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GEOMETRY OF THE MELNIKOV VECTOR

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ABSTRACT

GEOMETRY OF THE MELNIKOV VECTOR

By

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The Melnikov method is developed for higher dimensional systems, and the transversal and tangential intersection of the stable and unstable manifolds are discussed. Hamiltonian systems are discussed as a special case of the general theory. The theory is then extended to the case of a heteroclinic orbit to invariant tori which includes systems with quasi–periodic perturbations as a special case.

This thesis is dedicated

to the memory of my father,

Seiichi Yamashita

1912–1984

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TABLE OF CONTENTS

TABLE OF FIGURES

§1. INTRODUCTION

The notion of a homoclinic point was introduced by Poincare [18]. To recall this concept, consider a diffeomorphism in $\mathbb{R}^2$ with a hyperbolic fixed point p. A point q is called a homoclinic point of p if q is in the intersection of the stable and unstable manifolds of p. The point q is called a transversal homoclinic point of p if the intersection of the stable and unstable manifolds is transversal, i.e. the tangent spaces at q to the stable and unstable manifolds span the whole space. We note that if one homoclinic point exists, there must be infinitely many homoclinic points.

Poincare already observed that the existence of homoclinic points implies complexity of the dynamics of the diffeomorphism. Later G.D. Birkhoff [3] proved that every transversal homoclinic point of a two–dimensional diffeomorphism is accumulated by periodic orbits. The results by Smale [20], now called the Smale–Birkhoff theorem, extend the Birkhoff's results both in two dimentional and to higher dimensional cases and assert that the existence of a transversal homoclinic point implies the existence of an invariant Cantor set in which the periodic orbits are dense. See also Moser [14]. Moreover Newhouse [15] has proved that there is a much more complicated dynamical behavior associated with a homoclinic tangency. Thus the dynamics of diffeomorphisms with transversal or tangencial homoclinic points are fairly well understood.

However to apply the above abstract theories for diffeomorphisms to a system of differential equations, we need to know the existence of a homoclinic point of a diffeomorphism induced by this system. More precisely since we shall deal with an autonomous system with a time–periodic purturbation, the above diffeomorphism appears as a time–one map, called a Poincare map, induced by the flow of the system.

Our problem is the following: an autonomous system of ordinary differential equations with a time–periodic perturbation is given and assume that the unperturbed autonomous system has two hyperbolic critical point (not necessarily distinct) and a homoclinic or hetercolinic orbit connecting them. Find computable conditions under which the Poincare map induced by the perturbed system has a transversal homoclinic point. See §2 for more precise definitions of these notions and for a precise formulation of the problem.

Poincare [17], Melnikov [12] and Arnold [2] developed such conditions for two–dimensional analytic Hamiltonian systems and it is now called the Poincare–Melnikov–Arnold method or simply the Melnikov method. The Melnikov theory has been studied by several authors, e.g. Chow, Hale and Mallet–Paret [4], Holmes [9] and Palmer [16], and generalizations to higher dimensional cases have also been studied, e.g., Holmes and Marsden [10] and Gruendler [6]. The key of these theories is the use of the Melnikov function which measures the splitting distance between the perturbed stable and unstable manifolds.

One of the purposes of the present notes is to clarify the geometry of the Melnikov function (now should be called the Melnikov vector) in higher dimensional cases and to extend the previous theories for the two–dimensional case to higher dimensional cases.

Our theory is based on the theory of exponential dichotomy. We shall recall basic results on exponential dichotomy in §3. Palmer [16] showed that the linear variational system along the homoclinic orbit of the unperturbed autonomous system has exponential dichotomies on half-lines. Using this fact we shall derive explicit expressions of the local stable and unstable manifolds of the perturbed system. This is the content of §4. Then the Fredholm's alternative, given in Chow, Hale and Mallet-Paret [4] for the two-dimensional case, in Palmer [16] in heigher dimensional cases and explained in §5, is used to derive the Melnikov vector in §6. In §7 we examine conditions for a transversal homoclinic point and introduce the notion of the index of a homoclinic or heteroclinic orbit which is useful to classify the cases that can occur in higher dimensional cases. In §8 we discuss a relation between the dimension of the Melnikov vector and the index of the homoclinic or heteroclinic orbit. Numerical aspect of the Melnikov vector is discussed in §9. In §10 we consider several special cases in which the Melnikov vectors take simpler forms, and also we discuss the tangency. We apply these general theories to Hamiltonian systems in §11. In §12 we extend our theory to the case of a heteroclinic orbit to invariant tori and as a by-product we drive a formula which guarantees the transversal intersection of the stable and unstable manifolds of a two-dimensional system with a quasi-periodic perturbation. See also Meyer and Sell [13] and Wiggins [21]. Several interesting examples are discussed in §13 and finally in §14 we show a serious limitation of the Melnikov method by using an example for which the Melnikov method does not work. This difficulty comes from the nature of Melnikov method as a perturbation theory.

§2. FORMULATION OF THE PROBLEM

Consider a system of differential equations

$$(2.1) \qquad \dot{x} = f(x)$$

and its perturbed system

$$(2.2) \qquad \dot{x} = f(x) + \epsilon g(t,x)$$

where $x \in \mathbb{R}^n$, $t \in \mathbb{R}$, $\epsilon \in \mathbb{R}$ and $|\epsilon| << 1$. The vector fields f and g are assumed to be sufficiently smooth and bounded on bounded sets. The vector field g is periodic in t with the least period $T(>0)$.

Assume that system (2.1) has two hyperbolic critical points x_+ and x_- (not necessarily distinct). Also assume that there is an orbit $\gamma(t)$, $t \in \mathbb{R}$, of system (2.1) which connects the critical points x_+ and x_-. That is,

$$(2.3) \qquad \gamma(t) \to x_{\pm} \quad \text{as} \quad t \to \pm\infty.$$

If $x_+ = x_-$, the orbit γ is called a homoclinic orbit. Otherwise γ is called a heteroclinic orbit.

Let $x(t{:}x_0)$, $x_0 \in \mathbb{R}^n$, be the solution of system (2.1) with the initial data $x(0;\ t_0) = x_0$. The stable manifold $W^s(x_+)$ of the hyperbolic critical point x_+ of system (2.1) is defined by

$$(2.4) \qquad W^s(x_+) = \{x_0 \in \mathbb{R}^n: x(t;x_0) \to x_+ \quad \text{as} \quad t \to +\infty\},$$

and the unstable manifold $W^u(x_-)$ of the hyperbolic critical point x_- of the system (2.1) is defined by

$$(2.5) \qquad W^u(x_-) = \{x_0 \in \mathbb{R}^n: x(t;x_0) \to x_- \quad \text{as} \quad t \to -\infty\}.$$

Then we have

$$(2.6) \qquad \gamma \subset W^s(x_+) \cap W^u(x_-)$$

from the above assumption.

Since the critical points $x_\pm$ are hyperbolic and system (2.2) is periodic in t, there exist unique T–periodic solutions $\bar{x}_\pm(t;\epsilon)$ of system (2.2) such that

$$(2.7) \qquad \lim_{\epsilon \to 0} \bar{x}_\pm(t,\epsilon) = \bar{x}_\pm(t,0) = x_\pm$$

uniformly in t. For details see Hale [7].

It will be shown in the next section that there exist sets $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ in $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^n \times \mathbb{R}$, where $\mathbb{R}^n \times \mathbb{R}$ is the extended phase space of system (2.2), such that

(2.8) $$W^s_{loc}(\bar{x}_+,\epsilon) = \{(x_0,0)\epsilon\ \mathbb{R}^n\times\{0\}:|x(t;0,x_0) - \bar{x}_\pm(t;\epsilon)| \to 0 \text{ as } t \to +\infty \text{ and } x_0 \text{ is in a sufficiently small neighborhood of } \gamma\}$$

and

(2.9) $$W^u_{loc}(\bar{x}_-,\epsilon) = \{(x_0,0)\epsilon\ \mathbb{R}^n\times\{0\}:|x(t;0,x_0) - \bar{x}_-(t;\epsilon)| \to 0 \text{ and } x_0 \text{ is in a sufficiently small neighborhood of } \gamma\},$$

where $x(t;\tau,x_0)$ is the solution of system (3.2) with $x(\tau;\tau,x_0) = x_0$, $x_0\epsilon\ \mathbb{R}^n$.

If we define the time dependent stable and unstable manifolds, $\overline{W}^s(\bar{x}_+;\epsilon)$ and $\overline{W}^u(\bar{x}_-,\epsilon)$, of system (2.2) by

(2.10) $$\overline{W}^s(\bar{x}_+,\epsilon)=\{(x_0,\tau)\epsilon\ \mathbb{R}^n\times\mathbb{R}:\ |x(t;\tau,x_0) - \bar{x}_+(t;\epsilon)| \to 0 \text{ as } t \to +\infty\}$$

and

(2.11) $$\overline{W}^u(\bar{x}_-,\epsilon) = \{(x_0,\tau)\epsilon\ \mathbb{R}^n\times\mathbb{R}:\ |x(t;\tau,x_0) - \bar{x}_-(t;\epsilon)| \to 0 \text{ as } t \to -\infty\},$$

then $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ are the local cross sections at $t = 0$ of $\overline{W}^s(\bar{x}_+,\epsilon)$ and $\overline{W}^u(\bar{x}_-,\epsilon)$ respectively. That is,

$$W^s_{loc}(\bar{x}_+,\epsilon) \subset \overline{W}^s(\bar{x}_+,\epsilon) \cap (\mathbb{R}^n \times \{0\}) \tag{2.12}$$

and

$$W^u_{loc}(\bar{x}_-,\epsilon) \subset \overline{W}^u(\bar{x}_-,\epsilon) \cap (\mathbb{R}^n \times \{0\}). \tag{2.13}$$

Since system (2.2) is periodic in t, its extended phase space can be regarded as $\mathbb{R}^n \times S'$, where S^1 is the unit circle, and then $W^s_{loc}(\bar{x}_+\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ are the local stable and unstable manifolds of hyperbolic critical points $\bar{x}_\pm \equiv \bar{x}_\pm(0;\epsilon)$ of the Poincare map $\Pi_\epsilon\colon \mathbb{R}^n \to \mathbb{R}^n$, which is defined by the flow of system (2.2) as follows:

$$\Pi_\epsilon(x_0) = x(T;0,x_0), \quad x_0 \epsilon\ \mathbb{R}^n. \tag{2.14}$$

Now we state our problem.

<u>Problem I</u>. When does system (2.2) have an orbit $x(t)$, $t \epsilon\ \mathbb{R}$, so that $x(t) \to \bar{x}_\pm(t;\epsilon)$ as $t \to \pm\infty$?

Following the above argument, it is clear that Problem I is equivalent to

<u>Problem I'</u>. When do $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ defined above intersect each other?

Then next natural question would be

<u>Problem II</u>. When do $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ intersect transversally?

Here the transversal intersection means that tangent spaces to $W^s_{loc}(\bar{x}_+,\epsilon)$ and to $W^u_{loc}(\bar{x}_-,\epsilon)$ at a point of intersection span the whole space $\mathbb{R}^n$.

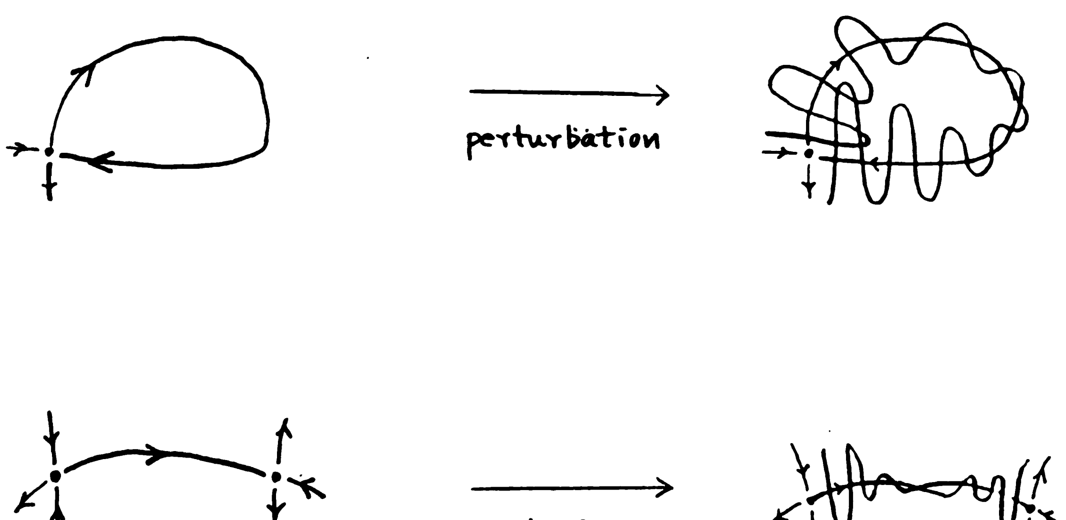


Figure 1

§3. BASIC RESULTS FROM THE THEORY OF EXPONENTIAL DICHOTOMY

In this section we recall the definition and basic results of exponential dichotomy which will play a key role throughout the paper. For details on the exponential dichotomy, see Coppel [5], Palmer [16] and Hale and Lin [8].

Consider the system

$$(3.1) \qquad \dot{z} = A(t)z, \; z \in \mathbb{R}^n$$

where $A(t)$ is assumed to be a continuous $n \times n$ real matrix function on $\mathbb{R}$. Denote by $\Phi(t,s)$ the transition matrix of system (3.1).

Definition 3.1. System (3.1) is said to have an exponential dichotomy on $[t_0,\infty)$, t_0 fixed, if there exists a projection $P: \mathbb{R}^n \to \mathbb{R}^n$, $K \geq 1$ and $a > 0$ such that

$$(3.2) \qquad |\Phi(t,t_0)P\Phi(t_0,s)| \leq Ke^{-a(t-s)}, \; t_0 \leq s \leq t,$$

and

$$(3.3) \qquad |\Phi(t,t_0)(I-P)\Phi(t_0,s)| \leq Ke^{-a(s-t)}, t_0 \leq t \leq s.$$

Similarly system (3.1) is said to have an exponential dichotomy on $(-\infty,t_0]$ if there exists a projection $Q: \mathbb{R}^n \to \mathbb{R}^n$, $L \geq 1$ and $b > 0$ such that

$$(3.4)\qquad |\Phi(t,t_0)Q\Phi(t_0,s)| \le Le^{-b(t-s)},\ s \le t \le t_0,$$

$$(3.5)\qquad |\Phi(t,t_0)(I-Q)\Phi(t_0,s)| \le Le^{-b(s-t)},\ t \le s \le t_0.$$

Roughly speaking, the exponential dichotomy is hyperbolicity on half-lines. More precisely, from (3.2) and (3.5), we see that the range of P, denoted by $\mathcal{R}(P)$, is the (exponentially) stable subspace at t_0:

$$(3.6)\qquad \mathcal{R}(P) = \{\xi \in \mathbb{R}^n:\ \Phi(t,t_0)\xi \to 0 \ \text{ as } \ t \to +\infty\},$$

and $\mathcal{R}(I-Q)$ is the unstable subspace at to:

$$(3.7)\qquad \mathcal{R}(I-Q) = \{\xi \in \mathbb{R}^n:\ \Phi(t,t_0)\xi \to 0 \ \text{ as } \ t \to -\infty\}.$$

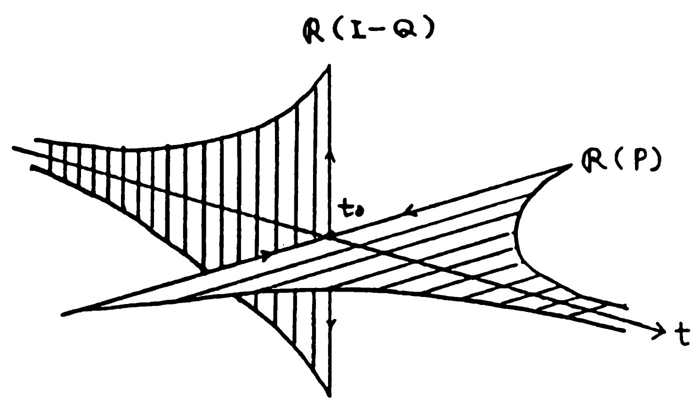


Figure 2

We also note that 'the stable projection' $\Phi(t,t_0)P\Phi(t_0,t)$ at time

$t(\geq t_0)$ is uniquely determined by P since $\Phi(t,t_0)P\Phi(t_0,t)$ is a solution of the matrix system.

$$\text{(3.8)} \qquad \dot{X} = A(t)X - XA(t)$$

with the initial value P at $t = t_0$.
The similar statement holds for the unstable projection $\Phi(t,t_0)(I-Q)\Phi(t_0,t)$ at time $t(\leq t_0)$.

The key fact on the exponential dichotomy which shall be used is the following: If system (3.1) possesses an exponential dichotomy on $[t_0,\infty)$ with projection P and if P' is a projection such that $\mathcal{R}(P) = \mathcal{R}(P')$, then system (3.1) also possesses an exponential dichotomy on $[t_0,\infty)$ with projection P'. Similarly if system (3.1) possesses an exponential dichotomy on $(-\infty,t_0]$ with projection Q and if Q' is a projection such that $\mathcal{R}(I-Q) = \mathcal{R}(I-Q')$, then system (3.1) also possesses an exponential dichotomy on $(-\infty,t_0]$ with projection Q'. Thus the stable subspace $\mathcal{R}(P)$ and the unstable subspace $\mathcal{R}(I-Q)$ are uniquely determined but their complementary subspaces can be any.

Next the adjoint system of system (3.1) is defined by

$$\text{(3.9)} \qquad \dot{\phi} + A^*(t)\phi = 0$$

where $A^*(t)$ is the transpose of $A(t)$.
If system (3.1) has an exponential dichotomy on $[t_0,\infty)$ with projection P, then the adjoint system (3.10) automatically has the exponential

dichotomy on $[t_0,\infty)$ with projection $I-P^*$ where P^* is the adjoint operator of P. That is, (3.2) and (3.3) imply

$$(3.10) \qquad |\Phi^*(t,t_0)^{-1}(I-P^*)\Phi^*(t_0,t)^{-1}| \le Ke^{-a(t-s)}, \; t_0 \le s \le t$$

and

$$(3.11) \qquad |\Phi^*(t,t_0)^{-1}P^*\Phi^*(t_0,t)^{-1}| \le Ke^{-a(s-t)}, \; t_0 \le t \le s.$$

Similarly system (3.1) has an exponential dichotomy on $(-\infty,t_0]$ with projection Q, then system (3.9) has the exponential dichotomy on $(-\infty,t_0]$ with the projection $I-Q^*$. That is, (3.4) and (3.5) imply

$$(3.12) \qquad |\Phi^*(t,t_0)^{-1}(I-Q^*)\Phi^*(t_0,t)^{-1}| \le Le^{-b(t-s)}, \; s \le t \le t_0$$

and

$$(3.13) \qquad |\Phi^*(t,t_0)^{-1}Q^*\Phi^*(t_0,t)^{-1}| \le Le^{-b(s-t)}, \; t \le s \le t_0.$$

Note that if $\Phi(t,s)$ is the transition matrix of system (3.1), then $\Phi^*(t,s)^{-1}$ is the transition matrix of system (3.9).

The key geometric fact, which will be used essentially in later sections, is the following relation between the space of bounded solutions of the adjoint system (3.9) and the spaces of bounded solutions on $[0,\infty)$ and on $(-\infty,0]$ of system (3.1).

$$(3.14) \qquad \{\eta \in \mathbb{R}^n : \ \Phi^*(t,t_0)^{-1}\ \eta \to 0 \quad \text{as} \quad t \to \pm\infty\}$$
$$= \mathcal{R}(I-P^*) \cap \mathcal{R}(Q^*)$$
$$= \{\mathcal{R}(P) + \mathcal{R}(I-Q)\}^{\perp}$$
$$= [\{\xi \in \mathbb{R}^n : \Phi(t,t_0)\xi \to 0 \quad \text{as} \quad t \to +\infty\} \cup$$
$$\{\xi \in \mathbb{R}^n : \ \Phi(t,to)\xi \to 0 \quad \text{as} \quad t \to -\infty\}]^{\perp}.$$

This is clear from (3.10) and (3.13).

§4. THE STABLE AND UNSTABLE MANIFOLDS

Since we would like to describe the local cross sections $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ of the time dependent stable and unstable manifolds $\overline{W}^s(\bar{x}_+,\epsilon)$ and $\overline{W}^u(\bar{x}_-,\epsilon)$ of system (2.2) as part of the 'perturbed manifolds' of $W^s(x_+)$ and $W^u(x_-)$ of system (3.1) respectively along the orbit $\gamma(t)$, we let, for a fixed $\alpha \epsilon \mathbb{R}$,

$$(4.1) \qquad x(t) = \gamma(t+\alpha) + \epsilon z(t+\alpha).$$

Then system (2.2) becomes

$$(4.2) \qquad \dot{z} = A(t)z + g(t-\alpha, \gamma(t)) + h(t,z,\alpha,\epsilon)$$

where

$$(4.3) \qquad A(t) = Df(\gamma(t)).$$

and

$$(4.4) \qquad h(t,z,\alpha,\epsilon) = \frac{1}{\epsilon}\{f(\gamma(t)+\epsilon z(t)) - f(\gamma(t)) - \epsilon Df(\gamma(t))z(t) + \epsilon g(t-\alpha,\gamma(t) + \epsilon z(t)) - \epsilon g(t-\alpha,\gamma(t))\}.$$

We note that

(4.5) $$|h(t,z,\alpha,\epsilon)| = 0(\epsilon) \quad \text{uniformly in } t,z \text{ and } \alpha.$$

Since $\bar{x}_+(0;\epsilon)$ is hyperbolic, by (2.3), (2.7) and (2.8)

(4.6) $$\gamma(\alpha) + \epsilon\xi \in W^s_{loc}(\bar{x}_+,\epsilon)$$

if and only if the solution $z(t;\alpha,\xi)$ of system (4.2) with $z(\alpha;\alpha,\xi) = \xi$ is bounded on the time interval $[\alpha,\infty)$. Thus we have, by changing $\alpha\epsilon\mathbf{R}$, that

(4.7) $$W^s_{loc}(\bar{x}_+,\epsilon) = \bigcup_{\alpha\epsilon\mathbf{R}} \{\gamma(\alpha) + \epsilon\xi^s\text{: the solution } z(t;\alpha,\xi^s) \text{ of system (4.2) is bounded on } [\alpha,\infty)\}.$$

Similarly we have

(4.8) $$W^u_{loc}(\bar{x}_-,\epsilon) = \bigcup_{\alpha\epsilon\mathbf{R}} \{\gamma(\alpha) + \xi^u\text{: the solution } z(t;\alpha,\xi^u) \text{ of system (4.2) is bounded on } (-\infty,\alpha]\}.$$

We remark that $\alpha\epsilon\mathbf{R}$ works as a 'sweeping' parameter along the orbit γ. See Figure 3.

