In Chapter 4 we study submanifolds $x : M^n \to E^m$ of a Euclidean space which satisfy $\Delta H = 0$ , where H is the mean curvature vector of the immersion. This condition is equivalent to $\Delta^2 x = 0$ . Minimal submanifolds being the trivial solution, the real problem is to find nonminimal examples, that is, those immersions which are biharmonic but not harmonic. While the construction of such examples (if they exist) seems difficult, we show that submanifolds satisfying $\Delta H = 0$ are necessarily minimal if any of the following conditions is satisfied

(1) $M^n$ has constant mean curvature .

(2) $M^n$ is a hypersurface of $E^{n+1}$ with at most two distinct principal curvatures .

(3) $M^n$ is a pseudoumbilical submanifold of $E^m$ ( $n \neq 4$ )

(4) $M^n$ is of finite type .

# CHAPTER 1

# PRELIMINARIES

The purpose of this introductory chapter is to supply necessary definitions and to outline ideas and some general techniques used in the subsequent chapters. We deem it good to have main facts that will be used assembled in one place for easy reference without having to digress from the main flow later. This overview is by no means supposed to be exhaustive, but rather to assist a potential reader in reading through the rest of the work without necessity to turn to the references frequently. Most of the material in this chapter, however, is well known.

## 1. Riemannian geometry and submanifolds

Standard references here are [K-N] and [C 1] . We assume elementary notions from the theory of differentiable manifolds (differentiable functions, vector fields, tensor and exterior algebras, connections, integration on compact manifolds, ...) known. All manifolds are real, and (with the possible exception of some Lie groups) will be assumed connected. A generic manifold is usually denoted by $M^n$, where n stands for the dimension, or simply by M. The word "differentiable" means "$C^\infty$- differentiable" and is synonymous with "smooth". All manifolds and geometric objects will be assumed smooth unless stated otherwise. The set of real-valued smooth functions on M is denoted by $C^\infty(M)$, and the algebra of differentiable functions in the neighborhood of p by $C^\infty_p(M)$.

A differentiable manifold locally looks like a Euclidean space of the same dimension. A *tangent vector* X to a manifold M at a point $p \in M$ is a linear map from $C_p^\infty(M)$ to R, which is a derivation of the algebra $C_p^\infty(M)$, that is

(1.1.1) $$X(fg) = (Xf)\, g + f\,(Xg) \quad , \qquad \text{for every} \quad f, g \in C_p^\infty(M) \ .$$

The set of all tangent vectors at p, with its natural vector space structure, is called the *tangent space of* M *at* p and is denoted by $T_pM$. It can be visualized as the set of tangent vectors at p to all curves in M passing through p. The set of all pairs $(p, T_pM)$ forms *the tangent bundle* TM which is a vector bundle over M. A smooth section of TM is just a vector field on M, and the set of those is denoted by $\Gamma(TM)$. For two vector fields X, Y, *the bracket* [X,Y] is the vector field defined as

(1.1.2) $$[X,Y]\, f = X(Yf) - Y(Xf) \ .$$

For every function $f \in C^\infty(M)$ we can define 1-form df, called the *differential of* f, by $df(X) = Xf$, for every $f \in \Gamma(TM)$. More generaly, for a map $f : M \to N$ between two manifolds and a point $p \in M$ we have the induced map $(f_*)_p : T_pM \to T_{f(p)}N$, called *differential of* f *at* p, defined as

$$(f_*(X))\, g = X\,(g \circ f) \quad , \quad \text{for every} \quad g \in C_{f(p)}^\infty(N) \quad \text{and } X \in T_pM \ .$$

The pull-back map $f^*$ (at f(p)) is the adjoint of this linear map.

An *(affine) connection* on M is a rule $\nabla$ which assigns to each vector field X a linear map $\nabla_X$ of the vector space $\Gamma(TM)$ into itself satisfying the following two conditions

(1.1.3)
$$\nabla_{fX+gY} = f\,\nabla_X + g\,\nabla_Y$$
$$\nabla_X(fY) = f\,\nabla_X Y + (Xf)\, Y$$

The operator $\nabla_X$ is called *covariant differentiation with respect to* X. $\nabla_X$ can be extended to arbitrary tensor fields in a natural way to produce derivation of the tensor algebra that commutes with contractions , e.g. for covariant 2-tensor T we have

$$(\nabla_X T)(Y, Z) = \nabla_X(T(Y,Z)) - T( \nabla_X Y, Z) - T(Y, \nabla_X Z) \ .$$

Given a coordinate neighborhood $(U, x^1, \dots, x^n)$ of a manifold $M^n$ , we have the coordinate vector fields $\partial_1 = \frac{\partial}{\partial x^1}, \dots, \partial_n = \frac{\partial}{\partial x^n}$ on U. In the presence of a connection $\nabla$ , we can define functions $\Gamma^k_{ij}$ called the *Christoffel symbols* by

$$\nabla_{\partial_i}(\partial_j) = \sum_k \Gamma^k_{ij}\partial_k \qquad (1.1.4)$$

Let $\gamma : I \rightarrow M$ be a curve in M. The tangent vector field to the curve, $T(t) = \gamma_*\left(\frac{d}{dt}\right)$, is called the *velocity vector field* of the curve $\gamma$ . The curve $\gamma$ is called a *geodesic* (of a connection $\nabla$) if $\nabla_T T = 0$ , i.e. the velocity vector field is parallel along the curve. Using the affine connection $\nabla$ on M we define two tensor fields, *curvature tensor* R and *torsion tensor* T by

$$R(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]} \qquad (1.1.5)$$

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \quad , \quad X, Y \in \Gamma(TM) \qquad (1.1.6)$$

A *Riemannian manifold* (M,g) is a differentiable manifold M equipped with a symmetric positive definite tensor field g of type (0,2), called the *Riemannian metric*. On a Riemannian manifold there exists a unique affine connection $\nabla$ which has zero torsion, $T \equiv 0$ , and such that the metric tensor is parallel, $\nabla g = 0$ . These two conditions are equivalent to

$$[X,Y] = \nabla_X Y - \nabla_Y X \qquad (1.1.7)$$

(1.1.8) $$Z\, g(X,Y) = g(\nabla_Z X\,,\, Y) + g(X\,,\, \nabla_Z Y)\ ,$$

for every vector fields X, Y, Z . This connection is called the *Levi-Civita* (or *Riemannian*) connection. The Christoffel symbols of this connection are computed in a local coordinate system $(U, x^1, \dots, x^n)$ as

(1.1.9) $$\Gamma_{ij}^{k} = \frac{1}{2} \sum g^{tk} \left( \frac{\partial g_{ti}}{\partial x^j} + \frac{\partial g_{tj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^t} \right)$$

As usual, $(g_{ij})$ denotes matrix of the metric tensor g and $(g^{ij})$ is its inverse matrix.

For each point $p \in M$ and each 2-plane $\Pi \subset T_pM$, the *sectional curvature* $K(\Pi)$ of $\Pi$ is defined by $K(\Pi) = g(R(X,Y)Y, X)$ , where X, Y are orthonormal vectors which span $\Pi$ (it is independent of the choice of such pair X,Y in $\Pi$). Given two vectors X and Y in $T_pM$ and an orthonormal basis $e_1, \dots, e_n$ of $T_pM$ we define the *Ricci tensor* S and the *scalar curvature* $\tau$ at p by

(1.1.10) $$S(X,Y) = \sum_i g(R(X,e_i)e_i\,,\, Y)$$

(1.1.11) $$\tau = \sum_i S(e_i,e_i)$$

If for a Riemannian manifold (M, g) the sectional curvature $K(\Pi)$ is constant for all planes $\Pi \subset T_pM$ and all points $p \in M$ , then M is called a *space of constant curvature* or a *space form.* Standard examples are : Euclidean space $E^m$ (sectional curvature is 0), Sphere $S^m(r)$ (curvature is $1/r^2 > 0$ ), and hyperbolic space $H^m$ (curvature $< 0$) . Under additional topological assumptions (completness, simply connectedness) these are the only ones. A manifold (M,g) is called *(locally) flat* if its sectional curvature is 0.

A map $f : (M, g) \to (N, h)$ between two Riemannian manifolds is called *conformal* if $f^*h = \phi g$ for some positive function $\phi$ on $M$. If $\phi$ is a positive constant $f$ is *homothetic*. If $\phi \equiv 1$ and $f$ is a diffeomorphism then $f$ is called an *isometry*. $(M,g)$ is called *conformally flat* if there is a metric on $M$ conformal to $g$ with respect to which $M$ is flat.

Let $(M, g)$ and $(N, h)$ be two Riemannian manifolds. Then one can define a Riemannian metric $g \times h$ on the product manifold $M \times N$ in the following way

$$(g \times h)(X, Y) = g(X_1, Y_1) + h(X_2, Y_2),$$

where $X = X_1 + X_2$ and $Y = Y_1 + Y_2$ are the decompositions of $X$ and $Y$ with respect to the sum $T_{(m,n)}(M \times N) \cong T_mM \oplus T_nN$.

Given a Riemannian manifold $(M, g)$ and a point $p \in M$. For each vector $X$ in $T_pM$ there is a unique geodesic $\gamma_X(t)$ defined in the neighborhood of $0$ such that $\gamma_X(0) = p$ and $\gamma_X'(0) = X$. We define $\exp_p X$ as the point in $M$ given by $\gamma_X(1)$ when $\gamma_X(1)$ is defined. The map $\exp_p$ is called the *exponential map* at p. For each $p \in M$, there is an open neighborhood $\bar{U}$ of $0 \in T_pM$ and an open neighborhood $U$ of $p \in M$ such that the exponential map $\exp_p : \bar{U} \to U$ is a diffeomorphism of $\bar{U}$ onto $U$. Let $\bar{U}$ and $U$ be as above, and let $e_1, \dots, e_n$ be an orthonormal basis of $T_pM$. For each $X \in \bar{U}$ we put $X = x^1e_1 + \dots + x^ne_n$. Then the components $x^1, \dots, x^n$ are called *normal coordinates* of the point $q = \exp_p X$ in $U$ (determined by the frame $e_1, \dots, e_n$). In the normal coordinate system $(U, x^1, \dots, x^n)$ we have $g_{ij}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$, i.e. $\nabla_{e_i}e_j(p) = 0$ for every i, j, k. A Riemannian metric is called *complete* if every geodesic can be extended indefinitely in both directions, equivalently, if $\exp_p X$ is defined for every point p and every vector $X \in T_pM$ .This corresponds to the topological completness of the metric space $M$, where the distance between two points is defined as the infimum of the lengths of curves joining the two points. Every compact Riemannian manifold is complete.

A map $x : M \to \tilde{M}$ is called an *immersion* if $(x_*)_p : T_pM \to T_{x(p)}\tilde{M}$ is injective for each $p \in M$. If, in addition, x itself is injective it is called an *embedding* . If (M, g) and $(\tilde{M}, \tilde{g})$ are both Riemannian manifolds , x is an *isometric immersion* if $x^*\tilde{g} = g$ . When this is the case we say that the metric on M is induced from that of $\tilde{M}$, and call M *submanifold* of $\tilde{M}$ . We shall identify X with its image $x_*(X)$ for any $X \in TM$. Corresponding to the orthogonal splitting

$$T_p\tilde{M} = T_pM \oplus T_p^{\perp}M \quad , \quad \text{for every } p \in M \tag{1.1.12}$$

we can write for (local) smooth vector fields X and Y on M

$$\tilde{\nabla}_{x_*X}\, x_*Y = x_*(\nabla_X Y) + h\,(X, Y) \ , \tag{1.1.13}$$

where $\nabla_X Y$ tangent to M and h (X, Y) is normal to M. Note that in general symbols with ~ denote objects on $\tilde{M}$ and without ~ objects on M. According to the convention above we will also supress writing $x_*$ in the sequel. We call $\nabla$ the *induced connection* of M (it is actually the Levi-Civita connection of (M, g) ), and normal bundle valued symmetric tensor field h we call the *second fundamental form* of the immersion. If $h \equiv 0$ , the submanifold M is called *totally geodesic* . An immersion x is said to be *full* if x(M) does not lie in any totally geodesic submanifold of $\tilde{M}$ . Let $\xi$ be a local normal vector field and X a vector field on M then we have the following orthogonal decomposition

$$\tilde{\nabla}_X \xi = - A_\xi X + D_X \xi \ , \tag{1.1.14}$$

where $- A_\xi X$ and $D_X\xi$ are the tangential and normal components of $\tilde{\nabla}_X\xi$ respectively. For every $\xi$ , $A_\xi$ is an endomorphism of tangent space of M at every point. It is known as the *Weingarten map* or *shape operator* of $\xi$ and is related to the second fundamental form h via

$$\tilde{g}\,(h(X,Y),\xi) = g\,(A_\xi X\,,Y)\quad. \tag{1.1.15}$$

$A_\xi X$ is a symmetric operator and as such can be diagonalized over the reals. Its eigenvectors are called *principal directions* of $\xi$ and its eigenvalues, *principal curvatures*.

Let $e_1, \dots, e_n, e_{n+1}, \dots, e_m$ be an *adapted frame*, i.e. local frame of orthonormal vector fields of $\tilde{M}$ along M such that the first n vectors are tangent to M and the remaining $m - n$ are normal to M. We adopt the following convention about the range of indices : $1 \le i, j, k, \dots \le n$ , $n + 1 \le r, s, \dots \le m$ and $1 \le A, B, C, \dots \le m$ . We define a normal vector field H by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i,e_i) = \frac{1}{n}\sum_{r=n+1}^{m} (\mathrm{tr}\ A_r)e_r \tag{1.1.16}$$

and call it the *mean curvature vector field* . A submanifold M (or an immersion x ) is called *minimal* if $H \equiv 0$ . If we choose $e_{n+1}$ to be in the direction of H, $e_{n+1} \parallel H$ , then $H = \alpha e_{n+1}$ for some real function $\alpha$ which is called the *mean curvature of* M. If $A_\xi = \rho\, I$ for some function $\rho$, we say that $\xi$ is an *umbilical* section. If every local normal section is umbilical, submanifold is called *(totally) umbilical.* Equivalently, a totally umbilical submanifold is characterized by the property $h(X, Y) = g(X, Y)\, H$ , for every $X,Y \in \Gamma(TM)$ . A submanifold is called *pseudoumbilical* if $A_H = \rho\, I$ . It is called *quasiumbilical* if there exists an orthonormal frame of local normal vector fields $e_{n+1}, \dots, e_m \in T^\perp M$ such that for every $r$ , all principal curvatures of $e_r$ , except possibly one, are equal.

The normal part of (1.1.14 ) , D , defines a metric connection in the normal bundle $T^\perp M$ i.e. $D_X(\tilde{g}(\xi,\eta)) = \tilde{g}\,(D_X\xi\,,\eta) + \tilde{g}(\xi, D_X\eta)$ . Its curvature will be denoted by $R^D$.

Let $e_1, \dots, e_m$ be a local orthonormal frame of vector fields defined on an open set U of a Riemannian manifold $\tilde{M}^m$. Denote by $\omega^1, \ldots, \omega^m$ the dual frame, and define $m^2$ *connection 1-forms* $\omega_A^B$ on U by

$$\tilde{\nabla}_X e_A = \sum_{B=1}^{m} \omega_A^B(X) e_B . \tag{1.1.17}$$

Then $\omega_A^B + \omega_B^A = 0$, and the following structural equations of Cartan hold

$$d\omega^A = -\sum \omega_B^A \wedge \omega^B \tag{1.1.18}$$

$$d\omega_B^A = -\sum \omega_C^A \wedge \omega_B^C + \Omega_B^A , \tag{1.1.19}$$

where $\Omega_B^A = \frac{1}{2} \sum \tilde{R}^A_{BCD}\, \omega^C \wedge \omega^D$ with $\tilde{R}^A_{BCD} = \tilde{g}\,(\tilde{R}(e_C, e_D)\, e_B, e_A)$. In the space of constant curvature, $\tilde{M}^m(c)$, we have $\Omega_B^A = c\, \omega^A \wedge \omega^B$.

Now if $M^n$ is a submanifold of $\tilde{M}^m$ and $e_1, \ldots, e_n, e_{n+1}, \ldots, e_m$ an adapted frame, then when the forms $\omega_A^B$ are restricted to M we see that $\omega_i^j$ are connection 1-forms of the induced connection $\nabla$, $\omega_r^s$ are connection 1-forms of the normal connection D, and $\omega_i^r$ determine the second fundamental form h. Moreover, by a lemma of Cartan

$$\omega_i^r = \sum_j h_{ij}^r \omega^j \quad , \quad \text{where } h_{ij}^r = \tilde{g}\,(h(e_i, e_j), e_r) . \tag{1.1.20}$$

Let $x : M^n \to \tilde{M}^m$ be an isometric immersion . Then the three fundamental equations of Gauss, Codazzi and Ricci "determine" immersion x (cf. [C 4], p.120). For the immersion into a space of constant curvature c, $x : M^n \to \tilde{M}^m(c)$, *equations of Gauss, Codazzi* and *Ricci* are respectively given by

$$R(X, Y; Z, W) = c\{\, g(X,W)g(Y, Z) - g(X, Z)g(Y, W)\} \tag{1.1.21}$$

$$+ \tilde{g}(h(X,W), h(Y,Z)) - \tilde{g}(h(X,Z), h(Y,W))$$

(1.1.22) $$(\bar{\nabla}_X h)(Y,Z) = (\bar{\nabla}_Y h)(X,Z)$$

(1.1.23) $$R^D(X,Y;\xi,\eta) = g([A_\xi, A_\eta]X, Y)$$

Here, $\bar{\nabla}$ is so called connection of van der Waerden - Bortolotti defined by

(1.1.24) $$(\bar{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) ,$$

and $R(X,Y;Z,W) = g(R(X,Y)Z, W)$ , $R^D(X,Y;\xi,\eta) = \tilde{g}(R^D(X,Y)\xi,\eta)$ . If M is a hypersurface of space of constant curvature c we have only Gauss and Codazzi equations which in this case read as

(1.1.25) $$R(X,Y) = c(X \wedge Y) + AX \wedge AY$$

(1.1.26) $$(\nabla_X A)Y = (\nabla_Y A)X \quad .$$

If $e_1, \dots, e_n$ is orthonormal basis of principal directions of A , $\lambda_1, \dots, \lambda_n$ respective principal curvatures and $\omega_i^j$ corresponding connection forms, then the Codazzi equation is equivalent to the following system of formulas

(1.1.27) $$(\lambda_j - \lambda_i)\,\omega_j^i(e_i) = e_j \lambda_i \ , \quad i \neq j$$

(1.1.28) $$(\lambda_j - \lambda_k)\,\omega_j^k(e_i) = (\lambda_i - \lambda_k)\,\omega_i^k(e_j) \ , \quad i \neq j \neq k \neq i$$

and no summation occurs on repeated indices .

## 2. Homogeneous spaces

For the basic facts about Lie groups we refer to [Wa], [He] and for homogeneous spaces to [K-N], [Ch-E], [Ch 1], [Be] .

A *Lie group* G is a smooth manifold (which we do not assume connected), which has the structure of a group in such a way that the map $\phi : G \times G \to G$ defined by $\phi(x, y) = x \cdot y^{-1}$ is smooth. The identity component of a Lie group is itself a Lie group. Readily available examples of Lie groups are classical groups GL(n), O(n), SO(n), U(n), Sp(n), ... etc. Also, the well known result of Myers and Steenrod asserts that the isometry group of any Riemannian manifold is a Lie group.

A Lie algebra over R is a real vector space V together with a bilinear map (called *bracket*) $[\ ,\ ] : V \times V \to V$ such that for any $x, y, z \in V$

(1.2.1) $$[x, y] = - [y, x]$$

(1.2.2) $$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0 \quad .$$

As an example, set of smooth vector fields on a manifold is (infinite dimensional) Lie algebra with the bracket operation defined in Section 1.

If $a \in G$ , then the left translation by $a$ and the right translation by $a$ are respectively the diffeomorphisms $L_a$ and $R_a$ of G defined by $L_a(x) = a\,x$ , $R_a(x) = xa$ . A vector field X on G is called *left invariant* if for each $a \in G$, $(L_a)_* \circ X = X \circ L_a$ . The set of left invariant vector fields on a Lie group G forms a Lie algebra called the *Lie algebra of G* and is denoted by $\mathfrak{g}$ . If we define a map $\alpha : \mathfrak{g} \to T_eG$ by $\alpha(X) = X(e)$ , then $\alpha$ is vector space isomorphism, so $\dim \mathfrak{g} = \dim G$. We can define [ , ] on $T_eG$ by requiring that $\alpha$ becomes Lie algebra isomorphism , thus identifying the tangent space at the identity of G with the Lie algebra of G.

A subspace $\boldsymbol{h}$ of $\boldsymbol{g}$ which is closed under [ , ] is called a *subalgebra* of $\boldsymbol{g}$. If $\boldsymbol{h}$ is a subalgebra of $\boldsymbol{g}$, then $\boldsymbol{h}$ defines an involutive distribution and the maximal connected integral manifold H through e is a subgroup of G (which, in general, is not a closed subset of G ). Conversely, if $H \subset G$ is a Lie subgroup, then the tangent space $\boldsymbol{h}$ of H at e is a subalgebra of $\boldsymbol{g}$ .

If we take $\boldsymbol{h}$ to be any 1-dimensional subspace of $\boldsymbol{g}$, then $[\boldsymbol{h}, \boldsymbol{h}] = 0 \subset \boldsymbol{h}$. The subgroup corresponding to such an $\boldsymbol{h}$ is called a 1-parameter subgroup. For any $X \in T_eG$ we have a natural homomorphism of Lie algebras $d\phi : R \to \boldsymbol{g}$ with $d\phi(d/dt) = X$, and hence a Lie group homomorphism $\phi : R \to G$ mapping R onto the integral curve through the origin of the left invariant vector field determined by X. We denote $\phi(1)$ by $\exp_e X$ and this coincides with usual exp defined before for smooth manifolds. The structures of $\boldsymbol{g}$ and G are related by the exponential mapping, in fact, the Lie algebra determines the Lie group in the sense that if G and G' are two simply connected Lie groups which have isomorphic Lie algebras then G and G' are isomorphic.

A Lie group G acts on itself on the left by inner automorphisms $\sigma_g : G \to G$, $g \in G$, defined by $\sigma_g(x) = g\, x\, g^{-1}$. The identity e is a fixed point of any such action. The map

$$g \to d\sigma_g|\, T_eG \cong \boldsymbol{g}$$

is a representation (i.e. homomorphism) of G into $\mathrm{Aut}(\boldsymbol{g}) = GL(\boldsymbol{g}) \cong GL(n)$. It is called the *adjoint representation* and is denoted by $\mathrm{Ad} : G \to \mathrm{Aut}(\boldsymbol{g})$. So $\mathrm{Ad}(g) = dR_g \circ dL_{g^{-1}}$. Define $\mathrm{ad} : \boldsymbol{g} \to gl(\boldsymbol{g})$ to be the differential of the adjoint representation, $\mathrm{ad} = d(\mathrm{Ad})$. Then $\mathrm{ad}X(Y) = [X, Y]$ for every $X, Y \in \boldsymbol{g}$, and by Jacobi identity (1.2.2), adX is a derivation of the Lie algebra $\boldsymbol{g}$, i.e. $\mathrm{ad}X([Y, Z]) = [\mathrm{ad}X(Y), Z] + [Y, \mathrm{ad}X(Z)]$.

Let K be a closed subgroup of a Lie group G, and let G/K denotes the space of cosets $\{gK \mid g \in G\}$. Let $\pi : G \to G/K$ denotes the natural projection $\pi(g) = gK$. Then G/K has a unique manifold structure such that $\pi : G \to G/K$ is smooth fibration, i.e. $\pi$ is smooth and there exist local smooth sections of G/K in G . We call G/K a *homogeneous space* . G acts naturally on G/K on the left by $g'\pi(g) = \pi(g'g)$ and this action is clearly transitive hence the name homogeneous space .

Let $\alpha : G \times M \to M$ be a smooth action of a Lie group G on M on the left and denote $\alpha(g, p) = \alpha_g(p)$. The action is called *transitive* if for any pair $x, y \in M$ there exists $g \in G$ such that $\alpha_g(x) = y$ . G acts *effectively* on M if $\alpha_g(p) = p$ for every $p \in M$ implies $g = e$ . Let $o \in M$ and let $K = \{g \in G \mid \alpha_g(o) = o\}$. K is a closed subgroup of G called the *isotropy group at* o. We now state the following theorem (see [Wa] ) .

**Theorem 1.2.1.** Let $\alpha : G \times M \to M$ be a transitive action of a Lie group G on a manifold M on the left . Let $o \in M$ , and let K be the isotropy group at o . Define a mapping $\beta : G/K \to M$ by $\beta(gK) = \alpha_g(o)$ . Then $\beta$ is a diffeomorphism .

For each $k \in K$ (= isotropy group at o ) the map $\rho : K \to GL(T_oM)$ defined by $\rho(k) = d\alpha_k|_{T_oM}$ is a representation of K ( [Wa], p.113) called the *linear isotropy representation* and the group $\rho(K)$ of linear transformations of $T_oM$ is called the *linear isotropy group at* o . Because of the Theorem 1.2.1 we adopt the following definition.

**Definition 1.2.1.** A Riemannian manifold (M, g) is called (*Riemannian*) *homogeneous space* if the group of isometries I(M) acts transitively on M .

Since there may be more than one Lie group acting transitively on a given homogeneous space we use the term G - homogeneous if G is a closed subgroup of I(M) which acts transitively on M. Note that M is compact if and only if G is compact. Since an isometry f is determined by giving only the image f(o) of a point o and the corresponding tangent map $df|_o$ , the linear isotropy representation of a Riemannian homogeneous space is faithfull (injective) orthogonal representation .

We recall that the projective spaces are homogeneous manifolds

$$RP^n = SO(n+1)/O(n) \ , \ CP^n = SU(n+1)/S(U(1)U(n)) \ ,$$

$$QP^n = Sp(n+1)/Sp(n)Sp(1) \quad , \quad CayP^2 = F_4/Spin(9) \ .$$

A homogeneous manifold $M = G/K$ is called *reductive* if there is an $Ad(K)$ - invariant subspace $\boldsymbol{m}$ of $\boldsymbol{g}$ that is complementary to $\boldsymbol{k}$, $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{m}$ , where $\boldsymbol{g}$ and $\boldsymbol{k}$ are the Lie algebras of G and K respectively. All homogeneous Riemannian manifolds are reductive ( see e.g. [T-V], pp 19-20 ). For the Levi - Civita connection and the curvature of a reductive homogeneous space see [K-N] and [Be] . Given a homogeneous space G/K we can define symmetric $Ad(G)$ - invariant bilinear form $B : \boldsymbol{g} \times \boldsymbol{g} \to R$ by $B(X, Y) = tr ( adX \circ adY)$ . B is called the *Killing - Cartan* form of $\boldsymbol{g}$ . For a reductive homogeneous space G/K , B is negative definite on $\boldsymbol{k}$ but $\boldsymbol{m}$ is not necessarily B orthogonal to $\boldsymbol{k}$ nor is B definite on $\boldsymbol{m}$ in general. We state the following theorem which can be found in [Ch], p. 48 or [O'N] , p. 311 .

**Theorem 1.2.2.** Let $M = G/K$ be a reductive homogeneous space with $Ad(K)$ - invariant splitting $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{m}$ . Then the linear isotropy group $\{d\alpha_k \mid k \in K\}$ acting on $T_0M$ corresponds under $d\pi$ to $Ad(K)$ on $\boldsymbol{m}$ ( $\pi$ is a natural projection $G \to G/K$ ) .

Next, we give basic facts about symmetric spaces. For thorough study see [He] .

