CONSTRUCTING SYMPLECTIC 4-MANIFOLDS

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ABSTRACT

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This thesis introduces a new technique for constructing symplectic 4-manifolds, generalizing the 3- and 4-fold sums introduced by Symington, and by McDuff and Symington.

We first define relative connect normal sums. This method allows one to join concave (or convex) fillings along complements of properly embedded symplectomorphic surfaces with boundary.

We then define the k-fold sum as follows. Given k pairs of symplectic surfaces, such that pairs are disjoint from one another, and the surfaces in each pair intersect ω-orthogonally once, we may remove neighbourhoods of the intersection points. We may then perform the relative connect normal sum k times to obtain a concave filling of a manifold that fibers over S^1 with torus fibers. We study when the resulting contact structure on the boundary is convexly fillable.

As an application of k-fold sums, we construct seven closed exotic symplectic manifolds, two of which violate the Noether inequality.

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Introduction

1.1 History

The fundamental question of smooth 4-dimensional topology asks: *how many distinct smooth structures exist on a given underlying topological* 4*-manifold?* This question is often modified by placing restrictions on the smooth structures being considered. We may ask to find irreducible or minimal smooth structures, or we may ask that the smooth structures admit some geometric property. For many topological manifolds, a basic version of this question is still open: *for a given topological manifold, is there more than one smooth structure?*

In 1987, Donaldson [7] provided the first examples of exotic smooth structures on a simply connected 4-manifold by demonstrating that the Dolgachev surfaces are not diffeomorphic to $\mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2$ (it follows by work of Freedman [22] that these manifolds are all homeomorphic). This was followed in 1989 by Kotschick's proof [37] that the Barlow surface is not diffeomorphic to $\mathbb{CP}^2 \sharp 8 \overline{\mathbb{CP}}^2$. Further progress in this direction was stymied by the difficulty in finding complex surfaces that are homeomorphic to standard manifolds (such as $\mathbb{CP}^2 \sharp k \overline{\mathbb{CP}}^2$).

However, it turns out that it is not necessary to look within complex manifolds to find examples of distinguishable smooth structures. This was first evidenced by Taubes [61], who showed that symplectic manifolds typically have non-trivial Seiberg-Witten invariants; one can therefore hope to find distinguishable exotic smooth structures by examining symplectic manifolds.

The symplectic category is larger and more malleable than the Kähler category. This was first seen by Thurston [63], who gave an example of a manifold that is symplectic but not Kähler. In 1995, Gompf [30] utilized a cut-and-paste technique, the *connect normal sum*, to show, for instance, that every finitely generated group appears as the fundamental group of a closed symplectic 4-manifold; this is not true for Kähler manifolds.

Other evidence that the symplectic category is much larger than the Kähler category was provided by Fintushel and Stern [19], who used rational blow-downs to show that there exist minimal symplectic manifolds with $c_1(X)^2 > 0$ that violate the Noether inequality $5c_1^2(X) - c_2(X) + 36 \geq 0$. In fact, they showed that there exists a simply-connected minimal symplectic manifold for every pair $(c_2(X), c_1(X)^2)$ satisfying $c_1^2 + c_2 \equiv 0 \bmod 12$, $c_1(X)^2 > 0$, and $5c_1(X)^2 - c_2(X) + 36 < 0$. Other constructions of symplectic manifolds violating the Noether inequality have been provided by Gompf [30] and Stipsicz [54].

Cut-and-paste techniques have since led to constructions of minimal exotic symplectic manifolds homeomorphic to $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$. In 2004 Park [50] constructed a minimal exotic symplectic manifold homoeomorphic to $\mathbb{CP}^2 \sharp 7\overline{\mathbb{CP}}^2$ via a rational blow-down (this was proven to be minimal by Ozsváth and Szabó in [48]). Since then, various cut-and-paste methods have been used to construct minimal exotic symplectic manifolds homeomorphic to $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$ for $2 \leq k \leq 9$ (see [1–4, 6, 17, 20, 52, 56]).

One such method that has proven to be helpful in constructing symplectic manifolds with small euler characteristic is the 3-fold sum. Using this, Fintushel and Stern have provided a systematic method for constructing $\mathbb{CP}^2 \sharp k\overline{\mathbb{CP}}^2$ for $2 \leq k \leq 7$ [20]. With this is mind, we shall re-examine the 3-fold sum and provide a generalization.

1.2 Outline

Throughout this thesis, X will denote a closed symplectic 4-manifolds with symplectic form ω.

The primary aim of this thesis is study a generalization of the *k-fold sum*. Before delving into this construction, however, we will review previous cut-and-paste constructions. First, we will review the connect normal sum. This method, described by Gromov [33] and Gompf [30], allows one to identify punctured neighbourhoods of symplectomorphic surfaces with opposite self-intersection numbers. Second, we will review symplectic gluing, a method described by McCarthy and Wolfson [42] that allows one to identify convex and concave symplectic fillings of a contact manifold along their boundaries. As an application of symplectic gluing, we will provide a new proof that rational blow-downs can be performed symplectically.

The k-fold sum, described by Symington [58, 59] and by McDuff and Symington [45], is a variation of the connect normal sum, where under certain conditions one can glue together the complement of either three or four disjoint pairs of transversely intersecting symplectic surfaces to create a new symplectic manifold. We will reinterpret this construction by first describing a relative version of the connect normal sum.

Theorem 1. *Let Σ_1 and Σ_2 be disjoint, properly embedded, symplectomorphic surfaces in a (possibly disconnected) convex (resp. concave) filling X. The connect sum of X along $\Sigma_1 \amalg \Sigma_2$ admits a convex (resp. concave) symplectic structure.*

We are thus able to construct new fillings from old.

In particular, we will re-interpret the k-fold sum not as a method of constructing closed manifolds, but as a method of constructing concave fillings of certain manifolds. We can first obtain concave fillings of $\amalg S^3$ by removing neighbourhoods of the intersection points of the pairs. We can then perform the relative connect normal sum on the newly punctured surfaces to obtain a concave filling of a manifold that fibers over S^1 with torus fiber. The induced contact structure on the boundary is universally tight, and it is straightforward to compute its Giroux torsion. Following Honda's classification of contact structures on such manifolds [35], we have completely specified the contact structure. If we then also have a preferred convex filling of this contact structure, we can symplectically glue these together to obtain a closed manifold.

The situations being considered by Symington and McDuff-Symington are concerned with two of the three uniform cases when the resulting boundary manifold is the unique fillable contact structure on T^3. In such a situation, the concave filling can be extended to a closed symplectic manifold by gluing it to the convex filling $T^2 \times \mathbb{D}^2$.

More generally, we can find convex fillings of other boundary manifolds that appear. Doing so allows us to construct certain minimal symplectic manifolds. In particular, we will attach convex fillings to k-fold sums that are taken along a pair of tori in $\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2$. These manifolds will provide another proof that:

Theorem 2. *There exist simply-connected minimal symplectic manifolds that violate the Noether inequality.*

These constructed manifolds have $b^+ > 1$. Besides the above-mentioned manifolds described in [19], [30], and [54], that violate the Noether inequality, other examples in literature of minimal symplectic manifolds with $b^+ > 1$ are provided in [1–4, 49, 51, 53, 55].

Chapter 2

Preliminaries

In this chapter, we discuss the necessary underpinnings of symplectic and contact topology that will be used in the subsequent chapters. For more detailed explanations, one may consult [26], [47], or [44]

2.1 Symplectic topology

Let X be a smooth n-dimensional manifold. A 2-form $\omega \in \Omega^2(X)$ is a *symplectic form* if it is non-degenerate and closed; i.e. $d\omega = 0$, and for every non-zero tangent vector v, there exists a tangent vector w such that $\omega(v, w) \neq 0$. Note that the existence of a non-degenerate skew-symmetric form on each tangent space necessitates that the manifold is even dimensional. Moreover, a symplectic form equips X with a preferred orientation $\bigwedge_{i=1}^n \omega$. Given a choice of symplectic form on X, the pair (X, ω) is called a *symplectic manifold*.

Example 1. Given $\mathbb{R}^{2n}$ with coordinates $(x_1, y_1, \ldots, x_n, y_n)$, the form $\sum_{i=1}^n dx_i \wedge dy_i$ is symplectic. This is often referred to as the standard symplectic structure on $\mathbb{R}^{2n}$.

Kähler manifolds provide one class of symplectic manifolds. Indeed, symplectic manifolds can be considered a weakening of the Kähler condition, in that we no longer require the complex structure to be integrable. In particular, symplectic manifolds still admit compatible almost-complex structures, meaning that $\omega(-, J-)$ is a Riemannian metric. The space of such compatible almost-complex structures is contractible. This implies that the chern classes of the almost-complex structure are invariants of the symplectic structure.

Similarly to complex manifolds, we will denote the first chern class of the cotangent bundle of a symplectic manifold using K.

There are multiple notions of equivalence between symplectic structures.

Definition 1. A diffeomorphism $\Phi : (X_1, \omega_1) \to (X_2, \omega_2)$ is a *symplectomorphism* if $\Phi^* \omega_2 = \omega_1$.

Symplectomorphisms do not exist between many symplectic manifolds that we may wish to consider equivalent. For instance, given a closed symplectic manifold (X, ω), ω defines a non-zero class in $H^2(X; \mathbb{R})$. It therefore follows that (X, ω) is not symplectomorphic to $(X, k \cdot \omega)$ for $k > 1$. To make examples such as these equivalent we introduce the notion of symplectic manifolds being deformation equivalent.

Definition 2. (X_1, ω_1) and (X_2, ω_2) are *deformation equivalent* if there exists a diffeomorphism $\Phi : X_1 \to X_2$ such that $\Phi^* \omega_2$ is isotopic to ω_1 through symplectic forms on X_1.

An important feature of symplectic topology is that there all symplectic manifolds locally look the same.

Theorem 3 (Darboux's Theorem). *Given $p \in (X, \omega)$, there exists a neighbourhood U that it symplectic to an open neighbourhood of $0 \in \mathbb{R}^{2n}$, equipped with the standard symplectic structure.*

This theorem extends to neighbourhoods of certain surfaces in symplectic 4-manifolds.

Definition 3. Given a symplectic manifold (X, ω), a surface $\Sigma \subset X$ is symplectic if $\omega|_\Sigma$ is a symplectic form on Σ.

Theorem 4 (Symplectic neighbourhood theorem, Weinstein [64])**.** *Suppose* $\Sigma_i \subset (X_i, \omega_i)$ *are closed symplectic surfaces such that* $\phi : (\Sigma_1, \omega_1|_{\Sigma_1}) \to (\Sigma_2, \omega_2|_{\Sigma_2})$ *is a symplectomorphism. Moreover, suppose that* $[\Sigma_1]^2 = [\Sigma_2]^2$*. There exists a symplectomorphism between tubular neighbourhoods of* Σ_i *that restricts to* ϕ*.*

One can always choose a compatible almost complex structure J on (X, ω) so that $TX|_\Sigma$ splits as complex bundles as $TX|_\Sigma = T\Sigma \oplus N\Sigma$. Here, $N\Sigma$ is the normal bundle of Σ. Applying the first chern class to this splitting when X is a 4-manifold, one has the *Adjunction formula* for symplectic surfaces:

$$-\chi(\Sigma) = [\Sigma]^2 + \langle K, [\Sigma] \rangle. \tag{2.1}$$

Another type of surface that interacts well with a symplectic structure is a Lagrangian surface.

Definition 4. Given a symplectic 4-manifold (X, ω), a surface $\Sigma \subset X$ is Lagrangian if $\omega|_\Sigma = 0$.

One method for constructing a Lagrangian submanifold, begins by examining the cotangent bundle of a surface. Let x_i be local coordinates for a surface Σ, and let y_i be the coordinates in the direction of dx_i in $T^*\Sigma$. We then have coordinate for $T^*\Sigma$ so that the zero section is given by $y_1 = y_2 = 0$. One can locally construct a symplectic form on $T^*\Sigma$ as $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$. It turns out that this form is independent of any choice of coordinates on Σ, and is therefore well defined. It is clear that the zero section is Lagrangian.

One has the following theorem for neighbourhoods of Lagrangian surfaces.

Theorem 5 (Lagrangian neighbourhood theorem, Weinstein [64])**.** *Let* $\Sigma \subset (X, \omega)$ *be a*

Lagrangian surface. There exists a tubular neighbourhood of Σ *that is symplectomorphic to a tubular neighbourhood of the zero section of* $(T^*\Sigma, \omega)$.

By examining the cotangent bundle of a surface, we also have an Adjunction-type equality for Lagrangian surfaces:

$$-\chi(\Sigma) = [\Sigma]^2. \tag{2.2}$$

Lastly, we wish to summarize certain properties about the Seiberg-Witten invariants of symplectic 4-manifolds. For the purpose of this thesis, we can treat the Seiberg-Witten invariants formally by simply using general properties of the Seiberg-Witten invariant. For simplicity, we will restrict our attention to manifolds with $b^+ > 1$. Recall that in the absence of 2-torsion, the *Spin*^{c}-structures on a manifold X are in bijective correspondence with characteristic classes $\kappa \in H^2(X)$. A cohomology class κ is *a basic class* if the Seiberg-Witten invariant associated to κ is non-zero. A manifold X is said to be of *simple type* if the expected dimension of the moduli spaces of Seiberg-Witten solutions associated to all basic classes of X is 0.

Theorem 6 (Taubes [62])**.** *Symplectic manifolds are of basic type. In particular, the basic classes* κ *of symplectic manifolds satisfy* $\kappa^2 = 3\sigma(X) + 2\chi(X)$.

Moreover, we are guaranteed that the Seiberg-Witten invariant of a symplectic manifold is non-trivial.

Theorem 7 (Taubes [60, 61])**.** *For a symplectic manifold* (X, ω)*, the canonical and anti-canonical classes* $\pm K = \pm c_1(T^*X)$ *are basic classes. Moreover, for any other basic class* κ*,* $|\kappa \cdot \omega| \leq |K \cdot \omega|$ *with equality if and only if* $\kappa = \pm K$.

Lastly, we wish to note two facts that basic classes can tell us about surfaces in X. The first is closely related to the Adjunction formula.