Now as the orbit γ is assumed to be a homoclinic or heteroclinic orbit to hyperbolic fixed points, the linearized sytem

(4.9) $$\dot{z} = A(t)z$$

of system (2.1) along the orbit γ has exponential dichotomies on $[\alpha,\infty)$ and on $(-\infty,\alpha]$. (See Palmer [16].) Let $P(\alpha)\colon \mathbb{R}^n \to \mathbb{R}^n$ and $Q(\alpha)\colon \mathbb{R}^n \to \mathbb{R}^n$ be projections for exponential dichotomies on $[\alpha,\infty)$ and on $(-\infty,\alpha]$ respectively.

Fix $\alpha\in\mathbb{R}$. Then from the variation of constant formula, the solution $z(t;\alpha,\xi)$ of system (4.2) satisfies

$$
\begin{aligned}
(4.10)\qquad z(t;\alpha,\xi) = {} & \Phi(t,\alpha)P(\alpha)\xi + \int_\alpha^t \Phi(t,\tau)S(\tau)\{g(\tau-\alpha,\gamma(\tau)) \\
& + h(\tau,z(\tau;\alpha,\xi),\alpha,\epsilon)\}d\tau + \Phi(t,\alpha)(I-P(\alpha))\xi \\
& + \int_\alpha^t \Phi(t,\tau)(I-S(\tau))\{g(\tau-\alpha,\gamma(\tau)) \\
& + h(\tau,z(\tau;\alpha,\xi),\alpha,\epsilon)\}d\tau
\end{aligned}
$$

where $\Phi(t,\tau)$ is the transition matrix of system (4.9), and $S(\tau)$ is defined by

$$
(4.11)\qquad S(\tau) = \Phi(\tau,\alpha)P(\alpha)\Phi(\alpha,\tau).
$$

It is easy to show that $z(t;\alpha,\xi^s)$ is a bounded solution of system (4.2) on $[\alpha,\infty)$ if and only if $z(t;\alpha,\xi^s)$ satisfies the following integral equation:

$$(4.12)\qquad z(t;\alpha,\xi^S) = \Phi(t,\alpha)P(\alpha)\xi^S$$
$$+ \Phi(\tau,\alpha)P(\alpha) \int_\alpha^t \Phi(\alpha,\tau)\{g(\tau-\alpha,\gamma(\tau))$$
$$+ h(\tau,z(\tau;\alpha,\xi^S),\alpha,\epsilon)\}d\tau$$
$$+ \Phi(t,\alpha)(I-P(\alpha)) \int_\infty^t \Phi(\alpha,\tau\{g(\tau-\alpha,\gamma(\tau))$$
$$+ h(\tau,z(\tau;\alpha,\xi^S),\alpha,\epsilon)\}d\tau.$$

Here we used

$$(4.13)\qquad \Phi(t,\tau)S(\tau) = \Phi(t,\alpha)P(\alpha)\Phi(\alpha,\tau).$$

Let $\eta^S = P(\alpha)\xi^S$. Then it can be shown by the contraction mapping principle that integral equation (4.12) has a unique solution $z(\eta^S)(t) \equiv z(t;\alpha,\xi^S(\eta^S))$ for $|\eta^S|<<|$ where $\xi^S = \xi^S(\eta^S)$ is a function of η^S. By letting $t = \alpha$, the function $\xi^S = \xi^S(\eta^S)$ is given by

$$(4.14)\qquad \xi^S = \eta^S + (I-P(\alpha))\{\int_\infty^\alpha \Phi(\alpha,\tau)g(\tau-\alpha,\gamma(\tau))d\tau$$
$$+ \int_\infty^\alpha \Phi(\alpha,\tau)h(\tau,z(\eta^S)(\tau),\alpha,\epsilon)d\tau\}.$$

We remark that

$$(4.15)\qquad \int_\infty^\alpha \Phi(\alpha,\tau)h(\tau,z(\eta^S)(\tau),\alpha,\epsilon)d\tau = 0(\epsilon).$$

uniformly in η^S. Similarly $z(t;\alpha,\xi^u)$ is a bounded solution of system (4.2) on $(-\infty,\alpha]$ if and only if

$$(4.16)\qquad \xi^u = \eta^u + Q(\alpha)\{\int_{-\infty}^{\alpha} \Phi(\alpha,\tau)g(\tau-\alpha,\gamma(\tau))d\tau + \int_{-\infty}^{\alpha} \Phi(\alpha,\tau)h(\tau,z(\eta^u)(\tau),\alpha,\epsilon)d\tau\}$$

where $\eta^u \in \mathscr{R}(I-Q(\alpha))$, $|\eta^u|<<1$ and $z(\eta^u)(t) \equiv z(t;\alpha,\xi^u(\eta^u))$ is the unique solution of

$$(4.17)\qquad \begin{aligned} z(t;\alpha,\xi^u) &= \Phi(t,\alpha)\eta^u \\ &+ \Phi(t,\alpha)(I-Q(\alpha)) \int_{\alpha}^{t} \Phi(\alpha,\tau)\{g(\tau-\alpha,\gamma(\tau)) \\ &+ h(\tau,z(\tau;\alpha,\xi^u),\alpha,\epsilon)\}d\tau \\ &+ \Phi(\tau,\alpha)Q(d) \int_{-\infty}^{t} \Phi(\alpha,\tau)\{g(T-\alpha,\gamma(\tau)) \\ &+ h(\tau,z(\tau;\alpha,\xi^u),\alpha,\epsilon)\}d\tau. \end{aligned}$$

We also remark that

$$(4.18)\qquad \int_{-\infty}^{\alpha} \Phi(\alpha,\tau)h(\tau,z(\eta^u)(\tau),\alpha,\epsilon)d\tau = 0(\epsilon).$$

Thus we have shown that $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ have the following expressions as functions of α, η^s or η^u.

Proposition 4.1.

(i) The local cross section $W^s_{loc}(\bar{x}_+,\epsilon)$ at $t = 0$ of the time dependent stable manifold $\overline{W}^s(\bar{x}_+,\epsilon)$ of system (2.2) is given by the following:

(4.19) $$W^s_{loc}(\bar{x}_+,\epsilon) = \bigcup_{\alpha\epsilon\mathbb{R}} \{\gamma(\alpha) + \epsilon M^s(\alpha,\eta^s,\epsilon)\}$$

where

(4.20) $$M^s(\alpha,\eta^s\epsilon) = \eta^s + (I-P(\alpha))[\int_{\infty}^{\alpha} \Phi(\alpha,\tau)g(\tau-\alpha,\gamma(\tau))d\tau + \int_{\infty}^{\alpha} \Phi(\alpha,\tau)h(\tau,z(\eta^s)(\tau),\alpha,\epsilon)d\tau]$$

and $\eta^s\epsilon\ \mathcal{R}P(\alpha)$, $|\eta^s|<<1$ and $z(\eta^s)$ is the solution of equation (4.12) with $\eta^s = P(\alpha)\xi^s$.

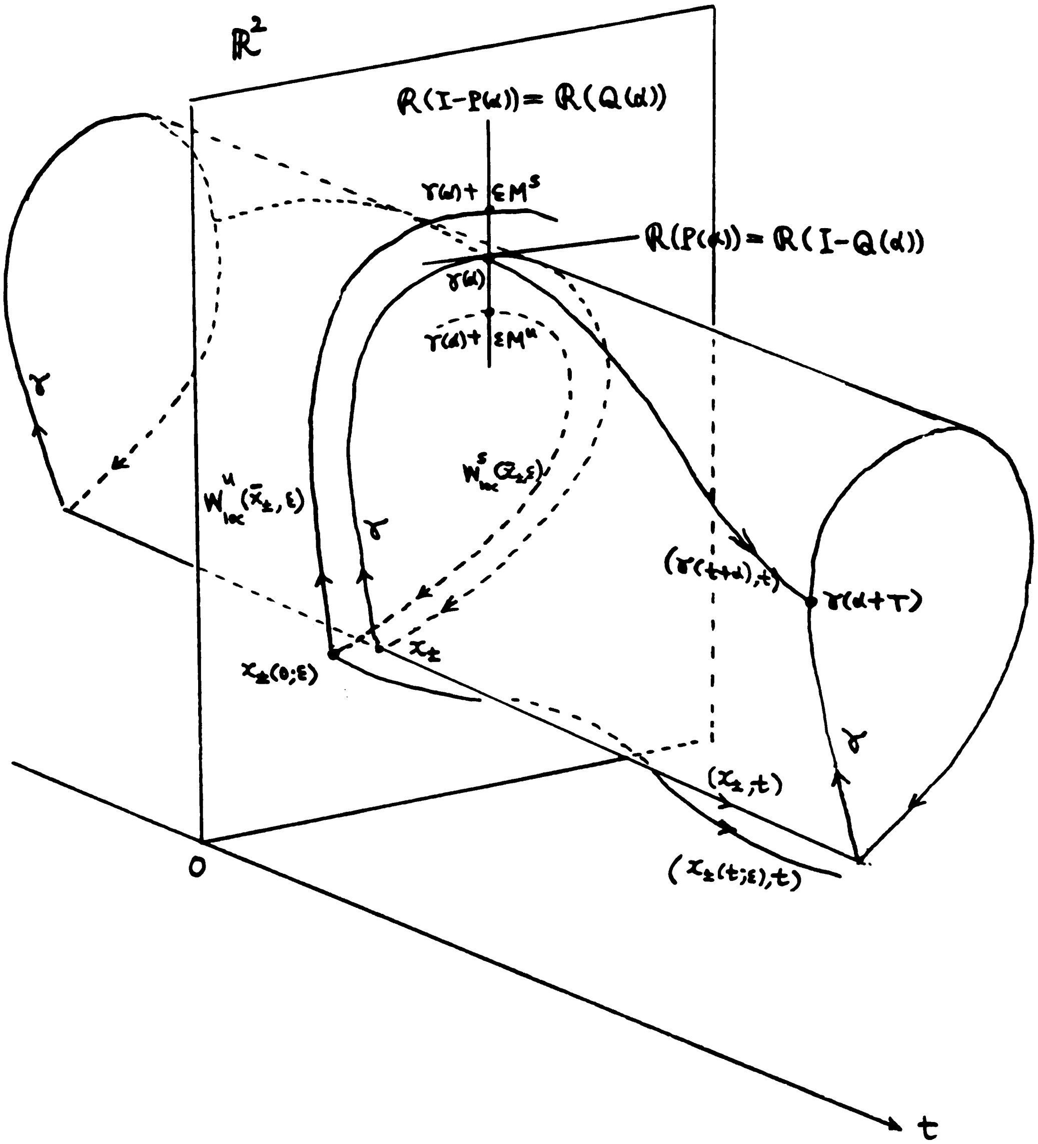
(ii) The local cross section $W^u_{loc}(\bar{x}_-,\epsilon)$ at $t = 0$ of the time dependent unstable manifold $\bar{W}^u(\bar{x}_-,\epsilon)$ of system (2.2) is given by the following:

(4.21) $$W^u_{loc}(\bar{x}_-,\epsilon) = \bigcup_{\alpha\epsilon\mathbb{R}} \{\gamma(\alpha) + \epsilon M^u(\alpha,\eta^u\epsilon)\}$$

where

(4.22) $$M^u(\alpha,\eta^u,\epsilon) = \eta^u + Q(\alpha)[\int_{-\infty}^{\alpha} \Phi(\alpha,\tau)g(\tau-\alpha,\gamma(\tau))d\tau + \int_{-\infty}^{\alpha} \Phi(\alpha,\tau)h(\tau,z(\eta^u)(\tau),\alpha,\epsilon)d\tau]$$

and $\eta^u\epsilon\mathcal{R}(I-Q(\alpha))$, $|\eta^u|<<1$ and $z(\eta^u)$ is the solution of equation (4.17). □



$\mathbb{R}^2$
$\mathbb{R}(I-P(\alpha)) = \mathbb{R}(Q(\alpha))$
$\gamma(\alpha) + \varepsilon M^s$
$\mathbb{R}(P(\alpha)) = \mathbb{R}(I-Q(\alpha))$
$\gamma(\alpha)$
$\gamma(\alpha) + \varepsilon M^u$
γ
$W^u_{loc}(\bar{x}_\pm, \varepsilon)$
$W^s_{loc}(\bar{x}_\pm, \varepsilon)$
γ
$(\gamma(t+\alpha), t)$
$\gamma(\alpha+T)$
$x_\pm(0;\varepsilon)$
$x_\pm$
γ
$(x_\pm, t)$
$(x_\pm(t;\varepsilon), t)$
0
t

Figure 3

Remark 4.2. Notice that $\dot{\gamma}(\alpha) \in \mathscr{R}(P(\alpha)) \cap \mathscr{R}(I-Q(\alpha))$. Since we consider the cross sections of the time dependent stable and unstable manifolds in a vicinity of the orbit γ, it is sufficient, by the tubular neighborhood theorem, to consider coordinates in the normal bundle $\bigcup_{\alpha \in \mathbb{R}} T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$ of the submanifold γ, where $T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$ stands for the normal vector subspace, in the tangent space $T_{\gamma(\alpha)}\mathbb{R}^n$, to the one-dimensional vectorsubspace spanned by $\dot{\gamma}(\alpha)$, i.e., $T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n \simeq T_{\gamma(\alpha)}\mathbb{R}^n/\mathrm{span}\{\dot{\gamma}(\alpha)\}$. Hence from now on, we assume, for $\eta^s \in T_{\gamma(\alpha)}\mathbb{R}^n$, $\eta^u \in T_{\gamma(\alpha)}\mathbb{R}^n$ and $\alpha \in \mathbb{R}$, that

$$(4.23) \qquad \eta^s \in T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n \cap \mathscr{R}P(\alpha)$$

and

$$(4.24) \qquad \eta^u \in T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n \cap \mathscr{R}(I-Q(\alpha)).$$

We also assume, for $\alpha \in \mathbb{R}$, that

$$(4.25) \qquad \mathscr{R}(I-P(\alpha)) \subset T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n, \quad \mathscr{R}Q(\alpha) \subset T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n.$$

Under these assumptions, each point in $W^s_{loc}(\bar{x}_+,\epsilon)$ or $W^u_{loc}(\bar{x}_-,\epsilon)$ is uniquely expressed in terms of the coordinates α, η^s or η^u.

Remark 4.3. The higher order terms

$$(I-P(\alpha)) \int_{\infty}^{\alpha} \Phi(\alpha,\tau)h(\tau,z(\eta^{s})(\tau),\alpha,\epsilon)d\tau \quad \text{and}$$

$$Q(\alpha) \int_{-\infty}^{\alpha} \Phi(\alpha,\tau)h(\tau,z(\eta^{u})(\tau),\alpha,\epsilon)d\tau$$

in (4.20) and (4.22) are of order ϵ uniformly in α. Though these terms include the solutions $z(\eta^{s})(\tau)$ and $z(\eta^{u})(\tau)$ of equations (4.12) and (4.18), these solutions can be approximated in an arbitralily high order of accuracy by an iterative scheme. In fact, second iteration is enough to obtain all information we need to determine the transversality of $W^{s}_{loc}(\bar{x}_{+},\epsilon)$ and $W^{u}_{loc}(\bar{x}_{-},\epsilon)$. (See §9).

§5. THE FREDHOLM'S ALTERNATIVE

Suppose that the following system

$$(5.1) \qquad \dot{z} = A(t)z, \quad z \in \mathbb{R}^n$$

has exponential dichotomies on $[0,\infty)$ with projection P and on $(-\infty,0]$ with projection Q. Consider the inhomogeneous system

$$(5.2) \qquad \dot{z} = A(t)\, z + g(t)$$

where $g(t)$ is bounded and continuous on $\mathbb{R}$.

Problem: find a condition under which system (5.2) has a bounded solution on $\mathbb{R}$.

Let $\Phi(t,s)$ be the fundamental matrix of system (5.1). Then it is known that the solution $z(t;0,\xi^s)$ of (5.2) is bounded on $[0,\infty)$ if and only if

$$(5.3) \qquad \xi^s = \eta^s + (I-P) \int_{\infty}^{0} \Phi(0,t)g(t)dt,$$

and the solution $z(t;0,\xi^u)$ of (5.2) is bounded on $(-\infty,0]$ if and only if

$$(5.4) \qquad \xi^u = \eta^u + Q \int_{-\infty}^{0} \Phi(0,t)g(t)dt,$$

where $\eta^s \in \mathcal{R}(P)$ and $\eta^u \in \mathcal{R}(I-Q)$.

Thus the set of initial data ξ^s (ξ^u respectively) which gives a bounded solution on $[0,\infty)$ ($(-\infty,0]$) constitutes the hyperplane which is a shift by the constant vector

$$(I-P) \int_{\infty}^{0} \Phi(0,t)g(t)dt \quad (Q \int_{-\infty}^{0} \Phi(0,t)g(t)dt)$$

from the unperturbed space $\mathcal{R}(P)$ ($\mathcal{R}(I-Q)$).

Let

$$\vec{d} = Q \int_{-\infty}^{0} \Phi(0,t)g(t)dt - (I-P) \int_{\infty}^{0} \Phi(0,t)g(t)dt. \tag{5.5}$$

Then we have the following lemma which is the starting point of the paper.

<u>Lemma 5.1</u>. System (5.2) has a bounded solution on $\mathbb{R}$ if and only if

$$\vec{d} \in \mathcal{R}(P) + \mathcal{R}(I-Q). \tag{5.6}$$

<u>Proof</u>. This is obvious because condition (5.6) is equivalent to say that two hyperplanes defined by (5.3) and (5.4) intersect. □

The geometrical statement in Lemma 5.1 can be expressed analytically by using bounded solutions of the adjoint system of (5.2):

(5.7) $\dot{\phi} = A^{*}(t)\ \phi$

where $A^{*}(t)$ is the transpose of $A(t)$.
Recall, from §3, that $\{\mathcal{R}(I-P^{*}) \cap \mathcal{R}(Q^{*})\} = \{\mathcal{R}(P) + \mathcal{R}(I-Q)\}^{\perp}$ and $\{\mathcal{R}(I-P^{*}) \cap \mathcal{R}(Q^{*})\}$ is the subspace of initial points at $t = 0$ of bounded solutions of the adjoint system (5.7). Therefore condition (5.6) is equivalent to say that $\vec{d}$ is annihilated by $\phi(0)$ where $\phi(t)$ is any bounded solution of the adjoint system (5.7).

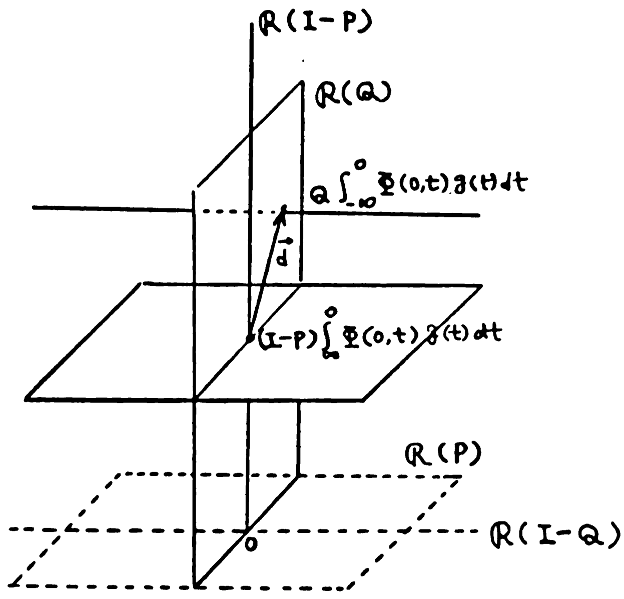


Figure 4

Let $m = \dim\{\mathcal{R}(P) + \mathcal{R}(I-Q)\}^{\perp}$ and let $\{\phi_1(t),\ldots,\phi_m(t)\}$ be a complete set of bounded solutions of system (5.7) which satisfies that

$$(5.8) \qquad \{\mathscr{R}(P) + \mathscr{R}(I-Q)\}^{\perp} = \text{span}\ \{\phi_1(0),\ldots,\phi_m(0)\}.$$

Then we have the following analytical restatement of Lemma 5.1.