A Riemannian manifold M is called a *symmetric space* if for every point $p \in M$ , there exists an involutive isometry $s_p$ with p as an isolated fixed point . Isometry $s_p$ is in fact geodesic symmetry at p, $s_p(\gamma(t)) = \gamma(-t)$ , for every geodesic $\gamma$ through $p = \gamma(0)$ . Every symmetric space M is a homogeneous space $M = G/K$ , where $G = I_0(M)$ is identity component of isometry group of M and K is a compact subgroup of G ([He], p. 208 ).

For a symmetric space $M = G/K$ , K isotropy group at o , we define involutive automorphism $\sigma : G \to G$ by $\sigma(g) = s_o g s_o$ . Then $G_\sigma^o \subseteq K \subseteq G_\sigma$ where $G_\sigma = \{g \in G \mid \sigma(g) = g\}$ and $G_\sigma^o$ is its identity component . Automorphism $\sigma$ induces involutive automorphism of $\boldsymbol{g}$ (by $d\sigma|_{T_0M}$) denoted by the same letter $\sigma$. We denote by $\boldsymbol{k}$ and $\boldsymbol{m}$ respectively +1 and -1 eigenspace of $\sigma$ . Then $\boldsymbol{k}$ is the Lie algebra of K , $\boldsymbol{m}$ can be identified with $T_0M$, and we have the following direct sum decomposition

$$(1.2.3) \qquad \boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{m} \ , \text{ with}$$
$$[\boldsymbol{k}, \boldsymbol{k}] \subset \boldsymbol{k} \ , \ [\boldsymbol{m}, \boldsymbol{m}] \subset \boldsymbol{k} \ , \ [\boldsymbol{k}, \boldsymbol{m}] \subset \boldsymbol{m} \ .$$

Decomposition (1.2.3) is called the *Cartan decomposition* of $\boldsymbol{g}$ with respect to $\sigma$.

Let ( $\boldsymbol{g}$ , $\boldsymbol{k}$ , $\sigma$) be a triple such that : (1) $\boldsymbol{g}$ is a Lie algebra (over R) ; (2) $\sigma$ is an involutive automorphism of G ; (3) $\boldsymbol{k} = F(\sigma, \boldsymbol{g})$, the fixed point set of $\sigma$, is compact subalgebra . Then ( $\boldsymbol{g}$ , $\boldsymbol{k}$ , $\sigma$) is called an *orthogonal symmetric Lie algebra* (o.s.L.a.). Obviously, for every symmetric space G/K we have an o.s.L.a. associated with it.

Let ( $\boldsymbol{g}$ , $\boldsymbol{k}$ , $\sigma$) be an o.s.L.a. with $\boldsymbol{g} = \boldsymbol{k} \oplus \boldsymbol{m}$ , and let $\boldsymbol{a}$ be a maximal abelian subspace of $\boldsymbol{m}$ . Then the dimension of $\boldsymbol{a}$ is called the *rank* of o.s.L.a. ( $\boldsymbol{g}$ , $\boldsymbol{k}$ , $\sigma$). Correspondingly, the *rank* of a symmetric space is the maximal dimension of a flat, totally geodesic submanifold (flat torus) of M. Compact rank one symmetric spaces are sphere and projective spaces. Compact symmetric spaces of rank two are used in construction of isoparametric spherical hypersurfaces ( see section 4 of this chapter).

Lie algebra $\boldsymbol{g}$ is *semisimple* if the Killing - Cartan form B is nondegenerate. An o.s.L.a. ( $\boldsymbol{g}$ , $\boldsymbol{k}$ , $\sigma$) with $\boldsymbol{g}$ semisimple is said to be *of compact type* if B is negative definite. In that case, - B restricted to $\boldsymbol{m}$ defines Ad(G) - invariant inner product .

### 3. Second standard immersion of a sphere

For a good exposition on this topic see [C 4] . On Euclidean space $E^m$ we have canonical inner product $<,>$ given by $< u, v > = u^t \cdot v$ , where vectors $u, v \in E^m$ are regarded as column matrices and $u^t$ is the transpose of u . The sphere of radius r centered at the origin is defined as $S^{m-1}(r) = \{ x \in E^m \mid < x, x > = r^2 \}$ . Hypersphere of unit radius centered at the origin will be simply denoted by $S^{m-1}$ .

Let $SM(m) = \{P \in GL(m;R) \mid P^t = P\}$ be the space of real symmetric $m \times m$ matrices. Since every symmetric matrix $P \in SM(m)$ has $m(m+1)/2$ independent entries, $SM(m)$ can be regarded as Euclidean space of dimension $N = m(m+1)/2$. Moreover, if we define metric g on SM(m) by

$$(1.3.1) \qquad g(P,Q) = \frac{1}{2}\,\mathrm{tr}\,(PQ) \;, \qquad P, Q \in SM(m)$$

then g is identified with the canonical metric on $E^N$. For computational purposes (multiplication of matrices), however, we view SM(m) as sitting in $E^{m^2}$. Consider now the mapping $f : S^{m-1} \to SM(m)$ defined by $f(u) = u \cdot u^t$ where $u \in S^{m-1} \subset E^m$ is a column vector in $E^m$ of unit length. Thus, if $u = (u_1, \ldots, u_m)^t$ we have

$$(1.3.2) \qquad f(u) = \begin{pmatrix} u_1^2 & u_1u_2 & \cdots & u_1u_m \\ u_2u_1 & u_2^2 & \cdots & u_2u_m \\ \cdots & \cdots & \cdots & \cdots \\ u_mu_1 & u_mu_2 & \cdots & u_m^2 \end{pmatrix}$$

We see that f is an isometric immersion by virtue of $f_*(X) = u\,X^t + X\,u^t$. It is in fact second standard immersion of $S^{m-1}$ and since $f(-u) = f(u)$ it gives an embedding of $RP^{m-1}$. Since $\mathrm{tr}\, f(u) = \sum_i u_i^2 = \langle u, u\rangle = 1$ and $f(u)^2 = u(u^tu)u^t = u\,u^t$ then by comparing the dimension we see that $f(S^{m-1}) = \{A \in SM(m) \mid A^2 = A \text{ and } \mathrm{tr}\, A = 1\}$. Thus the image $f(S^{m-1})$ is a real projective space lying fully in a hyperplane $E_1 = \{A \in SM(m) \mid \mathrm{tr}\, A = 1\}$ of $SM(m) = E^N$. We call $f(S^{m-1})$ a *Veronese submanifold.* Also we check that

$$g\left(A - \frac{I}{m}, A - \frac{I}{m}\right) = \frac{1}{2}\,\mathrm{tr}\left(A - \frac{I}{m}\right)^2 = \frac{m-1}{2m} \;,$$

where I is $m \times m$ identity matrix, so $f(S^{m-1})$ lies in a hypersphere $S^{N-1}_{I/m}(r)$ of SM(m) centered at I/m with radius $r = \sqrt{\frac{m-1}{2m}}$ .The mean curvature vector of $f : S^{m-1} \to SM(m)$

at $u \in S^{m-1}$ can be computed ([C 4]) as $\bar{H} = \frac{2}{m-1}(I - m f(u))$ which is parallel to the radius vector $f(u) - \frac{I}{m}$. Thus, $f(S^{m-1})$ is minimal submanifold of a hypersphere $S^{N-1}_{I/m}(r)$.

Tangent space and normal space of $f(S^{m-1})$ are given respectively by

(1.3.3) $$T_{f(u)}S^{m-1} = \{ P \in SM(m) \mid P f(u) + f(u) P = P \},$$

(1.3.4) $$T^{\perp}_{f(u)}S^{m-1} = \{ P \in SM(m) \mid P f(u) = f(u) P \},$$

or, equivalently,

(1.3.5) $$T^{\perp}_{f(u)}S^{m-1} = \{ P \in SM(m) \mid Pu = \mu u \text{, for some } \mu \in R \}.$$

If $\bar{\sigma}$ is the second fundamental form of f, then ( see [C 4], [R] )

(1.3.6) $$\bar{\sigma}(X, Y) = X Y^t + Y X^t - 2 \langle X, Y \rangle f(u), \qquad X, Y \in T_u S^{m-1}.$$

It is known that $\bar{\sigma}$ is parallel, i.e. $\bar{\nabla}\bar{\sigma} = 0$.

From (1.3.4) we see that both I and f(u) are normal to $S^{m-1}$ via f, also, for any tangent vector X to sphere, $X X^t$ is normal to $S^{m-1}$. We prove the following lemma that will be used in Chapter 3.

**Lemma 1.3.1.** For a standard hypersphere $u : S^{m-1} \to E^m$, let f be the second standard immersion $f : S^{m-1} \to SM(m)$ by $f(u) = u \cdot u^t$. If $e_1, \dots, e_{m-1}$ is a local orthonormal frame of tangent vetors to $S^{m-1}$ then $I = u u^t + \sum_{i=1}^{m-1} e_i e_i^t$, where I is $m \times m$ identity matrix.

**Proof.** Consider the following matrices : $u u^t$, $e_k e_k^t$ $(1 \le k \le m-1)$, $e_i e_j^t + e_j e_i^t$ $(1 \le i < j \le m-1)$. By (1.3.4) they all belong to the normal space $T^{\perp}_u S^{m-1}$, and there are

$1 + m - 1 + \binom{m-1}{2} = \frac{m^2-m+2}{2}$ of them in number. Also these vectors are linearly independent ( they are mutually orthogonal ) . On the other hand, $\dim SM(m) = \frac{m(m+1)}{2}$ and $\dim T_uS^{m-1} = m - 1$, so , $\dim T_u^{\perp}S^{m-1} = \frac{m(m+1)}{2} - m + 1 = \frac{m^2-m+2}{2}$ . We conclude therefore, that $T_u^{\perp}S^{m-1} = \mathrm{Span}\{u\,u^t,\ e_ke_k^t,\ e_ie_j^t + e_je_i^t\}$ . In particular, $I = a\,(u\,u^t) + \sum_i b_k(e_ke_k^t) + \sum_{i<j} c_{ij}(e_ie_j^t + e_je_i^t)$ . Using (1.3.1) , it is easy to see that $c_{ij} = 0$ and $a = b_k = 1$ for every k , proving the lemma.

◆

Standard embeddings of projective spaces can be realized in an analogous way using Hopf fibration . Namely, let F denote one of the fields R of real numbers, C of complex numbers or skew field Q of quaternions, and let $d = d(F)$ be the dimension of F over the reals. For a matrix A over F , $A^t$ and $\bar{A}$ denote transpose and conjugate matrix and let $A^* = \bar{A}^t$. $M(m; F)$ is the set of all $m \times m$ matrices over F and the set of Hermitian matrices is $H(m; F) = \{A \in M(m; F) \mid A^* = A\}$. $F^m$ is considered as an md - dimensional vector space over R with the usual Euclidean inner product $< z, w > = \mathrm{Re}(z^*w)$ . All vectors in $F^m$ are regarded as column matrices .

Projective space $FP^{m-1}$ is considered as the quotient of the unit hypersphere $S^{md-1} = \{ z \in F^m \mid z^*z = 1\}$ obtained by identifying z with $z\lambda$ where $\lambda \in F$ with $|\lambda| = 1$ . $FP^{m-1}$ is given canonical metric such that $\pi : S^{md-1} \to FP^{m-1}$ is a Riemannian submersion with totally geodesic fibers . Note that we have natural action of the unitary group $U(m; F)$ on $FP^{m-1}$ induced from the one on the sphere $S^{md-1}$ . Define the map

$$\phi : FP^{m-1} \to H(m; F) \qquad \text{by}$$
$$\phi(p) = z\,z^* , \qquad \text{where } z \in \pi^{-1}(p) .$$

This map is well defined and gives an embedding of $FP^{m-1}$ into $H(m; F)$ (the first standard embedding of a projective space). The image of $FP^{m-1}$ under this map is given as $\phi(FP^{m-1}) = \{A \in H(m; F) \mid A^2 = A$ and $trA = 1\}$ and lies as a minimal submanifold in a hypersphere of H(m; F) centered at I/m and with radius $r = \sqrt{\frac{m-1}{2m}}$ . The Cayley projective plane $CayP^2$ cannot be realized via Hopf fibration and is simply defined as $CayP^2 = \{A \in H(m; F) \mid A^2 = A$ and $trA = 1\}$ . Embedding $\phi$ was first studied by Tai [Tai] , who proved that the embedding $\phi$ is equivariant with respect to and invariant under the action of U(m; F) . For other properties of this map see also [S], [R], [C 3] and [C 4].

## 4. Isoparametric spherical hypersurfaces

In this section exposition follows essentially [Ce-Ry], [Car 2-5], [M], [F], [N 1-2] and also uses results of [T-Ta], [T 3], [H], [H-L], [A] .

Originally, a family of hypersurfaces $M^n_t$ in a real space form $\tilde{M}^{n+1}(c)$ of constant sectional curvature c is called *isoparametric* if each $M^n_t$ is equal to level hypersurface $f^{-1}(t)$ where f is a non - constant real valued function on $\tilde{M}^{n+1}(c)$ which satisfies system of differential equations of the form

$$\|\tilde{\nabla} f\|^2 = a(f) \qquad , \qquad \tilde{\Delta} f = b(f)$$

for some smooth real - valued functions a, b . Thus, the two classical Beltrami differential parameters, square of the norm of gradient and Laplacian, are functions of f itself, whence the name isoparametric. (For the shape operator and mean curvature of such level hypersurface in terms of a and b see [Ce-Ry] or [Ha] ). Equivalently, an isoparametric family of hypersurfaces can be characterized as a family of parallel hypersurfaces, each of which has constant principal curvatures ( [Car 2], [N 2] ). We will adopt the following definition .

**Definition 1.4.1** A (complete) hypersurface is called *isoparametric* if its principal curvatures (and their respective multiplicities) are constant.

Cartan [Car 2] established the following basic identity for principal curvatures of an isoparametric hypersurface of a space form $\tilde{M}^{n+1}(c)$ .

**Theorem 1.4.1** Suppose that an isoparametric hypersurface M has $\nu$ distinct principal curvatures $k_1, \ldots, k_\nu$ with respective multiplicities $m_1, \ldots, m_\nu$ . Then

$$\sum_{j \neq i} m_j \frac{c + k_i k_j}{k_i - k_j} = 0 \quad , \quad 1 \leq i \leq \nu . \tag{1.4.1}$$

Using this key identity Cartan was able to determine all isoparametric hypersurfaces in the cases $c \leq 0$ . Actually if $c \leq 0$ , then there are at most two distinct principal curvatures of M and M is either umbilical (one curvature), or standard spherical cylinder $S^k \times E^{n-k}$ (standard product $S^k \times H^{n-k}$ ) in $E^{n+1}$ for $c = 0$ (respectively in hyperbolic space form $H^{n+1}$ for $c = -1$ ).

For hypersurfaces of the sphere $S^{n+1}$ things are much more interesting, in particular number of principal curvatures can be greater then two. E. Cartan undertook study of the spherical isoparametric hypersurfaces in the series of papers [Car 2 - 5] . He classified isoparametric hypersurfaces of $S^{n+1}$ with two distinct principal curvatures as standard products of two spheres [Car 2] , and he found that those with three distinct principal curvatures are precisely the tubes of constant radius over the standard embeddings of $FP^2$ for F = R, C, Q (quaternions), O (Cayley octaves) in $S^4$, $S^7$, $S^{13}$, $S^{25}$ respectively [Car 3] . In each isoparametric family of parallel hypersurfaces there is a unique hypersurface which is minimal in sphere. It is easy to see that the principal curvatures of minimal isoparametric hypersurfaces with three principal curvatures are $\sqrt{3}$ , 0 , - $\sqrt{3}$ , i.e.

they are roots of the equation $x^3 - 3x = 0$ . Namely, from minimality and (1.4.1) with $c = 1$ we have

$$3\,k_i - k_i^3 - 2\det A = 0 \quad , \quad i = 1, 2, 3$$

from which $k_i \in \{\sqrt{3}\,,0\,,-\sqrt{3}\,\}$. We also used the fact that the multiplicities of principal curvatures for isoparametric spherical hypersurface with three curvatures are the same: 1, 2, 4 or 8 in dimensions 3, 6, 12, 24 respectively [Car 3] . Isoparametric spherical hypersurfaces with three principal curvatures are all homogeneous. They are identified as $SO(3)/Z_2 \times Z_2$ , $SU(3)/T^2$ , $Sp(3)/Sp(1)^3$ , $F_4/Spin(8)$ of dimensions 3, 6, 12, 24 respectively (see [H-L] , [T-Ta] ). The minimal hypersurface of the type $SO(3)/Z_2 \times Z_2$ in $S^4$ we call the *Cartan hypersurface* .

Cartan showed that any isoparametric family with $\nu$ distinct principal curvatures of the same multiplicity can be defined by the equation

$$F = \cos \nu\, t \qquad \text{(restricted to } S^{n+1}\text{)}$$

where F is a harmonic homogeneous polynomial of degree $\nu$ on $E^{n+2}$ satisfying

$$\|\mathrm{grad}\, F\|^2 = \nu^2\, r^{2\nu-2}\,,$$

where r is the distance from the origin and gradient is in $E^{n+2}$ . For example, for hypersurfaces with 3 principal curvatures polynomial F is given by (cf. [Car 3])

$$\text{(1.4.2)} \qquad F = u^3 - 3\,uv^2 + \frac{3}{2}\,u\,(X\bar{X} + Y\bar{Y} - 2\,Z\bar{Z}) + \frac{3\sqrt{3}}{2}\,v\,(X\bar{X} - Y\bar{Y}) + \frac{3\sqrt{3}}{2}\,(XYZ + \bar{Z}\bar{Y}\bar{X})\ .$$

In this formula u and v are real parameters, while X, Y, Z are coordinates in the algebra F = R, C, Q, O respectively for the cases corresponding to the multiplicities m = 1, 2, 4, 8

The sum $XYZ + \overline{Z}\overline{Y}\overline{X}$ is twice the real part of the product. In the case $m = 8$, multiplication is not associative but the real part of $XYZ$ is the same whether one interprets the product as $(XY)Z$ or $X(YZ)$.

In [Car 5], Cartan gave examples of two families of isoparametric hypersurfaces in $S^5$ and $S^9$ with four distinct principal curvatures of the same multiplicity ( respectively 1 and 2 ). The one in $S^5$ has particularly nice representation by the map

$$S^1 \times S_{3,2} \to S^5 \subset E^6 \qquad \text{given by}$$

(1.4.3) $$(\theta , (x,y)) \to e^{i\theta}(\cos t \; x + i \sin t \; y)$$

Here, $S_{3,2}$ denotes Stiefel manifold of orthonormal pairs of vectors in $E^3$ and $S^1$ is the unit circle . More precisely each isoparametric hypersurface $M_t^4 \subset S^5$ with four principal curvatures is the immage of the map (1.4.3) which doubly covers $M_t^4$. The minimal one is obtained when $t = \pi/8$ [N 2] . Nomizu used this map to construct infinite family of isoparametric hypersurfaces $M_t^{2n}$ with four principal curvatures of multiplicities 1, n-1 , 1 and n-1 . Takagi has shown ([T 3]) that any isoparametric hypersurface with four curvatures such that the multiplicity of one curvature is 1 is congruent to the example $M_t^{2n}$ of Nomizu for some n and t .

All examples of isoparametric spherical hypersurfaces known by Cartan are homogeneous. In fact each is the orbit of a point under an appropriate closed subgroup of $SO(n+2)$. Of course such orbit hypersurfaces have constant principal curvatures [T - Ta] . In particular, isoparametric hypersurfaces with four principal curvatures of the same multiplicity 1 or 2 mentioned above are $SO(2) \times SO(3)/Z_2$ , respectively $Sp(2)/T^2$. The minimal hypersurfaces in these two families have principal curvatures equal to $\sqrt{2} + 1$ , $\sqrt{2} - 1$ , $1 - \sqrt{2}$ , $-\sqrt{2} - 1$ (roots of $x^4 - 6x^2 + 1 = 0$ ) and they can be found in a similar way as was done in the case of hypersurface with three curvatures, using identity (1.4.1).

Cartan did not know what the possibilities were for the number $\nu$ of distinct principal curvatures, nor whether isoparametric hypersurface is necessarily homogeneous. Work on isoparametric spherical hypersurfaces was revived by Nomizu [N 1-2] and then several important results followed. Using classification of [H-L] Takagi and Takahashi determined all homogeneous hypersurfaces in sphere (including some with 6 curvatures) and found their principal curvatures [T-Ta]. Ozeki and Takeuchi ([O-T]) produced two infinite series of isoparametric hypersurfaces which are not homogeneous. Major results in the theory were obtained by H. F. Münzner . Through a geometric study of the focal submanifolds of an isoparametric family and their second fundamental form he reproved Cartan's identity (1.4.1) showing it to be equivalent to the minimality of focal submanifolds (Left hand side of (1.4.1) is trace of the shape operator of a focal submanifold.). He also proved the following theorem [M] .

**Theorem 1.4.2.** If $k_1 > k_2 > \dots > k_\nu$ are distinct principal curvatures of an isoparametric spherical hypersurface with respective multiplicities $m_1, m_2, \dots, m_\nu$ then

$$k_i = \cot \theta_i \quad , \quad 0 < \theta_1 < \dots < \theta_\nu < \pi$$

$$\text{where} \quad \theta_i = \theta_1 + \frac{i-1}{\nu}\pi \, , \quad 1 \le i \le \nu \;, \quad \text{with} \quad \theta_1 < \frac{\pi}{\nu} \;,$$

and the multiplicities satisfy $m_i = m_{i+2}$ (subscripts mod $\nu$ ) .

As a consequence, there are at most two different multiplicities $m_1, m_2$ for principal curvatures and if $\nu$ is odd then all multiplicities must be equal. (Münzner was also able to show that if $\nu = 6$ then $m_1 = m_2$ ). Using delicate cohomological arguments he also proved the following splendid result .

**Theorem 1.4.3.** The number $\nu$ of distinct principal curvatures of an isoparametric hypersurface satisfies $\nu = 1, 2, 3, 4$ or $6$ .

Generalizing Cartan's result, Münzner showed that the hypersurfaces of any isoparametric family with $\nu$ distinct principal curvatures in $S^{n+1}$ can be represented as open subsets of level hypersurfaces in $S^{n+1}$ of a homogeneous polynomial F of degree $\nu$ on $E^{n+2}$ which satisfies the differential equations (on $E^{n+2}$ )

$$\|\text{grad } F\|^2 = \nu^2 r^{2\nu-2}$$

(1.4.4)

$$\Delta F = \frac{\nu^2(m_2 - m_1)}{2} r^{\nu-2}$$

As a consequence, every isoparametric hypersurface is algebraic, and a piece of isoparametric hypersurface can always be extended to a complete one . Let us state also the following result of Abresch [A] who used refined techniques of Münzner to prove

**Theorem 1.4.4.** i) Given an isoparametric hypersurface in $S^{n+1}$ with $\nu = 4$ principal curvatures , let $m_1 \leq m_2$ be (possibly same) multiplicities of curvatures . Then the pair ( $m_1$ , $m_2$) satisfies one of the three conditions below

(a) $m_1 + m_2 + 1$ is divisible by $2^s := \min \{2^\sigma \mid 2^\sigma > m_1 , \sigma \in N \}$.

(b) $m_1$ is power of 2, and $2m_1$ divides $m_2 + 1$ .

(c) $m_1$ is power of 2, and $3m_1 = 2( m_2 + 1)$ .

Each condition corresponds to a topologically different kind of examples .

ii) Given an isoparametric hypersurface in $S^{n+1}$ with $\nu = 6$ then $m_1 = m_2 \in \{1, 2\}$ .

Regarding isoparametric hypersurfaces with four curvatures of the same multiplicity, Cartan asserts , without proof, that they have to be homogeneous [Car 5]. That was proved by Ozeki and Takeuchi if $m_1 = m_2 = 2$ [O-T]. However, in the light of the above theorem of Abresch we can easily prove that statement and moreover completely classify isoparametric hypersurface with four curvatures of the same multiplicity. Namely, if $m_1 = m_2$ then case (b) of the theorem gives $m_1 = m_2 = 1$ and then the results of Takagi [T 3] and Takagi and Takahashi [T-Ta] classify such hypersurface as $SO(2) \times SO(3)/Z_2$ . If case (c) occurs, then $m_1 = m_2 = 2$ hence by the result of Ozeki and Takeuchi [O-T] the hypersurface is homogeneous and therefore according to the list in [T-Ta] must be $Sp(2)/T^2$ . Therefore these hypersurfaces are exactly those two found by Cartan in [Car 5] .

Next, we give the list of all isoparametric hypersurfaces in sphere with three or four distinct principal curvatures of the same multiplicity. As remarked by Hsiang and Lawson [H-S] , homogeneous isoparametric hypersurfaces in sphere arise from isotropy representations of the corresponding symmetric spaces of rank 2. For our hypersurfaces, their isometry groups G, actions $\psi$ , principal isotropy groups H, common multiplicity of principal curvatures m and dimension n are given as follows (first four examples in the table have three curvatures, remaining two have four ).

Table 1. Isoparametric hypersurfaces in sphere with three or four principal curvatures of the same multiplicity

| G | $\psi$ | H | m | n |
|---|---|---|---|---|
| SO(3) | $S^2\rho_3 - \theta$ | $Z_2 \times Z_2$ | 1 | 3 |
| SU(3) | $Ad_{SU(3)}$ | $T^2$ | 2 | 6 |
| Sp(3) | $\wedge^2 v_3 - \theta$ | $Sp(1)^3$ | 4 | 12 |
| $F_4$ | $\phi_1$ | Spin(8) | 8 | 24 |
| SO(2) × SO(3) | $\rho_2 \otimes \rho_3$ | $Z_2$ | 1 | 4 |
| Sp(2) | Ad | $T^2$ | 2 | 8 |

Let us mention at the end that the theory of isoparametric hypersurfaces continues to be area of active research. Subsequent investigation exploited equations (1.4.4 ) of

Münzner - Cartan and new results were obtained using algebraic tools such as triple systems , Jordan algebras, Clifford systems (cf. [D-N], [F-K-M], [W 2] ) . For example, Ferus, Karcher and Münzner gave a construction of isoparametric hypersurfaces with $\nu = 4$ using representations of Clifford algebras which included all known examples, except two. Their method also exibited infinitely many series of infinite isoparametric families with four constant principal curvatures. However, the main problem of classification of isoparametric hypersurfaces in sphere still remains open. For isoparametric hypersurfaces in pseudo - Riemannian space forms see [Ha], [N 3] and [Ma] , and for real hypersurfaces with constant principal curvature in complex projective or complex hyperbolic spaces see [W 1], [T 2], [B] . One possible generalization to a submanifolds of higher codimension was dealt with in [Te] . See also [Pa-T] .

## 5. Finite type maps and submanifolds

For spectral geometry standard references are [B-G-M] , [Ch 2] and for finite type submanifolds [C 4] .

Let $(M^n, g)$ be a Riemannian manifold. *Laplacian* $\Delta$ acting on smooth functions is defined as

$$\Delta f = \sum_{i=1}^{n} [(\nabla_{e_i} e_i)f - e_i(e_i f)] \quad , \quad f \in C^\infty(M) \tag{1.5.1}$$

where $\{e_i\}$ denotes local orthonormal basis of tangent vectors ( $\Delta$ does not depend on the choice of such basis ). In local coordinates, $\Delta$ has the following expression

$$\Delta f = -\frac{1}{\sqrt{g}} \sum_{j,k} \partial_j (g^{jk} \sqrt{g}\, \partial_k f) \ , \text{ where } \ g = \det (g_{ij}) \ . \tag{1.5.2}$$

The following property of $\Delta$ acting on the product of two functions is well known

(1.5.3) $$\Delta(u\,v) = (\Delta u)\,v + u\,(\Delta v) - 2 < \nabla u, \nabla v > , \quad u, v \in C^{\infty}(M) .$$

The Laplacian is naturally extended to act on $E^m$ - valued maps (componentwise), so the rule above extends to inner product of vector functions U, V on M as follows

(1.5.4) $$\Delta < U, V > = < \Delta U, V > + < U, \Delta V > - 2 \sum_i < \tilde{\nabla}_{e_i} U, \tilde{\nabla}_{e_i} V >$$

(1.5.5) $$\Delta(f\,U) = (\Delta f)\,U + f\,(\Delta U) - 2 \sum_i (e_i f) \tilde{\nabla}_{e_i} U , \quad f \in C^{\infty}(M) .$$

Also, if $x : M^n \rightarrow E^m$ is an isometric immersion whose mean curvature vector is H, then the following formula holds (see e.g. [C 4], p.135)

(1.5.6) $$\Delta x = -n H .$$

An *eigenvalue* of $\Delta$ is any real number $\lambda$ for which there exists a smooth nonzero funcion f (called an *eigenfunction* ), so that $\Delta f = \lambda f$ . The set of all eigenfunctions of $\lambda$ , $V_\lambda$ , forms a vector space and its dimension (need not be finite) is called multiplicity of $\lambda$ . Clearly, for two different eigenvalues $\lambda_p , \lambda_q$ we have $V_p \cap V_q = \{0\}$ . The set of all eigenvalues taken with their multiplicities is called *spectrum* of M and denoted by Spec(M).

