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Shui-Nee Chow

SL-N-Liu

Major professor

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Kening Lu

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ABSTRACT

INVARIANT MANIFOLDS FOR FLOWS IN BANACH SPACES

By

Kening Lu

We consider the existence, smoothness and exponential attractivity of global invariant manifolds for flow in Banach Spaces. We show that every global invariant manifold can be expressed as a graph of a C^k map, provided that the invariant manifolds are exponentially attractive. Applications go to the Reaction–Diffusion equation, the Kuramoto–Sivashinsky equation, and singlular perturbed wave equation.

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§1 Introduction

In the study of dynamical systems in finite dimensional spaces or manifolds, the theory of invariant manifolds has proved to be a fundamental and useful idea. In recent years, the theory of invariant manifolds has been generalized to flows or semiflows in Banach spaces. See, for example, Babin and Vishik [2], Bates and Jones [3], Carr [4], Chow and Hale [5], Hale [16], [17], Hale and Lin [18], Henry [21], Marsden and Scheurle [26], Wells [32] and others. On the other hand, it is known that global compact attractors for many dissipative systems in Banach spaces have finite capacity or Hausdoff dimensions (see, for example, Mallet–Paret [23], Mane [25], Hale [17] Constantin, Foias and Temam [9], Babin and Vishik [2], Hale, Magalhaes and Oliva [19]). Recently, it has been found that in many cases these global compact attractors actually can be embedded in exponentially attractive finite dimensional invariant manifolds which we call inertial manifolds (see, for example, Conway, Hoff and Smoller [16], Constantin, Foias, Nicolaenko and Temam [8], Doering, Gibbon, Holm and Nicolaenko [11], Foias, Nicolaenko, Sell and Temam [13], Foias, Sell and Temam [14] and Mallet–Paret and Sell [24]). This supports the believe that the asymptotic behavior of solutions of many infinite dimensional dynamical systems resemble the behavior of solutions of finite dimensional dynamical systems. In most cases, the inertial manifolds are shown to be Lipschitz continuous. In Mallet–Paret and Sell [24] and Chow, Lu and Sell [7], it is shown that for a large class of evolution equations in Banach spaces, inertial manifolds are in fact C^1 with bounded C^1 norms. This smooth property is very important in applications. The smoothness proof is not trivial even in finite dimensional cases for the center manifold theorem (see, for example, van Gils and Vanderbauwhede [15] and Chow and Lu [6]).

In this paper, we present a theory of smooth invariant manifolds based on the classical method of Liapunov–Perron for continuous semiflows in Banach spaces. Basic hypotheses for these semiflows will be satisfied by semilinear parabolic equations on bounded or unbounded domains or hyperbolic equations. Examples of these continuous semigroups from evolution equations may be found in Bates and Jones [3]. The two basic theorems are stated for nonlinear integral equations. One is on the existence of smooth invariant manifolds (Theorems 4.4) and the other is on exponential attractivity of invariant manifolds (Theorem 5.1). In fact, Theorem 5.1 is related to the squeezing property in Foias Sell and Temam [14]. In §6 and §7, we show how our results are related the center manifold theorem and theorems on inertial manifolds.

In §8, we consider the question of continuous dependence on parameters for invariant manifolds or inertial manifolds. Since our existence theorem (theorem 4.4) is proved by using the uniform contraction theorem, the answer to the above question is obviously true provided the nonlinear equations depend smoothly on parameters. Hence, the interesting cases must involve equations which depend singularly on some parameters. As an example, we consider the following two scalar equations:

$$\epsilon^2 u_{tt} + u_t - u_{xx} = f(u) \tag{1.1}$$

$$u_t - u_{xx} = f(u) \tag{1.2}$$

on the interval $[0,\pi]$ with Dirichlet boundary conditions. We will show that under certain conditions on the nonlinear term f there are inertial manifolds $\mathcal{M}_\epsilon$ and $\mathcal{M}_p$

for equations (1.1) and (1.2) respectively, for all small ϵ. Moreover, dim $\mathscr{M}_\epsilon$ = dim $\mathscr{M}_p$ and $\mathscr{M}_\epsilon$ "approaches" $\mathscr{M}_p$ as $\epsilon \to 0$. In our proof, we use an equivalent inner product in the phase space for the damped wave equation to overcome some technical difficulties. This inner product was first used by Mora [28] and Mora and Sola–Morale [29]. Similar convergence results have also been independently obtained by Mora and Sola–Morales [30]. In Hale and Raugel [20], it is shown that the global attractor of (1.1) approaches that of (1.2). In fact, their results are valid for a much larger class of equations in several space variables.

§2. Notations

Let E_1, E_2 be Banach spaces and U be an open subset of E_1. For any integer $k \geq 0$, let

$$C^k(U,E_2) = \{f \mid f\colon U \to E_2 \text{ k–times differentiable}$$

$$\text{and } \sup |D^i f(x)| < \infty \text{ for } 1 \leq i \leq k\}$$

and

$$|f|_k = \sum_{i=o}^{k} \sup_{x \in U} |D^i f(x)|$$

where D^i is the i–th differentiation operator. Let

$$C^{k,1}(U,E_2) = \{f \mid f \in C^k(U,E_2) \text{ and}$$

$$\sup_{\substack{x,y \in U \\ x \neq y}} \frac{|D^k f(x) - D^k f(y)|}{|x - y|} < \infty\}$$

and $|f|_{k,1} = |f|_k + \text{Lip } D^k f$, where

$$\text{Lip } D^k f = \sup_{\substack{x \neq y \\ x,y \in U}} \frac{|D^k f(x) - D^k f(y)|}{|x - y|}$$

Clearly $C^k(U,E_2)$ and $C^{k,1}(U,E_2)$ are Banach spaces with norms $|\cdot|_k$ and $|\cdot|_{k,1}$.

Let $L^k(E_1,E_2)$ be the Banach space of all k multilinear continuous maps from E_1 into E_2. For $\lambda \in L^k(E_1,E_2,)$,

$$\|\lambda\|_k \text{ or } \|\lambda\|_{L^k(E_1,E_2)}$$

denotes the norm of λ. For notational simplicity, we will sometimes write $\|\lambda\|$ for $\|\lambda\|_k$ provided this will not cause confusion.

Let $J \subsetneq \mathbb{R}$ be an interval (in most cases, we will let $J = \mathbb{R}^- = (-\infty,0]$ or $J = \mathbb{R}^+ = [0,\infty)$). For any $\eta \in \mathbb{R}$ and any Banach space E, we denote by $C_\eta(J,E)$ the following Banach space

$$C_\eta(J,E) = \{f \mid f\colon J \to E \text{ is continuous and}$$

$$\sup_{t\in J} e^{-\eta t} |f(t)|_E < \infty \}$$

with norm

$$|f|_{C_\eta(J,E)} = \sup_{t\in J} e^{-\eta t} |f(t)|_E$$

§3. Linear and nonlinear integral equations.

Let X, Y and Z be Banach spaces. Suppose that $X \subseteq Y \subseteq Z$, X is continuously imbedded in Y and Y is continuously imbedded in Z. Let $S(t)$ ($t \geq 0$) be a strongly continuous semigroup of bounded linear operators on Z.

Consider the following assumptions:

(H_1) $Z = Z_1 \oplus Z_2$, where Z_1 and Z_2 are invariant linear subspaces under $S(t)$.

(H_2) $P_i S(t) = S(t) P_i \quad i = 1,2,$ where P_i is a projection from Z to Z_i.

(H_3) $P_i X$ and $P_i Y$ $(i = 1,2)$ are invariant under $S(t)$ and $S(t)Y \subseteq X$.

(H_4) $S(t)$ can be extended to a group on Z_1.

(H_5) There exist constants $\alpha, \beta, \gamma, \eta, M$ and M^* such that

$$\alpha > 0,\ \beta > 0,\ 0 \leq \gamma < 1,\ M \geq 1,\ M^* \geq 0\ ,$$

$$|e^{-\eta t} S(t) P_1 y|_X \leq M e^{\alpha t} |y|_Y \quad \text{for } t \leq 0,\ y \in Y, \tag{3.1}$$

$$|e^{-\eta t} S(t) P_2 x|_X \leq M e^{-\beta t} |x|_X \quad \text{for } t \geq 0 \text{ and } x \in X, \tag{3.2}$$

$$(3.3)\qquad |e^{-\eta t}S(t)P_2y|_X \le (Mt^{-\gamma}+M^*)e^{-\beta t}|y|_Y \text{ for } t>0 \text{ and } y \in Y.$$

By using (3.1) and (3.2), we have that for any $f(t) \in C_\eta(\mathbb{R}^-,Y)$, the integrals

$$\int_0^t S(t-s)P_1f(s)ds \text{ and } \int_{-\infty}^t S(t-s)P_2f(s)ds$$

exist for all $t \le 0$.. Hence, we can define the following linear operator

$$(3.4)\qquad \mathcal{T}f = \int_0^t S(t-s)P_1f(s)ds + \int_{-\infty}^t S(t-s)P_2f(s)ds\ .$$

Lemma 3.1 The operator $\mathcal{T}$ defined by (3.4) is a bounded linear operator from $C_{\eta+\epsilon}(\mathbb{R}^-,Y)$ to $C_{\eta+\epsilon}(\mathbb{R}^-,X)$ for every $\epsilon \in [0,\alpha)$ and the operator norm of $\mathcal{T}$ satisfies the following estimate

$$\|\mathcal{T}\| \le K(\alpha+\epsilon,\beta-\epsilon,\gamma)$$

where

$$(3.5)\qquad K(\alpha,\beta,\gamma) = M\left(\frac{1}{\alpha} + \frac{2-\gamma}{1-\gamma}\beta^{-1+\gamma}\right) + M^*\frac{1}{\beta}\ .$$

Proof: Obviously $\mathcal{T}$ is a linear operator. We will show that $\mathcal{T}$ is bounded and the estimate is valid. By using (3.1), (3.2), and (3.3), we have

$$|\mathscr{T}f|_{C_{\eta+\epsilon}(\mathbb{R}^-,X)} = \sup_{t\le 0}|e^{-(\eta+\epsilon)t}\mathscr{T}f(t)|_X$$

$$\le \sup_{t\le 0}\{\int_0^t |e^{-(\eta+\epsilon)t}S(t-s)P_1f(s)|_X ds + \int_{-\infty}^t |e^{-(\eta+\epsilon)t}S(t-s)P_2f(s)|_X ds$$

$$\le \{\sup_{t\le 0}\{M[|\int_0^t e^{(\alpha+\epsilon)(t-s)}ds + \int_{-\infty}^t (t-s)^{-\gamma}e^{-(\beta-\epsilon)s}ds|]$$

$$+ M^*\int_{-\infty}^t e^{-(\beta-\epsilon)s}ds\}\}|f|_{C_{\eta+\epsilon}(\mathbb{R}^-,Y)}$$

$$\le \{M[\frac{1}{\alpha+\epsilon} + \int_0^{\frac{1}{\beta-\epsilon}} s^{-\gamma}ds + \int_{\frac{1}{\beta-\epsilon}}^{\infty} (\beta+\epsilon)^{\gamma}e^{-(\beta+\epsilon)s}ds$$

$$+ M^*\int_{-\infty}^0 e^{-(\beta-\epsilon)s}ds\}|f|_{C_{\eta+\epsilon}(\mathbb{R}^-,Y)}$$

$$\le \{M[\frac{1}{\alpha+\epsilon} + \frac{2-\gamma}{1-\gamma}(\beta-\epsilon)^{\gamma-1}] + M^*\frac{1}{\beta-\epsilon}\}|f|_{C_{\eta+\epsilon}(\mathbb{R}^-,Y)} .$$

This completes the proof.

