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DIFFERENTIAL GEOMETRY
OF
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ABSTRACT

DIFFERENTIAL GEOMETRY OF SLANT SURFACES

By

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We consider immersions of differentiable manifolds into almost Hermitian manifolds. The Wirtinger angle is a quantity which measures how an immersed submanifold differs from a holomorphic submanifold. An immersion is called a slant immersion if the Wirtinger angle is constant. It is a generalization of holomorphic submanifolds and totally real submanifolds. A slant immersion which is neither holomorphic nor totally real is called a proper slant immersion.

In this article, we mainly consider slant surfaces of codimension 2. We first clarify the relation between 2–planes and complex structures of Euclidean 4–space $\mathbf{E}^4$ from the view point of multilinear algebra. Combining this with the Gauss map, we characterize slant surfaces in complex 2–space $\mathbb{C}^2$. We also show that a surface without complex tangent point in a 4–dimensional almost Hermitian manifold can be a slant surface with any given constant Wirtinger angle with respect to a suitable almost complex structure. This shows a big difference between almost Hermitian manifolds and Kähler manifolds.

Next we show that no compact proper slant submanifolds exist in any complex space $\mathbb{C}^m$. This is a similarity shared by proper slant submanifolds and holomorphic submanifolds.

Finally, under some additional conditions, we can determine the shapes of slant surfaces in $\mathbf{C}^2$. If a slant surface is contained in a 3–sphere S^3, then it is obtained from a kind of helix in S^3 or a great circle by left–translations along a curve in S^3. If a slant surface is contained in a 3–plane, or, more generally, the rank of the Gauss map is less than 2, then we can do the analogues of the classification of flat surfaces in Euclidean 3–space $\mathbf{E}^3$.

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INTRODUCTION

In this article, we consider slant immersions of differentiable manifolds into almost Hermitian manifolds.

The most natural submanifolds of an almost complex manifold are holomorphic submanifolds in the sense that they inherit both differentiable structure and almost complex structure from the ambient space, and the tangent spaces of the submanifolds are invariant under the almost complex structure. On the other hand, the notation of totally real immersions (or anti–invariant immersions, Lagrangean immersions) were introduced in the early 1970's. A submanifold of an almost Hermitian manifold is called totally real if each tangent space of the submanifold is mapped by the almost complex structure into the normal space.

Recently B.–Y. Chen defined the term slant immersions as a generalization of both holomorphic and totally real immersions ([BYC5]). An immersion of a differentiable manifold into an almost Hermitian manifold is defined to be a slant immersion if its Wirtinger angle α is constant (Ch1 §4). If $\alpha \equiv 0$, then the immersion is holomorphic and if $\alpha \equiv \pi/2$, then it is totally real.

In Chapters 2–5, we will state some properties of slant immersions which we have obtained up to now. We consider mainly slant surfaces in complex 2–space $\mathbb{C}^2$ expect for Chapter 2 §3 and Chapter 3. In Chapter 2 §3, the ambient space is an almost Hermitian manifold. Chapter 3 is about slant submanifolds in complex spaces of arbitrary dimension and codimension.

Chapter 1 is a preliminary. In §1 we recall the basic formulas of the differential geometry of submanifolds. §2 is about descriptions of the Grassmannian $G(2,4)$ as a product of two 2–spheres. The relation between two descriptions of $G(2,4)$, a quadric Q_2 in $\mathbf{C}P^2$ and a set $D_1(2,4)$ of unit decomposable 2–vectors in $\Lambda^2\mathbf{E}^4$, is clarified. §3 is a review of the generalized Gauss map. In §4 we introduce the definitions and basic properties of slant immersions written in [BYC5].

In Chapter 2, we consider slant surfaces with codimension 2. In §1, we investigate the relation between 2–planes in $\mathbf{E}^4$ and complex structures on $\mathbf{E}^4$ (Proposition 1). This provides a pointwise observation of slant surfaces. In §2, we consider the Gauss map and characterize slant surfaces in $\mathbf{C}^2$ (Proposition 2). Especially, we show that a non–minimal surface in $\mathbf{E}^4$ can be slant with respect to at most four complex structures on $\mathbf{E}^4$. In §3, we show that any surface without complex tangent points in an almost Hermitian manifold becomes a proper slant surface with any given constant Writinger angle with respect to a suitable almost complex structure (Proposition 3). §§2 and 3 show the difference between a Kähler manifold and an almost Hermitian manifold.

In Chapter 3, we show that a compact proper slant submanifold does not exist in complex space $\mathbf{C}^m$ (Proposition 4). This is a similarity of slant submanifolds and holomorphic submanifolds.

Most examples of slant surfaces in $\mathbf{C}^2$ which we have constructed up to now are doubly slant and have the rank of the Gauss map less than 2 and hence flat surfaces (Chapter 2 §2). So, it is natural to consider the problem of classifying flat slant surfaces in $\mathbf{C}^2$. Under some additional conditions, we can determine the shapes of flat slant surfaces (Chapters 4 and 5).

In Chapter 4, we consider slant surfaces contained in a 3–sphere S^3 in $\mathbf{C}^2$. Since S^3 is a Lie group of unit quaternions, the theory of curves and surfaces in S^3 is given a special development (§1). In §2, we define another Gauss map on S^3 using left invariant vector fields and characterize slant surfaces in S^3 (Proposition 5.). This is a spherical version of Proposition 2. In § 3, we determine proper slant surfaces in S^3 (Proposition 6).

In Chapter 5, we consider a slant surface with the rank of the Gauss map less than 2. Then, the surface becomes a flat slant ruled surface in $\mathbb{C}^2$ and we can do the analogy of the classical classification of flat surfaces in E^3 and determine the shapes of slant surfaces (Proposition 7). In particular, if a slant surface is contained in a 3–plane in $\mathbb{C}^2$, then its shape becomes more concrete (Proposition 8).

We are at the starting point of the differential geometry of slant immersions. We hope it will have a fertile development similar to the studies of holomorphic or totally real immersions.

CHAPTER 1.
PRELIMINARIES

In this chapter we review and arrange some well–known formulas and facts which we will use in this article. §1 is a list of formulas of differential geometry of submanifolds. In §2 we recall the description of the Grassmannian $G(2,4)$ as a product of two 2–spheres. § 3 is about the generalized Gauss map of submanifolds in Euclidean spaces. In §4 we introduce the definition and basic properties of slant immersions.

§ 1. Notations

We follow basically the definitions and notations of [CBY1] – [CBY3]. Differentiability always means differentiability of class C^{∞}. Listed below are some formulas which we will use in this article.

Let (M, g) be an n–dimensional Riemannian manifold with the Rienannian connection ∇, $\{e_i\}_{i=1}^{n}$ be a local orthonormal frame field and $\{\omega^i\}_{i=1}^{n}$ be its dual coframe field. The connection form $\{\omega^i_j\}$ is defined by

(1.1) $$\omega^i_j(X) = \omega^i(\nabla_X e_j) \qquad X \in TM$$

i.e.

(1.2) $$\nabla_X e_j = \Sigma\ \omega^i_j(X) e_i$$

The curvature form $\{\Omega^i_j\}$ is defined by

(1.3) $$\Omega^i_j = \frac{1}{2}\ \Sigma\ R^i_{jkl}\ \omega^k \wedge w^l$$

where

(1.4) $$R(e_j, e_k)e_l = \Sigma\ R^i_{ljk}\ e_i\ .$$

If we put

$$\nabla_{e_j} e_k = \Sigma\, \Gamma^i_{jk}\, e_i\,, \tag{1.5}$$

then

$$\omega^i_j = \Sigma\, \Gamma^i_{jk}\, \omega^k \tag{1.6}$$

$$\nabla_{e_j} \omega^k = -\Sigma\, \Gamma^k_{jl}\, \omega^l. \tag{1.7}$$

The structure equations are given by

$$d\omega^i = -\Sigma\, \omega^i_j \wedge \omega^j \tag{1.8}$$

$$d\omega^i_j = -\Sigma\, \omega^i_k \wedge \omega^k_j + \Omega^i_j \tag{1.9}$$

Let (M, g) and $(\tilde{M}, \tilde{g})$ be Riemannian manifolds of dimensions n and m respectively, and ∇ and $\tilde{\nabla}$ be their Riemannian connections. If

$$x:\ M \to \tilde{M} \tag{1.10}$$

is an isometric immersion, then

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad X, Y \in \mathfrak{X}(M) \tag{1.11}$$

$$\tilde{\nabla}_X \xi = -A_\xi Y + D_X \xi, \quad X \in \mathfrak{X}(M), \quad \xi \in \mathfrak{X}^\perp(M) \tag{1.12}$$

where h, A and D are the second fundamental form, the Weingarten map and the connection in the normal bundle. Let $\{\tilde{e}_A\}_{A=1}^m$ be a local adapted frame field, i.e., a local orthonormal frame field on $\tilde{M}$ and, if restricted to M, $\{\tilde{e}_i\}_{i=1}^n$ is an orthonormal frame field on M. Let $\{\tilde{\omega}^A\}$ and $\{\tilde{\omega}^A_B\}$ be its dual coframe field and connection form on $\tilde{M}$. If we put

$$e_A = \tilde{e}_{A|M} \tag{1.13}$$

$$\omega^i = \tilde{\omega}^i{}_{|M}, \quad i.e.,\ \omega^i = x^*\tilde{\omega}^i \tag{1.14}$$

$$\omega^A_B = \tilde{\omega}^A_{B|M} \tag{1.15}$$

where

$$i, j = 1,\dots,n; \quad A, B = 1,\dots,m$$

then $\{e_i\}$ and $\{\omega^i\}$ and dual, $\{\omega^i_j\}$ is the connection form with respect to $\{e_i\}$ and

$$D_X e_r = \sum_{s=n+1}^{m} \omega_r^s(X) e_s, \quad r = n+1,\dots,m. \tag{1.16}$$

If we put

$$h = \sum_{r=n+1}^{m} \sum_{i,j=1}^{n} h^r_{ij}\, \omega^i \otimes \omega^j \otimes e_r$$

i.e.,

$$\tilde{g}(h(e_i, e_j), e_r) = h^r_{ij}, \tag{1.17}$$

then

$$h^r_{ij} = h^r_{ji} \tag{1.18}$$

$$w^r_i = \Sigma\, h^r_{ij}\, w^j. \tag{1.19}$$

The mean curvature vector H is defined by

$$\begin{aligned} h &= \frac{1}{n}\, trace \quad h \\ &= \frac{1}{n} \sum_{r=n+1}^{m} \left(\sum_{i=1}^{n} h^r_{ii}\right) e_r. \end{aligned} \tag{1.20}$$

The equations of Gauss, Codazzi and Ricci are given respectively by

$$\begin{aligned} \tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) &+ \tilde{g}(h(X, Z), h(Y, W)) \\ &- \tilde{g}(h(X, W), h(Y, z)) \end{aligned} \tag{1.21}$$

$$(\tilde{R}(X, Y)Z)^{\perp} = (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) \tag{1.22}$$

$$R^D(X, Y, \xi, \eta) = R(X, Y, \xi, \eta) + g([A_\xi, A_\eta]X, Y) \tag{1.23}$$

where

$$(\tilde{\nabla}_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{1.24}$$

$$R^D(X, Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi. \tag{1.25}$$

§ 2. Geometry of G(2,4).

Let $E^m = (\mathbb{R}^m, < , >)$ be the Euclidean m–space with the canonical inner product $< , >$. Denote the canonical basis and orientation by

(2.1) $$\{\overset{\circ}{e}_A\}_{A=1}^m, \quad \overset{\circ}{e}_A=(0,\dots,0,\overset{(A)}{1},0,\dots,0),$$

(2.2) $$\omega = \overset{\circ}{e}_1 \wedge \dots \wedge \overset{\circ}{e}_m \in \Lambda^m E^m.$$

For each $n \in \{1,\dots,m\}$, the space $\Lambda^n E^m$ is a $\binom{m}{n}$–dimensional real vector space with the inner product, also denoted by $< , >$, defined by

(2.3) $$<X_1 \wedge \dots \wedge X_n, \; Y_1 \wedge \dots \wedge Y_n> \; = det\,[<X_i, Y_j>] \quad \textit{and bilinearity.}$$

Two spaces $\Lambda^n(E^m)^*$ and $(\Lambda^n E^m)^*$ are identified in a natural way, namely, for $\Omega \in \Lambda^n(E^m)^*$ and $X_1,\dots,X_n \in E^m$,

(2.4) $$\Omega(X_1 \wedge \dots \wedge X_n) = \Omega(X_1,\dots, X_n) \quad \textit{and linearity.}$$

Let, $G(n, m)$ denote the Grassmannian of oriented n–planes in E^m. Then, $G(n,m)$ is identified with the set $D_1(n,m)$ of unit decomposable n–vectors in $\Lambda^n E^m$. The correspondence ψ is given by

(2.5) $$\psi : G(n, m) \longrightarrow D_1(n,m)$$
$$\psi(V) = X_1 \wedge \dots \wedge X_n,$$

where $\{X_i\}_{i=1}^n$ is a positive orthonormal basis of V.

In particular, if $n = 2$ and $m = 4$, $G(2,4)$ is represented as a product of two 2–spheres as follows. The star operator

(2.6) $$* : \Lambda^2 E^4 \longrightarrow \Lambda^2 E^4$$

is defined by

(2.7) $$< *\xi, \eta > \omega = \xi \wedge \eta\,, \quad \textit{for} \quad \xi, \; \eta \in \Lambda^2 E^4.$$

For $V \in G(2,4)$, considered as an element of $D_1(2,4)$ through ψ,

(2.8) $$*V = V^{\perp}$$

where $V^{\perp}$ is the oriented orthogonal complement of V in $\mathbf{E}^4$. Since $*$ is a symmetric involution, $\Lambda^2\mathbf{E}^4$ is decomposed into an orthogonal direct sum

(2.9) $$\Lambda^2\mathbf{E}^4 = \Lambda^2_+\mathbf{E}^4 \oplus \Lambda^2_-\mathbf{E}^4$$

where $\Lambda^2_\pm\mathbf{E}^4$ are the eigenspaces of $*$ corresponding to the eigenvalues ± 1. Denote

(2.10) $$\pi_\pm : \Lambda^2\mathbf{E}^4 \to \Lambda^2_\pm\mathbf{E}^4$$

the projections of this decomposition. For a positive orthonormal basis $\{e_A\}_{A=1}^4$ of $\mathbf{E}^4$, put

(2.11) $$\begin{cases} \eta_1 = \dfrac{1}{\sqrt{2}}(e_1\wedge e_2 + e_3\wedge e_4) \\ \eta_2 = \dfrac{1}{\sqrt{2}}(e_1\wedge e_3 - e_2\wedge e_4) \\ \eta_3 = \dfrac{1}{\sqrt{2}}(e_1\wedge e_4 + e_2\wedge e_3) \end{cases}$$

(2.12) $$\begin{cases} \eta_4 = \dfrac{1}{\sqrt{2}}(e_1\wedge e_2 - e_3\wedge e_4) \\ \eta_5 = \dfrac{1}{\sqrt{2}}(e_1\wedge e_3 + e_2\wedge e_4) \\ \eta_6 = \dfrac{1}{\sqrt{2}}(e_1\wedge e_4 - e_2\wedge e_3) \end{cases}$$

then $\{\eta_1, \eta_2, \eta_3\}$ and $\{\eta_4, \eta_5, \eta_6\}$ are orthonormal bases of $\Lambda^2_+\mathbf{E}^4$ and $\Lambda^2_-\mathbf{E}^4$ respectively. In particular, $\{\mathring{\eta}_1, \mathring{\eta}_2, \mathring{\eta}_3\}$ and $\{\mathring{\eta}_4, \mathring{\eta}_5, \mathring{\eta}_6\}$ obtained from $\{\mathring{e}_A\}$ form canonical bases. For $\xi \in D_1(2,4)$,

(2.13) $$\begin{cases} \pi_+(\xi) = \frac{1}{2}(\xi + *\xi) \\ \pi_-(\xi) = \frac{1}{2}(\xi - *\xi) \end{cases}$$

and

(2.14) $$\|\pi_+(\xi)\| = \|\pi_-(\xi)\| = 1/\sqrt{2}.$$

Hence, if we denote by $S_{\pm}^{\ 2}$ the 2–spheres of radius $1/\sqrt{2}$ in $\Lambda_{\pm}^2 E^4$ centered at the origin, then

$$\pi_{\pm}: D_1(2,4) \longrightarrow S_{\pm}^{\ 2} \tag{2.15}$$

and actually this gives rise to a description

$$D_1(2,4) = S_+^2 \times S_-^{\ 2} \tag{2.16}$$

(cf [S–T] p 360). If we choose an adapted frame $\{e_A\}$ for V in $G(2,4)$; i.e., $\{e_A\}$ is a positive orthonormal basis of $\mathbf{E}^4$ such that $\{e_1, e_2\}$ is a positive basis of V, then

$$\begin{cases} \pi_+(V) = \pi_+(\psi(V)) = \frac{1}{2}(e_1 \wedge e_2 + e_3 \wedge e_4) \\ \pi_-(V) = \pi_-(\psi(V)) = \frac{1}{2}(e_1 \wedge e_2 - e_3 \wedge e_4) \end{cases} \tag{2.17}$$

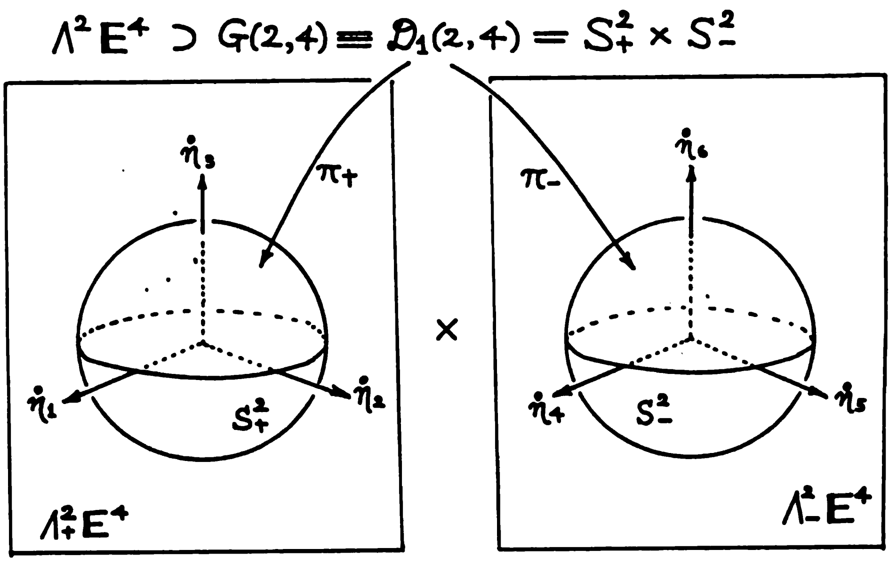


FIGURE 1

There is another description of $G(2,4)$ (cf [H–O1]). Let

$$[\ \ ] : \mathbb{C}^4 \to \mathbb{C}P^3 \tag{2.18}$$

be the canonical projection of the complex 4–space onto the complex projective space and define a quadric Q_2 by

$$\tilde{Q}_2=\{(z_1,\dots,z_4)\,|\,z_1^2+\dots+z_4^2=0\}\subset \mathbf{C}^4 \tag{2.19}$$

$$Q_2=\{[Z]\,|\,Z\in\tilde{Q}_2\}\subset \mathbf{C}\,\mathbf{P}^3.$$

We define a map

$$\Phi : G(2,4)\to Q_2 \tag{2.21}$$

as follows. For $V\in G(2,4)$, pick a positive orthonormal basis $\{X, Y\}$ of V and put $Z = X + iY$. Then, the complex vector Z is contained in $\tilde{Q}_2$. Put

$$\Phi(V) = [Z] \tag{2.22}$$

Then, Φ is well–defined and bijective and hence we may identify $G(2,4)$ with Q_2 through Φ ([H–O1] p6).