If M is compact, we can define natural $L^2$ - inner product ( , ) by $(f,g) = \int_M fg dV$ . In this case Laplacian is self adjoint strongly elliptic operator, all eigenvalues are nonnegative and the spectrum is discrete , $Spec(M) = \{ 0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots \uparrow \infty \}$. Multiplicity of each $\lambda$ (dim $V_\lambda$ ) is finite, dim $V_0 = 1$, and $\sum_{t=0}^{\infty} V_t$ is dense in $C^{\infty}(M)$ . Thus, we can write

(1.5.7) $$C^{\infty}(M) = \sum_{t=0}^{\infty} V_t \quad (\text{ in } L^2 \text{ - sense }) .$$

This direct sum decomposition is orthogonal with respect to ( , ) . According to (1.5.7) , any smooth function $f \in C^{\infty}(M)$ has the following spectral decomposition

$$(1.5.8) \qquad f = f_0 + \sum_{t=1}^{\infty} f_t \qquad (\text{ in } L^2 \text{ - sense}) ,$$

where $f_0$ is constant ( $\Delta f_0 = 0$ ) and $f_t$ is the projection of f onto $V_t$ , i.e. $\Delta f_t = \lambda_t f_t$ .

A map $f \in C^{\infty}(M)$ is said to be *finite type map* if its spectral decomposition (1.5.8) has finitely many nonzero terms. More precisely, f is of *k - type* if there are exactly k nonzero terms $f_{t_1}, \ldots, f_{t_k}$ ( $t_i \geq 1$, i = 1, ..., k ) in the decomposition (1.5.8). The set $[t_1, \ldots, t_k]$ ( also $[\lambda_{t_1}, \ldots, \lambda_{t_k}]$ ) is called the *order* of a map f . If f is not of finite type, that is, decomposition (1.5.8) has infinitely many nonzero terms, f is of *infinite type* .

Note that $\Delta$ can be naturally extended to $E^m$ - valued maps (by taking Laplacian componentwise), and accordingly, we extend the notion of finite type map as follows. For a smooth map $x : M^n \to E^m$ , $x = (f_1, \ldots, f_m)$ , we find spectral decomposition (1.5.8) of each $f_i$ and combine them to obtain spectral decomposition of a vector function x as

$$(1.5.9) \qquad x = x_0 + \sum_{t=1}^{\infty} x_t \qquad (\text{ in } L^2 \text{ - sense}) ,$$

where, $x_t = ((f_1)_t, \ldots, (f_m)_t)$ , i.e. $\Delta x_t = \lambda_t x_t$ . (Some of the $(f_i)_t$'s can be 0). Again , vector function x is called k - type if there are k nonzero vectors $x_t$ ($t \geq 1$) in decomposition (1.5.9). In particular, a submanifold of $E^m$ is of finite type (k - type) if the corresponding immersion is so. $x_0$ is always a constant vector, and if x is an isometric immersion of a compact manifold M, then $x_0$ is the center of mass of M in $E^m$ , i.e. $x_0 = \frac{1}{\text{vol}(M)} \int_M x$ .

If $x : M^n \rightarrow S_c^{m-1}(r) \subset E^m$ is an isometric immersion of a compact manifold into a sphere, then $M^n$ is called *mass - symmetric* in $S_c^{m-1}(r)$ if $x_0 = c$ , i.e. center of mass of M coincides with the center of sphere. For an 1 - type immersion $x : M^n \rightarrow E^m$, we have $x = x_0 + x_p$ with $x_0 = \text{const}$, $\Delta x_p = \lambda_p x_p$ . The well known theorem of Takahashi [Ta 1] can be stated in terms of 1 - type maps as follows

**Theorem 1.5.1.** Let M be a compact submanifold of $E^m$. Then M is of 1 - type if and only if M is a minimal submanifold of a hypersphere of $E^m$ .

If $x_1 : M^n \rightarrow E^{m1}$ and $x_2 : M^n \rightarrow E^{m2}$ are two isometric immersions, then the diagonal immersion $x = D(x_1, x_2) = \frac{1}{\sqrt{2}}(x_1, x_2)$ is of finite type if and only if both $x_1, x_2$ are of finite type. Let M be a compact, irreducible symmetric space and $p_1 < p_2 < ... < p_k$ any finite set of natural numbers. Then the diagonal immersion $D(x_{p_1}, ..., x_{p_k})$ of the standard immersions $x_{p_1}, ..., x_{p_k}$ is of k - type with order $[p_1, p_2, ..., p_k]$ . This shows that there are immersions of arbitrary high type. Also, if M is a compact homogeneous space which is equivariantly, isometrically immersed in $E^m$, then M is of k - type with $k \leq m$ ([C 4], p. 258 ; see also [Ta 2] and [D] ) . A closed curve C in $E^m$ is of finite type if and only if Fourier series expansion of each coordinate function of C has only finitely many nonzero terms ([C 4], p.283 ). We give the following criterion for finite type immersions [C 4] .

**Theorem 1.5.2.** Let $x : M \rightarrow E^m$ be an isometric immersion of a compact Riemannian manifold M into $E^m$. Then M is of finite type if and only if there is a non - trivial polinomial P(t) such that

$$P(\Delta)(x - x_0) = 0 . \qquad (1.5.10)$$

Moreover, M is of k - type if and only if polinomial P is of degree k having exactly k distinct (positive) roots and for any other polinomial Q that satisfies $Q(\Delta)(x - x_0) = 0$ , P is a factor of Q. The statement of the theorem remains the same if $x - x_0$ is replaced by the mean curvature vector H.

Let us note that the notion of finite type map and immersion make perfect sense also for a noncompact manifold, e.g. an immersion $x : M \to E^m$ is of k - type if we can write

$$(1.5.11) \qquad x = x_0 + x_{t_1} + \dots + x_{t_k} ,$$

where $x_0$ is a constant vector and $x_{t_1}, \dots , x_{t_k}$ are eigenvectors of the Laplacian corresponding to k different eigenvalues $\lambda_{t_1}, \dots , \lambda_{t_k}$. If M is noncompact, $\lambda$'s need not be positive, nor their multiplicities finite. Eigenspace $V_0$ (set of harmonic functions) is generaly of dimension > 1 (there may be nonconstant harmonic functions). If one of the eigenvalues $\lambda_{t_1}, \dots , \lambda_{t_k}$ corresponding to the decomposition (1.5.11) is 0, then the submanifold is said to be of *null k - type*. In this case $x_0$ is not uniquely determined ( for compact manifold, $x_0$ is always center of mass). The cylinder $x(\theta, u) = (\cos\theta, \sin\theta, u)$ is an example of noncompact null 2 - type submanifold.

Notions of order of a submanifold and submanifolds of finite type were first introduced by B.Y. Chen in [C 2] and the theory of finite type submanifolds has become an area of active reseach (see [C 4]). In particular, there is a problem of classification of low type submanifolds which lie in a hypersphere. By Theorem 1.5.1 , 1 - type submanifolds are characterized as being minimal in sphere and one can expect that 2 - type and higher type submanifolds are more general. Indeed classification of even 2 - type spherical submanifolds seems to be virtualy impossible.( Note, however, that the only compact 2 - type surface in $S^3$ is flat torus $S^1(a) \times S^1(b)$ , $a \neq b$ [B-C-G] ). On the other hand, studying finite type immersions of a spherical manifold into SM(m) via the second standard immersion of the sphere proved to be more manageable ( see [R], [B-C] ) . In Chapter 3 we study spherical hypersurfaces which are of 2 - type and those which are of 3 - type and mass - symmetric via the second standard immersion of the sphere.

# CHAPTER 2

## QUADRIC REPRESENTATION OF A SUBMANIFOLD

For an isometric immersion $x : M^n \to E^m$ of a Riemannian manifold into a Euclidean space, one defines the map $\tilde{x} : M^n \to SM(m)$ from M into the set of real symmetric $m \times m$ matrices by $\tilde{x} = x \cdot x^t$, where x is regarded as a column vector in $E^m$. Thus, if $x = (x_1, \dots, x_m)^t$ we have

$$\tilde{x} = \begin{pmatrix} x_1^2 & x_1x_2 & \cdots & x_1x_m \\ x_2x_1 & x_2^2 & \cdots & x_2x_m \\ \cdots & \cdots & \cdots & \cdots \\ x_mx_1 & x_mx_2 & \cdots & x_m^2 \end{pmatrix}$$

We call $\tilde{x}$ the *quadric representation* of a submanifold M. In this chapter we establish some general results about the quadric representation. First we prove a theorem about quadric representation being an isometric immersion.

**Theorem 2.1.** Let $x : M^n \to E^m$ be an isometric immersion of a Riemannian manifold into a Euclidean space. Then $\tilde{x} = x \cdot x^t$ is an isometric immersion if and only if $x(M^n) \subset S^{m-1}$, i.e. M is spherical. (In the case $n = 1$, a curve is assumed to be complete.)

**Proof.** First we prove the statement for a complete curve $x : C \to E^m$. Let $x(s) = (x_1(s), x_2(s), \dots, x_m(s))$ be the parametrization of the curve by its arclength. Then

$$d\tilde{x} \otimes d\tilde{x} = g(d\tilde{x}, d\tilde{x}) = \frac{1}{2}\mathrm{tr}(d\tilde{x}\, d\tilde{x}) = \frac{1}{2}\sum_{i,j}(x_i' x_j + x_i x_j')^2 ds^2 .$$

Since $\tilde{x}$ is assumed to be an isometry, tangent vector $d\tilde{x}/ds$ must have length 1, therefore we get

$$\begin{aligned}
1 &= \frac{1}{2}\sum_{i,j}(x_i' x_j + x_i x_j')^2 \\
&= 2\Big(\sum_i x_i x_i'\Big)^2 + \frac{1}{2}\sum_{i,j}(x_i' x_j - x_i x_j')^2 \\
&= 2\Big(\sum_i x_i x_i'\Big)^2 + \sum_{i<j}(x_i' x_j - x_i x_j')^2 \\
&= \frac{1}{2}[(\|x\|^2)']^2 + \|x \wedge x'\|^2 \\
&= \frac{1}{2}[(\|x\|^2)']^2 + \|x\|^2 - \frac{1}{4}[(\|x\|^2)']^2 \\
&= \|x\|^2 + \frac{1}{4}[(\|x\|^2)']^2 .
\end{aligned}$$

Here, $\wedge$ represents the usual wedge operation in the Grassmann algebra over $E^m$. Hence for $v_1, v_2 \in E^m$ we have

$$\|v_1 \wedge v_2\|^2 = \det(\langle v_i, v_j\rangle) = \|v_1\|^2\|v_2\|^2 - \langle v_1, v_2\rangle^2 .$$

Thus letting $u(s) = \|x\|^2$ we get the differential equation in u that separates the variables, $u + (1/4)(u')^2 = 1$ . One obvious solution is $u = 1$ , and there is no solution for $u > 1$ . If $u \neq 1$, solving the equation gives $u(s) = 1 - (c + s)^2$ , where c is an arbitrary constant. This solution, however, represents decreasing function of s and therefore, $u = \|x\|^2 < 0$ for sufficiently large s (curve is assumed to be complete) which is a contradiction. Therefore, $u = 1$ , i.e. curve C belongs to the unit sphere centered at the origin.

Now let $\tilde{x}$ be an isometric immersion for a manifold $M^n$ $(n > 1)$. Since $\tilde{x}$ preserves the first fundamental form of M, it also preserves the first fundamental form of any curve of M (isometry property is hereditary to a submanifold). Let $p \in M$ be an arbitrary point, and consider a small smooth loop based at p. Such loop can be chosen as the image of a circle passing through p in the normal neighborhood in the tangent space $T_pM$ via the exponential map. The restriction of $\tilde{x}$ to this loop is an isometry, and from the above we conclude that the loop belongs to the unit sphere centered at the origin and the same is true for point p. Since p is an arbitrary point of M , M is a spherical submanifold. The converse of the statement is well known. ♦

Actually, we have a similar result under weaker assumptions .

**Theorem 2.2.** Let $x : M^n \to E^m$ $(n > 1)$ be an isometric immersion. Then $\tilde{x}$ is a conformal map if and only if $M^n \subset S^{m-1}(r)$, in which case $\tilde{x}$ is homothety.

**Proof.** Let $\tilde{g}$ and $< , >$ be metrics on SM(m) and M respectively, and $\tilde{\nabla}$ and $\bar{\nabla}$ be Euclidean connections on SM(m) and $E^m$. If we set $\tilde{x} = (f_1, \ldots, f_N)$ , where $N = \dim SM(m)$ , then

$$d\tilde{x}\,(X) = (df_1, \ldots, df_N)X = (df_1(X), \ldots, df_N(X)) = (Xf_1, \ldots, Xf_N) = \tilde{\nabla}_X\tilde{x} .$$

Since $\tilde{\nabla}$ acts as a derivation on the set of smooth functions on M then the product rule extends also to the map $\tilde{x} = x{\cdot}x^t$ , namely, we have

$$\tilde{\nabla}_X\tilde{x} = \tilde{\nabla}_X(x{\cdot}x^t) = (\bar{\nabla}_X x))\, x^t + x\,(\bar{\nabla}_X x)^t = X\,x^t + x\,X^t .$$

If $\tilde{x}$ is a conformal map then $\tilde{x}^*g = \phi < , >$ for some positive function $\phi$ . In particular, $\tilde{x}$ maps a pair of perpendicular vectors into a pair of perpendicular vectors. Therefore, if $X \perp Y$ is a pair of perpendicular vectors of M , we have

$$\begin{aligned}
0 &= g(\tilde{x}_*X, \tilde{x}_*Y) \\
&= g(d\tilde{x}(X), d\tilde{x}(Y)) \\
&= g(\tilde{\nabla}_X\tilde{x}, \tilde{\nabla}_Y\tilde{x}) \\
&= g(X x^t + x X^t, Y x^t + x Y^t) \qquad (2.1) \\
&= \frac{1}{2}\operatorname{tr}(X x^t + x X^t)(Y x^t + x Y^t) \\
&= <x, X><x, Y> + <x, x><X, Y> \\
&= <x, X><x, Y>.
\end{aligned}$$

If X, Y is a pair of perpendicular unit vectors then X + Y and X - Y are also perpendicular, and from the equation above we obtain

$$0 = <x, X+Y> <x, X-Y> = <x, X>^2 - <x, Y>^2.$$

Then (2.1) implies $<x, X> = 0$, for every tangent vector X of M, and therefore $X<x, x> = 2<x, X> = 0$, that is, $<x, x> = r^2 =$ const, which shows that $x(M) \subset S^{m-1}(r)$. Converse is easy, because then $g(\tilde{x}_*X, \tilde{x}_*Y) = <x, x> <X, Y>$. ♦

Now we want to examine some relationships between the map $\tilde{x}$ and the condition of being of finite type. First, let us fix the notation. Let $M^n$ be a submanifold of the Euclidean space $E^m$. Suppose that $e_1, e_2, \dots, e_n, e_{n+1}, \dots, e_m$ are local orthonormal vector fields along M such that the first n vectors are tangent to M and the remaining m-n vectors normal to M. Let g and $\bar{\nabla}$ be the Euclidean metric and connection of $E^m$, and denote by $\nabla$, h, D, $A_\xi$ respectively, the induced connection, second fundamental form of M, connection in the normal bundle $T^\perp M$ and the Weingarten endomorphism relative to the normal direction $\xi$. The connection forms $\omega_i^j$ and the mean curvature vector H of M in $E^m$ are defined by $\nabla_{e_k} e_i = \sum_j \omega_i^j(e_k)e_j$ , $H = (1/n)\sum_r (\mathrm{tr}A_r)e_r$ . Here, indices i, j, k range from 1 to n and indices r,s range from n+1 to m . As usual, $\Delta$ denotes Laplacian on M . The metric on SM(m) is given by $\tilde{g}(P, Q) = \frac{1}{2}\,\mathrm{tr}\,(PQ)$ .

Since an 1-type map is next simplest to being harmonic , we start out by proving a theorem about $\tilde{x}$ being an 1-type map .

**Theorem 2.3.** For an isometric immersion $x : M^n \to E^m$ , $\tilde{x}$ is of 1-type if and only if $M^n$ is totally geodesic submanifold of the hypersphere $S^{m-1}(r) \subset E^m$ . In particular, if the immersion x is full and M complete, then $M = S^{m-1}(r)$ is the standard sphere.

**Proof.** Suppose that $\tilde{x}$ is 1-type map . Then we can write $\tilde{x} = \tilde{x}_0 + \tilde{x}_p$ , where $\tilde{x}_0$ is a constant vector and $\Delta\tilde{x}_p = \lambda_p \tilde{x}_p$ , i.e. $\tilde{x}_p$ is an eigenvector of the Laplacian. Thus ,

$$\Delta\tilde{x} = \lambda_p\tilde{x}_p = \lambda_p(\tilde{x} - \tilde{x}_0) . \tag{2.2}$$

On the other hand,

$$\Delta\tilde{x} = \Delta(x\cdot x^t) = (\Delta x)\, x^t + x\,(\Delta x)^t - 2\sum_i(\bar{\nabla}_{e_i}x)(\bar{\nabla}_{e_i}x)^t$$

$$= -\,n\,(H\,x^t + x\,H^t) - 2\sum_i e_i e_i^t \ . \tag{2.3}$$

Therefore, from (2.2) and (2.3) we have

$$-\,n\,(H\,x^t + x\,H^t) - 2\sum_i e_i e_i^t = \lambda_p(\tilde{x} - \tilde{x}_0) .$$

Differentiating this relation along an arbitrary vector field X of M, we get

$$
\begin{aligned}
& n[(A_H X)\, x^t + x\,(A_H X)^t] - [(D_X H)\, x^t + x\,(D_X H)^t] \\
& - n\,(H\, X^t + X\, H^t) - 2\sum_{i,j} \omega_i^j(X)(e_i e_j^t + e_j e_i^t) \\
& - 2\sum_i [h(X,e_i)e_i^t + e_i h(X,e_i)^t] \\
& = \lambda_p(X\, x^t + x\, X^t) \quad .
\end{aligned}
$$

Note that the first sum is equal to 0 , since $\omega_i^j(X)$ is antisymmetric in i and j whereas $e_i e_j^t + e_j e_i^t$ is symmetric in i, j. Also

$$
\begin{aligned}
\sum_i [h(X,e_i)e_i^t + e_i h(X,e_i)^t] &= \sum_{i,r} g(h(X,e_i),e_r)(e_i e_r^t + e_r e_i^t) \\
&= \sum_{i,r} g(A_r X,\, e_i)(e_i e_r^t + e_r e_i^t) \\
&= \sum_r [(A_r X)e_r^t + e_r(A_r X)^t] \quad ,
\end{aligned}
$$

and therefore, for every $X \in \Gamma(TM)$ we have

$$
\begin{aligned}
& n[(A_H X)\, x^t + x\,(A_H X)^t] - [(D_X H)x^t + x\,(D_X H)^t] \\
(2.4) \qquad & - n\,(H\, X^t + X\, H^t) - 2\sum_r [(A_r X)e_r^t + e_r(A_r X)^t] \\
& = \lambda_p(X\, x^t + x\, X^t) \quad .
\end{aligned}
$$

We now find $e_i e_r^t + e_r e_i^t$ component of (2.4) , i.e. apply $\tilde{g}(-\ ,\ e_i e_r^t + e_r e_i)$ to it :

$$n\, g(A_H X, e_i)\, g(x, e_r) - n\, g(D_X H, e_r)\, g(x, e_i)$$

$$- n\, g(H, e_r)\, g(X, e_i) - 2g(A_r X, e_i)$$

$$= \lambda_p\, g(X, e_i)\, g(x, e_r) .$$

Letting $X = e_i$ and summing on i we get

$$n(\mathrm{tr}A_H)\, g(x, e_r) - n\, g(D_{x_T} H, e_r) - n^2 g(H, e_r) - 2\, \mathrm{tr}A_r = n\, \lambda_p\, g(x, e_r) .$$

If we multiply this relation by $e_r$ and sum on r, we obtain

(2.5) $$(\mathrm{tr}A_H - \lambda_p)\, x_N - D_{x_T} H = (n+2)\, H .$$

Note that in general x is not perpendicular to M so we have normal and tangential component of x :

$$x_N = \sum_r g(x,e_r)e_r \quad , \quad x_T = \sum_i g(x,e_i)e_i \quad .$$

Finding $e_r e_r^t$ component of (2.4) and summing on r we get

(2.6) $$< D_X H , x_N > = 0 \qquad \text{i.e.} \quad D_X H \perp x_N \qquad \text{for every } X \in TM .$$

Finding $e_r e_s^t + e_s e_r^t$ component of (2.4) and summing on s (after multiplying by $e_s$) we obtain

$$< D_X H , e_r > x_N + < x , e_r > D_X H = 0 \quad ,$$

and by (2.6) we have

(2.7) $$< D_X H\,,\ e_r > x_N \;=\; < x\,,\ e_r > D_X H \;=\; 0\,, \quad \text{for every } r \text{ and } X \in TM.$$

Thus, at any given point of M we have

(2.8) $$x_N = 0 \quad \text{or} \quad x_N \neq 0 \;\text{ and }\; DH = 0$$

Next, by comparing $e_k e_k^t$ components of two sides of the equation (2.4), multiplying by $e_k$ and summing on k we get

(2.9) $$< n\,A_H X \;-\; \lambda_p X\,,\ x_T > \;=\; 0 \quad , \quad \text{for every } X \in TM\ ,$$

and by comparing $e_i e_k^t + e_k e_i^t$ components, summing on k and taking (2.9) into account we have

(2.10) $$[\, n < A_H X, e_i > - \lambda_p < X, e_i >]\, x_T \;=\; (\, n\,A_H X \;-\; \lambda_p X\,) < x, e_i > \;=\; 0 \quad ,$$

for every $i = 1, 2, \ldots, n$ and every $X \in TM$. Therefore, at any given point of M we have

(2.11) $$x_T = 0 \quad \text{or} \quad x_T \neq 0 \;\text{ and }\; n\,A_H = \lambda_p I\ .$$

Let $U = \{\, p \in M \mid x_T \neq 0 \;\text{ at } p \,\}$. Then U is an open subset of M, and on U we have by (2.11), $\operatorname{tr} A_H = \lambda_p$. Then (2.5) implies $D_{x_T} H = -(n+2)H$ on U. Now let V be an open subset of U defined by $V = \{\, p \in U \mid x_N \neq 0 \;\text{ at } p \,\}$. By (2.8) we have $DH = 0$ on V, and from the above we conclude $H = 0$ on V, i.e. V is the piece of M immersed minimally in $E^m$. Now we compute $\operatorname{tr}(\Delta \tilde{x})$ on V, noting that Laplacian commutes with trace since it is a linear operator.

$$\operatorname{tr}(\Delta\tilde{x}) = \Delta(\operatorname{tr}\tilde{x}) = \Delta < x, x >$$

$$= 2 < \Delta x, x > - 2\sum_i < e_i, e_i >$$

$$= -2n < H, x > -2n$$

$$= -2n$$

On the other hand, (2.2) yields

$$\operatorname{tr}(\Delta\tilde{x}) = \lambda_p(\operatorname{tr}\tilde{x} - \operatorname{tr}\tilde{x}_0) = \lambda_p(< x, x > - \operatorname{tr}\tilde{x}_0) .$$

Therefore, $\lambda_p(< x, x > - \operatorname{tr}\tilde{x}_0) = -2n$ , and since obviously $\lambda_p \neq 0$ we have

$$< x, x > = \operatorname{tr}\tilde{x}_0 - \frac{2n}{\lambda_p} = \text{const} \quad . \tag{2.12}$$

Consequently, $x(V) \subset S^{m-1}(r)$ and hence $0 = H = H' - \frac{x}{r^2}$ , where H' is the mean curvature vector of V in $S^{m-1}(r)$, which is a contradiction because $H' \perp x$ and $x \neq 0$. Therefore, we must have $V = \varnothing$ , and hence on U $x = x_T$ is tangential . Now on U as before we have ( note $H \perp x$ )

$$\operatorname{tr}(\Delta\tilde{x}) = -2n < H, x > -2n = -2n = \lambda_p(< x, x > - \operatorname{tr}\tilde{x}_0),$$

and therefore (2.12) holds again on U . So, $x(U) \subset S^{m-1}(r)$ but then $x_T = 0$ since x is normal to U for spherical submanifold and this is a contradiction. We conclude $U = \varnothing$ and $x = x_N$ is normal to submanifold M. Consequently, x immerses M into a hypersphere of $E^m$ centered at the origin , $x : M^n \to S^{m-1}(r) \subset E^m$ . In that case

$H = H' - \frac{x}{r^2}$ and $DH = DH' = D'H'$ . From (2.8) we get $DH = D'H' = 0$ and then from (2.5) it follows $(trA_H - \lambda_p)\, x = (n+2)(H' - \frac{x}{r^2})$ . Since $H' \perp x$ we see that $H' = 0$ i.e. M is minimal in the hypersphere .With these identities in effect, equation (2.4) becomes (we take $e_m = x/r$ )

$$\frac{2(n+1)}{r^2}(X\,x^t + x\,X^t) - 2\sum_{r=1}^{m-1}[(A_rX)e_r^t + e_r(A_rX)^t] = \lambda_p(X\,x^t + x\,X^t) \ ,$$

for every $X \in TM$ . Therefore, $\lambda_p = 2(n+1)/r^2$ and $A_s = 0$ for every $s = n+1, \ldots, m-1$. We conclude that $M^n$ is totally geodesic in $S^{m-1}(r)$, i.e. it is (a piece of ) standard $S^n(r)$ in $S^{m-1}(r)$ .

Conversely, if $M^n$ is totally geodesic $S^n(r) \subset S^{m-1}(r)$ than it is well known that $M^n$ is minimally immersed via $\tilde{x}$ (after scaling the metric in SM(m) with the factor $1/r^2$ ) as a Veronese submanifold in a hypersphere of SM(m) (see Ch.1, Sect.3). Then the well known theorem of Takahashi (Theorem 1.5.1) asserts that $\tilde{x}$ is of 1-type. As a matter of fact we have

$$\tilde{x} = \tilde{x}_0 + \tilde{x}_p = \frac{1}{n+1}(xx^t + r^2\sum_i e_ie_i^t) + \frac{1}{n+1}(n\,xx^t - r^2\sum_i e_ie_i^t) \ ,$$

where $(xx^t + r^2\sum_i e_ie_i^t)$ is a constant vector , actually equal to $r^2 I_{n+1}$ in SM(m) by Lemma 1.3.1 , and $(n\,xx^t - r^2\sum_i e_ie_i^t)$ is an eigenvector of the Laplacian corresponding to the eigenvalue $\lambda_p = 2(n+1)/r^2$ . Since this is the second nonzero eigenvalue of the sphere, it follows that $S^n(r)$ is of order [2] . ♦

It is known that a closed curve in $E^m$ is of finite type if and only if its Fourier series expansion has finitely many nonzero terms (see e.g. [C 4]).There are nonspherical closed curves in $E^m$ of finite (see [C 4], pp 288 - 289 and [C 5], pp 16 - 18). They are also of finite type ( $\tilde{x}$ not an isometric immersion) in SM(m) via $\tilde{x}$ since by the product formulas of trigonometry their Fourier series expansions still have finitely many nonzero terms. Also, given any finite type spherical submanifold M which is also of finite type via $\tilde{x}$ , translate M by any vector v , so that v + M again belongs to a sphere (now not centered at the origin). Quadric representation of such translated manifold will no longer be an isometric immersion, but it will still be of finite type. We also have the following example

**Example 2.1** Given two nonspherical finite type curves $C_1$, $C_2$ mentioned above, consider their product $C_1 \times C_2$. Such product does not belong to any sphere and its quadric representation is of finite type since the Laplacian of a product splits into the sum of Laplacians on the component manifolds .