<u>Lemma 5.2</u>. $\vec{d} \in \mathscr{R}(P) + \mathscr{R}(I-Q)$ if and only if

$$(5.9) \qquad \int_{-\infty}^{\infty} \phi_i^*(t)g(t)dt = 0, \quad i = 1,\ldots,m.$$

<u>Proof</u>. $\vec{d} \in \mathscr{R}(P) + \mathscr{R}(I-Q)$ if and only if $\phi_i^*(0)\,\vec{d} = 0$, $i = 1,\ldots,m.$

Since

$$Q^*\phi_i(0) = (I-P^*)\phi_i(0) = \phi_i(0)$$

and

$$\phi_i(t) = (\Phi^{-1}(t,0))^*\phi_i(0) = \Phi^*(0,t)\phi_i(0),$$

$$\begin{aligned}
0 &= \phi_i^*(0)\vec{d} \\
&= \phi_i^*(0)Q \int_{-\infty}^{0} \Phi(0,t)g(t)dt - \phi_i^*(0)(I-P) \int_{\infty}^{0} \Phi(0,t)g(t)dt \\
&= \int_{-\infty}^{0} \phi_i^*(0)\Phi(0,t)g(t)dt - \int_{\infty}^{0} \phi_i^*(0)\Phi(0,t)g(t)dt \\
&= \int_{-\infty}^{0} \phi_i^*(t)g(t)dt - \int_{\infty}^{0} \phi_i^*(t)g(t)dt \\
&= \int_{-\infty}^{\infty} \phi_i^*(t)g(t)dt. \qquad \square
\end{aligned}$$

We remark that Lemma 5.2 had been proved in Chow, Hale and Mallet–Paret [4] for two dimensional case and in Palmer [16] for general case. However our proof is different and more geometrical. We shall apply in the next section the method of proof of Lemma 5.2 to the tangent space at each point of a homodinic or heteroclinic oribit.

§6. THE MELNIKOV VECTOR

Our purpose in this section is to develop necessary concepts which are useful to derive computable conditions under which the purturbed stable and unstable manifolds intersect transversally. To do so, we would like to measure the 'distance' between $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$. Define, for simplicity, the following quantities in the expressions in (4.20) and (4.22):

(6.1) $$m^s(\alpha) = (I-P(\alpha)) \int_{\infty}^{\alpha} \Phi(\alpha,t)g(t-\alpha,\gamma(t))dt,$$

(6.2) $$\tilde{m}^s(\alpha,\eta^s,\epsilon) = (I-P(\alpha)) \int_{\infty}^{\alpha} \Phi(\alpha,t)h(t,z(\eta^s)(t),\alpha,\epsilon)dt,$$

(6.3) $$m^u(\alpha) = Q(\alpha) \int_{-\infty}^{\alpha} \Phi(\alpha,t)g(t-\alpha,\gamma(t))dt,$$

(6.4) $$\tilde{m}^u(\alpha,\eta^u,\epsilon) = Q(\alpha) \int_{-\infty}^{\alpha} \Phi(\alpha,t)h(t,z(\eta^u)(t),\alpha,\epsilon)dt.$$

Then we have

(6.5) $$M^s(\alpha,\eta^s,\epsilon) = \eta^s + m^s(\alpha) + \tilde{m}^s(\alpha,\eta^s\epsilon)$$

and

(6.6) $$M^u(\alpha,\eta^u,\epsilon) = \eta^u + m^u(\alpha) + \tilde{m}^u(\alpha,\eta^u,\epsilon).$$

Finally we define the distance vectors $\tilde{d}$ and $\vec{d}$ by

$$\tilde{d}(\alpha,\eta^s\eta^u,\epsilon) = M^u(\alpha,\eta^u,\epsilon) - M^s(\alpha,\eta^s,\epsilon) \tag{6.7}$$

and

$$\vec{d}(\alpha) = m^u(\alpha) - m^s(\alpha). \tag{6.8}$$

Recall that we are working on the normal bundle $\bigcup_{\alpha\in\mathbb{R}} T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$. Fix $\alpha\in\mathbb{R}$ and let $\eta^s,\eta^u\in T_{\gamma(\alpha)}\mathbb{R}^n$. Consider the following decomposition of $T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$:

$$\begin{aligned} T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n = &\{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}P(\alpha) \cap T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n\} \\ &\oplus \{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(I-P(\alpha)) \cap T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n)\} \\ &\oplus \{\mathcal{R}(Q(\alpha)) \cap \mathcal{R}P(\alpha) \cap T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n\} \\ &\oplus \{\mathcal{R}Q(\alpha) \cap \mathcal{R}(I-P(\alpha)) \cap T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n\}. \end{aligned} \tag{6.9}$$

According to this decomposition of $T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$, the vectors $M^u(\alpha,\eta^u,\epsilon)$ and $M^s(\alpha,\eta^s,\epsilon)$ are decomposed as follows:

$$M^u(\alpha,\eta^u,\epsilon) = (\eta_1^u,\eta_2^u,m_1^u + \tilde{m}_1^u, m_2^u + \tilde{m}_2^u) \tag{6.10}$$

and

$$M^s(\alpha,\eta^s,\epsilon) = (\eta_1^s,m_1^s + \tilde{m}_1^s,\eta_2^s,m_2^s + \tilde{m}_2^s). \tag{6.11}$$

See the diagram below.

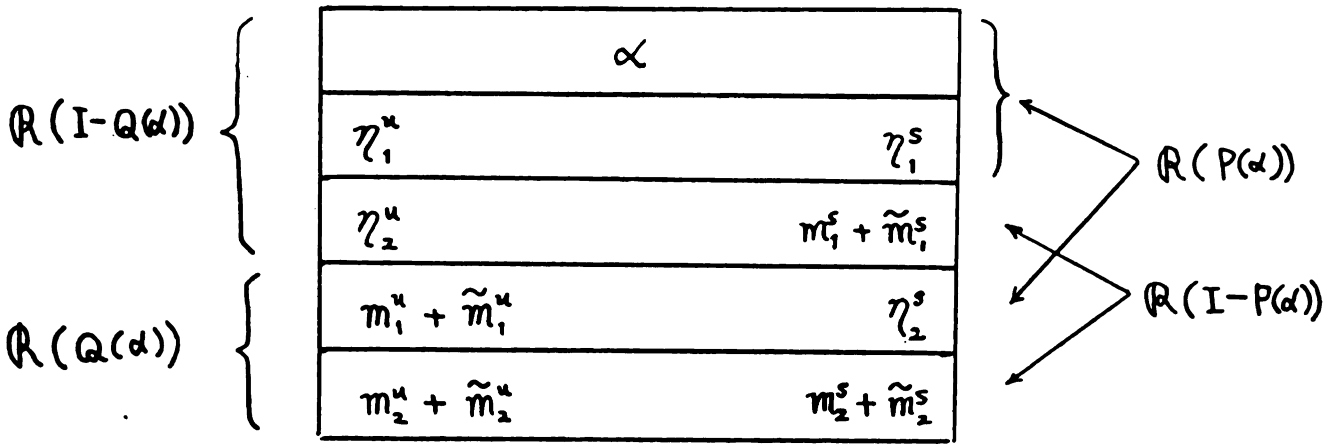


Figure 5

To get familiar with the decomposition defined above we give an example in Figure 6. Here consider a homoclinic orbit γ in $\mathbb{R}^3$ and assume $\dim \mathcal{R}\, P(\alpha) = 2$ and $\dim \mathcal{R}(I-Q(\alpha)) = 1$.

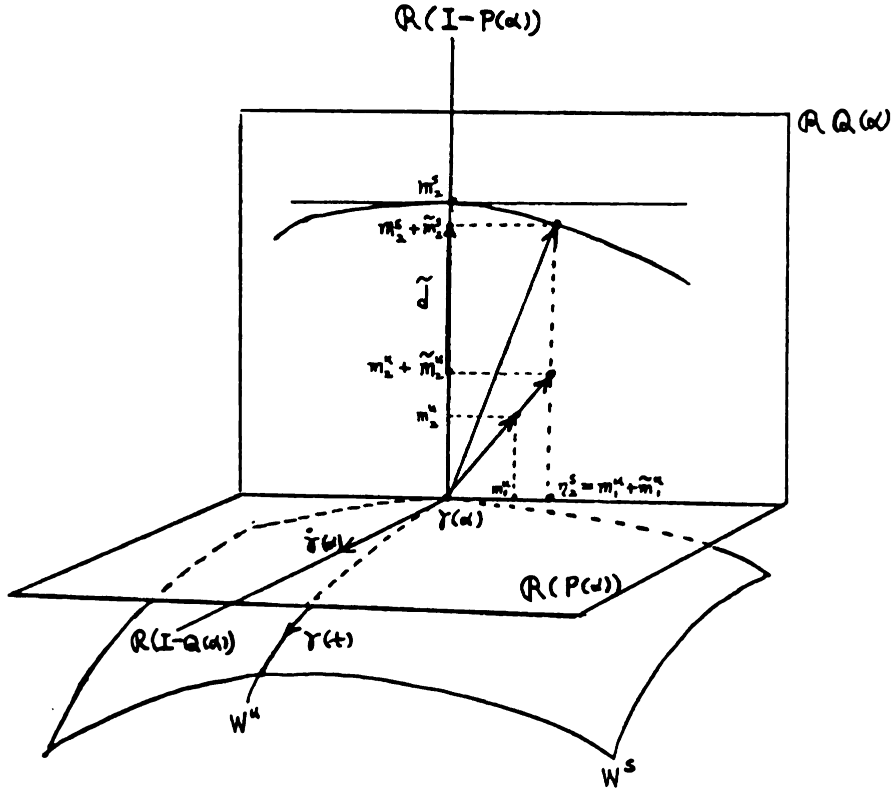

R(I-P(α))
R Q(α)
m_2^s
$m_2^s+\tilde{m}_2^s$
$\tilde{d}$
$m_2^u+\tilde{m}_2^u$
m_2^u
m_1^u
$\eta_2^s=m_1^u+\tilde{m}_1^u$
γ(α)
$\dot{\gamma}(\alpha)$
R(P(α))
R(I-Q(α))
γ(t)
W^u
W^s

Figure 6

Now it is clear from (4.19), (4.21) and (6.7) that $W^s_{loc}(\bar{x}_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ intersect each other in the hyperplane $\gamma(\alpha) + T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$ if and only if $\tilde{d}(\alpha,\eta^s,\eta^u,\epsilon) = 0$ for some α,η^s and η^u. Since

$$\tilde{d}(\alpha,\eta^s,\eta^u,\epsilon) = (\eta_1^u - \eta_1^s,\ \eta_2^u - (m_1^s+\tilde{m}_1^s),\ (m_1^u+\tilde{m}_1^u) - \eta_2^s,\ (m_2^u+\tilde{m}_2^u) - (m_2^s+\tilde{m}_2^s)), \tag{6.12}$$

$\tilde{d}(\alpha,\eta^s,\eta^u,\epsilon) = 0$ if and only if there exist $\alpha,\nu(= \eta_1^s = \eta_1^u)$, η_2^s and η_2^u such that the following three equations are satisfied.

$$\eta_2^u - \{m_1^s(\alpha) + \tilde{m}_1^s(\alpha,\nu,\eta_2^s,\epsilon)\} = 0, \tag{6.13}$$

$$\eta_2^s - \{m_1^u(\alpha) + \tilde{m}_1^u(\alpha,\nu,\eta_2^u,\epsilon)\} = 0, \tag{6.14}$$

$$\{m_2^u(\alpha)+\tilde{m}_2^u(\alpha,\nu,\eta_2^u,\epsilon)\} - \{m_2^s(\alpha)+\tilde{m}_2^s(\alpha,\nu,\eta_2^s,\epsilon)\} = 0. \tag{6.15}$$

From (6.13) and (6.14),

$$F(\eta_2^u,\alpha,\nu) \equiv \eta_2^u - \{m_1^s(\alpha)+\tilde{m}_1^s(\alpha,\nu,m_1^u(\alpha)+\tilde{m}_1^u(\alpha,\nu,\eta_2^u,\epsilon),\epsilon)\} = 0. \tag{6.16}$$

We notice that

$$\frac{\partial}{\partial\eta_2^u} F(\eta_2^u,\alpha,\nu) = I - \frac{\partial\tilde{m}_1^s}{\partial\eta_2^s}\frac{\partial\tilde{m}_1^u}{\partial\eta_2^u} \tag{6.17}$$

is nonsingular for ϵ small enough because

$$(6.18) \qquad \left| \frac{\partial \tilde{m}_1^s}{\partial \eta_2^s} \; \frac{\partial \tilde{m}_1^u}{\partial \eta_2^s} \right| = 0(\epsilon^2).$$

Hence, it follows from the implicit mapping theorem that

$$(6.19) \qquad \eta_2^u = \eta_2^u(\alpha,\nu,\epsilon)$$

for $|\nu|<<1$. Similarly we have

$$(6.20) \qquad \eta_2^s = \eta_2^s(\alpha,\nu,\epsilon)$$

for $|\nu|<<1$.

Therefore, by (6.15),

$$(6.21) \qquad \tilde{d}(\alpha,\eta^s\eta^u,\epsilon) = 0$$

if and only if

$$(6.22) \qquad \{m_2^u(\alpha) + \tilde{m}_2^u(\alpha,\nu,\eta_2^u(\alpha,\nu,\epsilon),\epsilon)\} - \{m_2^s(\alpha) + \tilde{m}_2^s(\alpha,\nu,\eta_2^s(\alpha,\nu,\epsilon),\epsilon)\} = 0.$$

To rewrite (6.22) in a more convenient form, we utilize bounded solutions of the adjoint system

$$(6.23) \qquad \dot{\phi} + A^*(t)\phi = 0$$

of system (4.9). Let

$$m = \dim\{\mathcal{R}(P(\alpha)) + \mathcal{R}(I-Q(\alpha))\}^{\perp} \tag{6.24}$$

and let $\{\phi_1(t),\ldots,\phi_m(t)\}$ be a complete set of bounded solutions of system (6.23) which satisfies

$$\{\mathcal{R}(P(\alpha)) + \mathcal{R}(I-Q(\alpha))\}^{\perp} = \mathrm{span}\{\phi_1(\alpha),\ldots,\phi_m(\alpha)\}. \tag{6.25}$$

Then (6.22) is equivalent to the following equations:

$$\begin{aligned} \phi_i^*(\alpha)[\{m_2^u(\alpha) + \tilde{m}_2^u(\alpha,\nu,\eta_2^u(\alpha,\nu,\epsilon)\epsilon)\} \\ - \{m_2^s(\alpha) + \tilde{m}_2^s(\alpha,\nu,\eta_2^s(\alpha,\nu,\epsilon),\epsilon)\}] = 0, \quad i=1,\ldots,m, \end{aligned} \tag{6.26}$$

where $\phi_i^*(\alpha)$ is the transpose of $\phi_i(\alpha)$. Since

$$\phi_i(t) = \Phi^*(\alpha,t)\phi_i(\alpha), \quad i=1,\ldots,m, \tag{6.27}$$

and

$$\phi_i^*(\alpha)Q(\alpha) = \phi_i^*(\alpha)(I-P(\alpha)) = \phi_i^*(\alpha), \quad i=1,\ldots,m, \tag{6.28}$$

(6.26) becomes

$$(6.29) \qquad 0 = \phi_i^*(\alpha)\{Q(\alpha) \int_{-\infty}^{\alpha} \Phi(\alpha,t)g(t-\alpha,\gamma(t))dt$$

$$+ Q(\alpha) \int_{-\infty}^{\alpha} \Phi(\alpha,t)h(t,z^u(\nu)(t),\alpha,\epsilon)dt$$

$$- (I-P(\alpha)) \int_{\infty}^{\alpha} \Phi(\alpha,t)g(t-\alpha,\gamma(t)dt$$

$$- (I-P(\alpha)) \int_{\infty}^{\alpha} \Phi(\alpha,t)h(t,z^s(\nu)(t),\alpha,\epsilon)dt\}$$

$$= \int_{-\infty}^{\infty} \phi_i^*(t)g(t-\alpha,\gamma(t))dt$$

$$+ \phi_i^*(\alpha)\{ \int_{-\infty}^{\alpha} \Phi(\alpha,t)h(t,z^u(\nu)(t),\alpha,\epsilon)dt$$

$$+ \int_{\alpha}^{\infty} \Phi(\alpha,t)h(t,z^s(\nu)(t),\alpha,\epsilon)dt\},$$

$i=1,\ldots,m$. Here $z^u(\nu)(t) \equiv z(t;\alpha,\xi^u(\nu,\eta_2^u(\alpha,\nu,\epsilon)))$ is the solution of (4.17) and η_2^u is given in (6.19). Similarly $z^s(\nu)(t) \equiv z(t;\alpha,\xi^s(\nu,\eta_2^s(\alpha,\nu,\epsilon)))$ is the solution of (4.12) and η_2^s is given in (6.20). Thus we have derived a bifurcation equation (6.29) and now it is reasonable to define the following quantities.

Definition 6.1. The Melnikov vector $M(\alpha,\nu,\epsilon)$ for system (2.2) is defined by

$$(6.30) \qquad M(\alpha,\nu,\epsilon) = (M_1(\alpha,\nu,\epsilon),\ldots,M_m(\alpha,\nu,\epsilon))$$

where

$$(6.31)\qquad M_i(\alpha,\nu,\epsilon) = \int_{-\infty}^{\infty} \phi_i^*(t)g(t-\alpha,\gamma(t))dt + \phi_i^*(\alpha)\{ \int_{-\infty}^{\alpha} \Phi(\alpha,t)h(t,z^u(\nu)(t),\alpha,\epsilon)dt + \int_{\alpha}^{\infty} \Phi(\alpha,t)h(t,z^s(\nu)(t),\alpha,\epsilon)dt\},$$

i=1,...,m.

Also the linear Melnikov vector $\hat{M}(\alpha)$ for system (2.2) is defined by

$$(6.32)\qquad \hat{M}(\alpha) = (\hat{M}_1(\alpha),\ldots,\hat{M}_m(\alpha))$$

where

$$(6.33)\qquad \hat{M}_i(\alpha) = \int_{-\infty}^{\infty} \phi_i^*(t)g(t-\alpha,\gamma(t))dt, \quad i=1,\ldots,m.$$

Remark 6.2. $M(\alpha,\nu,\epsilon) = \hat{M}(\alpha) + 0(\epsilon)$ uniformly in ν.

Remark 6.3. The above argument to derive the Melnikov vector is essentially the same as the Lyapunov–Schmidt reduction. However we employed the above more elementary and geometrical argument which will be useful when we derive the condition for the transversal intersection.

The next proposition follows from the definition of the Melnikov vector.

Proposition 6.4. $W^s(\bar{x}_+,\epsilon)$ and $W^u(\bar{x}_-,\epsilon)$ intersect each other if and only if $M(\alpha_0,\nu_0,\epsilon) = 0$ for some α_0 and ν_0.

Proof. If $M(\alpha_0,\mu_0,\epsilon) = 0$ for some α_0 and ν_0, then it is obvious from the definition of the Melnikov vector that $W^s_{loc}(x^-_+,\epsilon)$ and $W^u_{loc}(\bar{x}_-,\epsilon)$ intersect each other. Conversely once $W^s(\bar{x}_+,\epsilon)$ and $W^u(\bar{x}_-,\epsilon)$ intersect, then there is a bi–infinite sequence $\{p_i\}_{i=-\infty}^{\infty}$ of points of intersection which approaches $\bar{x}_+$ and $\bar{x}_-$ as $x \to +\infty$ and $x \to -\infty$ respectively. Hence for sufficiently large $|i|$, $p_i \epsilon W^s_{loc}(\bar{x}_+,\epsilon) \cap W^u_{loc}(\bar{x}_-,\epsilon)$ which implies that $M(\alpha_0,\nu_0,\epsilon) = 0$ for some α_0 and ν_0. □

§7. TRANSVERSAL INTERSECTION OF THE STABLE AND UNSTABLE MANIFOLDS

In this section we will prove our main result which gives conditions for transversal intersection of the stable and unstable manifolds.

Recall from Remark 4.2 that $W^u_{loc}(\bar{x}_-,\epsilon)$ and $W^s_{loc}(\bar{x}_+,\epsilon)$ are diffeomorphic to the graphs $F^u(\alpha,\eta^u_1,\eta^u_2)$ and $F^s(\alpha,\eta^s_1,\eta^s_2)$ respectively in a tabular neighborhood of γ which are given by

$$(7.1)\quad F^u(\alpha,\eta^u_1,\eta^u_2) = \begin{bmatrix} \alpha \\ \eta^u_1 \\ \eta^u_2 \\ m^u_1(\alpha)+\bar{m}^u_1(\alpha,\eta^u_1,\eta^u_2) \\ m^u_2(\alpha)+\bar{m}^u_2(\alpha,\eta^u_1,\eta^u_2) \end{bmatrix} \begin{matrix} \}\ \epsilon\ \mathbb{R} \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\}\ \epsilon\ \text{Range}(I-Q(\alpha)) \\ \left.\begin{matrix} \\ \\ \end{matrix}\right\}\ \epsilon\ \text{Range } Q(\alpha) \end{matrix}$$

and

$$(7.2)\quad F^s(\alpha,\eta^s_1,\eta^s_2) = \begin{bmatrix} \alpha \\ \eta^s_1 \\ m^s_1(\alpha)+\bar{m}^s_1(\alpha,\eta^s_1,\eta^s_2) \\ \eta^s_2 \\ m^s_2(\alpha)+\bar{m}^s_2(\alpha,\eta^s_1\eta^s_2) \end{bmatrix} \begin{matrix} \}\ \epsilon\ \mathbb{R} \\ \text{rows 2, 4}\ \epsilon\ \text{Range } P(\alpha) \\ \text{rows 3, 5}\ \epsilon\ \text{Range } (I-P(\alpha)) \end{matrix}$$

Hence to show the existence of transversal intersection, it is sufficient to show that column vectors in the following matrices $D_{(\alpha,\eta^u_1,\eta^u_2)}F^u$ and

$D_{(\alpha,\eta_1^s,\eta_2^s)} F^s$ span the whole space $\mathbb{R}^n$.

$$
(7.3)\qquad D_{(\alpha,\eta_1^u,\eta_2^u)} F^u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ \frac{\partial}{\partial\alpha}(m_1^u+\tilde{m}_1^u) & \frac{\partial}{\partial\eta_1^u}\tilde{m}_1^u & \frac{\partial}{\partial\eta_2^u}\tilde{m}_1^u \\ \frac{\partial}{\partial\alpha}(m_2^u+\tilde{m}_2^u) & \frac{\partial}{\partial\eta_1^u}\tilde{m}_2^u & \frac{\partial}{\partial\eta_2^u}\tilde{m}_2^u \end{bmatrix} \begin{matrix} \\ \\ (a) \\ (b) \\ (c) \end{matrix}
$$

$$
\begin{matrix} (1) & \qquad (2) & \qquad (3) \end{matrix}
$$

$$
(7.4)\qquad D_{(\alpha,\eta_1^s,\eta_2^s)} F^s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ \frac{\partial}{\partial\alpha}(m_1^s+\tilde{m}_1^s) & \frac{\partial}{\partial\eta_1^s}\tilde{m}_1^s & \frac{\partial}{\partial\eta_2^s}\tilde{m}_1^s \\ 0 & 0 & I \\ \frac{\partial}{\partial\alpha}(m_2^s+\tilde{m}_2^s) & \frac{\partial}{\partial\eta_1^s}\tilde{m}_2^s & \frac{\partial}{\partial\eta_2^s}\tilde{m}_2^s \end{bmatrix} \begin{matrix} \\ \\ (d) \\ (e) \\ (f) \end{matrix}
$$

$$
\begin{matrix} (4) & \qquad (5) & \qquad (6) \end{matrix}
$$

Now we have the main result in this paper.