Theorem 8 (Adjunction inequality, Kronheimer and Mrowka [38])**.** *If κ is a basic class of X, any embedded surface $\Sigma \in X$ that is not a sphere must satisfy $-\chi(\Sigma) \geq [\Sigma]^2 + |\langle \kappa, [\Sigma] \rangle|$.*

We can therefore use basic classes to provide lower bounds on genera of surfaces, or use surfaces in X to provide bounds on potential basic classes.

Moreover, the basic classes can help identify when a manifold is not minimal.

Theorem 9 (Fintushel and Stern [18])**.** *Suppose that $X \cong Z \sharp \overline{\mathbb{CP}}^2$, where Z is of simple type. Let $\{\kappa_i\}$ be the basic classes of Z. The basic classes of X are $\{\kappa_i \pm e\}$, where e is the Poincaré dual to the -1-sphere in $\overline{\mathbb{CP}}^2$.*

2.2 Contact structures on 3-manifolds

To construct new symplectic manifolds, it is natural to consider boundary conditions that would allow one to glue together symplectic forms. One could, of course, consider symplectic manifolds that have symplectomorphic open subsets: if $U_1 \subset X_1$ is symplectomorphic to $U_2 \subset X_2$ such that $X = X_1 \cup_{U_1 = U_2} X_2$ is a manifold, then X inherits a symplectic structure. However, guaranteeing symplectomorphic subsets requires extensive knowledge of the symplectic structures. Such a gluing therefore tends to rely on neighbourhood theorems such as Theorems 4 or 5. Instead, a more useful gluing principle can be obtained by considering symplectic manifolds that naturally endow their boundaries with *contact structures*.

A 2-plane field ξ on a 3-manifold Y is *nowhere integrable* if there does not exist an embedding $\mathbb{D}^2 \hookrightarrow Y$ such that the tangent planes of $\mathbb{D}^2$ agree with ξ. By Frobenius' Integrability Theorem, the condition that ξ is nowhere integrable is locally equivalent to ξ being the kernel of a 1-form $\alpha \in \Omega^1(Y)$ satisfying $\alpha \wedge d\alpha \neq 0$.

Definition 5. Let Y be a 3-manifold. A *contact structure* ξ is a 2-plane field of TY that is nowhere integrable. Given a choice of contact structure ξ on Y, the pair (Y, ξ) is called a *contact manifold.*

Throughout this thesis we are only concerned with contact structures for which α can be defined globally. Such contact structures are called *co-oriented.* For a co-oriented contact structure, a choice of global 1-form is called a *contact form.* Note that a contact form α induces a preferred contact structure ξ, but the converse is not true; for instance, one may multiply α by any nowhere-zero function to construct a new contact form inducing the same contact structure. When we wish to emphasize the role of a chosen contact form, we will write the pair (Y, α) in place of (Y, ξ).

Since ξ is an oriented 2-plane field over Y, it is naturally a complex line bundle. In particular, the invariant $c_1(\xi)$ of ξ is well-defined.

There are multiple notions of equivalence between contact structures.

Definition 6. Two contact structures on Y, ξ_1 and ξ_2, are *isotopic* if there exists a diffeomorphism $\phi : Y \to Y$ that is isotopic to the identity such that $\phi_* \xi_1 = \xi_2$.

Equivalently, two contact structures ξ_1 and ξ_2 are isotopic if there exists a homotopy from ξ_1 to ξ_2 (as 2-plane fields) through contact structures [32].

Definition 7. Two contact manifolds (Y_1, ξ_1) and (Y_2, ξ_2) are *contactomorphic* if there exists a diffeomorphism $\phi : Y_1 \to Y_2$ such that $\phi^* \xi_2 = \xi_1$.

Example 2. Consider the 1-form $\alpha = dz + xdy - ydx$ on $\mathbb{R}^3$. Since $\alpha \wedge d\alpha = 2dx \wedge dy \wedge dz$, it follows that the kernel of α is a contact structure. This contact structure is often called the *standard contact structure on* $\mathbb{R}^3$.

Example 3. In cylindrical coordinates, consider the 1-form $\beta = \cos r dz + r \sin r d\theta$. Since $\beta \wedge d\beta = (r + \sin r \cos r) dz \wedge dr \wedge d\theta \neq 0$, the kernel of β is a contact structure.

Note that a co-oriented contact structure equips Y with a preferred orientation $\alpha \wedge d\alpha$; for 3-manifolds, multiplying α by a non-zero function will not affect this orientation.

Definition 8. An embedded curve $K \subset (Y, \xi)$ in a contact 3-manifold (Y, ξ) is *Legendrian* if its tangent space $T_p K$ lies in ξ_p for all p.

Note that Legendrian knots admit a canonical framing; since $\xi|_K$ is trivial, we can choose a vector field $v \in \xi_K \backslash TK$. Any such choice induces the same framing of K. Call the framing the *Legendrian framing.* When K is null-homologous, we can compare this framing to the Seifert framing.

Definition 9. Recall that framings of a knot are an affine $H^1(S^1) \cong \mathbb{Z}$. For a null-homologous Legendrian knot, the *Thurston-Bennequin* invariant is the integer specifying the Legendrian framing relative to the Seifert framing. Denote this number by $tb(K)$.

Since the Seifert framing is independent of the choice of Seifert surface, $tb(K)$ is also independent of this choice.

Another invariant of a null-homologous Legendrian knot is the rotation number.

Definition 10. For a null-homologous Legendrian knot with Seifert surface Σ, the *rotation number* is the first chern class of $\xi|_\Sigma$ relative to the Legendrian framing. Denote this number by $rot_\Sigma(K)$.

Unlike the Thurston-Bennequin invariant, the rotation number depends on the choice of orientation of Σ, and hence of K. Moreover, $rot_\Sigma(K)$ may depend on the choice of Σ itself.

Given two Seifert surfaces for K, one has that

$$rot_{\Sigma_1}(K) - rot_{\Sigma_2}(K) = \langle c_1(\xi), [\Sigma_1 - \Sigma_2] \rangle. \tag{2.3}$$

If $c_1(\xi) \neq 0$, this difference may be non-zero. If $c_1(\xi) = 0$, the rotation number is often denoted more simply as $rot(K)$.

Example 4. The contact structure associated to β in Example 3 is spanned by $\langle \frac{\partial}{\partial r}, r \sin r \frac{\partial}{\partial z} - \cos r \frac{\partial}{\partial \theta} \rangle$ away from the locus $r = 0$. The unknot U parameterized by $z = 0$, $r = \pi$, and $0 \leq \theta \leq 2\pi$ is therefore Legendrian. Moreover, $\frac{\partial}{\partial r}$ serves as both the Legendrian framing and the Seifert framing, and so $tb(U) = 0$.

Since $r \frac{\partial}{\partial r}$ is a section of $\xi|_{\mathbb{D}^2}$, where $\mathbb{D}^2$ is the obvious Seifert surface, we see that $rot(K) = \pm 1$, depending on the orientation of U.

There is a fundamental dichotomy of contact structures involving the above example.

Definition 11. A contact structure is *over-twisted* if there exists an unknot U with $tb(U) = 0$. If no such disk exists, the contact structure is *tight*.

Over-twisted contact structures up to isotopy are in bijective correspondence with co-oriented 2-plane fields. [9].

Tight contact structures ξ are precisely those whose surfaces satisfy a certain adjunction-type inequality called the *Thurston-Bennequin inequality*. Given a surface $\Sigma \subset Y$ with Legendrian boundary K, one always has that

$$-\chi(\Sigma) \geq tb(K) + |rot_{\Sigma}(K)| \tag{2.4}$$

if ξ is tight.

There is a refinement of tight contact structures.

Definition 12. A contact structure ξ on Y is *universally tight* if the pullback of ξ to the universal cover of Y is also tight. A contact structure is said to be *virtually overtwisted* if it is tight, but lifts to an overtwisted contact structure under some finite cover.

We will see examples of universally tight contact structures in Example 10.

2.3 Interactions between contact topology and symplectic topology

One can always build a symplectic manifold from any co-oriented contact manifold (Y, α). Let $SY = \mathbb{R} \times Y$. Equip SY with the 2-form $\omega_\alpha = d(e^s\alpha)$, where s parameterizes the $\mathbb{R}$ direction. Clearly ω_α is closed, and a simple calculation shows that ω_α is non-degenerate. Moreover, the symplectic orientation on SY agrees with the product orientation.

Definition 13. The above-constructed symplectic manifold (SY, ω_α) is called the *symplectization* of (Y, α).

Since the symplectic form ω_α is exact, there exists a vector field v that recovers the preferred primitive of ω_α: $\iota_v\omega_\alpha = e^s\alpha$. Since ω_α is closed, v solves $\mathcal{L}_v\omega_\alpha = \omega_\alpha$.

Definition 14. A vector field v defined on an open set $U \subseteq X$ of a symplectic manifold (X, ω) is called a *Liouville vector field* if $\mathcal{L}_v\omega|_U = \omega|_U$.

Since ω is non-degenerate and closed, Liouville vector fields v are in bijective correspondence with primitives $\alpha_v = \iota_v\omega$ of ω. Furthermore, if v is transverse to a hyperplane Y, we see that $\alpha_v \wedge d\alpha_v = \iota_v\omega \wedge \omega$. Since $\omega \wedge \omega > 0$ and v is transverse to Y, $\alpha_v|_Y$ is a contact form on Y. This provides a methodology for finding contact 3-manifolds within a symplectic manifold.

For example, in the symplectization (SY, ω_α), $v = \frac{\partial}{\partial s}$is a Liouville vector field that is transverse to the hypersurfaces $\{s_0\} \times Y$. These hypersurfaces are then equipped with the contact form $\iota_v \omega_\alpha|_{\{s_0\}\times Y} = e^s \alpha$.

Definition 15. A hypersurface $Y \subset (X, \omega)$ is said to be of *contact-type* if Y is contained in an open set $U \subseteq X$ that admits a Liouville vector field transverse to Y.

When Y is compact, we may restrict the open subset U containing the contact-type hyperplane Y to a set symplectomorphic to $(-\epsilon, \epsilon) \times Y$. Here, the interval is parameterized using the flow of v. We can then symplectomorphically identify U with a subset of the symplectization of (Y, α).

Let (X, ω) be a symplectic manifold with connected boundary Y. Suppose that U is a neighbourhood of Y that admits a Liouville vector field v that is transverse to Y. Using the flow of v, we can symplectomorphically identify an open subset of U with either $(-\epsilon, 0] \times Y$ or $[0, \epsilon) \times Y$, depending on whether v is outward-pointing or inward-pointing along X.

Definition 16. A symplectic manifold (X, ω) is a *convex filling* of ∂X if there exists a Liouville vector field defined in a neighbourhood of ∂X that is outward-pointing along the boundary.

Definition 17. A symplectic manifold (X, ω) is a *concave filling* of ∂X if there exists a Liouville vector field defined in a neighbourhood of ∂X that is inward-pointing along the boundary.

In literature, convex fillings are often called *strong fillings*. This is contrasted with *weak fillings*.

Definition 18. A symplectic manifold (X, ω) is a *weak filling* of $(\partial X, \xi)$ if $\omega|_\xi > 0$.

A symplectic manifold may be a weak filling for multiple contact structures on ∂X (c.f. [11]). Throughout this thesis, fillings will be synonymous with either convex or concave fillings.

Example 5. Consider the unit sphere S^3 in $(\mathbb{R}^4, \omega_{std})$. The vector field

$$v = \frac{1}{2}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + w\frac{\partial}{\partial w}\right) \tag{2.5}$$

is a Liouville vector field for ω. Since this vector field is radially pointing outward, it follows that $(\mathbb{D}^4, \omega)$ is a convex filling of $(S^3, \iota_v\omega|_{S^3})$. This contact structure is called the *standard contact structure on* S^3, and is denoted by ξ_{std}.

One can also examine this contact structure by identifying $\mathbb{R}^4$ with the quaternions, and hence identifying S^3 with the group of unit-length quaternions. Note that the above contact form scales to

$$\alpha = x\ dy - y\ dx + z\ dw - w\ dz. \tag{2.6}$$

Let i,j, and k denote the left-invariant vector fields on S^3 that restrict in the obvious way on T_1S^3. Then i, j and k are given at $(x, y, z, w) \in S^3$ by

$$\begin{aligned} i &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - w\frac{\partial}{\partial z} + z\frac{\partial}{\partial w} \\ j &= -z\frac{\partial}{\partial x} + w\frac{\partial}{\partial y} + x\frac{\partial}{\partial z} - y\frac{\partial}{\partial w} \\ k &= -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w} \end{aligned} \tag{2.7}$$

We therefore have that ξ_{std} is spanned by j and k. On the other hand, i is the *Reeb vector field* of α: this is the vector field specified by the equations $i_v d\alpha = 0$ and $\alpha(v) = 1$.

Example 6. Let (X, ω) be a closed symplectic 4-manifold. Let $p \in X$. By Darboux's Theorem, there exists a neighbourhood of p that is symplectomorphic to $(\mathbb{D}^4_{1+\epsilon}, \omega)$, where ω is defined as in Example 5. Since there exists a Liouville vector field defined on $\mathbb{D}^4_{1+\epsilon}$, $X \backslash \mathbb{D}^4_1$ is a concave filling of the standard contact structure on S^3.

Example 7 (McDuff [43])**.** Let L be a complex line bundle over the symplectic surface (Σ, ω), and write $c_1(L) = \frac{c}{2\pi}\omega$ for some $c \in \mathbb{R}$. Let $\beta \in \Omega^1(P; i\mathbb{R})$ be a connection 1-form of a hermitian connection on the principal circle bundle associated to L, so that $\beta(\frac{\partial}{\partial\theta}) = i$. Set $\alpha = -i\beta$. One then has that $\alpha(\frac{\partial}{\partial\theta}) = 1$ and $d\alpha = -2\pi c_1(L) = -c\,\omega$. Moreover, one can extend α to a 1-form on L^*.