However, we are able to prove the following theorem for minimal submanifolds .

**Theorem 2.4.** For a minimal immersion $x : M^n \to E^m$ , quadric representation $\tilde{x}$ is of infinite type .

**Proof.** Supose $\tilde{x}$ is k-type map where k is finite. Then we can decompose $\tilde{x}$ as

$$\tilde{x} = \tilde{x}_0 + \tilde{x}_{t_1} + \tilde{x}_{t_2} + \dots + \tilde{x}_{t_k} , \text{ where } \tilde{x}_0 = \text{const and } \Delta \tilde{x}_{t_i} = \lambda_{t_i} \tilde{x}_{t_i} .$$

Finding successively iterated Laplacians of $\tilde{x}$ we obtain

$$\Delta \tilde{x} = \lambda_{t_1} \tilde{x}_{t_1} + \lambda_{t_2} \tilde{x}_{t_2} + \dots + \lambda_{t_k} \tilde{x}_{t_k}$$

$$\vdots$$

$$\Delta^k \tilde{x} = \lambda_{t_1}^k \tilde{x}_{t_1} + \lambda_{t_2}^k \tilde{x}_{t_2} + \dots + \lambda_{t_k}^k \tilde{x}_{t_k} .$$

Eliminating $\tilde{x}_{t_1}, \tilde{x}_{t_2}, \dots , \tilde{x}_{t_k}$ from these k+1 equations we get

$$(2.13)\qquad \sigma_k(\tilde{x}-\tilde{x}_0)+\sigma_{k-1}\Delta(\tilde{x}-\tilde{x}_0)+\dots+\sigma_1\Delta^{k-1}(\tilde{x}-\tilde{x}_0)+\Delta^k(\tilde{x}-\tilde{x}_0)=0$$

where $\sigma_i$ is the $i^{th}$ elementary symmetric function of $\lambda_{t_1}, \lambda_{t_2}, \dots, \lambda_{t_k}$ that is

$$\sigma_1 = -(\lambda_{t_1}+\dots+\lambda_{t_k})$$

$$\vdots$$

$$\sigma_{k-1} = (-1)^{k-1}\sum_j \lambda_{t_1}\dots\hat{\lambda}_{t_j}\dots\lambda_{t_k} \qquad (\wedge \text{ denotes omission })$$

$$\sigma_k = (-1)^k\lambda_{t_1}\dots\lambda_{t_k}$$

As before we find $tr(\Delta\tilde{x})$ to be

$$tr(\Delta\tilde{x}) = \Delta(tr\,\tilde{x}) = \Delta\langle x, x\rangle = 2\langle\Delta x, x\rangle - 2\sum_i\langle e_i, e_i\rangle = -2n$$

and by iterating we get $tr(\Delta^i\tilde{x}) = 0$ for $i \geq 2$. Hence if we take trace of (2.13) we obtain

$$(2.14)\qquad \sigma_k(tr\,\tilde{x} - tr\,\tilde{x}_0) - 2n\,\sigma_{k-1} = 0\ .$$

1°. If $\sigma_k \neq 0$ i.e. submanifold is not of null k-type then

$$\langle x, x\rangle = tr\,\tilde{x} = \frac{2n\sigma_{k-1}}{\sigma_k} + tr\,\tilde{x}_0 = \text{const}\ ,$$

so $x(M^n) \subset S^{m-1}(r)$. But spherical submanifold cannot be minimal in ambient Euclidean space, therefore we have a contradiction.

2°. If $\sigma_k = 0$, then one eigenvalue, say $\lambda_{t_1}$, must be zero. If $k \geq 2$ we conclude from (2.14) that also $\sigma_{k-1} = 0$. That implies that another eigenvalue, say $\lambda_{t_2}$, is zero which is a contradiction since $\lambda_{t_1}$ and $\lambda_{t_2}$ are two different eigenvalues. If $k = 1$ (and $\sigma_1 = 0$) then $\tilde{x} = \tilde{x}_0 + \tilde{x}_p$ with $\Delta\tilde{x} = 0$, so by taking trace, $0 = tr(\Delta\tilde{x}) = -2n$ again contradiction. We conclude that $\tilde{x}$ cannot be finite type map. ♦

If $x : M^n \to E^m$ is spherical submanifold, i.e. submanifold of the unit hypersphere centered at the origin, then $\tilde{x}$ is also an isometric immersion by virtue of $\tilde{x}_*(X) = X x^t + x X^t$. It is interesting to see how certain properties of the immersion x are reflected in the immersion $\tilde{x}$ and vice versa. To that end we prove the following

**Theorem 2.5.** Let $x : M^n \to S^{m-1} \subset E^m$ be an isometric immersion and let $\tilde{x} : M^n \to SM(m)$ be its quadric representation. Symbols with ~ are related to the immersion $\tilde{x}$, those without ~ to the immersion into $E^m$ and symbols with ' relate to the immersion into $S^{m-1}$. Then

*i* ) $\quad \| \tilde{h} \| = \text{const} \iff \| h \| = \text{const}$

$\qquad \| \tilde{H} \| = \text{const} \iff \| H \| = \text{const}$ .

*ii* ) $M^n$ is pseudoumbilical in SM(m) via $\tilde{x} \iff M^n$ is pseudoumbilical in $E^m$ via x .

*iii* ) $\tilde{D}\tilde{H} = 0 \iff h' = 0$ , i.e. $M^n$ is totally geodesic in $S^{m-1}$ .

*iv* ) $\tilde{\nabla}\tilde{h} = 0 \iff h' = 0$ .

**Proof.** *i* ) Since $\tilde{x}$ is an isometric immersion we have $\Delta\tilde{x} = -n\tilde{H}$ , and using (2.3) we get

$$\tilde{H} = (H x^t + x H^t) + \frac{2}{n} \sum_i e_i e_i^t \quad . \tag{2.15}$$

Using the fact that H = H' - x , we obtain

$$\| \tilde{H} \|^2 = \tilde{g}(\tilde{H}, \tilde{H}) = \frac{1}{2} \operatorname{tr}(\tilde{H}^2) = \| H' \|^2 + 2 + \frac{2}{n} = \| H \|^2 + \frac{2+n}{n} \ .$$

This proves second equivalence of *i* ) ( cf. [C 4], Lemma 4.6.4 , p. 152 ). The first equivalence can be proved using similar reasoning. In fact

$$\tilde{h}(X, Y) = \tilde{x}_*(h'(X, Y)) + XY^t + YX^t - 2 < X, Y > xx^t ,$$

and hence $\| \tilde{h} \|^2 = \| h \|^2 + n^2 + 2n$ .

*ii* ) We differentiate (2.15) along vector field $X \in TM$ to get

$$\tilde{\nabla}_X \tilde{H} = \tilde{\nabla}_X (H x^t + x H^t) + \frac{2}{n} \sum_i \tilde{\nabla}_X (e_i e_i^t)$$

$$= -[(A_H X) x^t + x (A_H X)^t] + [(D_X H) x^t + x (D_X H)^t]$$

$$+ (H X^t + X H^t) + \frac{2}{n} \sum_{r=n+1}^{m} [e_r (A_r X)^t + (A_r X) e_r^t]$$

We simplify this by choosing $x = e_m$, and observing that $H = H' - x$ we obtain

$$\tilde{\nabla}_X \tilde{H} = -[\tilde{x}_*(A_H X) + \frac{n+2}{n} \tilde{x}_*(X)]$$

(2.16)
$$+ [(D_X H') x^t + x (D_X H')^t] + (H' X^t + X H'^t)$$

$$+ \frac{2}{n} \sum_{r=n+1}^{m-1} [e_r (A_r X)^t + (A_r X) e_r^t] .$$

On the other hand, $\tilde{\nabla}_X \tilde{H} = -\tilde{A}_{\tilde{H}} X + \tilde{D}_X \tilde{H}$ , so by comparing components of (2.16) which are tangent and normal to M we obtain

(2.17)
$$\tilde{A}_{\tilde{H}} X = A_H X + \frac{n+2}{n} X$$

(2.18)
$$\tilde{D}_X \tilde{H} = [(D_X H') x^t + x (D_X H')^t]$$

$$+ (H' X^t + X H'^t) + \frac{2}{n} \sum_{r=n+1}^{m-1} [e_r (A_r X)^t + (A_r X) e_r^t] .$$

The equation (2.17) proves part *ii* ). Note that in (2.18) the first line of the right hand side is the component tangent to the sphere $S^{m-1}$ via $\tilde{x}$ and the second line represents the component of $\tilde{D}_X\tilde{H}$ which is normal to $S^{m-1}$ . Therefore, if $\tilde{D}_X\tilde{H} = 0$ , we see that $D'H' = 0$ and

$$(2.19) \qquad (H' X^t + X H'^t) + \frac{2}{n} \sum_{r=n+1}^{m-1} [\, e_r (A_r X)^t + (A_r X) e_r^t \,] = 0 \quad .$$

Using $H' = \frac{1}{n} \sum_{r=n+1}^{m-1} (\operatorname{tr} A_r) e_r$ and substituting into (2.19) we obtain

$$(2.20) \qquad \frac{1}{n} \sum_{r=n+1}^{m-1} \{ e_r [(\operatorname{tr} A_r) X + 2\,(A_r X)]^t + [(\operatorname{tr} A_r) X + 2\,(A_r X)]\, e_r^t \} = 0 \quad ,$$

for every $X \in TM$ . From here we have $(\operatorname{tr} A_r)\, I + 2\, A_r = 0$ and taking trace of this relation we get $(n + 2) \operatorname{tr} A_r = 0$ , i.e. $\operatorname{tr} A_r = 0$ . Putting this back into the relation we conclude $A_r = 0$ for every $r = n+1, \dots, m-1$ , or equivalently $h' = 0$ , which means that $M^n$ is totally geodesic in $S^{m-1}$ . Conversely, if $h' = 0$ , then $\tilde{x}$ immerses M as a minimal submanifold of a hypersphere of SM(m) centered at I/m (Veronese submanifold) so that $\tilde{H} \parallel (\tilde{x} - I/m)$ and therefore $\tilde{D}\tilde{H} = 0$ proving *iii* ).

Part *iv* ) follows from *iii* ) because $\tilde{\nabla} \tilde{h} = 0$ implies $\tilde{D}\tilde{H} = 0$ . Namely,

$$0 = \sum_i (\tilde{\nabla}_Z \tilde{h})(e_i, e_i) = \sum_i \tilde{D}_Z\, \tilde{h}(e_i, e_i) - 2 \sum_i \tilde{h}(\nabla_Z e_i, e_i)$$

$$= n\, \tilde{D}_Z \tilde{H} - 2 \sum_{i,j} \omega_i^j(Z) h(e_j, e_i) \quad ,$$

and the second sum is equal to zero.

Finally, let us compute second iterated Laplacian of the quadric representation because it sets the stage for the investigation in Chapter 3 .

Recall that we computed

$$\Delta\tilde{x} = -n(H x^t + x H^t) - 2\sum_i e_i e_i^t \quad . \tag{2.3}$$

To find $\Delta^2\tilde{x}$ we first find $-\sum_i \Delta(e_i e_i^t)$ and then $\Delta(H x^t + x H^t)$ . We can assume that at given point p we have $(\nabla_{e_k} e_j)(p) = 0$ (normal coordinate system ). Then the Laplacian becomes $\Delta f = -\sum_k e_k e_k f$ at point p , so we have first

$$\sum_i \tilde{\nabla}_{e_k}(e_i e_i^t) = \sum_i [h(e_k,e_i)\, e_i^t + e_i\, h(e_k,e_i)^t] \quad ,$$

and then at p

$$-\sum_i \Delta(e_i e_i^t) = \sum_{k,i} \tilde{\nabla}_{e_k}\tilde{\nabla}_{e_k}(e_i e_i^t))$$

$$= -\sum_{k,i} \{[A_{h(e_k,e_i)}e_k]\, e_i^t + e_i\, [A_{h(e_k,e_i)}e_k]^t\}$$

$$+ \sum_{k,i} \{[D_{e_k}h(e_k,e_i)]e_i^t + e_i\, [D_{e_k}h(e_k,e_i)]^t\} \tag{2.21}$$

$$+ 2\sum_{k,i} h(e_k,e_i)h(e_k,e_i)^t \quad .$$

Now we compute each sum separately, first

$$2\sum_{k,i} h(e_k,e_i)h(e_k,e_i)^t = 2\sum_{k,i,r,s} g(h(e_k,e_i),e_r)\, g(h(e_k,e_i),e_s)\, e_r e_s^t$$

$$= 2 \sum_{r} \sum_{s} \sum_{k,i} g(A_r e_i, e_k) g(A_s e_i, e_k)\, e_r e_s^t$$

$$= 2 \sum_{r} \sum_{s} \sum_{i} g(A_r e_i,\ A_s e_i)\, e_r e_s^t$$

$$= 2 \sum_{r,s} tr(A_r A_s)\, e_r e_s^t$$

$$= \sum_{r,s} tr\,(A_r A_s)[e_r e_s^t + e_s e_r^t] \quad ,$$

then

$$\sum_{k,i} \{[A_{h(e_k,e_i)} e_k]\, e_i^t + e_i\, [A_{h(e_k,e_i)} e_k]^t\}$$

$$= \sum_{i,k,r} [g(A_r e_k, e_i)\, (A_r e_k) e_i^t \;+\; g(A_r e_k, e_i)\, e_i (A_r e_k)^t]$$

$$= 2 \sum_{k,r} (A_r e_k)(A_r e_k)^t \quad ,$$

and finaly, using the Codazzi equation ,

$$\sum_{k,i} \{[D_{e_k} h(e_k, e_i)] e_i^t + e_i\, [D_{e_k} h(e_k, e_i)]^t\}$$

$$= \sum_{i,k} \{[(\bar{\nabla}_{e_k} h)(e_k, e_i)]\, e_i^t \;+\; e_i\, [(\bar{\nabla}_{e_k} h)(e_k, e_i)]^t\}$$

$$= \sum_{i,k} \{ [(\overline{\nabla}_{e_i} h)(e_k, e_k)]\, e_i^t + e_i\, [(\overline{\nabla}_{e_i} h)(e_k, e_k)]^t \}$$

$$= \sum_{k,i} \{ [D_{e_i} h(e_k, e_k)] e_i^t + e_i\, [D_{e_i} h(e_k, e_k)]^t \}$$

$$= n \sum_{i} [(D_{e_i} H)\, e_i^t + e_i\, (D_{e_i} H)^t] \quad .$$

Substituting these formulas into (2.21) and putting it together we see that at point p the following equation holds

$$- \sum_i \Delta(e_i e_i^t) = n \sum_i [(D_{e_i} H)\, e_i^t + e_i\, (D_{e_i} H)^t]$$

$$(2.22) \qquad + \sum_{r,s} \operatorname{tr}(A_r A_s)[e_r e_s^t + e_s e_r^t]$$

$$- 2 \sum_{k,r} (A_r e_k)(A_r e_k)^t \quad .$$

Neither left hand side nor right hand side of (2.22) depend on the adapted frame chosen, so the formula is true for any (local) frame at any point of M.

Next we compute $\Delta(H\, x^t + x\, H^t)$ using product formula for the Laplacian

$$\Delta(H\, x^t + x\, H^t) = [(\Delta H)\, x^t + x(\Delta H)^t] + [H(\Delta x)^t + (\Delta x) H^t]$$

$$- 2 \sum_i [(\overline{\nabla}_{e_i} H)(\overline{\nabla}_{e_i} x)^t + (\overline{\nabla}_{e_i} x)(\overline{\nabla}_{e_i} H)^t]$$

$$(2.23) \qquad = [(\Delta H)\, x^t + x(\Delta H)^t] - 2n\, HH^t$$

$$+ 2 \sum_i [(A_H e_i)e_i^t + e_i\,(A_H e_i)^t]$$

$$- 2 \sum_i [(D_{e_i}H)\, e_i^t + e_i\,(D_{e_i}H)^t] \quad .$$

combining (2.22), (2.23) and (2.3) we finaly obtain the following formula for $\Delta^2 \tilde{x}$ .

$$\Delta^2 \tilde{x} = -n\,[(\Delta H)\, x^t + x(\Delta H)^t] + 2n^2\, HH^t$$

$$- 2n \sum_i [(A_H e_i)e_i^t + e_i\,(A_H e_i)^t]$$

$$(2.24) \qquad + 4n \sum_i [(D_{e_i}H)\, e_i^t + e_i\,(D_{e_i}H)^t]$$

$$+ 2 \sum_{r,s} \mathrm{tr}\,(A_r A_s)[e_r e_s^t + e_s e_r^t] - 4 \sum_{k,r} (A_r e_k)(A_r e_k)^t \quad .$$

Expression (2.24) can be further broken down into components using the following formula for $\Delta H$ due to B. Y. Chen ([C 4], p. 271)

$$(2.25) \qquad \Delta H = \Delta^D H + \| A_{n+1} \|^2 H + a\,(H) + \mathrm{tr}\,(\bar{\nabla} A_H) \ ,$$

where $e_{n+1} \parallel H$ , $\Delta^D$ is the Laplacian of the normal bundle, $a\,(H) = \sum_{r=n+2}^{m} \mathrm{tr}(A_H A_r)e_r$ is so called *allied mean curvature vector* and $\mathrm{tr}\,(\bar{\nabla} A_H) = \mathrm{tr}\,(\nabla A_H) + \mathrm{tr}\,(A_{DH})$ .

# CHAPTER 3

# SPHERICAL HYPERSURFACES WHICH ARE OF LOW TYPE VIA THE SECOND STANDARD IMMERSION OF THE SPHERE

In the previous chapter we classified submanifolds of a Euclidean space whose quadric representation is of 1 - type as those which are totally geodesic in a hypersphere centered at the origin. Of course the same is true when a submanifold is assumed from the outset to be a hypersurface of the sphere. In this chapter we consider a hypersurface of the unit sphere centered at the origin (henceforth called spherical) and study those which are of 2 - or 3 - type via the second standard immersion of the sphere. Throughout, we generally assume that the dimension of a submanifold is greater than one. For finite type curves in general see [C 4], [C-D-V] .

## 1. Spherical hypersurfaces which are of 2 - type via $\tilde{x}$

In Chapter 2 we derived the formula for the second iterated Laplacian of the quadric representation $\tilde{x}$ of a submanifold $M^n$ (formula (2.24)). If $M^n$ is spherical submanifold, then the Laplacian of the mean curvature vector H of $M^n$ in $E^m$ can be computed as ([C 4], Lemma 6.4.2 , p. 273 )

(3.1.1) $$\Delta H = \Delta^{D'}H' + a'(H') + \mathrm{tr}\,(\bar{\nabla} A_H) + (\|A_\xi\|^2 + n)H' - n\alpha^2 x \ ,$$

where, as usual, symbols with ' denote objects and quantities related to the immersion of $M^n$ into the hypersphere $S^{m-1}$, thus, $\Delta^{D'}$ is the Laplacian of the normal connection of M in $S^{m-1}$, and $a'(H')$ is the allied mean curvature vector in $S^{m-1}$. The mean curvature of $M^n$ in $E^m$ is denoted by $\alpha$ and the one in $S^{m-1}$ by $\alpha'$. They are related via $\alpha^2 = \alpha'^2 + 1$. $\xi$ is a local unit normal vector field of M in $S^{m-1}$ such that $\xi \parallel H'$, hence $H' = \alpha'\xi$. We also have

$$\mathrm{tr}\,(\bar{\nabla} A_H) = \mathrm{tr}\,(\nabla A_H) + \mathrm{tr}\,(A_{DH})$$

(3.1.2) $$= \sum_{i=1}^{n} (\nabla_{e_i} A_H) e_i + \sum_{i=1}^{n} A_{D_{e_i}H}\, e_i \ ,$$

where $e_1, \dots, e_n$ is a local orthonormal frame of tangent vectors of $M^n$.

If M is now spherical hypersurface, then $\Delta^{D}H' = (\Delta\alpha')\xi$, $a'(H') = 0$ and by one result of [C 6] ( see also [C 5], p. 21 ) we have

(3.1.3) $$\mathrm{tr}\,(\bar{\nabla} A_H) = n\,\alpha'\nabla\alpha' + 2\,A(\nabla\alpha')$$

Putting this back into (3.1.1) and combining with (2.24) we have

$$\Delta^2\tilde{x} = -\,n\,[\,\Delta\alpha' + \alpha'(\,\|A\|^2 + 3n + 4\,)]\,(\xi x^t + x\xi^t\,) - n\,(\,Wx^t + xW^t)$$

$$+\ 2n\,(n\alpha^2 + n + 2)\,xx^t \ +\ 2\,(n^2\alpha'^2 + 2\|A\|^2)\,\xi\xi^t$$

(3.1.4) $$+\ 4n\,[\,\xi(\nabla\alpha')^t + (\nabla\alpha')\xi^t\,] \ -\ 4(n+1)\sum_i e_i e_i^t$$

$$-\ 2n\,\alpha' \sum_i [(Ae_i)e_i^t + e_i(Ae_i)^t] \ -\ 4\sum_i (Ae_i)(Ae_i)^t \quad ,$$

where $A = A_\xi$ and $W = \operatorname{tr}(\bar{\nabla} A_H) = n\alpha'\nabla\alpha' + 2A(\nabla\alpha')$ .

Suppose that $\tilde{x} = xx^t$ is of 2 - type. Then we have $\tilde{x} = \tilde{x}_0 + \tilde{x}_p + \tilde{x}_q$ and hence

$$\Delta^2\tilde{x} - (\lambda_p + \lambda_q)\Delta\tilde{x} + \lambda_p\lambda_q(\tilde{x} - \tilde{x}_0) = 0 . \tag{3.1.5}$$

In order to eliminate constant vector $\tilde{x}_0$ from this equation we find the directional derivative $\tilde{\nabla}_X$ of (3.1.5) . First, using straightforward but long computation we obtain

$$\begin{aligned} \tilde{\nabla}_X(\Delta^2\tilde{x}) = & - n\langle X, \nabla\rho + AW + 4\nabla\alpha'\rangle(\xi x^t + x\xi^t) \\ & + 2n\langle X, W + 2n\alpha'\nabla\alpha'\rangle xx^t \\ & + \langle X, 4n^2\alpha'\nabla\alpha' + 4\nabla\|A\|^2 + 8n A(\nabla\alpha')\rangle \xi\xi^t \\ & + 2(n^2\alpha^2 + n^2 + 4n + 2)(Xx^t + xX^t) \\ & + n(\rho + 4\alpha')[(AX)x^t + x(AX)^t] \\ & + 4[(A^2X)x^t + x(A^2X)^t] - n[(\nabla_X W)x^t + x(\nabla_X W)^t] \\ & - n\rho(X\xi^t + \xi X^t) + 4n[(\nabla_X(\nabla\alpha'))\xi^t + \xi(\nabla_X(\nabla\alpha'))^t] \\ & - 2(n^2\alpha'^2 + 2\|A\|^2 + 2n + 2)[(AX)\xi^t + \xi(AX)^t] \\ & - 4n\alpha'[(A^2X)\xi^t + \xi(A^2X)^t] - 4[(A^3X)\xi^t + \xi(A^3X)^t] \\ & - n(XW^t + WX^t) - 4n[(AX)(\nabla\alpha')^t + (\nabla\alpha')(AX)^t] \\ & - 2n\langle X, \nabla\alpha'\rangle \sum_i [(Ae_i)e_i^t + e_i(Ae_i)^t] \end{aligned} \tag{3.1.6}$$

$$- 2n\,\alpha' \sum_i [((\nabla_X A)e_i)e_i^t + e_i((\nabla_X A)e_i)^t]$$

$$- 2 \sum_i [((\nabla_X A^2)e_i)e_i^t + e_i((\nabla_X A^2)e_i)^t] \quad ,$$

where $\|A\|^2 = \mathrm{tr}\, A^2$ , and $\rho = \Delta\alpha' + \alpha'(\|A\|^2 + 3n + 4)$ . Also, we easily obtain

(3.1.7) $$\tilde{\nabla}_X \tilde{x} = \tilde{\nabla}_X(xx^t) = X x^t + x X^t$$

(3.1.8) $$\tilde{\nabla}_X(\Delta\tilde{x}) = 2(n+1)(X x^t + x X^t) - n\langle X, \nabla\alpha'\rangle(\xi x^t + x\xi^t)$$
$$+ n\,\alpha'[(AX)x^t + x(AX)^t] - n\,\alpha'(\xi X^t + X\xi^t)$$
$$- 2[(AX)\xi^t + \xi(AX)^t] \ .$$

Denote the left hand side of (3.1.5) by $Q(\tilde{x})$ , i.e.

(3.1.9) $$Q(\tilde{x}) = \Delta^2\tilde{x} - (\lambda_p + \lambda_q)\Delta\tilde{x} + \lambda_p\lambda_q(\tilde{x} - \tilde{x}_0) \ .$$

From $\tilde{g}(\tilde{\nabla}_X[Q(\tilde{x})], xx^t) = 0$ using (3.1.6) - (3.1.8) we obtain $W + 2n\alpha'\nabla\alpha' = 0$ and combining with (3.1.3) we have

(3.1.10) $$A(\nabla\alpha') = -\frac{3}{2}\, n\alpha'\nabla\alpha' \ ,$$

and therefore (3.1.3) yields

(3.1.11) $$W = \mathrm{tr}(\bar{\nabla}A_H) = - 2n\,\alpha'\nabla\alpha' \ .$$

From $\tilde{g}(\tilde{\nabla}_X[Q(\tilde{x})], \xi\xi^t) = 0$ , we get $4n^2\alpha'\nabla\alpha' + 4\nabla\|A\|^2 + 8n\,A(\nabla\alpha') = 0$ , and therefore, using (3.1.10) , we have

$$\nabla \|A\|^2 = 2n^2 \alpha' \nabla \alpha' \quad , \tag{3.1.12}$$

or $\|A\|^2 = n^2 \alpha'^2 - c$ ( c is a constant ), which implies that the scalar curvature of M is constant , since $n^2 \alpha'^2 - \|A\|^2 = \tau - n(n+1)$ by the Gauss equation .