<u>Theorem 7.1</u>. Assume that system (2.1) has two hyperbolic critical points x_+ and x_- (not necessarily distinct) and has an orbit γ connecting them: $\gamma(t) \to x_+$ as $t \to \infty$ and $\gamma(t) \to x_-$ as $t \to -\infty$.

Let $k = \dim\{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(P(\alpha))\}$, $m = \dim\{\mathcal{R}(I-Q(\alpha)) + \mathcal{R}(P(\alpha))\}^{\perp}$ and let $\nu = (\nu_1,\ldots,\nu_{k-1}) \in \{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(P(\alpha))\} \cap T^{\perp}_{\gamma(\alpha)}\mathbb{R}^n$, where $P(\alpha)$ and $Q(\alpha)$ are respectively the projections of exponential dichotomies on $(-\infty,\alpha]$ and on $[\alpha,\infty)$ of system (4.9) and satisfy the conditions in (4.25). Consider the perturbed system (2.2) and define the Melnikov vector $M(\alpha,\nu,\epsilon)$ and the linear Melnikov vector $\hat{M}(\alpha)$ by (6.30) – (6.33). Then

(i) the cross sections $W^s(\bar{x}_+,\epsilon)$ and $W^u(\bar{x}_-,\epsilon)$ at $t = 0$ of the time dependent stable and unstable manifolds of system (2.2) intersect each other if and only if $M(\alpha_0,\nu_0,\epsilon) = 0$ for some α_0,ν_0 and small ϵ.

In this case,

(ii) the intersection of $W^u_{loc}(\bar{x}_-,\epsilon)$ and $W^s_{loc}(\bar{x}_+,\epsilon)$ are transversal if there exist m nonzero column vectors in the $m\times k$ matrix

$$\left[\frac{d}{d\alpha}\hat{M}(\alpha_0) \quad \frac{\partial}{\partial\nu} M(\alpha_0,\nu_0,\epsilon)\right].$$

<u>Proof</u>. (i) This is Proposition 6.4.

(ii) Consider column vectors in matrices (7.3) and (7.4).

Firstly it is clear that all column vectors in blocks (3) and (6) are always linearly independent. Secondly by (6.26), $M(\alpha_0,\nu_0,\epsilon) = 0$ implies that

$$(7.5) \qquad \begin{aligned} |m^u_2(\alpha_0) - m^s_2(\alpha_0)| &= |\tilde{m}^s_2(\alpha_0,\nu_0,\eta^s_2(\alpha_0,\epsilon),\epsilon) \\ &\quad -\tilde{m}^u_2(\alpha_0\nu_0,\eta^u_2(\alpha_0,\nu_0,\epsilon),\epsilon)| = 0(\epsilon). \end{aligned}$$

So

$$(7.6)\quad \frac{\partial}{\partial\alpha}\hat{M}(\alpha_0)= \begin{bmatrix} \dot{\phi}_1^*(\alpha_0)(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \\ \vdots \\ \dot{\phi}_m^*(\alpha_0)(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \end{bmatrix} + \begin{bmatrix} \phi_1^*(\alpha_0)\frac{\partial}{\partial\alpha}(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \\ \\ \phi_m^*(\alpha_0)\frac{\partial}{\partial\alpha}(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1^*(\alpha_0)\frac{\partial}{\partial\alpha}(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \\ \vdots \\ \phi_m^*(\alpha_0)\frac{\partial}{\partial\alpha}(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \end{bmatrix} + 0(\epsilon).$$

Therefore if $\frac{\partial}{\partial\alpha}\hat{M}(\alpha_0) \neq 0$, then $\frac{\partial}{\partial\alpha}(m_2^u(\alpha_0)-m_2^s(\alpha_0)) \neq 0$ and hence column vectors (1) and (4) are linearly independent at the point of intersection. Finally

$$(7.7)\qquad \frac{\partial}{\partial\nu_j} M(\alpha_0,\nu_0,\epsilon)$$

$$= \begin{bmatrix} \phi_1^*(\alpha_0)\frac{\partial}{\partial\nu_j}\{\tilde{m}_2^u(\alpha_0,\nu_0,\eta_2^u(\alpha_0,\nu_0,\epsilon)\epsilon)-\tilde{m}_2^s(\alpha_0,\nu_0,\eta_2^s(\alpha_0,\nu_0,\epsilon)\epsilon))\} \\ \vdots \\ \phi_m^*(\alpha_0)\frac{\partial}{\partial\nu_j}\{\tilde{m}_2^u(\alpha_0,\nu_0,\eta_2^u(\alpha_0,\nu_0,\epsilon)\epsilon)- \hat{m}_2^s(\alpha_0,\nu_0,\eta_2^s(\alpha_0,\nu_0,\epsilon)\epsilon)\} \end{bmatrix},$$

$j=1,\dots,k-1$.

Therefore if $\frac{\partial}{\partial\nu_j} M(\alpha_0,\nu_0,\epsilon) \neq 0$, then $\frac{\partial}{\partial\nu_j}(\tilde{m}_2^u-\tilde{m}_2^s) \neq 0$ and hence j–th column vectors in block (2) and (5) are linearly independent at the point of intersection.

Now consider the following decomposition of $T_{\gamma(\alpha)}\mathbb{R}^n$.

$$(7.8)\qquad T_{\gamma(\alpha)}\mathbb{R}^n = \{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(P(\alpha))\}\oplus\{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(I-P(\alpha))\}$$
$$\oplus\{\mathcal{R}(Q(\alpha)) \cap \mathcal{R}(P(\alpha))\}\oplus\{\mathcal{R}(Q(\alpha)) \cap \mathcal{R}(I-P(\alpha))\}$$

with the following dimensions:

$$\dim\{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(P(\alpha))\} = k,$$
$$\dim\{\mathcal{R}(I-Q(\alpha)) \cap \mathcal{R}(I-P(\alpha))\} = n-n_--k,$$
$$\dim\{\mathcal{R}(Q(\alpha)) \cap \mathcal{R}(P(\alpha))\} = n_+-k,$$
$$\dim\{\mathcal{R}(Q(\alpha)) \cap \mathcal{R}(I-P(\alpha))\} = m,$$

where $n_- = \dim W^s(x_-)$ and $n_+ = \dim W^s(x_+)$.
See the diagram below.

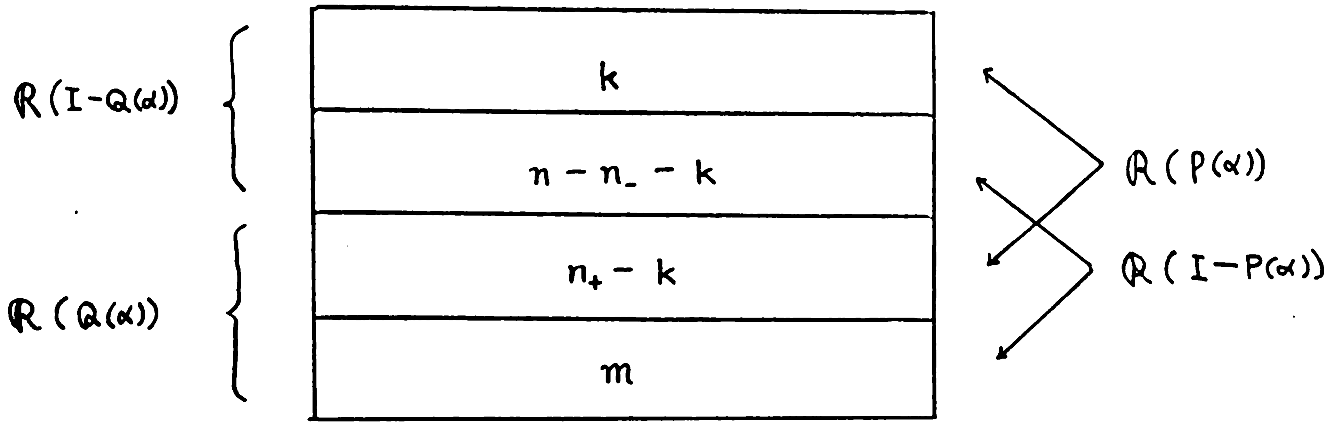


Figure 7

From the above argument, $(n-n_--k) + (n_+-k)(= n-m-k)$ column vectors in block (3) and (6) are linearly independent. The condition of the theorem and the above argument imply that there are $(k-m) + 2m = k + m$ column vectors in block (1), (2), (4) and (5) which are linearly independent and so we have $(n-m-k) + (k+m) = n$ linearly independent column vectors in matrices (7.3) and (7.4) □

The condition for transversal intersections in part (ii) of the theorem can not be, in general, the necessary and sufficient condition for transversal intersections since the Melnikov vector is the projection of the distance vector (6.12) to the subspace $\mathcal{R}(Q(\alpha)) \cap \mathcal{R}(I-P(\alpha))$. See the decomposition in (7.8). In other words, the Melnikov vector contributes only to the last components of column vectors, i.e. components (c) and (f), in matrices (7.3) and (7.4). See example 3 in Section 13. However if

(7.9) $\mathcal{R}(P(\alpha)) = \mathcal{R}(I-Q(\alpha))$,

the components (a), (b), (d) and (e) of column vectors of matrices in (7.3) and (7.4) are missing and hence the Melnikov vector can be used to give a necessary and sufficient condition for transversal intersection. Thus we have the following

<u>Corollary 7.2</u>. Under the same assumption as in Theorem 7.1 and assumption (7.9), the intersection of $W^u_{loc}(\bar{x}_-,\epsilon)$ and $W^s_{loc}(\bar{x}_+,\epsilon)$ is transversal if and only if $M(\alpha_0,\nu_0,\epsilon) = 0$ for some α_0 and ν_0 and there exist m nonzero column vectors in the m×k matrix $[\frac{d}{d\alpha}\hat{M}(\alpha_0)\ \frac{\partial}{\partial\nu}M(\alpha_0,\nu_0,\epsilon)]$. □

We remark that condition (7.9) implies $m = n - k$.

<u>Remark 7.3</u>. It can be easily shown, by the implicit mapping theorem, that if $\hat{M}(\alpha_1) = 0$ and $\frac{\partial}{\partial\alpha}\hat{M}(\alpha_1) \neq 0$ for some α_1, then $M(\alpha,\nu,\epsilon) = 0$ for some α and ν.

Now we define the following quantity. Let $\gamma \subset W^u(x_-) \cap W^s(x_+)$ be a homoclinic or heteroclinic orbit in $\mathbb{R}^n$.

Definition 7.4. The splitting index $\delta(\gamma)$ of γ is defined by

(7.10) $\delta(\gamma) = \dim W^s(x_-) - \dim W^s(x_+)$.

Notice from (6.24) that

(7.11) $m = n - [\{n-\dim W^s(x_-)\} + \dim W^s(x_+) - k]$.

Thus m,k and $\delta(\gamma)$ satisfy the following relation.

(7.12) $m = k + \delta(\gamma)$.

Remark 7.5. If γ is a homoclinic orbit, then the splitting index $\delta(\gamma)$ is always zero. Thus by theorem 7.1(ii), all of k column vectors in $[\frac{\partial}{\partial\alpha}\hat{M} \quad \frac{\partial}{\partial\nu}M]$ must be nonzero to guarantee the transversal intersection. This situation is only the case in the homoclinic case.

Remark 7.6. Assume that γ is a heteroclinic orbit in $\mathbb{R}^n$. Then we have three different situations.

(i) $\delta(\gamma) > 0$, i.e., $\dim W^s(x_+) < \dim W^s(x_-)$. Then $m = k + \delta(\gamma) > k$. Thus theorem 7.1(ii) implies that that there is no transversal intersection because the matrix $[\frac{\partial}{\partial\alpha}\hat{M} \quad \frac{\partial}{\partial\nu}M]$ is of the size $m \times k$. A reason for this is that $\dim W^s(x_+) < \dim W^s(x_-)$ is equivalent to say that $\dim W^u(x_-) + \dim W^s(x_+) < n$.

(ii) $\delta(\gamma) = 0$, i.e., $\dim W^s(x_+) = \dim W^s(x_-)$. In this case we have the same situation as in the homoclinic case.

(iii) $\delta(\gamma) < 0$, i.e., $\dim^s(x_+) > \dim W^s(x_-)$. Then $m = k + \delta(\gamma) < k$. Thus transversal intersection is possible.
In this way, we can classify the possibility and impossibility of transversal intersection by using the splitting index $\delta(\gamma)$.

§8 THE INDEX OF γ, THE FREDHOLM INDEX AND THE DIMENSION OF THE MELNIKOV VECTOR

Consider system (2.1) with the same assumptions in §2 and system

$$\dot{x} = f(x) + \epsilon g(t). \tag{8.1}$$

The linearized system of (8.1) along γ is given by

$$\dot{z} = A(t)z + \epsilon g(t), \tag{8.2}$$

and the system $\dot{z} = A(t)z$ has exponential dichotomies on half-lines $[0,\infty)$ and $(-\infty,0]$ with projections P and Q respectively.

We recall that the dimension m of the (linear) Melnikov vector of system (8.2) is given by

$$\begin{aligned} m &= \dim\{\mathscr{R}\, P + \mathscr{R}\,(I-Q)\}^{\perp} \\ &= \dim\{\mathscr{R}\,(I-P^{*}) \cap \mathscr{R}\, Q^{*}\}, \end{aligned} \tag{8.3}$$

which says that the dimension of the Melnikov vector is the same as the number of independent bounded solutions of the adjoint system $\dot{\phi} + A^{*}(t)\phi = 0.$

We also defined the splitting index $\delta(\gamma)$ by

$$\delta(\gamma) = \dim W^{s}(x_{-}) - \dim W^{s}(x_{+}). \tag{8.4}$$

Though the splitting index $\delta(\gamma)$ is defined by local data, that is, the dimensions of stable manifolds of hyperbolic critical points, $\delta(\gamma)$ is global in nature since it can be used to distinguish homoclinic and heteroclinic orbits, and also used to classify heteroclinic orbits. Furthermore the relation

$$(8.5) \qquad m = k + \delta(\gamma)$$

shows how the dimension of the Melnikov vector depends on γ.
In this section we shall clarify the relationship between $m, \delta(\gamma)$ and index L which is the Fredholm index defined as follows.
Define an operator $L\colon BC^1(\mathbb{R},\mathbb{R}^n) \to BC^o(\mathbb{R},\mathbb{R}^n)$ by

$$(8.6) \qquad (Lz)(t) = \dot{z}(t) - A(t)z,$$

where $BC^1(\mathbb{R},\mathbb{R}^n)$ is the space of bounded C^1 functions from $\mathbb{R}$ to R^n and $BC^o(\mathbb{R},\mathbb{R}^n)$ is the space of bounded continuous functions from $\mathbb{R}$ to $\mathbb{R}^n$. As we shall show, L is a Fredholm operator. See also Palmer [16]. The Fredholm index is defined by

$$(8.7) \qquad \text{index } L = \dim(\ker L) - \text{codim } (\text{Range } L)$$

<u>Proposition 8.1</u>. $\delta(\gamma) = -\text{ index } L.$

<u>Proof</u>. Define a bounded linear operator $A\colon BC^0(\mathbb{R},\mathbb{R}^n) \to \mathbb{R}^m$

by

$$(8.8) \qquad Ag = \left(\int_{-\infty}^{\infty} \phi_1^*(t)g(t)dt, \ldots, \int_{-\infty}^{\infty} \phi_m^*(t)g(t)dt\right)$$

where ϕ_i, $i = 1,\ldots,m$ are independent bounded solutions of the adjoint system $\dot\phi + A^*(t)\phi = 0$.

Then by Lemma 5.10, z is a bounded solution of (8.2) if and only if $g \in \ker A$. Thus Range L = ker A, which means that Range L is closed and

$$(8.9) \qquad \text{codim (Range L)} = m,$$

and hence L is a Fredholm operator.

Thus

$$\begin{aligned} \text{index } L &= \dim(\ker L) - \text{codim (Range L)} \\ &= k - m \\ &= -\delta(\gamma). \qquad \square \end{aligned}$$

<u>Remark 8.2</u>. The splitting index $\delta(\gamma)$ in (8.4) was defined in Sacker [19] and Proposition 8.1 was also proved there. However his definition is for linear systems. Our definition of the splitting index is to relate a local information about eigervalues to a global information about a homoclinic or a heteroclinic orbit.

§9 COMPUTATION OF HIGHER ORDER TERMS

In the case of $\dim\{\mathscr{R}(P)(\alpha)) \cap \mathscr{R}(I-Q(\alpha))\} > 1$ for n–dimensional systems ($n \geq 3$), we need to know nonlinear terms in expression (6.31) of the Melnikov vector to examine the transversality condition. To this end, we consider again bounded solutions on $[\alpha,\infty)$ and on $(-\infty,\alpha]$ of system (5.2). We use the same assumption of exponential dichotomics as in §4. These bounded solutions are given as unique solutions of integral equations (4.12) and (4.17) respectively.

Let $\eta^s \epsilon \mathscr{R}(P(\alpha))$, $|\eta^s|<<1$, and let $z(\eta^s)(t)$ be the unique solution of (4.12) which is guaranteed by the contraction mapping principle. That is, $z(\eta^s)(t)$ is the solution of the following integral equation:

$$z(t) = \mathscr{T}_s(\eta^s)(g(t-\alpha,\gamma(t)) + h(t,z(t),\alpha,\epsilon)), \tag{9.1}$$

where the operator $\mathscr{T}_s(\eta^s)$ is defined by

$$\begin{aligned} &\mathscr{T}_s(\eta^s)(g(t-\alpha,\gamma(t))+h(t,z(t),\alpha,\epsilon)) \\ &= \Phi(t,\alpha)\eta^s + \Phi(t,\alpha)P(\alpha) \int_\alpha^t \Phi(\alpha,\tau)\{g(\tau-\alpha,\gamma(\tau)) + h(\tau,z(\tau),\alpha,\epsilon)\}d\tau \\ &\quad + \Phi(t,\alpha)(I-P(\alpha)) \int_\infty^t \Phi(\alpha,\tau)\{g(\tau-\alpha,\gamma(\tau)) + h(\tau,z(\tau),\alpha,\epsilon)\}d\tau. \end{aligned} \tag{9.2}$$

To approximate $z(\eta^s)(t)$, we use the following interation scheme

$$z_s^{(n+1)}(\eta^s)(t) = \mathscr{T}_s(\eta^s)(g(t-\alpha,\gamma(t)) + h(t,z_s^{(n)}(\eta^s)(t),\alpha,\epsilon)). \tag{9.3}$$

Set $z_s^{(o)}(\eta^s)(t) \equiv 0$. Then

$$z_s^{(1)}(\eta^s)(t) = \mathcal{T}_s(\eta^s)(g(t-\alpha,\gamma(t))$$

and

$$z_s^{(2)}(\eta^s)(t) = \mathcal{T}_s(\eta^s)(g(t-\alpha,\gamma(t)) + h(t,\mathcal{T}_s(\eta^s)(g(t-\alpha,\gamma(t)),\alpha,\epsilon)).$$

Notice that $\mathcal{T}_s(\eta^s)(h(t,\mathcal{T}_s(\eta^s)(g(t-\alpha,\gamma(t),\alpha,\epsilon)) = 0(\epsilon)$ and hence

$$z_s^{(2)}(\eta^s)(t) = z_s^{(1)}(\eta^s)(t) + \epsilon\tilde{z}_s^{(1)}(\eta^s)(t)$$

for some function $\tilde{z}_s^{(1)}(\eta^s)(t)$. The true solution $z(\eta^s)(t)$ of (8.1) satisfies

$$\begin{aligned} (9.4) \qquad z(\eta^s)(t) &= z_s^{(2)}(\eta^s)(t) + 0(\epsilon^2) \\ &= z_s^{(1)}(\eta^s)(t) + \epsilon\tilde{z}_s^{(1)}(\eta^s)(t) + 0(\epsilon^2), \ t \geq \alpha. \end{aligned}$$

Apparently $z_s^{(1)}(\eta^s)(t)$ is the bounded solution on $[\alpha,\infty)$ of linear system $\dot{z} = A(t)z + g(t-\alpha,\gamma(t))$.