If we define a map

$$\varphi : \mathbf{C}\times\mathbf{C}\longrightarrow \mathbf{C}^4 \tag{2.23}$$

by

$$\begin{aligned}\varphi(w_1, w_2) &= (\varphi_1(w_1, w_2),\dots,\varphi_4(w_1,w_2))\\ &= (1 + w_1w_2,\ i(1 - w_1w_2),\ w_1 - w_2,\ -i(w_1 + w_2)),\end{aligned} \tag{2.24}$$

then φ satisfies

$$\varphi_1^2 + \cdots + \varphi_4^2 = 0 \tag{2.25}$$

and hence $[\varphi(w_1, w_2)]\in Q_2$. On $\varphi(\mathbf{C}\times\mathbf{C})$, φ^{-1} is given by

$$\varphi^{-1}(z_1,\dots,z_4) = \left(\frac{z_3+iz_4}{z_1-iz_2}, \frac{-z_3+iz_4}{z_1-iz_2}\right) \tag{2.26}$$

and the biholomorphic map $[\varphi]$ from $\mathbf{C}\times\mathbf{C}$ into Q_2 extends to a biholomorphic map of $\mathbf{CP}^1\times\mathbf{CP}^1$ onto Q_2, when we consider $(w_1, w_2)\in\mathbf{C}\times\mathbf{C}$ as inhomogeneous coordinates on $\mathbf{CP}^1\times\mathbf{CP}^1$

$$[\varphi] : \mathbf{CP}^1\times\mathbf{CP}^1\to Q_2. \tag{2.27}$$

We consider the metric on Q_2 induced from the Fubini–study metric of $\mathbf{C}P^3$ with constant holomorphic curvature 2. We also consider a metric on $\mathbf{C}P^1$ defined, using inhomogeneous coordinate w, by

$$ds^2 = \frac{2|dw|^2}{(1+|w|^2)^2}. \tag{2.28}$$

Then, $[\varphi]$ becomes an isometry and hence Q_2 is considered a product of two 2–spheres of radius $1/\sqrt{2}$ through $[\varphi]$.

$$[\varphi] : \mathbf{C}P^1 \times \mathbf{C}P^1 = S_1^2 \times S_2^2 \to Q_2. \tag{2.29}$$

(For details, cf [H–O1]§2)

The following lemma shows the relation between the two descriptions of the Grassmannian $G(2,4)$ as a product of two 2–spheres.

Lemma 1.

The bijection

$$\Psi \circ \Phi^{-1} \circ [\varphi] : S_1^2 \times S_2^2 \to S_+^2 \times S_-^2 \tag{2.30}$$

is an isometry. S_1^2 *is mapped onto* S_+^2 *and* S_2^2 *is mapped onto* S_-^2.

(Proof)

Let $V \in G(2,4)$ and put

$$\Psi(V) = (\xi_+, \xi_-), \quad \xi_\pm \in S_\pm^2 \tag{2.31}$$

$$\Phi(V) = [\varphi(w_1, w_2)] \tag{2.32}$$

Express $\xi_\pm$ with the coordinates with respect to the canonical basis $\{\overset{\circ}{\eta}_A\}_{A=1}^6$.

$$\begin{cases} \xi_+ = (\xi_1, \xi_2, \xi_3), & \xi_+ = \sum_{j=1}^{3} \xi_j \overset{\circ}{\eta}_j \\ \xi_- = (\xi_4, \xi_5, \xi_6), & \xi_- = \sum_{k=4}^{6} \xi_k \overset{\circ}{\eta}_k \end{cases} \tag{2.33}$$

Put

$$A + iB = \varphi(w_1, w_2) \qquad i.e.\ [A + iB] = \Phi(V) \tag{2.34}$$

Since $|A| = |B| \neq 0$ *and* $\langle A, B\rangle = 0$,

$$\Psi(V) = \frac{1}{|A|^2} A \wedge B \tag{2.35}$$

Hence

$$\begin{aligned} \xi_1 &= \langle \Psi(V), \overset{\circ}{\eta}_1 \rangle \\ &= \frac{1}{|A|^2} \langle A \wedge B, \overset{\circ}{\eta}_1 \rangle \\ &= \frac{1}{\sqrt{2}(1+u_1^2+v_1^2)} (a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3) \end{aligned} \tag{2.36}$$

where $A = \Sigma a_A \overset{\circ}{e}_A$, $B = \Sigma b_A \overset{\circ}{e}_A$, and $w_j = u_j + iv_j$, $j = 1,2$.

Therefore,

$$\xi_1 = \frac{1 - u_1^2 - v_1^2}{\sqrt{2}\,(1+u_1^2 + v_1^2)}$$

By similar calculations,

$$\begin{cases} \xi_+ = \dfrac{1}{\sqrt{2}\,(1+u_1^2 + v_1^2)} (1 - u_1^2 - v_1^2 \ , 2v_1 \ , -2u_1) \\ \xi_- = \dfrac{1}{\sqrt{2}\,(1+u_2^2 + v_2^2)} (1 - u_2^2 - v_2^2 \ , -2v_2 \ , -2u_2) \end{cases} \tag{2.37}$$

This shows that the mapping

$$(u_1, u_2) \longrightarrow (\xi_1, \xi_2, \xi_3) \tag{2.38}$$

is the composition of the stereographic projection, the homothety with ratio $1/\sqrt{2}$ and a exchange of the coordinate axes. Note that this is the same way, except for the change of axes, in which $S^2 = \mathbb{C}\,\mathbf{P}^1$ was parametrized and given the metric. Hence, $\Psi \circ \Phi^{-1} \circ [\varphi]$ maps S_1^2 isometrically onto S_+^2. Similar for S_2^2.

Q.E.D.

In this sense, we identify

$$\text{(2.39)} \qquad \begin{cases} S_+^2 \times S_-^2 = D_1(2,4) = G(2,4) = Q_2 = S_1^2 \times S_2^2 \\ S_+^2 = S_1^2, \ S_-^2 = S_2^2 \end{cases}$$

We choose orientations on $\Lambda_+^2\mathbb{R}^4$ and $\Lambda_+^2\mathbb{R}^4$ such that $\{\mathring{\eta}_1, \mathring{\eta}_2, \mathring{\eta}_3\}$ and $\{\mathring{\eta}_4, \mathring{\eta}_5, \mathring{\eta}_6\}$ are positive basis respectively and also orientations on S_+^2 and S_-^2 corresponding to the exterior normal vectors.

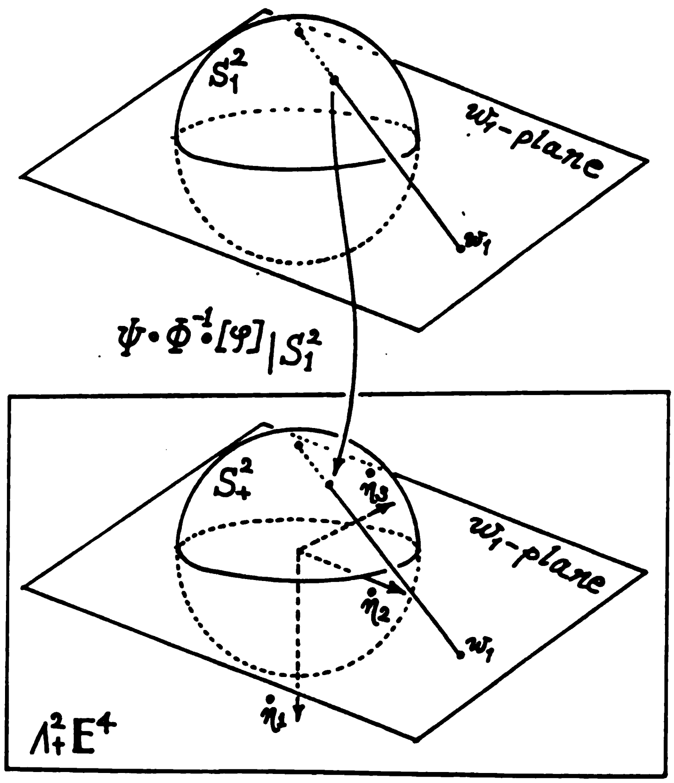


FIGURE 2

§ 3. The Gauss Map.

Let x be an immersion of an n-dimensional oriented differentiable manifold M into Euclidean m-space

(3.1) $$x : M \longrightarrow \mathbf{E}^m.$$

We always identify the tangent spaces of $\mathbf{E}^m$ with $\mathbf{E}^m$ itself. Then the Gauss map, or the generalized Gauss map,

(3.2) $$\nu : M \longrightarrow G(n, m)$$

is defined by

(3.3) $$\nu(p) = T_pM \subset T_p\mathbf{E}^m \equiv \mathbf{E}^m.$$

Let $\{e_A\}_{A=1}^m$ be an adapted local frame field on $\mathbf{E}^m$, i.e., a local positive orthonromal frame field so that $\{e_i(p)\}_{i=1}^n$ is a positive basis of T_pM if $p \in M$. Then, identifying $G(n, m)$ with $D_1(n, m)$,

(3.4) $$\nu(p) = (e_1 \wedge \ldots . \wedge e_n)(p)$$

If M is compact, then the Gauss image $\nu(M)$ is mass-symmetric in the unit sphere of $\Lambda^n \mathbf{E}^m$, according to Chen–Piccini ([CBY–P] Lemma 3.1), namely

(3.5) $$\int_{p \in M} \nu(p) dV_M = 0$$

where ν is considered as a $\Lambda^n \mathbf{E}^m$-valued function on M, and dV_M is the volume element of M with respect to the metric induced by the immersion x.

(3.6) $$\nu : M \longrightarrow G(n,m) = D_1(n,m) \subset S_1^{N-1}(0) \subset \Lambda^n\mathbf{E}^m = (\Lambda^n\mathbb{R}^m, < , >)$$
$$= \mathbf{E}^N,\ N = \binom{m}{n}$$

We rewrite this as follows for later use.

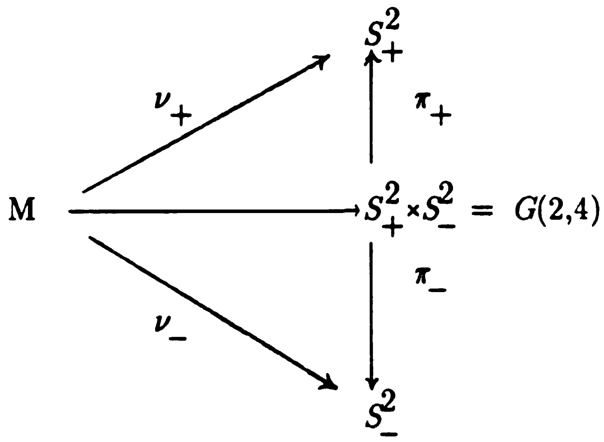
Lemma 2.

If M is compact, then

(3.7) $$\int_M < \nu, \xi > dV_M = 0$$

for any n-vector $\xi \in \Lambda^n\mathbf{E}^m$.

In the case of $n = 2$ and $m = 4$, we put $\nu_{\pm} = \pi_{\pm} \cdot \nu$ where $\pi_{\pm}$ are the projections defined in §1.



These maps ν_+ and ν_- are related to the Gaussian curvature G of M with respect to the induced metric and the normal curvature G^D of the immersion x as seen in the following lemma. This lemma is stated by Hoffman–Osserman ([H–O2] Proposition 4.5) in terms of the decomposition $G(2,4) = Q_2 = S^2_1 \times S^2_2$ but we prove it here in our notations.

Lemma 3.

(3.9)
$$\begin{cases} \textit{Jacobian of } \nu_+ = \frac{1}{2}(G + G^D) \\ \textit{Jacobian of } \nu_- = -\frac{1}{2}(G - G^D) \end{cases}$$

(Proof)

Let $\{\overset{\circ}{e}_A\}_{A=1}^4$ be the canonical basis of E^4 and $\{\tilde{e}_A\}_{A=1}^4$ be a positive local adapted frame field such that restricted to M, $\{\tilde{e}_i\}_{i=1}^2$ is a positive frame on M. *Let* $\{\tilde{\omega}^A\}$, $\{\tilde{\omega}^A_B\}$, $\{e_A\}$, $\{\omega^i\}$, $\{\omega^A_B\}$ be as in §1. Put

(3.11)
$$e_A = \Sigma\, \beta^B_A \overset{\circ}{e}_B$$

where (β_A^B) is a $SO(4)$–valued local function of $\mathbf{E}^4$. By (3.4) and (3.11)

(3.12)
$$\nu(p) = e_1(p) \wedge e_2(p) = \sum_{A<B} \xi^{AB}(p)\, \mathring{e}_A \wedge \mathring{e}_B$$

where

(3.13)
$$\xi^{AB} = \beta_1^A \beta_2^B - \beta_1^B \beta_2^A$$

are local functions on M. If we denote by (y^{AB}) $1 \le A < B \le 4$ the coordinate of $\Lambda^2\mathbf{E}^4$ with respect to the canonical basis $\{\mathring{e}_A \wedge \mathring{e}_B\}$ $1 \le A < B \le 4$, then, for a function φ on $\mathbf{E}^4$ and $X \in TM$,

(3.14)
$$\begin{aligned}(\nu_* X)\varphi &= X(\varphi \cdot \nu) \\ &= \sum_{A<B} \frac{\partial\varphi}{\partial y^{AB}} (X\, \xi^{AB}) \\ &= (\Sigma(X\xi^{AB}) \frac{\partial}{\partial y^{AB}})\, \varphi\end{aligned}$$

Since $\Lambda^2\mathbf{E}^4$ is a vector space, $\frac{\partial}{\partial y^{AB}}$ is identified with $\mathring{e}_A \wedge \mathring{e}_B$ and hence

(3.15)
$$\begin{aligned}\nu_* X &= \sum_{A<B} (X\, \xi^{AB})\, \mathring{e}_A \wedge \mathring{e}_B \\ &= \sum_{A<B} [(X\, \beta_1^A)\beta_2^B + \beta_1^A(X\, \beta_2^B) - (X\, \beta_1^B)\beta_2^A - \beta_1^B(X\beta_2^A)]\mathring{e}_A \wedge \mathring{e}_B \\ &= \sum_{A,B=1}^{4} (X\, \beta_1^A)\beta_2^B\ \mathring{e}_A \wedge \mathring{e}_B + \sum_{A,B=1}^{4} \beta_1^A(X\beta_2^B)\ \mathring{e}_A \wedge \mathring{e}_B \\ &= (\Sigma\ (X\, \beta_1^A)\mathring{e}_A) \wedge (\ \Sigma\ \beta_2^B\, \mathring{e}_B) + (\Sigma\ \beta_1^A \mathring{e}_A) \wedge (\ \Sigma(X\, \beta_2^B)\, \mathring{e}_B) \\ &= (\tilde{\nabla}_X\, e_1) \wedge e_2 + e_1 \wedge (\tilde{\nabla}_X\, e_2)\end{aligned}$$

Hence by (1.2)

(1.16)
$$\nu_* X = (\Sigma\ \tilde{\omega}_1^A(X)e_A) \wedge e_2 + e_1 \wedge (\Sigma\ \tilde{\omega}_2^B(X)e_B)$$

since $X \in TM$

(3.17)
$$\begin{aligned}\nu_* X = \Sigma\ &\omega_2^3(X)e_1 \wedge e_3 + \omega_2^4(X)e_1 \wedge e_4 \\ &- \omega_1^3(X)e_2 \wedge e_3 - \omega_1^4(X)e_2 \wedge e_4\end{aligned}$$

We use here $\{\eta_a\}_{a=1}^{6}$ defined by (2.11) and (2.12). Then

(3.18) $$\nu_* X = \frac{1}{\sqrt{2}}[(\omega_1^4 + \omega_2^3)(X)\eta_2 + (-\omega_1^3 + \omega_2^4)(X)\eta_3 + (-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6]$$

Hence

(3.19) $$\begin{cases} \nu_{+*}X = \frac{1}{\sqrt{2}}[(\omega_1^4 + \omega_2^3)(X)\eta_2 - (-\omega_1^3 + \omega_2^4)(X)\eta_3] \\ \nu_{-*}X = \frac{1}{\sqrt{2}}[(-\omega_1^4 + \omega_2^3)(X)\eta_5 + (\omega_1^3 + \omega_2^4)(X)\eta_6] \end{cases}$$

By (1.19)

(3.20) $$\begin{cases} \nu_{+*}(e_1) = \frac{1}{\sqrt{2}}[(h_{11}^4+h_{21}^3)\eta_2 + (-h_{11}^3+h_{21}^4)\eta_3] \\ \nu_{+*}(e_2) = \frac{1}{\sqrt{2}}[(h_{12}^4+h_{22}^3)\eta_2 + (-h_{12}^3+h_{22}^4)\eta_3] \end{cases}$$

(3.21) $$\begin{cases} \nu_{-*}(e_1) = \frac{1}{\sqrt{2}}[(-h_{11}^4+h_{21}^3)\eta_5 + (h_{11}^3+h_{21}^4)\eta_6] \\ \nu_{-*}(e_2) = \frac{1}{\sqrt{2}}[(-h_{12}^4+h_{22}^3)\eta_5 + (h_{12}^3+h_{22}^4)\eta_6] \end{cases}$$

Since at each $p \in M$

(3.22) $$\nu_+(p) = \frac{1}{\sqrt{2}}(e_1 \wedge e_2 + e_3 \wedge e_4)(p) = \frac{1}{\sqrt{2}}\eta_1(p),$$

$\eta_1(p)$ is the exterior normal of S_+^2 at $\nu_+(p)$ and hence $\{\eta_2, \eta_3\}$ is a positive orthonormal frame field on S_+^2 with respect to the orientation defined in §1.