Let $U = \{ p \in M \mid \nabla(\alpha')^2 \neq 0 \text{ at } p \}$. Then U is an open (possibly empty) subset of M, and on U we obviously have also $\alpha' \neq 0$ and $\nabla\alpha' \neq 0$ . If U is nonempty , then by (3.1.10) we see that $\nabla\alpha'$ is an eigenvector of the shape operator A on U with the eigenvalue $-\frac{3}{2} n\alpha'$ . Now on U we choose unit tangent vector $e_1$ to be in the direction of $\nabla\alpha'$ , i.e. $e_1 = \nabla\alpha'/\|\nabla\alpha'\|$ . We find $e_1 e_1^t$ component of $\tilde{\nabla}_X[Q(\tilde{x})]$ on U setting first $X = \nabla\alpha'$ . Combining (3.1.6) , (3.1.10) and (3.1.11) we get the following by exploiting $\tilde{g}( \tilde{\nabla}_X[Q(\tilde{x})], e_1 e_1^t ) = 0$

$$\begin{aligned} 0 &= 16 n^2\alpha' \, \tilde{g}( (\nabla\alpha')(\nabla\alpha')^t , e_1 e_1^t ) - 2n \|\nabla\alpha'\|^2 \langle Ae_1, e_1\rangle \\ &\quad - 2n\,\alpha' \langle (\nabla_{\nabla\alpha'} A)e_1 , e_1\rangle - 2 \langle (\nabla_{\nabla\alpha'} A^2)e_1 , e_1\rangle \\ &= 8\, n^2\alpha' \|\nabla\alpha'\|^2 + 3\, n^2\alpha'\|\nabla\alpha'\|^2 \\ &\quad - 2n\,\alpha' \langle \nabla_{\nabla\alpha'}(Ae_1) - A(\nabla_{\nabla\alpha'} e_1) , e_1\rangle \\ &\quad - 2 \langle \nabla_{\nabla\alpha'}(A^2 e_1) - A^2(\nabla_{\nabla\alpha'} e_1) , e_1\rangle \ . \end{aligned}$$

Note that $\langle\nabla_{\nabla\alpha'} e_1 , e_1\rangle = 0$ , and therefore also $\langle A(\nabla_{\nabla\alpha'} e_1) , e_1\rangle = 0$ and $\langle A^2(\nabla_{\nabla\alpha'} e_1) , e_1\rangle = 0$ . Hence, the calculation above continues as

$$0 = 11 n^2\alpha'\|\nabla\alpha'\|^2 + 3n^2\alpha'(\nabla\alpha')(\alpha') - \frac{9}{2} n^2 (\nabla\alpha')(\alpha'^2) = 5n^2\alpha'\|\nabla\alpha'\|^2 .$$

From this we conclude $\alpha' = 0$ or $\nabla\alpha' = 0$ at any point of U. However, this is a contradiction, and hence U must be empty. This means that $\nabla(\alpha')^2 = 0$ everywhere on

M , i.e. $\alpha' = \text{const}$ . Therefore, a hypersurface of $S^{n+1}$ which is of 2 - type via $\tilde{x}$ must have constant mean curvature $\alpha'$ in sphere .

Let us remark that in order to find different components of $\tilde{\nabla}_X[Q(\tilde{x})]$ it is not absolutely necessary to use long formula (3.1.6). We can also find those components indirectly, for example, $\xi\xi^t$ - component can be found in the following way . Let $Q'(\tilde{x}) = \Delta^2\tilde{x} - (\lambda_p + \lambda_q)\Delta\tilde{x} + \lambda_p\lambda_q\tilde{x}$ . Then

$$
\begin{aligned}
0 &= \langle \tilde{\nabla}_X[Q(\tilde{x})], \xi\xi^t \rangle \\
&= X \langle Q'(\tilde{x}), \xi\xi^t \rangle - \langle Q'(\tilde{x}), \tilde{\nabla}_X(\xi\xi^t) \rangle \\
&= X \langle Q'(\tilde{x}), \xi\xi^t \rangle + \langle Q'(\tilde{x}), (AX)\xi^t + \xi(AX)^t \rangle \\
&= X(n^2\alpha'^2 + 2\,\|A\|^2) + 4n \langle \nabla\alpha', AX \rangle \\
&= \langle X,\ 2n^2\alpha'\nabla\alpha' + 2\nabla\|A\|^2 + 4n\,A(\nabla\alpha') \rangle ,
\end{aligned}
$$

so that $n^2\alpha'\nabla\alpha' + \nabla\|A\|^2 + 2n\,A(\nabla\alpha') = 0$ as before . Similarly for $xx^t$ - and $(\nabla\alpha')(\nabla\alpha')^t$ - component .

We are ready now to prove the following classification result

**Theorem 3.1.1.** Let $x: M^n \to S^{n+1}$ be an isometric immersion of a compact n - dimensional Riemannian manifold M into $S^{n+1}$ $(n \geq 2)$ . Then $\tilde{x} = xx^t$ is of 2 - type if and only if either

(1) M is a small hypersphere of $S^{n+1}$ of radius $r < 1$ , or

(2) $M = S^p(r_1) \times S^{n-p}(r_2)$ , with the following possibilities for the radii $r_1$ and $r_2$ :

i) $r_1^2 = \frac{p+1}{n+2}$ , $r_2^2 = \frac{n-p+1}{n+2}$ ; ii) $r_1^2 = \frac{p+2}{n+2}$ , $r_2^2 = \frac{n-p}{n+2}$ ; iii) $r_1^2 = \frac{p}{n+2}$ , $r_2^2 = \frac{n-p+2}{n+2}$

The immersions in (1) and (2) are given in a natural way.

**Proof.** If M is one of the submanifolds described in (1) and (2), then M is of 2-type via the second standard immersion of the sphere as shown in [B-C]. Conversely, let us assume that for a spherical hypersurface $x : M^n \to S^{n+1}$ the quadric representation $\tilde{x}$ is of 2-type. Then (3.1.5) holds, and from the above we see that the mean curvature $\alpha'$ of x is constant. In that case $\nabla\alpha' = W = \nabla \|A\|^2 = \nabla\rho = 0$, and the formula (3.1.6) simplifies, so that the part of (3.1.6) which is tangent to M reduces to

$$2\,(n^2\alpha^2 + n^2 + 4n + 2)\,(X x^t + x X^t)$$
$$+\ n\,(\rho + 4\,\alpha')[\,(AX)x^t + x(AX)^t]$$
$$+\ 4\,[\,(A^2X)x^t + x(A^2X)^t] \quad ,$$

where $\rho = \alpha'\,(\|A\|^2 + 3n + 4)$ is constant. Let $e_k$, $k = 1, 2, \ldots, n$ be a local orthonormal vector fields which are eigenvectors of A (principal directions) and let $\mu_k$ be the corresponding principal curvatures. We set $X = e_k$ in (3.1.6) and compute the component tangential to M. Then from $\tilde{g}\,(\tilde{\nabla}_{e_k}[Q(\tilde{x})],\ xe_k^t + e_kx^t) = 0$ we obtain

$$0 = [\,2\,(n^2\alpha^2 + n^2 + 4n + 2) - 2(n+1)(\lambda_p + \lambda_q) + \lambda_p\lambda_q\,]$$
(3.1.13)
$$+\ n\,[\,\rho - (\lambda_p + \lambda_q)\alpha'\,]\,\mu_k + 4\,\mu_k^2 \quad .$$

This is a quadratic equation in $\mu_k$ with constant coefficients which do not depend on k and the equation is not trivial ( 0 = 0 ) because of the term $4\,\mu_k^2$. We conclude, therefore, that each principal curvature is constant and that there are at most two distinct principal curvatures. If M has only one principal curvature, i.e. if it is umbilical, then M is a small hypersphere of $S^{n+1}$. If M has two distinct (constant) principal curvatures then M is the standard product of two spheres, $M = S^p(r_1) \times S^{n-p}(r_2)$ with $r_1^2 + r_2^2 = 1$ (see [Car 2], or

[Ry] ). Then, according to [B-C] (Lemma 3), such product will be of 2 - type via $\tilde{x}$ if and only if the radii satisfy precisely those three possibilities listed in (2) ♦

Theorem 3.1.1 is a generalization of a result of M. Barros and B.Y. Chen, who proved a similar theorem assuming M to be mass - symetric (cf. [B-C], Theorem 3 ).

## 2. Minimal spherical hypersurfaces which are of 3 - type and mass - symmetric via $\tilde{x}$

Since computations for the third iterated Laplacian of $\tilde{x}$ become more involved and considerably more difficult to handle we restrict our investigation to minimal spherical hypersurfaces which are of 3 - type and mass - symmetric via $\tilde{x}$ . For minimal hypersurfaces in sphere, calculations from before give

(3.2.1) $$\Delta\tilde{x} = \Delta(xx^t) = 2n\, xx^t - 2\sum_i e_i e_i^t$$

(3.2.2) $$-\Delta\Big(\sum_i e_i e_i^t\Big) = 2n\, xx^t + 2\,\|A\|^2 \xi\xi^t - 2\sum_i e_i e_i^t - 2\sum_k (Ae_k)(Ae_k)^t$$

(3.2.3) $$\Delta(\xi\xi^t) = 2\,\|A\|^2 \xi\xi^t - 2\sum_k (Ae_k)(Ae_k)^t$$

(3.2.4) $$\Delta^2\tilde{x} = 4n(n+1)\, xx^t + 4\,\|A\|^2 \xi\xi^t - 4(n+1)\sum_i e_i e_i^t - 4\sum_k (Ae_k)(Ae_k)^t .$$

We also have the following lemma, which can be proved by direct computation in a similar fashion as was done to prove formula (2.22) .

**Lemma 3.2.1.** If $e_1, \dots, e_n$ is a local orthonormal basis of tangent vector fields of

M , and $\Delta A = \sum_{i=1}^{n} [ \nabla_{\nabla_{e_i} e_i} A - \nabla_{e_i}(\nabla_{e_i} A)]$ is the trace Laplacian of the shape operator, then

$$
\begin{aligned}
-\Delta \{ \sum_i (Ae_i)(Ae_i)^t \} = & -\sum_i [((\Delta A)e_i)(Ae_i)^t + (Ae_i)((\Delta A)e_i)^t] \\
& -2 \sum_i (Ae_i)(Ae_i)^t - 2 \sum_i (A^2 e_i)(A^2 e_i)^t \\
& + 2(\mathrm{tr}\, A^2)\, xx^t + 2(\mathrm{tr}\, A^4)\, \xi\xi^t - 2\, (\mathrm{tr}\, A^3)( x\xi^t + \xi x^t ) \\
& - 2[(\mathrm{tr} \nabla A^2) x^t + x\, (\mathrm{tr} \nabla A^2)^t] \\
& + [\, 2\, (\mathrm{tr} \nabla A^3) - A^2 (\mathrm{tr} \nabla A)]\, \xi^t + \xi [\, 2\, (\mathrm{tr} \nabla A^3) - A^2 (\mathrm{tr} \nabla A)]^t \\
& + 2 \sum_{i,k} [(\nabla_{e_k} A) e_i][(\nabla_{e_k} A) e_i]^t \quad .
\end{aligned}
\tag{3.2.5}
$$

Each sum here is independent of the frame $\{e_i\}$ chosen .

One of the results of K. Nomizu and B. Smyth in [N-S] is computation of $\Delta A$ for spherical hypersurface with $\mathrm{tr}\, A = \mathrm{const}$ . Namely,

$$\Delta A = (\mathrm{tr}\, A^2 - n)\, A + (\mathrm{tr}\, A)\, I - (\mathrm{tr}\, A)\, A^2 \quad . \tag{3.2.6}$$

Because we assume $\mathrm{tr}\, A = 0$ we will have

$$\Delta A = (\mathrm{tr}\, A^2 - n)\, A \quad . \tag{3.2.7}$$

Now taking Laplacian of (3.2.4) and taking into account (3.2.1 - 3) and (3.2.7) we get the following formula for $\Delta^3 \tilde{x}$ .

$$\Delta^3 \tilde{x} = 8[n(n+1)^2 + \mathrm{tr}\, A^2]\, xx^t$$

$$+\ 4\,[\,\Delta(\mathrm{tr}\,A^2) + 2\,(\mathrm{tr}\,A^2)^2 + 2\,(n+1)\mathrm{tr}\,A^2 + 2\,\mathrm{tr}\,A^4]\,\xi\xi^t$$

$$-\ 8\,(\mathrm{tr}\,A^3)(x\xi^t + \xi x^t)$$

(3.2.8)

$$-\ 4\,[\,(\nabla(\mathrm{tr}\,A^2))x^t + x(\nabla(\mathrm{tr}\,A^2))^t\,]$$

$$+\ [\,\tfrac{8}{3}\nabla(\mathrm{tr}\,A^3) + 12\,A\,(\nabla(\mathrm{tr}\,A^2))]\,\xi^t + \xi[\,\tfrac{8}{3}\nabla(\mathrm{tr}\,A^3) + 12\,A\,(\nabla(\mathrm{tr}\,A^2))]^t$$

$$-\ 8\,(n+1)^2\sum_i e_i e_i^t - 16\,(1+\mathrm{tr}\,A^2)\sum_i (Ae_i)(Ae_i)^t - 8\sum_i (A^2e_i)(A^2e_i)^t$$

$$+\ 8\sum_{i,k}[(\nabla_{e_k}A)e_i][(\nabla_{e_k}A)e_i]^t \quad .$$

Each sum in this formula is independent of the frame chosen.

Suppose now that $M^n$ is mass - symmetric and of 3 - type via $\tilde{x}$ so that $\tilde{x}_0 = \frac{I}{n+2}$ and

(3.2.9) $$\Delta^3\tilde{x} + a\,\Delta^2\tilde{x} + b\,\Delta\tilde{x} + c\,(\,xx^t - \frac{I}{n+2}\,) = 0 \;,$$

where a, b and c are constants.(They are equal to elementary symmetric functions of three eigenvalues of the Laplacian which arise from the decomposition $\tilde{x} = \tilde{x}_0 + \tilde{x}_p + \tilde{x}_q + \tilde{x}_r$.) Using (3.2.1), (3.2.4) and (3.2.8) we find different components of (3.2.9) such as $xx^t$ component, $\xi\xi^t$ component, $x\xi^t + \xi x^t$ component etc. For example, comparing $x\xi^t + \xi x^t$ components of left and right hand side of (3.2.9) we see easily that $\mathrm{tr}\,A^3 = 0$. Comparing $xx^t$ components in (3.2.9) we obtain

$$8[n(n+1)^2 + \mathrm{tr}\,A^2] + 4an\,(n+1) + 2\,bn + c\,(1 - \frac{1}{n+2})\,,$$

and consequently $\mathrm{tr}\,A^2 = \mathrm{const}$. Similarly, from $\xi\xi^t$ component of (3.2.9) we have

$$4\,[\,\Delta(\mathrm{tr}\,A^2) + 2\,(\mathrm{tr}\,A^2)^2 + 2\,(n+1)\mathrm{tr}\,A^2 + 2\,\mathrm{tr}\,A^4] + 4a\,\|A\|^2 - c\,\frac{1}{n+2} = 0\ ,$$

and hence $\mathrm{tr}\,A^4 = \mathrm{const}$ as well. We conclude,therefore, that for the dimension $n \leq 4$ minimal spherical hypersurface which is of 3 - type and mass - symmetric via $\tilde{x}$ must be isoparametric, i.e. its principal curvatures must be constant.

Because of this obvious importance of isoparametric spherical hypersurfaces for our investigation we consider next some examples.

**Example 3.2.1. Cartan hypersurface**

According to Cartan theory ( [Car 2, 3] ), there is only one (up to congruences of the sphere) compact minimal isoparametric hypersurface $M^3$ of $S^4$ with three principal curvatures. This hypersurface is a tube about Veronese surface and is usually called the *Cartan hypersurface*. It is a homogeneous space of type $SO(3)/Z_2 \times Z_2$ and an algebraic manifold whose equation is

$$2\,x_5^3 + 3(\,x_1^2 + x_2^2\,)\,x_5 - 6\,(x_3^2 + x_4^2)\,x_5 + 3\sqrt{3}\,(\,x_1^2 - x_2^2\,)\,x_4 + 6\sqrt{3}\,x_1x_2x_3 = 2\ ,$$

with $\sum_i x_i^2 = 1$ ( see Ch.1, Sect.4 ). The Cartan hypersurface has three distinct principal curvatures $k_1 = -\sqrt{3}$ , $k_2 = 0$ and $k_3 = \sqrt{3}$ , hence by the Gauss equation the scalar curvature is equal to $0$ .

We are now going to show that the Cartan hypersurface $M^3 = SO(3)/Z_2 \times Z_2$ is an example of minimal spherical hypersurface which is of 3 - type and mass-symmetric via $\tilde{x}$. Let $\{e_i\}$, $i = 1, 2, 3$ be an orthonormal basis of principal directions. Then, for the Cartan hypersurface, equations (3.2.1) and (3.2.4) become respectively

(3.2.10) $$\Delta\tilde{x} = 6\,xx^t - 2\,(\,e_1e_1^t + e_2e_2^t + e_3e_3^t\,)$$

(3.2.11) $$\Delta^2\tilde{x} = 48\,xx^t + 24\,\xi\xi^t - 12\,(\,e_1e_1^t + e_3e_3^t\,) - 16\,(\,e_1e_1^t + e_2e_2^t + e_3e_3^t\,)\,.$$

In [Car 2], Cartan also computed the connection of $M^3$, namely

$$\omega_2^3 = -\omega^1 \ , \ \omega_3^1 = \frac{1}{2}\omega^2 \text{ and } \omega_1^2 = -\omega^3 \ ,$$

where the connection forms are computed with respect to the basis $\{e_i\}$ of principal directions. Substituting this into (3.2.8) we obtain

$$(3.2.12) \quad \Delta^3 \tilde{x} = 432\, xx^t + 624\, \xi\xi^t - 408\,( e_1e_1^t + e_3e_3^t ) - 80\,( e_1e_1^t + e_2e_2^t + e_3e_3^t )$$

From Lemma 1.3.1 we have $I = xx^t + \xi\xi^t + e_1e_1^t + e_2e_2^t + e_3e_3^t$ , and combining with (3.2.10 - 12) we have

$$(3.2.13) \qquad \Delta^3 \tilde{x} - 34\,\Delta^2 \tilde{x} + 328\,\Delta \tilde{x} - 960\,( \tilde{x} - \frac{I}{5} ) = 0 .$$

It follows that the Cartan hypersurface is mass - symmetric and of 3 - type via $\tilde{x}$ since it cannot be of 1- or 2 - type by the classification in Section 1 of this chapter. Moreover, we easily find the three eigenvalues determining the order to be $\lambda_p = 6$ , $\lambda_q = 8$ , $\lambda_r = 20$ . As a byproduct, we found three eigenvalues of $\Delta$ for the Cartan hypersurface. As a matter of fact, the spectrum of the Cartan hypersurface was computed in [M-O-U] , from which we determine its order via $\tilde{x}$ to be [2, 3, 8] .

Note also that, according to [H-L], the Cartan hypersurface arises from the isotropy representation of the symmetric space of rank two which in this case is SU(3)/SO(3) . Namely, the Lie algebra ***su(3)*** decomposes into a direct sum of the subalgebra ***so(3)*** and the vector space ***m*** which is identified with the set of $3 \times 3$ real symmetric matrices with zero trace, $\boldsymbol{m} = \{ iA \mid A \in SM(3) ,\ \mathrm{tr}\, A = 0 \}$. Using $\langle X, Y \rangle = -\mathrm{tr}\,(XY)$ as an inner product on ***su(3)*** , this decomposition is orthogonal, and SO(3) acts isometrically on the Euclidean 5 - space $\boldsymbol{m}$ by inner automorphisms (see [F], [Ko], [M-O-U], [Ce-Ry], p. 298 ). The Cartan hypersurface is the orbit of the point

$$x = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} \in \boldsymbol{m} \quad .$$

**Example 3.2.2. Minimal isoparametric hypersurface in $S^5$ with 4 principal curvatures**

As discussed in Section 4 of Chapter I , there is only one minimal isoparametric hypersurface $M^4$ in $S^5$ with four curvatures; it is the image of the following map

$$S^1 \times S_{3,2} \to S^5 \subset E^6$$

$$(\theta \, , (x,y)) \to z = e^{i\theta} (\cos t \, x + i \sin t \, y) \qquad , \tag{3.2.14}$$

for $t = \pi/8$ . In general, (3.2.14) defines the isoparametric family studied by Cartan [Car 3] and Nomizu [N 1-2] . It is an algebraic family defined as [Car 3]

$$\cos 4t = (x_1^2 + x_2^2 + \dots + x_6^2)^2 - 2\,(x_3^2 - x_4^2 - 2x_1x_5 + 2x_2x_6)^2$$
$$- 2\,(2x_3x_4 - 2x_1x_6 - 2x_2x_5)^2 \qquad , \qquad \sum_i x_i^2 = 1$$

To parametrize the Stiefel manifold $S_{3,2}$ choose x to be an arbitrary vector of the sphere $S^2$ , i. e. $x = (\cos\alpha \cos\beta \, , \cos\alpha \sin\beta \, , \sin\alpha)$ , and choose vectors u and v of $S^2$ that span the plane perpendicular to x , e.g. $u = (-\sin\beta \, , \cos\beta \, , 0)$ and $v = u \times x$ , thus $v = (\sin\alpha \cos\beta \, , \sin\alpha \sin\beta \, , -\cos\alpha)$ . For any vector $y \perp x$, $y = \cos\phi \, u + \sin\phi \, v$ , so $y = (-\sin\beta \cos\phi + \sin\alpha \cos\beta \sin\phi \, , \cos\beta \cos\phi + \sin\alpha \sin\beta \sin\phi \, , -\cos\alpha \sin\phi)$

Denote $r = \cos t$ and $s = \sin t$ . Then from (3.2.14) and the consideration above we have the following parametrization of $M^4$

$$
\begin{aligned}
z_1 &= r\cos\theta\cos\alpha\cos\beta - s\sin\theta\,(-\sin\beta\cos\phi + \sin\alpha\cos\beta\sin\phi) \\
z_2 &= r\cos\theta\cos\alpha\sin\beta - s\sin\theta\,(\cos\beta\cos\phi + \sin\alpha\sin\beta\sin\phi) \\
z_3 &= r\cos\theta\sin\alpha + s\sin\theta\cos\alpha\sin\phi \\
z_4 &= r\sin\theta\cos\alpha\cos\beta + s\cos\theta\,(-\sin\beta\cos\phi + \sin\alpha\cos\beta\sin\phi) \\
z_5 &= r\sin\theta\cos\alpha\sin\beta + s\cos\theta\,(\cos\beta\cos\phi + \sin\alpha\sin\beta\sin\phi) \\
z_6 &= r\sin\theta\sin\alpha - s\cos\theta\cos\alpha\sin\phi
\end{aligned} \tag{3.2.15}
$$

We differentiate z to get basis vector fields $\partial_1 = \frac{\partial}{\partial\theta}$ , $\partial_2 = \frac{\partial}{\partial\alpha}$ , $\partial_3 = \frac{\partial}{\partial\beta}$ , $\partial_4 = \frac{\partial}{\partial\phi}$ as follows

$$
\begin{aligned}
\frac{\partial}{\partial\theta} = \big( &-r\sin\theta\cos\alpha\cos\beta - s\cos\theta\,(-\sin\beta\cos\phi + \sin\alpha\cos\beta\sin\phi), \\
&-r\sin\theta\cos\alpha\sin\beta - s\cos\theta\,(\cos\beta\cos\phi + \sin\alpha\sin\beta\sin\phi), \\
&-r\sin\theta\sin\alpha + s\cos\theta\cos\alpha\sin\phi, \\
&r\cos\theta\cos\alpha\cos\beta - s\sin\theta\,(-\sin\beta\cos\phi + \sin\alpha\cos\beta\sin\phi), \\
&r\cos\theta\cos\alpha\sin\beta - s\sin\theta\,(\cos\beta\cos\phi + \sin\alpha\sin\beta\sin\phi), \\
&r\cos\theta\sin\alpha + s\sin\theta\cos\alpha\sin\phi \big).
\end{aligned} \tag{3.2.16}
$$

Note that $\frac{\partial}{\partial\theta} = (-z_4, -z_5, -z_6, z_1, z_2, z_3)$

$$
\begin{aligned}
\frac{\partial}{\partial\alpha} = \big( &-\cos\beta\,(r\cos\theta\sin\alpha + s\sin\theta\cos\alpha\sin\phi), \\
&-\sin\beta\,(r\cos\theta\sin\alpha + s\sin\theta\cos\alpha\sin\phi), \\
&r\cos\theta\cos\alpha - s\sin\theta\sin\alpha\sin\phi, \\
&\cos\beta\,(-r\sin\theta\sin\alpha + s\cos\theta\cos\alpha\sin\phi), \\
&\sin\beta\,(-r\sin\theta\sin\alpha + s\cos\theta\cos\alpha\sin\phi),
\end{aligned} \tag{3.2.17}
$$

$$r \sin\theta \cos\alpha \;-\; s \cos\theta \sin\alpha \sin\phi \;) \;.$$

$$\frac{\partial}{\partial\beta} = (\; \sin\beta\,(-r\cos\theta\cos\alpha + s\sin\theta\sin\alpha\sin\phi) + s\sin\theta\cos\beta\cos\phi\;,$$

$$-\cos\beta\,(-r\cos\theta\cos\alpha + s\sin\theta\sin\alpha\sin\phi) + s\sin\theta\sin\beta\cos\phi\;,\; 0\;,$$

(3.2.18)
$$-\sin\beta\,(\; r\sin\theta\cos\alpha + s\cos\theta\sin\alpha\sin\phi) - s\cos\theta\cos\beta\cos\phi\;,$$

$$\cos\beta\,(\; r\sin\theta\cos\alpha + s\cos\theta\sin\alpha\sin\phi) - s\cos\theta\sin\beta\cos\phi\;,\; 0\;)$$

$$\frac{\partial}{\partial\phi} = (\; -s\sin\theta\,(\sin\beta\sin\phi + \sin\alpha\cos\beta\cos\phi)\;,$$

$$-s\sin\theta\,(-\cos\beta\sin\phi + \sin\alpha\sin\beta\cos\phi)\;,$$

$$s\sin\theta\cos\alpha\cos\phi\;,$$

(3.2.19)
$$s\cos\theta\,(\sin\beta\sin\phi + \sin\alpha\cos\beta\cos\phi)\;,$$

$$s\cos\theta\,(-\cos\beta\sin\phi + \sin\alpha\sin\beta\cos\phi)\;,$$

$$-\; s\cos\theta\cos\alpha\cos\phi\;) \qquad .$$

We compute componenets of the metric tensor as $g_{ij} = \langle \partial_i, \partial_j \rangle$ to get the following matrix $G = (\, g_{ij}\,)$ of the metric tensor

$$G = \begin{pmatrix} 1 & 2rs\sin\phi & -2rs\cos\alpha\cos\phi & 0 \\ 2rs\sin\phi & r^2+s^2\sin^2\phi & -s^2\cos\alpha\cos\phi\sin\phi & 0 \\ -2rs\cos\alpha\cos\phi & -s^2\cos\alpha\cos\phi\sin\phi & s^2+\cos^2\alpha\,(r^2 - s^2\sin^2\phi) & -s^2\sin\alpha \\ 0 & 0 & -s^2\sin\alpha & s^2 \end{pmatrix}$$

The determinant of this matrix is computed to be $\det G = r^2s^2(1 - 4r^2s^2)\cos^2\alpha$ . We find the inverse matrix of G to be $G^{-1} = \frac{1}{\det G} B$ , i.e. $g^{ij} = \frac{1}{\det G} b^{ij}$, where $B = (b^{ij})$ is symmetric matrix with the following entries

$$b^{11} = r^2s^2\cos^2\alpha \ , \ b^{12} = b^{21} = -2r^3s^3\sin\phi\cos^2\alpha \ , b^{13} = b^{31} = 2r^3s^3\cos\alpha\cos\phi \ ,$$

$$b^{14} = b^{41} = 2r^3s^3\sin\alpha\cos\alpha\cos\phi \ , b^{22} = s^2\cos^2\alpha\,[r^2 + s^2(1 - 4r^2)\cos^2\phi] \ ,$$

$$b^{23} = b^{32} = s^4(1 - 4r^2)\cos\alpha\cos\phi\sin\phi \ , b^{24} = b^{42} = s^4(1 - 4r^2)\sin\alpha\cos\alpha\cos\phi\sin\phi \ ,$$

$$b^{33} = s^2[r^2 + s^2(1 - 4r^2)\sin^2\phi] \ , b^{34} = b^{43} = s^2\sin\alpha\,[r^2 + s^2(1 - 4r^2)\sin^2\phi] \ ,$$

$$b^{44} = s^2(1 - 4r^2s^2) + s^4(4r^2 - 1)\cos^2\phi\sin^2\alpha + (r^2 - s^2)(1 - 4r^2s^2)\cos^2\alpha \ .$$