Similarly define $\mathcal{T}_u(\eta^u)$ by

$$\begin{aligned} (9.5) \qquad & \mathcal{T}_u(\eta^u)(g(t-\alpha,\gamma(t) + h(t,z(t),\alpha,\epsilon)) \\ &= \Phi(t,\alpha)\eta^u + \Phi(t,\alpha)(I-Q(\alpha)) \int_\alpha^t \Phi(\alpha,\tau)\{g(\tau-\alpha,\gamma(\tau)) + h(\tau,z(\tau),\alpha,\epsilon)\}d\tau \\ &\quad + \Phi(t,\alpha)Q(\alpha) \int_{-\infty}^t \Phi(\alpha,\tau)\{g(\tau-\alpha,\gamma(\tau)) + h(\tau,z(\tau),\alpha,\epsilon)\}d\tau, \end{aligned}$$

where $\eta^u \epsilon \mathcal{R}(I-Q(\alpha))$, $|\eta^u| << 1$ and let $z(\eta^u)(t)$ be the unique solution of (4.17). That is, $z(\eta^u)(t)$ is the unique solution of

$$(9.6) \qquad z(t) = \mathcal{T}_u(\eta^u)(g(t-\alpha,\gamma(t)) + h(t,z(t),\alpha,\epsilon)).$$

We use the following iteration scheme.

$$(9.7) \qquad z_u^{(n+1)}(\eta^u)(t) = \mathcal{T}_u(\eta^u)(g(t-\alpha,\gamma(t)) + h(t,z_u^{(n)}(\eta^u)(t),\alpha,\epsilon)).$$

By setting $z_u^{(o)}(\eta^u)(t) \equiv 0$, we have

$$z_u^{(1)}(\eta^u)(t) = \mathcal{T}_u(\eta^u)(g(t-\alpha,\gamma(t)),$$

$$z_u^{(2)}(\eta^u)(t) = z_u^{(1)}(\eta^u)(t) + \epsilon\tilde{z}_u^{(1)}(\eta^u)(t),$$

and the true solution $z(\eta^u)(t)$ of (8.6) satisfies

$$(9.8) \qquad z(\eta^u)(t) = z_u^{(1)}(\eta^u)(t) + \epsilon\tilde{z}_u^{(1)}(\eta^u)(t) + 0(\epsilon^2),\ t \geq \alpha.$$

We notice that, by taking $\nu \equiv \eta^s = \eta^u \epsilon \mathcal{R}(P(\alpha)) \cap \mathcal{R}(I-Q(\alpha))$, $z_s^{(1)}(\nu)(t)$ and $z_u^{(1)}(\nu)$ give the linear Melnikov vector, and $\tilde{z}_s^{(1)}(\nu)(t)$ and $\tilde{z}_u^{(1)}(\nu)(t)$ give the term of order ϵ or higher in the Melnikov vector. Thus we have derived the following expression of the Melnikov vector:

$$(9.9) \qquad M_i(\alpha,\nu) = \hat{M}_i(\alpha) + \epsilon\phi_i^*(\alpha)\{\int_{-\infty}^{\alpha} \Phi(\alpha,t)h(t,z_n^{(1)}(\nu)(t),\alpha,\epsilon)dt + \int_{\alpha}^{\infty} \Phi(\alpha,t)h(t,z_s^{(1)}(\nu)(t),\alpha,\epsilon)dt\} + 0(\epsilon^2).$$

In this way we can compute the Melnikov vector in arbitralily high order of accuracy. It is also clear that it is sufficient to consider the first two terms in expression (9.9) for the transversality condition in Theorem 7.1. More accurate expressions than (9.9) are needed for the tangency condition.

§10 THE LINEAR MELNIKOV VECTOR

In this section we consider special cases in which the linear Melnikov vector gives a sufficient information for transversal and tangential intersection of the stable and unstable manifolds. We consider system (2.1) and (2.2) under the same assumption as in §2.

Case (i). Suppose

$$\text{(10.1)} \qquad k = 1 \quad \text{and} \quad \delta(\gamma) = 0.$$

This means that $\mathcal{R}(P(\alpha)) \cap \mathcal{R}(I-Q(\alpha)) = \mathrm{span}\{\dot{\gamma}(\alpha)\}$ and $m = 1$.

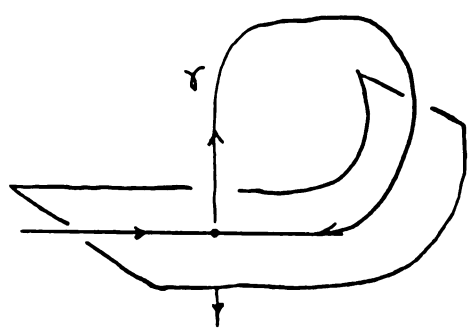


Figure 8

Note that the Melnikov function in this case is

$$\text{(10.2)} \qquad M(\alpha,\epsilon) = \hat{M}(\alpha) + 0(\epsilon).$$

Proposition 10.1. Assume (10.1) and suppose that there exists $\alpha_0 \in \mathbb{R}$ such that

(10.3) $\hat{M}(\alpha_0) = 0$ and $\frac{d}{d\alpha}\hat{M}(\alpha_0) \neq 0.$

Then $W^u_{loc}(x_-,\epsilon)$ and $W^s_{loc}(x_+,\epsilon)$ of system (2.2) have a point of transversal intersection.

Proof. By the implicit function theorem, condition (10.3) combining (10.2) implies $M(\alpha,\epsilon) = 0$ and $\frac{d}{d\alpha} M(\alpha,\epsilon) \neq 0$ for some α near α_0. Hence this proposition follows from Theorem 7.1. □

Note: Condition (10.3) can not be a necessary and sufficient condition for transversal intersection.

Apparently the two–dimensional case satisfies condition (10.1). In this special case we have the following corollary.

Corollary 10.2. Suppose that system (3.1) and (3.2) are two–dimensional. Then $W^s_{loc}(x_+,\epsilon)$ and $W^u_{loc}(x_-,\epsilon)$ of system (2.2) have a point of transversal intersection if and only if there exists $\alpha_0 \in \mathbb{R}$ so that

(10.4) $\hat{M}(\alpha_0) = 0$ and $\frac{d}{d\alpha}\hat{M}(\alpha_0) \neq 0.$

Proof. 'If' part is a special case of Proposition 10.1. Conversely if transversal intersection exists, then by Corollary 7.2, there exists α_1 such that $M(\alpha_1,\epsilon) = 0$ and $\frac{d}{d\alpha} M(\alpha_1,\epsilon) \neq 0$. Using (10.2), the conclusion follows from the implicit function theorem. □

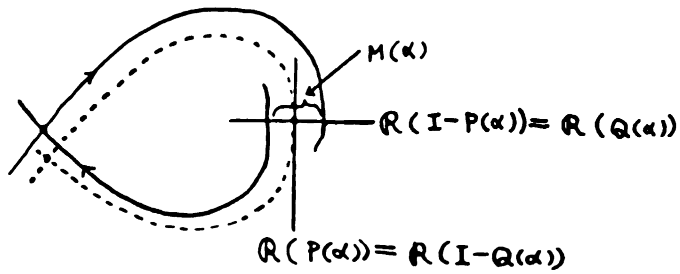


Figure 9

Case (ii). Suppose that

$$\text{(10.5)} \qquad W^u(x_-) \cap W^s(x_+) = \{\gamma(t,\nu):\ t\in\mathbb{R},\ \nu\in S \subset \mathbb{R}^{k-1}\}$$

where S is a open subset of $\mathbb{R}^{k-1}$ and $\gamma(t,\nu)$ is a homoclinic or heteroclinic orbit connecting x_- and x_+ for each $\nu\in S$.
In other words the 'homoclinic or heteroclinic manifold' $W^u(x_-) \cap W^s(x_+)$ is parametrized by $(t,\nu)\in \mathbb{R}\times S$. This case can occur when the system and its perturbation have some symmetric properties. See example 2 in §13.

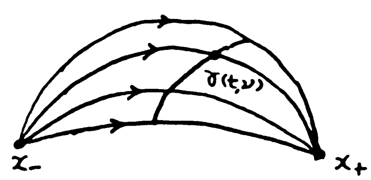


Figure 10

In this case the linear Melnikov vector has the form

(10.6) $$\hat{M}_i(\alpha,\nu) = \int_{-\infty}^{\infty} \phi_i^*(t)g(t-\alpha,\gamma(t,\nu))dt, \quad i=1,\ldots,m.$$

Note that

(10.7) $$M(\alpha,\nu,\epsilon) = \hat{M}(\alpha,\nu) + 0(\epsilon).$$

Hence we have

Proposition 10.3. Assume (10.5) and suppose that there exist α_0 and ν_0 such that

(10.8) $$\hat{M}(\alpha_0,\nu_0) = 0$$

and

(10.9) $$\text{rank } [\tfrac{d}{d\alpha} \hat{M}(\alpha_0,\nu_0) \ \ \tfrac{\partial}{\partial\nu} \hat{M}(\alpha_0,\nu_0)] = m.$$

Then $W^u_{loc}(x_-,\epsilon)$ and $W^s_{loc}(x_+,\epsilon)$ of system (2.2) have a point of transversal intersection.

Proof. By the implicit mapping theorem, we have $M(\alpha_1,\nu_1,\epsilon) = 0$ and rank $[\frac{\partial}{\partial\nu} M(\alpha_1,\nu_1,\epsilon) \ \ \frac{\partial}{\partial\nu} M(\alpha_1,\nu_1,\epsilon)] = m$ for (α_1,ν_1) near (α_0,ν_0). Then the statement follows from Theorem 7.1. □

Remark 10.4. In the case $m = 1$, the rank condition (10.9) gives a necessary and sufficient condition for transversal intersection.

Next we turn to the tangency condition. Here a tangential intersection of the stable and unstable manifolds means that the tangent spaces of the stable and unstable manifolds at a point of intersection do not span the whole space. Our discussion of tangency is based on Corollary 7.2. Since Corollary 7.2 gives a necessary and sufficient condition for transversality, we consider the situations in which the condition in Corollary 7.2 is violated.

We consider the following system with parameters.

$$(10.10) \qquad \dot{x} = f(x) + \epsilon g(t,x,\mu)$$

where $x \in \mathbb{R}^n$, $\mu \in \mathbb{R}^N$, $\epsilon << 1$, f and g are sufficiently smooth in all arguments, and g is periodic in t. Assume, as before, that the unperturbed system (ϵ=0) has a homoclinic or heteroclinic orbit $\gamma(t)$.

Recall, first of all, that

$m = k + \delta(\gamma)$.

Thus it is clear that if $\delta(\gamma) > 0$, then intersection is always tangential (see Remark 7.6).

Assume $\delta(\gamma) \le 0$ and assume that

$$(10.11) \qquad \mathcal{R}(P(\alpha)) = \mathcal{R}(I-Q(\alpha)) \ (=k).$$

We consider only several special cases here. Extension to more general cases is straightforward.

(i) Assume $m = 1$.

This case includes e.g. $k = 1$, $\delta(\gamma) = 0$ in $\mathbb{R}^2$ and $k = 2$, $\delta(\gamma) = -1$ in $\mathbb{R}^3$. We also assume that $N \geq k$.

Proposition 10.5. Suppose that

$$\hat{M}(\alpha_0,\mu_0) = \frac{\partial}{\partial\alpha}\hat{M}(\alpha_0,\mu_0) = 0, \tag{10.12}$$

$$\frac{\partial^2}{\partial\alpha^2}\hat{M}(\alpha_0,\mu_0) \neq 0 \tag{10.13}$$

and

$$\frac{\partial}{\partial\mu}\hat{M}(\alpha_0,\mu_0) \text{ has rank } k. \tag{10.14}$$

Then there exists a point of tangential intersection for sufficiently small ϵ.

Proof. Define

$$F(\alpha,\nu,\mu,\epsilon) = \left(M(\alpha,\nu,\mu,\epsilon), \frac{\partial}{\partial\alpha}M(\alpha,\nu,\mu,\epsilon), \frac{\partial}{\partial\nu}M(\alpha,\nu,\mu,\epsilon)\right).$$

Note that $F\colon \mathbb{R}^{2k+1} \to \mathbb{R}^{k+1}$ and $F(\alpha_0,0,\mu_0,0) = 0$.

Since conditions (10.12), (10.13) and (10.14) imply the matrix

$$D_{(\alpha,\mu)}F(\alpha_0,0,\mu_0,0) = \begin{bmatrix} \frac{\partial}{\partial\alpha}\hat{M}(\alpha_0,\mu_0) & \frac{\partial}{\partial\mu}\hat{M}(\alpha_0,\mu_0) \\ \frac{\partial^2}{\partial\alpha^2}\hat{M}(\alpha_0,\mu_0) & \frac{\partial^2}{\partial\alpha\partial\mu}\hat{M}(\alpha_0,\mu_0) \end{bmatrix}$$

has rank (k+1), by the implicit mapping theorem there exist functions $\alpha(\nu,\epsilon)$ and $\mu(\nu,\epsilon)$ such that

$$F(\alpha(\nu,\epsilon),\ \nu,\ \mu(\nu,\epsilon)) = 0$$

for sufficiently small ν and ϵ. Hence the condition in Corollary (7.2) is violated and the statement follows. □

See Wiggins and Holmes [22] for a similar result.

(ii) Assume that $m = 2$ and $k = 2$ (and hence $\delta(\gamma) = 0$). Assume also that $N \geq 3$.

Proposition 10.6. Suppose that

$$\hat{M}(\alpha_0,\mu_0) = \frac{\partial}{\partial\alpha}\hat{M}(\alpha_0,\mu_0) = 0 \tag{10.15}$$

and the matrix

$$\begin{bmatrix} \frac{\partial}{\partial\alpha}\hat{M}(\alpha_0,\mu_0) & \frac{\partial}{\partial\mu}\hat{M}(\alpha_0,\mu_0) \\ \frac{\partial^2}{\partial\alpha^2}\hat{M}(\alpha_0,\mu_0) & \frac{\partial^2}{\partial\alpha\partial\mu}\hat{M}(\alpha_0,\mu_0) \end{bmatrix} \tag{10.16}$$

has rank 4.

Then there exists a point of tangential intersection for sufficiently small ϵ.

Proof. Define

$$F(\alpha,\nu,\mu,\epsilon) = (M(\alpha,\nu,\mu,\epsilon),\ \frac{\partial}{\partial\alpha}M(\alpha,\nu,\mu,\epsilon)).$$

Then the proof is identical to the one in Proposition 10.5. □

Next we consider a more special case.

(ii)' Assume that $m = 2$, $k = 2$ and $N \geq 2$, and assume condition (10.5). Thus the Melnikov vector satisfies (10.7).

Proposition 10.7. Suppose that

$$\hat{M}(\alpha_0,\nu_0,\mu_0) = \frac{\partial}{\partial\alpha}\hat{M}(\alpha_0,\nu_0,\mu_0) = 0 \tag{10.17}$$

and the matrix

$$\begin{bmatrix} \frac{\partial}{\partial\alpha}\hat{M} & \frac{\partial}{\partial\nu}\hat{M} & \frac{\partial}{\partial\mu}\hat{M} \\ \\ \frac{\partial^2}{\partial\alpha^2}\hat{M} & \frac{\partial^2}{\partial\nu\partial\alpha}\hat{M} & \frac{\partial^2}{\partial\mu\partial\alpha}\hat{M} \end{bmatrix} \text{ or } \begin{bmatrix} \frac{\partial}{\partial\alpha}\hat{M} & \frac{\partial}{\partial\nu}\hat{M} & \frac{\partial}{\partial\mu}\hat{M} \\ \\ \frac{\partial^2}{\partial\alpha\partial\nu}\hat{M} & \frac{\partial^2}{\partial\nu^2}\hat{M} & \frac{\partial^2}{\partial\mu\partial\nu}\hat{M} \end{bmatrix} \tag{10.18}$$

has rank 4 at (α_0, ν_0, μ_0).

Then there exists a point of tangential intersection for sufficiently small ϵ.

Proof. Similar to Proposition 10.6. □

§11 HAMILTONIAN SYSTEMS

In this section we assume that the unperturbed system

$$(11.1) \qquad \dot{x} = X_H(x)$$

is completely integrable, and we consider its non–Hamiltonian and Hamiltonian perturbations

$$(11.2) \qquad \dot{x} = X_H(x) + \epsilon g(t,x),$$

and

$$(11.3) \qquad \dot{x} = X_H(x) + \epsilon X_G(t,x).$$

We shall derive the Melnikov vectors for system (11.2) and (11.3).

We first recall some basic facts from Hamiltonian systems. Let $H \epsilon C^\infty(\mathbb{R}^{2n})$. Then the Hamiltonian vector field X_H with Hamiltonian H on $\mathbb{R}^{2n}$ is defined by

$$(11.4) \qquad X_H(x) = J\nabla H(x)$$

where $J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ and $\nabla H(x) = \begin{bmatrix} \frac{\partial}{\partial x_1} H(x) \\ \vdots \\ \frac{\partial}{\partial x_{2n}} H(x) \end{bmatrix}.$

Let F_1, $F_2 \epsilon C^\infty(\mathbb{R}^{2n})$. The Poisson bracket $\{F_1,F_2\}$ of functions F_1 and F_2 is defined by

$$\{F_1,F_2\}(x) = dF_1(x)X_{F_2}(x), \quad x\epsilon\mathbb{R}^{2n} \tag{11.5}$$

where dF_1 is a differential 1–form on $\mathbb{R}^{2n}$.

One of the key facts on the Poisson bracket is the following:

$\{F_1,F_2\} = 0$ if and only if F_i is invariant under the flow of X_{F_j} where $(i,j) = (1,2)$ or (2.1).

We suppose that system (11.1) has two hyperbolic critical points x_- and x_+ (not necessarily distinct) joined by an orbit $\gamma(t)$ of system (11.1): $\lim_{t\to\pm\infty} \gamma(t) = x_\pm$. Then the linearized system of (11.1) along the orbit γ is given by

$$\dot{x} = A(t)z, \quad A(t) = DX_H(\gamma(t)). \tag{11.6}$$

Since $A(t) = JD^2H(\gamma(t))$, $A(t)$ is infinitesimally symplectic for each $t \epsilon \mathbb{R}$. Namely

$$A^*(t)J + JA(t) \equiv 0, \quad t\epsilon\mathbb{R} \tag{11.7}$$

where $A^*(t)$ is the transpose of $A(t)$. Now let us define the adjoint system of (11.6) by

$$\dot{\phi} = -\phi A(t) \tag{11.8}$$

where $\phi(t) \epsilon T^{*}_{\gamma(t)}\mathbb{R}^{2n} \simeq (\mathbb{R}^{2n})^{*}$.

We note that $z(t)$ is a solution of (11.6) if and only if $\phi(t) = (J^{-1}z(t))^{*}$ is a solution of (11.8). This is clear from (11.7). We have the following

Proposition 11.1. Let $F \epsilon C^{\infty}(\mathbb{R}^{2n})$. If $\{F,H\} = 0$, then $X_F(\gamma(t))$ is a solution of (11.6) and hence $dF(\gamma(t))$ is a solution of (11.8).

Proof. Let $< \cdot, \cdot >$ be the standard inner product on $\mathbb{R}^{2n}$. Then

$$\{F,H\} = < \nabla F, J\nabla H >.$$

Since

$$\begin{aligned} 0 &= \nabla < \nabla F, J\nabla H > \\ &= D^2F\ J\nabla H - D^2HJ\nabla F \\ &= D^2F\ X_H - D^2H\ X_F, \end{aligned}$$

we have

$$DX_FX_H = DX_HX_F.$$

Hence

$$\begin{aligned} \frac{d}{dt} X_F(\gamma(t)) &= DX_F(\gamma(t))X_H(\gamma(t)) \\ &= DX_H(\gamma(t))X_F(\gamma(t)) \\ &= A(t)X_F(\gamma(t)). \end{aligned}$$

Finally

$$(J^{-1}X_F(\gamma(t)))^* = dF(\gamma(t)). \qquad \square$$

One of the special situations of the Hamiltonian nature appears in the splitting index $\delta(\gamma)$ of γ. Since $DX_H(x_+) = \lim_{t\to\infty} A(t)$, $DX_H(x_+)$ is infinitesimally symplectic. Hence if $\lambda\epsilon\sigma(DX_H(x_+))$, then $\bar{\lambda}$, $-\lambda$, $-\bar{\lambda}\epsilon\sigma(DX_H(x_+))$, where $\sigma(DX_H(x_+))$ is the spectrum of $DX_H(x_+)$ and $\bar{\lambda}$ is the complex conjugate of λ. This symmetry property implies that both of the stable and the unstable subspaces of $DX_H(x_+)$ have dimension n. Similarly the stable and the unstable subspaces of $DX_H(x_-) = \lim_{t\to -\infty} A(t)$ have dimension n and hence we have

Proposition 11.2. Suppose that Hamiltonian system (11.1) has a homoclinic or heteroclinic orbit γ. Then $\delta(\gamma) = 0$. $\square$

Now we suppose that Hamiltonian system (11.1) is completely integrable. That is, there exist n C^∞ functions $F_1 = H, F_2,\ldots,F_n$ on $\mathbb{R}^{2n}$ which are in involution, namely $\{F_i,F_j\} = 0$ for $1 \le i,j \le n$, and dF_i, $i = 1,\ldots,n$, are linearly independent everywhere in $\mathbb{R}^{2n} - \{x_\pm\}$. We recall that $dH(x_\pm) = 0$ in our case.

Let $F_i(\gamma(t)) = f_i \ \epsilon\ \mathbb{R}$, $i=1,\ldots,n$, and define the set $M_f = \{x \ \epsilon\ \mathbb{R}^{2n} - \{x_\pm\}\colon\ F_i(x) = f_i,\ i=1,\ldots,n\}$. Then the Liuville integrability theorem (cf. Arnold [1]) asserts that M_f is a n–dimensional smooth manifold in $\mathbb{R}^{2n}$ which is invariant under the flow of each

Hamiltonian vector field X_{F_i}, i=1,...,n. Therefore the components of the stable and unstable manifolds both of which contain the orbit γ coincide each other and it is contained in M_f. Furthermore $X_{F_i}(\gamma(t))$, i=1,...,n, constitute a basis of $T_{\gamma(t)}M_f$.