Define

$$\omega' = d\left((r^2 - \frac{1}{c})\alpha\right) = (1 - cr^2)\omega + 2rdr \wedge \alpha. \tag{2.8}$$

For small enough r, ω' is a symplectic form on L that induces the same orientation as ω, and it restricts to ω on the zero section. Moreover, for $c \neq 0$, the vector $v = \frac{1}{2r}(r^2 - \frac{1}{c})\frac{\partial}{\partial r}$ is a Liouville vector field. Note that if $c_1(L) > 0$ (resp. $c_1(L) < 0$), then v is inward pointing (resp. outward pointing), and so the circle bundle in L, defined using a small enough radius, is a contact-type hypersurface.

Using Theorem 4, one therefore has that if a symplectic surface $\Sigma \subset (X, \omega)$ has $[\Sigma]^2 < 0$ (resp. $[\Sigma]^2$), then it admits a convex (resp. concave) neighbourhood.

More generally, Gay and Stipsicz [24] have shown that a tubular neighbourhood of ω-orthogonally symplectic surfaces is convex if the intersection form of the neighbourhood is negative-definite.

Example 8. Let Σ be a Lagrangian surface in a symplectic 4-manifold. By Theorem 5, there exists a neighbourhood $\nu\Sigma$ of Σ that is symplectomorphic to the zero section of $T^*\Sigma$. Recall that the symplectic form on $T^*\Sigma$ is locally given by $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ where y_i are he coordinates in the direction dx_i, so that Σ is given by $y_1 = y_2 = 0$. A simple calculation reveals that $v = y_1 \frac{\partial}{\partial y_i}$ is a Liouville vector field for ω. Since v is radially outward-pointing, it follows that $\partial \nu \Sigma$ is a contact-type hyperplane in X with convex filling $\nu \Sigma$.

More generally, Etnyre has shown that a union of embedded Lagrangian surfaces also admits a convex neighbourhood so long as all intersections are transverse [14].

Example 9. A wealth of examples of convex fillings are *Stein fillings*. A *Stein surface* is a complex surface X that admits a strictly pluri-subharmonic function $\phi : X \to \mathbb{R}$. The triple

$$\big(X,\ \omega_\phi = -dJ^* d\phi,\ g_\phi = \omega_\phi(-, J-)\big) \tag{2.9}$$

is a Kähler manifold, and the gradient of ϕ with respect to g_ϕ is a Liouville vector field. Therefore, if c is a regular value of ϕ, $Y = \phi^{-1}(c)$ is a contact-type hypersurface. Moreover, since ϕ is pluri-subharmonic, $\phi^{-1}((\infty, c])$ is compact, and hence a convex filling of Y. These fillings are often called *Stein fillings*.

In practice, all Stein fillings can be built from the $\mathbb{D}^4$ filling of (S^3, ξ_{std}) (c.f. Example 5) by attaching 1-handles and 2-handles along Legendrian knots [10, 31]. A 2-handle attached along K must have framing $tb(K) - 1$. In particular, since any unknot in ξ_{std} can be perturbed to a Legendrian knot with any value of $tb(K) \leq -1$, one can construct Stein fillings by attaching along unknots with framing any value less than or equal to -2. Similarly, the right-handed trefoil can be perturbed to a Legendrian knot with any value of $tb(K) \leq 1$, and so one can attach 2-handles to this trefoil with framing any values less than or equal to 0.

Using the identification of neighbourhoods of the boundaries of fillings with neighbourhoods of symplectizations, one can perform *symplectic gluing*: one can glue together a convex filling and a concave filling of the same contact structure.

Theorem 10 (McCarthy and Wolfson [42])**.** *Let (X_1, ω_1) have convex boundary component (Y_1, ξ_1). Let (X_2, ω_2) have concave boundary component (Y_2, ξ_2) that is contactomorphic to (Y_1, ξ_1). The manifold X obtained by identifying Y_1 to Y_2 via the contactomorphism admits a symplectic structure ω. We may assume that $\omega|_{X_1} = \omega_1$.*

This method of constructing closed symplectic manifolds therefore leads to two obvious questions:

1. Which contact structures admit convex and concave fillings?
2. Can we classify convex and concave fillings of a given contact manifold?

It was proven by Etnyre and Honda that every contact structure admits infinitely many concave fillings (with b^+ arbitrarily large) [16].

On the other hand, contact structures that admit convex fillings are quite restricted. Eliashberg and Gromov have shown that convexly fillable contact structures are necessarily tight [12]. It therefore follows that the standard contact structure, which is the only tight contact structure on S^3, is the unique fillable contact structure on S^3 [8].

Moreover, the non-vanishing of *Giroux torsion* has been proven by Gay [23] to obstruct a contact structure admitting a convex filling.

Definition 19. Let ξ_n be the contact structure defined by $\cos 2\pi n z\ dx - \sin 2\pi n z\ dy$ on $T^2 \times I$ for $n \geq 1$ (here, z parameterizes the I direction). The *Giroux torsion* of (Y, ξ) is

$$Tor(Y,\xi) = \sup\left\{n\middle|\text{ there is a contactomorphic embedding from } (T \times I, \xi_n) \text{ into } (Y,\xi)\right\}. \tag{2.10}$$

By convention, if no such embedding exists, the Giroux torsion of (Y, ξ) is 0.

Due to these results, the term *fillable contact structure* is used often used to specify that a contact structure is *convexly fillable*.

Example 10. The contact structures used to define Giroux torsion are closely related to the tight contact structures on T^3. The tight contact structures on T^3 up to contactomorphism are the contact structures ζ_n, for $n \in \mathbb{N}$, given by $\cos 2\pi nz \; dx - \sin 2\pi nz \; dy$; here, z is parameterizing a circle of length 1 [27, 36]. It is easy to see that these are tight since they are all universally tight; they induce the standard contact structure on $\mathbb{R}^3$.

Giroux torsion defines a bijection between contactomorphism classes of tight contact structures on T^3 and $\mathbb{Z}_{\geq 0}$. The maximal k for which $(T \times I, \xi_k)$ contactomorphically embeds into (T^3, ζ_n) is $k = n - 1$, and so $Tor(T^3, \zeta_n) = n - 1$. Moreover, since non-trivial Giroux torsion obstructs the existence of a convex filling, the only contact structure that is possibly fillable is ζ_1. We have already seen a filling for this manifold; since the cotangent bundle of T^2 is trivial, Example 8 shows that $T^2 \times \mathbb{D}^2$ is a filling of a contact structure on T^3. Since the disks $pt \times \mathbb{D}^2$ lie in the Lagrangian fibers in T^*T, their boundaries are Legendrian curves in (T^3, ζ_1). One such choice is to map the boundaries to the circles parameterized by z. This is not the only choice, however. The contactomorphisms of (T^3, ζ_1) lie precisely in the class of automorphisms that stabilizes the image of $H_1(T^2)$ under the previous identification of $\partial(T^2 \times \mathbb{D}^2)$ with T^3 [13]. We can therefore choose a representative of any such class to identify the contact structures. Effectively, we can perform Luttinger surgery [5, 41] on the torus in $T^2 \times \mathbb{D}^2$ to obtain different identifications.

Chapter 3

Constructing symplectic manifolds

3.1 Rational blow-downs via symplectic gluing

Theorem 10 states that a convex filling of a contact manifold can be *symplectically glued* to a concave filling of the same contact manifold. It turns out that many symplectic cut-and-paste techniques can be described in this manner. As an example of this, we will provide a new proof that the rational blow-down process is symplectic. Rational blow-downs were first described by Fintushel and Stern in [19].

For $p \geq 2$, let C_p be the configuration of transverse spheres specified by Figure 3.1. The

Figure 3.1 The configuration C_p

spheres represent the homology classes $u_1, \ldots, u_{p-1} \in H_2(\nu C_p)$ with $u_i^2 = -2$ for $i \leq p-2$ and $i_{p-1}^2 = -p-2$. By performing the slam-dunk handlebody move along the chain of -2-circles, one sees that the boundary of νC_p is diffeomorphic to $L(p^2, 1-p)$. $L(p^2, 1-p)$ also bounds a rational homology ball B_p, which can be described as follows. Let $\mathbb{F}_{p+1}$ be the rational ruled surface whose negative section, s_-, has square $-(p+1)$. Let s_+ denote a positive section, and let f denote a fiber. The homology classes $[s_- + f]$ and $[s_+]$ can then be represented by spheres.Call this configuration A_p. The oriented boundary of νA_p is

$L(p^2, p-1)$, and so the complement is a rational ball with the same boundary as νC_p. Call this rational ball B_p.

Definition 20. Let X be a 4-manifold that contains C_p. The *rational blow-down* of X along C_p is the manifold X_p obtained by removing νC_p and gluing in B_p.

Since all diffeomorphisms of ∂B_p extend over B_p [19], it follows that X_p is well-defined. When the spheres are symplectic, this process can be done symplectically:

Theorem 11 (Symington [57])**.** *Let (X, ω) be a symplectic 4-manifold that contains C_p as a configuration of symplectic spheres that are perpendicular with respect to ω. The rational blow-down X_p admits a symplectic structure ω_p satisfying $\omega_p|_{X_p \setminus B_p} = \omega|_{X \setminus \nu C_p}$.*

Proof. We will present a new proof by seeing that νC_p and B_p are convex fillings for the contactomorphic contact structure on their boundaries. The theorem then follows from Theorem 10. Since νC_p is a negative-definite plumbing of symplectic manifolds, it admits a convex structure (c.f. Example 7).

We wish to realize $\nu C_p \cup \nu A_p$ as a closed symplectic manifold. Consider the configuration of a positive section s_+, a fiber f, and a negative section s_- in $\mathbb{F}_{p+1}$, as in Figure 3.2. Note that $\mathbb{F}_{p+1}$ splits into convex and concave fillings as neighbourhoods of the s_- and s_+

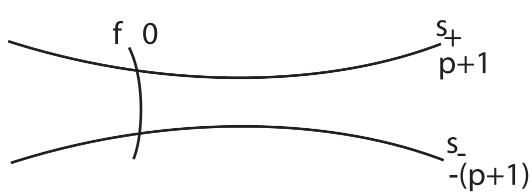

Figure 3.2 Symplectic curves in $\mathbb{F}_{p+1}$

respectively. Blow up $\mathbb{F}_{p+1}$ along $s_- \cap f$. Label their proper transforms again by s_- and f. The proper transform s_- has self-intersection $-(p+2)$, and f has self-intersection -1. The exceptional divisor e_1 intersects both s_- and f positively once, as depicted in Figure 3.3.

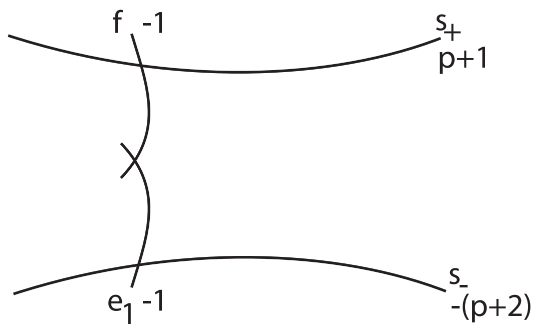

Figure 3.3 Symplectic curves in $\mathbb{F}_{p+1}\sharp\overline{\mathbb{CP}}^2$

We can now perform a series of $p-2$ blow ups at the intersection of f and the most recent exceptional divisor, to obtain the following intersection of curves in $\mathbb{F}_{p+1}\sharp(p-1)\overline{\mathbb{CP}}^2$.

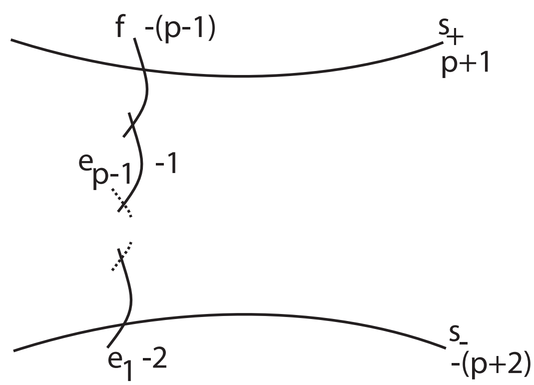

Figure 3.4 Symplectic curves in $\mathbb{F}_{p+1}\sharp(p-1)\overline{\mathbb{CP}}^2$

Note that a neighbourhood of $s_- \cup e_1 \cup \ldots \cup e_{p-2}$ is symplectomorphic to νC_p. Furthermore, removing νC_p cuts all regular fibers in half, and cuts e_{p-1} to split $\mathbb{F}_{p+1}\sharp(p-1)\overline{\mathbb{CP}}^2$ as $\nu C_p \cup \nu A_p$. Since νC_p admits a convex structure, νA_p is endowed with a concave structure. Moreover, this concave structure embeds into $\mathbb{F}_{p+1}$, and so B_p is endowed with a convex structure for the same contact structure as C_p. This completes the proof. □

3.2 The connect normal sum

3.2.1 The (absolute) connect normal sum

In [30], Gompf described a symplectic cut-and-paste technique called the *connect normal sum*. Suppose that X is a (possibly disconnected) 4-manifold containing disjoint closed surfaces $\Sigma_1, \Sigma_2 \subset X$ such that Σ_1 is diffeomorphic to Σ_2, and $[\Sigma_1]^2 + [\Sigma_2]^2 = 0$. An orientation-preserving diffeomorphism $\phi : \Sigma_1 \to \Sigma_2$ then lifts to an orientation-reversing diffeomorphism $\Phi : \partial\nu\Sigma_1 \to \partial\nu\Sigma_2$ that is orientation-reversing on each fiber. The choices of such lifts are affinely indexed by $H^1(\Sigma_1)$.

Definition 21. Let X, Σ_1, Σ_2 be as above. The *connect normal sum* of X along $\Sigma = \Sigma_1 \amalg \Sigma_2$ is the manifold $\sharp_\Sigma X$ defined by $(X_1 \backslash \nu\Sigma_1) \cup_\Phi (X_2 \backslash \nu\Sigma_2)$.

The connect normal sum is a symplectic construction.