On the other hand, by definitions of G and G^D ([CBY2]), and Gauss' and Ricci's equations,

(3.23) $$G = R(e_1, e_2; e_2, e_1) = \sum_{r=3}^{4}(h_{11}^r h_{22}^r - (h_{12}^r)^2)$$

(3.24) $$G^D = R^D(e_1, e_2; e_4, e_3) = h_{12}^3(h_{22}^4 - h_{11}^4) - h_{12}^4(h_{22}^3 - h_{11}^3).$$

By (3.20)–(3.24), we get (3.9) and (3.10) .

Q.E.D.

§ 4. Slant Immersions.

The notion of slant immersions has been defined recently by B.Y.Chen as a generalization of both holomorphic and totally real immersions ([CBY5]). In this section we introduce the definitions and some basic properties of slant immersions written in [CBY5].

Let

(4.1) $$x : M \to \tilde{M}$$

be an isometric immersion of a Riemannian manifold (M, g) with a Riemannian metric g into an almost Hermitian manifold $(\tilde{M}, \tilde{g}, \tilde{J})$ with an almost complex structure $\tilde{J}$ and almost Hermitian metric $\tilde{g}$.

For each nonzero vector X tangent to M at p the angle $\theta(X)$ between $\tilde{J}X$ and the tangent space T_pM of M at p is called the Wirtinger angle.

(4.2) $$\theta(X) = \measuredangle(\tilde{J}X,\ T_pM), \quad X \in T_pM$$

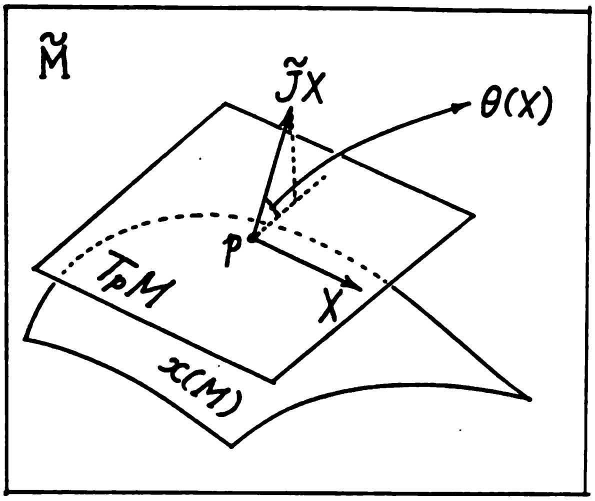


FIGURE 3

If $\theta(X)$ is constant, we call immersion x a general slant immersion.

If x is a totally real immersion, i.e.,

(4.3) $$\tilde{J}(T_pM) \subset T_p^{\perp}M \quad \text{for} \quad \forall\ p \in M$$

where T_pM denotes the normal space of M in $\tilde{M}$, then x is a general slant immersion with $\theta(X) \equiv \pi/2$.

If M is also an almost Hermitian manifold (M,g,J) and x is an holomorphic immersion, i.e.,

(4.4) $$x_*(JX) = \tilde{J}(x_*X) \quad \textit{for} \quad \forall\ X \in TM$$

where x_* is the differential of x, then x is a general slant immersion with $\theta(X) \equiv 0$. Similarly, an anti–holomorphic immersion satisfying

(4.5) $$x_*(JX) = -\tilde{J}(x_*X) \quad \textit{for} \quad \forall\ X \in TM$$

is a general slant immersion with $\theta(X) \equiv 0$.

In this sense, a general slant immersion is a generalization of totally real or holomorphic immersions. A general slant immersion with $\theta(X) \neq 0$ is called a slant immersion and the angle $\theta(X)$ is called the slant angle.

A slant immersion with $\theta(X) \neq \pi/2$ is called a proper slant immersion.

For any vector X tangent to M, we put

(4.5) $$JX = PX + FY, \qquad PX \in TM, \qquad FX \in T^{\perp}M.$$

Then, P is a (1,1)–tensor field on M and F is a $T^{\perp}M$–valued 1–form on M. A proper slant submanifold which satisfied $\nabla P \equiv 0$ is called a Kählerian slant submanifold. If (M, g) is a Kählerian slant submanifold with the slant angle θ, then, with respect to an almost complex structure $\mathfrak{J}$ defined by

(4.6) $$\mathfrak{J} = (\sec\ \theta)\ P,$$

$(M, g, \mathfrak{J})$ becomes a Kähler manifold.

Listed below are some of the lemmas, propositions and theorems in [CBY2] for later use.

(a) (Lemma 2) Let M be a submanifold of an almost Hermitian manifold $\tilde{M}$. Then, $\nabla P \equiv 0$ if and only if M is locally the Rienannian product $M_1 \times \dots \times M_k$, where each M_i is either a Kähler submanifold, a totally real submanifold, or a Kählerian slant submanifold.

This lemma comes from the decomposition $T_pM = D_p^1 \oplus \cdots \oplus D_p^{k(p)}$, where D_p^j's are the eigenspaces of the self-adjoint operator P^2.

(b) (Lemma 5) If M is a slant surface in $\mathbf{C}^2$, then $G = G^D$ identically. G, G^{D} are as in §3. Hence in this case ν_- is degenerated.

(c) (Proposition 5) Let M be a compact surface. Then we have

1. If the Euler number $\chi(M) \neq 0$, then M admits no slant embedding in $\mathbf{C}^2$.
2. If $\chi(M) = 0$, then every slant immersion of M in $\mathbf{C}^2$ is regularly homotopic to an embedding.
3. If M has positive (or negative) Gauss curvature, then M admits no slant immersion into $\mathbf{C}^2$.

(d) (Lemma 6) If M is a holomorphic surface in $\mathbf{C}^2$, then, for any constant α, $0 < \alpha \leq \pi/2$, M is a slant surface in (E^4, J_α) with slant angle α, where J_α is the compatible almost complex structure on E^4 defined by

$$J_\alpha(a,b,c,d) = (\cos\alpha)(-c,-d,a,b) + (\sin\alpha)(-b,a,d,-c) \tag{4.7}$$

In [CBY5] and [CBY3], and also in this §4, $\mathbb{C}^m$ is regarded as follows.

$$\mathbb{C}^m = (\mathbb{R}^{2m}, <\ ,\ >,\ J_o)$$

(4.7) $$J(x_1,\dots,x_m\ y_1,\dots,y_m) = (-y_1,\dots,-y_m,\ x_1,\dots,x_m)$$

(e) (Theorem 1) Let M be an oriented surface in $\mathbb{C}^2$. Then there is a compatible complex structure $\mathfrak{J}$ on E^4 such that M is holomorphic in $(E^4, \mathfrak{J})$ if and only if M is minimal.

(f) (Theorem 3) Let M be a proper slant surface in $\mathbb{C}^2$. Then there is a compatible almost complex structure J_1 on E^4 so that M is totally real in (E^4, J_1) if and only if M is minimal.

(g) (Theorem 4) Let M be a totally real surface in $\mathbb{C}^2$. Then there is a compatible almost complex structure J_1 on E^4 so that M is a proper slant surface in (E^4, J_1) if and only if M is minimal.

The properties (*b*), (*d*) (*e*), (*f*) and (*g*) are all explained by a simple characterization of slant surfaces as we will see in Ch 2. We will deal with the compact case (c) in Ch 3.

We also list here the examples of slant surfaces in [CBY5]. These examples have a common interesting feature as seen in Ch 2.

(Eg 1) For any non-zero constants a and b,

(4.8) $$x\ (u,\ v) = (a\cos u,\ b\cos v,\ a\sin u,\ b\sin v)$$

gives a compact totally real surface in $\mathbb{C}^2$ with $\tilde{\nabla} h = 0$.

Here $\tilde{\nabla}h$ is defined by (1.24).

(Eg 2) For any $a>0$

(4.9) $$x(u, v) = (a \cos u, v, a \sin u, 0)$$

defines a non–compact totally real surface in $\mathbb{C}^2$ with $\tilde{\nabla}h = 0$.

(Eg 3) For any α, $0<\alpha\leq\pi/2$,

(4.10) $$x(u, v) = (u \cos \alpha, u \sin \alpha, v, 0)$$

defines a slant plane with slant angle α in $\mathbb{C}^2$.

(Eg 4) For any positive constant k,

(4.11) $$x(u, v) = (e^{ku} \cos u \cos v, e^{ku} \sin u \cos v, e^{ku} \cos u \sin v, e^{ku} \sin u, \sin v)$$

defines a complete, non–minimal pseudo–umbilical proper slant surface in $\mathbb{C}^2$ with slant angle $\cos^{-1}(k \sqrt{1+k^2})$ and with mean curvature $e^{-ku}/\sqrt{1+k^2}$.

(Eg 5) For any positive number k,

(4.12) $$x(u, v) = (u, k \cos v, v, k \sin v)$$

defines a complete, flat, non–minimal and non–pseudo–umbilical, proper slant surface with slant angle $\cos^{-1}(k \sqrt{1+k^2})$ and constant mean curvature $k/2(1+k^2)$ and with non–parallel mean curvature vector.

(Eg 6) Let k be any positive number and $(g(s), h(s))$ a unit speed plant curve. then

(4.13) $$x\,(u,\ v) = (-ks\ sin\ u,\ g(s),\ ks\ cos\ u,\ h(s))$$

defines a non–minimal, flat, proper slant surface with slant angle $(k\ \sqrt{1+k^2})$.

There is another example of a Kählerian slant submanifold in $\mathbb{C}^4$.

CHAPTER 2.

A CHARACTERIZATION OF SLANT SURFACES

In [CBY–M] B.–Y. Chen and J.–M. Morvan characterized holomorphic surfaces and totally real surfaces in $\mathbb{C}^2$ using the description $G(2,4)=S^2_+ \times S^2_-$ as follows. Let $x : M \to \mathbf{E}^4=(\mathbb{R}^4, <\ ,\ >)$ be an isometric immersion of an oriented Riemannian surface into $\mathbf{E}^4$ and $\nu = (\nu_+, \nu_-)$ be its Gauss map defined in Ch 1 §3. Then,

(a) x is an holomorphic immersion with respect to some complex structure J on $\mathbf{E}^4$ compatible with $<\ ,\ >$ if and only if $\nu_+(M)$ is a singleton.

(b) x is a totally real immersion with respect to some complex structure J on $\mathbf{E}^4$ compatible with $<\ ,\ >$ if and only if $\nu_+(M)$ is contained in some great circle in S^2_+.

The purpose of this chapter is to show that (a) and (b) have a natural generalization to the case of slant immersions.

In §1 we consider the relation between 2–dimensional linear subspaces of $\mathbf{E}^4$ and complex structures on $\mathbf{E}^4$ compatible with $<\ ,\ >$. In § 2 we combine this with the Gauss map and characterize slant surfaces in $\mathbb{C}^2$. In § 3 we show that *most* surfaces in an almost Hermitian manifold $(\tilde{M}, \tilde{g}, \tilde{J})$ can be slant surfaces with any given slant angle with respect to some almost complex structures $\tilde{J}_1$'s so that $(\tilde{M}, \tilde{g}, \tilde{J}_1)$'s are almost Hermitian. This shows that the argument about slant surfaces in 4–dimensional almost Hermitian manifolds does not have much significance and also that there is a big difference between almost Hermitian manifolds and Kähler manifolds.

§ 1. Complex structures on $\mathbf{E}^4$

Let $\mathbb{C}^m$ be the complex m–space with the canonical complex structure J_o.

(1.1) $$\mathbb{C}^m = (\mathbb{R}^{2m}, <\ ,\ >, J_o)$$

where

(1.2) $$J_o(x_1, y_1, \dots, x_m, y_m) = (-y_1, x_1, \dots, -y_m, x_m).$$

If we use the canonical basis $\{\overset{\circ}{e}_A\}_{A=1}^{2m}$ *of* $\mathbf{E}^{2m} = (\mathbb{R}^{2m}, <\ ,\ >)$ then

(1.3) $$J_o \overset{\circ}{e}_{2A-1} = \overset{\circ}{e}_{2A} \quad \textit{and hence} \quad J_o \overset{\circ}{e}_{2A} = -\overset{\circ}{e}_{2A-1} \quad \textit{for} \quad A=1,\dots,m.$$

Note that this is different from [CBY3], [KN] and Ch 1 §4, and this is the only difference between our notations and those of [CBY3]. J_o is an orientation–preserving isomorphism of $\mathbf{E}^{2m}$. In this section we consider the case $m = 2$.

We denote by $\mathcal{J}$ the set of all complex structures on $\mathbf{E}^4$ compatible with $<\ ,\ >$, i.e.,

(1.4) $$\mathcal{J} = \{J : \mathbf{E}^4 \to \mathbf{E}^4 \,|\, \textit{linear},\ J^2 = -id,$$
$$<JX, JY> = <X, Y> \quad \textit{for} \ \forall\, X, Y \in \mathbf{E}^4\}.$$

For each $J \in \mathcal{J}$, we can always choose a J–basis $\{e_A\}_{A=1}^4$, i.e., an orthonormal basis satisfying

(1.5) $$Je_1 = e_2, \quad Je_3 = e_4.$$

Two J–bases of the same J have the same orientation. Hence using the canonical orientation $\omega = \overset{\circ}{e}_1 \wedge \cdots \wedge \overset{\circ}{e}_4$. We divide $\mathcal{J}$ into two disjoint subsets:

(1.6) $$\begin{cases} \mathcal{J}^+ = \{J \in \mathcal{J} \,|\, J\text{–bases are positive}\} \\ \mathcal{J}^- = \{J \in \mathcal{J} \,|\, J\text{–bases are negative}\} \end{cases}$$

For each $J \in \mathcal{J}$, we determine a unique 2–vector $\zeta_J \in \Lambda^2 E^4$ as follows. Let Ω_J be the Kähler form of J.

(1.7) $$\Omega_J(X, Y) = \langle X, JY \rangle \quad X, Y \in E^4, \quad \Omega_J \in \Lambda^2(E^{4*})$$

Since $\Lambda^2(E^{4*})$ is identified with $(\Lambda^2 E^4)^*$ by Ch 1(2.4), we can set ζ_J to be the metric dual of $-\Omega_J \in (\Lambda^2 E^4)^*$ with respect to the metric $\langle\ ,\ \rangle$ of $\Lambda^2 E^4$ defined by Ch 1(2.3). Hence, for $X, Y \in E^4$,

(1.8) $$\begin{aligned} \langle \zeta_J , X \wedge Y \rangle &= -\Omega_J(X \wedge Y) \\ &= -\Omega_J(X, Y) \\ &= -\langle X, JY \rangle \\ &= \langle JX, Y \rangle \end{aligned}$$

We have the following lemma.

Lemma 4

The mapping

(1.9) $$\zeta : \mathcal{J} \to \Lambda^2 E^4 \ ; \quad J \to \zeta_{\mathcal{J}}$$

determines bijections

(1.10) $$\begin{cases} \zeta : \mathcal{J}^+ \to S^2_+(\sqrt{2}) \\ \zeta : \mathcal{J}^- \to S^2_-(\sqrt{2}) \end{cases}$$

where $S^2_\pm(\sqrt{2})$ *are 2–spheres with radius* $\sqrt{2}$ *centered at the origin in* $\Lambda^2_\pm E^4$.