Now since system (11.6) has exponential dichotomies on $[\alpha,\infty)$ and on $(-\infty,\alpha]$, $\alpha\in\mathbb{R}$, we denote projections at $t=\alpha$ by $P(\alpha)$ and $Q(\alpha)$ respectively. Then we have the following

Proposition 11.3. Suppose that Hamiltonian system (11.1) is completely integrable. Then

(i) $\mathcal{R}(P(\alpha)) = \mathcal{R}(I-Q(\alpha)) = T_{\gamma(\alpha)}M_f$,

and

(ii) $\{dF_i(\gamma(t));\ i=1,\dots,n\}$ forms a complete set of bounded solutions of the adjoint system (11.8).

Proof. These are clear from the above argument and proposition 11.1. □

Thus for completely integrable Hamiltonian systems in $\mathbb{R}^{2n}$, we have $k = m = n$ where $k = \dim\{\mathcal{R}(P(\alpha)) \cap \mathcal{R}(I-Q(\alpha))\}$ and $m = \dim\{\mathcal{R}(P(\alpha)) + \mathcal{R}(I-Q(\alpha))\}^{\perp}$.

Now we will give special forms for the Melnikov vector of system (11.2) and (11.3). Let $x(t) = \gamma(t+\alpha) + \epsilon z(t+\alpha)$. then system (11.2) and (11.3) become

(11.9) $$\dot{z} = A(t)z + g(t-\alpha,\gamma(t)) + h(t,z,\alpha,\epsilon)$$

and

(11.10) $$\dot{z} = A(t)z + X_G(t-\alpha,\gamma(t)) + X_{\tilde{G}}(t,z,\alpha,\epsilon)$$

respectively. Here

$$A(t) = JD^2H(\gamma(t)),$$
$$h(t,z,\alpha,\epsilon) = \frac{1}{\epsilon}\{f(\gamma(t)+\epsilon z) - f(\gamma(t)) - \epsilon Df(\gamma(t))z$$
$$+ \epsilon g(t-\alpha,\gamma(t) + \epsilon z) - \epsilon g(t-\alpha,\gamma(t))\},$$
$$\tilde{G}(t,z,\alpha,\epsilon) = \frac{1}{\epsilon}\{H(\gamma(t) + \epsilon z) - H(\gamma(t)) - \epsilon DH(\gamma(t))z$$
$$+ \epsilon G(t-\alpha,\gamma(t) + \epsilon z) - \epsilon G(t-\alpha,\gamma(t))\}.$$

Theorem 11.4. For system (11.2), the linear Melnikov vector $\hat{M}(\alpha)$ and the Melnikov vector $M(\alpha,\nu,\epsilon)$ are given by

(11.11) $$\hat{M}_i(\alpha) = \int_{-\infty}^{\infty} dF_i(\gamma(t))g(t-\alpha,\gamma(t))dt, \quad i=1,\dots,n$$

and

$$(11.12)\qquad M_i(\alpha,\nu,\epsilon) = \hat{M}_i(\alpha) + \int_{-\infty}^{\alpha} dF_i(\gamma(t))h(t,z^u(\nu)(t),\alpha,\epsilon)dt$$

$$+ \int_{\alpha}^{\infty} dF_i(\gamma(t))h(t,z^s(\nu)(t),\nu,\epsilon)dt, \quad i=1,\ldots,n.$$

where $z^u(\nu)(t)$ and $z^s(\nu)(t)$ are bounded solutions on $(-\infty,\alpha]$ and on $[\alpha,\infty)$ respectively of system (11.9).

Proof. These are simple consequences of Proposition 11.3 (ii) and Definition 6.1 of the Melnikov vector. □

Corollary 11.5. For system (11.3), the linear Melnikov vector $\hat{M}(\alpha)$ and the Melnikov vector $M(\alpha,\nu,\epsilon)$ are given by

$$(11.13)\qquad \hat{M}_i(\alpha) = \int_{-\infty}^{\infty} \{F_i,G(t-\alpha,\cdot)\}(\gamma(t))dt, \quad i=1,\ldots,n$$

and

$$(11.14)\qquad M_i(\alpha,\nu,\epsilon) = \hat{M}_i(\alpha) + \int_{-\infty}^{\alpha} dF_i(\gamma(t))X_{\tilde{G}}(t,z^u(\nu)(t),\alpha,\epsilon)dt$$

$$+ \int_{\alpha}^{\infty} dF_i(\gamma(t))X_{\tilde{G}}(t,z^s(\mu)(t),\alpha,\epsilon)dt, \quad i=1,\ldots,n$$

where $z^u(\nu)(t)$ and $z^s(\mu)(t)$ are bounded solutions on $(-\infty,\alpha]$ and on $[\alpha,\infty)$ respectively of system (11.10).

Proof. These are simple consequences of Proposition 11.4 and the definition of the Poisson bracket. □

The expression of the Melnikov vector in a more special case was given in Holmes and Marsden [10]. See also Wiggins [21].

Remark 11.6. The dimension of the Melnikov vector for a completely integrable Hamiltonian system is the same as the degree of freedom of that system.

Remark 11.7. Since the linear Melnikov vector $\hat{M}_i(\alpha)$ can be written as

$$(11.15) \qquad \hat{M}_i(\alpha) = \int_{-\infty}^{\infty} \{F_i, G(\tau,\cdot)\}(\gamma(t+\alpha))d\tau,$$

we have

$$(11.16) \qquad \frac{d}{d\alpha}\hat{M}_i(\alpha) = \int_{-\infty}^{\infty} \frac{\partial}{\partial\alpha}\{F_i, G(\tau,\cdot)\}(\gamma(t+\alpha))d\tau$$

$$= \int_{-\infty}^{\infty} \{H,\{F_i, G(\tau,\cdot)\}(\tau(t+\alpha))d\tau$$

$$= \int_{-\infty}^{\infty} \{H,\{F_i, G(t-\alpha,\cdot)\}(\gamma(t))dt.$$

Here we used the following fact: for $f \in C^\infty(\mathbb{R}^{2n})$,

$$(11.17) \qquad \frac{d}{dt}(\overleftarrow{F}_t f) = \{H, \overleftarrow{F}_t f\}$$

where F_t is the flow of X_H and $\overleftarrow{F}_t f$ is the pullback of f by F_t. More generally we have

$$(11.18) \qquad \frac{d^k}{d\alpha^k} \hat{M}_i(\alpha) = \int_{-\infty}^{\infty} \underbrace{\{H,\{H,\ldots,\{H,}_{k\text{-times}}\{F_i, G(t-\alpha,\cdot)\}\}(\gamma(t))dt.$$

§12 A HETEROCLINIC ORBIT TO INVARIANT TORI

In this section we extend our theory developed in previous sections to the case of a heteroclinic orbit to invariant tori. The system we consider is

$$(12.1) \qquad \dot{x} = f(x),\ x \in \mathbb{R}^n$$

and its perturbed system

$$(12.2) \qquad \dot{x} = f(x) + \epsilon g(x), \quad |\epsilon| << 1$$

where f and g are sufficiently smooth and are bounded on bounded sets. We assume that system (12.1) has two normally hyperbolic smooth invariant tori M^1 and M^2 on which the flow of f is quasi-periodic, and also that system (12.1) has a heteroclinic orbit γ to M^1 and M^2. That is, there exist orbits $\{x^i(t)\colon t \in \mathbb{R}\} \subset M^i$, $i=1,2$, such that

$$|\gamma(t) - x^1(t)| \to 0 \quad \text{as} \quad t \to -\infty$$

and

$$|\tau(t) - x^2(t)| \to 0 \quad \text{as} \quad t \to +\infty.$$

If $M^1 = M^2$, we have a homoclinic orbit to invariant torus M^1 as a special case. We remark that M^1 and M^2 could be any normally hyperbolic smooth invariant sets with no stationary points. However we assume the above conditions for its simplicity and applications. We also remark that the normal hyperbolicity is nothing but the hyperbolicity to the normal direction.

To develop an anologous theory to the one in previous sections, we need the expressions of the stable and the unstable manifolds. For this reason we shall decompose system (12.2), in neighborhoods of invariant tori M^i, into the tangential and the normal components and apply the theory of exponential dichotomy to the normal components.

Let $\dim M^i = d_i$ $(i=1,2)$ and we assume for $i=1,2$ that M^i is given by the embedding

$$(12.3) \qquad u^i:\ T^i \to M^i \subset \mathbb{R}^n$$

where $T^i = S^1 \times \ldots \times S^1$ (d_i–times) is a standard d_i–dimensional torus with coordinates $\theta^i = (\theta^i_1, \ldots, \theta^i_{d_i})$. From now on we supress i and assume that M stands for M^1. The case of M^2 is done exactly in the same fashion.

Let $T\mathbb{R}^n|M$ be the ristriction of the tangent bundle $T\mathbb{R}^n$ to M. Then

$$(12.4) \qquad (T\mathbb{R}^n|M)_x = T_xM \oplus T_x^{\perp}M, \ x \in M$$

where T_xM and $T_x^{\perp}M$ are respectively the tangent space and the normal space to M at x.

Now by the tubular neighbourhood theorem, there exist a neighborhood U of T in $T \times \mathbb{R}^{n-d}$, a neighbourhood V of M in $T^{\perp}M$ and a linear map $N(\theta)$: $\mathbb{R}^{n-d} \to \mathbb{R}^n$ for each $\theta \in T$ such that the vector bundle map F: $U \subset T \times \mathbb{R}^{n-d} \to V \subset T^{\perp}M$ defined by

$$(12.5) \qquad F(\theta,z) = u(\theta) + \epsilon N(\theta) z$$

is a diffeomorphism. Here N depends on θ smoothly. Clearly we have

$$(12.6) \qquad (Du(\theta))^{*} N(\theta) = 0$$

and

$$(12.7) \qquad N^{*}(\theta) N(\theta) = I^{n-d}$$

where $(Du(\theta))^{*}$ and $N^{*}(\theta)$ are transposes of $Du(\theta)$ and $N(\theta)$ respectively. By using F, we transform the vector field $f + \epsilon g$ on $\mathbb{R}^n$ to the vector field $\overleftarrow{F}(f+\epsilon g)$ on $T \times \mathbb{R}^{n-d}$ where $\overleftarrow{F}$ is the pullback by F. We set

$$(12.8) \qquad \overleftarrow{F}(f+\epsilon g) = \sum_{j=1}^{d} Aj \frac{\partial}{\partial \theta_j} + \sum_{\ell=1}^{n-d} B_{\ell} \frac{\partial}{\partial z_{\ell}} .$$

That is,

$$(12.9)\qquad f(u(\theta) + \epsilon N(\theta)z) + \epsilon g(u(\theta) + \epsilon N(\theta)z)$$
$$= [Du(\theta) + \epsilon \frac{\partial}{\partial\theta} N(\theta)z]\begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix} + \epsilon N(\theta)\begin{bmatrix} B_1 \\ \vdots \\ B_{n-d} \end{bmatrix}.$$

By using (12.6) and (12.7) we have

$$(12.10)\qquad \begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix} = (Du(\theta))^* f(u(\theta)) + \epsilon(Du(\theta))^* [-\frac{\partial}{\partial\theta}(N(\theta)z)Du(\theta)f(u(\theta))$$
$$-Df(u(\theta))N(\theta)z - g(u(\theta))] + 0(\epsilon)$$
$$\equiv \omega + \epsilon\Theta(\theta,z) + 0(\epsilon^2).$$

Here we assumed that that $(Du(\theta))^* f(u(\theta)) = \omega$, $\omega = (\omega_1,\dots,\omega_d)$ are rationally independent. Under this assumption we also have

$$(12.11)\qquad \begin{bmatrix} B_1 \\ \vdots \\ B_{n-d} \end{bmatrix} = N^*(\theta)[Df(u(\theta))N(\theta)z+g(u(\theta)) + 0(\epsilon) - \frac{\partial}{\partial\theta} N(\theta)z\begin{bmatrix} A_1 \\ \vdots \\ A_d \end{bmatrix}]$$
$$= N^*(\theta)Df(u(\theta))N(\theta)z - N^*(\theta)(\frac{\partial}{\partial\theta} N(\theta)z)\omega$$
$$+ N^*(\theta)g(u(\theta)) + 0(\epsilon)$$
$$\equiv A(\theta)z + N^*(\theta)g(u(\theta)) + 0(\epsilon).$$

Note that $A(\theta)$ is a linear mapping for each θ.

To obtain the differential equations in $T\times\mathbb{R}^{n-d}$, we need a scale change in time. This is because ω will be changed after perturbation. Thus we set

$$(12.12)\qquad x((1+\epsilon\beta)t+\epsilon a) = F(\theta(t),z(t)),$$

where $\beta\epsilon\,\mathbb{R}$ and $a\epsilon\,\mathbb{R}$. By using (12.12) we have the following system in $T\times\mathbb{R}^{n-d}$.

$$(12.13)\qquad \begin{aligned} \dot\theta &= (1+\epsilon\beta)[\omega+\epsilon\Theta(\theta,z) + 0(\epsilon^2)] = \omega + \epsilon(\beta\omega+\Theta(\theta,z)) + 0(\epsilon^2) \\ \dot z &= (1+\epsilon\beta)[A(\theta)z+N^*(\theta)g(u(\theta)) +0(\epsilon)] \\ &= A(\theta)z+N^*(\theta)g(u(\theta)) + 0(\epsilon). \end{aligned}$$

Now we choose $t^1 <<-1$ so that $\gamma(t^1)$ is enoughly close to M. Then from (12.5), there exist unique $\alpha^1 \equiv \alpha^1(t^1)\epsilon T$ and $w^1 = w^1(t^1)\epsilon\,\mathbb{R}^{n-d}$ such that

$$(12.14)\qquad \gamma(t^1) = u(\alpha^1) + \epsilon N(\alpha^1)w^1.$$

Hereafter we use α and w instead of α^1 and w^1.

Consider system (12.13) and let $z = \bar z(\theta,t)$ where $\bar z$ is a bounded function for $t\epsilon(-\infty,t^1]$ which will be determined later. Then the 'θ-equation' in (12.13) becomes

$$(12.15)\qquad \dot\theta = \omega + \epsilon(\beta\omega+\Theta(\theta,\bar z(\theta,t))) + 0(\epsilon^2).$$

Let $\bar\theta(t) \equiv \bar\theta(t;t^1,\alpha;\bar z)$ be the solution of (12.15) with $\bar\theta(t^1) = \alpha$. By using $\bar\theta(t)$ for θ, the 'Z-equation' in (12.13) becomes

$$(12.16) \qquad \dot{z} = A(\bar{\theta}(t))z + N^*(\bar{\theta}(t)g(u(\bar{\theta}(t))) + 0(\epsilon)$$

with $z(t^1) = \bar{z}(\alpha,t^1)$. Since M is normally hyperbolic, the linear system $\dot{z} = A(\bar{\theta}(t))z$ has an exponential dichotomy on $(-\infty,t^1]$. This means that there exist a projection $Q\colon \mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$, constants $K \geq 1$ and $L > 0$ such that

$$(12.17) \qquad |\Phi(t,t^1)Q\Phi(t^1,s)| \leq Ke^{-L(t-s)}, \; s \leq t \leq t^1,$$

and

$$(12.18) \qquad |\Phi(t,t^1)(I-Q)\Phi(t^1,s)| \leq Ke^{-L(s-t)}, \; t \leq s \leq t^1,$$

where $\Phi(t,s)$ is the transition matrix of $\dot{z} = A(\bar{\theta}(t))z$. Thus $z(t;t^1,\bar{z}(\alpha,t^1))$ is a bounded solution of (12.16) on $(-\infty,t^1]$ if and only if

$$\begin{aligned}(12.19) \qquad z(t;t^1,\bar{z}(\alpha,t^1)) &= \Phi(t,t^1)(I-Q)\bar{z}(\alpha,t^1) \\ &+\Phi(t,t^1)(I-Q) \int_{t^1}^{t} \Phi(t^1,s)\{N^*(\bar{\theta}(t))g(u(\theta(\bar{\theta}(t))) \\ &+ 0(\epsilon)\}ds + \Phi(t,t^1)Q \int_{-\infty}^{t} \Phi(t^1,s)\{N^*(\bar{\theta}(t))g(u(\bar{\theta}(t))) \\ &+ 0(\epsilon)\}ds.\end{aligned}$$

By letting $t = t^1$, we have

$$\bar{z}(\alpha,t^1) = \eta^u + Q \int_{-\infty}^{t^1} \Phi(t^1,s)\{N^*(\bar{\theta}(s;t^1,\alpha;\bar{z}))g(u(\bar{\theta}(s;t^1,\alpha;\bar{z}))) + 0(\epsilon)\}ds \tag{12.20}$$

where $\eta^u \in \mathcal{R}(I-Q)$. We can show, by the contraction mapping principle, that equation (12.20) has a unique solution $\bar{z}(\eta_u)(\alpha,t^1)$ for each η_u which is enoughly small. Thus (12.20) gives a local expression of the unstable manifold of system (12.13) near $\alpha\epsilon T$ in the space $T\times\mathbf{R}^{n-d}$. (12.20) has the folliwing expression in the 'original space' $\mathbf{R}^n$. (We use $i = 1$ this time).

$$\begin{aligned} \xi^u &\equiv u^1(\alpha^1) + \epsilon N^1(\alpha^1)\bar{z}^1(\alpha^1,t^1) \\ &= u^1(\alpha^1) \\ &+ \epsilon[\bar{\eta}^u + N^1(\alpha^1)Q \int_{-\infty}^{t^1} \Phi(t^1,s)\{N^{1*}(\bar{\theta}^1(s;t^1,\alpha^1;\bar{z}^1))\, g(u^1(\bar{\theta}^1(s;t^1,\alpha^1;\bar{z}^1))) \\ &+ 0(\epsilon)\}ds] \end{aligned} \tag{12.21}$$

where $\bar{\eta}^u = N^1(\alpha^1)\eta^u \in \mathcal{R}(N^1(\alpha^1)(I-Q))$.

Next we choose $t^2 >> 1$ so that $\gamma(t^2)$ is enoughly close to M^2. Then there exist unique $\alpha^2 = \alpha^2(t^2)\epsilon T^2$ and $w^2 = w^2(t^2) \epsilon \mathbf{R}^{n-d_2}$ such that

$$\gamma(t^2) = u^2(\alpha^2) + \epsilon N^2(\alpha^2)w^2. \tag{12.22}$$

In exactly the same fashion as before we have the following local expression of the stable manifold of system (12.13) near $u^2(\alpha^2) \epsilon M^2$ in the 'original space' $\mathbf{R}^n$.

$$(12.23)\qquad \xi^s \equiv u^2(\alpha^2) + \epsilon N^2(\alpha^2)\bar{z}^2(\alpha^2,t^2)$$
$$= u^2(\alpha^2)$$
$$+ \epsilon[\bar{\eta}^s + N^2(\alpha^2)(I-P)\int_\infty^{t^2} \Phi^2(t^2,s)\,\{N^{2*}(\bar{\theta}^2(s;t^2,\alpha^2;\bar{z}^2))g(U^2(\bar{\theta}^2(s;t^2,\alpha^2,\bar{z}^2)))$$
$$+ 0(\epsilon)\}ds]$$

where $\bar{\eta}^s \in (N^2(\alpha^2)P)$ and P is a projection of the exponential dichotomy on $[t^2,+\infty)$.

To measure the distance between the sections of the stable and unstable manifolds given in (12.21) and (12.23), we shall 'carry' the section of the unstable manifold given in (12.21) to the hyperplane $\gamma(t^2) + \mathcal{R}(N^2(\alpha^2))$ by the flow of (12.1). Hence we shall need the following expressions of the unstable and stable manifolds. From (12.14) and (12.22), we have

$$(12.24)\qquad \xi^u = \gamma(t^1) + \epsilon N^1(\alpha^1)[-w^1+\eta^u$$
$$+ Q\int_\infty^{t^1} \Phi^1(t^1,s)\{N^{1*}(\bar{\theta}^1(s;t^1,\alpha^1;\bar{z}^1))g(u^1(\bar{\theta}^1(s;t^1,\alpha^1;\bar{z}^1)))$$
$$+ 0(\epsilon)\}ds]$$
$$\equiv \gamma(t^1) + \epsilon M^u(\eta^u)$$

and

$$(12.25)\qquad \xi^s = \gamma(t^2) + \epsilon N^2(\alpha^2)[-w^2+\eta^s$$
$$+(I-P)\int_{\infty}^{t^2} \Phi^2(t^2,s)\{N^{2*}(\bar{\theta}^2(s;t^2,\alpha^2,\bar{z}^2))g(u^2(\bar{\theta}^2(s;t^2,\alpha^2;\bar{z}^2)))$$
$$+ 0(\epsilon)\}ds]$$
$$\equiv \gamma(t^2) + \epsilon M^s(\eta^s).$$

Now we consider system (12.2) along the heteroclinic orbit γ. Let $\beta(t)$ and $a(t)$ be bump functions such that

$$(12.26)\qquad \beta(t) \equiv \begin{cases} \beta^1 & \text{for } t \le t^1 \\ \beta^2 & \text{for } t \ge t^2 \end{cases} \quad \text{and} \quad a(t) \equiv \begin{cases} a^1 & \text{for } t \le t^1 \\ a^2 & \text{for } t \ge t^2 \end{cases},$$

and let

$$(12.27)\qquad x((1+\epsilon\beta(t))t + \epsilon a(t)) = \gamma(t) + \epsilon y(t).$$

Then system (12.2) becomes

$$(12.28)\qquad \dot{y} = \beta(t)y + g(\gamma(t)) + \{\beta(t) + \dot{a}(t) + \dot{\beta}(t)t\}f(\gamma(t)) + 0(\epsilon)$$

where $B(t) = Df(\gamma(t))$. Now let $X(t,s)$ be the transition matrix of $\dot{y} = B(t)y$ and let $y(\eta^u)(t)$ be the solution of (12.28) with the initial condition $y(\eta^u)(t^1) = M^u(\eta^u)$. Also define $\tau^2(\eta^u)$ so that

$$(12.29)\qquad y(\eta^u)(\tau^2(\eta^u)) \in \gamma(t^2) + \mathcal{R}(N^2(\alpha^2)).$$

We note that $\tau^2(\eta^u) = t^2 + 0(\epsilon)$ because of the continuous dependence of solutions on initial conditions. Thus

$$(12.30) \quad \begin{aligned} y(\eta^u)(\tau^2(\eta^u)) &= X(\tau^2(\eta^u),t^1)M^u(\eta^u) \\ &\quad + \int_{t^1}^{\tau^2(\eta^u)} X(\tau^2(\eta^u),s)\{g(\gamma(s)) + (\beta(s) + \dot{a}(s) \\ &\qquad + \dot{\beta}(s)s)f(\gamma(s)) + 0(\epsilon)\}ds \\ &= X(t^2,t^1)M^u(\eta^u) + \int_{t^1}^{t^2} X(t^2,s)\{g(\gamma(s)) \\ &\qquad + (\beta(s) + \dot{a}(s) + \dot{\beta}(s)s)f(\gamma(s))\}ds + 0(\epsilon). \end{aligned}$$

To measure the distance between $y(\eta^u)(\tau^2(\eta^u))$ and $M^s(\eta^s)$, we use linearly independent bounded solutions on $\mathbb{R}$ of the adjoint system

$$(12.31) \quad \dot{\phi} + B^*(t)\phi = 0.$$

Let $\{\phi_1,\ldots,\phi_m\}$ be a complete set of linearly independent bounded solutions on $\mathbb{R}$ of system (12.31) which satisfy

$$(12.32) \quad \phi_i(t^2)\epsilon\{\mathcal{R}\ (P) \cap \mathcal{R}(N^2(\alpha^2))\}^\perp, \quad i=1,\ldots,m.$$

Define

$$(12.33) \quad M_i = \phi_i^*(t^2)\ [y(\eta^u)(\tau^2(\eta^u)) + \tfrac{1}{\epsilon}(\gamma(\tau^2(\eta^u)) - \gamma(t^2)) - M^s(\eta^s)], \quad i=1,\ldots,m.$$

By using (12.24), (12.25) and (12.30), M_i is computed as follows.