Next we compute Christoffel's symbols. Nonzero ones are given as follows

$$\Gamma^1_{23} = \Gamma^1_{32} = -\frac{rs(r^2 - s^2)}{1 - 4r^2s^2}\sin\alpha\cos\phi \quad , \quad \Gamma^1_{24} = \Gamma^1_{42} = \frac{rs(r^2 - s^2)}{1 - 4r^2s^2}\cos\phi \ ,$$

$$\Gamma^1_{33} = -\frac{2rs(r^2 - s^2)}{1 - 4r^2s^2}\sin\alpha\cos\alpha\sin\phi \quad , \quad \Gamma^1_{34} = \Gamma^1_{43} = \frac{rs(r^2 - s^2)}{1 - 4r^2s^2}\cos\alpha\sin\phi \ ,$$

$$\Gamma^2_{13} = \Gamma^2_{31} = -\frac{s}{r}\sin\alpha\cos\phi \quad , \quad \Gamma^2_{14} = \Gamma^2_{41} = \frac{s}{r}\cos\phi \ ,$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{s^2(r^2 - s^2)}{1 - 4r^2s^2}\sin\alpha\sin\phi\cos\phi \ , \Gamma^2_{24} = \Gamma^2_{42} = -\frac{s^2(r^2 - s^2)}{1 - 4r^2s^2}\sin\phi\cos\phi \ ,$$

$$\Gamma^2_{33} = \sin\alpha\cos\alpha\left[1 + \frac{2s^2(r^2 - s^2)}{1 - 4r^2s^2}\sin^2\phi\right] \ , \ \Gamma^2_{34} = \Gamma^2_{43} = -\frac{s^2(r^2 - s^2)}{1 - 4r^2s^2}\sin^2\phi\cos\alpha \ ,$$

$$\Gamma^3_{13} = \Gamma^3_{31} = -\frac{s}{r}\tan\alpha\sin\phi \ , \ \Gamma^3_{14} = \Gamma^3_{41} = \frac{s}{r}\frac{\sin\phi}{\cos\alpha} \ ,$$

$$\Gamma^3_{23} = \Gamma^3_{32} = -\tan\alpha\left[1 + \frac{s^2(r^2 - s^2)}{1 - 4r^2s^2}\cos^2\phi\right] \ , \ \Gamma^3_{24} = \Gamma^3_{42} = \frac{s^2(r^2 - s^2)}{1 - 4r^2s^2}\frac{\cos^2\phi}{\cos\alpha} \ ,$$

$$\Gamma^3_{33} = -\frac{2s^2(r^2-s^2)}{1-4r^2s^2}\sin\alpha\sin\phi\cos\phi \quad , \quad \Gamma^3_{34} = \Gamma^3_{43} = \frac{s^2(r^2-s^2)}{1-4r^2s^2}\sin\phi\cos\phi \ ,$$

$$\Gamma^4_{12} = \Gamma^4_{21} = -\frac{r}{s}\cos\phi \ , \ \Gamma^4_{13} = \Gamma^4_{31} = -\frac{\sin\phi}{rs\cos\alpha}[\, s^2 + (r^2-s^2)\cos^2\alpha] \ ,$$

$$\Gamma^4_{14} = \Gamma^4_{41} = \frac{s}{r}\tan\alpha\sin\phi \ , \ \Gamma^4_{22} = -\sin\phi\cos\phi \, ,$$

$$\Gamma^4_{23} = \Gamma^4_{32} = -\frac{1}{\cos\alpha}[\, \sin^2\phi + \frac{r^2(r^2-s^2)}{1-4r^2s^2}\sin^2\alpha\cos^2\phi\,] \, ,$$

$$\Gamma^4_{24} = \Gamma^4_{42} = \frac{s^2(r^2-s^2)}{1-4r^2s^2}\tan\alpha\cos^2\phi \, , \ \Gamma^4_{33} = \sin\phi\cos\phi\,[\, 1 - \frac{r^2-s^2}{1-4r^2s^2}\cos^2\alpha] \ ,$$

$$\Gamma^4_{34} = \Gamma^4_{43} = \frac{s^2(r^2-s^2)}{1-4r^2s^2}\sin\alpha\sin\phi\cos\phi \ .$$

We want to find the shape operator A of the hypersurface and the basis of principal directions . But first we need to find the unit normal direction $\xi$ . It turns out that $\xi$ is obtained by differentiating z with respect to t , i.e. take $\xi = -\frac{\partial}{\partial t}$ . So we get

$$\begin{aligned} \xi = (\ & \cos\beta\,(s\cos\theta\cos\alpha + r\sin\theta\sin\alpha\sin\phi) - r\sin\theta\sin\beta\cos\phi \ , \\ & \sin\beta\,(s\cos\theta\cos\alpha + r\sin\theta\sin\alpha\sin\phi) + r\sin\theta\cos\beta\cos\phi \ , \\ & s\cos\theta\sin\alpha - r\sin\theta\cos\alpha\sin\phi \ , \\ & \cos\beta\,(s\sin\theta\cos\alpha - r\cos\theta\sin\alpha\sin\phi) + r\cos\theta\sin\beta\cos\phi \ , \\ & \sin\beta\,(s\sin\theta\cos\alpha - r\cos\theta\sin\alpha\sin\phi) - r\cos\theta\cos\beta\cos\phi \ , \\ & s\sin\theta\sin\alpha + r\cos\theta\cos\alpha\sin\phi \ ) \ . \end{aligned} \tag{3.2.20}$$

For every $i, j = 1,2,3,4$ we can compute $< A(\partial_i), \partial_j > = -< \tilde{\nabla}_{\partial_i}\xi, \partial_j >$ and find the matrix of A in the basis $\{\partial_i\}$ . We get

$$A(\partial_1) = \frac{r^2 - s^2}{1 - 4r^2s^2}\left\{-2\,rs\,\partial_1 + \sin\phi\,\partial_2 - \frac{\cos\phi}{\cos\alpha}\,\partial_3 - \tan\alpha\,\cos\phi\,\partial_4\right\},$$

$$A(\partial_2) = \frac{r^2 - s^2}{1 - 4r^2s^2}\left\{\sin\phi\,\partial_1 + \frac{s}{r}(\cos^2\phi - 2r^2)\partial_2 + \frac{s}{r}\,\frac{\sin\phi\,\cos\phi}{\cos\alpha}\partial_3 + \frac{s}{r}\tan\alpha\,\sin\phi\,\cos\phi\,\partial_4\right\},$$

$$A(\partial_3) = \frac{r^2 - s^2}{1 - 4r^2s^2}\left\{-\cos\alpha\,\cos\phi\,\partial_1 + \frac{s}{r}\cos\alpha\,\sin\phi\,\cos\phi\,\partial_2 + \frac{s}{r}(\sin^2\phi - 2r^2)\partial_3 - \frac{1}{rs}\sin\alpha(r^2 - s^2\sin^2\phi)\,\partial_4\right\},$$

$$A(\partial_4) = \frac{r}{s}\,\partial_4 \;.$$

Note that even though A is symmetric operator, the matrix of A in this basis is not symmetric since $\{\partial_i\}$ is not an orthonormal basis . Minimal hypersurface in the family (3.2.14) is obtained when $t = \pi/8$. In that case $r = \cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$ , $s = \sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$ . Principal curvatures of minimal $M^4$ are given as follows :

$$k_1 = \sqrt{2} + 1 \;\;,\;\; k_2 = -\sqrt{2} - 1 \;\;,\;\; k_3 = \sqrt{2} - 1 \;\;,\;\; k_4 = 1 - \sqrt{2}$$

That follows from the Cartan's identity (1.5.1) or Münzner's Theorem 1.5.2 . Next we find the orthonormal basis of principal directions by diagonalizing matrix of A in the basis $\{\partial_i\}$. We get the following principal directions corresponding respectively to the curvatures $k_1, k_2, k_3, k_4$.

$$e_1 = \sqrt{4+2\sqrt{2}}\;\frac{\partial}{\partial\phi}\;,$$

$$e_2 = \frac{\sqrt{4+2\sqrt{2}}}{2}\left\{\frac{\partial}{\partial\theta} - \sin\phi\frac{\partial}{\partial\alpha} + \frac{\cos\phi}{\cos\alpha}\frac{\partial}{\partial\beta} + \tan\alpha\cos\phi\frac{\partial}{\partial\phi}\right\},$$

$$e_3 = \frac{\sqrt{4-2\sqrt{2}}}{2}\left\{-\frac{\partial}{\partial\theta} - \sin\phi\frac{\partial}{\partial\alpha} + \frac{\cos\phi}{\cos\alpha}\frac{\partial}{\partial\beta} + \tan\alpha\cos\phi\frac{\partial}{\partial\phi}\right\},$$

$$e_4 = \sqrt{4-2\sqrt{2}}\left\{\cos\phi\frac{\partial}{\partial\alpha} + \frac{\sin\phi}{\cos\alpha}\frac{\partial}{\partial\beta} + \tan\alpha\sin\phi\frac{\partial}{\partial\phi}\right\}.$$

To check if $M^4$ is of 3 - type via $\tilde{x}$ or not we find connection coefficients with respect to the basis $\{e_i\}$. For example, we compute

$$\omega_1^2(e_3) = <\nabla_{e_3}e_1 , e_2> = 0 , \ \omega_1^2(e_4) = \sqrt{2-\sqrt{2}} , \ldots \text{ etc} .$$

But combining the equations of Gauss, Codazzi and condition (3) of Theorem 3.2.1 below it follows that in order that $M^4$ be mass - symmetric and of 3 - type via $\tilde{x}$ we must have

$$[\,\omega_1^2(e_3)]^2 = [\,\omega_1^2(e_4)]^2 = \frac{2-\sqrt{2}}{2} \quad , \text{ and } [\,\omega_3^4(e_1)]^2 = [\,\omega_3^4(e_2)]^2 = \frac{2+\sqrt{2}}{2} \quad .$$

Therefore, $M^4$ is not mass - symmetric and of 3 - type via $\tilde{x}$ .

Now we prove the following characterization of minimal spherical hypersurfaces which are mass - symmetric and of 3 - type via $\tilde{x}$ .

**Theorem 3.2.1** Let $x : M^n \to S^{n+1}$ be an isometric immersion of a compact manifold $M^n$ as a minimal hypersurface of $S^{n+1}$ . If $\tilde{x}$ is mass - symmetric and of 3 - type then

(1) $\operatorname{tr} A = \operatorname{tr} A^3 = 0$ ,

(2) $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$ are constant ,

(3) $\operatorname{tr} (\nabla_X A)^2 = <A^2X, A^2X> + p<AX, AX> + q<X, X>$ , $X \in TM$

where p and q are constants (depending on the order of M , $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$ ) .

Conversely, if (1), (2) and (3) hold then M is mass - symmetric and of 1 - , 2 - , or 3 - type via $\tilde{x}$ .

**Proof.** Suppose that $\tilde{x}$ is mass - symmetric and of 3 - type so that (3.2.9) holds. As before, from $x\xi^t + \xi x^t$ component of (3.2.9) we get $\operatorname{tr} A^3 = 0$ , and $xx^t$ and $\xi\xi^t$ components give respectively

$$8[\, n(n+1)^2 + \operatorname{tr} A^2\,] + 4an\,(n+1) + 2\,bn + c\,\frac{n+1}{n+2} = 0\,, \tag{3.2.21}$$

$$8\,[\,(\operatorname{tr} A^2)^2 + (n+1)(\operatorname{tr} A^2) + \operatorname{tr} A^4] + 4a\,(\operatorname{tr} A^2) - c\,\frac{1}{n+2} = 0\,, \tag{3.2.22}$$

Obviously $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$ are constant, and (3.2.8) simplifies to

$$\begin{aligned}\Delta^3 \tilde{x} \;=\; & 8[n(n+1)^2 + \operatorname{tr} A^2\,]\,xx^t + 8\,[\,(\operatorname{tr} A^2)^2 + (n+1)(\operatorname{tr} A^2) + \operatorname{tr} A^4]\xi\xi^t \\ & - 8\,(n+1)^2 \sum_i e_i e_i^t - 16\,(1 + \operatorname{tr} A^2) \sum_i (Ae_i)(Ae_i)^t \\ & - 8 \sum_i (A^2 e_i)(A^2 e_i)^t + 8 \sum_{i,k} [(\nabla_{e_k} A)e_i][(\nabla_{e_k} A)e_i]^t \quad .\end{aligned} \tag{3.2.23}$$

We readily observe that $\Delta^3 \tilde{x}$ , $\Delta^2 \tilde{x}$, $\Delta \tilde{x}$ , I are all normal to hypersphere $S^{n+1}$ ( follows e.g. from the proof of Lemma 1.5.1 ). Next, we find $X\,Y^t + Y\,X^t$ component of (3.2.9) for arbitrary pair X , Y of vector fields on M. Observe first that

$$\begin{aligned}\sum_{i,k} g((\nabla_{e_k} A)e_i, X)\; g((\nabla_{e_k} A)e_i, Y) &= \sum_{i,k} g(e_i, (\nabla_{e_k} A)X)\; g(e_i, (\nabla_{e_k} A)Y) \\ &= \sum_k g((\nabla_X A)e_k\,, (\nabla_Y A)e_k) \\ &= \operatorname{tr}\,(\nabla_X A)\circ(\nabla_Y A)\end{aligned}$$

by the Codazzi equation and symmetry of the operator $\nabla_{e_k} A$ .

Now applying $\tilde{g}\,( - \ , XY^t + YX^t )$ to (3.2.9) and taking into account (3.2.1), (3.2.4) and (3.2.23) we get

$$- \ 8(n+1)^2 < X, Y > - \ 16\,(1 + \operatorname{tr} A^2) < AX, AY > \ - \ 8 < A^2X, A^2Y >$$

$$+ \ 8 \operatorname{tr} (\nabla_X A)\circ(\nabla_Y A) - \ 4a < AX, AY > \ - \ 4a\,(n+1) < X, Y >$$

$$- \ 2b < X, Y > \ - c\,\frac{1}{n+2} < X, Y > \ = \ 0 \ , \quad \text{from where}$$

(3.2.24) $$\operatorname{tr} (\nabla_X A)\circ(\nabla_Y A) \ = \ < A^2X, A^2Y > + \ p < AX, AY > + \ q < X, Y > ,$$

where p and q are constants given by

(3.2.25) $$p \ = \ \frac{a}{2} \ + \ 2(1 + \operatorname{tr} A^2)$$

(3.2.26) $$q \ = \ (n+1)^2 + \ \frac{a}{2}(n+1) \ + \ \frac{b}{4} \ + \ \frac{c}{8(n+2)} \ .$$

It is easy to see that (3.2.24) is equivalent, by linearization, to

(3.2.27) $$\operatorname{tr} (\nabla_X A)^2 = < A^2X, A^2X > + \ p < AX, AX > + \ q < X, X > \ ,$$

for any $X \in TM$. Therefore, we proved necessity of the conditions (1), (2), (3). Conversely, given (1), (2) and (3), we have to show that we can find constants a, b and c so that (3.2.9) holds. That boils down to solving the system of the following four equations (3.2.21), (3.2.22), (3.2.25) and (3.2.26) for a, b, c. This system of four linear equations in three unknowns can be uniquely solved if the eliminate is zero, i.e.

(3.2.28) $$\operatorname{tr} A^4 \ + \ p \operatorname{tr} A^2 \ + \ qn \ + \ (\,n - \operatorname{tr} A^2) \operatorname{tr} A^2 = 0 \ .$$

But this formula is always satisfied under our conditions (1) - (3), by virtue of

$$0 \ = \ \frac{1}{2}\,\Delta(\operatorname{tr} A^2\,) \ = \ \operatorname{tr} (\Delta A)A \ - \ \|\nabla A\|^2$$

( cf.[N-S], p. 369) . Therefore $P(\Delta)(\tilde{x} - \tilde{x}_0) = 0$, where $P(t) = t^3 + a t^2 + bt + c$. Note that M need not be exactly of 3 - type, i.e. can be of 1 - or 2 - type, for example if there is a factor P' of P of degree 1 or 2 so that $P'(\Delta)(\tilde{x} - \tilde{x}_0) = 0$. ♦

We now prove the following characterization of the Cartan hypersurface .

**Theorem 3.2.2.** Let $x : M^n \to S^{n+1}$ be a compact minimal hypersurface of $S^{n+1}$ of dimension $n \le 5$ . Then $\tilde{x}$ is mass - symmetric and of 3 - type if and only if $n = 3$ and $M^3 = SO(3)/Z_2 \times Z_2$ is the Cartan hypersurface .

**Proof.** From the Example 3.2.1. we know that the Cartan hypersurface is mass - symmetric and of 3 - type via $\tilde{x}$ . Conversely, suppose that $\tilde{x}$ is mass - symmetric and of 3 - type. We will show that $M^n$ is necessarily isoparametric. From the computation carried out before that is already clear for $n \le 4$ . If we compute $\Delta(\mathrm{tr}\, A^m)$ we obtain

$$\Delta(\mathrm{tr}\, A^m) = m\,(\mathrm{tr}\, A^2 - n)(\mathrm{tr}\, A^m) - \sum_i \sum_{j\neq k} \mathrm{tr}(A \circ \ldots \circ A \circ \overset{j}{\nabla_{e_i} A} \circ A \circ \ldots \circ A \circ \overset{k}{\nabla_{e_i} A} \circ \ldots \circ A) \quad (3.2.29)$$

In particular, for m = 3 we have

$$\Delta(\mathrm{tr}\, A^3) = 3\,(\mathrm{tr}\, A^2 - n)(\mathrm{tr}\, A^3) - 6 \sum_i \mathrm{tr}\,[(\nabla_{e_i} A)^2 \circ A] \quad . \quad (3.2.30)$$

Since $\mathrm{tr}\, A^3 = 0$ by Theorem 3.2.1 , we will have ( $\{e_i\}$ is chosen to be the basis of principal directions )

$$0 = \sum_i \mathrm{tr}\,[(\nabla_{e_i} A)^2 \circ A]$$

$$= \sum_{i,k} g((\nabla_{e_i}A)^2 A e_k, e_k)$$

$$= \sum_{i,k} g((\nabla_{e_i}A)^2(\lambda_k e_k), e_k)$$

$$= \sum_{i,k} g((\nabla_{e_i}A)(\lambda_k e_k), (\nabla_{e_i}A)e_k) \quad , \quad \text{since } \nabla_{e_i}A \text{ is symmetric}$$

$$= \sum_{i,k} \lambda_k\, g((\nabla_{e_i}A)e_k, (\nabla_{e_i}A)e_k) \quad , \quad \text{since } \nabla_{e_i}A \text{ is a tensor}$$

$$= \sum_{i,k} \lambda_k\, g((\nabla_{e_k}A)e_i, (\nabla_{e_k}A)e_i) \quad , \quad \text{by Codazzi equation}$$

$$= \sum_{k} \lambda_k \operatorname{tr} (\nabla_{e_k}A)^2$$

$$= \sum_{k} \lambda_k ( \lambda_k^4 + p\lambda_k^2 + q ) \quad , \text{ by condition (3) of the Theorem 3.2.1}$$

$$= \operatorname{tr} A^5 + p \operatorname{tr} A^3 + q \operatorname{tr} A$$

$$= \operatorname{tr} A^5 \quad .$$

Therefore, conditions (1) - (3) of the Theorem 3.2.1 imply also $\operatorname{tr} A^5 = 0$ . We conclude that for $n \leq 5$ the hypersurface M has to be isoparametric. If M has only one curvature it has to be umbilical in $S^{n+1}$ and therefore (since it is minimal) great hypersphere which is of 1 - type via $\tilde{x}$ . If M has two distinct principal curvatures and is minimal it must be Clifford minimal hypersurface $M = M_{p,n-p} = S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}})$ ([C 1], pp 87, 97 ). But the product of spheres that satisfies the conditions of our Theorem 3.2.1 must be of 2 - type as can be seen from the following argument.

Suppose $\lambda_1$ and $\lambda_2$ are the two principal curvatures of multiplicities $m_1$ and $m_2$ respectively . Then $\operatorname{tr}A = \operatorname{tr} A^3 = 0$ imply $m_1\lambda_1 + m_2\lambda_2 = m_1\lambda_1^3 + m_2\lambda_2^3 = 0$ . Also, we have $1 + \lambda_1\lambda_2 = 0$ (by e.g. (1.4.1) ). Using this to eliminate $m_1$, $m_2$ and $\lambda_2$ we obtain $\lambda_1^2 = \lambda_1^6 = q/p$ . Thus, $p = q = n/2$ , $\lambda_1 = \pm 1$ and $\lambda_2 = \mp 1$ . So , $n = p + q$ has to be even, $p = n - p$ , $p/n = 1/2$ and $S^p(\sqrt{\frac{p}{n}}) \times S^{n-p}(\sqrt{\frac{n-p}{n}}) = S^p(\sqrt{\frac{1}{2}}) \times S^p(\sqrt{\frac{1}{2}})$ . This hypersurface is mass - symetric and of 2 - type by Lemma 3 (case II) of [B-C] for $n = 2p$ . If M has three curvatures, then according to the classification of Cartan M is the Cartan hypersurface which indeed is mass - symmetric and of 3 - type via $\tilde{x}$ . If M has four principal curvatures, then the result of Takagi [T 3] classifies such hypersurface as the one considered in Example 3.2.2 which is not of 3 - type via $\tilde{x}$ . Finaly, M cannot have five principal curvatures by the result of Münzner (Theorem 1.4.3). This completes the proof of the theorem. ♦

**Remark.** The proof above does not a priori exclude the case $n = 1$ . Actually, if $n = 1$ , there are no minimal curves in $S^2$ which are of 3 - type in SM(3) via $x^\sim$ because such a curve is automatically a great circle of $S^2$ (totally geodesic), and therefore of 1 - type via $\tilde{x}$ . Namely, if $x : C \to S^2$ is a minimal curve parametrized by the arclength s, we have $\Delta x = x$ , i.e. $x'' + x = 0$ and hence $x(s) = a \sin s + b \cos s$ , $a, b \in E^3$ . From $\langle x, x \rangle = 1$ we get $|a| = |b| = 1$ and $\langle a, b \rangle = 0$ . A spherical curve C with these properties is the great circle lying in the plane perpendicular to the (constant) vector $x' \times x = a \times b$ .

Theorem 3.2.2 gives a new characterization of the Cartan hypersurface in terms of the spectrum of its Laplacian . For other characterizations see [P-T], [T 1], [Ki-Na] .

In dimensions greater than 5 there are other examples of spherical hypersurfaces which are of 3 - type and mass - symmetric via $\tilde{x}$ . In fact every minimal isoparametric spherical hypersurface with exactly three different principal curvatures is of 3 - type (see below). It would be interesting to decide if any spherical hypersurface which is of 3 - type and mass - symmetric via $\tilde{x}$ is necessarily isoparametric.

**Lemma 3.2.2.** If $M^n \subset S^{n+1}$ is a compact minimal isoparametric hypersurface which is mass - symmetric and of 3 - type via $\tilde{x}$ then $M^n$ is necessarily homogeneous with $\nu = 3$ or $4$ distinct principal curvatures.

**Proof.** First, we saw before, from the proof of Theorem 3.2.2, that if $\nu = 1$ or $2$ then $\tilde{x}$ is not of 3 - type . If there are six distinct principal curvatures, then by Theorem 1.4.2 the curvatures $k_i$ have the same multiplicities and they are given as

$$\cot\theta \ , \ \cot(\theta + \tfrac{\pi}{6}) \ , \ \cot(\theta + \tfrac{\pi}{3}) \ , \ \cot(\theta + \tfrac{\pi}{2}) \ , \ \cot(\theta + \tfrac{2\pi}{3}) \ , \ \cot(\theta + \tfrac{5\pi}{6}) \ .$$

From minimality condition we obtain $\theta = \frac{\pi}{12}$ and then find curvatures to be (in descending order)

$$2+\sqrt{3} \ , \ 1 \ , \ 2-\sqrt{3} \ , \ -(2-\sqrt{3}) \ , \ -1 \ , \ -(2+\sqrt{3}) \ .$$

We see that these hypersurfaces satisfy conditions (1) and (2) of Theorem 3.2.1 and to determine if they are of 3 - type and mass - symmetric via $\tilde{x}$ one needs to check the condition (3) . It is likely (but still not known) that all isoparametric spherical hypersurfaces with six curvatures are homogeneous. That is proved when $m = 1$ ( [D-N 1] ), classifying such hypersurface as $G_2/SO(4)$ , but not yet for $m = 2$ . If $\nu = 4$ , then

$$k_1 = \cot\theta \ , \ k_2 = \cot(\theta + \tfrac{\pi}{4}) \ , \ k_3 = \cot(\theta + \tfrac{\pi}{2}) \ , \ k_4 = \cot(\theta + \tfrac{3\pi}{4}) \ ,$$

and there are at most two different multiplicities $m_1$ ( of $k_1$ and $k_3$ ) and $m_2$ ( of $k_2$ and $k_4$ ). Then from $\operatorname{tr} A = 0$ and $\operatorname{tr} A^3 = 0$ we get respectively

$$m_1 \frac{\cos 2\theta}{\sin 2\theta} - m_2 \frac{\sin 2\theta}{\cos 2\theta} = 0 \ , \quad \text{i.e.} \quad \tan^2 2\theta = \frac{m_1}{m_2} \ ,$$

$$m_1 \frac{\cos 2\theta\,(4 - \sin^2 2\theta)}{\sin^3 2\theta} - m_2 \frac{\sin 2\theta\,(3 + \sin^2 2\theta)}{\cos^3 2\theta} = 0 \text{ , from where}$$

$\frac{m_1}{m_2} = \tan^4 2\theta \, \frac{3 + \sin^2 2\theta}{4 - \sin^2 2\theta}$ . Let $r = \frac{m_1}{m_2}$ . Then from these two equations we get

$r = r^2 \frac{3 + \sin^2 2\theta}{4 - \sin^2 2\theta}$ , which implies $\sin^2 2\theta = \frac{4 - 3r}{r + 1}$ hence $r = \tan^2 2\theta = \frac{4 - 3r}{4r - 3}$ .

From the last relation we have $r = 1$, i.e. $m_1 = m_2$ so multiplicities of all four curvatures are equal. We also get $\theta = \frac{\pi}{8}$, and four curvatures to be $k_1 = \sqrt{2} + 1$ , $k_2 = \sqrt{2} - 1$ , $k_3 = 1 - \sqrt{2}$ , $k_4 = -\sqrt{2} - 1$ . Therefore, as argued in Sect.4 of Ch.1, Theorem 1.4.4 of Abresch implies that the common multiplicity of curvatures is 1 or 2 . If the common multiplicity is 1 than $M^4$ has to be the hypersurface considered in Example 3.2.2 which is not of 3 - type via $\tilde{x}$ . If the common multiplicity is 2 , then $M^8$ is minimal homogeneous hypersurface in $S^9$ of type $Sp(2)/T^2$ . In the next lemma we show that all minimal isoparametric spherical hypersurfaces with $\nu = 3$ are indeed of 3 - type via $\tilde{x}$ . ♦

**Lemma 3.2.3.** If $M^n \subset S^{n+1}$ is a compact minimal isoparametric spherical hypersurface with exactly three distinct principal curvatures then $M^n$ is mass - symmetric and of 3 - type via $\tilde{x}$ .