Let $F \in C^k(X,Y)$ and $\varphi \in C^0(\mathbb{R}^-,X)$. Consider the following nonlinear integral equation

$$\varphi(t) = S(t)\xi + \int_0^t S(t-s)P_1F(\varphi(s))ds + \int_{-\infty}^t S(t-s)P_2F(\varphi(s))ds \quad (3.6)$$

Set $(\mathscr{S}(\xi))(t) = S(t)\xi$ and $\mathscr{F}(\varphi)(t) = F(\varphi(t))$ and rewrite (3.6) in the following abstract form:

$$\varphi = \mathscr{S}(\xi) + \mathscr{T}(\mathscr{F}(\varphi)) \ , \tag{3.7}$$

where $\xi \in P_1X$.

<u>Lemma 3.2</u> If $F \in C^1(X,Y)$ and $K(\alpha,\beta,\gamma)(\text{Lip } F) < 1$, then there exists $0 < \epsilon_0 < \alpha$ such that for each $0 \le \epsilon \le \epsilon_0$ and $\xi \in P_1X$, $K(\alpha+\epsilon,\beta-\epsilon,\gamma)(\text{Lip } F) < 1$ and (3.7) has a unique solution $\varphi(\xi) \in C_{\eta+\epsilon}(\mathbb{R}^-,X)$. Moreover, $\varphi : P_1X \to C_{\eta+\epsilon}(\mathbb{R}^-,X)$ is Lipschitz and $\varphi(\xi)$ is independent of $\epsilon \in [0,\epsilon_0]$.

<u>Proof</u>: By the continuity of $K(\alpha,\beta,\gamma)$, there exists $\epsilon_0 > 0$ such that $K(\alpha+\epsilon,\beta-\epsilon,\gamma)(\text{Lip } F) < 1$ for every $\epsilon \in [0,\epsilon_0]$. By (3.1), we have that $\mathscr{S}$ is a bounded linear operator from P_1X to $C_{\eta+\epsilon}(\mathbb{R}^-,X)$ for every $\epsilon \in [0,\epsilon_0]$. Set

$$\mathscr{J}(\varphi,\xi) = \mathscr{S}(\xi) + \mathscr{T}(\mathscr{F}(\varphi)) \ . \tag{3.8}$$

Since $\mathscr{S}$ and $\mathscr{T}$ are bounded linear operators and $F \in C^1$, for any $\varphi_1, \varphi_2 \in C_{\eta+\epsilon}(\mathbb{R}^-,X)$ and $\xi \in P_1X$, we have

$$|\mathscr{J}(\varphi_1,\xi)-\mathscr{J}(\varphi_2,\xi)|_{C_{\eta+\epsilon}(\mathbb{R}^-,X)} \tag{3.9}$$

$$\le |\mathscr{T}(\mathscr{F}(\varphi_1)) - \mathscr{T}(\mathscr{F}(\varphi_2))|_{C_{\eta+\epsilon}(\mathbb{R}^-,X)}$$

$$\leq \|\mathcal{T}\|(\text{Lip } F)|\varphi_1 - \varphi_2|_{C_{\eta+\epsilon}(\mathbb{R}^-,X)}$$

$$\leq K(\alpha-\epsilon,\beta+\epsilon,\gamma)(\text{Lip } F)|\varphi_1 - \varphi_2|_{C_{\eta+\epsilon}(\mathbb{R}^-,X)} .$$

Since $K(\alpha+\epsilon,\beta-\epsilon,\gamma)(\text{Lip } F) < 1$, this implies that $\mathcal{J}$ is a uniform contraction with respect to the variable ξ. By using uniform contraction theorem, we have that for any $\xi \in P_1X$, $\mathcal{J}(\varphi,\xi)$ has a unique fixed point $\varphi_\epsilon(\xi) \in C_{\eta+\epsilon}(\mathbb{R}^-,X)$. It is clear that $\mathcal{J}$ is Lipschitz continuous. Hence, $\varphi_\epsilon(\xi)$ is Lipschitz as a mapping from P_1X into $C_{\eta+\epsilon}(\mathbb{R}^-,X)$. Since $C_{\eta+\epsilon}(\mathbb{R}^-,X) \subsetneq C_\eta(\mathbb{R}^-,X)$, by uniqueness of the fixed point of $\mathcal{J}$, we have $\varphi_\epsilon(\xi) = \varphi_0(\xi)$ for any $\epsilon \in [0,\epsilon_0]$. Define $\varphi(\xi) = \varphi_0(\xi)$. This completes the proof of Lemma 3.2.

Lemma 3.3 (Fiber contraction theorem) Let E_1 and E_2 be Banach spaces and $\overline{U} \subsetneq E_1$ be a closed subset. Suppose that

$$\mathcal{B}: \overline{U} \to \overline{U} \quad ,$$

$$\mathcal{A}_x: E_2 \to E_2 \, , \quad x \in \overline{U}$$

and

$$\mathcal{C}(x,y) = (\mathcal{B}(x), \mathcal{A}_x(y)), \quad x \in \overline{U}, \, y \in E_2$$

are continuous maps. Suppose that $\mathcal{B}$ is a contraction and

$$\sup \{\text{Lip}(\mathcal{A}_x) : x \in \overline{U}\} < 1 \quad .$$

Let the unique fixed point of $\mathscr{B}$ be u and the unique fixed of $\mathscr{A}_u$ be v. Then (u,v) is an attractive fixed point of $\mathscr{C}$.

Proof: See Hirsch and Pugh [22].

Lemma 3.4 Let $k \geq 1$ be an integer and $F \in C^k(X,Y)$. If $\eta < 0$, $\beta+(k-1)\eta > 0$ and $K(\alpha,\beta+(k-1)\eta,\gamma)(\text{Lip } F) < 1$, then the unique solution $\varphi(\xi)$ of equation (3.7) is C^k as a mapping from P_1X into $C_{k\eta}(\mathbb{R}^-,X)$.

Proof: By the definition of $K(\alpha,\beta,\gamma)$ there exists $\epsilon_1 > 0$ such that $\epsilon_1 < \alpha$ and for every $\epsilon \in [0,\epsilon_1]$

$$K(\alpha+\epsilon,\beta+(k-1)\eta-\epsilon,\gamma)(\text{Lip } F) < 1. \tag{3.11}$$

By using Lemma 3.2, equation (3.7) has a unique solution $\varphi(\xi) \in C_{\eta+\epsilon}(\mathbb{R}^-,X)$ for any $\epsilon \in [0,\epsilon_1]$ and $\varphi(\xi)$ is $C^{0,1}$ from P_1X to $C_{\eta+\epsilon}(\mathbb{R}^-,X)$. We will first show that $\varphi(\xi) : P_1X \to C_\eta(\mathbb{R}^-,X)$ is in fact C^1.

Let $\psi(\xi) : P_1X \to C_{\eta+\epsilon_1}(\mathbb{R}^-,X)$ be continuous. Set

$$(\mathscr{T}_1\psi)(t,\xi) = \int_0^t S(t-s)P_1F(\psi(\xi)(s))ds, \quad t \leq 0.$$

The following smoothness properties are needed. Choose an arbitrary but fixed infinite sequence :

$$\epsilon_1 > \delta_1 > \delta_2 > \cdots > 0.$$

<u>Claim 1</u> If $\psi : P_1X \to C_{\eta+\delta_1}(\mathbb{R}^-,X)$ is C^1, then $\mathcal{T}_1\psi : P_1X \to C_{\eta+\delta_2}(\mathbb{R}^-,X)$ is C^1.

<u>Proof of claim 1</u>: Let $\psi(\xi) \in C_{\eta+\delta_1}(\mathbb{R}^-,X)$ be fixed. Assume that $\psi(\xi)$ is C^1 with respect to ξ. Define

$$(3.13)\qquad (\mathcal{D}_1^1(\psi)\cdot\zeta)(t,\xi) = \int_0^t S(t-s)P_1D^1F(\psi(\xi))(D\psi(\xi)\cdot\zeta)(s)ds$$

where $\zeta \in P_1X$ and $D\psi(\xi)$ is the direvative of $\psi(\xi)$ with respect to ξ evaluated at ξ. Let ξ_1 and $\xi_2 \in P_1X$ be given. We have

$$(3.14)\quad I = |e^{-(\eta+\delta_2)t}[(\mathcal{T}_1\psi)(t,\xi_1)-(\mathcal{T}_1\psi)(t,\xi_2)-(\mathcal{D}_1^1(\psi)\cdot(\xi_1-\xi_2))(t,\xi_2))]|$$

$$\le I_1 + I_2 \;,$$

where

$$I_1 = |e^{-(\eta+\delta_2)t}\int_T^t S(t-s)[F(\psi(s,\xi_1)$$

$$-F(\psi(s,\xi_2)-DF(\psi(s,\xi_2))(D\psi(\xi_2)\cdot(\xi_1-\xi_2))(s)]ds|$$

$$I_2 = |e^{-(\eta+\delta_2)t}\int_0^T S(t-s)[F(\psi(s,\xi_1)-F(\psi(s,\xi_2)$$

$$-DF(\psi(s,\xi_2))(D\psi(\xi_2)\cdot(\xi_1-\xi_2))(s)]ds|$$

and $T < 0$ is a fixed constant which is to be chosen later. Since $\mathscr{D}_1^1$ is linear and continuous in $\zeta \in P_1X$, it suffices to show that $I = o(|\xi_1-\xi_2|)$ as $|\xi_1-\xi_2| \to 0$. In other words, it is sufficient to show that for any given $\epsilon > 0$ there exists $\delta > 0$ such that $I \le \epsilon|\xi_1-\xi_2|$ for all $|\xi_1- \xi_2| < \delta$. Let $\epsilon > 0$ be given. Choose $T < 0$ so that

$$\frac{1}{\alpha-\delta_2} e^{(\delta_1-\delta_2)T} 2|F|_1|\psi|_1 < \epsilon/2 .$$

If $t \ge T$, it is clear that $I = o(|\xi_1-\xi_2|)$ as $|\xi_1-\xi_2| \to 0$. Let $t < T < 0$. We have

$$(3.15) \qquad I_1 \le \left|\int_T^t e^{(\alpha-\delta_2)(t-s)+(\delta_1-\delta_2)s} 2|F|_1 \cdot \|D\psi(\xi_2)\|_{L^1(P_1X,C_{\eta+\delta_1}(\mathbb{R}^-,X))} |\xi_1-\xi_2|ds\right|$$

$$\le \frac{1}{\alpha-\delta_2} e^{(\delta_1-\delta_2)T} 2|F|_1|\psi|_1|\xi_1-\xi_2|$$

$$\le \frac{\epsilon}{2}|\xi_1-\xi_2| .$$

Since $T < 0$ is finite, it is not hard to see that $I_2 < \frac{\epsilon}{2} |\xi_1-\xi_2|$ if $|\xi_1-\xi_2|$ is sufficiently small. This proves the claim.