(Proof)

Let $J \in \mathcal{J}$ and $\{e_A\}_{A=1}^4$ be a J–basis. If $J \in \mathcal{J}^+$ (or $\mathcal{J}^-$), then $\{e_A\}$ is a positive (or negative respectively) basis and vice versa. By (1.7)

(1.11) $$\zeta_J = e_1 \wedge e_2 + e_3 \wedge e_4$$

Hence, by Ch 1(2.3)

(1.12) $$\|\zeta_J\| = \sqrt{2}$$

By (1.10), if $J \in \mathcal{J}^{\pm}$ then $*\zeta_J = \pm\zeta_J$ and so $\zeta_J \in \Lambda^2_{\pm}\mathbf{E}^4$, and hence $\zeta_J \in S^2_{\pm}(\sqrt{2})$. The injectivity of the map ζ is clear.

Conversely, let $\xi \in S^2_+(\sqrt{2})$. Then, $\frac{1}{2}\xi \in S^2_+$ and hence we can pick an oriented 2–plane V such that

(1.13) $$V \in \pi_+^{-1}(\tfrac{1}{2}\xi) \subset G(2,4).$$

Choose a positive adapted frame $\{e_A\}_{A=1}^4$ of V in $\mathbf{E}^4$ and define a complex structure J on $\mathbf{E}^4$ by

(1.14) $$Je_1 = e_2, \quad Je_2 = -e_1, \quad Je_3 = e_4, \quad Je_4 = -e_3.$$

Then, $J \in \mathcal{J}^+$ and $\zeta_J = \xi$. Similarly for $\mathcal{J}^-$.

(1.14′) $$\mathcal{J}^- e_1 = e_2, \quad \mathcal{J}^- e_2 = -e_1, \quad \mathcal{J}^- e_3 = -e_4, \quad \mathcal{J}^- e_4 = e_3.$$

Q. E. D.

Through the bijection of Lemma 4, we can identify these sets.

(1.15) $$\mathcal{J} \equiv S^2_+(\sqrt{2}) \cup S^2_-(\sqrt{2})$$
$$S^2_+(\sqrt{2}) \equiv \mathcal{J}^+, \; S^2_-(\sqrt{2}) \equiv \mathcal{J}^-.$$

Next, we consider slant 2–planes in $(\mathbf{E}^4, <\,,\,>, J)$ for $J \in \mathcal{J}$. Before that we deform the definition of the slant angle slightly.

Definition 1.

For $V \in G(2,4)$ *and* $J \in \mathcal{J}$, *put*

(1.16) $$\alpha_J(V) = \cos^{-1}(-\Omega_J(V)) \in [0,\pi]$$

and call V *to be a–slant with respect to* J *if* $\alpha_J(V) = a$.

The relation between $\theta(X)$ of Ch 1(4.2) and $\alpha_J(V)$ is as follows. Let $x : M \to (\tilde{M},\tilde{g},\tilde{J})$ be an immersion of a 2–dimensional differentiable manifold M into an 4–dimensional almost Hermitian manifold $\tilde{M}$. Then, regarding $(T_p\tilde{M}, \tilde{g}, \tilde{J}) \equiv (\mathbf{E}^4, <\,,\,>, \tilde{J})$,

$$\theta(X) = \min\{\alpha_{\tilde{J}}(T_pM), \pi - \alpha_{\tilde{J}}(T_pM)\} \qquad \text{for } X \in T_pM. \tag{1.17}$$

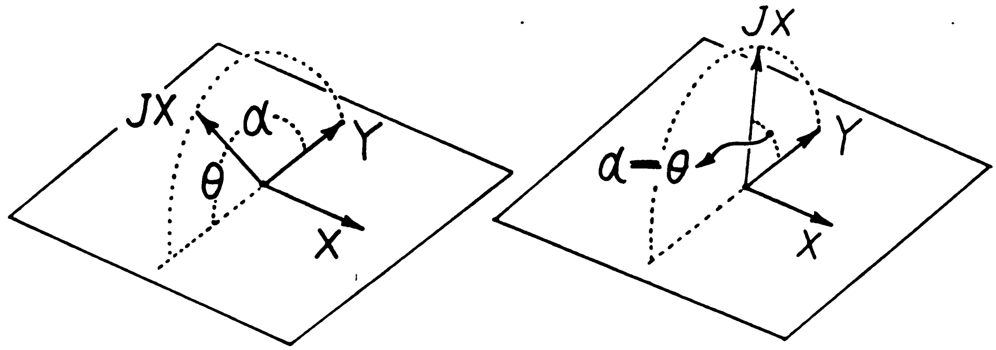


FIGURE 4

If M is oriented, then it has a unique complex structure J determined by the orientation and the induced metric, with respect to it M is a Kähler manifold. Hence,

(1.18) x is a holomorphic immersion

$$\rightleftarrows \tilde{J}\, x_*X = x_*JX \quad \forall\, X \in TM$$

$$\rightleftarrows \langle \tilde{J}\, x_*X,\, x_*JX\rangle = 1, \quad X \in TM,\ \|X\|=1$$

$$\rightleftarrows \Omega_{\tilde{J}}(x_*X \wedge x_*(JX)) = -1, \quad X \in TM,\ \|X\|=1$$

$$\rightleftarrows \alpha_{\tilde{J}}(T_pM) = 0$$

Similarly

(1.19) *x; anti-holomorphic immersion*

$$\rightleftarrows \alpha_{\tilde{J}}(T_pM) = \pi$$

(1.20) *x; totally real immersion*

$$\rightleftarrows \alpha_{\tilde{J}}(T_pM) = \pi/2$$

This argument holds also for dim $\tilde{M} > 4$, and we note here that the angle α_J coincides with the angle defined by Chern and Wolfson in [CSS–W1], although they look quite different.

The angle α_J can be also described in the following way.

Lemma 5.

If $J \in \mathcal{J}^+$, *then* $\alpha_J(V)$ *is the angle between* $\pi_+(V)$ *and* ζ_J. *If* $J \in \mathcal{J}^-$, *then* $\alpha_J(V)$ is the *angle between* $\pi_-(V)$ *and* ζ_J.

(Proof)

Let $J \in \mathcal{J}^+$. Then by (1.16), (1.7)

$$\begin{aligned} \cos(\alpha_J(V)) &= -\Omega_J(V) \\ &= \langle \zeta_J , V \rangle \\ &= \langle \zeta_J , \pi_+(V)+\pi_-(V) \rangle \\ &= \langle \zeta_J , \pi_+(V) \rangle \end{aligned} \tag{1.21}$$

since $\zeta_J \in S^2_+(\sqrt{2}) \subset \Lambda^2_+ E^4$. Note that $\|\zeta_J\| = \sqrt{2}$ and $\|\pi_+(V)\| = 1/\sqrt{2}$. Similarly for $J \in \mathcal{J}^-$. Q.E.D.

For each $a \in [0,\pi]$ and $J \in \mathcal{J}$, we define $G_{V,a}$ to be the set of all oriented 2–planes in E^4 which are a–slant with respect to J, i.e.,

$$G_{J,a} = \{ V \in G(2,4) \mid \alpha_J(V) = a \} \tag{1.22}$$

and also, for each $a \in [0,\pi]$ and $V \in G(2,4)$, we put $\mathcal{J}_{V,a}$ to be the set of all complex structures on E^4 compatible with the metric with respect to which V is a–slant, i.e.,

$$\mathcal{J}_{V,a} = \{ J \in \mathcal{J} \mid \alpha_J(V) = a \}. \tag{1.23}$$

Put

$$\mathcal{J}^{\pm}_{V,a} = \mathcal{J}_{V,a} \cap \mathcal{J}^{\pm}.$$

Then we can "visualize" these sets as follows.

Proposition 1.

(i) *If* $J \in \mathcal{J}^+$, *then*

$$G_{J,a} \doteq S^1_{J,a} \times S^2_-$$

where $S^1_{J,a}$ *is the circle on* S^2_+ *consisting of the* 2*–vectors which have the angle* a *between* ζ_J. *If* $J \in \mathcal{J}^-$, *then*

$$G_{J,a} = S^2_+ \times S^1_{J,a}$$

where $S^1_{J,a}$ *is a circle on* S^2_- *defined similarly.*

(ii) *Under the identification of* (1.15), $\mathcal{J}^+_{V,a}$ *is a circle on* $S^2_+(\sqrt{2})$ *consists of the* 2*–vectors which have the angle* a *between* $\pi_+(V)$.

$\mathcal{J}^-_{V,a}$ *is a circle on* $S^2_-(\sqrt{2})$ *defined similarly by* $\pi_-(V)$.

(Proof)

Direct from Lemma 5.

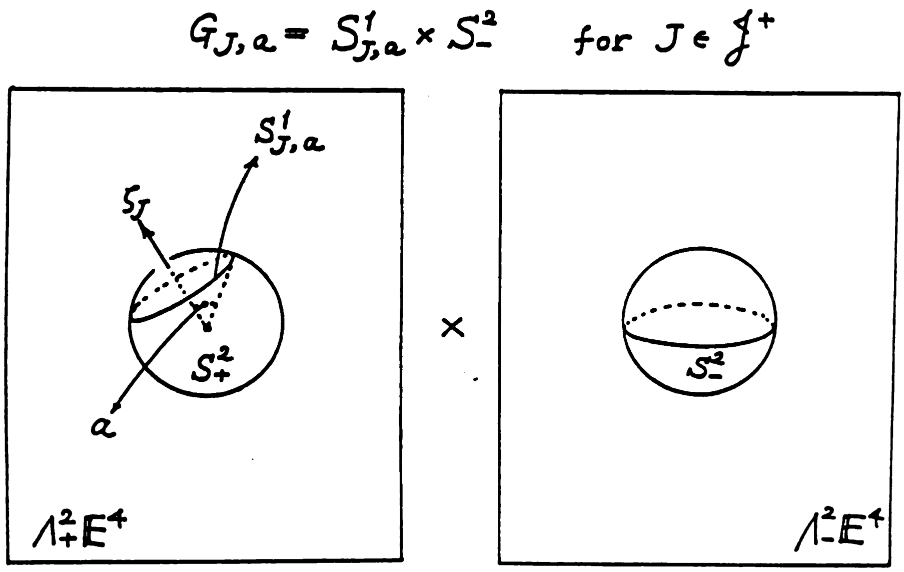


FIGURE 5

$\mathcal{J} = \mathcal{J}^+ \cup \mathcal{J}^- \xrightarrow[=]{\zeta} S^2_+(\sqrt{2}) \cup S^2_-(\sqrt{2})$

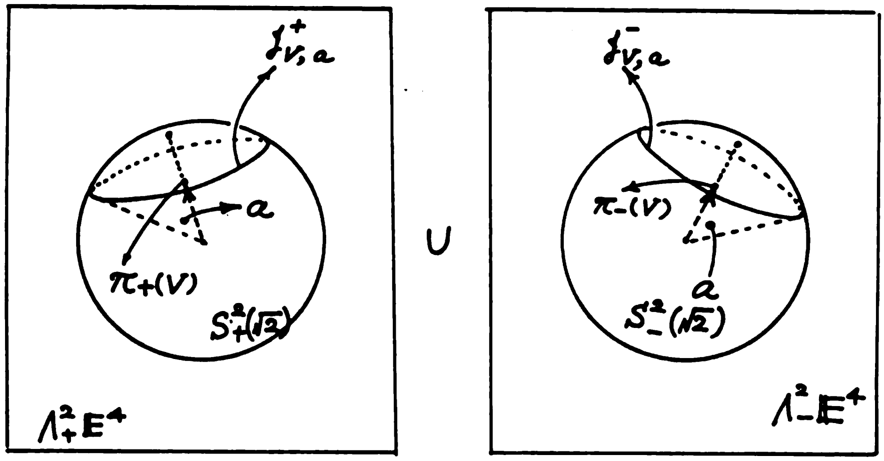


FIGURE 6

We state the following lemma which we need in Chapter 4 and 5.

Lemma 6

Let $W_0 \in G(3,4)$ *and* $V \in G(2,4)$ *such that* $V \subset W_0$. *Then* V *is* a*–slant with respect to* $J \in \mathcal{J}$ *if and only if*

(1.24) $$<\hat{\nu},\ J\eta> \ = \ - \cos a$$

where $\hat{\nu}$ *and* η *are positive unit normal vectors of* V *and* W_0 *in* W_0 *and* E^4 *respectively.*

(**Proof**)

We put $\tilde{W} = W \cap JW$. Then, $\tilde{W}$ is a 2–dimensional J–invariant linear subspace of E^4. We choose an orthonormal J–basis $\{e_A\}$ of E^4 such that

$$(1.25)\qquad \begin{cases} e_1,\ e_2 = Je_1 \in \tilde{W}, \\ e_4 = Je_3 \ = \eta \end{cases}$$

Then $e_3 \in \tilde{W}^{\perp} \cap W$ *and* $\{e_1, e_2, e_3\}$ is a positive orthonormal basis of W. Let $\{X_1, X_2\}$ be a positive orthonormal basis of V. Since $V \subset W$, $X_1 \wedge X_2$ is spanned by $\{e_1\wedge e_2, e_1\wedge e_3, e_2\wedge e_3\}$ and hence by (1.21) and (1.11)

$$(1.26)\qquad \begin{aligned} cos(\alpha_J(v)) &= <\zeta_J, V> \\ &= <e_1\wedge e_2+e_3\wedge e_4, X_1\wedge X_2> \\ &= <e_1\wedge e_2, X_1\wedge X_2> \end{aligned}$$

Since W is a Euclidean 3–space, the wedge product $\wedge$ in W is identified with the usual vector product $\times$, more precisely, the map defined by

$$(1.27)\qquad \begin{cases} f : \Lambda^2(W) \to W \\ f(X \wedge Y) = X \times Y \quad and \quad linear \end{cases}$$

is an a isomorphism preserving the inner product. Therefore,

$$(1.28)\qquad \begin{aligned} cos(\alpha_J(v)) &= <e_3, \hat{\nu}> \\ &= -< J\eta, \hat{\nu}> \end{aligned}$$

Q.E.D.

We also use the following notation. For $V \in G(2,4)$, we denote by J_V^+ and J_V^- the complex structures determined by

$$(1.29)\qquad J_V^{\pm} = \zeta^{-1}(\pi_{\pm}\ (V))$$

where ζ is the bijection in Lemma 4 and hence $J_V^{\pm} \in \mathcal{J}^{\pm}$ respectively.

§ 2. Slant Surfaces in $\mathbb{C}^2$

Let x be an immersion of a surface M into a 4–dimensional almost Hermitian manifold $(\tilde{M}, \tilde{g}, \tilde{J})$. If we fix a point p in M, then we can use the argument in § 1 about the slant angle of T_pM in $(T_p\tilde{M}\,,\,\tilde{g}_p)$ with respect to $\tilde{J}_p$. But, in order to compare the situations at different points, we need some global structure.

In this section we assume $\tilde{M} = \mathbf{E}^4$ and choose the parallel displacement in $T\tilde{M}$, i.e., the identification of $T_p\mathbf{E}^4$ and $\mathbf{E}^4$, as a "global structure". In short we use the Gauss map. We note that the argument in this section also holds when $\tilde{M}$ is a Riemannian quotient $\mathbf{E}^4/\Gamma$ by some discontinuous group, since it is parallelizable. In Ch 4 we will consider a different "global structure" using left invariant vector fields. Another interesting example of this "global structure" has been given by Micallef and Wolfson ([*M–W*]): if $\tilde{M}$ is a Ricci flat K3 surface then $\Lambda^2_+(T\tilde{M})$ is a flat bundle over $\tilde{M}$ and we can use parallel displacement in $\Lambda^2_+(T\tilde{M})$ instead of $T\tilde{M}$.

A slant surface in $\mathbb{C}^2$ is characterized as follows.

Proposition 2.

Let x be an immersion of a surface M into $\mathbf{E}^4$.

(2.1) $$x : M \longrightarrow \mathbf{E}^4$$

Then, x is a–slant with respect to $J \in \mathcal{J}^+$ if and only if

(2.2) $$\nu_+(M) \subset S^1_{J,a} \subset S^2_+$$

where $S^1_{J,a}$ is the circle in S^2_+ defined in Proposition 1. *The same holds replacing $+$ with $-$.*

(**Proof**)

Direct from Proposition 1 and the definition of $\nu_\pm$.

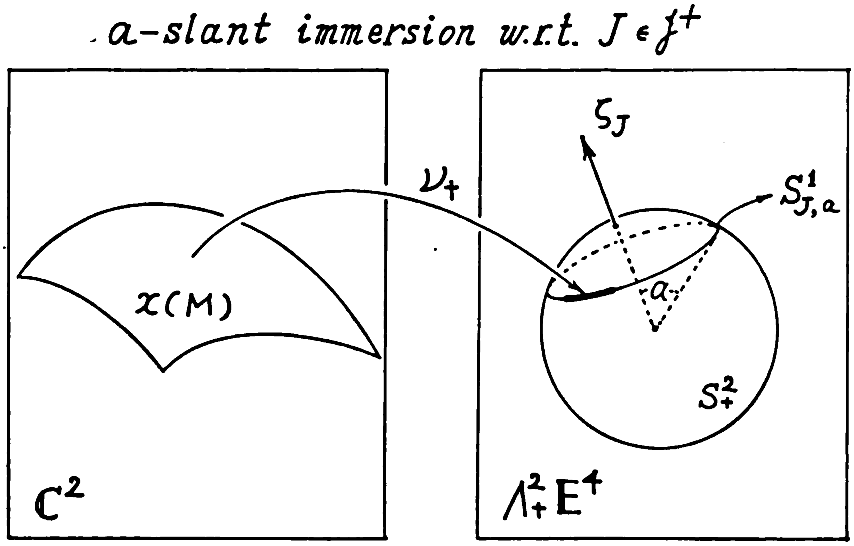


FIGURE 7

The following lemma is shown in [CBY 5] (in the proof of Theorem 1). We state and prove it again in our notation.