(12.34)
$$\phi_i^*(t^2)\,[y(\eta^u)(\tau^2(\eta^u)) + \tfrac{1}{\epsilon}(\gamma(\tau^2(\eta^u)) - \gamma(t^2))]$$
$$= \phi_i^*(t^2)\,[X(t^2,t^1)M^u(\eta^u) + \int_{t^1}^{t^2} X(t^2,s)\{g(\gamma(s))$$
$$+ \beta(s) + \dot{a}(s) + \dot{\beta}(s)s)f(\gamma(s))\}ds + \tfrac{1}{\epsilon}(\gamma(\tau^2(\eta^u))$$
$$- \gamma(t^2)) + 0(\epsilon)]$$
$$=\phi_i^*(t^1)M^u(\eta^u) + \int_{t^1}^{t^2} \phi_i^*(s)\{g(\gamma(s)) +(\beta(s) + \dot{a}(s)$$
$$+ \dot{\beta}(s)s)f(\gamma(s))\}ds + 0(\epsilon)$$
$$= \phi_i^*(t^1)M^u(\eta^u) + \int_{t^1}^{t^2} \phi_i^*(s)g(\gamma(s))ds + 0(\epsilon)$$
$$= \phi_i^*(t^1)N^1(\alpha^1)[-w^1 + \eta^u$$
$$+ Q\int_{-\infty}^{t^1} \Phi^1(t^1,s)\{N^{1*}(\overline{\theta}^1(s))g(u^1(\overline{\theta}^1(s))$$
$$+ 0(\epsilon)\}ds] + \int_{t^1}^{t^2} \phi_i^*(s)g(\gamma(s))ds + 0(\epsilon)$$
$$= \int_{\infty}^{t^1} \Psi_i^{1*}(s)N^{1*}(\overline{\theta}^1(s))g(u^1(\overline{\theta}^1(s)))ds + \int_{t^1}^{t^2} \phi_i^*(s)g(\gamma(s))ds$$
$$+ 0(\epsilon)$$
$$= \int_{-\infty}^{t^1} \phi_i^*(s)g(u^1(\overline{\theta}^1(s)))ds + \int_{t^1}^{t^2} \phi_i^*(s)g(\gamma(s))ds + 0(\epsilon)$$
$$= \int_{-\infty}^{t^1} \phi_i^*(s)g(\gamma(s))ds + \int_{t^1}^{t^2} \phi_i^*(s)g(\gamma(s)))ds + 0(\epsilon)$$
$$= \int_{-\infty}^{t^2} \phi_i^*(s)g(\gamma(s))ds + 0(\epsilon).$$

Here we let $\phi_i(t) = \Psi_i^1(t)N^{1^*}(\overline{\theta}^1(t)), \quad t \le t^1.$

$$(12.35) \qquad \phi_i^*(t^2)\ M^s(\eta^s)$$

$$= \phi_i^*(t^2)N^2(\alpha^2)[-w^2 + \eta_s + (I-P)\int_{\infty}^{t^2} \Phi^2(t^2,s)\{N^{2^*}(\overline{\theta}^2(s))g(u^2(\overline{\theta}^2(s))) + 0(\epsilon)\}\, ds$$

$$= \int_{\infty}^{t^2} \Psi_i^{2^*}(s)N^{2^*}(\overline{\theta}^2(s))g(u^2(\overline{\theta}^2(s)))ds + 0(\epsilon)$$

$$= \int_{\infty}^{t^2} \phi_i^*(s)g(u^2(\overline{\theta}^2(s)))ds + 0(\epsilon)$$

$$= \int_{\infty}^{t^2} \phi_i^*(s)g(\gamma(s))ds + 0(\epsilon).$$

Here we let $\phi_i(t) = \Psi_i^2(t)N^{2^*}(\overline{\theta}^2(t)), \quad t \ge t^2.$

Thus we have

$$(12.36) \qquad M_i = \int_{-\infty}^{\infty} \phi_i^*(s)g(\gamma(s))ds + 0(\epsilon)$$

and we have proved the following

Theorem 12.1. The linear Melnikov vector $\hat{M} = (\hat{M}_1,\ldots,\hat{M}_m)$ for system (12.2) is given by

$$(12.37) \qquad \hat{M}_i = \int_{-\infty}^{\infty} \phi_i^*(t)g(\gamma(t))dt, \qquad i=1,\ldots,m,$$

where m is the number of linearly independent bounded solutions of $\dot{\phi} + [Df(\gamma(t))]^* \phi = 0$. □

Remark 12.2. Since the heteroclinic orbit γ is contained both in the unstable manifold $W^u(M^1)$ of M^1 and the stable manifold $W^s(M^2)$ of M^2, it is interesting to consider how these manifolds intersect each other along the heteroclinic orbit γ. Consider the case of a time–independent perturbation and define

$$(12.38) \qquad k = \dim[T_{\gamma(t)} W^u(M^1) \cap T_{\gamma(t)} W^s(M^2)]$$

where $T_{\gamma(t)} W^w(M^1)$ and $T_{\gamma(t)} W^s(M^2)$ are tangent spaces at $\gamma(t)$ to $W^u(M^1)$ and $W^s(M^2)$ respectively. Then k and the dimension m of the Melnikov vector have the following relation.

$$(12.39) \qquad m = n - [\dim W^u(M^1) + \dim W^s(M^2) - k],$$

where

$$(12.40) \qquad \dim W^u(M^1) = \dim \mathcal{R}(I-Q_1) + d_1$$

and

$$(12.41) \qquad \dim W^s(M^2) = \dim \mathcal{R}(P_2) + d_2.$$

Note that $\dim W^u(M^1) = n - [\dim W^s(M^1) - d_1]$.

If we define the splitting index $\delta(\gamma)$ of γ by

$$(12.42) \qquad \delta(\gamma) = \dim W^s(M^1) - \dim W^s(M^2),$$

we have, from (12.40), the following relation

$$(12.43) \qquad m = k + \delta(\gamma) + d_1$$

which is a generalization of (7.12).

Now we go into a special case to which Theorem 12.1 can easily be applied. consider a system with a quasi periodic perturbation

$$(12.44) \qquad \dot{z} = f(z) + \epsilon g(z,\omega_1 t,\ldots,\omega_d t), \quad z\epsilon\mathbb{R}^n$$

where g is periodic in each 't' argument and $\omega_1,\ldots,\omega_d$ are rationally independent, see Meyer and Sell [13]. We assume that the unperturbed system $\dot{z} = f(z)$ has a homoclinic or heteroclinic orbit γ to hyperbolic critical point(s). System (12.44) is equivallent to the following system on the torus T^d.

$$(12.45) \qquad \begin{aligned} \dot{z} &= f(z) + \epsilon g(z,\theta) \\ \dot{\theta} &= \omega \end{aligned}$$

where $\theta = (\theta^1,...,\theta^d)$, $\omega = (\omega_1,...,\omega_d)$ and $(z,\theta)\epsilon\ \mathbb{R}^n\times T^d$. This is a special case of system (12.2) in the sense that the 'z–dynamics' of the unperturbed system of (12.45) is globally defined in the normal bundle of T^d. By using the homoclinic orbit γ, the homoclinic orbit $\bar{\gamma}$ of system (12.45) to the torus T^d is given by

$$(12.46) \qquad \bar{\gamma}(t) = (\gamma(t),\ \omega_1 t + \theta_1,...,\omega_d t + \theta_d)$$

where $\theta_i \epsilon [0,2\pi)$, i=1,...,d. This is because the 'z–dynamics' and 'θ–dynamics' of the unperturbed system of (12.45) are completely decoupled. By Theorem 12.1, we have the following corollary in this case.

<u>Corollary 12.2</u>. The linear Melnikov vector $\hat{M}(\theta_1,...,\theta_d) = (\hat{M}_1(\theta_1,...,\theta_d),...,\hat{M}_m(\theta_1,...,\theta_d))$ for system (12.44) is given

$$(12.47) \qquad \hat{M}_i(\theta_1,...,\theta_d) = \int_{-\infty}^{\infty} \phi_i^*(t) g(\gamma(t),\omega_1 t + \theta_1,...,\omega_d t + \theta_d)dt,$$

i=1,...,m. Here $\{\phi_1,...,\phi_m\}$ is a set of linearly independent bounded solutions of $\dot{\phi} + [Df(\gamma(t))]^* \phi = 0$. □

As a special case, we shall prove the following proposition for two–dimensional systems. See also Meyer and Sell [13] and Wiggins [21].

<u>Proposition 12.3</u>. Consider system (12.44) with the same assumption as before and let $n = 2$ and $d \geq 2$. Then the stable and

unstable manifolds of system (12.44) intersect transversally if and only if for the linear Melnikov function $\hat{M}(\theta_1,\dots,\theta_d)$ defined in (12.47) (i = 1 in this case), there exist $(\bar{\theta}_1,\dots,\bar{\theta}_d)$ such that

$$\hat{M}(\bar{\theta}_1,\dots,\bar{\theta}_d) = 0 \tag{12.48}$$

and

$$(\mathcal{L}_\omega \hat{M})(\bar{\theta}_1,\dots,\bar{\theta}_d) \neq 0 \tag{12.49}$$

where $\omega = (\omega_1,\dots,\omega_d)$ and $\mathcal{L}_\omega \hat{M}$ is the Lie derivative of $\hat{M}$ with respect to ω.

Proof. Let $\theta_i(\alpha) = \bar{\theta}_i - \omega_i\alpha$, i = 1,...,d, $\alpha \in \mathbb{R}$ and define

$$D(\alpha) = \hat{M}(\theta_1(\alpha),\dots,\theta_d(\alpha)).$$

Then we have

$$\begin{aligned} D(\alpha) &= \int_{-\infty}^{\infty} \phi^*(t)g(\gamma(t),\omega_1(t-\alpha) + \bar{\theta}_1,\dots,\omega_d(t-\alpha) + \bar{\theta}_d)dt \\ &= \int_{-\infty}^{\infty} \phi^*(t+\alpha)g(\gamma(t+\alpha),\omega_1 t + \bar{\theta}_1,\dots,\omega_d t + \bar{\theta}_d)dt \end{aligned} \tag{12.50}$$

and

$$(12.51) \qquad D(0) = \hat{M}(\bar{\theta}_1,\ldots,\bar{\theta}_d) = 0.$$

From (12.50) α works as a 'sweeping' parameter along γ. See Figure 11. Since $D(\alpha)$ changes along $\gamma(\alpha)$, $\alpha \in \mathbb{R}$ and since the difference of the true distance of the stable and unstable manifolds and D(s) is of order ϵ, the implicit mapping theorem implies, using (12.51), that the transversal intersection exists if and only if $D(0) = 0$ and $D'(0) \neq 0$. Finally

$$(12.52) \qquad \begin{aligned} D'(0) &= \sum_{i=1}^{d} \frac{\partial \hat{M}}{\partial \theta_i} (\bar{\theta}_1,\ldots,\bar{\theta}_d) \frac{d\theta_i}{d\alpha}(0) \\ &= -\sum_{i=1}^{d} \omega_i \frac{\partial \hat{M}}{\partial \theta_i} (\bar{\theta}_1,\ldots,\bar{\theta}_d) = -(\mathcal{L}_\omega \hat{M})(\bar{\theta}_1,\ldots,\bar{\theta}_d). \end{aligned} \qquad \square$$

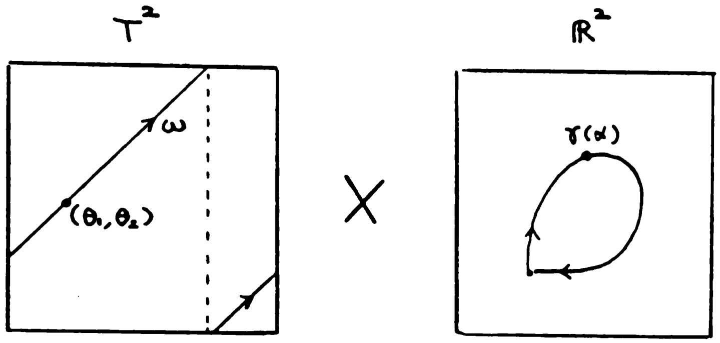


Figure 11

Remark 12.4. We note that the case of two-dimensional systems with periodic perturbations in §10 is a special case of this proposition. That is, $\frac{d}{d\alpha} \hat{M}(\alpha_0)$ in Corollary 10.2 is generalized to $(\mathcal{L}_\omega \hat{M})(\bar{\theta}_1,\ldots,\bar{\theta}_d)$.

§13. THREE EXAMPLES

In this section we apply the methods developed in previous sections to three examples. We shall exame (1) a two–dimensional system which has transversal intersections, (2) a four–dimensional system which has both of transversal and tangential intersections and (3) a system for which condition (ii) in Theorem 7.1 is not satisfied but transversal intersection exists.

Example 1 (Chow, Hale and Mallet–Paret [4])

We consider the following second order equation

$$(13.1) \qquad \ddot{x} - x + \frac{3}{2} x^2 = \epsilon \text{ cost},$$

where ϵ is sufficiently small. That is,

$$(13.2) \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = f(x,y) + \epsilon g(t)$$

$$\equiv \begin{bmatrix} y \\ x - \frac{3}{2}x^2 \end{bmatrix} + \epsilon \begin{bmatrix} 0 \\ \text{cost} \end{bmatrix}.$$

We notice that the unperturbed system (i.e. ϵ=0) has a homoclinic orbit,

$$(13.3) \qquad \gamma(t) = \begin{bmatrix} P(t) \\ \dot{p}(t) \end{bmatrix} \equiv \begin{bmatrix} \text{sech}^2(t/2) \\ -4\text{sech}^2(t/2)\tanh(t/2) \end{bmatrix}$$

to the origin. See the figure below.

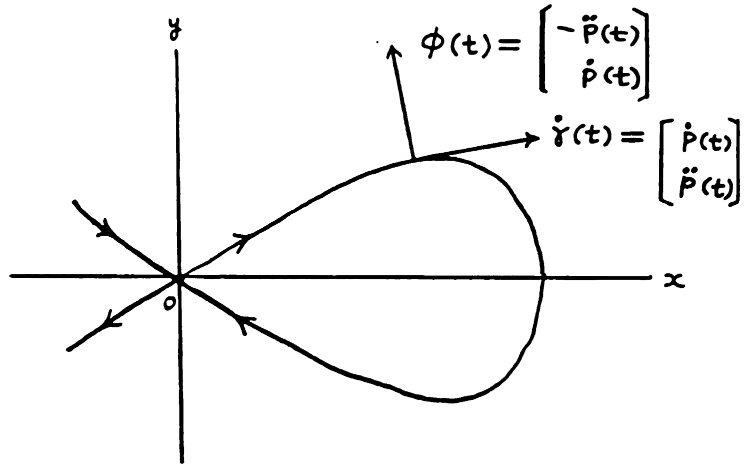


Figure 12

The linearized system along γ is

(13.4) $\dot{z} = A(t)z$

where

(13.5) $A(t) = Df(\gamma(t)) = \begin{bmatrix} 0 & 1 \\ 1-3p(t) & 0 \end{bmatrix}.$

The adjoint system $\dot{\phi} + A^*(t)\phi = 0$ has only one linearly independent bounded solution

(13.6) $\phi(t) = \begin{bmatrix} -\ddot{p}(t) \\ \dot{p}(t) \end{bmatrix}$

and hence the (linear) Melnikov function is

$$\text{(13.7)} \qquad \hat{M}(\alpha) = \int_{-\infty}^{\infty} \phi^{*}(t)g(t-\alpha)dt$$
$$= \int_{-\infty}^{\infty} \dot{p}(t)\cos(t-\alpha)dt$$
$$= -\,c \sin\alpha$$

where $c > 0$ is a constant.

Since

$$\text{(13.8)} \qquad \frac{d}{d\alpha}\hat{M}(n\pi) = (-1)^{n+1}c, \quad n = 0, \pm 1, \pm 2, \ldots,$$

the perturbed stable and unstable manifolds always intersect transversally and so tangential intersections never occur.

Example 2 (Gruendler [6])

In this example we consider the following system of two second order equations.

$$\text{(13.9)} \qquad \ddot{x}_1 = x_1 - 2x_1(x_1^2+x_2^2) + \epsilon\{-3\mu_1 x_1 - \mu_2 \dot{x}_1 + \frac{2\mu_3}{1+\omega^2}(3x_1^2+x_2^2)\cos\omega t + \frac{4\mu_4}{1+\omega^2} x_1 x_2 \cos\omega t\}$$
$$\ddot{x}_2 = x_2 - 2x_2(x_1^2+x_2^2) + \epsilon\{-\mu_1 - x_2 - \mu_2 \dot{x}_2 + \frac{4\mu_3}{1+\omega^2} x_1 x_2 \cos\omega t + \frac{2\mu_4}{1+\omega^2}(x_1^2+3x_2^2)\cos\omega t\}.$$

Here μ_1, μ_2. μ_3 and μ_4 are parameters, and ϵ is assumed to be sufficiently small. We consider first the unperturbed system ($\epsilon = 0$).

As easily seen, this unperturbed system is a Hamiltonian system. Let $\dot{x}_1 = x_3$ and $\dot{x}_2 = x_4$, and let $x = (x_1, x_2, x_3, x_4)$. Then the unperturbed system becomes

$$(13.10) \qquad \dot{x} = X_H(x)$$

where the Hamiltonian function $H(x)$ is given by

$$(13.11) \qquad H(x_1,x_2,x_3,x_4) = \tfrac{1}{2}\,(x_1^2 + x_2^2) + \tfrac{1}{2}(x_1^2+x_2^2)^2.$$

Furthermore system (13.10) has one more first integral

$$(13.12) \qquad F(x_1,x_2x_3x_4) = x_1x_4 - x_2x_3$$

which results from the conservation of the angular momentum.
Since

$$(13.13) \qquad \begin{aligned} \{F,H\}(x) &= dF(x)X_H(x) \\ &= [x_4 - x_3 - x_2 x_1] \begin{bmatrix} x_3 \\ x_4 \\ x_1 - 2x_1(x_1^2+x_2^2) \\ x_2 - 2x_2(x_1^2+x_2^2) \end{bmatrix} \\ &= 0 \end{aligned}$$

and since $dF(x)$ and $dH(x)$ are linearly independent for any $x \epsilon \mathbb{R}^4 \setminus \{0\}$, unperturbed system (13.10) is a completely integrable system in $\mathbb{R}^4\setminus\{0\}$. So we shall utilize this special structure (see Proposition 11.3(ii) and Theorem 11.4) to derive the Melnikov vector even through the perturbed system is not a Hamiltonian system.

Next we notice that the unperturbed system (13.10) has a homoclinic orbit $\gamma(t,o)$ to the origin.

$$(13.14) \qquad \gamma(t,o) = (p(t),\ o,\ \dot{p}(t),\ o)$$

where $p(t) = \text{sech}\ t$.

In terms of the complete integrability, we know that the stable and the unstable manifolds of system (13.10), both of which have dimension two, must coinside along $\gamma(t,o)$ and in fact, by using a symmetry property of X_H, this 'homoclinic manifold' can be expressed as

$$(13.15) \qquad \gamma(t,\nu) = (p(t)\cos\nu,\ p(t)\sin\nu,\ \dot{p}(t)\cos\nu,\ \dot{p}(t)\sin\nu),$$
$$\nu\epsilon[o,2\pi),\ t\epsilon\mathbb{R}.$$

That is, system (13.10) has a family of homoclinic orbits parametrized by ν. Thus system (13.10) is an example to case (ii) in Section 10.

Now we go back to the original perturbed system (13.8).