**Proof.** From (1.1.5) and the Gauss equation (1.1.27) we obtain the following for principal directions $e_i$, $e_k$ and corresponding curvatures $\lambda_i$, $\lambda_k$ $(i \neq k)$

$$\begin{aligned} R(e_i, e_k, e_k, e_i) = 1 + \lambda_i \lambda_k = \; & e_i(\omega_k^i(e_k)) - e_k(\omega_k^i(e_i)) \\ & + \sum_j \omega_k^j(e_k)\omega_j^i(e_i) - \sum_j \omega_k^j(e_i)\omega_j^i(e_k) \\ & - \sum_j \omega_k^j(e_i)\omega_k^i(e_j) + \sum_j \omega_i^j(e_k)\omega_k^i(e_j) \quad . \end{aligned} \tag{3.2.31}$$

For an isoparametric hypersurface, the Codazzi equation $(\nabla_{e_i}A)e_k = (\nabla_{e_k}A)e_i$ is equivalent to the following

(3.2.32) $$(\lambda_k - \lambda_j)\,\omega_k^j(e_i) = (\lambda_i - \lambda_j)\,\omega_i^j(e_k) \quad , \text{ for every } i, j, k \quad ,$$

and hence

(3.2.33) $$\omega_i^j(e_k) = 0 \quad , \qquad \text{for} \quad \lambda_k = \lambda_j \neq \lambda_i \; .$$

Therefore, if $\lambda_i \neq \lambda_k$ formula (3.2.31) reduces to

(3.2.34) $$1 + \lambda_i \lambda_k = - \sum_j \omega_k^j(e_i)\omega_j^i(e_k) - \sum_j \omega_k^j(e_i)\omega_k^i(e_j) + \sum_j \omega_i^j(e_k)\omega_k^i(e_j) \; .$$

All four minimal isoparametric spherical hypersurfaces with three distinct principal curvatures have curvatures equal to $-\sqrt{3}$ , 0 and $\sqrt{3}$ , and the common multiplicity m satisfies $m \in \{1, 2, 4, 8\}$ , so that $\mathrm{tr}\, A = \mathrm{tr}\, A^3 = 0$ . In order to prove that these hypersurfaces are of 3 - type and mass - symmetric it is enough to check condition (3) of Theorem 3.2.1 , which can be also written as

(3.2.35) $$\mathrm{tr}\,(\nabla_{e_i}A)^2 = \lambda_i^4 + p\,\lambda_i^2 + q \qquad ,$$

where $e_i$ is a principal direction , $\lambda_i$ corresponding principal curvature, and p and q constants. We transform $\mathrm{tr}\,(\nabla_{e_i}A)^2$ as

(3.2.36) $$\mathrm{tr}\,(\nabla_{e_i}A)^2 = \sum_{k,j} (\lambda_k - \lambda_j)^2[\omega_k^j(e_i)]^2 \quad .$$

Let $e_1, \dots, e_m$ be the set of principal directions that correspond to $-\sqrt{3}$ eigenvalue, $e_{m+1}, \dots, e_{2m}$ the set of principal directions that correspond to 0 eigenvalue, and

$e_{2m+1}, \dots, e_{3m}$ those corresponding to $\sqrt{3}$ eigenvalue. We use the boldface type to denote the following set of indices

$$\mathbf{1} = \{ 1, \dots , m \} \;, \quad \mathbf{2} = \{ m+1, \dots , 2m \} \quad \text{and} \quad \mathbf{3} = \{ 2m+1, \dots , 3m \} \;.$$

Let $i \in \mathbf{1}$, $k \in \mathbf{2}$ be any two indices so that $e_i, e_k$ are two principal directions corresponding to the curvatures $-\sqrt{3}$, 0 respectively . Then from (3.2.34) using (3.2.33) we obtain

$$(3.2.37) \quad 1 = 1 + \lambda_i \lambda_k = - \sum_{j\in 3} \omega_k^j(e_i)\omega_j^i(e_k) - \sum_{j\in 3} \omega_k^j(e_i)\omega_k^i(e_j) + \sum_{j\in 3} \omega_i^j(e_k)\omega_k^i(e_j) \;.$$

From Codazzi equation (3.2.32) we get

$$\sqrt{3}\;\omega_j^k(e_i) = -\sqrt{3}\;\omega_i^k(e_j) \quad , \quad 2\sqrt{3}\,\omega_j^i(e_k) = \sqrt{3}\,\omega_k^i(e_j) \quad , \quad \text{so that}$$

$$(3.2.38) \qquad \omega_k^j(e_i) = \omega_i^k(e_j) \quad , \quad \omega_i^j(e_k) = \frac{1}{2}\,\omega_i^k(e_j) \;.$$

Now in (3.2.37) we express everything in terms of $\omega_i^k(e_j)$ using (3.2.38) and simplifying to get

$$(3.2.39) \qquad \sum_{j\in 3} [\omega_i^k(e_j)]^2 = 1 \quad , \quad \text{for every } i \in \mathbf{1},\; k \in \mathbf{2} \;.$$

By a similar computation, using expressions for $1 + \lambda_i \lambda_k$, where $i \in \mathbf{2}$, $k \in \mathbf{3}$ and, respectively, $i \in \mathbf{1}$, $k \in \mathbf{3}$, we obtain

$$(3.2.40) \qquad \sum_{j\in 1} [\omega_i^k(e_j)]^2 = 1 \quad , \quad \text{for every } i \in \mathbf{2},\; k \in \mathbf{3}$$

$$(3.2.41) \qquad \sum_{j\in 2} [\omega_i^k(e_j)]^2 = \frac{1}{4} \quad , \quad \text{for every } i \in \mathbf{1},\; k \in \mathbf{3} \;.$$

Next, we compute $\operatorname{tr}(\nabla_{e_i}A)^2$ from (3.2.36) to get for $i \in \mathbf{1}$ :

$$\operatorname{tr}(\nabla_{e_i}A)^2 = 3 \sum_{\substack{j\in \mathbf{2}\\ k\in \mathbf{3}}} [\omega_i^k(e_j)]^2 + 12 \sum_{\substack{j\in \mathbf{3}\\ k\in \mathbf{2}}} [\omega_i^k(e_j)]^2$$

$$= 3\,m + 12\,\frac{m}{4} = 6\,m \ .$$

Similar computation can be carried out for $i \in \mathbf{2}$ and $i \in \mathbf{3}$, yielding the same result, so

(3.2.42) $$\operatorname{tr}(\nabla_{e_i}A)^2 = 6\,m \quad , \text{ for every } \ i = 1, \dots, 3m \ .$$

Therefore, we see that (3.2.35) is satisfied with $p = -3$ and $q = 6m$. We conclude that all minimal isoparametric spherical hypersurfaces with three curvatures are of 3 - type via $\tilde{x}$ . As a matter of fact we can show that

$$\Delta^3 \tilde{x} + a\,\Delta^2 \tilde{x} + b\,\Delta\,\tilde{x} + c\,(\tilde{x} - \frac{I}{3m+2}) = 0$$

is satisfied for $a = -(10 + 24m)$ , $b = 4\,[(3m + 1)(15m + 6) - 2\,]$ and $c = -48m\,(3m + 1)(3m + 2)$ , so that the three eigenvalues of the Laplacian arising from the decomposition $\tilde{x} = \tilde{x}_0 + \tilde{x}_p + \tilde{x}_q + \tilde{x}_r$ are $\lambda_p = 6m$ , $\lambda_q = 2(3m + 1)$ and $\lambda_r = 4\,(3m + 2)$ . ♦

**Remark.** From the above we know three eigenvalues of any isoparametric spherical hypersurface with three curvatures. Even though the spectrum of the Cartan hypersurface is known ( [M-O-U] ), not much information is available about eigenvalues of the Laplacian for other isoparametric hypersurfaces with three curvatures. Also, it is known that any minimal spherical hypersurface with $\operatorname{tr} A^2 = \text{const}$ has $n$, $\operatorname{tr} A^2$ and $n + \operatorname{tr} A^2$ as three eigenvalues of the Laplacian ( cf. [M-O-U] ). See also [Ko], [Mu] .

In order to chech which minimal isoparametrtic spherical hypersurfaces (or at least homogeneous ones) with four or six principal curvatures are mass - symmetric and of 3 - type via $\tilde{x}$ , one has to check the condition (3) of Theorem 3.2.1 . That can be done (for homogeneous ones) by the methods of [T-Ta], considering the action of the Lie group $K = Sp(2)$ or $G_2$ on the Euclidean space $\boldsymbol{m}$ arising from the Cartan decomposition $\boldsymbol{g} = \boldsymbol{k} + \boldsymbol{m}$ of the corresponding orthogonal symmetric Lie algebra $(\boldsymbol{g}, \boldsymbol{k}, \sigma)$, but the computations involved are rather long. First, one has to choose a point $P \in \boldsymbol{a}$ ( a 2 - dimensional abelian subspace of $\boldsymbol{m}$ ) so that the orbit of P under the adjoint action of K is minimal in sphere. That requires some manipulation with the roots of the Lie algebra determined by $\boldsymbol{a}$ . Second, one needs to find the principal directions for the shape operator and compute the connection coefficients. The shape operator of an orbit hypersurface is given by $AX = -[Y, \xi]$ , where $\xi$ is the unit normal to the hypersurface in sphere ( $\xi$ is perpendicular to P , and $\xi$ and P span $\boldsymbol{a}$ ) , and $Y \in \boldsymbol{k}$ is a vector such that $X = Y_P^* = [Y, P]$ ( cf. [T-Ta]) .

Also, it would be important to resolve if any minimal spherical hypersurface which is of 3 - type and mass - symmetric via $\tilde{x}$ is necessarily isoparametric. Techniques used in this chapter can be modified to study hypersurfaces of a projective space which are of low type via the first standard embedding of a projective space.

# CHAPTER 4

# SUBMANIFOLDS OF $E^m$ WITH HARMONIC MEAN CURVATURE VECTOR

In this chapter we discuss certain aspects of the following problem proposed by B.Y. Chen [C 7] .

**Problem:** Classify or characterize submanifolds $x : M^n \to E^m$ which satisfy

(4.1) $$\Delta H = 0 ,$$

where, as usual, H denotes the mean curvature vector of the immersion and $\Delta$, the Laplacian of M acting on smooth functions, naturally extended to act on $E^m$ - valued maps .

Obviously, every minimal submanifold in $E^m$ satisfies $\Delta H = 0$ , so the real problem is if there are other submanifolds, besides minimal, that satisfy this equation. In view of the formula $\Delta x = - n H$, the equation (4.1) becomes $\Delta^2 x = 0$ , that is, we want to study immersions which are biharmonic (but not harmonic).

The well known theorem of Takahashi (Theorem 1.5.1) asserts that if $\Delta x = \lambda x$ then M is minimal in $E^m$ if $\lambda = 0$ , or M is minimal in hypersphere of $E^m$ centered at the origin, if $\lambda > 0$ ( $\lambda < 0$ cannot occur here ). An analogous problem to this would be to consider the equation $\Delta H = \lambda H$ and see what it implies for submanifold M. In particular, for $\lambda = 0$ we have the problem above. If M is compact , $\Delta H = \lambda H$ implies $\Delta(\Delta x - \lambda x) = 0$ , so we get $\Delta x - \lambda x = c = const$ . Further, if $\lambda = 0$ , by integrating we have $c = 0$ , and therefore $\Delta x = 0$ , which means that the immersion M is minimal. But it is well known that there are no compact minimal submanifolds of $E^m$. In case $\lambda \neq 0$, we get $\Delta(x + c/\lambda) = \lambda(x + c/\lambda)$, so submanifold is minimal in hypersphere centered at $-c/\lambda$ ( i.e. it is of 1 - type ). It is easy to see, using induction, that condition $\Delta^k H = 0$ ( k nonnegative integer) is possible only on a noncompact manifold (cf. [C 4], Corollary 8.7.2., p. 302 ), and that is what makes our problem difficult since analysis on noncompact manifolds is not so well understood. While constructing examples (if they exist ) of nonminimal submanifolds which satisfy $\Delta H = 0$ seems to be reasonably difficult , we prove that under various additional conditions on the immersion x , a submanifold satisfying (4.1) is necessarily minimal. Let us note that there are known examples of submanifolds in pseudo - Euclidean spaces satisfying $\Delta H = 0$ [Ho] . In fact, Houh gave characterization of spacelike surfaces in pseudo sphere satisfying (4.1) in terms of Weingarten maps.

First we consider a curve case ( n = 1).

**Theorem 4.1.** If $x : C \to E^m$ is a curve with mean curvature vector H satisfying $\Delta H = 0$ , then the curve is a straight line, i.e. totally geodesic in $E^m$ .

**Proof.** Let s be a natural parameter of the curve . Then the Laplacian becomes $\Delta = -d^2/ds^2$, and we have $0 = \Delta H = -\Delta^2 x = -\frac{d^4x}{ds^4}$ . Hence, x has to be cubic polynomial in s, $x = \frac{1}{3} a s^3 + \frac{1}{2} b s^2 + c s + d$ , where a, b, c, d are constant vectors. Since s is the natural parameter we have

$$1 = \left< \frac{dx}{ds}, \frac{dx}{ds} \right>$$

$$= < as^2 + bs + c , as^2 + bs + c >$$

$$= |a|^2 s^4 + 2 <a, b> s^3 + (2 <a, c> + |b|^2) s^2 + 2 <b, c> s + |c|^2.$$

On the right hand side we have a polynomial in s , so we must have $a = b = 0$ , $|c|^2 = 1$ . In other words, $x(s) = cs + d$ with $|c|^2 = 1$, and therefore the curve is a straight line . ♦

From now on we assume that the dimension $n \geq 2$ . We use fundamental formula (2.25) of B.Y. Chen

$$\Delta H = \Delta^D H + \| A_{n+1} \|^2 H + a(H) + \mathrm{tr}(\bar{\nabla} A_H) \ , \tag{2.25}$$

where $e_{n+1} \parallel H$, $a(H) = \sum_{r=n+2}^{m} \mathrm{tr}(A_H A_r) e_r$ , and $\mathrm{tr}(\bar{\nabla} A_H) = \mathrm{tr}(\nabla A_H) + \mathrm{tr}(A_{DH})$.

$\{e_1, \ldots , e_n, e_{n+1}, \ldots , e_m\}$ will denote an adapted frame with the usual range for indices. The mean curvature $\alpha$ is defined by $H = \alpha\, e_{n+1}$ . Now we proceed with the computation of $\Delta^D H$ and $\mathrm{tr}(\bar{\nabla} A_H)$ .

$$\Delta^D H = \sum_{i=1}^{n} ( D_{\nabla_{e_i} e_i} H - D_{e_i} D_{e_i} H )$$

$$= \sum_{i=1}^{n} [ D_{\nabla_{e_i} e_i} (\alpha e_{n+1}) - D_{e_i} D_{e_i} (\alpha e_{n+1}) ]$$

$$= \sum_{i=1}^{n} [(\Delta\alpha)\, e_{n+1} + \alpha\, D_{\nabla_{e_i} e_i} e_{n+1} - 2(e_i \alpha)\, D_{e_i} e_{n+1} - \alpha\, D_{e_i} D_{e_i} e_{n+1}] \ ,$$

so that

$$< \Delta^D H , e_{n+1} > = \Delta\alpha - \sum_i < D_{e_i} D_{e_i} e_{n+1} , e_{n+1} >$$

$$= \Delta\alpha + \sum_i < D_{e_i} e_{n+1} , D_{e_i} e_{n+1} > \tag{4.2}$$

$$= \Delta\alpha + \alpha \| D e_{n+1} \|^2 \ .$$

Also, we have

$$\mathrm{tr}(\bar{\nabla} A_H) = \mathrm{tr}(\nabla A_H) + \mathrm{tr}(A_{DH})$$

$$= \sum_{i=1}^{n} [(\nabla_{e_i} A_H)e_i + A_{D_{e_i}H} e_i]$$

$$= \sum_{i=1}^{n} [(\nabla_{e_i}(\alpha A_{n+1}))e_i + A_{D_{e_i}(\alpha e_{n+1})} e_i]$$

(4.3)

$$= \sum_{i=1}^{n} [\, 2(e_i\alpha)(A_{n+1}e_i) + \alpha\,(\nabla_{e_i} A_{n+1})e_i \,] + \alpha \operatorname{tr} A_{De_{n+1}}$$

$$= 2A_{n+1}(\nabla\alpha) + \alpha \operatorname{tr} A_{De_{n+1}} + \alpha \sum_{i=1}^{n} (\nabla_{e_i} A_{n+1})e_i$$

$$= n\alpha\nabla\alpha + 2\alpha \operatorname{tr} A_{De_{n+1}} + 2A_{n+1}(\nabla\alpha) \ ,$$

by virtue of the Codazzi equation

(4.4)
$$(\nabla_X A_\xi)Y - (\nabla_Y A_\xi)X = A_{D_X\xi}\, Y - A_{D_Y\xi}\, X \ .$$

Namely, for any $X \in TM$ we have (see also [C 6])

$$\sum_i \langle (\nabla_{e_i} A_{n+1})e_i, X \rangle = \sum_i \langle e_i, (\nabla_{e_i} A_{n+1})X \rangle$$

$$= \sum_i \langle e_i, (\nabla_X A_{n+1})\, e_i + A_{D_{e_i}e_{n+1}} X - A_{D_X e_{n+1}} e_i \rangle$$

$$= \operatorname{tr}(\nabla_X A_{n+1}) + \sum_i \langle e_i, A_{D_{e_i}e_{n+1}} X \rangle$$

$$= \nabla_X(\operatorname{tr} A_{n+1}) + \langle \operatorname{tr} A_{De_{n+1}}, X \rangle$$

$$= \langle n\nabla\alpha + \operatorname{tr} A_{De_{n+1}}, X \rangle \ .$$

Assume now that $\Delta H = 0$ on a manifold. Then by separating off tangential part, normal part in the direction of $e_{n+1}$ and normal part perpendicular to $e_{n+1}$ we get respectively

$$\operatorname{tr}(\bar{\nabla} A_H) = 0 \ ,$$

$$< \Delta^D H , e_{n+1} > + \alpha \ \| \underset{=}{A_{n+1}} \|^2 = 0 \ ,$$

$$\operatorname{tr}(A_H A_r) + \ < \Delta^D H , e_r > = \ 0 \ , \ r = n+2, \dots, m \ ,$$

or, due to the calculations above we see that the condition $\Delta H = 0$ is equivalent to the following system of equations

$$2A_{n+1}(\nabla\alpha) + \ n\alpha\nabla\alpha \ + \ 2\alpha \operatorname{tr} A_{De_{n+1}} \ = \ 0 \ , \tag{4.5}$$

$$\Delta\alpha \ + \ \alpha \, \| De_{n+1} \|^2 \ + \ \alpha \, \| \underset{=}{A_{n+1}} \|^2 \ = \ 0 \ , \tag{4.6}$$

$$\alpha \operatorname{tr}(A_{n+1}A_r) \ + \ \alpha < \Delta^D e_{n+1} \, , e_r > - \ 2 < D_{\nabla\alpha} e_{n+1} \, , e_r > = \ 0 \ , \ r = n+2, \dots, m. \tag{4.7}$$

Because of the equation (4.6) we readily obtain the following lemma .

**Lemma 4.1.** Let $x : M^n \to E^m$ be an isometric immersion and assume that the mean curvature $\alpha$ is constant . Than if $\Delta H = 0$ it follows that $\alpha = 0$ , that is the submanifold is minimal.

System (4.5) - (4.7) in general is difficult system of PDE's to solve, but if $De_{n+1} = 0$ , in particular if M is hypersurface, the system simplifies to (last equation is not present in hypersurface case )

$$A_{n+1}(\nabla\alpha) = -\frac{n}{2}\alpha\nabla\alpha \tag{4.8}$$

$$\Delta\alpha \ + \ \alpha \ \| \underset{=}{A_{n+1}} \|^2 = 0 \tag{4.9}$$

$$\alpha \operatorname{tr}(A_{n+1}A_r) \ = \ 0 \ , \quad r = n+2, \dots, m. \tag{4.10}$$

From the equation (4.8) it follows that on the open (possibly empty) set $\{ \nabla\alpha \neq 0 \}$ of M, $\nabla\alpha$ is a principal direction of $e_{n+1}$ and $-\frac{n}{2}\alpha$ corresponding principal curvature .

**Theorem 4.2.** Let $x : M^n \to E^{n+1}$ be a hypersurface of $E^{n+1}$ with at most two distinct principal curvatures. Then the condition $\Delta H = 0$ implies $H = 0$, that is M is a minimal submanifold of $E^{n+1}$.

**Proof.** For a hypersurface the condition $\Delta H = 0$ is equivalent to the system

$$\Delta\alpha + \alpha \| A \|^2 = 0 \, , \tag{4.11}$$

$$A(\nabla\alpha) = -\frac{n}{2}\alpha\nabla\alpha \, . \tag{4.12}$$

Let us also recall the Codazzi equations in the form

$$(\lambda_j - \lambda_i)\,\omega_j^i(e_i) = e_j \lambda_i \, , \quad i \neq j \tag{1.1.29}$$

$$(\lambda_j - \lambda_k)\,\omega_j^k(e_i) = (\lambda_i - \lambda_k)\,\omega_i^k(e_j) \, , \quad i \neq j \neq k \neq i \, . \tag{1.1.30}$$

Let U be an open set of M defined by $U = \{p \in M \mid \nabla\alpha^2 \neq 0 \text{ at } p\}$, and let $\{e_i\}$, $i = 1,\dots,n$ be the basis of principal directions on U so that $e_1 = \frac{\nabla\alpha}{\|\nabla\alpha\|}$ is the eigenvector of the shape operator corresponding to eigenvalue $\lambda_1 = -\frac{n}{2}\alpha$. Then $e_j\alpha = 0$ for $j \geq 2$. If the multiplicity of $\lambda_1$ is at least 2, i.e. if $\lambda_i = \lambda_1$ for some $i \geq 2$ then $e_1\alpha = 0$. That follows from the equation (1.1.29) putting $j = 1$. In that case $\alpha = \text{const}$, and by Lemma 4.1 we conclude that $\alpha = 0$ on U and as well on entire manifold M. If the multiplicity of $\lambda_1$ is one, then since there are at most two distinct principal curvatures and since $\operatorname{tr} A = n\alpha$ we have

$$\lambda_1 = -\frac{n}{2}\alpha \, , \quad \lambda_2 = \lambda_3 = \dots = \lambda_n = \frac{3n\alpha}{2(n-1)} \, . \tag{4.13}$$

In the rest of the proof, all computations will be done on the set U.

We compute $\| A \|^2 = \operatorname{tr} A^2 = \frac{(n+8)n^2\alpha^2}{4(n-1)}$. Since $\lambda_1 \neq \lambda_j$ and $e_j\alpha = 0$ for $j \geq 2$, we get from (1.1.29)

(4.14) $\omega_1^j(e_1) = 0$ , for all $j = 1, \ldots, n$ , i.e. $\nabla_{e_1}e_1 = 0$ ,

which means that the integral curves of $e_1$ are geodesics on U. For $j = 1$ and $i \geq 2$ (1.1.29) gives $(\lambda_1 - \lambda_i)\,\omega_1^i(e_i) = e_1\lambda_i$ , with $\lambda_1 - \lambda_i = -\frac{n(n+2)\alpha}{2(n-1)}$ . Therefore,

$$\omega_1^i(e_i) = \frac{e_1\lambda_i}{\lambda_1-\lambda_i} = -\frac{3(e_1\alpha)}{(n+2)\alpha} , \quad \text{for } i \geq 2 . \tag{4.15}$$

For $j, k \geq 2$ , $j \neq k$ and $i = 1$ we have $\lambda_j - \lambda_k = 0 \neq \lambda_1 - \lambda_k$ so the formula (1.1.30) yields

$$\omega_1^k(e_j) = 0 , \quad \text{for } j, k \geq 2 , j \neq k . \tag{4.16}$$

Combining (4.14), (4.15) and (4.16) we get the expression for $\omega_1^k$ as follows

$$\omega_1^k = -\frac{3(e_1\alpha)}{(n+2)\alpha}\,\omega^k , \quad \text{for } k \geq 2 . \tag{4.17}$$

Let us compute now the Laplacian of the mean curvature

$$\begin{aligned}
\Delta\alpha &= \sum_i [\,(\nabla_{e_i}e_i)\alpha - e_i e_i \alpha\,] \\
&= \sum_i [\,\sum_k \omega_i^k(e_i)\, e_k \alpha - e_i e_i \alpha\,] \\
&= [\,\sum_i \omega_i^1(e_i)\,]\, e_1\alpha - e_1 e_1 \alpha \\
&= \frac{3(n-1)}{(n+2)\alpha}(e_1\alpha)^2 - e_1 e_1 \alpha , \quad \text{by (4.15)} .
\end{aligned}$$

Therefore, (4.11) becomes

$$\frac{3(n-1)}{(n+2)\alpha}(e_1\alpha)^2 - e_1e_1\alpha + \frac{(n+8)n^2\alpha^3}{4\,(n-1)} = 0 \quad , \text{ or}$$

(4.18) $$\alpha'' - \frac{3(n-1)}{(n+2)\alpha}(\alpha')^2 - \frac{(n+8)n^2}{4\,(n-1)}\alpha^3 = 0 \quad ,$$

where ' denotes derivative with respect to $e_1$. Formula (4.17) can be rewritten as

(4.17) $$(n+2)\,\alpha\,\omega^1_k = 3\alpha'\,\omega^k \,, \qquad \text{for } k \geq 2 \;.$$

Differentiating this relation we get

(4.19) $$(n+2)\,d\alpha \wedge \omega^1_k + (n+2)\,\alpha\,d\omega^1_k = 3\,d\alpha' \wedge \omega^k + 3\,\alpha'\,d\omega^k \;.$$

Using Cartan structural equations (1.1.20), (1.1.21) and (1.1.22) we have

$$d\omega^1_k = \omega^{n+1}_1 \wedge \omega^{n+1}_k + \sum_{j=1}^{n} \omega^j_1 \wedge \omega^j_k$$

$$= \lambda_1\lambda_k\,\omega^1 \wedge \omega^k + \sum_{j=2}^{n}\sum_{r=1}^{n} \omega^j_1(e_j)\,\omega^j_k(e_r)\,\omega^j \wedge \omega^r$$

$$= -\frac{3n^2\alpha^2}{4\,(n-1)}\,\omega^1 \wedge \omega^k - \frac{3\alpha'}{(n+2)\alpha}\sum_{j=2}^{n}\sum_{r=1}^{n} \omega^j_k(e_r)\,\omega^j \wedge \omega^r \;,$$

and

$$d\omega^k = \sum_{j=1}^{n} \omega^j_k \wedge \omega^j = \frac{3\alpha'}{(n+2)\alpha}\,\omega^k \wedge \omega^1 + \sum_{j=2}^{n} \omega^j_k \wedge \omega^j \;.$$

Because of these calculations, the left hand side of (4.19) takes the following form

$$\frac{3\alpha'^2}{\alpha}\,\omega^1 \wedge \omega^k + (n+2)\,\alpha\left[-\frac{3n^2\alpha^2}{4\,(n-1)}\,\omega^1 \wedge \omega^k - \frac{3\alpha'}{(n+2)\alpha}\sum_{j=2}^{n}\sum_{r=1}^{n} \omega^j_k(e_r)\,\omega^j \wedge \omega^r\right],$$

and the right hand side of (4.19) reads as

No

(4

$$3\,\alpha''\,\omega^1\wedge\omega^k + 3\sum_{j=2}^{n} e_j(\alpha')\,\omega^j\wedge\omega^k - \frac{9\alpha'^2}{(n+2)\alpha}\,\omega^1\wedge\omega^k + 3\alpha'\sum_{j=2}^{n}\omega_k^j\wedge\omega^j\ .$$

Now comparing the coefficients of $\omega^1 \wedge \omega^k$ term ($k \geq 2$) of the left and right hand side of (4.19) we obtain

$$\frac{3\alpha'^2}{\alpha} - \frac{3(n+2)n^2\alpha^3}{4\,(n-1)} = 3\,\alpha'' - \frac{9\alpha'^2}{(n+2)\alpha}\ , \quad \text{that is}$$

(4.20)
$$\alpha'' - \frac{n+5}{(n+2)\alpha}\,\alpha'^2 + \frac{(n+2)n^2}{4(n-1)}\,\alpha^3 = 0\ .$$

Eliminating $\alpha'' = e_1e_1\alpha$ from (4.18) and (4.20) we get

(4.21)
$$\frac{2(n-4)}{n+2}\,\alpha'^2 + \frac{(n+5)n^2}{2(n-1)}\,\alpha^4 = 0\ .$$

Clearly, if $n \geq 4$ from (4.21) we conclude $\alpha = 0$ on U which is a contradiction unless U is empty . In any case we can solve the equations (4.18) and (4.20) explicitly. Namely, for any equation of the form $y'' = f(y, y')$ , we introduce the substitution $v = y'$ , regarding v as a function of y . Then $y'' = v\,\frac{dv}{dy}$ , and the equation becomes the first order differential equation in v(y) . Regarding the equation (4.18) , let $z = (\alpha')^2$ . Then from the above we have

$$\frac{dz}{d\alpha} = 2\alpha'\frac{d\alpha'}{d\alpha} = 2\alpha'\frac{\alpha''}{\alpha'} = 2\alpha''\ ,$$

and the equation (4.18) becomes

$$\frac{dz}{d\alpha} - \frac{6(n-1)}{(n+2)\alpha}\,z - \frac{n^2(n+8)}{2(n-1)}\,\alpha = 0\ ,$$

which is first order linear differential equation whose solution is given by

$$(4.22) \qquad z = \alpha'^2 = \frac{n^2(n+2)(n+8)}{4(n-1)(7-n)}\alpha^4 + c\,\alpha^{\frac{6(n-1)}{n+2}} , \quad c = \text{const} .$$

Using the same method for solving (4.20) we get

$$(4.23) \qquad \alpha'^2 = k\,\alpha^{\frac{2(n+5)}{n+2}} - \frac{n^2(n+2)^2}{4(n-1)^2}\alpha^4 , \quad k = \text{const} .$$

Considering all possibilities for exponents of $\alpha$ in (4.22) and (4.23) we see that these two equations contradict each other on the set U, and therefore U must be empty, which means that the mean curvature $\alpha$ is constant and hence by Lemma 4.1 submanifold M is minimal.