Lemma 9.

Let x *be an immersion of a surface* M *into* $\mathbf{E}^4$. *Then,* x *is minimal and slant with respect to some* $J \in \mathcal{J}^+$ *if and only if* $\nu_+(M)$ *is a singleton. The same holds replacing* $+$ *with* $-$.

(Proof)

Assume x is minimal. Then ν_1 and ν_2 are both anti-holomorphic ([CSS 1] and also cf [H–O3]). In particular ν_1 and ν_2 are open maps if they are not constant. Hence by Lemma 1, ν_+ and ν_- are open maps if they are not constant. Furthermore, if x is slant with respect to $J \in \mathcal{J}^+$, then by Proposition 2 ν_+ cannot be an open map and hence $\nu_+(M)$ is a singleton.

Conversely, let $\nu_+(M)$ be a singleton $\{\xi\}$. Then $2\xi \in S^2_+(\sqrt{2})$ and we can choose a complex structure $J = \zeta^{-1}(2\xi) \in \mathcal{J}^+$ determined by Lemma 4. By Proposition 2, x is 0–slant, i.e., holomorphic with respect to J and hence minimal.

Q. E. D.

By Proposition 2 and Lemma 9, we can say:

(h) The following are equivalent:

(i) x is minimal and slant with respect to some $J \in \mathcal{J}^+$ (or $\mathcal{J}^-$)

(ii) $\nu_+(M)$ (or $\nu_-(M)$) is a singleton.

(iii) x is holomorphic with respect to some $J \in \mathcal{J}^+$ (or $\mathcal{J}^-$)

(iv) For any $a \in [0, \pi]$, there exists $J_a \in \mathcal{J}^+$ (or $\mathcal{J}^-$) such that x is a–slant with respect to J_a.

(ii)⇒(iv) is shown as follows. If $\nu_+(M)$ is a singleton, we choose some $V \in \pi_+^{-1}(\nu_+(M))$ and x is a–slant with respect to any $J \in \mathcal{J}^+_{V,a}$
(iv)⇒(ii) follows from (i) below.

If x is a non–minimal a–slant immersion with respect to $J \in \mathcal{J}^+$, then $\nu_+(M)$ contains a 1–dimensional portion of the circle $S^1_{J,a}$, hence we have;

(i) If x is not minimal, then x can be slant with respect to at most two complex structures $\pm J \in \mathcal{J}^+$, and at most two complex structures $\pm J' \in \mathcal{J}^-$.

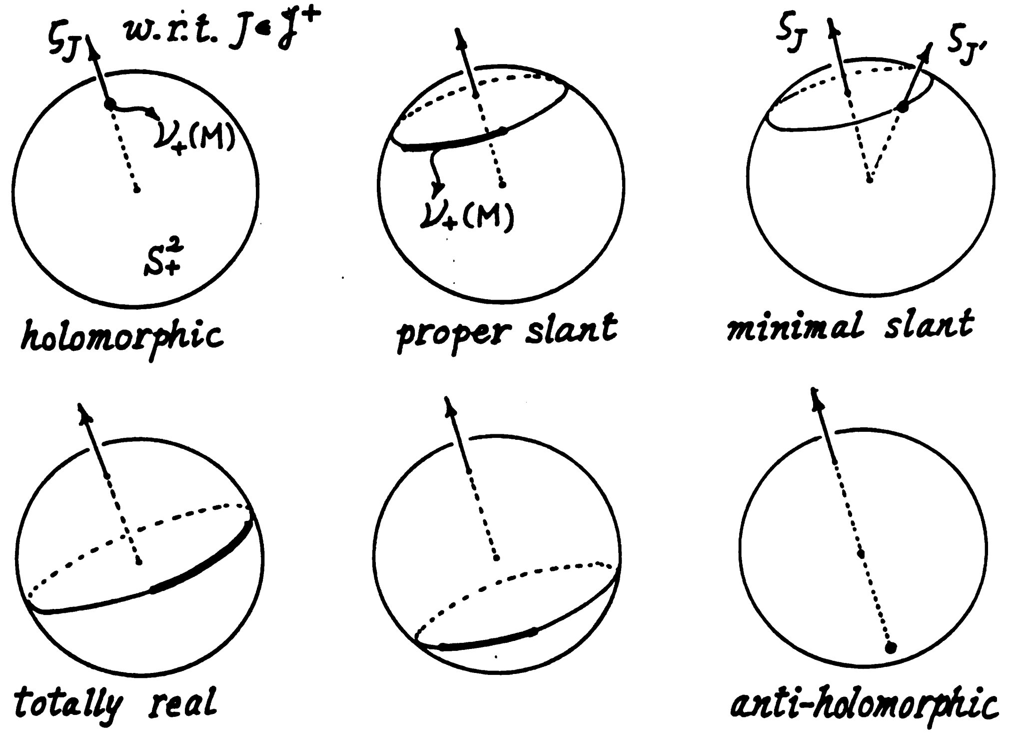


FIGURE 8

These facts give a clear geometric image to (*b*), (*d*), (*e*), (*f*) and (*g*) in Ch 1 §4. And they show that minimal surfaces are completely atypical (cf [H–O3] p 731) also from the view point of slant immersions.

We define the following term, because this property is common in examples (Eg 1) – (Eg 6) in Ch 1 § 4.

Definition 2.

An immersion $x : M^2 \to E^4$ *is called doubly slant if it is slant with respect to one complex structure* $J \in \mathcal{J}^+$ *and at the same time slant with respect to another complex structure* $J' \in \mathcal{J}^-$.

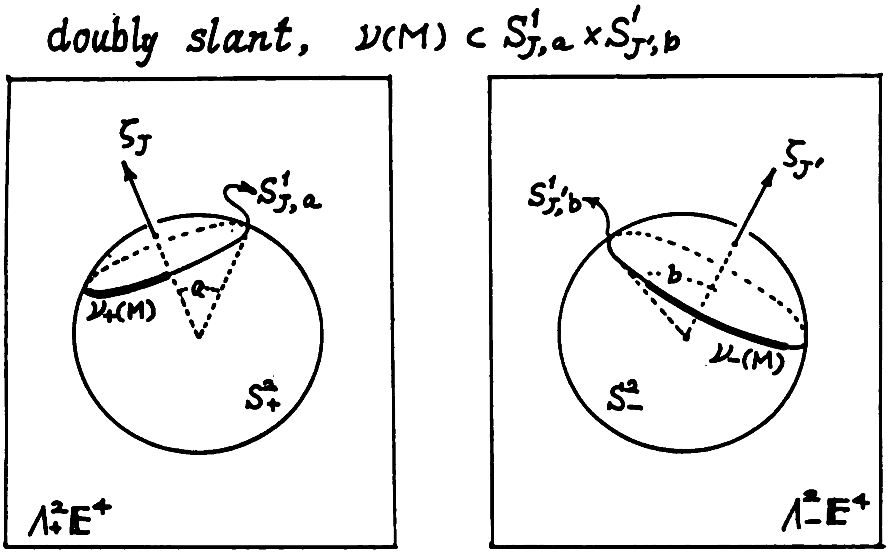


FIGURE 9

Or equivalently we can say, x is doubly slant if and if there exists $V \in G(2,4)$ such that x is slant with respect to J^+_V and J^-_V, where $J^\pm_V$ is defined by (1.29).

Lemma 10.

If $x : M^2 \to E^4$ *is a doubly slant immersion, then* $G \equiv G^D \equiv 0$.

(Proof)

Direct from Lemma 3 and Proposition 2.

Q.E.D.

The next lemma shows how to calculate slant angles and ranks of ν, ν_+ and ν_-.

Lemma 11.

Let $x : M^2 \to E^4$ *be an immersion of an oriented surface and put*

$$\text{(2.3)} \qquad \begin{cases} a(p)=\alpha_{J_o}(T_pM) \\ b(p)=\alpha_{J_1}(T_pM) \end{cases}$$

where $J_0=J^+_{\mathring{e}_1\wedge\mathring{e}_2}$ *and* $J_1=J^-_{\mathring{e}_1\wedge\mathring{e}_2}$ *defined by* (1.29). Let $\{e_1, e_2\}$ *be a positive orthonormal frame field on* M *and put*

$$\text{(2.4)} \qquad e_i = \sum_{B=1}^{4} \beta_i^B \mathring{e}_B , \qquad i = 1,2$$

Then

$$\text{(2.5)} \qquad \cos a = (\beta_1^1\beta_2^2-\beta_1^2\beta_2^1) + (\beta_1^3\beta_2^4-\beta_1^4\beta_2^3)$$

$$\text{(2.6)} \qquad \cos b = (\beta_1^1\beta_2^2-\beta_1^2\beta_2^1) - (\beta_1^3\beta_2^4-\beta_1^4\beta_2^3)$$

$$\text{(2.7)} \qquad \nu_* e_i = \sum_{A<B} [e_i(\beta_1^A\beta_2^B-\beta_1^B\beta_2^A)] \, \mathring{e}_A \wedge \mathring{e}_B$$

(Proof)

By (1.21), (1.11) and Ch 1 (3.12), (3.13),

$$\text{(2.8)} \qquad \begin{aligned} \cos a &= < \zeta_{J_o} , \nu(p)> \\ &= <\mathring{e}_1\wedge\mathring{e}_2 + \mathring{e}_3\wedge\mathring{e}_4, \sum_{A<B} \xi_{AB}\mathring{e}_A\wedge\mathring{e}_B > \\ &= \xi_{12} + \xi_{34} \\ &= (\beta_1^1\beta_2^2-\beta_1^2\beta_2^1) + (\beta_1^3\beta_2^4-\beta_1^4\beta_2^3) \end{aligned}$$

Similarly for *cos b*. (2.7) is Ch 1 (3.5).

Q.E.D.

The following is an example of a doubly slant surface with given slant angles $a, b \in (0, \pi/2)$.

(Eg 7)

Let p, q be a non-zero real numbers and define an immersion x by

(2.9) $$x : \mathbb{R} \times (0, \infty) \to \mathbf{E}^4$$

$$x(u,v) = (pv \sin u, pv \cos u, v \sin qu, v \cos qu)$$

Then, using the notations of Lemma 11, x is doubly slant with respect to J_0 and J_1 and the slant angles are given by

(2.10) $$\begin{cases} \cos a = (p^2+q)/\sqrt{(p^2+q)(p^2+1)} \\ \cos b = (p^2-q)/\sqrt{(p^2+q)(p^2+1)} \end{cases}$$

and

(2.11) $$\text{rank } \nu \equiv \text{rank } \nu_+ \equiv \text{rank } \nu_- \equiv 1$$

We can also show by an elementary calculation that for any a and b in $(0, \pi/2)$ there exist non-zero real numbers p and q which satisfy (2.10).

In fact Eg 1 – Eg 6 of [CBY 5] stated in Ch 1 §4 are all doubly slant as is seen in the following table.

	slant angles	*rank* ν	*rank* ν_+	*rank* ν_-
Eg 1	$a=b=\pi/2$	2	1	1
Eg 2	$a=b=\pi/2$	1	1	1
Eg 3	$a=b\in[0,\pi/2]$	0	0	0
Eg 4	$a\in(0,\pi/2], b=\pi/2$	2	1	1
Eg 5	$a=b\in(0,\pi/2)$	1	1	1
Eg 6	$a=b\in(0,\pi/2)$	not constant	1	1
Eg 7	$a, b\in(0,\pi/2)$	1	1	1

Except for Eg 4, a and b are slant angles with respect to J_0 and J_1 of Lemma 11. In Eg 4, a = slant angle with respect to J_0 and b = slant angle with respect to $J^-_{e_1 \wedge e_3}$. In Eg 6, rank $\nu = 1$ on $M' = \{p \in M | g'(p) = h'(p) = 0\}$ and rank $\nu = 2$ on $M \backslash M'$. Note that all examples are flat, and proper slant examples have rank $\nu < 2$. Note also that if an immersion x is a–slant with respect to J then x is $(\pi - a)$–slant with respect to $-J$.

Eg 1 – Eg 6 are adjusted here to match our complex structure by

(2.12) $$(x_1, x_2, x_3, x_4) \to (x_1, x_3, x_2, x_4).$$

§ 3. Slant Surfaces in 4–dimensional Almost Hermitian Manifolds.

Consider an immersion x of a differentiable manifold M into an almost complex manifold $(\tilde{M}, J)$. Then, a point $p \in M$ is called a complex tangent point if the tangent space T_pM of M at p is invariant in $T_p\tilde{M}$ under the action of J. The purpose of this section is to show the following proposition.

Proposition 3.

Let x be an embedding of an oriented surface into a 4–dimensional almost Hermitian manifold $(\tilde{M}, g, J)$. Assume that x has no complex tangent point. Then, for any angle $a \in (0, \pi)$ there exists an almost complex $\tilde{J}$ on $\tilde{M}$ satisfying the following conditions.

(i) *$(\tilde{M}, g, \tilde{J})$ is almost Hermitian manifold.*

(ii) *x is an a–slant immersion with respect to* $\tilde{J}$.

(Proof)

We follow several steps.

Step 1. An almost complex structure on $\tilde{M}$ is considered as a cross–section of a sphere bundle $\hat{S}^2_+(\tilde{M})$.

Step 2. The tangent bundle TM of M corresponds to a cross–section of the pull–back $\hat{S}^2_+(M) = x^*(\hat{S}^2_+(\tilde{M}))$.

Step 3. We construct a suitable cross–section σ of $\hat{S}^2_+(M)$.

Step 4. We extend σ to a cross–section $\hat{\sigma}$ of $\hat{S}^2_+(\tilde{M})$ to obtain a desired almost complex structure.

Step 1

M has the natural orientation determined by J. We note again that at each point $p \in \tilde{M}$, $(T_p\tilde{M}, g_p)$ is a Euclidean 4–space and we can apply the argument of §1.

The bundle $\Lambda^2(\tilde{M})$ of 2–vectors is a direct sum of two bundles.

(3.1) $$\Lambda^2(\tilde{M}) = \Lambda^2_+(\tilde{M}) \oplus \Lambda^2_-(\tilde{M})$$

where

(3.2) $$\Lambda^2(T_p\tilde{M}) = \Lambda^2_+(T_p\tilde{M}) \oplus \Lambda^2_-(T_p\tilde{M}), \quad \forall \ p \in M$$

in the sense of Ch 1(2.9) and

(3.2) $$\Lambda^2_\pm(\tilde{M}) = \bigcup_{p \in \tilde{M}} \Lambda^2_\pm(T_p\tilde{M}).$$

We define two bundles over $\tilde{M}$ by

(3.3) $$S^2_+(\tilde{M}) = \{\xi \in \Lambda^2_+(\tilde{M}) | \ |\xi| = 1/\sqrt{2}\}$$

(3.4) $$\hat{S}^2_+(\tilde{M}) = \{\xi \in \Lambda^2_+(\tilde{M}) | \ |\xi| = \sqrt{2}\}$$

Then a cross–section

(3.5) $$\rho : \tilde{M} \to \hat{S}^2_+(\tilde{M})$$

determines at each point $p \in M$ a complex structure $(J_\rho)_p \in \mathcal{J}^+$ on $T_p\tilde{M}$ by Lemma 4 compatible with g_p in the following sense

(3.6) $$g_p((J_\rho)_p X, (J_\rho)_p Y) = g_p(X, Y), \quad \forall\ X, Y \in T_p\tilde{M}.$$

And hence ρ determines an almost complex structure J_ρ so that $(\tilde{M}, g, J_\rho)$ is an almost Hermitian manifold, and vice versa.

Step 2

We consider the pull-backs of these bundles by the immersion x.

(3.7) $$\Lambda^2_+(M) = x^*(\Lambda^2_+(\tilde{M}))$$

(3.8) $$S^2_+(M) = x^*(S^2_+(\tilde{M}))$$

(3.9) $$\hat{S}^2_+(M) = x^*(\hat{S}^2_+(\tilde{M}))$$

Then the tangent bundle TM of M determines a cross-section τ of $S^2_+(M)$, i.e.,

(3.10) $$\tau : M \to S^2_+(M)$$

(3.11) $$\tau(p) = \pi_+(T_pM) \qquad \text{for } \forall\ p \in M$$

where π_+ is a projection of $\Lambda^2(T_p\tilde{M})$ onto $\Lambda^2_+(T_p\tilde{M})$. Note that 2τ is a cross-section of $\hat{S}^2_+(M)$.

(3.12) $$2\tau : M \to \hat{S}^2_+(M)$$

and we have another cross-section $x^*\rho$, which we also denote by ρ, of $\hat{S}^2_+(M)$.

(3.13) $$\rho : M \to \hat{S}^2_+(M)$$

Step 3

Let $p \in M$. By the assumption that p is not a complex tangent point, (3.14) $$\rho(p) \neq \pm 2\ \tau(p)$$

hence $\rho(p)$ and $2\tau(p)$ determine a 2–plane in $(\Lambda^2_+(M))p = \Lambda^2_+(T_p\tilde{M})$ which intersects the circle $(\mathcal{J}^+_{\tau,a})_p$ at two points, where $(\mathcal{J}^+_{\tau,a})_p$ is the circle on $(\hat{S}^2_+(M))_p$ determined by Proposition 1 setting $V=T_pM$. We define $\sigma(p)$ to be the one of these intersecting points such that $\rho(p)$, $2\tau(p)$ and $\sigma(p)$ lie on an open hemi–sphere of $(\hat{S}^2_+(M))_p$ as indicated in the figure. Since ρ and τ are differentiable, the cross–section

(3.15) $$\sigma : M \to \hat{S}^2_+(M)$$

is differentiable.