Let
$$g(t_1 x_, \mu) = (0,0,-3\mu_1 x_1 - \mu_2 x_3 + \frac{2\mu_3}{H\omega^2}(3x_1^2+x_2^2)\cos\omega t + \frac{4\mu_4}{1+\omega^2} x_1 x_2 \cos\omega t,$$
$$-\mu_1 x_2 - \mu_2 x_4 + \frac{4\mu_3}{1+\omega^2} x_1 x_2 \cos\omega t + \frac{2\mu_4}{1+\omega^2}(x_1^2+3x_2^2)\cos\omega t),$$

where $\mu = (\mu_1,\mu_2,\mu_3,\mu_4)$. Then system (13.9) has the form

$$(13.16) \qquad \dot{x} = X_H(x) + \epsilon g(t,x,\mu).$$

As we mentioned in case (ii) in Section 10, the linear Melnikov vector can be used to detect a point of transversal or tangential intersection of the perturbed stable and unstable manifolds of system (13.16). Furthermore by virtue of Proposition 11.3 (ii), bounded solutions of the adjoint system of the linearized system of system (13.10) along an orbit $\gamma(t,\nu)$ are given by

$$(13.17) \qquad dH(\gamma(t,\nu)) = (-p(t)\cos\nu + 2p(t)^3\cos\nu, -p(t)\sin\nu + 2p(t)^3\sin\nu, \dot{p}(t)\cos\nu, \dot{p}(t)\sin\nu)$$

and

$$(13.18) \qquad dF(\gamma(t,\nu)) = (\dot{p}(t)\sin\nu, -\dot{p}(t)\cos\nu, -p(t)\sin\nu, p(t)\cos\nu).$$

It is easily shown that $dH(\gamma(t,\nu))$ and $dF(\gamma(t,\nu))$ are linearly independent.

We can now compute the linear Melnikov vector $\hat{M}(\alpha,\nu) = (\hat{M}_1(\alpha,\nu), \hat{M}_2(\alpha,\nu))$ as follows.

$$(13.19) \qquad \begin{aligned} \hat{M}_1(\alpha,\nu) &= \int_{-\infty}^{\infty} dH(\gamma(t,\nu))g(t-\alpha,\gamma(t,\nu),\mu)dt \\ &= \int_{-\infty}^{\infty} \{-\mu_1(3\cos^2\nu + \sin^2\nu)p(t)\dot{p}(t) - \nu_2\dot{p}(t)^2 \\ &\qquad + \frac{6}{1+\omega^2}(\nu_3\cos\nu + \mu_4\sin\nu)p(t)^2\dot{p}(t)\cos\omega(t-\alpha)\}dt \\ &= -\frac{2}{3}\mu_2 - \pi\omega\,\mathrm{sech}(\frac{\pi\omega}{2})(-\mu_3\sin\nu + \mu_4\cos\nu)\cos\omega\alpha. \end{aligned}$$

Let $c = \pi \operatorname{sech}(\frac{\pi\omega}{2})$. Then the Melnikov vector becomes

$$(13.21) \qquad \hat{M}(\alpha,\nu,\mu) = \begin{bmatrix} -\frac{2}{3}\mu_2 - c\omega(\mu_3\cos\nu + \mu_4\sin\nu)\sin\omega\alpha \\ 2\mu_1\sin 2\nu + c(-\mu_3\sin\nu + \nu_4\cos\nu)\cos\omega\alpha \end{bmatrix}.$$

To find points of intersection, consider, for example, the case $\nu = 0$. In this case the Melnikov vector becomes

$$(13.22) \qquad \hat{M}(\alpha,0;\mu) = \begin{bmatrix} -\frac{2}{3}\mu_2 - c\omega\mu_3\sin\omega\alpha \\ c\mu_4\cos\omega\alpha \end{bmatrix}.$$

Solving $\hat{M}(\alpha,0;\mu) = 0$, we have the following bifurcation set S in the parameter space $(\mu_1,\mu_2,\mu_3,\mu_4)$.

$$(13.23) \qquad S = A \cup B \cup C,$$

where
$$A = \{(\mu_1,\mu_2,\mu_3,\mu_4)\colon \mu_2 = \pm\tfrac{3}{2}c\omega\mu_3,\ \mu_4 \neq 0,\ \mu_1,\mu_2 \in \mathbb{R}\},$$
$$B = \{(\mu_1,\mu_2,\mu_3,\mu_4)\colon |\mu_2| < |\tfrac{3}{2}c\omega\mu_3|,\ \mu_4 = 0,\ \mu_1,\mu_3 \in \mathbb{R}\},$$
$$C = \{(\mu_1,\mu_2,\mu_3,\mu_4)\colon \mu_2 = \pm\tfrac{3}{2}c\omega\mu_3,\ \mu_4 = 0,\ \mu_1,\mu_3 \in \mathbb{R}\}.$$

See Figure 13.

Next we examine the transversality and the tangency of intersection in the case $\nu = 0$. The derivatives of $\hat{M}$ are given by

$$(13.24) \qquad \frac{\partial}{\partial\alpha}\hat{M}(\alpha,0;\mu) = \begin{bmatrix} -c\omega^2\mu_3\cos\omega\alpha \\ -c\omega\mu_4\sin\omega\alpha \end{bmatrix}$$

and

(13.25) $$\frac{\partial}{\partial\nu}\hat{M}(\alpha,0;\mu) = \begin{bmatrix} -c\omega\mu_4\sin\omega\alpha \\ 4\mu_1 - c\mu_3\cos\omega\alpha \end{bmatrix}.$$

Since, in this example, the stable and the unstable manifolds of the unperturbed system coincide and constitute a two dimensional manifold in $\mathbb{R}^4$, rank $[\frac{\partial}{\partial\alpha}\hat{M}(\alpha,0;\mu) \ \frac{\partial}{\partial\nu}\hat{M}(\alpha,0;\mu)] = 2$ implies a transversal intersection. (see Proposition 10.3.)

(i) Let $\mu\epsilon A$. Since

rank $[\frac{\partial}{\partial\alpha}\hat{M}(\alpha,0,\mu) \ \frac{\partial}{\partial\nu}\hat{M}(\alpha,0,\mu]$ = rank $\begin{bmatrix} 0 & \pm c\omega\mu 4 \\ \pm c\omega\mu 4 & 4\mu 1 \end{bmatrix} = 2$,

intersection is always transversal.

(ii) Let $\mu\epsilon B$. In this case we have

$$[\frac{\partial}{\partial\alpha}\hat{M}(\alpha,0,\mu) \ \frac{\partial}{\partial\nu}\hat{M}(\alpha,0,\mu)] = \begin{bmatrix} -c\omega^2\mu_3\cos\omega\alpha & 0 \\ 0 & 4\mu_1 - c\mu_3\cos\omega\alpha \end{bmatrix}.$$

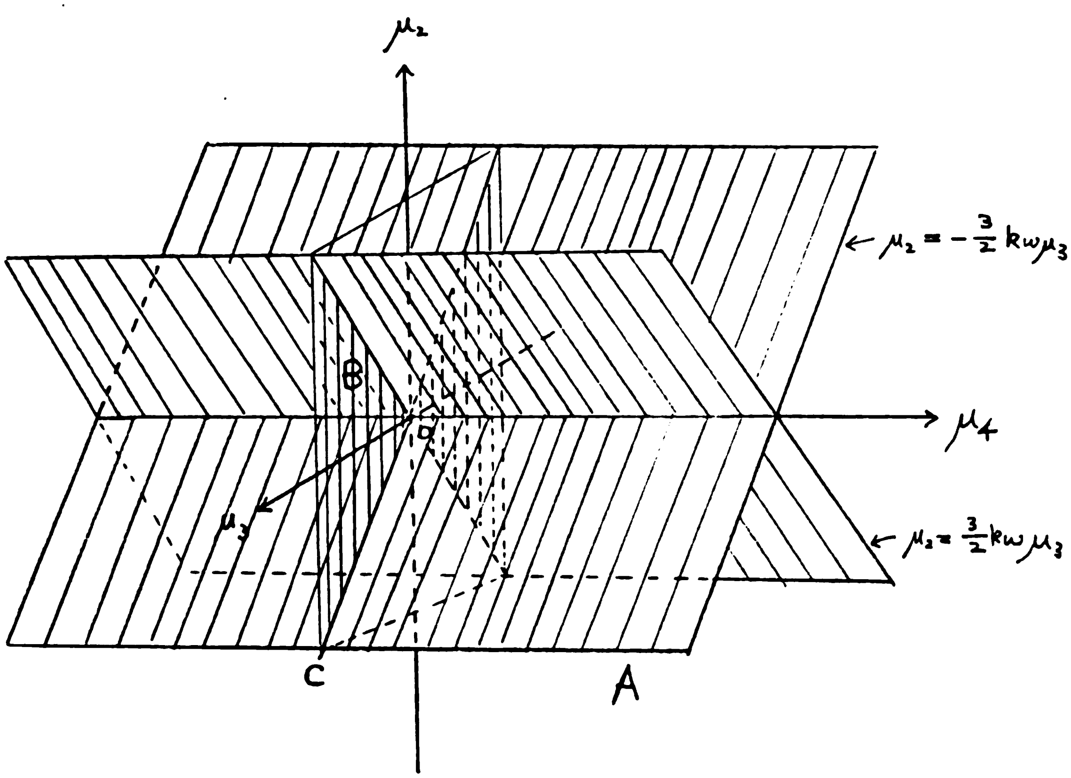
(ii–1) If $\mu_1 \neq \frac{1}{4}c\mu_3\cos\omega\alpha$, then the rank of the above matrix is 2 and so we have a transversal intersection.

(ii–2) If $\mu_1 = \frac{1}{4}c\mu_3\cos\omega\alpha$, then $\frac{\partial}{\partial\nu}\hat{M}(\alpha,0,\mu) = 0$. By computing the rank of (10.18), we have a tangential intersection.

(iii) Let $\mu\epsilon C$. In this case

$$[\frac{\partial}{\partial\alpha}\hat{M}(\alpha,0,\mu) \ \frac{\partial}{\partial\nu}\hat{M}(\alpha,0,\mu] = \begin{bmatrix} 0 & 0 \\ 0 & 4\mu_1 \end{bmatrix}.$$

By computing the rank of (10.18), we have a tangencial intersection if $\mu_1 \neq 0$, $\mu_3 \neq 0$.



Bifurcation Set A∪B∪C (for a fixed μ_1)

Figure 13

Example 3

The aim of this example is to give an example in which condition (ii) of Theorem 7.1 is not satisfied but the stable and unstable manifolds intersect transversally. To this end we modify the system in Example 2 slightly and consider the following system.

$$(13.26)\qquad \begin{aligned} \dot{x}_1 &= x_3 \\ \dot{x}_2 &= x_4 \\ \dot{x}_3 &= x_1 - 2x_1(x_1^2+x_2^2) + \epsilon\{-3\mu_1 x_1 - \mu_2 x_3 + \frac{2\mu_3}{1+\omega^2}(3x_1^2+x_2^2)(\cos\omega t \\ &\quad + \frac{4\mu_4}{1+\omega^2} x_1 x_2 \cos\omega t\} \\ \dot{x}_4 &= x_2 - 2x_2(x_1^2+x_2^2) + \epsilon\{-\mu_1 x_2 - \mu_2 x_4 + \frac{4\mu_3}{1+\omega^2} x_1 x_2 \cos\omega t \\ &\quad + \frac{2\mu_4}{1+\omega^2}(x_1^2+3x_2^2)\cos\omega t\}. \\ \dot{y} &= y + \epsilon \cos\omega t. \end{aligned}$$

Notice that the unperturbed system ($\epsilon=0$) has the following stable and unstable manifolds.

$$(13.27)\qquad W^u = (p(\alpha)\cos\nu,\ p(\alpha)\sin\nu,\ \dot{p}(\alpha)\cos\nu,\ \dot{p}(\alpha)\sin\nu, y),$$

$$(13.28)\qquad W^s = (p(\alpha)\cos\nu,\ p(\alpha)\sin\nu,\ \dot{p}(\alpha)\cos\nu,\ \dot{p}(\alpha)\sin\nu, 0)$$

where $p(t) = \operatorname{sech} t$ and $\alpha, y \epsilon \mathbb{R}$, $\nu \epsilon [0,2\pi]$.

Note that $\dim W^u = 3$ and $\dim W^s = 2$, and the 'homoclinic manifold' is $W^u \cap W^s$.

From (13.17) and (13.18), it is clear that the adjoint system of the linearized system of the unperturbed system has two linearly independent bounded solutions on $\mathbb{R}$ which are given by

$$(13.29) \qquad \phi_1^*(t) = (-p(t)\cos\nu + 2p(t)^3\cos\nu, -p(t)\sin\nu + 2p(t)^3\sin\nu, \dot{p}(t)\cos\nu, \dot{p}(t)\sin\nu, 0)$$

and

$$(13.30) \qquad \phi_2^*(t) = (\dot{p}(t)\sin\nu, -\dot{p}(t)\cos\nu, -p(t)\sin\nu, p(t)\cos\nu, 0).$$

Hence the Melnikov vector for system (13.26) is precisely the same as before. Consider the case $\nu = 0$. From now on we assume that

$$(13.31) \qquad \mu \epsilon C \text{ and } \mu_1 \neq 0.$$

Then we know that $\frac{\partial}{\partial\alpha}\hat{M}(\alpha,0;\mu) = 0$ and $\frac{\partial}{\partial\nu}\hat{M}(\alpha,0;\mu) \neq 0$. Thus condition (ii) in Theorem 7.1 is not satisfied. However we shall show that there exists the transversal intersection of the stable and unstable manifolds of system (13.26). First recall, by using the notation in (7.3) and (7.4), that we have the following situation.

$\mathcal{R}(I-Q(\alpha))$	α,ν		$\mathcal{R}(P(\alpha))$
	μ_2^u	$m_1^s(\alpha,\nu)$	$\mathcal{R}(I-P(\alpha))$
$\mathcal{R}(Q(\alpha))$	$m_2^u(\alpha,\nu,\mu_2^u)$	$m_2^s(\alpha,\nu)$	

(13.32) $$DF^u = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{\partial}{\partial\alpha} m_2^u & \frac{\partial}{\partial\nu} m_2^u & \frac{\partial}{\partial\mu_2^u} m_2^u \end{bmatrix},$$

(13.33) $$DF^s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\partial}{\partial\alpha} m_1^s & \frac{\partial}{\partial\nu} m_2^s \end{bmatrix}.$$

Therefore if $\frac{\partial}{\partial\alpha} m_1^s \neq 0$, we have the transversal intersection even though $\frac{\partial}{\partial\alpha} \hat{M}(\alpha,0:\mu) = 0$ which means that $\frac{\partial}{\partial\alpha} m_2^u = \frac{\partial}{\partial\alpha} m_2^s$. Note that the linearized system of the unperturbed system of (12.26) has an unbounded solution $(0,0,0,0,ce^t)$ on $[\alpha,\infty)$ where C is a nonzero constant. Let

(13.34) $$\phi_3^*(t) = (0,0,0,0,Ce^{-t}).$$

Then

(13.35) $$m_1^s(\alpha) = \phi_3^*(\alpha)m^s(\alpha).$$

Referring (6.1) and the proof of Lemma 4.2, we have

$$(13.36) \qquad m_1^s(\alpha) = \int_{\infty}^{\alpha} ce^{-t} \cos\omega(t-\alpha)dt = \frac{C}{1+\omega^2} e^{-\alpha}.$$

So we have

$$(13.37) \qquad \frac{d}{d\alpha} m_1^s(\alpha) = \frac{C}{1+\omega^2} e^{-\alpha} \neq 0 \quad \text{for any} \quad \alpha.$$

Thus in this example, there exists the transversal intersection but the condition by the Melnikov vector can not be used to show it.

Let us summarize these analysis. In these examples the Hamiltonian nature of the unperturbed systems is effectively used even though the perturbations are not Hamiltonian. See Proposition 11.3 and Theorem 11.4. Example 1 is a standard two-dimensional case which gives the simplest case to which the linear Melnikov function can be easily applied. We note that the linear Melnikov function gives a necessary and sufficient condition of the transversal intersection and hence it also can be used to show the tangential intersection. See Proposition 10.2. Example 2 gives a higher dimensional case to which the linear Melnikov vector can be used to detect the transversal and tangential intersection. See Propositions 10.3 and 10.7. In Example 3, we consider a case to which the Melnikov vector can not give a complete information about the transversal intersection. This limitation of the Melnikov vector in higher dimensional cases comes from the fact that the Melnikov vector is the projection of the real distance between the stable and unstable manifolds to the space of the completely unbounded solutions, i.e., the complement subspace of $\mathcal{R}(P(\alpha)) + \mathcal{R}(I-Q(\alpha))$. Therefore the Melnikov vector drops the information about the projection of the real distance to other subspaces. See the decomposition in (6.9).

§14. EXPONENTIALLY SMALL SPITTING OF STABLE AND UNSTABLE MANIFOLDS

In this section we examine an example for which the Melnikov function can not be applied to detect the intersection of the stable and unstable manifolds. Before doing this, we recall Example 1 in §13.

$$(14.1) \qquad x - x + \frac{3}{2}x^2 = \epsilon \cos t.$$

The linear Melnikov function of this system was

$$(14.2) \qquad \hat{M}(\alpha) = -c \sin\alpha$$

where $c \cdot 0$ is a constant. The reason for that $\hat{M}(\alpha)$ can be used to detect the intersection of the stable and unstable manifolds of this system is that the distance $\tilde{d}$ between the stable and unstable manifolds is expressed as

$$(14.3) \qquad \tilde{d} = \epsilon(\hat{M}(\alpha) + 0(\epsilon)).$$

That is, the linear Melnikov function $\hat{M}(\alpha)$ constitutes the leading term.

Now we consider the following rapidly forced system.

$$(14.4) \qquad x - x + \frac{3}{2}x^2 = \epsilon \cos \left(\frac{t}{\epsilon_1}\right)$$

where $\epsilon << 1$ and $\epsilon_1 << 1$. In this case the linear Melnikov function takes the form

$$\hat{M}(\alpha,\epsilon) = -\frac{\pi}{\epsilon}\,\text{cosech}\,(\frac{\pi}{\epsilon})\,\sin\,(\frac{\alpha}{\epsilon}). \tag{14.5}$$

Hence $\hat{M}(\alpha,\epsilon)$ can not be the leading term of the expansion of $\tilde{d}$ in terms of ϵ. See also Holmes, Marsden and Scheule [11]. This is a serious limitation of the perturbation method used in the theory of the Melnikov function we developed before and in fact it relates to one of the fundamental problems in dynamics since the time of Poincare. Resolution of this difficulty has to wait for future study.

REFERENCES

[1] Arnold, V.I., Mathematical Methods of Classical Mechanics, Springer–Verlag, Berlin, New York, 1978.

[2] Arnold, V.I., Instability of Dynamical Systems with Several Degrees of Freedom, Sov. Math. Dokl., 5 (1964), 581–585.

[3] Birkhoff, G.D., Nouvelles recherches sur les systemes dynamiques, Mem. Pont. Acad. Sci. Novi Lyncaei, 1 (1935), 85–216.

[4] Chow, S.N., Hale, J.K., and Mallet–Paret, J., An Example of Bifurcation to Homoclinic Orbits, J. Differential Equations, 37 (1980), 351–373.

[5] Coppel, W.A., Dichotomies in Stability Theory, Lecture Notes in Mathematics No. 629, Springer–Verlag, Berlin, New York, 1978.

[6] Gruendler, J., The Existence of Homoclinic Orbits and The Method of Melnikov for Systems in $\mathbb{R}^n$, SIAM J. Mathematical Analysis, Vol. 16, No. 5 (1985), 907–931.

[7] Hale, J.K., Ordinary Differential Equations, Krieger, 1980.

[8] Hale, J.K. and Lin, X.B., Heteroclinic Orbits for Retarded Functional Differential Equations, J. Differential Equations, 65, (1986), 175–202.

[9] Holmes, P.J., Averaging and Chaotic motions in Forced Oscillations, SIAM J. Applied Mathematics, 38, (1980), 65–80.

[10] Holmes, P.J. and Marsden, J.E., Melnikov Method and Arnold Diffusion for Perturbations of Integrable Hamiltonian Systems, J. Mathematical Physics, Vol. 4 (1982), 669–675.

[11] Holmes, P., Marsden, J. and Scheurle, J., Exponentially Small Splitting of Separatrices, Preprint 1987.

[12] Melnikov, V.K., On the Stability of the Center for Time Periodic Purturbations, Trans. Moscow Math. Soc., 12 (1963), 1–57.

[13] Meyer, K.R. and Sell, G.R., Melnikov Transforms, Bernoulli Bundles, and Almost Periodic Perturbations, in Oscillation, Bifurcation and Chaos, CMS Conference Proceedings, Vol. 8, 1987.

[14] Moser, J. Stable and Random Motions in Dynamical Systems. Princeton University Press, 1973.

[15] Newhouse, S.E., Diffeomorphisms with infitely many sinks, Topology 13, (1974), 9–18.

[16] Palmer, K.J., Exponential Dichotomies and Transversal Homoclinic Points, J. Differential Equations, 55 (1984), 225–256.

[17] Poincare, H., Sur le Probleme des Trois Corps et les Equations de la Dynamique, Acta Math. 13 (1890) 1–270.

[18] Poincare, H., Les Methodes Nouvelles de la Mecanique celeste, t.III, Gauthier–Villars, Paris 1899.

[19] Sacker, R.J., The Splitting Index for Linear Differential Systems, J. Differential Equations, 33 (1979), 368–405.

[20] Smale, S., Diffeomorphisms with many periodic points. In Differential and Combinatorial Topology, S.S. Chern (ed.), 63–80. Princeton University Press, 1963.

[21] Wiggins, S., A Generalization of the Method of Melnikov for Detecting Chaotic Invariant Sets, Preprint.

[22] Wiggins, S. and Holmes, P., Homoclinic Orbits in Slowly Varying Oscillators, Preprint.