Theorem 12 (Gompf [30])**.** *Let (X, ω) be a symplectic manifold that contains disjoint closed symplectomorphic surfaces $\Sigma_1, \Sigma_2 \subset X$ such that $[\Sigma_1]^2 + [\Sigma_2]^2 = 0$. Then $\sharp_\Sigma X$ admits a symplectic structure.*

McCarthy and Wolfson first realized that most cases of the connect normal sum are a special case of symplectic gluing [42]. Suppose that Σ_i lie in separate components and $[\Sigma_1]^2 < 0$. Then, the complements of $\nu\Sigma_1$ and $\nu\Sigma_2$ admit concave and convex structures respectively (c.f. Example 7). The connect normal sum along $\Sigma_1 \amalg \Sigma_2$ can be obtained by symplectically gluing the complements together.

The more important case for the intent of this thesis is the situation when $[\Sigma_i]^2 = 0$; this will serve as the blueprint for the proof of the *relative connect normal sum* (c.f. Section 3.2.2). In this case, the above proof is no longer applicable, as the neighbourhoods of Σ_i

are neither convex nor concave. However, the punctured neighbourhood $\Sigma_1 \times \mathbb{D}^*$ admits a symplectomorphism onto itself that reverses the orientation of the boundary of the fibers. Using polar coordinates on $\mathbb{D}$, define $\phi : \Sigma_1 \times \mathbb{D}^*_\epsilon \to \Sigma_2 \times \mathbb{D}^*_\epsilon$ by mapping (p, r, θ) to $(p, \sqrt{\epsilon^2 - r^2}, -\theta)$. We can then symplectically define the connect sum by identifying the punctured neighbourhoods of Σ_1 and Σ_2 using ϕ:

$$\sharp_\Sigma X = (X \backslash (\Sigma_1 \amalg \Sigma_2)) \big/ \phi(p, r, \theta) \sim (p, r, \theta).$$

The symplectic form is well-defined since we are gluing together X using a symplectomorphism defined on an open region.

3.2.2 The relative connect normal sum

If X is a (possibly disconnected) manifold with boundary, the topological construction for the connect normal sum along properly embedded surfaces with boundary continues to makes sense; we still define $\sharp_\Sigma X$ to be $(X \backslash \nu\Sigma_1) \cup_\Phi (X_2 \backslash \nu\Sigma_2)$. This construction depends upon a chosen orientation-reversing identification of $\partial\nu\Sigma_1$ and $\partial\nu\Sigma_2$. This identification is affinely equivalent to a choice of trivialization of $\nu\Sigma_1$, which is affinely indexed by $H^1(\Sigma_1)$. Note that all choices of trivialization for the normal bundle of a surface with boundary induce the same trivialization of the normal bundle of the boundary of the surface. This can be seen by examining the relative cohomology long exact sequences:

$$H^1(\Sigma_1, \partial\Sigma_1) \xrightarrow{\cong} H^1(\Sigma_1) \xrightarrow{0} H^1(\partial\Sigma_1). \tag{3.1}$$

In particular, the boundary of $\sharp_\Sigma X$ is independent of the choices of trivializations.

The next theorem establishes a convex (resp. concave) symplectic structure on $\sharp_\Sigma X$ when X is a convex (resp. concave) symplectic manifold.

Theorem 13. *Let X be a (possible disconnected) convex (resp. concave) symplectic manifold with boundary. Let Σ_1 and Σ_2 be disjoint properly embedded symplectomorphic surfaces with boundary in X. Then $\sharp_\Sigma X$ admits a convex (resp. concave) symplectic structure.*

Proof. For simplicity, assume that X is convex. The proof for when X is concave simply requires one to adjust notation. We first wish to construct a sufficiently nice neighbourhood of a properly embedded symplectic surface Σ. Split $TX|_{\nu\Sigma} = T\Sigma \oplus N\Sigma$, where $N\Sigma$ is the normal bundle of Σ that is defined using ω. Recall that ∂X admits a neighbourhood that is symplectomorphic to a neighbourhood of the symplectization of ∂X. Symplectically attach $[0,\infty) \times \partial X$. In this enlarged neighbourhood, we can find a graph of ∂X such that $N\Sigma|_{\nu\partial\Sigma}$ lies parallel to the graph. Cut along this graph to form a new boundary (with the same induced contact structure), so that the symplectic tubular neighbourhood embeds into X. We can then express $\omega|_{\nu\partial\Sigma} = \omega_\Sigma + 2rdr \wedge d\theta$, where $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$ span $N\Sigma|_{\nu\partial\Sigma}$. Using Moser's trick, we can then guarantee that $\nu\Sigma \cong (\Sigma \times \mathbb{D}^2, \omega_\Sigma + 2rdr \wedge d\theta)$.

In $\nu\partial X$, ω admits a primitive $\alpha = \alpha_\Sigma + r^2 d\theta$, where α_Σ is a local primitive for ω_Σ. The corresponding, outward-pointing, Liouville vector field is $v = v_\Sigma + \frac{r}{2}\frac{\partial}{\partial r}$.

Returning to the relative connect normal sum construction, we wish to perform the above procedure to both Σ_1 and Σ_2. Topologically, redefine $\sharp_\Sigma X$ as

$$\big(X \backslash (\nu\Sigma_1^{[0,\frac{\epsilon}{2}]} \amalg \nu\Sigma_1^{[0,\frac{\epsilon}{2}]})\big)/\Phi(p,r,\theta) \sim (p,r,\theta) \tag{3.2}$$

Here $\nu\Sigma_i^I$ denotes the region of the tubular neighbourhood with radius in I. The identifying map Φ is given by

$$\begin{aligned} \Phi : \nu\Sigma_1^{(\frac{\epsilon}{2},\frac{\sqrt{3}\epsilon}{2})} &\longrightarrow \nu\Sigma_2^{(\frac{\epsilon}{2},\frac{\sqrt{3}\epsilon}{2})} \\ (p,r,\theta) &\longmapsto (p,\sqrt{\epsilon^2-r^2},-\theta) \end{aligned} \tag{3.3}$$

Problematically, $\Phi^*\omega \neq \omega$, and so ω does not immediately extend to $\sharp_\Sigma X$. Instead, we will alter ω in the punctured neighbourhood of Σ_i as follows. Choose $f : (\frac{\epsilon}{2}, 2\epsilon) \to (\frac{-3\epsilon^2}{4}, 4\epsilon^2)$ satisfying:

1. $f'(r) > 0$,
2. $f(r) = r^2$ on $(\epsilon, 2\epsilon)$, and
3. $f(r) = r^2 - \frac{\epsilon^2}{2}$ on $(\frac{\epsilon}{2}, \frac{\sqrt{3}\epsilon}{2})$.

Define ω' on $\nu\Sigma_i^{(\frac{\epsilon}{2},2\epsilon)}$ by $\omega' = \omega + f'(r)dr\wedge d\theta$. Similarly, on $\nu\Sigma^{(\frac{\epsilon}{2},2\epsilon)} \cap \nu\partial X$, define a primitive $\alpha' = \alpha_\Sigma + f(r)d\theta$ for ω'. The corresponding Liouville vector field, $v' = v_\Sigma + \frac{f(r)}{f'(r)}\frac{\partial}{\partial r}$ remains outward-pointing.

Using Property 2, we see that ω' agrees with ω on $\nu\Sigma_i^{(\epsilon,2\epsilon)}$. In particular, we can extend ω' to the rest of X using ω. Similarly, α' extends to a primitive of ω' everywhere that α is defined. By Property 1, we see that ω' is symplectic. Using Property 3, $\Phi^*\omega' = \omega'$ and $\Phi^*\beta' = \beta'$. We therefore immediately have that ω' defines a symplectic form on $\sharp_\Sigma X$, and β' is a primitive of ω' near $\partial(\sharp_\Sigma X)$. Moreover, $\Phi_*^{-1}v' = v'$, so v' glues together to show that $\sharp_\Sigma X$ is a convex filling of its boundary. $\square$

This construction should be compared to a construction of Geiges [25], which describes how to glue contact structures together along transverse knots that are equipped with an arbitrary framing. This is precisely the situation that is occurring along the boundary (where the framing is induced by trivializing $\nu\Sigma_i$).

Using Geiges' construction, one can begin with any surgery diagram, and build a contact manifold by perturbing all knots to be transverse, and gluing together the complement of the transverse link in (S^3, ξ_{std}) with the complements of unknots in (S^3, ξ_{std}). We can therefore build 3-manifolds that do not admit tight contact structures, such as $-\Sigma(2,3,5)$ [15], using fillable structures. We therefore see that gluing together tight (or fillable) structures does not necessarily result in a tight structure.

Contrasting this, if we perform the relative connect normal sum to glue together convex fillings, the result is again a convex filling, and so the induced contact structure on the boundary is tight.

When performing the relative connect normal sum with concave fillings, the situation is not so clear cut. While there are currently no examples of constructing concave fillings of over-twisted contact structures from concave fillings of tight contact structures, it is not clear that this cannot happen. Moreover, as we will see in Chapter 4, one can perform the relative connect normal sum of concave fillings of (S^3, ξ_{std}) and construct concave fillings of tight contact structures that are not convexly fillable (the resulting contact structures have non-trivial Giroux torsion).

Nevertheless, in certain cases, we understand the contact structures on the constructed manifolds well enough to guarantee that resulting contact structure is fillable. We will see this in practice when examining the k-fold sum in the following section.

3.3 The k-fold sum

A generalization of the symplectic connect normal sum, called the generalized connect normal sum, was first proposed by Symington [58, 59].

Definition 22. Let $\mathcal{C}$ be a collection of intersecting immersed symplectic surfaces in a possibly disconnected symplectic 4-manifold (X, ω). Let $\widehat{X}$ be the symplectic manifold with boundary that is associated to $X \backslash \mathcal{C}$. Assume that any intersections amongst surfaces in $\mathcal{C}$ are ω-orthogonal. A closed symplectic manifold $(\widetilde{X}, \widetilde{\omega})$ is a *generalized symplectic sum of X along $\mathcal{C}$* if there exists a symplectic embedding $\phi : X \backslash \mathcal{C} \to \widetilde{X}$ which extends to a surjective symplectic immersion $\widehat{\phi} : \widehat{X} \to \widetilde{X}$.

In [58, 59] and [45], Symington, and McDuff and Symington, provided criteria for constructing certain generalized connect normal sums, called 3- and 4-fold sums respectively.

Theorem 14. *Let $\{S_i, T_i\}_{i=1}^{3}$ be a collection of surfaces such that S_i and T_i are disjoint from both S_j and T_j for $i \neq j$, and S_i intersects T_i ω-orthogonally once. Assume that $[T_i]^2 + [S_{i+1}]^2 = -1$ for each i, and that T_i is symplectomorphic to S_{i+1}. The result of identifying a punctured neighbourhood of T_i with a punctured neighbourhood of S_{i+1} is a generalized symplectic sum.*

Theorem 15. *Let $\{S_i, T_i\}_{i=1}^{4}$ be a collection of surfaces such that S_i and T_i are disjoint from both S_j and T_j for $i \neq j$, and S_i intersects T_i ω-orthogonally once. Assume that $[T_i]^2 + [S_{i+1}]^2 = 0$ for each i, and that T_i is symplectomorphic to S_{i+1}. The result of identifying a punctured neighbourhood of T_i with a punctured neighbourhood of S_{i+1} is a generalized symplectic sum.*

To understand these theorems, we will consider a generalization where we allow for arbitrary fixed k, and we remove any requirement on $[T_i]^2 + [S_{i+1}]^2$. To that end, we will make the following definition (which redefines 3- and 4-fold sums).

Definition 23. Let $\{S_i, T_i\}_{i=1}^{k}$ be a collection of closed surfaces in a (possibly disconnected) closed symplectic manifold (X, ω) such that S_i and T_i are disjoint from both S_j and T_j for

$i \neq j$, and S_i intersects T_i ω-orthogonally once. Assume T_i is symplectomorphic to S_{i+1}. The manifold with boundary that is obtained by removing neighbourhoods of the k intersection points, and identifying a punctured neighbourhood of T_i with a punctured neighbourhood of S_{i+1} is a *k-fold sum.*

We will first understand the underlying topological construction of the k-fold sum (see also [21]). Doing so allows us to see that the boundaries are T^2 bundles over S^1. We therefore understand the boundary once we understand the monodromy of the boundary.

Moreover, as we will see in Section 3.3.2, we can interpret this construction as providing a concave filling. When the surfaces satisfy the hypotheses of Theorems 14 or 15, the boundary is T^3, equipped with the unique fillable contact structure. We can then symplectically glue in $T^2 \times \mathbb{D}^2$, to reobtain the conclusions of these theorems (up to deformation equivalence). We will adopt the convention that the boundaries of the disks $\{pt\} \times \mathbb{D}^2 \subset T^2 \times \mathbb{D}^2$ are identified with the Legendrian foliation of the boundary constructed by closing the intervals in $T^2 \times I$ to circles using the trivial monodromy. Note that this convention is not uniquely specified (c.f. Example 10).

In Chapter 4, we will make use of the k-fold sum, and glue in convex fillings of other boundary manifolds. The constructed manifolds should satisfy the definition of a generalized symplectic sum, up to deformation equivalence, but more general constructions involving the k-fold sum should not.

3.3.1 Topology of the k-fold sum

Consider a collection of closed surfaces $\{S_i, T_i\}_{i=1}^k$ in a (possibly disconnected) 4-manifold X such that S_i and T_i are disjoint from both S_j and T_j for $i \neq j$, and S_i intersects T_i

transversely at a single point p_i. Orient S_i and T_i so that the intersection point is positive. Moreover, assume that T_i is diffeomorphic to S_{i+1}. We will denote the self-intersection of S_i as m_i, and the self-intersection of T_i as n_i.

At each point p_i, choose a neighbourhood νp_i that intersects both S_i and T_i in disks D_{S_i} and D_{T_i} respectively. Call $\partial\nu p_i$ the sphere S_i^3. Remove these balls to get a manifold with boundary $\amalg_k S_i^3$. Label $S_i^0 = S_i \backslash D_{S_i}$ and $T_i^0 = T_i \backslash D_{T_i}$. Since S_i and T_i intersect positively in X_i, the boundaries of S_i^0 and T_i^0 intersect $\partial\nu p_i$ as a positive Hopf link (orient the components of the link as the oriented boundaries of the disks D_{S_i} and D_{T_i} in νp_i). We then form Z by perform the relative connect normal sum k times along T_i^O and S_{i+1}^O. Topologically, choose tubular neighbourhoods νS_i^0 of S_i^0 and νT_i^0 of T_i^0 that are small enough so that they intersect $\partial\nu p_i$ in disjoint solid tori. Remove these neighbourhoods. We now form Z by identifying each $\partial\nu T_i^0$ to $\partial\nu S_{i+1}^0$ by using a lift of an orientation-preserving diffeomorphism from T_i^0 to S_{i+1}^0 that is orientation-reversing on the fiber circle.