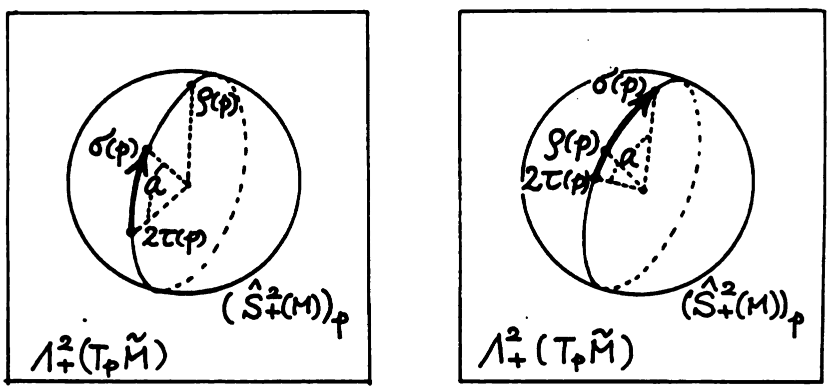


FIGURE 10

Step 4.

For each $p \in M$, we choose an open neighborhood U_p of p in $\tilde{M}$ such that $\sigma|(U_p \cap M)$ can be extended to a cross–section of $S^2_+(\tilde{M})$ on U_p.

(3.16) $$\sigma_p : U_p \to \hat{S}^2_+(\tilde{M})|U_p.$$

We can do this by the local triviality of $\hat{S}^2_+(\tilde{M})$. We identify here the manifold M and the embedded image $x(M) \subset \tilde{M}$. We put

(3.17) $$U = \bigcup_{p \in M} U_p$$

and pick a locally finite countable refinement $\{U_i\}_{i=1}^{\infty}$ of the open covering $\{U_p | p \in M\}$ of U. For each i we pick $p \in M$ such that $U_i \subset U_p$ and put

(3.18) $$\sigma_i = \sigma_p | U_i \,.$$

We choose a differentiable partition of unity $\{\rho_i\}$ on U subordinate to the covering $\{U_i\}$ and define a cross-section $\overline{\sigma}$ of $\Lambda^2_+(M)) | U$ by

(3.19) $$\overline{\sigma} : U \to \Lambda^2_+(\tilde{M})) | U \,; \quad \overline{\sigma} = \Sigma \, \rho_i \sigma_i \,.$$

Note that by constructions of σ_i and $\overline{\sigma}$

(3.20) $$\overline{\sigma} | \mathrm{M} = \sigma.$$

Since $\measuredangle(\sigma(p), \rho(p)) < \pi$ and $\sigma_i(p) = \sigma(p)$ for $\forall \; p \in M$,

(3.21) $$\measuredangle(\sigma_i(p), \rho(p)) < \pi \quad \text{for } \forall_p \in U_i \cap M \quad \forall_i \,.$$

Since $\overline{\sigma}(p)$ is a finite linear combination of $\sigma_i(p)$'s with positive coefficients, (3.21) means

(3.22) $$\overline{\sigma}(p) \neq 0, \quad \measuredangle(\overline{\sigma}(p), \rho(p)\,) < \pi \quad \text{for } \forall \; p \in M,$$

By the continuity of $\overline{\sigma}$, we can pick an open neighborhood W of M contained in U such that

(3.23) $$\overline{\sigma}(q) \neq 0, \quad \measuredangle(\overline{\sigma}(q), \rho(a)) < \pi \quad \text{for } \forall \; q \in W,$$

and we can define a cross-section $\hat{\sigma}$ of $\hat{S}^2_+(\tilde{M}) | W$ by

(3.24) $$\hat{\sigma} : W \to \hat{S}^2_+(\tilde{M}) | W \,; \; \hat{\sigma} = \overline{\sigma}/\sqrt{2}|\overline{\sigma}|$$

satisfying

(3.25) $$\measuredangle(\hat{\sigma}(q), \;\; \rho(a)) < \pi \;\; \text{for } \forall \; q \in M.$$

Finally we consider an open covering $\{W, \; \tilde{M} - M\}$ of M and local cross-sections

$$\left\{\begin{array}{l}\hat{\sigma}: W \to \hat{S}^2_+(\tilde{M})\,|\,W \\ \rho : \tilde{M}-M \to \hat{S}^2_+(\tilde{M})\,|\,\tilde{M}-M\end{array}\right. \tag{3.26}$$

and repeat the same argument using a partition of unity subordinate to $\{W, \tilde{M}-M\}$ and get a cross-section $\tilde{\sigma}$

$$\tilde{\sigma} : \tilde{M} \to \hat{S}^2_+(\tilde{M}) \tag{3.27}$$

satisfying

$$\tilde{\sigma}_{|M} = \sigma \tag{3.28}$$

Note that this is possible by (3.25).

Now, it is clear that the almost complex structure $\tilde{J}$ corresponding to $\tilde{\sigma}$ in the sense of Step 1 is the desired one.

Q. E.D.

CHAPTER 3

COMPACT SLANT SUBMANIFOLDS IN $\mathbb{C}^m$

In this chapter we prove the following.

Proposition 4.

Let x *be a general slant immersion of* l*-dimensional differentiable manifold* M *into the complex* m*-space* $\mathbb{C}^m$. *If* M *is compact, then* x *is a totally real immersion.*

In other words, there exists no compact proper slant submanifold in $\mathbb{C}^m$ just as in the case of holomorphic submanifolds:

(j) There exists no compact holomorphic submanifold in $\mathbb{C}^m$.

Since a slant submanifold is a generalization of both holomorphic and totally real submanifolds, this similarity is no surprise. (j) is shown in several ways. From the view point of complex manifolds, (j) is a consequence of the maximal modulus principle. From the view point of differential geometry, (j) comes from

(k) Holomorphic submanifolds are minimal.

and

(l) There exists no compact minimal submanifold in $\mathbf{E}^k$,

and (l) is shown by the maximal principle of harmonic functions or by the existence of a point where the mean curvature vector does not vanish.

We note that our proof of Proposition 4 contains another proof of (j) basically based on Stokes' Theorem.

We note also that there is a compact proper slant surface in a complex torus and, as is well–known, a compact totally real submanifold in complex spaces as seen in the following examples.

(Eg 8) Let $x : \mathbb{R}^2 \to \mathbb{C}^2 = (\mathbf{E}^4, J)$ be the proper slant plane of Ch 1 §4 Eg 3. Let $\{e_A\}_{A=1}^4$ be a basis of $\mathbf{E}^4$ such that $e_1 = x^*(\partial/\partial u)$ and $e_2 = x^*(\partial/\partial v)$. Let Γ be the lattice generated by $\{e_A\}$. Then, x induces a proper slant immersion of $S^1 \times S^1 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ into $\mathbb{C}^2/\Gamma$.

(Eg 9) Let $1 \le l \le m$. For each $j = 1,\ldots,l$, choose a closed curve $C_j : S^1 \to (x_j, y_j)$–plane. Then $C_1 \times \cdots\cdots \times C_l : S^1 \times \cdots\cdots \times S^1 \to \mathbb{C}^m$ is a totally real immersion.

To prove Proposition 4, we consider two cases:

(Case 1) $l = m = 2$

(Case 2) general dimensions.

In case 1, Proposition 4 comes directly from Proposition 2 and Lemma 2. In case 2, the idea is the same but we need some lemmas.

(Proof of case 1)

Let $\mathbb{C}^2 = (\mathbf{E}^4, J)$ and ζ_J be the 2–vector in $\Lambda_+^2 \mathbf{E}^4$ defined by Ch 2 (1.7) and (1.8). By Lemma 2 and Ch 2 (1.21)

(1)
$$\begin{aligned} 0 &= \int_{p \in M} \langle \nu(p), \zeta_J \rangle \; dV_M \\ &= \int_{p \in M} \cos(\alpha_J(\nu(p))) \; dV_M \\ &= \mathrm{vol}\,(M)\, \cos(\alpha_J(\nu)) \end{aligned}$$

because x is slant and hence $\alpha_J(\nu)$ is constant. Hence $\alpha_J(\nu) = \pi/2$, which means x is a totally real immersion.

(case 1 Q.E.D.)

To consider the second case, we set some notations similar to the ones in case 1. We put

(2) $$\mathbb{C}^m = (\mathbb{R}^{2m}, <\ ,\ >,\ J\ )$$

(3) $$\Omega(X,\ Y) = <X,\ JY>,\quad \Omega \in \Lambda^2(\mathrm{E}^{2m})^*$$

Then by the identification of Ch 1(2.4)

(4) $$\Omega^n \in \Lambda^{2n}(\mathrm{E}^{2m})^* = (\Lambda^{2n}\mathrm{E}^{2m})^*,$$

and we define ζ_J to be the metric dual of $(-\Omega)^n$ in $\Lambda^{2n}\mathrm{E}^{2m}$ with respect to the inner product $<\ ,\ >$ defined by Ch 1(2.3)

(5) $$<\zeta_J,\ \eta> = (-1)^n\Omega^n(\eta) \qquad \text{for } \forall\ \eta \in \Lambda^{2n}\mathrm{E}^{2m}.$$

For $V \in G(l,\ 2m)$ and $a \in [0,\ \pi/2]$, we call V a-slant if

(6) $$\measuredangle(JX,\ V) = a \qquad \text{for } X \in V,\ X \neq 0.$$

Lemma 12

(7) $$\Omega^n(X_1 \wedge \ldots \wedge X_{2n}) = \frac{1}{(2n)!} \sum_{\sigma \in S_{2n}} sgn(\sigma)\Omega(X_{\sigma(1)},\ X_{\sigma(2)}) \ldots \Omega(X_{\sigma(2n-1)},\ X_{\sigma(2n)})$$
$$\text{for } X_1, \ldots, X_{2n} \in \mathrm{E}^{2m}$$

where S_{2n} is the symmetric group of order $2n$ and sgn denotes the signature of permutations.

(Proof)

Let $\{e_A\}_{A=1}^{2m}$ be an orthonormal frame of E^{2m} and $\{\omega^A\}$ be its dual coframe. Put

$$(8) \qquad \Omega = \sum_{A,B=1}^{2m} \varphi_{AB}\omega^A \wedge \omega^B, \qquad \varphi_{AB} = \varphi_{BA}.$$

Then

$$(9) \qquad \Omega(X, Y) = \sum_{A,B=1}^{2m} \varphi_{AB}\omega^A(X)\omega^B(Y)$$

and

$$(10) \quad \Omega^n(X_1,\dots,X_{2n})$$
$$= [(\Sigma\varphi_{A_1A_2}\omega^{A_1} \wedge \omega^{A_2}) \wedge \cdots\cdots \wedge (\Sigma\varphi_{A_{2n-1}A_{2n}}\omega^{A_{2n-1}} \wedge \omega^{A_{2n}})](X_1,\dots,X_{2n})$$
$$= \sum_{A_1,\dots A_{2n}=1}^{2m} \varphi_{A_1A_2}\cdots\cdots\varphi_{A_{2n-1}A_{2n}}[\omega^{A_1} \wedge \cdots\cdots \wedge \omega^{A_{2n}}(X_1,\dots,X_{2n})]$$
$$= \sum_{A_1,\dots A_{2n}=1}^{2m} \varphi_{A_1A_2}\cdots\cdots\varphi_{A_{2n-1}A_{2n}} \times$$
$$[\frac{1}{(2n)!} \sum_{\sigma\in S_{2n}} sgn(\sigma)\omega^{A_1}(X_{\sigma(1)})\cdots\cdots\omega^{A_{2n}}(X_{\sigma(2n)})]$$
$$= \frac{1}{(2n)!} \sum_{\sigma} sgn(\sigma)[(\Sigma\varphi_{A_1A_2}\omega^{A_1}(X_{\sigma(1)})\omega^{A_2}(X_{\sigma(2)})) \times$$
$$\cdots \times (\Sigma\, \varphi_{A_{2n-1}A_{2n}}\omega^{A_{2n-1}} (X_{\sigma(2n-1)}) \,\omega^{A_{2n}}(X_{\sigma(2n)}))]$$

By (9), (10) and Ch 1(2.4), we get (7).

Q.E.D.

The following Lemma 13 is shown in [CBY5] but we prove here again to make our argument clear.

Lemma 13

Let $V \in G(l, 2m)$ *and* $\pi_V : \mathrm{E}^{2m} \to V$ *be the orthogonal projection. If* V *is* a*–slant in* $\mathbb{C}^m = (\mathrm{E}^{2m}, J)$ *with* $a \neq \pi/2$, *then the linear endomorphism* J_V *of* V *defined by*

$$(11) \qquad J_V = (sec\ a)(\pi_V \circ J_{|V})$$

is a complex structure compatible with $<, >|\, V$. *In particular,* l *is even.*

(Proof)

Put

(12) $P = \pi_V \circ (J_{|V}) : \ V \to V$

(13) $N = J_{|V} - P : V \to V^{\perp}$

(14) $S = P^2 : V \to V^{\perp}$

Then

(15) $$J_{|V} = P + N.$$

By a simple computation using (15), we get

(16) $$<SX, \ Y> = <X, SY>,$$

(17) $$<PX, Y> = -<X, PY> \qquad \text{for } \forall \ X, \ Y \in V.$$

Since V is a–slant

(18) $$a = \measuredangle(JX, \ V) = \measuredangle(JX, PX), \quad X \in V, \ X \neq 0,$$

and hence

(19) $$\|PX\| = \|X\| \ \cos a \qquad \text{for } \forall \ X \in V.$$

By (16) S has real eigenvalues $\{\lambda_i\}_{i=1}^{l}$. Let $\{e_i\}_{i=1}^{l}$ be corresponding orthonormal eigenvectors, i.e.,

(20) $$S(e_i) = \lambda_i e_i \ \ \forall i$$

Put

(21) $$P(e_i) = \Sigma \ P_{ij} \ e_j$$

then by (17)

(22) $$P_{ij} = -P_{ji}$$

Hence

(23) $$\begin{aligned} \lambda_i &= <S(e_i), e_i> \\ &= <P^2(e_i), e_i> \\ &= - \sum_j (P_{ij})^2 \leq 0. \end{aligned}$$

On the other hand, by (19), (20) and (14)

(24) $$|\lambda_i| = \|S(e_i)\| \; cos^2 a \qquad \text{for } \forall i,$$

i.e.,

(25) $$\lambda_i = -cos^2 a \qquad \forall i,$$

(26) $$SX = -cos^2 aX \qquad \text{for } \forall \; X \in V$$

and hence J_V defined by (11) is a complex structure on V. Since

(27) $$\|J_V X\|^2 = sec^2 a \; \|PX\|^2 = \|X\|^2 \qquad \text{for } \forall \; X \in V$$

namely, J_V is compatible with $< \, , \, >| V$.

Q.E.D.

Lemma 14

Let $V \in G(2n,2m)$. If V is a–slant with $a \neq \pi/2$, then

(29) $$< \zeta_J \, , \, V > = \mu_n \; cos^n a$$

where μ_n is a non–zero constant determined by n.

(Proof)

Let J_V be the complex structure on V defined by (11). Let X be a unit vector in V and put $Y = J_V X \in V$. Then, using P defined by (12);

$$\begin{aligned} \Omega(X, J_V X) &= \Omega(X,Y) \\ &= <X, \; JY> \\ &= <-J_V Y, \; JY> \\ &= - \|J_V Y\| \; \|JY\| \; cos \measuredangle(J_V Y, \; JY) \\ &= - \; cos \; \measuredangle(PY, JY) \\ &= - \; cos \; \measuredangle(V, \; JY) \\ &= - \; cos \; a \end{aligned}$$

i.e.

(30) $$\Omega(X, J_V X) = - \; cos \; a \qquad \text{for } \forall \; X \in V \; \text{ with } \|X\| = 1$$

If $Z \in V$, $Z \perp J_V X$,

$$\begin{aligned}\Omega(X,Z) &= \langle X,\ JZ \rangle \\ &= \langle X,\ PZ \rangle \\ &= \cos a \langle X,\ J_V Z \rangle = 0\end{aligned}$$

i.e.,

(31) $\Omega(X,Z) = 0$ for $X, Z \in V$ with $Z \perp J_V X$.

We choose an orthonormal J_V-basis $\{e_a\}_{a=1}^{2n}$ on V, i.e.,

(32) $e_{2k} = J_V\, e_{2k-1}$, $k = 1,\ldots,n$

(33) $V = e_1 \wedge \ldots \wedge e_{2n}$.