<u>Claim 2</u>: If $\psi : P_1X \to C_{i\eta+\delta_i}(\mathbb{R}^-,X)$ is C^i, then $\mathscr{T}_1\psi : P_1X \to$

$C_{i\eta+\delta_{i+1}}(\mathbb{R}^-,X)$ is C^i for $i = 1,2,\cdots k$.

<u>Proof of claim 2</u> We will prove the claim by induction. Assume that for $i = 1,2,\cdots,k-1$ the claim is true. By induction, we can compute the $(k-1)$th derivative of $\mathcal{T}_1\psi$ with respect to ξ, $D^{k-1}(\mathcal{T}_1\psi)$. It is not hard to see that $D^{k-1}(\mathcal{T}_1\psi)$ has the same integral form as $\mathcal{T}_1$. Using the same argument, we have $D^{k-1}(\mathcal{T}_1\psi)$ is C^1 from P_1X to $C_{k\eta}(\mathbb{R},X)$. This proves the claim.

Let $\psi(\xi) : P_1X \to C_{\eta+\epsilon_1}(\mathbb{R}^-,X)$ be continuous. Similarly set

$$\mathcal{T}_2\psi(\xi) = \int_{-\infty}^{t} S(t-s)P_2F(\psi(s,\xi))ds \quad .$$

<u>Claim 3</u> If $\psi(\xi)$ is C^i from P_1X to $C_{i\eta+\delta_i}(\mathbb{R}^-,X)$ then $\mathcal{T}_2\varphi(\xi)$ is C^i from P_1X to $C_{i\eta+\delta_{i+1}}(\mathbb{R}^-,X)$ for $1 \le i \le k$.

Claim 3 is similar to claim 2 and the proof is omitted.

Now, we will now prove that the unique solution $\varphi : P_1X \to C_\eta(\mathbb{R}^-,X)$ is C^1. Since differentiability is a local property, it is sufficient to show that $\varphi : B \to C_\eta(\mathbb{R}^-,X)$ is C^1 for any fixed but arbitrary bounded ball in P_1X. Let $E_1 = C^0(B,C_\eta(\mathbb{R}^-,X))$ and $E_2 = C^0(B,L^1(P_1X,C_\eta(\mathbb{R}^-,X)))$. Let $\psi \in E_1$ and $\Psi \in E_2$. Define

$$\mathcal{B}(\psi)(t,\xi) = S(t)\xi + \int_0^t S(t-s)P_1F(\psi(\xi))ds + \int_{-\infty}^t S(t-s)P_2F(\psi(\xi))ds.$$

and

$$\mathscr{A}_{\psi}(\Psi) = S(t)\xi + \int_0^t S(t-s)P_1 DF(\psi(s,\xi))\Psi ds + \int_{-\infty}^t S(t-s)P_2 DF(\psi(s,\xi))\Psi ds .$$

In the definition of $\mathscr{A}_{\psi}(\Psi)$, we assume that for every $\zeta \in P_1X$, $\mathscr{A}_{\psi}(\Psi)\cdot\zeta \in C^0(B,C_{\eta}(\mathbb{R}^-,X))$ and is defined by

$$(\mathscr{A}_{\psi}(\Psi)\cdot\zeta)(t,\xi) = S(t)\xi + \int_0^t S(t-s)P_1 DF(\psi(\xi))(\Psi\cdot\zeta)(s,\xi)ds$$

$$+ \int_{-\infty}^t S(t-s)P_2 DF(\psi(\xi))(\Psi\cdot\zeta)(s,\xi)ds .$$

Since $(\mathrm{Lip}\ F)K(\alpha,\beta,\gamma) < 1$, $\mathscr{A}_{\psi}(\cdot)$ is a uniform contraction. Hence, $\mathscr{A}_{\psi}(\cdot)$ has a unique fixed point Ψ_{ψ} for every $\psi \in E_1$.

By Lemma 3.2, $\mathscr{B}$ is a contraction in E_1. Let $\varphi(\xi)$ be the unique fixed point of $\mathscr{B}$ and $\Phi \in E_2$ the unique fixed point of $\mathscr{A}_{\varphi}(\cdot)$. We claim that $\Phi = D_{\xi}\varphi$. To prove our claim, let

$$\mathscr{C}(\psi,\Psi) = (\mathscr{B}(\psi),\mathscr{A}_{\psi}(\Psi)) .$$

By fiber contraction theorem, (φ,Φ) is an attractive fixed point of $\mathscr{C}$. This says that for every $\psi \in E_1$ and $\Psi \in E_2$, we have

$$\mathscr{C}^n(\psi,\Psi) \to (\varphi,\Phi) \text{ as } n \to \infty.$$

where $\mathscr{C}^n$ denotes the nth iterate of $\mathscr{C}$.

Fixed $\psi \in C^1(B, C_{\eta+\delta_1}(\mathbb{R}^-, X))$. By claims 2 and 3, $\mathcal{T}\psi \in C^1(B, C_{\eta+\delta_2}(\mathbb{R}^-, X))$ and

$$(D\mathcal{T}\psi\cdot\zeta)(t,\xi) = \int_0^t S(t-s)P_1DF(\psi(\xi))(D\psi(\xi)\cdot\zeta)(s)ds$$

$$+\int_{-\infty}^t S(t-s)P_2DF(\psi(\xi))(D\psi(\xi)\cdot\zeta)(s)ds\,.$$

This implies that $D\mathcal{T}\psi \in E_2$ because $C_{\eta+\delta}(\mathbb{R}^-, X) \subseteq C_\eta(\mathbb{R}^-, X)$ for all $\delta > 0$. Thus,

$$\mathscr{C}(\psi, D\psi) = (\mathscr{B}(\psi), \mathscr{A}_\psi(D\psi)) = (\mathscr{B}(\psi), D\mathscr{B}(\psi));$$

$$\mathscr{C}^2(\psi, D\psi) = (\mathscr{B}^2(\psi), \mathscr{A}_{\mathscr{B}(\psi)}\circ D\mathscr{B}(\psi)) = (\mathscr{B}^2(\psi), D\mathscr{B}^2(\psi));$$

...

and

$$\mathscr{C}^n(\psi, D\psi) = (\mathscr{B}^n(\psi), \mathscr{A}_{\mathscr{B}^{n-1}(\psi)}\circ\cdots\circ\mathscr{A}_{\mathscr{B}(\psi)}\circ D\mathscr{B}(\psi)) = (\mathscr{B}^n(\psi), D\mathscr{B}^n(\psi))\,.$$

We note that,

$$\mathscr{A}_{\mathscr{B}^{n-1}(\psi)}\circ\cdots\circ\mathscr{A}_{\mathscr{B}(\psi)}\circ D\mathscr{B}(\psi) \in E_2\,.$$

By the attractivity of the pair (φ,Φ), we have $\mathscr{B}^n(\psi) \to \varphi$ and $D\mathscr{B}^n(\psi) \to \Phi$ as $n \to \infty$. This implies $\Phi = D\varphi$ and φ is C^1..

Next we assume the theorem is true up to k–1 and we will use induction. By claims 1 and 2, we have

$$D^i\varphi(\xi) \in C^0(B,C_{\eta_i}(\mathbb{R}^-,X)),\ \ \eta_i = i\eta-(k-i)\epsilon_1 k^{-1},$$

for $i = 1, \cdots k-1$. Let $E_1^k = C^0(B,L^{k-1}(P_1X,C_\eta(\mathbb{R}^-,X)))$ and $E_2^k = C^0(B,L^k(P_1X,C_\eta(\mathbb{R}^-,X)))$. By differentiating $\mathscr{A}$ and $\mathscr{B}$ formally, we define for $\omega \in E_1^k$ and $\Omega \in E_2^k$ the following functions:

$$\mathscr{A}_{k,\omega}(\Omega) = \int_0^t S(t-s)P_1DF(\varphi(\xi))\Omega(\xi)ds$$

$$+ \int_0^t S(t-s)P_1D^2F(\varphi(\xi))[(k-1)D\varphi(\xi)\omega(\xi)+\omega(\xi)D\varphi(\xi)]ds$$

$$+ R_1^{k-1} + \int_{-\infty}^t S(t-s)P_2DF(\varphi(\xi))\Omega(\xi)ds$$

$$+ \int_{-\infty}^t S(t-s)P_2D^2F(\varphi(\xi))[(k-1)D\varphi(\xi)\omega(\xi)+\omega(\xi)D\varphi(\xi)]ds + R_2^{k-1}$$

and

$$\mathscr{B}_k(\omega) = \int_0^t S(t-s)P_1DF(\varphi(\xi))\rho(\xi)ds + R_3^{k-1}$$

where R_i^{k-1}, $i = 1,2,3$, is an appropriate term involving derivatives of φ with respect to ξ of order at most k–1. By using the same arguments as in the case $k = 1$, we have that $D^{k-1}\varphi : P_1X \to L^{k-1}(P_1X, C^0(B,C_\eta(\mathbb{R},X))$ is continuously differentiable.

This completes the proof of Lemma 3.3.

§4. Invariant manifolds.

Let $F \in C^k(X,Y)$. Consider the ~~following~~ integral equation

$$(4.1) \qquad x(t) = S(t-t_0)x(t_0) + \int_{t_0}^{t} S(t-t_0)F(x(s))ds$$

where $x(t)$ is a map from an interval $J \subseteq \mathbb{R}$ to X.

<u>Definition 4.1</u>: If $x\colon J \to X$ is continuous and satisfies (4.1) for all $t_0, t \in J$, $t_0 \le t$, then we call $x(t)$ a solution of (4.1) on J. For $x_0 \in X$, we denote by $x(t,x_0)$ a solution of (4.1) which equals to x_0 at $t = 0$.

<u>Lemma 4.2</u> Let $\eta < 0$. Assume that (H_1)–(H_5) are satisfied. Let $x(t)$ be a solution of (4.1) on $\mathbb{R}^-$. Then the following properties are equivalent

(i) $$P_2x(t) \in C^0(\mathbb{R}^-,P_2X) .$$

(ii) $$x(t) \in C_\eta(\mathbb{R}^-,X) .$$

(iii) $x(t)$ can be expressed as

$$x(t) = S(t)x(0) + \int_0^t S(t-s)P_1F(x(s))ds + \int_{-\infty}^{t} S(t-s)P_2F(x(s))ds .$$

<u>Proof</u>. First we prove that (i) implies (ii). Since

$$C^0(\mathbb{R}^-,X) \subseteq C_\eta(\mathbb{R}^-,X)$$

we have that (i) implies $P_2x(t) \in C_\eta(\mathbb{R}^-,X)$. Since $x(t)$ is a solution of (4.1) on $\mathbb{R}^-$, By using (H_4) and (4.1) we have

$$P_1x(t) = S(t)P_1x(0) + \int_0^t S(t-s)P_1F(x(s))ds . \tag{4.2}$$

By using (3.1), we have that $P_1x(t) \in C_\eta(\mathbb{R}^-,X)$. Hence (i) implies (ii).