The boundary of Z consists of a union of pieces $S_i^3 \backslash \left(\partial S_i^0 \times \mathbb{D}^2 \amalg \partial T_i^0 \times \mathbb{D}^2\right)$. Since each piece is the complement of a thickened Hopf link, it is diffeomorphic to $T^2 \times I$. Moreover, the identifications of $\partial\nu T_i^0$ with $\partial\nu S_{i+1}^0$ glue boundary tori together, and so ∂Z is a torus bundle over S^1. To understand the topology of ∂Z, it therefore suffices to understand the monodromy of this fibration.

We will compute the monodromy that specifies $-\partial Z$ as an oriented manifold, since we ultimately wish to view Z as a concave filling.

To compute the monodromy, it suffices to compute its action on the first homology of the fiber. Consider the ordered basis for the first homology of each fiber given by $\langle \sigma, \tau \rangle$ where

$$\sigma_i = [\partial D_{T_i}] = [\text{fiber of } \partial \nu S_i^O \text{ over } \partial S_i] \quad \text{and} \quad \tau_i = [\partial D_{S_i}] = [\text{fiber of } \partial \nu T_i^O \text{ over } \partial T_i]. \tag{3.4}$$

We first wish to compute the 'local monodromy', meaning the induced map from the homology of a fiber in S_i^3 to that of a fiber in S_{i+1}^3 with respect to the above basis on both fibers. The action of the total monodromy on $H_1(T)$ is then a composition of k of these maps. Under this convention, the monodromy ϕ identifies $-\partial Z$ as $T^2 \times I$ under the identification $(x, 1) \sim (\phi(x), 0)$.

We will express the local monodromy as a composition of three maps:

1. Push the torus fiber in S_i^3 to $\partial T_i^0 \times S^1$, and express the basis $\langle \sigma_i, \tau_i \rangle$ in terms of $\langle [\partial T_i^0], [S^1] \rangle$.
2. Apply the gluing of $T_i^0 \times S^1$ (using homology basis $\langle [\partial T_i^0], [S^1] \rangle$) to $S_{i+1}^0 \times S^1$ (using homology basis $\langle [S^1], [\partial S_{i+1}^0] \rangle$ that preserves the boundary of the surface, and is orientation-reversing on the fiber.
3. Push $\partial S_{i+1}^O \times S^1$ to a torus fiber in S_{i+1}^3, expressing the basis $\langle [S^1], [\partial S_{i+1}^0] \rangle$ in terms of $\langle \sigma_{i+1}, \tau_{i+1} \rangle$.

The first map is the clutching map that identifies trivial circle bundles over D_{T_i} and T_i^0 to obtain a bundle over T_i with euler class m_i; this map is

$$\begin{bmatrix} -1 & 0 \\ n_i & 1 \end{bmatrix}. \tag{3.5}$$

The second map is the fiber-reversing gluing, and so it is given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.6}$$

The last map is the same as the inverse of the first, composed with a transposition matrix due to the change of ordering of the basis. It is therefore given by

$$\begin{bmatrix} 1 & m_{i+1} \\ 0 & 1 \end{bmatrix}. \tag{3.7}$$

The local monodromy is therefore given by

$$\phi_{n_i+m_{i+1}} = \begin{bmatrix} n_i + m_{i+1} & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.8}$$

For the remainder of this subsection, we will only consider collections of configurations where $N = n_i + m_{i+1}$ is equal for all i. In this case, the monodromy of $-\partial Z$ is ϕ_N^k.

The boundary is T^3 if and only if ϕ_N^k is the identity. In this case, we can attach $T^2 \times D$ to obtain a closed manifold. For this to occur, the eigenvalues of ϕ_N must be k^{th} roots of unity. This occurs precisely when N is $-2, -1, 0, 1,$ or 2. The matrices ϕ_{-2} and ϕ_2 have infinite order. The matrices ϕ_{-1}, ϕ_0, and ϕ_1 have orders 3, 4, and 6 respectively. We have therefore shown the following.

Proposition 1. *Let Z be as above. Then $-\partial Z$ is a 3-torus precisely when:*

1. *$N = -1$, and k is a multiple of 3,*
2. *$N = 0$, and k is a multiple of 4, and*

3. $N = 1$*, and* k *is a multiple of* 6*.*

More generally, one can consider other values for N and k that result in other tractable 3-manifolds. The next proposition uses the convention that $Y(e_0; r_1, \ldots, r_l)$ is the Seifert fibered space given by the following Kirby diagram.

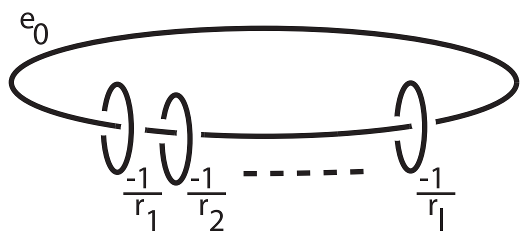

Figure 3.5 The Seifert fibered manifold $Y(e_0; r_1, \ldots, r_l)$

Proposition 2. *Let* Z *be as above. For certain values of* N *and* k*,* $-\partial Z$ *is given in the following table:*

N	k	$-\partial Z$
2	k	euler class $-k$ bundle over T^2
-2	$2l$	euler class $2l$ bundle over T^2
1	$6l$	T^3
1	$6l+1$	$Y(0; \frac{1}{2}, \frac{-1}{3}, \frac{-1}{6})$
1	$6l+2$	$Y(0; \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3})$
1	$6l+3$	$Y(0; \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$
1	$6l+4$	$Y(0; \frac{-2}{3}, \frac{1}{3}, \frac{1}{3})$
1	$6l+5$	$Y(0; \frac{-1}{2}, \frac{1}{3}, \frac{1}{6})$

N	k	$-\partial Z$
-1	$3l$	T^3
-1	$3l+1$	$Y(0; \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3})$
-1	$3l+2$	$Y(0; \frac{-2}{3}, \frac{1}{3}, \frac{1}{3})$
0	$4l$	T^3
0	$4l+1$	$Y(0; \frac{1}{2}, \frac{-1}{4}, \frac{-1}{4})$
0	$4l+2$	$Y(0; \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$
0	$4l+3$	$Y(0; \frac{-1}{2}, \frac{1}{4}, \frac{1}{4})$

Table 3.1 Boundaries of k-fold sums

Proof. When the monodromy has finite order, one has a foliation of $-\partial Z$ into circles, and so the boundary is a Seifert fibered manifold. This is precisely the cases when $|N| \leq 1$. The Seifert invariants of these manifolds are computed as in [46].

When the monodromy has trace 2, the monodromy is conjugate to a matrix of the form

$$\begin{bmatrix} 1 & e \\ 0 & 1 \end{bmatrix}. \tag{3.9}$$

When written in this manner, we have a preferred factoring of the torus fibers into two circles. Since the monodromy acts trivially on the first factor, we can recognize the manifold as a circle bundle over T^2 with euler class e.

The monodromy ϕ_N^k has trace 2 precisely when $N = 2$, in which case the euler class is $-k$, or when $N = -2$ and k is even, in which case the euler class is k. □

3.3.2 Contact geometry and the k-fold sum

Suppose that the X is symplectic, and that the collection of surfaces $\{S_i, T_i\}_{i=1}^k$ are symplectic such that T_i is symplectomorphic to S_{i+1}. By removing the convex neighbourhoods νp_i, we obtain a concave filling of $\amalg_k\ S_i^3$. Moreover, each pair of surfaces $\{T_i^O, S_{i+1}^O\}$ satisfies the conditions of Theorem 13. By performing the relative connect normal sum with each pair, we therefore realize Z as a concave filling of its boundary.

As mentioned in section 3.2.2, without understanding the contact structure on $-\partial Z$, it is unclear when we can extend Z to a closed symplectic manifold. However, for boundaries of k-fold sums, identifying the contact structure on the boundary is tractable. We will show the following:

Lemma 1. *Let (Z, ω) be a k-fold sum. The induced contact structure on $-\partial Z$ is universally tight.*

Lemma 2. *Let (Z, ω) be a k-fold sum that is constructed by gluing along surfaces T_i and*

S_{i+1} *such that* $[T_i]^2 + [S_{i+1}]^2 = N$*. The Giroux torsion of* $(-\partial Z, \xi)$ *is given by*

$$Tor(-\partial Z) = \begin{cases} 0 & \text{for } N \geq 2 \\ \lfloor \frac{k-1}{6} \rfloor & \text{for } N = 1 \\ \lfloor \frac{k-1}{4} \rfloor & \text{for } N = 0 \\ \lfloor \frac{k-1}{3} \rfloor & \text{for } N = -1 \\ \lfloor \frac{k}{2} \rfloor & \text{for } N \leq -2 \end{cases} \tag{3.10}$$

Using these lemmas together with the classification of contact structures on torus bundles over the circle [35], we can identify the contact structure (see also [28], which classifies universally contact structures on these manifolds).

At this point, when $Tor(\partial Z) = 0$ we can hope to extend Z to a closed symplectic manifold by symplectically gluing Z to a convex filling of the specified contact structure.

In particular, we have the following theorem.

Theorem 16. *Let* $\{S_i, T_i\}_{i=1}^k$ *be a collection of closed surfaces in the closed, possibly disconnected, symplectic manifold* (X, ω) *such that* S_i *and* T_i *are disjoint from both* S_j *and* T_j *for* $i \neq j$*, and* S_i *intersects* T_i ω*-orthogonally once. Assume that* T_i *is symplectomorphic to* S_{i+1}*, and let* $N = T_i^2 + S_{i+1}^2$*. Moreover, assume that* k *and* N *satisfy one of the following:*

1. $k = 3,\ N = -1$
2. $k = 4,\ N = 0$
3. $k = 6,\ N = 1$.

The k*-fold sum taken along these surfaces extends to a closed symplectic manifold.*

Case 1 is equivalent to Theorem 14, up to deformation equivalence. Case 2 is equivalent to Theorem 15, up to deformation equivalence. Case 3 is new.

Proof. Using Proposition 1, we have that the boundary is diffeomorphic T^3. Moreover, using the previous two lemmas, we see that the induced contact structure is the unique fillable contact structure on T^3 (c.f. [11] or [35]), and so we can glue in the convex filling $T^2 \times \mathbb{D}^2$ to obtain closed symplectic manifolds. □

It now remains to prove Lemmas 1 and 2.

Proof of Lemma 1. Note that the torus fibers, considered in S^3, can be expressed as

$$T_\eta = \frac{1}{\sqrt{1+\eta^2}}(\eta \cos t, \eta \sin t, \cos s, \sin s) \tag{3.11}$$

for $s, t \in \mathbb{R}/2\pi$ and $\eta \in (0, \infty)$. The tangent space to T_η, expressed using the quaternions (c.f. Example 5)

$$T_{(s,t)}T_\eta = \langle i, \sin(s+t)j - \cos(s+t)k \rangle. \tag{3.12}$$

In particular, these tori are foliated by the Reeb orbits. Examining the construction of the contact form obtained by identifying a neighbourhood of $\partial T_i^O = \{(\cos(t), \sin(t), 0, 0)\}$ with $\partial S_{i+1}^O = \{(0, 0, \cos(s), \sin(s))\}$, we see that this continues to be the case for $-\partial Z$. In particular, all Reeb orbits on $-\partial Z$ are homotopically non-trivial. Since any contact form associated to an overtwisted contact structure necessarily admits a homotopically trivial Reeb orbit (proven by Hofer [34]), the constructed contact structures are all tight.

Moreover, note that all contact structures formed on the boundary of k-fold sums will pullback to contactomorphic contact structures on $\mathbb{R} \times T^2$, and hence to their universal cover $\mathbb{R}^3$. Tight contact structures on T^3 are known to be universally tight (proven independently by Giroux [28] and Kanda [36]). It therefore follows that all constructed contact structures are universally tight. □

Proof of Lemma 2. It follows from the classification of universally tight contact structures that it suffices to compute the Giroux torsion in a neighbourhood of the fiber. To compute the Giroux torsion, we wish to find a maximal neighbourhood of the fiber that is contactomorphic to ξ_n as in Definition 19. To this end, it suffices to find a maximal region $I \times T$ such that each curve in the I direction is Legendrian, and each torus is foliated by Legendrian curves.

To do this, consider the embedding of $(0, \infty) \times T$ into S^3 given by

$$\frac{1}{\sqrt{1+\eta^2}}(\eta \cos t, \eta \sin t, \cos s, \sin s) \tag{3.13}$$

so that the η parameterizes the fibers. Moreover, the curves parameterized by η (fixing s and t) are tangent to $\cos(s+t)j + \sin(s+t)k$, and are therefore Legendrian. Moreover, as we have seen in the proof of Lemma 1, the tori are foliated by curves tangent to $\sin(s+t)j - \cos(s+t)k$, which are again Legendrian.

Fixate on the curve specified by $s = t = 0$. The tangent space of T_η at this curve is spanned by

$$\left\langle v_1 = \frac{\eta}{\sqrt{1+\eta^2}}\frac{\partial}{\partial y}, v_2 \frac{1}{\sqrt{1+\eta^2}}\frac{\partial}{\partial w} \right\rangle. \tag{3.14}$$

We choose this normalization of the vectors so that the canonical framing of η, constructed by taking the tangent vectors to a fixed circle in each T_η, will have coefficients independent of η. Note that for each fiber, v_1 is tangent to a circle representing σ, and v_2 is tangent to a circle representing τ.

In this basis, the Legendrian framing, which is tangent to the Legendrian foliation of each T_η, is given by $v_1 - \eta v_2$. We therefore see that as η traverses from 0 to ∞, the Legendrian sweeps that fourth quadrant of the (v_1, v_2)-plane from v_1 to $-v_2$. Define the canonical framing using the circle parameterized by t. The canonical framing is therefore given by v_2.