We fix a notation for indices by

(34) $\overline{2k} = 2k-1$, $\overline{2k-1} = 2k$ for $k = 1,\ldots,n$

Then, by (30) and (31),

(35) $\Omega(e_a, e_b) = -\delta_{\bar{a}b} \cos a$ for $a < b$

Using (7), (33) and (35), we compute $\Omega^n(V)$ as follows

(36) $$\begin{aligned}(2n)!\ \Omega^n(V) &= (2n!)\ \Omega\ (e_1 \wedge \ldots \wedge e_{2n}) \\ &= \sum_{\sigma \in S_{2n}} sgn(\sigma) \Omega(e_{\sigma(1)}, e_{\sigma(2)}) \ldots\ldots \Omega(e_{\sigma(2n-1)}, e_{\sigma(2n)}) \\ &= \sum_{a_1,\ldots,a_{2n}=1}^{2n} \delta^{1\ldots\ldots 2n}_{a_1 \ldots\ldots a_{2n}} \Omega(e_{a_1}, e_{a_2}) \ldots\ldots \Omega(e_{a_{2n-1}}, e_{a_{2n}}) \\ &= \sum_{a_1,\ldots,a_n=1}^{2n} \delta^{1\ 2 \ldots\ldots 2n}_{a_1 \bar{a}_1 \ldots a_n \bar{a}_n} \Omega(e_{a_1}, e_{\bar{a}_1}) \ldots\ldots \Omega(e_{a_n}, e_{\bar{a}_n}) \\ &= 2^n \sum_{a_1 < \bar{a}_1} \ldots\ldots \sum_{a_n < \bar{a}_n} \delta^{1 \ldots\ldots 2n}_{a_1 \bar{a}_1 \ldots a_n \bar{a}_n} \Omega(e_{a_1}, e_{\bar{a}_1}) \ldots\ldots \Omega(e_{a_n}, e_{\bar{a}_n}) \\ &= 2^n(-\cos a)^n \sum_{a_1 < \bar{a}_1} \ldots\ldots \sum_{a_n < \bar{a}_n} \delta^{1 \ldots\ldots 2n}_{a_1 \bar{a}_1 \ldots a_n \bar{a}_n} \\ &= 2^n(-\cos a)^n\ n! \sum_{a_1 < \bar{a}_1 < \ldots < a_n < \bar{a}_n} \delta^{1 \ldots\ldots 2n}_{a_1 \bar{a}_1 \ldots a_n \bar{a}_n} \\ &= 2^n(-\cos a)^n\ n!\end{aligned}$$

Hence

(37) $$\Omega^n(V) = (-1)^n \mu_n \cos^n a$$

where

(38) $$\mu_n = 2^n n!/(2n)!$$

Note that, if $n = 1$, then $\mu_n = 1$ and (37) is nothing but Ch 2 (1.16).

Q.E.D.

Now we can prove the second case of Proposition 4.

(Proof of case 2)

Assume that the immersion x is not totally real. Then, by Lemma 13, l is even. Put $l = 2n$. By Lemma 2.

(39) $$0 = \int_{p\in M} < \nu(p),\ \zeta_J > dV_M$$

Let a be the slant angle of x. By assumption $a \neq \pi/2$. So by Lemma 14

(40) $$\begin{aligned} 0 &= \int_M \mu_n \cos^n a \ dV_M \\ &= \mu_n \mathrm{vol}(M) \ \cos^n a, \end{aligned}$$

but this contradicts to $\cos a \neq 0$. Hence x is a totally real immersion.

Q.E.D.

CHAPTER 4
SPHERICAL SLANT SURFACES IN $\mathbb{C}^2$

As we have seen in Ch 2 §2, examples Eg1–Eg7 are all flat slant surfaces. In Chapters 4 and 5 we consider flat slant surfaces under slightly stronger assumptions, spherical slant surfaces and slant surfaces with the rank of the Gauss map less than 2. Both surfaces are flat slant surfaces. Under these additional assumptions, the shapes of slant surfaces become clearer.

In this chapter we consider spherical slant surfaces in $\mathbb{C}^2$, namely a slant surface contained in a 3–sphere in $\mathbb{C}^2$. Slant angles are invariant under parallel translations and homotheties, so, without loss of generality, we can assume that the 3–sphere is the unit sphere centered at the origin. Our argument depends on a special structure of S^3—the Lie group of unit quaternions. We review this in §1. In §2, we define a map analogous to the Gauss map to characterize spherical slant surfaces. In §3, we will see spherical proper slant surfaces are two families of surfaces which we will temporarily call helical cylinders and circular cylinders in S^3.

§ 1. Geometry of S^3.

This section is a short review of the geometry of the 3–sphere written in [S1] vol. 4 Ch 7.

$\mathbb{R}^4$ is considered the non–commutative division algebra of quanternions generated by $\{1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$

$$1 = (1,0,0,0),\ \boldsymbol{i} = (0,1,0,0), \qquad (1.1)$$
$$\boldsymbol{j} = (0,0,1,0),\ \boldsymbol{k} = (0,0,0,1)$$

satisfying

$$(1.2)\qquad \begin{cases} 1 = \text{unit} \\ \boldsymbol{i}\cdot\boldsymbol{j} = \boldsymbol{k} = -\boldsymbol{j}\cdot\boldsymbol{i},\ \ \boldsymbol{j}\cdot\boldsymbol{k} = \boldsymbol{i} = -\boldsymbol{k}\cdot\boldsymbol{j},\ \ \boldsymbol{k}\cdot\boldsymbol{i} = \boldsymbol{j} = -\boldsymbol{i}\cdot\boldsymbol{k} \\ \boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1 \end{cases}$$

Then, S^3 is the lie group consisting of quaternions of norm 1. We can also regard S^3 as a subgroup of $O(4)$ through the identification

$$(1.3)\qquad a+b\boldsymbol{i}+c\boldsymbol{j}+d\boldsymbol{k} \longrightarrow \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}$$

For each $p \in S^3$, the left translation L_p

$$(1.4)\qquad L_p : S^3 \to S^3;\ q \to p\cdot q$$

is an isometry satisfying

$$(1.5)\qquad d(L_p q,\ q) = const \quad for\ \forall\ q \in S^3$$

where d is the distance on S^3 induced from $\mathbf{E}^4$, and hence L_p is the analogous of a translation in $\mathbf{E}^3$.

We identify $T_p\mathbf{E}^4$ with $\mathbf{E}^4$ as usual. We put

$$(1.6)\qquad X_1 = (0,1,0,0),\quad X_2 = (0,0,1,0),\quad X_3 = (0,0,0,1)$$

then $\{X_i\}_{i=1}^3$ is an orthonormal basis of T_1S^3. For $Y=(0,\ y_1,\ y_2,\ y_3) \in T_1S^3$ and $p = (a,\ b,\ c,\ d) \in S^3$, $L_{p*}Y$ is calculated by

$$(1.7)\qquad {}^t(L_{p*}Y) = \begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

where ${}^t(\ \ )$ denotes the transposed matrix. Let $\{\tilde{X}_i\}$ be the left invariant extension of $\{X_i\}$, i.e., (1.8) $$\tilde{X}_i(p) = L_{p*}X_1, \quad p \in S^3$$

then, every tangent vector of S^3 is spanned by $\{\tilde{X}_i\}$.

Since S^3 is a space of constant curvature, the analogy of the curve theory in Euclidean space holds. Suppose $c(s)$ is a curve in S^3 parametrized by arclength. We denote by ∇' the Riemannian connection of S^3 and put

(1.9) $$\boldsymbol{t}(s) = \dot{c}(s) = c_*(\partial/\partial s)$$

(1.10) $$\kappa(s) = \|\nabla'_{\boldsymbol{t}(s)}\boldsymbol{t}\|$$

and if $\kappa \neq 0$, put

(1.11) $$\boldsymbol{n}(s) = (1/\kappa(s))\nabla'_{\boldsymbol{t}(s)}\boldsymbol{t}$$

(1.12) $$\boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s)$$

(1.13) $$\tau(s) = -\langle \nabla'_{\boldsymbol{t}(s)}\boldsymbol{b}, \boldsymbol{n}(s) \rangle$$

where $\times$ denotes the vector product in $T_{c(s)}S^3$ determined by the metric and orientation. Then the Frenet–Serret formulas hold. If we set

(1.14) $$\boldsymbol{t}(s) = \sum_{i=1}^{3} f_i(s)\, \tilde{X}_i(c(s)),$$

then we have the following equations.

(1.15) $$\Sigma f_i f_i' = 0$$

(1.16) $$k = \Sigma((f_i')^2)^{1/2}$$

(1.17) $$\boldsymbol{n} = \kappa^{-1}\Sigma f_i'\, \tilde{X}_i$$

(1.18) $$\boldsymbol{b} = \kappa^{-1}\Sigma g_i \tilde{X}_i$$

where

(1.19) $$\begin{cases} g_1 = f_2 f_3' - f_3 f_2' \\ g_2 = f_3 f_1' - f_1 f_3' \\ g_3 = f_1 f_2' - f_2 f_1' \end{cases}$$

(1.20) $$\nabla'_{\boldsymbol{t}}\boldsymbol{b} = -\boldsymbol{n} + \Sigma\, (g_i/\kappa)'\tilde{X}_i$$

and also the following hold.

(m) If $\tau \equiv 1$, then b is left invariant along c. If $\tau \equiv -1$, then b is right invariant along c.

(n) A flat surface M in S^3 is "in general" a translation surface

(1.21) $$\{c(s)\cdot\gamma(t)\}$$

where c and γ are curves in S^3 parametrized by arclength satisfying one of the following conditions:

(i) Torsions of c and γ are $+1$ and -1 respectively, $c(0)=\gamma(0)$ and the osculating planes at $s=t=0$ coincide. (In this case, at each point $c(s_0)\cdot\gamma(t_0)$ the binormals of the s-curve $c(s)\cdot\gamma(t_0)$ at $s=s_0$ and the t-curve $c(s_0)\cdot\gamma(t)$ at $t=t_0$ are normal to M in S^3.)

(ii) c has torsion $\tau=1$ and γ is geodesic. (In this case, at each point $c(s_0)\cdot\gamma(t_0)$ the binormal of the s-curve $c(s)\cdot\gamma(t_0)$ at $s=s_0$ is normal to M in S^3.)

(ii′) c is a geodesic and γ has torsion $\tau=-1$. (In this case, at each point $c(s_0)\cdot\gamma(t_0)$ the binormal of the t-curve $c(s_0)\cdot\gamma(t)$ at $t=t_0$ is normal to M in S^3.)

(iii) c and γ are both (distinct) geodesics. (In this case, s-curves and t-curves are geodesics intersecting at a constant angle.)

Here, "in general" means that we avoid the case in which the curvature κ of c or γ has isolated zeros.

§ 2 Another Gauss Map

Let x be a spherical immersion of an oriented surface M into $\mathbb{C}^2 = (\mathbf{E}^4, J_o)$

$$x : M \to S^3 = S_0^3(1) \subset \mathbb{C}^2 \tag{2.1}$$

We define $\mathcal{J}_o$, ψ, η and ν as follows. Let $\mathcal{J}_o$ be the complex structure $\mathcal{J}_V \in \mathscr{J}^-$ defined by Ch 1 (1.29) with $V = \mathring{e}_1 \wedge \mathring{e}_2$, where $\{\mathring{e}_A\}$ is the canonical basis of $\mathbf{E}^4$, i.e.,

$$\mathcal{J}_o : \mathring{e}_1 \to \mathring{e}_2,\ \mathring{e}_2 \to -\mathring{e}_1,\ \mathring{e}_3 \to -\mathring{e}_4,\ \mathring{e}_4 \to \mathring{e}_3. \tag{2.2}$$

Let ψ be an isometry of $\mathbf{E}^4$ defined by

$$\psi(a, b, c, d) = (a, b, d, c) \tag{2.3}$$

Let η, $\hat{\nu}$ be the positive unit normal vector field of S^3 in $\mathbf{E}^4$ and of $x(M)$ in S^3 respectively.

$$\eta = \textit{unit normal of } S^3 \subset \mathbf{E}^4 \tag{2.4}$$

$$\hat{\nu} = \textit{unit normal of } x(M) \subset S^3 \tag{2.5}$$

Using the notation of § 1, we state the following Lemmas.

Lemma 15

$$J_o\eta(q) = R_{q*}X_1, \tag{2.6}$$

$$\mathcal{J}_o\eta(q) = L_{q*}X_1 = \tilde{X}_1(q) \tag{2.7}$$

$$\textit{for } \forall\, q \in S^3$$

Hence, $J_0\eta$ and $\mathcal{J}_0\eta$ are right–and left–invariant respectively.

(Proof)

Let $q = (a, b, c, d) \in S^3$. Then

$$J_o\eta(q) = (-b, a, -d, c) \tag{2.8}$$

$$\mathcal{J}_o\eta(q) = (-b, a, d, -c) \tag{2.9}$$

Define a curve γ in S^3 by

(2.10) $$\gamma(s) = \cos s + \sin s\, \mathbf{i}$$

Then

(2.11) $$\dot{\gamma}(0) = X_1$$

and hence

(2.12) $$\begin{aligned} R_{q*}X_1 &= [\tfrac{d}{ds}(R_q \circ c(s))]_{s=0} \\ &= \tfrac{d}{ds}[(\cos s + \sin s\, \mathbf{i})(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})]_{s=0} \\ &= (-b,\ a,\ -d,\ c) \end{aligned}$$

(2.13) $$\begin{aligned} L_{q*}X_1 &= \tfrac{d}{ds}[(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})(\cos s + \sin s\, \mathbf{i})]_{s=0} \\ &= (-b,\ a,\ -d,\ c) \end{aligned}$$

Hence we get (2.6) and (2.7)

Q.E.D.

Lemma 16

The following are equivalent.

(i) x *is* a*–slant with respect to* J_o

(ii) $< \hat{\nu}(p), J_o\eta(x(p)) > = -\cos a$ *for* $\forall\, p \in M.$

(iii) $<\psi_* \hat{\nu}(p), \tilde{X}_1(\psi \circ x(p)) > = -\cos a$ *for* $\forall\, p \in M.$

(iv) $\psi \circ x$ *is* $(\pi - a)$*–slant with respect to* J_o^-.

(Proof)

If we apply Lemma 6 pointwise, we get the equivalence of (i) and (ii). Since ψ is an orientation–reversing isometry, $-\psi_*\hat{\nu}$ is the positive unit normal of $\psi \circ x(M)$ in S^3. Hence, using (2.7), we apply again Lemma 6 pointwise and get the equivalence of (iii) and (iv). By (2.3), (2.8), (1.7) and (1.8)

(2.14) $$\psi_*(J_o\eta(q)) = \tilde{X}_1(\psi(q)) \qquad \text{for } \forall\ q \in S^3,$$

which shows the equivalence of (ii) and (iii).

Q.E.D.

Now we define two maps analogous to the Gauss map in $\mathbf{E}^3$ which translate each unit normal of a surface in S^3 to the unit tangent sphere at the unit $1 = (1,0,0,0)$ using left–translations instead of parallel translations. Recall that left–translations in S^3 are the analogues of parallel translations in $\mathbf{E}^3$ (§1).

Definition 3

Let x be a spherical immersion $x : M \to S^3 \subset \mathbf{E}^4$ of an oriented surface M. We define two maps $g_\pm$ from M to the unit 2–sphere S^2 in T_1S^3

(2.15) $$g_\pm : M \to S^2 \subset T_1S^3$$

by

(2.16) $$g_+(p) = (L_{\psi(x(p))*})^{-1}(\psi_*\hat{\nu}(p))$$

(2.17) $$g_-(p) = (L_{x(p)*})^{-1}(\hat{\nu}(p))$$

for $\forall\ p \in M$.

We show here two examples.

(Eg 10)

Let $M = S^1\times S^1$ be a flat torus in S^3 parametrized by

(2.18) $$x(u,v) = \frac{1}{\sqrt{2}}\,(\cos u,\ \sin u,\ \cos v,\ \sin v)$$

Then

$$\hat{\nu}(u, v) = \frac{1}{\sqrt{2}} (\cos u, \sin u, -\cos v, -\sin v) \tag{2.19}$$

$$\psi_* \hat{\nu}(u, v) = \frac{1}{\sqrt{2}} (\cos u, \sin u, -\sin v, -\cos v) \tag{2.20}$$

Hence, the images of $g_\pm$ are great circles perpendicular to X_1.

(Eg 11)

Let $M = S^2$ be a totally geodesic 2-sphere in S^3 parametrized by

$$x(u,v) = (\cos u \cos v, \sin u \cos v, \sin v, 0) \tag{2.23}$$

Then

$$\hat{\nu}(u,v) = (0,0,0,1) \tag{2.24}$$

$$\psi_* \hat{\nu}(u,v) = (0,0,1,0) \tag{2.25}$$

Therefore

$$g_+(u,v) = (0, \sin v, \cos u \cos v, -\sin u \cos v) \tag{2.26}$$

$$g_-(u,v) = (0, -\sin v, \sin u \cos v, \cos u, \cos v) \tag{2.27}$$

Hence, g_+ and g_- are isometries.

Now, we can state the spherical version of Proposition 2. As before, we define circles in $S^2 \subset T_1 S^3$ perpendicular to X_1:

$$S_a^1 = \{X \in T_1 S^3 \mid \|X\| = 1, \; \langle X, X_1 \rangle = -\cos a\}, \quad a \in [0, \pi] \tag{2.28}$$

Then, the following proposition characterizes spherical slant surfaces.

Proposition 5.

Let $x : M \to S^3 \subset E^4$ be a *spherical immersion of an oriented surface* M. *Then*

(i) x *is* a*-slant with respect to* J_0 *if and only if*

$$g_+(M) \subset S_a^1 \subset T_1 S^3. \tag{2.29}$$

(ii) x *is* a*–slant with respect to* J_0^- *if and only if*

(2.30) $$g_-(M) \subset S^1_{(\pi-a)} \subset T_1 S^3.$$

(Proof)

Direct from Lemma 16, (2.7) and Definition 3.

Q.E.D.