Next we show that (ii) implies (iii). By (4.1), we have (4.2) and

$$P_2x(t) = S(t-t_0)P_2x(t_0) + \int_{t_0}^t S(t-s)F_2(x(s))ds . \tag{4.3}$$

$$\begin{aligned} |S(t-t_0)P_2x(t_0)|_X &\le Me^{-(\beta+\eta)(t-t_0)}|x(t_0)|_X \\ &\le Me^{-(\beta+\eta)(t-t_0)+\eta t_0}|x|_{C_\eta(\mathbb{R}^-,X)} \\ &\le Me^{-\beta t_0-(\beta+\eta)t}|x|_{C_\eta(\mathbb{R}^-,X)} . \end{aligned} \tag{4.4}$$

Letting $t_0 \to -\infty$ in (4.3) and using (4.4), we have

$$P_2x(t) = \int_{-\infty}^t S(t-s)P_2F(x(s))ds .$$

Hence,

$$x(t) = S(t)P_1x(0) + \int_0^t S(t-s)P_1F(x(s))ds$$

$$+ \int_{-\infty}^t S(t-s)P_2F(x(s))ds \ .$$

Finally, we show that (iii) implies (i). Since F is bounded, we have

$$|P_2x(t)| \le |F|_0\{M\int_{-\infty}^t (t-s)^{-\gamma} e^{-(t-s)(\beta-\eta)}ds + N\int_{-\infty}^t e^{-(t-s)(\beta-\eta)}ds\}$$

$$\le |F|_0\{M\,\frac{2-\gamma}{1-\gamma}\,(\beta-\eta)^{-1+\gamma} + N\frac{1}{\beta-\eta}\}$$

Hence,

$$P_2x(t) \in C^0(\mathbb{R}^-,P_2\,X).$$

This completes the proof.

Theorem 4.4. Let $\eta < 0$. Assume that (H_1)–(H_5) are satisfied. If $F \in C^k(X,Y)$, $\beta+(k-1)\eta > 0$ and

$$K(\alpha,\beta+(k-1)\eta,\gamma)(\mathrm{Lip}\ F) < 1,$$

then there exists a C^k invariant manifold $\mathcal{M}$ for the flow defined by (4.1) and $\mathcal{M}$ satisfies

(i) $\mathscr{M} = \{x_0 \mid x(t,x_0)$ is defined for all $t \in \mathbb{R}^-$ and $P_2x(t,x_0) \in C^0(\mathbb{R}^-,X)\}$

(ii) $$\mathscr{M} = \{\xi + h(\xi) \mid \xi \in P_1X\}$$

where $h\colon P_1X \to P_2X$ is C^k.

<u>Proof</u>: Let

$$h(\xi) = \int_{-\infty}^{0} S(t-s)P_2F(\varphi(\xi)(s))ds,$$

where $\varphi(\xi)$ is the unique solution of (3.7). By using Lemma 3.4, we have $h(\xi) = \varphi(\xi)(0) - S(0)\xi$ is C^k from P_1X to P_2X. To prove that $\mathscr{M}$ is invariant, let $x_0 \in \mathscr{M}$. Since F is globally Lipschitz and $x_0 \in \mathscr{M}$, the unique solution $x(t,x_0)$ is defined for all $t \in \mathbb{R}$ and $x(\cdot,x_0) \in C_\eta(\mathbb{R}^-,X)$. Furthermore, $x(t,x_0) \in \mathscr{M}$ for all $t \le 0$. Let $t_1 > 0$.. Since $x(t,x_0)$ is a solution of (4.1), $y(t) = x(t+t_1,x_0)$ satisfies

$$y(t) = S(t-t_0)y(t_0) + \int_{t_0}^{t} S(t-s)F(y(s))ds$$

for all $-\infty < t_0 \le t \le 0$. Since $x(\cdot,x_0) \in C_\eta(\mathbb{R}^-,X)$, $y(\cdot) = x(\cdot+t_1,x_0) \in C_\eta(\mathbb{R}^-,X)$. Hence, $y(0) = x(t_1,x_0) \in \mathscr{M}$. This completes the proof of the theorem.

§5. Exponential attractivity

In this section, we will prove that the invariant manifold $\mathcal{M}$ obtained in Theorem 4.4 is exponentially attractive. More precisely, we have the following.

Theorem 5.1 Let $\eta < 0$. Assume (H_1)–(H_5) are satisfied. If $F \in C^1(X,Y)$, $K(\alpha,\beta,\gamma)(\text{Lip } F) < 1$ and

$$(5.1) \qquad \frac{MK(\alpha,\beta,\gamma)\text{Lip}(F)}{1 - K(\alpha,\beta,\gamma)\text{Lip}(F)} < 1,$$

then for any solution $x(t,x_0)$ of (4.1) on $[0,\infty)$, there exists a unique $x_0^* \in \mathcal{M}$ such that

$$\sup_{t\geq 0} e^{-\eta t}|(x(t,x_0)-x(t,x_0^*)|_X < +\infty \ .$$

Proof: By Theorem 4.4, $\mathcal{M}$ is a C^1 invariant manifold. Let x and x^* be any two solutions of (4.1) on $(0,\infty)$ and $w = x^*-x$. Hence, $w(t)$ satisfies the following equation

$$(5.2) \qquad w = S(t-t_0)w(t_0) + \int_{t_0}^{t} S(t-s)(F(w+x)-F(x))ds \ .$$

As in §4, it can be shown that if w is a solution of (5.2) then $w \in C_\eta(\mathbb{R}^+,X)$ if and only if

$$w = S(t)\omega_2 + \int_0^t S(t-s)P_2(F(w+x)-F(x))ds$$

$$+ \int_\infty^t S(t-s)P_1(F(w+x)-F(x))ds$$

where $\omega_2 = P_2w(0) = P_2x^*(0)-P_2x(0) = \xi_2^* - \xi_2$.

Let $\mathscr{S}(\omega_2) = S(t)\omega_2$ and

$$\mathscr{Y}(w,x) = \int_0^t S(t-s)P_2(F(w+x)-F(x))ds$$

$$+ \int_\infty^t S(t-s)P_1(F(w+x)-F(x))ds.$$

Clearly $\mathscr{S}$ is a bounded linear operator from P_2X to $C_\eta(\mathbb{R}^+,X)$ and $\mathscr{Y}$ takes $C_\eta(\mathbb{R}^+,X)$ into itself. For any w_1 and $w_2 \in C_\eta(\mathbb{R}^+,X)$, we have

$$|e^{\eta t}(\mathscr{Y}(w_1,x)-\mathscr{Y}(w_2,x))|_X$$

$$\leq |e^{\eta t}\int_0^t S(t-s)P_2(F(w_1+x)-F(w_1+x)-F(w_2+x))ds|_X$$

$$+|e^{\eta t}\int_0^t S(t-s)P_1(F(w_1+x)-F(w_2+x))ds|_X$$

$$\leq \{M[\tfrac{1}{\alpha} + \tfrac{2-\gamma}{1-\gamma}\beta^{\gamma-1}] + M^*\beta^{-1}\}(\mathrm{Lip}\ F)|w_1- w_2|_{C_\eta(\mathbb{R}^+,X)}$$

Hence, $\mathscr{L}(\omega_2,w,x) = \mathscr{A}(\omega_2) + \mathscr{Y}(w,x)$ is a uniform contraction with respect to x and ω_2. By using uniform contraction theorem, we have that for any $\omega_2 \in P_2X$, and any solution x(t) of (4.1) $\mathscr{L}$ has a unique fixed point $\hat{w}(x,w_2)$.

Furthermore, if $\omega_1 = P_1\hat{w}(x,\omega_2)(0)$, then

$$(5.3) \qquad \omega_1 = \int_{+\infty}^{0} S(-s)P_1[F(\hat{w}(x,\omega_2)+x(s))-F(x))]ds = g(x,\omega_2).$$

Note that g is C^{k-1} and $\hat{w} = x^* - x$. Let $P_1x^*(0) = \xi_1^*$ and $P_1x(0) = \xi_1$. Thus, $x^* \in \mathscr{M}$ if and only if $\xi_2^* = h(\xi_1^*)$, where h is given in Theorem 4.4. By using (5.3), $x^* \in \mathscr{M}$ if and only if

$$(5.4) \qquad \xi_1^* = \xi_1 + g(x,h(\xi_1^*) - \xi_2) .$$

Since Lip(g) < 1 and Lip(h) < 1, by using condition (5.1) we have that for every solution x(t) of (4.1) on $[0,\infty)$ equation (5.4) has a unique solution ξ_1^*. This proves the theorem.

§6 Semilinear evolution equations

As a simple application of the results in §4, we will show how one can obtain C^k global center unstable manifolds for abstract semilinear evolution equations in Banach spaces. We will not prove the existence of local C^k center unstable manifolds since they can be obtained by using a cut off function. We refer the readers to Carr [4] for more detail.

Consider the following semilinear evolution equation in the Banach space $Z = Z_1 \oplus Z_2$, where Z_1 and Z_2 are subspaces of Z.

$$(6.1) \qquad \begin{cases} \dot{x} + Ax = f(x,y) \\ \dot{y} + By = g(x,y) \end{cases}$$

where $x \in Z_1$ and $y \in Z_2$, A and B are linear operator from their domains $\mathscr{D}(A)$ and $\mathscr{D}(B)$ into Z_1 and Z_2 respectively, and f and g are nonlinear maps. We assume that B is a sectorial operator [21]. For $o \leq \gamma \leq 1$, let B^γ be the γ–fractional power of B. The domain of B^γ is $\mathscr{D}(B^\gamma) = Z_2^\gamma$. It is well known that Z_2^γ is a Banach Space with norm $|x|_\gamma = |B^\gamma x|$. Note that $Z_2^0 = Z_2$.

Let $0 \leq \gamma < 1$ be fixed. Assume that $f\colon Z_1 \times Z_2^\gamma \to Z_1$ and $g\colon Z_1 \times Z_2^\gamma \to Z_2$ satisfy the following conditions

$$f(x,y) = 0(|x|^2 + |y|_\alpha^2) \text{ and } g(x,y) = 0(|x|^2 + |y|_\alpha^2)$$

as $(x,y) \to (0,0)$. Assume that the spectra $\sigma(A)$ and $\sigma(B)$ of A and B satisfy the following conditions

$$\begin{cases} \mathrm{Re}(\sigma(A)) \le \lambda_1 \le 0 \\ \mathrm{Re}(\sigma(B)) > \lambda_2 > 0 \end{cases} . \tag{6.2}$$

We also assume

$$A : X_1 \to X_1 \text{ is bounded.} \tag{6.3}$$

Let $\mathcal{M}$ be an invariant manifold of (7.1). $\mathcal{M}$ is called a global center unstable manifold if it is the graph of a C^1 map $h: Z_1 \to Z_2$ which satisfies $h(0) = 0$ and $Dh(0) = 0$.