When identifying a punctured neighbourhood of ∂T_i^O to a punctured neighbourhood of ∂S_i^O, we may see the change in framing by seeing the image of v_1 and v_2 under the local monodromy

$$\phi_N = \begin{bmatrix} N & -1 \\ 1 & 0 \end{bmatrix}. \tag{3.15}$$

Note that the Legendrian framing is reset to v_1, while the canonical framing is sent to $\phi_M v_1$ (and subsequently $\phi_N^i v_1$). To determine the maximal n such that $(I \times T, \zeta_n)$, it therefore suffices to determine how many times $\phi_n^i v_1$ enters the fourth quadrant of the (v_1, v_2) plane, as it moves from v_1 to ϕ_N^k.

The cases when N is -1, 0, or 1 are straightforward, since ϕ_N has finite order. For instance, when $N = -1$, the canonical framing cyclically jumps from v_1 to $-v_1 + v_2$ to v_2, and we see that in this case the Giroux torsion is $\lfloor \frac{k-1}{3} \rfloor$. The other finite order cases are similar.

For the remaining cases, note that ϕ_N^i is of the form

$$\begin{bmatrix} \psi_i & -\psi_{i-1} \\ \psi_{i-1} & -\psi_{i-2} \end{bmatrix} \tag{3.16}$$

where $\psi_{-1} = 0$, $\psi_0 = 1$, and $\psi_1 = N$. We can see this inductively using the fact that ϕ_N^i will commute with ϕ_N. Moreover, ψ_i satisfies the recurrence relation $\psi_i = N \cdot \psi_{i-1} + \psi_{i-2}$.

A simple inductive argument shows that $\psi_i \geq \psi_{i-1}$ for $N \geq 2$. We therefore have that the vector $\phi_N^i v_1 = \binom{\psi_N}{\psi_{N-1}}$ lies in the first quadrant for all i. The canonical framing therefore never passes the Legendrian framing, and so the Giroux torsion is 0.

Using the same recurrence relation, another inductive argument shows that

$(-1)^i\psi_i \geq (-1)^{i-1}\psi_{i-1}$ for $N \leq -2$. This implies that $\psi_i < 0$ when i is odd, and $\psi_i > 0$ when i is even, and so $\phi_N^i v_1$ lies in the second quadrant when i is odd, and it lies in the fourth quadrant when i is even. We therefore have that the Giroux torsion is $\lfloor \frac{k}{2} \rfloor$. $\square$

Chapter 4

Manifolds violating the Noether inequality

As an example of the efficacy of the k-fold sum, we will construct a collection of minimal symplectic manifolds with $c_1^2 > 0$ that do not satisfy the Noether inequality.

A standard method of organizing questions within 4-dimensional topology is via "geography problems", which asks what values of $(c_2 = \chi, c_1^2 = 3\sigma + 2\chi) \in \mathbb{Z}^2$ are realizable by 4-manifolds satisfying some criterion. It is a classic result that minimal simply-connected Kähler manifolds must either have $c_1^2 = 0$ and $c_2 \geq 3$ (consisting of rational surfaces, ruled surfaces, $K3$ surfaces, and elliptic surfaces), or must satisfy $c_1^2 > 0$, the Bogomolov-Miyaoko-Yau inequality $3c_2 \geq c_1^2$, and the Noether inequality $5c_1^2(X) - c_2(X) + 36 \geq 0$.

The existence of symplectic manifolds not satisfying the inequality therefore demonstrates a difference between the Kähler and symplectic categories.

Examples of minimal symplectic manifolds that do not satisfy the Noether inequality exist in literature. In fact, using the rational blow-down technique, Fintushel and Stern [19] have proven that there exists minimal symplectic manifolds covering all integral points satisfying $c_1^2 + c_2 \equiv 0 \bmod 12$ within the region between the Noether line and the line $c_2 = 0$. Additional constructions have been provided by Gompf [30], and by Stipsicz [54] when c_1^2 is even. It is not known if any of these constructions, nor the one presented below, provide diffeomorphic manifolds.

Liu [40] has shown that simply-connected minimal symplectic manifolds must satisfy $c_1^2 \geq 0$. It is currently unknown whether minimal symplectic manifolds must satisfy the Bogomolov-Miyaoko-Yau inequality.

Theorem 17. *There exist minimal symplectic manifolds homeomorphic to* $(1+2n)\mathbb{CP}^2\sharp(9+9n)\overline{\mathbb{CP}}^2$ *for* $1 \leq n \leq 6$.

The basic building block for these manifolds is constructed by considering $\Sigma_1 = 3h - \sum_{i=1}^{9} e_i$ and $\Sigma_2 = 3h - \sum_{i=1}^{8} e_i$ in $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}}^2$ so that $[\Sigma_1]^2 = 0$, $[\Sigma_2]^2 = 1$, and $[\Sigma_1]\cdot[\Sigma_2] = 1$. We can arrange Σ_1 and Σ_2 so that they are represented by symplectomorphic tori that intersect transversely in a single point. Let $\overline{X}_k$ be k-fold sum along k copies of this configuration. Following section 3.3.1, the monodromy defining $-\partial\overline{X}_k$ is

$$\phi_k = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}^k.$$

Note that ϕ_1 has order six., and so we are constructing sequences of concave fillings for six different topological manifolds.

In section 4.1, we will construct convex fillings C_k for each of these six boundary manifolds, as well as the two remaining manifolds described in Proposition 2($Y(0; \frac{1}{2}, \frac{-1}{4}, \frac{-1}{4})$ and $Y(0; \frac{-1}{2}, \frac{1}{4}, \frac{1}{4})$). These will be fillings for the unique universally tight, Giroux torsion 0 contact structure on these manifolds. Following section 3.3.2, we can guaranteed that $\widetilde{X}_k = C_k \cup \overline{X}_k$ is a closed symplectic manifold for $k \leq 6$.

In section 4.2, we will compute $\chi(\widetilde{X}_k)$, $\sigma(\widetilde{X}_k)$, and $\pi_1(\widetilde{X}_k)$, as well as show that $\widetilde{X}_k$ is odd. This will show that $\widetilde{X}_k$ is homeomorphic to the manifolds listed in Theorem 17. Finally, in section 4.3, we will examine the potential Seiberg-Witten basic classes of $\widetilde{X}_k$. While the Seiberg-Witten invariant is not completely computed, we can still verify that $\widetilde{X}_k$ is minimal, completing the proof of Theorem 17.

Numerical data for these manifolds is provided in table 4.1. Note that Theorem 2 follows as an immediate corollary.

k	$\sigma(\widetilde{X}_k)$	$\chi(\widetilde{X}_k)$	$c_1^2(\widetilde{X}_k)$	$5 \cdot c_1^2(\widetilde{X}_k) - \chi(\widetilde{X}_k) + 36$	Homeomorphism Type
1	-15	23	1	18	$3\mathbb{CP}^2 \sharp 18\overline{\mathbb{CP}}^2$
2	-22	34	2	12	$5\mathbb{CP}^2 \sharp 27\overline{\mathbb{CP}}^2$
3	-29	45	3	6	$7\mathbb{CP}^2 \sharp 36\overline{\mathbb{CP}}^2$
4	-36	56	4	0	$9\mathbb{CP}^2 \sharp 45\overline{\mathbb{CP}}^2$
5	-43	67	5	-6	$11\mathbb{CP}^2 \sharp 54\overline{\mathbb{CP}}^2$
6	-50	78	6	-12	$13\mathbb{CP}^2 \sharp 63\overline{\mathbb{CP}}^2$

Table 4.1 Numerical properties of $\widetilde{X}_k$

4.1 Convex fillings

Since we understand the monodromy defining the boundary manifolds, we can identify these manifolds as certain Seifert fibered spaces using Proposition 2. We will explicitly construct convex fillings by considering weak fillings for these manifolds. McCarthy and Wolfson noted that negatively plumbed trees corresponding to these Seifert fibered spaces are equipped with a symplectic structure, making them a weak filling for all contact structures that are transverse to the circle fibration [42]. Lisca and Matić have shown that these contact structures are precisely the universally tight ones [39]. Moreover, since blow-downs of these symplectic structures can be described as Stein fillings, they are therefore convex fillings of the unique universally tight, Giroux torsion 0 contact structure on each boundary manifold.

These Stein fillings will be determined by altering the original Kirby diagrams for the boundaries, given in Figure 3.5, to a Stein handlebody diagram. For a reference about Stein structures and Kirby calculus, see [31, 47]. Throughout all diagrams in this section, we will use the Seifert framing convention rather than the Legendrian framing convention.

The constructions of the convex fillings will make repeated use of the following sequence of moves. The only exception to this is when $k \equiv 0 \bmod 6$, in which case the boundary manifold is T^3, and we can use the convex filling $T^2 \times \mathbb{D}^2$.

Suppose that a Kirby diagram of a 3-manifold contains the following sub-diagram:

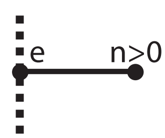

Figure 4.1 Sub-diagram of a Kirby diagram

One can then perform a sequence of blow-ups, followed by a single blow-down to alter the diagram as follows:

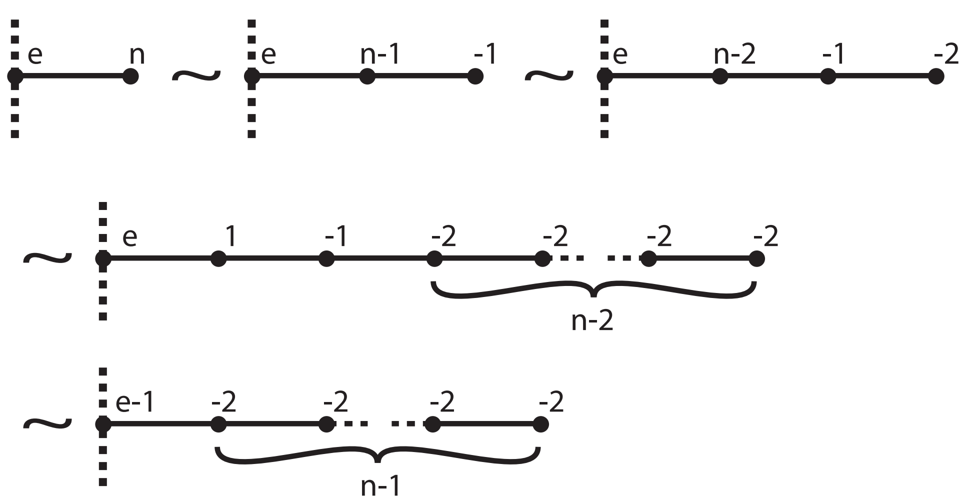

Figure 4.2 Replacing a positive sphere with -2 spheres

When $k \equiv 1 \mod 6$, the boundary is $Y(0; \frac{1}{2}, \frac{-1}{3}, \frac{-1}{6})$. Using the previous move, we immediately see that $Y(0; \frac{1}{2}, \frac{-1}{3}, \frac{-1}{6})$ is fillable by a $-E_9$ plumbing of spheres.

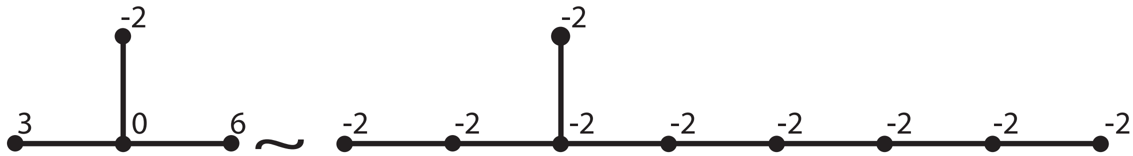

Figure 4.3 Stein filling of $Y(0; \frac{1}{2}, \frac{-1}{3}, \frac{-1}{6})$

When $k \equiv 2 \mod 6$, the boundary is $Y(0; \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3})$. In this case, we obtain the following Stein filling.

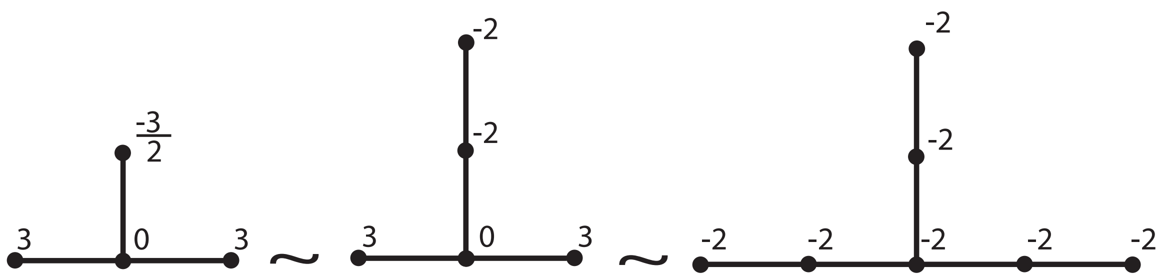

Figure 4.4 Stein filling of $Y(0; \frac{2}{3}, \frac{-1}{3}, \frac{-1}{3})$

When $k \equiv 3 \mod 6$, the boundary is $Y(0; \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$, which admits the following Stein filling.