**Corollary 4.1.** Any surface in $E^3$ which satisfies $\Delta H = 0$ is minimal.

**Corollary 4.2.** If $M^n$ is a quasiumbilical hyperesurface of $E^{n+1}$ which satisfies $\Delta H = 0$, then M is necessarily minimal and therefore generalized catenoid (see [Bl]).

By a result of Cartan [C 1], a hypersurface $M^n \subset E^{n+1}$ is quasiumbilical if and only if it is conformally flat (for $n \geq 4$), so the Corollary 4.2 can be appropriately stated for conformally flat hypersurfaces ( except when $n = 3$ ).

Next we have following theorem for pseudoumbilical submanifolds

**Theorem 4.3.** Let $x : M^n \to E^m$ be a pseudoumbilical submanifold, that is $A_{n+1} = \alpha I$. If $\Delta H = 0$ and $n \neq 4$ then M is minimal in $E^m$.

**Proof.** From the equation (4.5) and pseudoumbilicity we obtain

$$(4.24) \qquad \frac{n+2}{2}\nabla\alpha + \operatorname{tr} A_{De_{n+1}} = 0 , \quad \text{for } \alpha \neq 0 ,$$

or, equivalently,

$$\frac{n+2}{2}(e_i\alpha) + \sum_{k=1}^{n}\sum_{r=n+2}^{m}\omega_{n+1}^{r}(e_k)\,h_{ki}^{r} = 0 \ , \text{ for every } i = 1,2, \ldots, n \ . \tag{4.25}$$

Using the Codazzi equation (1.1.24) we have

$$(\bar\nabla_{e_i}h)(e_j, e_j) = (\bar\nabla_{e_j}h)(e_i, e_j) \quad , \qquad i \neq j \ . \tag{4.26}$$

We fix index $i \in \{1, \ldots , n\}$ and let r denote any index $\geq 2$ . By comparing terms in the direction of $e_{n+1}$ on both sides of (4.26) we obtain

$$D_{e_i}(h_{jj}^{n+1}e_{n+1} + h_{jj}^{r}e_r) = D_{e_j}(h_{ij}^{r}e_r) - \omega_i^j(e_j)\,h_{jj}^{n+1}e_{n+1} - \omega_j^i(e_j)\,h_{ii}^{n+1}e_{n+1} \ .$$

Note that $h_{jj}^{n+1} = \alpha$ for any j , and $h_{ij}^{n+1} = 0$ for $i \neq j$ . Therefore, $e_{n+1}$ components give

$$(e_i\alpha) + \sum_r h_{jj}^{r}\,\omega_r^{n+1}(e_i) = \sum_r h_{ij}^{r}\,\omega_r^{n+1}(e_j) \quad , \qquad i \neq j \ . \tag{4.27}$$

Summing over all $j \neq i$ , and observing that $0 = \operatorname{tr} A_r = \sum_{j=1}^{n} h_{jj}^{r}$ , that is $\sum_{j\neq i}^{n} h_{jj}^{r} = - h_{ii}^{r}$ , we get from (4.27)

$$(n-1)(e_i\alpha) - \sum_r h_{ii}^{r}\,\omega_r^{n+1}(e_i) = \sum_r\sum_{j\neq i} h_{ij}^{r}\,\omega_r^{n+1}(e_j) \quad , \text{ that is}$$

$$(n-1)(e_i\alpha) + \sum_{k=1}^{n}\sum_{r=n+2}^{m}\omega_{n+1}^{r}(e_k)\,h_{ki}^{r} = 0 \ . \tag{4.28}$$

Comparing (4.25) and (4.28) we see that if $n \neq 4$ , $e_i\alpha = 0$ for every $i = 1, \ldots , n$ which shows that $\alpha$ = const and therefore equal to zero. ♦

**Lemma 4.2.** Let $x : M^n \to E^m$ be an isometrically immersed submanifold which satisfies $\Delta H = 0$ and $< x, H > =$ const . Then M is minimal in $E^m$ .

**Proof.** By formula (1.5.4) we have

$$\Delta < x, H > = -n < H, H > + < x, \Delta H > + 2 \operatorname{tr} A_H = -n\alpha^2 + 2n\alpha^2 = n\alpha^2 .$$

Therefore, if $< x, H > =$ const , we get $\alpha = 0$ . ♦

As a corollary, any cone in $E^m$ that satisfies $\Delta H = 0$ must be minimal . Namely, without loss of generality, we can assume that a cone has the vertex at the origin so that $< x, H > =$ const holds .

Next we show that if $\Delta H = 0$ for a submanifold $M \subset E^m$ and M is of finite type, then $H = 0$ , so M is minimal again. First, using induction we can easily prove the following

**Lemma 4.3.** If M is a manifold and $\Delta$ the Laplacian acting on smooth functions of M , then no eigenfunction ( not identically 0 ) of $\Delta$ can be represented as the sum of k ( $k \geq 2$) other eigenfunctions from k different eigenspaces.

**Theorem 4.4.** Suppose that $\Delta^r H = 0$ holds for a submanifold $x : M^n \rightarrow E^m$ , for some positive integer r . If M is of finite type it follows that M is minimal , i.e. of null 1 - type .

**Proof.** Suppose that M is of k - type so that we have

$$x = x_0 + x_{t_1} + \dots + x_{t_k} , \tag{4.29}$$

with $x_0 =$ const and $\Delta x_{t_i} = \lambda_{t_i} x_{t_i}$ , $i \geq 1$ . Then taking $\Delta^{r+1}$ of (4.29) we have

$$0 = -n \, \Delta^r H = \Delta^{r+1} x = \lambda_{t_1}^{r+1} x_{t_1} + \dots + \lambda_{t_k}^{r+1} x_{t_k} ,$$

By Lemma 4.3 , this is possible only when there is only one nonzero $x_{t_i}$ in this sum , and the corresponding eigenvalue $\lambda_{t_i}$ is zero . This means $\Delta x = 0$ and the submanifold M is of null 1 - type (minimal) . ♦

**Corollary 4.3.** Suppose that $x : M^n \to E^m$ is a submanifold such that the component functions of x are eigenfunctions of the Lapacian. If $\Delta H = 0$ then M is minimal.

**Proof.** If $x = (x_1, \dots , x_m)$ and $\Delta x_i = \lambda_{t_i} x_i$ then x is of finite type, actually of type $\leq m$ since

$$x = (x_1, 0, \dots , 0) + (0, x_2, \dots , 0) + (0, \dots , 0, x_m) .$$

Then the Theorem 4.4 proves the claim. ♦

Surfaces of revolution in $E^3$ which have the property described in previous corollary were studied in [G] .

In view of Theorem 4.2 it seems probable that a hypersurface of $E^m$ which satisfies $\Delta H = 0$ is minimal since there is no "room" in the normal space. (There is also strong indication that that is so for any 3 - dimensional hypersurface in $E^4$.) If a codimension is higher, it is possible to have a nonminimal submanifold which satisfies (4.1), but construction of such submanifolds seems to be difficult. If $H = (h_1, \dots , h_m)$ and $\Delta H = 0$, then each $h_i$ is harmonic. For a harmonic map on a manifold there are Liouville type theorems. For example, if M is a complete Riemannian manifold of nonnegative Ricci curvature, then any bounded harmonic function on M is a constant function. The same conclusion holds if a harmonic map grows slower than linear growth or have a finite energy (see [Y], [Che]) . So, if such submanifold satisfies (4.1) and the mean curvature $\alpha$ is bounded then $\alpha = 0$ . Also a bounded harmonic map on a simple Riemannian manifold is necessarily constant (see [Hi]) . A manifold M is *simple* if it is topologically $R^n$ with metric for which $\Delta$ is uniformly elliptic.

# SUMMARY

For an isometric immersion $x : M^n \to E^m$ of a Riemannian manifold into a Euclidean space, one defines the map $\tilde{x} = x \cdot x^t$ ( x regarded as column vector) from M into SM(m), the set of $m \times m$ symmetric matrices, which we call *quadric representation* of M. If M is submanifold of the unit hypersphere centered at the origin (henceforth called spherical), then it is well known that $\tilde{x}$ is an isometric immersion (via 2nd standard immersion of sphere). It appears, however, that this map has not been studied in general.

A smooth map $f : M^n \to R$ is said to be of k-type if it can be decomposed as $f = f_0 + \sum f_t$ (k nonzero terms in the sum) , where $f_0 = \text{const}$ and $\Delta f_t = \lambda_t f_t$ i.e. $f_t$'s are eigenfunctions of Laplacian on M. This naturally extends to an $E^m$-valued map. In particular, a manifold immersed into Euclidean space is of k-type if the corresponding immersion is so.

In Chapter 2 we proved some general results about quadric representation. First we showed that $\tilde{x}$ is an isometric immersion if and only if M is spherical. The same conclusion if $\tilde{x}$ is conformal ($n \geq 2$) (see Theorems 2.1 - 2). Submanifolds for which $\tilde{x}$ is 1-type map are classified as totally geodesic spherical submanifolds (Theorem 2.3) . While it is relatively easy to construct nonspherical submanifolds for which $\tilde{x}$ is finite type (Example 2.1), we prove that for minimal submanifold of $E^m$ the quadric representation is of infinite type (Theorem 2.4 ). For a spherical submanifold, certain relationships between the immersions x and $\tilde{x}$ can be shown as exemplified by

**Theorem 2.5.** Let $x : M^n \to S^{m-1} \subset E^m$ be an isometric immersion and let

$\tilde{x} : M^n \rightarrow SM(m)$ be its quadric representation. Symbols with ~ are related to the immersion $\tilde{x}$ , those without ~ to the immersion into $E^m$ and symbols with ' relate to the immersion into $S^{m-1}$ . Then

*i* ) $$\| \tilde{h} \| = \text{const} \Leftrightarrow \| h \| = \text{const} ,$$
$$\| \tilde{H} \| = \text{const} \Leftrightarrow \| H \| = \text{const} .$$

*ii* ) $M^n$ is pseudoumbilical in SM(m) via $\tilde{x}$ $\Leftrightarrow$ $M^n$ is pseudoumbilical in $E^m$ via x .

*iii* ) $\tilde{D}\tilde{H} = 0 \Leftrightarrow h' = 0$ , i.e. $M^n$ is totally geodesic in $S^{m-1}$ .

*iv* ) $\tilde{\nabla}\tilde{h} = 0 \Leftrightarrow h' = 0$ .

In Chapter 3 we study compact spherical hypersurfaces which are of low type via the quadric representation. We have the following classification result for those which are of 2 - type via $\tilde{x}$ , thus generalizing similar result of M. Barros and B.Y. Chen [B-C].

**Theorem 3.1.1.** Let $x : M^n \rightarrow S^{n+1}$ be an isometric immersion of a compact n - dimensional Riemannian manifold M into $S^{n+1}$ $(n \geq 2)$ . Then $\tilde{x} = xx^t$ is of 2 - type if and only if either

(1) M is a small hypersphere of $S^{n+1}$ of radius $r < 1$ , or

(2) $M = S^p(r_1) \times S^{n-p}(r_2)$ , with the following possibilities for the radii $r_1$ and $r_2$ :

$$\text{i) } r_1^2 = \frac{p+1}{n+2} , \; r_2^2 = \frac{n-p+1}{n+2} ; \text{ ii) } r_1^2 = \frac{p+2}{n+2} , \; r_2^2 = \frac{n-p}{n+2} ; \text{ iii) } r_1^2 = \frac{p}{n+2} , \; r_2^2 = \frac{n-p+2}{n+2}$$

The immersions in (1) and (2) are given in a natural way .

Next we compute the third iterated Laplacian and undertake study of minimal spherical hypersurfaces which are mass - symmetric and of 3 - type via $\tilde{x}$ . We obtain the following characterization

**Theorem 3.2.1** Let $x : M^n \to S^{n+1}$ be an isometric immersion of a compact manifold $M^n$ as a minimal hypersurface of $S^{n+1}$. If $\tilde{x}$ is mass - symmetric and of 3 - type then

(1) $\operatorname{tr} A = \operatorname{tr} A^3 = 0$ ,

(2) $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$ are constant ,

(3) $\operatorname{tr} (\nabla_X A)^2 = < A^2X, A^2X > + p < AX, AX > + q < X, X >$ , $X \in TM$

where p and q are constants (depending on the order of M , $\operatorname{tr} A^2$ and $\operatorname{tr} A^4$ ). Conversely, if (1), (2) and (3) hold then M is mass - symmetric and 1 - , 2 - , or 3 - type via $\tilde{x}$ .

The main result of Chapter 3 is the classification of compact minimal spherical hypersurfaces which are of 3-type and mass - symmetric via $\tilde{x}$ in dimensions $n \leq 5$ , thus giving a new characterization of the Cartan hypersurface $SO(3)/Z_2 \times Z_2$ in terms of its spectral behavior. Namely,

**Theorem 3.2.2.** Let $x : M^n \to S^{n+1}$ be compact minimal hypersurface of $S^{n+1}$ of dimension $2 \leq n \leq 5$ . Then $\tilde{x}$ is mass - symmetric and of 3 - type if and only if $n = 3$ and $M^3 = SO(3)/Z_2 \times Z_2$ is the Cartan hypersurface .

Actually, all minimal isoparametric spherical hypersurfaces with three distinct principal curvatures are of 3-type and mass-symmetric via $\tilde{x}$ (Lemma 3.2.3).

In Chapter 4 we study submanifolds $x : M^n \to E^m$ of a Euclidean space with harmonic mean curvature vector , i.e. those that satisfy $\Delta H = 0$ , or equivalently $\Delta^2 x = 0$ . Minimal submanifolds being the trivial solution, the real problem is to find nonminimal examples, that is, those immersions which are biharmonic but not harmonic. While the construction of such examples (if they exist) seems difficult, we show that submanifolds satisfying $\Delta H = 0$ are necessarily minimal if any of the following conditions is satisfied

(1) $M^n$ has constant mean curvature .

(2) $M^n$ is a hypersurface of $E^{n+1}$ with at most two distinct principal curvatures .

(3) $M^n$ is conformally flat hypersurface of $E^{n+1}$ ( $n \neq 3$) .

(3) $M^n$ is a pseudoumbilical submanifold of $E^m$ ( $n \neq 4$ ) .

(4) $M^n$ is of finite type .

# BIBLIOGRAPHY

## BIBLIOGRAPHY


[A] Abresch, U., Isoparametric hypersurfaces with four or six distinct principal curvatures, Math. Ann. 264(1983), 283 - 302 .

[B-C] Barros, M. and Chen, B.Y., Spherical submanifolds which are of 2- type via the second standard immersion of the sphere, Nagoya Math. J. 108 (1987), 77 - 91 .

[B-C-G] Barros, M., Chen, B.Y. and Garay, O., Spherical finite type hypersurfaces, Algebras, Groups, Geom., 4(1987), 58 - 72 .

[B-U] Barros, M. and Urbano, F., Spectral geometry of minimal surfaces in the sphere, Tohoku Math. J., 39(1987), 575 - 588 .

[B-G-M] Berger, M., Gauduchon, P. and Mazet, M., Le spectre d'une variete Rieman - niene , Lecture Notes in Math., No 194, Springer, Berlin , 1971.

[B] Berndt, J., Real hypersurfaces with constant principal curvatures in complex hyperbolic space, J. reine angew. Math. 395(1989), 132 - 141 .

[Be] Besse, A.L., Einstein manifolds , Springer, Berlin, 1987 .

[Bl] Blair, D., On a generalization of the catenoid , Canad. J. Math.27, No2 (1975), 231 - 236 .

[Car 1] Cartan, E., La déformation des hypersurfaces dans l'espace conformément réel à n ≥ 5 dimensions , Bull. Soc. Math. France 45(1917), 57 - 121 .

[Car 2] Cartan, E., Familles de surfaces isoparamétriques dans le espaces à courbure constante , Ann. Mat. Pura Appl. 17(1938), 177 - 191 .

[Car 3] Cartan, E., Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques , Math. Z. 45(1939), 335 - 367 .

[Car 4] Cartan, E., Sur quelques familles remarquables d'hypersurfaces , C. R. Congrès Math. Liège , 1939 .

[Car 5] Cartan, E., Sur des familles d'hypersurfaces isoparamétriques des espaces spheriques a 5 et a 9 dimensions , Revista Univ. Tucuman, Serie A, 1 (1940), 5 - 22 .

[Ce-Ry] Cecil, T.E. and Ryan, P.J., Tight and taut immersions of manifolds, Research notes in mathematics No. 107, Pitman, Boston, 1985.

[Ch 1] Chavel, I., Riemannian symmetric spaces of rank one, Marcel Dekker, New York, 1972.

[Ch 2] Chavel, I., Eigenvalues in Riemannian geometry, Academic Press, New York, 1984.

[Ch-E] Cheeger, J. and Ebin, D., Comparison theorems in Riemannian geometry, North Holland, Amsterdam, 1975.

[C 1] Chen, B.Y., Geometry of submanifolds, Marcel Dekker, New York, 1973.

[C 2] Chen, B.Y., On total curvature of immersed manifolds, IV, Bull. Math. Acad. Sinica 7(1979), 301 - 311

[C 3] Chen, B.Y., On the first eigenvalue of Laplacian of compact minimal submanifolds of rank one symmetric spaces, Chinese J. Math. 11(1983), 259 - 273.

[C 4] Chen, B.Y., Total mean curvature and submanifolds of finite type, World Scientific, Singapore, 1984.

[C 5] Chen, B.Y., Finite type submanifolds and generalizations, Instituto Matematico "Guido Castelnuovo", Roma, 1985.

[C 6] Chen, B.Y., 2-Type submanifolds and their applications, Chinese J. Math. 14 (1986), 1 - 14.

[C 7] Chen, B.Y., Open problems and conjectures on finite type submanifolds, Unpublished.

[C-D-V] Chen, B.Y., Deprez, J., Dillen, F., Verstraelen, L. and Vrancken, L., Curves of finite type, to appear.

[Che] Cheng, S.Y., Liouville theorem for harmonic maps, Proc. of Symposium in Pure Math. 36(1980), 147 - 151.

[Ch-L] Chern, S.S. and Lashof, R.K., On the total curvature of immersed manifolds, II, Michigan Math. J., 5(1958), 5 - 12.

[D] Deprez, J., Immersions of finite type of compact, homogeneous, Riemannian manifold, Thesis, K. U. Leuven (Belgium) 1988.

[D-N] Dorfmeister, J. and Neher, E., An algebraic approach to isoparametric hypersurfaces I, II, Tohoku Math. J. 35(1983), 187 - 224 and 225 - 247.

[D-N 1] Dorfmeister, J. and Neher, E., Isoparametric hypersurfaces, case g = 6, m = 1, Comm. in Algebra, 13(11) (1985), 2299 - 2368.

[F] Ferus, D., Notes on isoparametric hypersurfaces, Universidade Estadual de Campinas, Brasil, 1980.

[F-K-M] Ferus, D., Karcher, H. and Münzner, H.F., Cliffordalgebren und neue isoparametrische Hyperflächen , Math. Z. 177(1981), 479 - 502 .

[G] Garay, O., On a certain class of finite type surfaces of revolution, Kodai Math. J. 11(1988), 25 - 31 .

[Ha] Hahn, J., Isoparametric hypersurfaces in the pseudo - Riemannian space forms Math. Z. 187(1984), 195 - 208 .

[He] Helgason, S., Differential geometry, Lie groups, and symmetric spaces, Academic press , New York, 1978 .

[Ho] Houh, C.S., Some low type spacelike surfaces in a Minkowski space - time, Soochow J. Math. 15, No 2 (1988), 167 - 178 .

[Hi] Hildebrandt, S., Liouville theorems for harmonic mappings, and an approach to Bernstein theorems, Seminar on Differential Geometry ( S.T. Yau editor ) Annals of Mathematical studies No. 102 , pp. 107 - 131 , Princeton, New Jersey 1982 .

[H] Hsiang, W.Y., Remarks on closed minimal submanifolds in the standard Riemannian m - sphere , J. Differential Geometry 1(1967), 257 - 267 .

[H-L] Hsiang, W.Y. and Lawson, H.B., Minimal submanifolds of low cohomogeneity , J. Differential Geometry 5(1971), 1 - 38 .

[Ki-Na] Ki, U.H. and Nakagawa, H., A characterization of the Cartan hypersurface in a sphere , Tohoku Math. J. 39(1987), 27 - 40 .

[K-N] Kobayashi, S. and Nomizu, K., Foundations of differential geometry, Vol I and II , Wiley , New York , 1963 and 1969 .

[Ko] Kotani, M., The first eigenvalue of homogeneous minimal hypersurfaces in a unit sphere $S^{n+1}(1)$ , Tohoku Math. Journ., 37(1985), 523 - 532 .

[Ma] Magid, M., Lorentzian isoparametric hypersurfaces, Pacific J. Math.118(1985) 165 - 197 .

[M] Münzner, H.F., Isoparametrische Hyperflächen in Sphären , I, Math. Ann. 251(1980), 57 - 71, II, ibid. 256(1981), 215 - 232.

[Mu] Muto, H., The first eigenvalue of the Laplacian of an isoparametric minimal hypersurface in a unit sphere, Math. Z. 197(1988), 531 - 549 .

[M-O-U] Muto, H., Ohnita, Y. and Udagawa, S., Homogeneous minimal hypersurfaces in the unit spheres and the first eigenvalue of their Laplacian, Tohoku Math. Journ. 36(1984), 253 - 267 .

[N 1] Nomizu, K., Some results in E. Cartan's theory of isoparametric families of hypersurfaces, Bull. Amer. Math. Soc. 79(1973), 1184 - 1188 .

[N 2] Nomizu, K., Elie Cartan's work on isoparametric families of hypersurfaces Proc. Symposia in Pure Math., Amer. Math. Soc. 27(Part I),(1975), 191 - 200 .

[N 3] Nomizu, K., On isoparametric hypersurfaces in the Lorentzian space forms , Japan. J. Math. 7(1981), 217 - 226 .

[N-S] Nomizu, K. and Smyth, B., A formula of Simons' type and hypersurfaces with constant mean curvature , J. Differential Geometry 3(1969), 367 - 377 .

[O'N] O'Neill, B., Semi - Riemannian geometry with applications to relativity, Acad - emic Press, New York 1983 .

[O-T] Ozeki, H. and Takeuchi, M., On some types of isoparametric hypersurfaces in spheres, I, Tohoku Math. J. 27(1975), 515 - 559, II, ibid. 28(1976), 7 - 55 .

[Pa-T] Palais, R. and Terng, C.L., Critical point theory and submanifold geometry, Lecture notes in mathematics 1353 , Springer Verlag , Berlin 1988 .

[P] Park, K.S., Isoparametric families on projective spaces, Math. Ann. 284 (1989), 503 - 513 .

[P-T] Peng, C.K. and Terng, C.L., Minimal hypersurfaces of spheres with constant scalar curvature , Seminar on minimal submanifolds , Ann. of Math. Studies No 103 , Princeton Univ. Press , 1983, 177 - 205 .

[P-H] Peng, C.K. and Hou, Z., A remark on the isoparametric polynomials of degree 6 , in Differential geometry and topology (Proc., Tianjin 1986-87) , Lecture notes in mathematics 1369 , Springer Verlag, Berlin 1989 .

[R] Ros, A., Eigenvalue inequalities for minimal submanifolds and P-manifolds, Math. Z. 187(1984), 393 - 404 .

[Ry] Ryan, P.J., Homogeneity and some curvature conditions for hypersurfaces , Tohoku Math. J. 21(1969), 363 - 388 .

[S] Sakamoto, K., Planar geodesic immersions , Tohoku Math. J. 291(1977), 25 - 56 .

[Tai] Tai, S.S., Minimum imbeddings of compact symmetric spaces of rank one , J. Differential Geometry 2(1968), 55 - 66 .

[T 1] Takagi, R., Homogeneous hypersurfaces in a sphere with the type number 2 , Tohoku Math. J. 23(1971), 49 - 58 .

[T 2] Takagi, R., Real hypersurfaces in a complex projective space with constant principal curvatures I,II J. Math. Soc. Japan 27(1975), 43 - 53 , 507 - 516 .

[T 3] Takagi, R., A class of hypersurfaces with constant principal curvatures in a sphere , J. Differential Geometry 11(1976), 225 - 233 .

[T-Ta] Takagi, R. and Takahashi, T., On the principal curvatures of homogeneous hypersurfaces in a sphere , Differential geometry in honor of K. Yano , Kinokuniya, Tokyo, 1972, 469 - 481 .

[Ta 1] Takahashi, T., Minimal immersions of Riemmanian manifolds , J. Math. Soc. Japan 18(1966), 380 - 385 .

[Ta 2] Takahashi, T., Isometric immersion of Riemannian homogeneous manifold, Tsukuba J. Math. 12(1988), 231 - 233 .

[Te] Terng, C.L., Isoparametric submanifolds and their Coxeter groups , J. Dif - ferential Geometry 21(1985), 79 - 107 .

[T-V] Tricerri, F. and Vanhecke, L., Homogeneous structures on Riemannian manifolds, London Math. Soc. Lect. Note Series No. 83 Cambridge University Press, Cambridge 1983 .

[U] Udagawa, S., Spectral geometry of compact Hermitian symmetric submanif- olds, Math. Z. 192(1986), 57 - 72 .

[W 1] Wang, Q.M., Isoparametric hypersurfaces in complex projective spaces , Proc. 1980 Beijing Symposium on Differential geometry and Differential equ- ations , 1509 - 1523 .

[W 2] Wang, Q.M., On the topology of Clifford isoparametric hypersurfaces , J. Differential Geometry 27(1988), 55 - 66 .

[Wa] Warner, F.W., Foundations of differentiable manifolds and Lie groups , Springer - Verlag, New York 1983 .

[Y] Yau, S.T., Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math., 28(1975), 201 - 228 .