Since A is bounded and B is sectorial, the linear operator

$$\begin{bmatrix} -A & 0 \\ 0 & -B \end{bmatrix}$$

generates an analytic semigroup $S(t)$ on Z. Set $X = Z_1 \oplus Z_2^\gamma$ and $Y = Z$. It can be shown [21] that (H_1)–(H_4) are satisfied. We will see that (H_5) is also satisfied. Since (6.2) and (6.3) are true and B is a sectorial operator, there exists a constant $\omega_2 > 0$ such that for every small $\omega_1 > 0$ there exists $M \ge 1$, such that

$$|e^{-tA}| \le Me^{-\omega_1 t}, \quad t \le 0 \tag{6.4}$$

$$|e^{-tB}| \le Me^{-\omega_2 t}, \quad t \ge 0 \tag{6.5}$$

$$|B^{\gamma}e^{-tB}| \le M\frac{1}{t^{\gamma}}e^{-\omega_2 t}, \quad t > 0 \ . \tag{6.6}$$

Let $\omega_1 < \eta < \omega_2$, $\alpha = \eta - \omega_1 > 0$, and $\beta = \omega_2 - \eta > 0$. We have that (H_5) is satisfied. By using Theorems 4.4 and 5.1, we have the following center unstable manifold theorem.

Theorem 6.1: Assume that conditions (6.2) and (6.3) are satisfied. Assume that $0 < \gamma < 1$. For any integer $k \ge 1$, if

$$f \in C^k(Z_1 \oplus Z_2^{\gamma}, Z_1), \; g \in C^k(Z_1 \oplus Z_2^{\gamma}, Z_2),$$

$$k\eta < \omega_2 \text{ and } K(\eta-\omega_1, \omega_2-k\eta, \gamma)(\mathrm{Lip}(f) + \mathrm{Lip}(g)) < 1 \ ,$$

then (6.1) has a C^k global center unstable manifold $\mathcal{M}$. Furthermore, if $|\mathrm{Lip}(f)+\mathrm{Lip}(g)|$ is sufficiently small, then $\mathcal{M}$ is exponentially attractive.

Remark 6.2 In Theorem 6.1, we do not require C^k norms of f and g to be small.

§7 C^k inertial manifolds.

Consider the following equation in the Banach Space Z

$$
\begin{cases} \frac{dz}{dt} + Az + R(z) = 0 \\ z(0) = z_0 \end{cases} \tag{7.1}
$$

where A is a sectorial operator on Z, $R(z)$ is a nonlinear map from X^{ρ_1} to X^{ρ_2} where the exponents ρ_1 and ρ_2 satisfy either $1 \geq \rho_1 \geq \rho_2 > 0$, or $1 > \rho_1 \geq \rho_2 \geq 0$.

An invariant manifold $\mathscr{M}$ of (7.1) is called an inertial manifold of (7.1) if it is a finite dimensional Lipschitz manifold and is globally attractive. In this section we will applied the results obtained in §4 and §5 to the abstract nonlinear evolution equations (7.1) to obtain C^k inertial manifolds. Applications will also considered.

Assume that the spectrum of A, $\sigma(A)$, satisfies the following conditions

$$
\sigma(A) = \sigma_1(A) \cup \sigma_2(A) \quad , \tag{7.2}
$$

$$
\lambda_1 = \sup\{\mathrm{Re}\lambda : \lambda \in \sigma_1(A)\} < \inf\{\mathrm{Re}\lambda : \lambda \in \sigma_2(A)\} = \lambda_2 \ . \tag{7.3}
$$

(7.4) $\sigma_1(A)$ consists of only eigenvalues with finite, multiplicities and is a finite set

$$\mathrm{Re}\,\sigma(A) > 0 \,. \tag{7.5}$$

Let P_1 be the projection associated with $\sigma_1(A)$ and $P_2 = I - P_1$. Then there exist constants $M \geq 1$, $\omega_1 > 0$ and $\omega_2 > 0$ such that

$$|P_1 e^{-At} x|_{\rho_1} \leq M e^{\omega_1 |t|} |x|_{\rho_2}, \quad \text{for all } t \in \mathbb{R} \tag{7.6}$$

$$|P_2 e^{-At} x|_{\rho_1} \leq M e^{-\omega_2 t} |x|_{\rho_1}, \quad t \geq 0 \tag{7.7}$$

$$|P_2 e^{-At} x|_{\rho_1} \leq M t^{\rho_2 - \rho_1} e^{-\omega_2 t} |x|_{\rho_2}, \quad t > 0 \,. \tag{7.8}$$

Let $\omega_1 < \eta < \omega_2$, $\alpha = \eta - \omega_1$, $\beta = \omega_2 - \eta$, and $\gamma = \rho_1 - \rho_2$. Then hypotheses (H_1)–(H_5) are satisfied. By using Theorems 4.4 and 5.1, we have

Theorem 7.1: If $R \in C^k(X^{\rho_1}, X^{\rho_2})$ and there exists $\eta > 0$ such that $\omega_1 < \eta < k\eta < \omega_2$, $\mathrm{Lip}(F) K(\alpha, \beta - (k-1)\eta, \gamma) < 1$ and

$$\frac{M K(\alpha,\beta,\gamma) \mathrm{Lip}(F)}{1 - K(\alpha,\beta,\gamma) \mathrm{Lip}(F)} < 1,$$

then (7.1) has a C^k inertial manifold.

Example 7.2 Let Z be a Hilbert space. Consider the following problem [14]:

$$\begin{cases} \dfrac{du}{dt} + Au + R(u) = 0 \\ u(0) = u_0 \end{cases} \tag{7.9}$$

where $u \in Z$, A is positive self–adjoint linear operator with domain $\mathscr{D}(A)$ dense in Z,

$$R(u) = Cu + B(u,u) + f$$

where C is linear, B is bilinear and $f \in Z$ is fixed.. Assume A has a compact inverse A^{-1}. Hence, the spectrum of consists of only eigenvalues λ_i, $i = 1,2,...,$ satisfying:

$$\lambda_1 \leq \lambda_2 \leq \cdot \ldots \cdot \leq \lambda_i \to \infty, \quad \text{as } i \to \infty \quad .$$

Let $e_i \in Z$, $i = 1,2,...,$ be the eigenvector of A corresponding to eigenvalues λ_i. Let $N > 0$ be an integer and P_1 be the projection from H into $\text{span}\{e_1, \ldots ,e_N\}$ and $P_2 = I - P_1$. Furthermore, we assume $f \in D(A^{1/2})$, C and B satisfy the following conditions

$$\text{(7.10)} \qquad |A^{1/2}B(u,v)| \leq c_1|Au||Av| \quad \text{for all } u,v \in D(A^{1/2})$$

$$\text{(7.11)} \qquad |A^{1/2}C\,u| \leq c_2|Au| \quad \text{for all } u \in D(A^{1/2})$$

where $c_1, c_2 \geq 0$ are constants.

Since A is a positive self–adjoint operator with compact inverse in the Hilbert space Z, we have the following properties.

$$\text{(7.12)} \qquad |P_1 e^{-At}| \leq e^{\lambda_N|t|}, \quad \text{for } t \in \mathbb{R}$$

$$(7.13) \qquad |A^{1/2}e^{-At}P_1| \le \lambda_N^{1/2}e^{\lambda_N|t|}, \quad \text{for } t \in \mathbb{R}$$

$$(7.14) \qquad |P_2e^{-At}| \le e^{-\lambda_{N+1}t}, \quad \text{for } t \ge 0$$

$$(7.15) \qquad |A^{1/2}P_2e^{-At}| \le (t^{-1/2} + \lambda_{N+1}^{1/2})e^{-\lambda_{N+1}t}, \quad \text{for } t > 0 \,.$$

For many equations in applications, e.g., 2D Navier–Stokes equations [9], the flows are dissipative, i.e., there exists a bounded ball in an appropriate function space such that every solution will eventually enter the bounded ball and stay there for all future time. Hence, the study of asymptotic behavior of solutions can be reduced to the study of a modified equation:

$$(7.16) \qquad \frac{du}{dt} + Au + F(u) = 0$$

where

$$F(u) = \theta_\epsilon(|Au|)R(u),\ \theta_\epsilon(s) = \theta(\tfrac{s}{\epsilon}),\ \theta(s) \in C_0^\infty(\mathbb{R})\ 0 \le \theta(s) \le 1$$

$$\theta(s) = 1 \ \text{ for } \ |s| \le 1,\ \ \theta(s) = 0 \ \text{ for } \ |s| \ge 2.$$

and $\epsilon > 0$ is some constant. Since $\theta \in C_0^\infty(R)$ and the norm of Hilbert space is smooth,

$$F(u) \in C^k(Z^1, Z^{1/2})$$

for any integer $k \geq 0$. For more detail, see [8], [13] and [14].

By Theorem 7.1, equation (7.16) has a C^1 inertial manifold provided

$$\operatorname{Lip}(F)K(\alpha,\beta,\gamma) < 1.$$

Since Lip(F) is only a finite number, we need to have $K(\alpha,\beta,\gamma)$ small. Recall that we may choose

$$\alpha = \beta = \frac{\lambda_{N+1} - \lambda_N}{2} \quad .$$

It is not difficult to see that $K(\alpha,\beta,1/2) \to 0$ as $(\lambda_{N+1}^{1/2} - \lambda_N^{1/2}) \to \infty$. This says that if the gap $(\lambda_{N+1}^{1/2} - \lambda_N^{1/2})$ is sufficiently large, then equation (7.16) has a C^1 inertial manifold.

<u>Example 7.3</u> Consider the Kuramoto–Sivashinsky equation [12], [13] and [27]:

$$\frac{\partial u}{\partial t} + \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x} = 0 \quad \text{in } [0,\pi]\times\mathbb{R}^+ \tag{7.17}$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{7.18}$$

and

(7.19) $$\frac{\partial}{\partial x}u(o,t) = \frac{\partial^2}{\partial x^2}u(\pi,t) = o \quad .$$

Let $A = \frac{\partial^4}{\partial x^4}$, $B(u,v) = u\frac{\partial v}{\partial u}$ and $Cu = \frac{\partial u}{\partial x^2}$.

The operator A with boundary condition (7.18), (7.19) has eigenvalues

$$\lambda_k = k^4, \ k = 1,2,3,\dots \quad .$$

It is not hard to see that Example 7.2 is applicable in this example provided the flow is dissipative.

Example 7.4: Consider the following reaction–diffusion equation

(7.20) $$u_t - u_{xx} = f((u) \qquad 0 \le x \le \pi$$

with boundary condition

(7.21) $$u(0,t) = u(\pi,t) = 0 \quad .$$

For simplicity, we assume $f \in C^1(L^2(0,\pi),L^2(0,\pi))$. Since the eigenvalues of operator $A = -\partial^2/\partial x^2$ with boundary condition (7.20) are $\lambda_n = n^2$, $n = 1,2,\cdots$, Example 7.2 is again applicable in this case provided the flow (7.20) (7.21) is dissipative.

§8. Singularly perturbed wave equation.

In this section, we will consider a scalar semilinear parabolic equation in the interval $[0,\pi]$:

$$(8.1)\qquad \begin{cases} U_t - U_{xx} = f(U) & 0 \le x \le \pi \\ U(t,0) = U(t,\pi) = 0 \\ U(0,x) = U_0(x)\,. \end{cases}$$

and a singularly perturbed scalar semilinear wave equation in $[0,\pi]$:

$$(8.2)\qquad \begin{cases} \frac{\epsilon^2}{4} u_{tt} + u_t - u_{xx} = f(u), \quad 0 \le x \le \pi \\ u(t,0) = u(t,\pi) = 0 \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x)\,. \end{cases}$$

where $U_0 \in L^2(0,\pi)$, $u_0 \in H_0^1(0,\pi)$, $u_1 \in L^2(0,\pi)$ and f is C^1 from $\mathbb{R}$ into itself. In this section, we will show that under some conditions, for sufficiently small ϵ (8.2) has an inertial manifold $\mathcal{M}_\epsilon$ which "approaches" to an inertial manifold $\mathcal{M}$ of (8.1) as ϵ approaches 0. Precise convergence statements are given in Theorems 8.6 and 8.8.