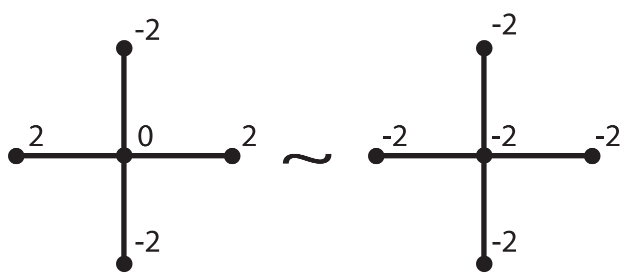

Figure 4.5 Stein filling of $Y(0; \frac{1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{-1}{2})$

When $k \equiv 4 \mod 6$, the boundary is $Y(0; \frac{-2}{3}, \frac{1}{3}, \frac{1}{3})$. To obtain a Stein filling, we will perform a sequence of blow-downs. We see that:

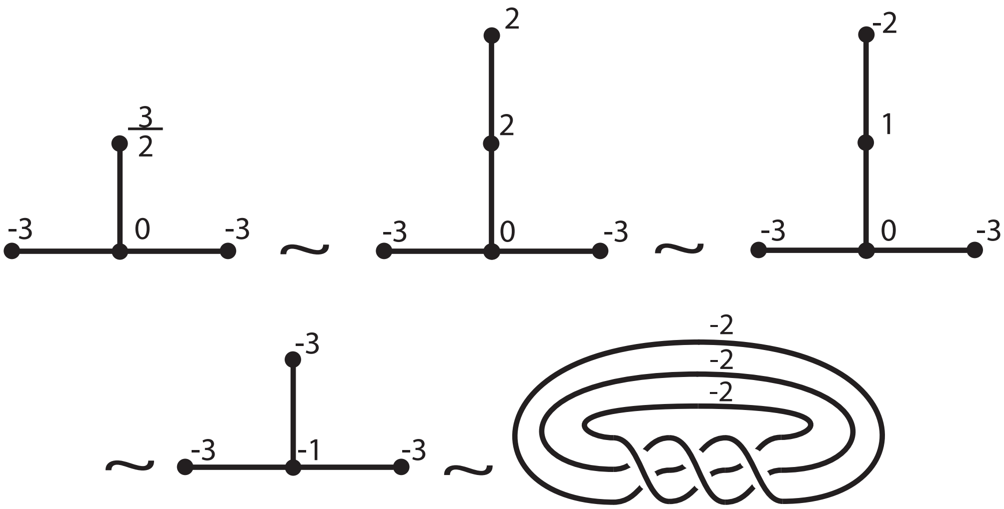

Figure 4.6 Kirby calculus applied to $Y(0; \frac{-2}{3}, \frac{1}{3}, \frac{1}{3})$

We can realize this last diagram as a Stein filling by:

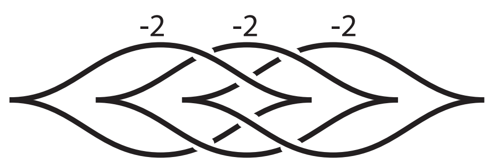

Figure 4.7 Stein filling of $Y(0; \frac{-2}{3}, \frac{1}{3}, \frac{1}{3})$

Lastly, when $k \equiv 5 \bmod 6$, the boundary is $Y(0; \frac{-1}{2}, \frac{1}{3}, \frac{1}{6})$. A sequence of blow-downs alters the initial diagram:

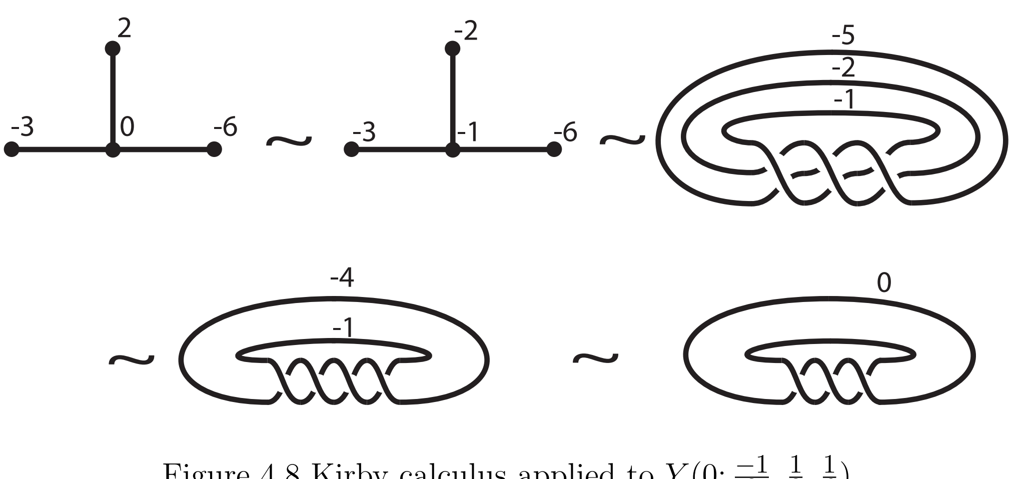

Figure 4.8 Kirby calculus applied to $Y(0; \frac{-1}{2}, \frac{1}{3}, \frac{1}{6})$

It follows that $Y(0; \frac{-1}{2}, \frac{1}{3}, \frac{1}{6})$ is obtained by 0-surgery on the right-handed trefoil. We can realize this as a Stein filling by:

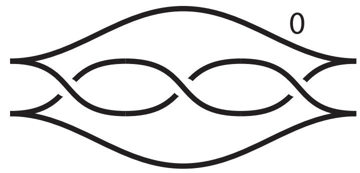

Figure 4.9 Stein filling of $Y(0; \frac{-1}{2}, \frac{1}{3}, \frac{1}{6})$

Note that for all k, $H_2(C_k)$ is generated by either spheres of self-intersection -2 or tori of self-intersection 0. It follows that $c_1(C_k) = 0$.

The signature and euler characteristic of each of these convex fillings is organized below in Table 4.2.

k	$\sigma(C_k)$	$\chi(C_k)$
1	-8	10
2	-6	8
3	-4	6
4	-2	4
5	0	2
0	0	0

Table 4.2 Invariants of C_k

The manifold $Y(0; \frac{1}{2}, \frac{-1}{4}, \frac{-1}{4})$ appears as the boundary (with opposite orientation) of the k-fold sum when $N = 0$ and $k \equiv 1 \bmod 4$. In this case, we immediately obtain the following Stein filling.

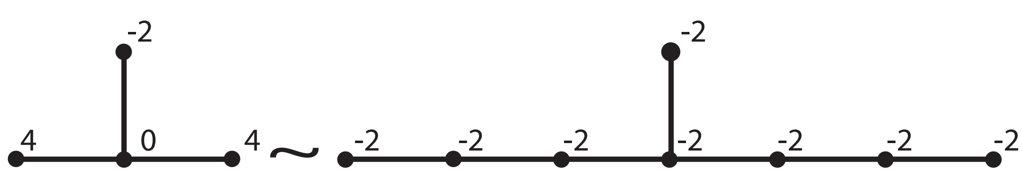

Figure 4.10 Stein filling of $Y(0; \frac{1}{2}, \frac{-1}{4}, \frac{-1}{4})$

Note that this filling has signature -7 and euler characteristic 9.

The manifold $Y(0; \frac{-1}{2}, \frac{1}{4}, \frac{1}{4})$ appears as the boundary (with opposite orientation) of the k-fold sum when $N = 0$ and $k \equiv 3 \mod 4$. To obtain a Stein filling, we will perform a sequence of blow-downs. We see that:

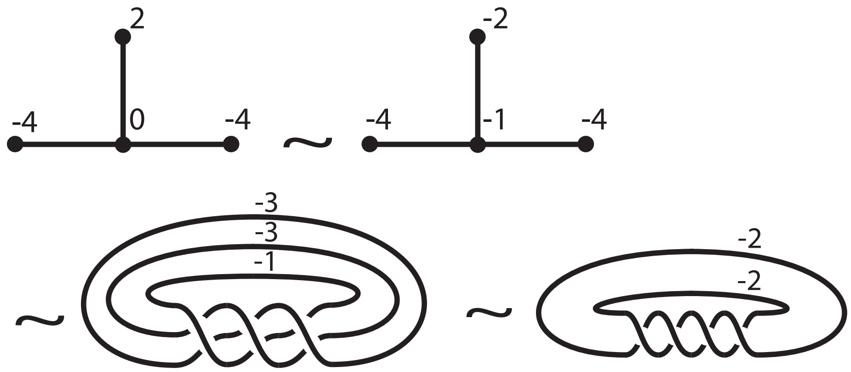

Figure 4.11 Kirby calculus applied to $Y(0; \frac{-1}{2}, \frac{1}{4}, \frac{1}{4})$

We can realize this last diagram as a Stein filling by:

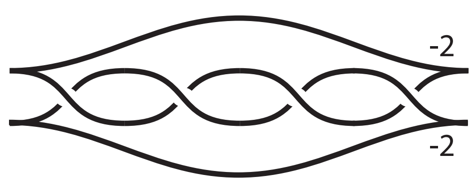

Figure 4.12 Stein filling of $Y(0; \frac{-1}{2}, \frac{1}{4}, \frac{1}{4})$

Note that this filling has signature -1 and euler characteristic 3.

4.2 Algebraic invariants of $\widetilde{X}_k$

To show that the manifolds are homeomorphic to those specified in Theorem 17, it suffices, by work of Freedman [22], to see that they have the same euler characteristics, signatures, that are all odd, and that they have trivial fundamental groups. To compute the signature, we will explicitly compute the $H_2(\overline{X}_k)$.

Let ν represent a neighbourhood of $\Sigma_1 \cup \Sigma_2$, so that ν is homotopy equivalent to the wedge product of two tori. By examining the relative long exact sequence of $(\nu, \partial\nu)$, and computing the intersection form of ν, we have:

$$H_2(\nu) \xrightarrow{\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]} H_2(\nu, \partial\nu) \xrightarrow{0} H_1(\partial\nu) \cong \mathbb{Z}^4 \xrightarrow{\cong} H_1(\nu) \cong \mathbb{Z}^4 \longrightarrow 0. \tag{4.1}$$

It therefore follows that $H_1(\partial\nu)$ is naturally isomorphic to $H_1(\nu)$. Let X_1 be the complement of ν in $(\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2)$. Examining the corresponding Mayer-Vietoris sequence of this splitting, we then have that:

$$0 \longrightarrow H_2(\partial\nu) \longrightarrow H_2(\nu) \oplus H_2(X_1) \longrightarrow H_2(\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2) \longrightarrow 0 \tag{4.2}$$

and

$$0 \longrightarrow H_1(\partial\nu) \xrightarrow{\cong} H_1(\nu) \oplus H_1(X_1) \longrightarrow 0 \tag{4.3}$$

Thus $H_1(X_1) = 0$ and $H_2(X_1) \cong \mathbb{Z}^4 \oplus \mathbb{Z}^8$. More precisely, $H_2(X_1)$ is generated by $H_2(\partial\nu)$, which consists of four Lagrangian tori that are lifts of simple curves lying on Σ_1 and Σ_2, and the annihilator of $\langle [\Sigma_1], [\Sigma_2] \rangle$ in $H_2(\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2)$. Since $[\Sigma_2] - [\Sigma_1] = [e_9]$, this is isomorphic to the annihilator of $\langle [\Sigma_1] \rangle$ in $H_2(\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2)$; this subgroup is generated by a $-E_8$ configuration of symplectic spheres (of the form $[e_i - e_{i+1}]$ and $[e_6 - e_7 - e_8 - h]$).

Topologically, we can construct $\overline{X}_k$ by taking the union of k copies of X_1 glued cyclically by identifying $\overline{\Sigma}_2 \times S^1 \subset \partial X_1$ in one copy of X_1 with $\overline{\Sigma}_1 \times S^1 \subset \partial X_1$ in its cyclic successor (here, $\overline{\Sigma}$ is a punctured torus). We therefore have that

$$\chi(\overline{X}_k) = k \cdot (\chi(\mathbb{CP}^2 \sharp 9\overline{\mathbb{CP}}^2) - \chi(\Sigma_1 \vee \Sigma_2)) = 13k. \tag{4.4}$$

Define X_k inductively by $X_k = X_{k-1} \cup_{\overline{\Sigma} \times S^1} X_1$.

The corresponding Mayer-Vietoris sequence then inductively shows that $H_1(X_k)$ and $H_3(X_k)$ are trivial. The remaining portion of the sequence is:

$$0 \longrightarrow H_2(\overline{\Sigma} \times S^1) \longrightarrow H_2(X_{k-1}) \oplus H_2(X_1) \longrightarrow H_2(X_k) \longrightarrow H_1(\overline{\Sigma} \times S^1) \longrightarrow 0 \tag{4.5}$$

We therefore get that $H_2(X_k) \cong \mathbb{Z}^{13k-1}$. Moreover, we can generate $H_2(X_k)$ by:

- k $-E_8$ configurations of symplectic spheres,
- $2(k-1)$ pairs of 2 Lagrangian tori (at each of the $k-1$ places where copies of X_1 are glued together, the Lagrangian tori sitting near $\overline{\Sigma}_2$ are identified with the Lagrangian tori sitting near $\overline{\Sigma}_1$),
- 4 Lagrangian tori that are supported on ∂X_k, and
- $3(k-1)$ classes formed when gluing copies of X_1 together.

These last $3(k-1)$ can be explained as follows. Choose curves α_2 and β_2 on Σ_2 that generate $H_1(\Sigma_2)$. Let α_1 and β_1 be curves on Σ_1 that are the image of α_2 and β_2 respectively under the chosen identification of Σ_2 with Σ_1. Lifts of α_i and β_i to $\partial\nu$ bound chains in X_1. At each identification amongst the X_1s, these chains glue together to form closed cycles. Label these classes S_α and S_β, respectively. Label the Lagrangian tori associated to α and β by T_α and T_β respectively.

Since these homology classes are formed by choosing chains that have α and β as boundaries (after the identification), S_α will intersect T_β once, and S_α can be seen not to intersect T_α (by defining T_α using a push-off of α). Similarly, S_β will intersect T_β once, and will not intersect T_α.

Lastly, note that in $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}}^2$, $U = e_9$ intersects Σ_1 once, and does not intersect Σ_2. Similarly, $V = 3h - \sum_{i=1}^{9} e_i$ intersects Σ_2 once, but does not intersect Σ_1. Thus, when removing ν from $\mathbb{CP}^2\sharp 9\overline{\mathbb{CP}}^2$, both S and T will be punctured once. For $k > 1$, we can choose the identification of $\overline{\Sigma}_2 \times S^1$ with $\overline{\Sigma}_1 \times S^1$ to that S and T glue together to form a symplectic torus of self-intersection $[S]^2 + [T]^2 = -1$. When $k = 1$, the $S + T$ will glue to itself to form a genus 2 surface of self-intersection $[S + T]^2 = 1$. Call such classes S_γ.