Recently, it is shown by Hale and Raugel [20] and Babin and Vishik [2] that under some mild conditions on f, for all $\epsilon > 0$ there exists a compact (global) attractor for equation (8.2). Moreover, for sufficiently small $\epsilon > 0$, these attractors are uniformly bounded. Thus, we may assume without loss of generality that equations (8.1) and (8.2) are modified equations (see §7). Hence, we assume $f \in C^1(L^2(0.\pi),L^2(0.\pi))$, i.e, the mapping $v(x) \to f(v(x))$, $0 \le x \le \pi$, is C^1 as a mapping from $L^2(0,\pi)$ into itself and has bounded C^1 norm.

We will rewrite equation (8.2) as a system of first order equations. For technical reasons, we consider the following change of variables:

$$u_t = -2\epsilon^{-2}u + 2\epsilon^{-1}v, \quad \text{and} \quad w = (u,v).$$

We can rewrite equation (8.2) as a system:

$$w_t = C_\epsilon w + 2\epsilon^{-1}\hat{f}(w), \tag{8.3}$$

where

$$C_\epsilon = -2\epsilon^{-2}I + 2\epsilon^{-1}\begin{bmatrix} 0 & 1 \\ \epsilon^{-2}-A & 0 \end{bmatrix}, \; A = -\partial^2/\partial x^2 \text{ and } \hat{f}(w) = \begin{bmatrix} 0 \\ f(u) \end{bmatrix}$$

Let $X = H_0^1(0,\pi)\times L^2(0,\pi)$ and $N > 0$ be an integer. Set

$$X_N = \text{span}\left\{\binom{\sin px}{0}, \binom{0}{\sin px} : P = N+1,N+2,\ldots\right\} \tag{8.4}$$

$$X_N^\perp = \text{span}\left\{\binom{\sin x}{0},\ldots,\binom{\sin Nx}{0},\binom{0}{\sin x},\ldots,\binom{0}{\sin Nx}\right\}. \tag{8.5}$$

Clearly $X = X_N \oplus X_N^\perp$, X_N is orthogonal to $X_N^\perp$ and $\dim X_N^\perp = 2N$.. Moreover, both X_N and $X_N^\perp$ are invariant subspaces of the operator C_ϵ. We also note that the spectrum of C_ϵ consists of only eigenvalues.

Define an equivalent inner product in $H_0^1(0,\pi)$ by

$$<u,v> = ((A+(\frac{1}{\epsilon^2}-2(N+1)^2))^{\frac{1}{2}}u, (A+(\frac{1}{\epsilon^2}-2(N+1)^2))^{\frac{1}{2}}v)_{L^2}$$

where $(\, , \,)_{L^2}$ is the usual inner product in L^2. By using the above inner product in $H_0^1(0,\pi)$, we define the following equivalent inner product in the product space $X = H_0^1(0,\pi)\times L^2(0,\pi)$ by

$$<< w_1,w_2 >> = < u_1,u_2 > + (v_1,v_2)_{L^2}$$

where $w_i = (u_i,v_i)$, $i = 1,2$. The norm induced by $<<\cdot,\cdot>>$ will be denoted by $\|\cdot\|$.

<u>Lemma 8.1</u> There exist an ϵ dependent decomposition $X = X_N \oplus X_N^- \oplus X_N^+$ with projections P_N, P_N^-, P_N^+ respectively, where X_N is as in (8.4) and $X_N^\pm \subseteq X_N^\perp$ (see (8.5)) such that

(i) X_N, X_N^- and X_N^+ are invariant subspaces of C_ϵ

(ii) $$\|e^{C_\epsilon t}P_N\| \le e^{\frac{-2+2(1-\epsilon^2(N+1)^2)^{1/2}}{\epsilon^2}t}, \qquad t \ge 0$$

$$\|e^{C_\epsilon t}P_N^-\| \le \|P_N^-\|e^{\frac{-2+2(1-\epsilon^2(N+1)^2)^{1/2}}{\epsilon^2}t}, \qquad t \ge 0$$

$$\|e^{C_\epsilon t}P_N^+\| \le \|P_N^+\| e^{\frac{-2+2(1-\epsilon^2N^2)^{1/2}}{\epsilon^2}t}, \qquad t \le 0$$

where $\|\cdot\|$ denotes the operator norm in the Hilbert space $(X,<<\cdot,\cdot>>)$.

(iii) $\|P_N\| = 1$ and there exists $\epsilon_0 > 0$ such that for every $0 < \epsilon < \epsilon_0$, $\|P_N^-\| \le 2$ and $\|P_N^+\| \le 2$.

Proof We have that $X = X_N \oplus X_N^\perp$ and $X_N, X_N^\perp$ are invariant subspaces of X. By restricting C_ϵ to X_N^1, we find that the eigenvalues of $C_\epsilon|_{X_N^\perp}$ are:

$$\lambda_n^\pm = \frac{-2\pm2(1-\epsilon^2n^2)^{1/2}}{\epsilon^2}, \quad n = 1,2,\dots,N \tag{8.6}$$

and corresponding eigenvectors are

$$\begin{bmatrix} \sin nx \\ \lambda_n^\pm \sin nx \end{bmatrix}, \quad n = 1,2,\dots,N .$$

Let

$$X_N^- = \text{span}\left\{\begin{bmatrix} \sin nx \\ \lambda_n^- \sin nx \end{bmatrix} : n = 1,\dots,N\right\}$$

$$X_N^+ = \text{span}\left\{\begin{bmatrix} \sin nx \\ \lambda_n^+ \sin nx \end{bmatrix} : n = 1,\dots,N\right\} .$$

Obviously, $X_N^\perp = X_N^- \oplus X_N^+$ and X_N^-, X_N^+ are invariant subspaces of C_ϵ and

$$(8.7)\qquad \ll \begin{bmatrix} \sin nx \\ \lambda_n^\pm \sin nx \end{bmatrix}, \begin{bmatrix} \sin mx \\ \lambda_m^\pm \sin mx \end{bmatrix} \gg = 0 \quad \text{for } m \neq n.$$

Note that X_N^- is not orthogonal to X_N^+. Hence, $X = X_N \oplus X_N^- \oplus X_N^+$ and (i) holds.

Let P_N^-, and P_N^+ be the corresponding spectral projections [31] and P_N be the unique orthogonal projection onto X_N. Obviously, we have $\|P_N\| = 1$. By using (8.7) we have that

$$\|e^{C_\epsilon t} P_N^-\| \leq \|P_N^-\| e^{\frac{-2+2(1-\epsilon^2(N+1)^2)^{1/2}}{\epsilon^2} t}, \quad \text{for } t \geq 0$$

and

$$\|e^{C_\epsilon t} P_N^+\| \leq \|P_N^+\| e^{\frac{-2+2(1-\epsilon^2 N^2)^{1/2}}{\epsilon^2} t}, \quad \text{for } t \leq 0.$$

Now we consider $C_\epsilon|_{X_N}$. For any $w \in X_N$

$$\ll \left(-\frac{2}{\epsilon^2}(1-\epsilon^2(N+1)^2)^{1/2} I + \frac{2}{\epsilon}\begin{bmatrix} 0 & I \\ \epsilon^{-2} - A & 0 \end{bmatrix}\right) w, w \gg$$

$$= -\frac{2}{\epsilon^2}(1-\epsilon^2(N+1)^2)^{1/2}\left[((A+(\frac{1}{\epsilon^2}-2(N+1)^2)u,u)_{L^2} + (v,v)_{L^2}\right]$$

$$+\frac{4}{\epsilon}\left(\frac{1}{\epsilon^2}-(N+1)^2\right)(u,v)_{L^2}$$

$$\leq -\frac{2}{\epsilon^2}\,(1-\epsilon^2(N+1)^2)^{1/2}[<u,u> + (v,v)_{L^2}]$$

$$+\frac{2}{\epsilon^2}\,(1-\epsilon^2(N+1)^2)^{1/2}[<u,u> + (v,v)_{L^2})$$

$$= 0 \ .$$

This says that the operator:

$$-\frac{2}{\epsilon^2}(1-\epsilon^2(N+1)^2)^{1/2}I + \frac{2}{\epsilon}\begin{bmatrix} 0 & I \\ \epsilon^{-2}-A & 0 \end{bmatrix}$$

is dissipative (see Pazy [31]). By the Lumer–Phillips theorem [31], the above linear operator generates a contraction semigroup. Thus, we have

$$\|e^{\frac{2}{\epsilon}\begin{bmatrix} 0 & I \\ \epsilon^{-2}-A & 0 \end{bmatrix}t}\| \leq e^{2\epsilon^{-2}(1-\epsilon^2(N+1)^2)^{1/2}t} , \qquad t \geq 0 \ .$$

Hence

$$\|e^{C_\epsilon t}P_N\| \leq e^{\frac{-2+2(1-\epsilon^2(N+1)^2)^{1/2}}{\epsilon^2}t} , \quad t \geq 0 \ .$$

We will now get the estimates for P_N^- and P_N^+. For any $w \in X_N^\perp$, $w = w_2 + w_3$ where $w_2 \in X_N^-$ and $w_3 \in X_N^+$. We claim that

$$\cos\theta = \frac{\langle\langle\, w_2, w_3 \,\rangle\rangle}{\|w_2\|\ \|w_3\|} \to 0 \quad \epsilon \to 0$$

where θ is the angle between w_2 and w_3. Suppose

$$w_2 = \begin{bmatrix} \sin nx \\ \lambda_n^- \sin nx \end{bmatrix}, \quad w_3 = \begin{bmatrix} \sin mx \\ \lambda_n^+ \sin mx \end{bmatrix},$$

Then $\cos\theta = 0$ if $n \neq m$. If $n = m$, then

$$\cos\theta = \frac{\langle\langle\, w_2, w_3 \,\rangle\rangle}{|w_2|\,|w_3|}$$

$$= \frac{\langle \sin nx\ ,\ \sin nx\rangle + \lambda_n^-\lambda_n^+}{(\langle \sin nx\ ,\ \sin nx\rangle + (\lambda_n^+)^2)^{1/2}\ (\langle \sin nx\ ,\ \sin nx\rangle + (\lambda_n^-)^2)^{1/2}}$$

$$\leq \frac{n^2+\epsilon^{-2}-2(N+1)^2+4\epsilon^{-2}n^2}{(n^2+\frac{1}{\epsilon^2}-2(N+1)^2+(\frac{-2+2(1-\epsilon^2n^2)^{1/2}}{\epsilon^2})^2)^{1/2}(n^2+\frac{1}{\epsilon^2}-2(N+1)^2+\frac{1}{\epsilon^4})^{1/2}}$$

$$\to 0 \qquad \text{as } \epsilon \to 0\,.$$

This proves our claim. Since X_N^- and X_N^+ are finite dimensional vector spaces, there exists $\epsilon_0 > 0$ such that if $0 < \epsilon \le \epsilon_0$ $|\cos\theta| \le \frac{1}{2}$. Hence

$$<< w,w >> = << w_2,w_2 >> + << w_3,w_3 >> + 2 << w_2,w_3 >>$$

$$\ge << w_2,w_2 >> + << w_3,w_3 >> - \|w_2\| \, \|w_3\|$$

$$\ge \frac{1}{2}(<< w_2,w_2 >> + << w_3,w_3 >>)$$

This implies (iii) and completes the proof.