We can therefore reorganize $H_2(X_k)$ as a direct sum of:

- k $-E_8$ configurations of spheres,
- $2(k-1)$ hyperbolic pairs($\langle S_\alpha, T_\beta\rangle$ and $\langle S_\beta, T_\alpha\rangle$),
- 4 Lagrangian tori that are supported on ∂X_k, and
- $k-1$ S_γ classes.

Finally, express $\overline{X}_k$ as $X_k \cup_{\overline{\Sigma}\times S^1\times\{0,1\}} \overline{\Sigma} \times S^1 \times I$. We get that $H_1(\overline{X}_k) \cong \mathbb{Z}$, and the second homology changes by identifying the pairs of Lagrangian tori on ∂X_k (reducing to two Lagrangian tori in $\overline{X}_k$), and creating one more of each of the S_α, S_β, and S_γ classes.

Note that since S_γ has odd square, $\overline{X}_k$, and hence $\widetilde{X}_k$, is odd.

Moreover, we have that the intersection form of $\overline{X}_k$ is

$$Q(\overline{X}_k) \cong k \cdot (-E_8) \oplus 2k \cdot H \oplus Q_{\gamma,k} \tag{4.6}$$

where $Q_{\gamma,k}$ is the intersection form restricted to the S_γ classes.

For $k = 1, 2$, $Q_{\gamma,k}$ is non-generic. When $k = 1$, the sole S_γ class has self-intersection 1, and so $Q_{\gamma,1} = \langle 1\rangle$. When $k = 2$, there will be two S_γ classes, and they will intersect twice, once in each copy of X_1 (in each X_1, the classes will intersect $[S] \cdot [T] = 1$ time). Therefore,

$$Q_{\gamma,2} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}. \tag{4.7}$$

For $k > 2$, the classes S_γ will intersect their cyclic predecessor and successor once. Therefore,

$$Q_{\gamma,k} = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & & 0 \\ 0 & \ddots & & & \ddots & \vdots \\ \vdots & & \ddots & & & 0 \\ 0 & & & \ddots & \ddots & 1 \\ 1 & 0 & \cdots & 0 & 1 & -1 \end{bmatrix}. \tag{4.8}$$

We can write $Q_{\gamma,k}$ as $A - I$ where A is the adjacency matrix for the cyclic graph on k vertices. The eigenvectors of A are well understood [29], and these are necessarily the eigenvectors of $Q_{\gamma,k}$. Thus, the eigenvalues of $Q_{\gamma,k}$ will be one less than the eigenvalues of A; the eigenvalues of $Q_{\gamma,k}$ are $\tau + \tau^{-1} - 1$, where τ runs through the k^{th} roots of unity.

To compute $\sigma(Q_{\gamma,k})$, it therefore suffices to count the k^{th} roots of unity with argument in $(\frac{-\pi}{3}, \frac{\pi}{3})$, and subtract the count of k^{th} roots of unity with argument in $(\frac{\pi}{3}, \frac{5\pi}{3})$. The number of positive eigenvalues is $2\lceil\frac{k}{6}\rceil - 1$. If 6 does not divide k, the number of negative eigenvalues is $k - (2\lceil\frac{k}{6}\rceil - 1)$ since $e^{\frac{\pi i}{3}}$ is not a k^{th} root of unity. If 6 does divide k, the number of negative eigenvalues is $k - (2\lceil\frac{k}{6}\rceil - 1) - 2$. We therefore have that the signature of $Q_{\gamma,k}$ is given by

$$\sigma(Q_{\gamma,k}) = \begin{cases} 4\lceil\frac{k}{6}\rceil - 2 - k & \text{if 6 does not divide } k \\ 4\lceil\frac{k}{6}\rceil - k & \text{if 6 divides } k \end{cases} \tag{4.9}$$

Specializing this formula to the six congruence classes mod 6, we can rewrite this formula as:

$$\sigma(Q_{\gamma,k}) = \begin{cases} \frac{-k}{3} + \frac{4}{3} & \text{if } k \equiv 1(6) \\ \frac{-k}{3} + \frac{2}{3} & \text{if } k \equiv 2(6) \\ \frac{-k}{3} & \text{if } k \equiv 3(6) \\ \frac{-k}{3} - \frac{2}{3} & \text{if } k \equiv 4(6) \\ \frac{-k}{3} - \frac{4}{3} & \text{if } k \equiv 5(6) \\ \frac{-k}{3} & \text{if } k \equiv 0(6) \end{cases} \tag{4.10}$$

Define $\widetilde{X}_k = C_k \cup \overline{X}_k$. We can compute the signature and euler characteristic of this manifold by adding the signature and euler characteristic of $\overline{X}_k$, provided in equations 4.10 and 4.4, to those of C_k, provided in table 4.2. This computation is provided below in Table 4.3.

k(mod 6)	$\sigma(\overline{X}_k)$	$\chi(\overline{X}_k)$	$\sigma(C)$	$\chi(C)$	$\sigma(\widetilde{X}_k)$	$\chi(\widetilde{X}_k)$
1	$\frac{-25k}{3} + \frac{4}{3}$	$13k$	-8	10	$\frac{-25k}{3} - \frac{20}{3}$	$13k + 10$
2	$\frac{-25k}{3} + \frac{2}{3}$	$13k$	-6	8	$\frac{-25k}{3} - \frac{16}{3}$	$13k + 8$
3	$\frac{-25k}{3}$	$13k$	-6	4	$\frac{-25k}{3} - \frac{12}{3}$	$13k + 6$
4	$\frac{-25k}{3} - \frac{2}{3}$	$13k$	-2	4	$\frac{-25k}{3} - \frac{8}{3}$	$13k + 4$
5	$\frac{-25k}{3} - \frac{4}{3}$	$13k$	0	2	$\frac{-25k}{3} - \frac{4}{3}$	$13k + 2$
0	$\frac{-25k}{3}$	$13k$	0	0	$\frac{-25k}{3}$	$13k$

Table 4.3 Computation of $\sigma(\widetilde{X}_k)$ and $\chi(\widetilde{X}_k)$

We now wish to show that $\pi_1(\widetilde{X}_k)$ is trivial.

Referring again to the splitting $\mathbb{CP}^2 \sharp 9 \overline{\mathbb{CP}}^2 = \nu \cup X_1$, we see that $\pi_1(X_1)$ is generated by meridional curves of Σ_1 and Σ_2. Call such curves γ_1 and γ_2 respectively. Since the sphere e_9 intersects $\nu\Sigma_1$ in γ_1, and is disjoint from Σ_2, we see that $\pi_1(X_1)$ is generated by γ_2, and so it must be cyclic. Thus, since $H_1(X_1) \cong 0$, it follows that $\pi_1(X_1) \cong 1$. Inductively applying the Seifert-Van Kampen Theorem to the splitting $X_k = X_{k-1} \cup X_1$ shows that $\pi_1(X_k)$ must also be trivial. Next, we wish to compute $\pi_1(\overline{X}_k)$. Since Homotopically, $\overline{X}_k$ is equivalent to $X_k \cup \overline{\Sigma} \times S^1$, and we can therefore build $\overline{X}_k$ from X_k by adding one 1-handle, four 2-handles, and one 3-handle. In particular, $\pi_1(\overline{X}_k)$ can be expressed at a group with a single generator, and therefore it is also cyclic. Since $H_1(\overline{X}_k) \cong \mathbb{Z}$, we therefore have that $\pi_1(\overline{X}_k) \cong \mathbb{Z}$. Moreover, using the commutative diagram

$$\begin{tikzcd}
\pi_1(\partial\overline{X}_k) \arrow[r] \arrow[d] & \pi_1(\overline{X}_k) \arrow[d, "\cong"] & \\
H_1(\partial\overline{X}_k) \arrow[r, two heads] & H_1(\overline{X}_k) \arrow[r] & H_1(\overline{X}_k, \partial\overline{X}_k) \cong 0
\end{tikzcd} \tag{4.11}$$

we see that induced morphism from $\pi_1(\partial\overline{X}_k)$ to $\pi_1(\overline{X}_k)$ is surjective.

If $k \not\equiv 0 \bmod 6$, the convex filling C has trivial fundamental group. This is easily seen since the given handle body diagrams of the fillings consist only of 0- and 2-handles. The Seifert-Van Kampen Theorem therefore shows that

$$\pi_1(\widetilde{X}_k) \cong \pi_1(\overline{X}_k)/\pi_1(\partial\overline{X}_k) \cong 1. \tag{4.12}$$

If $k \equiv 0 \bmod 6$, then $\partial\overline{X}_k$ is T^3, and the convex filling is $T^2 \times \mathbb{D}^2$. Moreover, following our convention, we will identify the boundary circles $\partial\mathbb{D}^2$ with is the Legendrian sections of the torus fibration. Since the generator of $\pi_1(\widetilde{X}_k)$ is the image of this simple Legendrian curve (c.f. Example 10), the Seifert-Van Kampen Theorem again shows that $\pi_1(\widetilde{X}_k)$ is trivial. This follows because the maps from $\pi_1(T^3)$ to $\pi_1(\overline{X}_k)$ and $\pi_1(T^2 \times \mathbb{D}^2)$ are both surjective, and generators of $\pi_1(T^3)$ map to 1 under one of these two maps.

We therefore have that $\widetilde{X}_k$ is simply-connected for all k. Since $\widetilde{X}_k$ is odd and the signature and euler characteristic match those manifolds listed in Theorem 17, it therefore follows that the manifolds $\widetilde{X}_k$ are homeomorphic to those manifolds. Note that since the manifolds $\widetilde{X}_k$ are symplectic and yet are homeomorphic to $a\mathbb{CP}^2\sharp b\overline{\mathbb{CP}}^2$ for $a > 1$, they are necessarily exotic.

4.3 $\widetilde{X}_k$ is minimal

Lastly, to prove Theorem 17, it remains to shown that $\widetilde{X}_k$ is minimal. To demonstrate that $\widetilde{X}_k$ is minimal, we will examine its Seiberg-Witten basic classes. While we are not able to completely determine the basic classes, we can sufficiently identify potential basic classes. Doing so allows us to see that the Seiberg-Witten invariant of $\widetilde{X}_k$ cannot be structured as the Seiberg-Witten invariant of a blown-up manifold.

Note that since $\widetilde{X}_k$ is symplectic and $b^+(\widetilde{X}_k) > 1$, the canonical class K is a basic class [61].

As mentioned in Section 4.1, the canonical class restricted to C_k is trivial. We therefore have that K is supported in $H_2(X)$.

Recall from (4.6) that

$$Q(\overline{X}_k) \cong k \cdot (-E_8) \oplus 2kH \oplus Q_{\gamma,k}. \tag{4.13}$$

The $-E_8$ configurations consist of symplectic spheres. Label these spheres as $\{U_1, \ldots, U_{8k}\}$. The hyperbolic pairs consist of Lagrangian tori, $T_{\alpha,i}$ or $T_{\beta,i}$, that are dual to surfaces $S_{\beta,i}$ or $S_{\alpha,i}$ respectively. The $Q_{\gamma,k}$ configuration consists of k symplectic tori $S_{\gamma,i}$ of self-intersection -1, organized in a cyclic manner.

Write the Poincaré dual of K by

$$PD(K) = \sum_{i=1}^{8k} \alpha_i [U_i] + \sum_{i=1}^{k} \beta_i [S_{\alpha,i}] + \gamma_i [S_{\beta,i}] + \delta_i [T_{\alpha,i}] + \epsilon_i [T_{\beta,i}] + \zeta_i [S_{\gamma,i}]. \tag{4.14}$$

Applying the adjunction formula to U_i we see that

$$\begin{aligned} -2 &= -2 + K \cdot [U_i] \\ &= -2 + \sum_{i=1}^{8} \alpha_j \cdot (-E_8)_{ji} \end{aligned} \tag{4.15}$$

and so $\alpha_i = 0$ for all i.

Similarly, we have that

$$0 = -1 + K \cdot [S_{\gamma,i}] \tag{4.16}$$

and so $\xi_i = 1$ for all i. Applying the adjunction inequality to $T_{\alpha,i}$, we see that

$$0 \geq |K \cdot [T_{\alpha,i}]| = |\gamma_i|. \tag{4.17}$$

Similarly, $\beta_i = 0$.

We can therefore express the Poincaré dual of K as

$$PD(K) = \sum_{i=1}^{k} [S_{\gamma,i}] + \sum_{i=1} \delta_i [T_{\alpha,i}] + \epsilon_i [T_{\beta,i}]. \tag{4.18}$$

Since K is characteristic, δ_i and ϵ_i are even integers.

By Theorem 7, all basic classes must therefore be of the form

$$PD(\kappa) = \sum_{i=1}^{k} \pm[S_{\gamma,i}] + \sum_{i=1}^{k} d_i[T_{\alpha,i}] + e_i[T_{\beta,i}] \tag{4.19}$$

for $d_i, e_i \in 2\mathbb{Z}$. Moreover, since such a basic class must satisfy

$$\kappa^2 = 3\sigma(\widetilde{X}_k) + 2\chi(\widetilde{X}_k) = k, \tag{4.20}$$

the basic classes satisfy

$$\pm PD(\kappa) = \sum_{i=1}^{k}[S_{\gamma,i}] + \sum_{i=1}^{k} d_i[T_{\alpha,i}] + e_i[T_{\beta,i}]. \tag{4.21}$$

According to the Seiberg-Witten blow-up formula, any homology class that is represented by a -1 sphere will be realized as $\frac{1}{2}(\kappa - \kappa')$ for basic classes κ and κ'. Examining the above potential basic classes, $\frac{1}{2}(\kappa - \kappa')$ takes either the value

$$\frac{1}{2}\sum_{i=1}^{k}(d_i - d_i')[T_{\alpha,i}] + (e_i - e_i')[T_{\beta,i}] \tag{4.22}$$

or

$$\pm\sum_{i=1}^{k}[S_{\gamma,i}] + \frac{1}{2}\sum_{i=1}^{k}(d_i - d_i')[T_{\alpha,i}] + (e_i - e_i')[T_{\beta,i}]. \tag{4.23}$$

The first class has square 0. The second class has square $k > 0$. We therefore have that $\widetilde{X}_k$ is minimal, which completes the proof of Theorem 17.

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