Lemma 8.2 Let

$$K^*(\epsilon,N) = 2\left(\frac{1}{\eta-\omega_1} + \frac{1}{\omega_2-\eta}\right)\frac{1}{(1-2\epsilon(N+1)^2)^{1/2}}$$

where

$$\omega_1 = \frac{2-2(1-\epsilon^2N^2)^{1/2}}{\epsilon^2}, \quad \omega_2 = \frac{2-2(1-\epsilon^2(N+1)^2)^{1/2}}{\epsilon^2}, \quad \eta = \frac{(N+1)^2+N^2}{2} \quad .$$

Then there exist $\epsilon_0 > 0$, $0 < c < 1$ and an integer $N > 0$ such that

$$K^*(\epsilon,N)\mathrm{Lip}(f) < c < 1 \quad . \tag{8.8}$$

Proof We have that $\alpha \to N^2$ and $\beta \to (N+1)^2$ as $\epsilon \to 0$. This implies

$$K^*(\epsilon,N) \to 2\left[\frac{1}{\frac{(N+1)^2-N^2}{2}}+\frac{1}{\frac{(N+1)^2-N^2}{2}}\right], \text{ as } \epsilon \to 0.$$

We can chose N so large that the above limit is strictly less than one. Thus, the lemma follows directly from the continuity of K^*.

Theorem 8.3 If $f \in C^1(L^2,L^2)$ and $N > 0$ satisfies the following gap condition:

$$\left[\frac{1}{\frac{(N+1)^2-N^2}{2}}+\frac{1}{\frac{(N+1)^2-N^2}{2}}\right] < \frac{1}{24\mathrm{Lip}(f)}, \tag{8.9}$$

then there exists $\epsilon_0 = \epsilon_0(N) > 0$ such that for every $\epsilon_0 > \epsilon > 0$ equation (8.3) has a C^1 inertial manifold $\mathcal{M}_\epsilon$ with $\dim \mathcal{M}_\epsilon = N$.

Proof By (8.9) and Lemma 8.2, there exists $\epsilon_0 = \epsilon_0(N) > 0$ such that condition (8.8) is satisfied for all $0 < \epsilon < \epsilon_0$. Let $\alpha = \eta-\omega_1$ and $\beta = \omega_2-\eta$. It is not hard to see that hypotheses (H_1)–(H_5) are satisfied because $\|P_N^-\|$ and $\|P_N^+\|$ are uniformly bounded in $0 < \epsilon < \epsilon_0$. Next we note that if $w = (u,v) \in X$, then by the definition of the norm $\|\cdot\|$ if

$$c_1 = \frac{1}{(1-2\epsilon_0(N+1)^2)^{1/2}}$$

then $|u|_{L^2} \le \epsilon c_1\|w\|$. This implies that if $w_i = (u_i,v_i)$, $i = 1,2$, then

$$\|\epsilon^{-1}[\hat{f}(w_1)-\hat{f}(w_2)]\| = \|\epsilon^{-1}\begin{bmatrix} 0 \\ f(u_1)-f(u_2)\end{bmatrix}\|$$

$$\leq \epsilon^{-1}\mathrm{Lip}(f)|u_1-u_2|_{L^2} \leq c_1\mathrm{Lip}(f)\|w_1-w_2\|.$$

This says that the coefficient ϵ^{-1} of the nonlinear function $\hat{f}$ in equation (8.3) will be canceled by the ϵ term in the norm $\|\cdot\|$. Hence, by using the above estimate, Lemma 8.1, Theorems 4.4 and 5.1 (with $k = 1$, $M = 3$, $M^* = 0$ and $\gamma = 0$ in condition (H_5)) and the gap condition (8.9), we obtain the desired result. We note that $K^*(\epsilon,N)$ takes the role of $K(\alpha,\beta,\gamma)$ in Theorems 4.4 and 5.1

We note that the inertial manifold $\mathcal{M}_\epsilon$ in Theorem 8.3 can be written as:

$$\mathcal{M}_\epsilon = \{P_N^+ w + h_\epsilon(P_N^+ w) : w \in X\}$$

where

$$h_\epsilon(P_N^+ w) = \int_{-\infty}^{0} e^{-C_\epsilon s}(P_N+P_N^-)\hat{f}(w_\epsilon(P_N^+ w))ds \tag{8.10}$$

and $w_\epsilon(P_N^+ w)(\cdot)$ is the unique solution of equation (3.6) with $\xi = P_N^+ w$, $S(t) = \exp(C_\epsilon t)$, $F = \hat{f}$, $P_1 = P_N^+$ and $P_2 = P_N^-+P_N$.

The following follows from Example 7.4.

<u>Theorem 8.4</u> If $f \in C^1(L^2,L^2)$ and $N > 0$ satisfies the gap condition (8.9), then equation (8.1) has a C^1 inertial manifold

$$\mathcal{M}_p = \{U_0 : U(t,U_0) \in C_\eta(R^-,L^2) \text{ and satisfies } (8.1)\}$$

$$= \{\xi + h(\xi) : \xi \in Q_N L^2\}$$

where Q_N is the orthogonal projection from L^2 to span$\{\sin x, \cdots, \sin Nx\}$ and

$$h(\xi) = \int_{-\infty}^{0} e^{As}(I - Q_N) f(W(\xi)) ds \tag{8.11}$$

where $W(\xi)(\cdot)$ is the unique solution of equation (3.6) with $S(t) = e^{-At}$, $\xi \in Q_N L^2$, $F = f$, $P_1 = Q_N$ and $P_2 = I - Q_N$..

Lemma 8.5 Suppose that the conditions in Theorem 8.4 are satisfied. For each $R > 0$, there exists $M_1 > 0$ such that if $|\xi|_{L^2} \le R$ and $\xi \in Q_N L^2$, then

(i) $$|e^{\eta t} U_t(t, \xi + h(\xi))|_{L^2} \le M_1 \quad \text{for } t \in \mathbb{R}^-$$

(ii) $$|e^{\eta t} U_{tt}(t, \xi + h(\xi))|_{L^2} \le M_1 \quad \text{for } t \in \mathbb{R}^-$$

(iii) $$|e^{\eta t} A^{1/2} U(t, \xi + h(\xi))|_{L^2} \le M_1.$$

where U is the unique solution of equation (8.1) with $U_0 = \xi + h(\xi)$ and h is given by (8.11)

Proof For each $U_0 = \xi + h(\xi)$, we have

$$U(t, U_0) = U_N(t, \xi) + h(U_N(t, \xi))$$

where $U_N(t, \xi)$ is the solution of the following initial value problem:

$$(8.12) \qquad (U_N)_t = AU_N+Q_Nf(U_N+h(U_N)) \quad U_N(0) = \xi$$

Since equation (8.12) is finite dimensional and f is globally Lipschitz, U_N exists for all t. From our choice of N and the spectral property of $A|_{Q_N}$, we obtain from Gronwall's inequality and equation (8.12) that there exists M_1' such that

$$(8.13) \qquad |e^{\eta t}U_N(t)|_{L^2} \le M_1' \quad \text{for all } |\xi|_{L^2} \le R \text{ and } \xi \in Q_N L^2.$$

Since h is C^1, we have

$$U_t(t,U_0) = (U_N)_t(t,\xi)+Dh(U_N)\cdot(U_N)_t(t,\xi).$$

By (8.12) and (8.13), we have

$$|e^{\eta t}U_t|_{L^2} \le (1+\mathrm{Lip}(h))(M_1'\|AQ_N\|+\|Q_N\|\,|f|_0) \ .$$

Since Q_N is an orthogonal projection, $\|Q_N\| = 1$. Thus, (i) follows from the above inequality.

Next, U_t satisfies the variational equation:

$$\begin{cases} W_t = AW + Df(U)W \\ W(0) = U_t(0) \end{cases} .$$

The above equation is linear and nonautonomous. If we consider $Df(U)W$ as a perturbation to the autonomous equation:

$$W_t = AW$$

then by condition (8.9) one can prove exactly as in §4 and §5 the existence of a time varying C^1 finite dimensional invariant manifold (see Henry [21]). Thus, we can prove (ii) by using exactly the same method as in (i).

To show (iii), we note that $AU = U_t - f(U) \in C_\eta(R^-, L^2)$. Since $U(t) \in L^2$ and $AU(t) \in L^2$ for every $t \le 0$, by a well-known interpolation theorem (Adams [1], p.75) we have $A^{1/2}U \in C_\eta(R^-, L^2)$. This completes the proof.

Let

$$\bar{\mathcal{M}}_p = \{(U_0, U_t(0,U_0)) : U_0 \in \mathcal{M}_p\} \text{ and}$$

$$\bar{\mathcal{M}}_{p,R} = \{(U_0, U_t(0,U_0)) : U_0 = \xi + h(\xi) \in \mathcal{M}_p,\ |\xi|_{L^2} < R\} \tag{8.14}$$

where $U(\cdot, U_0)$ is the unique solution of the initial value problem (8.1) and R is an arbitrary constant. We have the following theorem.

<u>Theorem 8.6</u> Suppose that $f \in C^1(L^2, L^2)$ and $N > 0$ satisfies the gap condition (8.9). Then for each $R > 0$, we have

$$\lim_{\epsilon \to 0}\{\sup_{W_0 \in \bar{\mathcal{M}}_{p,R}} (\inf_{w \in \mathcal{M}_\epsilon} |W_0 - w|_{H_0^1 \times L^2})\} = \lim_{\epsilon \to 0}\{\sup_{W_0 \in \bar{\mathcal{M}}_{p,R}} \operatorname{dist}(W_0, \mathcal{M}_\epsilon)\} = 0.$$

where $\mathcal{M}_\epsilon$ is the inertial manifold given by (8.10) and $\overline{\mathcal{M}}_{p,R}$ is as in (8.14).

<u>Proof</u> For each $W_0 \in \overline{\mathcal{M}}_{p,R}$, we have $W_0 = (U_0, U_t(0,U_0))$ and

$$U_t = U_{xx} + f(U), \quad U(0) = U_0 \quad .$$

Define

$$W(t) = (U(t)\ , \tfrac{\epsilon}{2}U_t(t) - \tfrac{1}{\epsilon}U(t))$$

where $U(t) = U(t,U_0)$. Thus $W(t) = (U(t),V(t))$ satisfies the following perturbation of equation (8.3):

$$W_t = C_\epsilon W + 2\epsilon^{-1}\hat{f}(W) + \tfrac{\epsilon}{2}\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}.$$

Let $w \in \mathcal{M}_\epsilon$ be a solution of (8.3) and $0 < \epsilon < \epsilon_0$ (see Theorem 8.3). Let $z(t) = W(t) - w(t)$. Hence, $z(t)$ satisfies the following equation:

$$z_t = C_\epsilon z + 2\epsilon^{-1}\{\hat{f}(z+w_\epsilon) - \hat{f}(w_\epsilon)\} + \tfrac{\epsilon}{2}\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}.$$

By (i), (ii) and (iii) of Lemma 8.5, we have that $z \in C_\eta(R^-,X)$. By Lemma 4.2, we have

$$z(t) = e^{C_\epsilon t} P_N^+ z(0) + \int_0^t e^{C_\epsilon(t-s)} P_N^+\{2\epsilon^{-1}[\hat{f}(z+w_\epsilon)-\hat{f}(w_\epsilon)]+\frac{\epsilon}{2}\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}\}$$

$$+ \int_{-\infty}^t e^{C_\epsilon(t-s)} (P_N+P_{\overline{N}})\{2\epsilon^{-1}[\hat{f}(z+w_\epsilon)-\hat{f}(w_\epsilon)]+\frac{\epsilon}{2}\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}\} .$$

Since $\mathscr{M}_\epsilon$ is a graph over the finite dimensional subspace P_N^+X ($X = H_0^1(0,\pi)\times L^2(0,\pi)$), we may choose $w_\epsilon(0)$ so that $P_n z(0) = 0$. Note that

$$\left\|\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}\right\| = |U_{tt}|_{L^2} .$$

Hence,

$$|z|_{C_\eta(\mathbb{R}^-,X)} = \sup_{t\le 0} e^{\eta t}\left\|\int_0^t e^{C_\epsilon(t-s)} P_N^+\{2\epsilon^{-1}[\hat{f}(z+w_\epsilon)-\hat{f}(w_\epsilon)]+\frac{\epsilon}{2}\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}\}\right.$$

$$\left. + \int_{-\infty}^t e^{C_\epsilon(t-s)} (P_N+P_{\overline{N}})\{2\epsilon^{-1}[\hat{f}(z+w_\epsilon)-\hat{f}(w_\epsilon)]+\frac{\epsilon}{2}\begin{bmatrix} 0 \\ U_{tt} \end{bmatrix}\}\right\|$$

$$\le \mathrm{Lip}(f)K^*(\epsilon,N)|z|_{C_\eta(\mathbb{R}^-,X)} + \frac{1}{2}M_1K^*(\epsilon,N)\epsilon. \quad \text{(Lemma 8.5)}$$

By Lemma 8.2, $K^*(\epsilon,N)\mathrm{Lip}(f) < c < 1$ for all $0 < \epsilon < \epsilon_0$, where c is some fixed constant. It follows from the above inequality

$$|z|_{C_\eta(\mathbb{R}^-,X)} \le \frac{cM_1}{2(1-c)\,\mathrm{Lip}(f)}\,\epsilon \quad .$$

Hence,

$$|z(0)|_{H_0^1\times L^2} \le \|z(0)\| \le \frac{cM_1}{2(1-c)\,\mathrm{Lip}(f)}\,\epsilon$$

This implies the theorem.

Lemma 8.7 Assume $f(0) = 0$. Assume that f is C^1 from L^2 into itself and $N > 0$ satisfies the gap condition (8.9). Then for any $R > 0$, there exists $M_2 > 0$ such that if $\|\zeta\| \le R$ and $\zeta \in P_N^+X$, then the following inequalities are satisfied by any solution $w(t)$ of (8.3) on the inertial manifold $\mathcal{M}_\epsilon$, $0 < \epsilon < \epsilon_0$ (see Theorem 8.3):

(8.15) $$\|e^{\eta t}w(t)\| \le M_2 \qquad t \le 0$$

(8.16) $$\|e^{\eta t}w_t(t)\| \le M_2 \qquad t \le 0$$

where $w(0) = \zeta + h_\epsilon(\zeta)$ and h_ϵ is given by (8.10).

Proof: Since w is a solution of (8.3) on the inertial manifold $\mathcal{M}_\epsilon$, by Lemma 4.2, we have

(8.17) $$w(t) = e^{C_\epsilon t}\zeta + \int_0^t e^{(t-s)C_\epsilon}\, P_N^+ \frac{2}{\epsilon}\,\hat{f}(w)ds$$

$$+\int_{-\infty}^{t} e^{(t-s)C_\epsilon}(P_N+P_{\overline{N}})\,\frac{2}{\epsilon}\,\hat{f}(w)ds\ .$$

Since f(0) = 0, we obtain as in the proof of Theorem 8.3 that

$$\|\epsilon^{-1}\hat{f}(w)\| \le \epsilon^{-1}\mathrm{Lip}(f)\ |u|_{L^2} \le c_1\mathrm{Lip}(f)\|w\| \tag{8.18}$$

where w = (u,v) and

$$c_1 = \frac{1}{(1-2\epsilon_0(N+1)^2)^{1/2}}.$$

By the gap condition (8.9), we have $\mathrm{Lip}(f)K^*(\epsilon,N) < c < 1$ for some fixed c (Lemma 8.2). By Lemma 8.1, (8.18), the definition of K^* and equation (8.17), we have

$$|w|_{C_\eta(R^-,x)} \le 3\|\zeta\| + c_1\mathrm{Lip}(f)2\Big(\frac{1}{\eta-\omega_1}+\frac{1}{\omega_2-\eta}\Big)|w|_{C_\eta(R^-,x)}$$

$$\le 3\|\zeta\| + \mathrm{Lip}(f)K^*(\epsilon,N)|w|_{C_\eta(R^-,x)}.$$

Hence,

$$|w|_{C_\eta(R^-,x)} \le \frac{3}{1-c}R\ .$$

This implies (8.15). Since $\mathcal{M}_\epsilon$ is invariant and is the graph of h_ϵ, we have $w(t) = \zeta(t) + h_\epsilon(\zeta(t))$, $\zeta(0) = \zeta$, $\zeta(t) \in P_N^+X$ for all t. Furthermore, $\zeta(t)$ satisfies

$$(8.19) \qquad \zeta_t = C_\epsilon\zeta + P_N^+[\tfrac{2}{\epsilon}\hat{f}(\zeta+h_\epsilon(\zeta))] \ .$$

We note that P_N^+X is invariant under C_ϵ and (8.19) is finite dimensional. By (8.18), (8.19) and (iii) of Lemma 8.1, we have

$$|\zeta_t|_{C_\eta(\mathbb{R}^-,X)} \le \{\|C_\epsilon|_{P_N^+X}\| + 4c_1\mathrm{Lip}(f)(1+\mathrm{Lip}(h_\epsilon))\}|\zeta|_{C_\eta(\mathbb{R}^-,X)} \cdot$$

Since $\|C_\epsilon|_{P_N^+X}\| \le \sup_{1\le i\le N}\{|\lambda_i^+|\}$ and $\lambda_i^+ \to i$ as $\epsilon \to 0$ (see (8.6)), there exists a constant M_3 independent of $\epsilon \in (0,\epsilon_0)$ such that

$$|\zeta_t|_{C_\eta(\mathbb{R}^-,X)} \le M_3.$$

This implies (8.16) and completes the proof.

Let

$$\overline{\mathcal{M}}_\epsilon = \{u : w = (u,v) \in \mathcal{M}_\epsilon \text{ for some } v \in L^2\}$$

and

(8.20) $$\overline{\mathcal{M}}_{\epsilon,R} = \{u \in \overline{\mathcal{M}}_\epsilon : w = (u,v) = \zeta + h_\epsilon(\zeta),\ \|\zeta\| \le R\}.$$

<u>Theorem 8.8</u> Assume $f(0) = 0$. Suppose that $f \in C^1(L^2,L^2)$ and $N > 0$ satisfies the gap condition (8.9). Then for each $R > 0$, we have

$$\lim_{\epsilon\to 0}\{\sup_{u\in\overline{\mathcal{M}}_{\epsilon,R}}(\inf_{U\in\mathcal{M}_p}|u-U|_{L^2})\} = \lim_{\epsilon\to 0}\{\sup_{u\in\overline{\mathcal{M}}_{\epsilon,R}}\mathrm{dist}(u,\mathcal{M}_p)\} = 0.$$

where $\mathcal{M}_p$ is the inertial manifold given by Theorem 8.4 and $\overline{\mathcal{M}}_{\epsilon,R}$ is as in (8.20).

<u>Proof</u> Let $w_0 = (u_0,v_0)$ and $u_0 \in \overline{\mathcal{M}}_{\epsilon,R}$. Let $w(t) = (u(t),v(t))$ bet the unique solution of (8.3) with $w(0) = w_0$. Since $\mathcal{M}_\epsilon$ is invariant, $u(t)$ satisfies the following equation:

$$\begin{cases} u_t + Au = f(u) - \frac{\epsilon^2}{4} u_{tt} \\ u(0) = u_0 \end{cases}.$$

Let $Z(t) = U(t) - u(t)$, where $U(t)$ is the unique solution of (8.1) on the inertial manifold $\mathcal{M}_p$ with initial data $U(0) = U_0 \in L^2$. Thus, $Z(t)$ satisfies

$$Z_t + AZ = f(Z+u(t)) - f(u(t)) - \frac{\epsilon}{4}u_{tt} \quad .$$

Since $U(\cdot) \in \mathcal{M}_p$ and $w(\cdot) \in \mathcal{M}_\epsilon$, we have $Z(\cdot) \in C_\eta(\mathbb{R}^-,L^2)$. By using Lemma 4.2 and (8.16), we have

$$Z(t) = e^{-At}Q_N Z(0) + \int_0^t e^{-A(t-s)}\{Q_N[f(Z(s)+u(s))-f(u(s))] - \frac{\epsilon^2}{4} u_{tt}\}ds$$

$$+ \int_{-\infty}^t e^{-A(t-s)}\{[I-Q_N][f(Z(s)+u(s))-f(u(s))] - \frac{\epsilon^2}{4} u_{tt}\}ds \ .$$

As in the proof of Theorem 8.6, we may assume without loss of generality that $Q_N Z(0) = 0$. As in the proof of Theorem 8.6, we have

$$|Z|_{C_\eta(\mathbb{R}^-,L^2)} \le \frac{M_2}{2N-3}\epsilon^2 \ .$$

This completes the proof.

Remark 8.9 Consider the damped sine–Gordon equation

(8.21) $$\frac{1}{4}\epsilon^2 u_{tt} + u_t - u_{xx} = \sin u$$

with boundary conditions

(8.22) $$u(t,0) = u(t,\pi) = 0.$$

Theorem 8.3 is not applicable in this case because $f(u) = \sin u$ is not a C^1 map from L^2 into itself (see Henry [21]). However, (8.21) (8.22) defines a C^0 nonlinear semigroup on $(H^2 \cap H_0^1) \times H_0^1$ (Hale [17], Theorem 7.5 in Chapter 4) and $f(u) = \sin u$ is C^1 from H_0^1 into itself. If we define the following inner product in $(H^2 \cap H_0^1) \times H_0^1$:

$$<<< w_1, w_2 >>> = (A(A + \frac{1}{\epsilon^2} - 2(N+1)^2)u_1, u_2)_{L^2} + (Av_1, v_2)_{L^2}$$

where $w_i = (u_i, v_i)$, $i = 1,2$, $\in (H^2 \cap H_0^1) \times H_0^1$, then we can get the same results as Theorem 8.3, Theorem 8.6 and Theorem 8.8 for the damped sine–Gordon equation (8.21) (8.22) by using the same arguments.

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