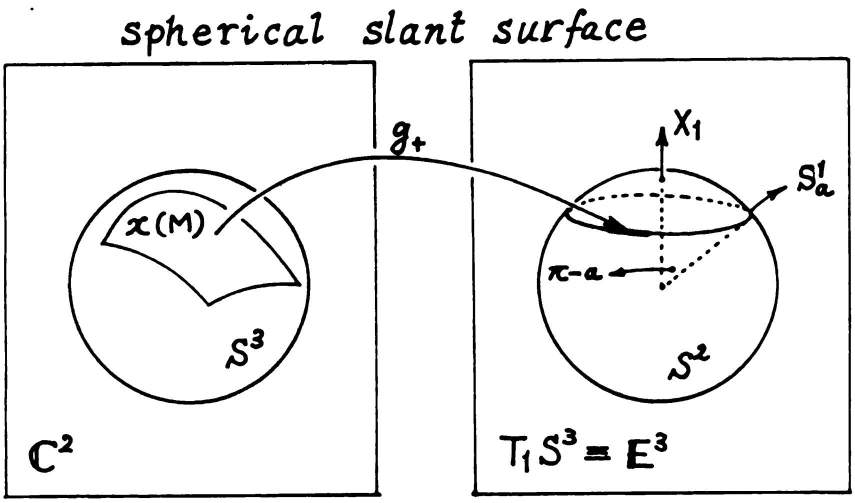


FIGURE 11

Corollary 1.

If $g_+(M)$ *is contained in a circle* S^1 in S^3, *then* x *is slant with respect to a complex structure* J_V^+ *determined by* Ch 2(1.29), *where*

(2.31) $$V = (1, 0, 0, 0) \wedge Z$$

(2.32) $$Z \in T_1 S^3,\ Z \perp S^1,\ \|Z\| = 1$$

Similar for $g_-(M)$.

(Proof)

If suffices to consider an orthogonal transformation of E^4 which leaves (1, 0, 0, 0) fixed and maps (0, 1, 0, 0) to Z.

Q.E.D.

Corollary 2

$g_+(M)$ or $g_-(M)$ cannot be a singleton.

(Proof)

If $g_+(M)$ is a singleton, then by Corollary 2 x is a holomorphic immersion with respect to some complex structure on $\mathbf{E}^4$ and hence $x(M)$ is minimal in $\mathbf{E}^4$. But spherical submanifolds cannot be minimal in $\mathbf{E}^4$.

Q.E.D.

§ 3 Classification of Spherical Slant Surfaces

As will be seen in Proposition 6, spherical proper slant surfaces are something like "helical cylinders". A generalized helix in Euclidean 3-space $\mathbf{E}^3$ is a curve such that the angle between its tangent vector $\dot{c}(s)$ and a fixed vector $\mathbf{v} \in \mathbf{E}^3$ is constant. In other words, if we extend $\mathbf{v} \in \mathbf{E}^3 \equiv T_o\mathbf{E}^3$ to a global vector field $\tilde{\mathbf{v}}$ on $\mathbf{E}^3$ by parallel translations, then a general helix is a curve $c(s)$ satisfying

(3.1) $$< \dot{c}(s), \tilde{\mathbf{v}}(c(s)) > = const.$$

We will define a "helix" in S^3 replacing parallel translations with left translations. A cylinder in $\mathbb{R}^3$ is a surface obtained by parallel translations of a curve along a straight line. We will define a "cylinder" in S^3 as a surface obtained by left translations along another curve in S^3.

Let

(3.2) $$c : I \to S^3$$

be a curve parametrized by arc length s and put

(3.3) $$\dot{c}(s) = t(s) = \sum_{i=1}^{3} f_i(s) \ \tilde{X}_i(c(s))$$

as (1.14).

Definition 4

(1) *We call* $c(s)$ *a helix in* S^3 *with the axis vector* $\tilde{X}_1$ *if*

(3.4) $$\begin{cases} f_1(s) = b \\ f_2(s) = a \cos(a' s + s_0) \\ f_3(s) = a \sin(a' s + s_0) \end{cases}$$

where a, b, a', s_0 *are constants satisfying*

(3.5) $$a^2 + b^2 = 1$$

(ii) *We call an immersion* $x : D \to S^3$ *of a domain* D *of* $\mathbb{R}^2$ *a helical cylinder if* $x(t,s)$ *is a flat translation surface* $\gamma(t) \cdot c(s)$ *described in* §1 (n) *and* $c(s)$ *is a helix in* S^3 *with the axis vector* $\tilde{X}_1$ *defined in* (i).

(iii) *We call an immersion* $x : D \to S^3$ *of a domain* D *of* $\mathbb{R}^2$ *a circular cylinder in* S^3 *if* $x(t,s)$ *is a flat translation surface* $c(s)\cdot\gamma(t)$ *of type* (ii′) *in* (n) *which satisfies for some* t_0

(3.6) $$<L_{c(s)*}b(t_0),\ J_0\eta(c(s)\cdot\gamma(t_0))> = -\cos\alpha \quad \text{for } \forall s$$

where α *is a constant with* $\cos\alpha \neq 0, \pm 1$ *and* $b(t)$ *is the binormal of* $\gamma(t)$.

Of course we can define a helix with an arbitrary left invariant vector filed as its axis but we don't need it in this article. The following lemma shows the existence of helixes in S^3. Then, the existence of a helical cylinder reduced to the existence of the curve $\gamma(t)$ satisfying the conditions of §1(n), but this is guaranteed by the Existence and Uniqueness Theorem of the curve theory in S^3 ([S1] vol. 4 p 35).

The existence of a circular cylinder in S^3 is a pending problem.

Lemma 17

Let I *be an open interval containing* 0 *and* $f_i(s)$, $i = 1,2,3,$ *be differentiable functions on* I *satisfying*

(3.5) $$f_1^2 + f_2^2 + f_3^2 = 1$$

Then, for any point $p_0 \in S^3$, *there exists a curve* $c(s)$ *defined on a neighborhood* I' *of* 0 *in* I *satisfying*

(3.6) $$c'(s) = \sum_{i=1}^{3} f_i(s) \tilde{X}_i(c(s)).$$

(Proof)

Considering the curve $L_{p_0}^{-1} \circ c$ if necessary, we can aassume without loss of generality $p_0 = 1 = (1, 0, 0, 0)$.

First we assume such $c(s)$ exists and put

(3.7) $$c(s) = (x(s), y(s), z(s), w(s)) \in S^3$$

Then

(3.8) $$t(s) = (x'(s), y'(s), z'(s), w'(s))$$

By (1.8) and (1.6)

(3.9) $$\begin{aligned}\Sigma f_i(s) \tilde{X}_i(c(s)) &= \Sigma f_i(s) L_{c(s)*} X_i \\ &= L_{c(s)*}(\Sigma f_i(s) X_i) \\ &= L_{c(s)*}(0, f_1(s), f_2(s), f_3(s))\end{aligned}$$

Hence, by Ch 1 (5.7)

$$(3.10)\quad \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \begin{bmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{bmatrix} \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Conversely, consider a system of 1st order ordinary linear differential equation (2.10) with the initial condition

$$(3.11)\quad (x(0),\ y(0),\ z(0),\ w(0)) = (1,0,0,0) = 1 \in S^3.$$

Then, it has a unique solution in a neighborhood I^* of 0 in I. Put

$$(3.12)\quad c(s) = (x(s),\ y(s),\ z(s),\ w(s)),\quad s \in I^*$$

then $c(s)$ is a curve in E^4 with $c(0) \in S^3$. Hence, in neighborhood I' of 0 in I^*,

$$(3.13)\quad (x(s))^2 + \cdots\cdots + (w(s))^2 \neq 0,\quad s \in I'.$$

Put

$$(3.14)\quad \lambda = \lambda(s) = (x(s)^2 + \cdots\cdots + (w(s))^2)^{1/2}$$

Then, $(1/\lambda)(x,y,z,w) \in S^3$ and hence

$$(3.15)\quad \frac{1}{\lambda} \begin{bmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{bmatrix} \in O(4)$$

Put this matrix A and multiply $A^{-1} = {}^tA$ from the left on both sides of (3.10), then

$$(3.16)\quad \begin{bmatrix} x & -y & -z & -w \\ y & x & -w & z \\ z & w & x & -y \\ w & -z & y & x \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \\ w' \end{bmatrix} = \lambda^2 \begin{bmatrix} 0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

Picking up the first component,

(3.17) $$xx' + yy' + zz' + ww' = 0$$

hence

(3.18) $$\frac{d}{ds}(x^2 + y^2 + z^2 + w^2) = 0$$

which means, together with (3.11),

(3.19) $$x^2 + y^2 + z^2 + w^2 = 1$$

Therefore $c_{|I'}$ is a curve on S^3. Tracing back (3.10) and (3.9), $c(s)$ satisfies (3.6)

Q.E.D.

Lemma 18

Let $c(s)$ be a curve parametrized by arclength. Then, the following are equivalent.

(A) *$c(s)$ is a helix with the axis vector $\tilde{X}_1$ of the form*

(3.20) $$\begin{cases} f_1(s) = b \\ f_2(s) = a\cos((-2/b)s + s_0) \\ f_3(s) = a\sin((-2/b)s + s_0) \end{cases}$$

where

(3.21) $$a^2 + b^2 = 1, \quad ab \neq 0$$

(B) *$c(s)$ satisfies*

(3.22) $$\tau(s) \equiv -1$$

(3.23) $$\langle b(s), \tilde{X}_1(c(s))\rangle \equiv a, \qquad a \neq \pm 1, 0$$

(Proof)

((B) ⇒(A))

Assume (B) holds. By Frenet–Serret formulas and (3.22)

(3.24) $$\nabla'_t b = n$$

By (3.24), (1.20) and (1.17),

(3.25) $$(g_i/\kappa)' = 2f_i'/\kappa, \ i = 1,2,3$$

By (3.23) and (1.18)

(3.26) $$a = g_1/\kappa$$

By (3.25) and (3.26)

(3.27) $$2f_i'/\kappa = (a)' = 0$$

Hence f_1 = const. Put

(3.28) $$b = f_1 \ (const).$$

By (1.18), (3.26), (1.19) and (3.28)

(3.29) $$\mathbf{b} = a\tilde{X}_1 - (bf_3'/\kappa)\tilde{X}_2 + (bf_2'/\kappa)\tilde{X}_3.$$

By (1.16) and (3.28)

(3.30) $$\kappa^2 = (f_2')^2 + (f_3')^2$$

By (1.17) and (3.28)

(3.31) $$\mathbf{n} = (f_2'/\kappa)\tilde{X}_2 + (f_3'/\kappa)\tilde{X}_3$$

By $\|\mathbf{b}\| = 1$, (3.29) and (3.30),

(3.32) $$a^2 + b^2 = 1$$

By $\|\mathbf{t}\| = 1$, (1.14), (3.28) and (3.21)

(3.33) $$f_2^2 + f_3^2 = a^2$$

By (1.15) and (3.28)

(3.34) $$f_2 f_2' + f_3 f_3' = 0$$

By (1.18) and (3.29)

(3.35) $$g_1 = a\kappa, \qquad g_2 = -bf_3', \qquad g_3 = bf_2'$$

By (3.33) we can put

(3.36) $$\begin{cases} f_2 = f_2(s) = a \cos \theta, & \theta = \theta(s) \\ f_3 = f_3(s) = a \sin \theta \end{cases}$$

By (3.30) and (3.36)

(3.37) $$\kappa = |a\theta'| \neq 0$$

because we are considering the curve c with binormal b. By (3.35) and (3.36)

(3.38)
$$\begin{cases} g_1 = a\ |a\theta'| \\ g_2 = a\ b\ \cos\ \theta\cdot\theta' \\ g_3 = -a\ b\ \sin\ \theta\cdot\theta' \end{cases}$$

By (3.25), (3.37) and (3.38)

(3.39)
$$(-a\ b\ \cos\ \theta\cdot\theta'/|a\theta'|)' = -2\ a\ \sin\ \theta\cdot\theta'/|a\theta'|$$

Hence

(3.40)
$$\sin\ \theta\cdot\ (b\theta' + 2) = 0.$$

By (3.37) $\sin\ \theta(s)$ has isolated zeros so by continuity,

(3.41)
$$b\theta' + 2 = 0$$

By assumption $a \neq \pm 1$ and hence by (3.32) $b \neq 0$, so

(3.42)
$$\theta = -\frac{2}{b}\ s + s_o, \quad s_o = const.$$

Hence, by (3.28), (3.36) and (3.42) we get (3.20) together with (3.21) = (3.32).

((A) $\Rightarrow$ (B)) straightforward.

Q.E.D.

Now, we can state and prove the following proposition which determine spherical proper slant surfaces.

Proposition 6

Let x be a spherical proper slant immersion of an oriented surface M into complex 2–space.

$$x : M \to S^3 = S_0^3(1) \subset \mathbb{C}^2 = (\mathbf{E}^4,\ J_0)$$

Then, $x(M)$ is locally a helical cylinder in S^3 or a circular cylinder in S^3. Conversely, a helical cylinder and a circular cylinder (if exits) in S^3

are proper slant surfaces in $\mathbf{C}^2$.

(Proof)

First we note that the isometry ψ has the following properties:

(3.44) $\psi(p\cdot q)=\psi(q)\cdot\psi(p)$ for p, q $\in S^3$

(3.45) $X \in \mathfrak{X}(S^3)$ is left (or right) invariant.

$\rightleftarrows$ $\psi_* X$ is right (or left respectively) invariant.

(3.46) $\tau_{(\psi\circ c)}=-\tau_c$ for a curve c in S^3

(3.47) b is the binormal of c in S^3.

$\Longrightarrow$ $-\psi_* b$ is the binormal of $\psi\circ c$.

Assume x is proper slant immersion with the slant angle α. Since x is spherical, its normal curvature G^D vanishes and hence by Lemma 3 and Proposition 2, $x(M)$ is a flat surface in S^3. Therefore $x(M)$ is locally a flat translation surface

(3.48) $$x(M) = \{c(s)\cdot\gamma(t)\}$$

described in §1 (n). We follow the four cases in (n).

Cases (i) and (ii)

With a suitable choice of orientations we can assume that the binormal of s-curves are positive unit normals of $x(M)$ in S^3. Let $b(s)$ be the binormal of $c(s)$. Then,

(3.49) $$\hat{\nu}(c(s)) = b(s).$$

By Lemma 16 (iii) and (3.49),

(3.50) $$<\psi_*\, b(s),\ \tilde{X}_1(\psi\circ c(s))> \ = -\cos\alpha$$

Let $\bar{c} = \psi\circ c$ and $\bar{b}$ be the binormal of $\bar{c}$. Then, by (3.47) and (3.50),

(3.51) $$<\bar{b}(s),\ \tilde{X}_1(\bar{c}(s))> \ = \cos\alpha$$

By $\tau_c=1$ and (3.46)

(3.52) $$\tau_{\bar{c}}=-1$$

Since x is proper slant,

(3.53) $$\cos\alpha \neq 0, \pm 1$$

By (3.51), (3.52), (3.53) and Lemma 18, $\bar{c}$ is a helix in S^3 with a, b, a' in (3.4) determined by

(3.54) $$a = \cos\alpha, \quad b = \sin\alpha, \quad a' = -2/\sin\alpha.$$

By (3.49)

(3.55) $$\begin{aligned}(\psi\circ x)(M) &= \{\psi\,(c(s)\cdot\gamma(t))\}\\ &= \{\psi(\gamma(t))\cdot\psi(c(s))\}\\ &= \{(\psi\circ\gamma)(t)\cdot\bar{c}(s)\}\end{aligned}$$

Since $(\psi\circ x)(M)$ is also flat, (3.55) shows that $(\psi\circ x)(M)$ is a helical cylinder. Note that if γ has torsion -1, then $\psi\circ\gamma$ has torsion 1 by (3.46) and if γ is a geodesic, then $\psi\circ\gamma$ is also a geodesic.

Conversely, if $(\psi\circ x)(M)$ is a helical cylinder

(3.56) $$(\psi\circ x)(M) = \{\bar{\gamma}(t)\cdot\bar{c}(s)\}$$

such that $\bar{c}$ is a helix satisfying (3.54) and $\bar{\gamma}$ is a geodesic or has torsion $\tau = +1$.

Put $c = \psi\circ\bar{c}$, $\gamma = \psi\circ\bar{\gamma}$. Then by (3.44),

$$x(M) = \{c(s)\cdot\gamma(t)\}$$

and $\tau_c=1$. Let b and $\bar{b}$ be the binormals of c and $\bar{c}$. By Lemma 18,

(3.57) $$<\bar{b}(s), \tilde{X}_1(\bar{c}(s))> = \cos\alpha$$

By (3.47)

(3.58) $$<b(s), \psi_*(\tilde{X}_1(\bar{c}(s)))> = -\cos\alpha$$

Since $\psi_*\tilde{X}_1 = J_0\eta$,

(3.59) $$<b(s),\ J_0\eta(c(s)> \ = -\cos\ \alpha$$

Since the binormals of s–curves are the normals $\hat{\nu}$ of $x(M)$ in S^3 and $x(M)$ is a translation surface,

(3.60) $$\hat{\nu}(c(s)\cdot\gamma(t))=R_{\gamma(t)*}b(s)$$

Since $J_0\eta$ is right invariant, (3.55) and (3.56) means

(3.61) $$<\hat{\nu}(c(s)\cdot\gamma(t)),\ J_0\eta\ (c(s)\cdot\gamma(t))> \ = -\cos\ \alpha$$

which shows, by Lemma 16, that $x(M)$ is α–slant.

Case (ii′)

Let $x(M) = \{c(s)\cdot\gamma(t)\}$ where c is a geodesic and γ has torsion $\tau=-1$. With suitable choice of orientations, the binormal of any t–curve is the positive unit normal vector of $x(M)$, i.e.,

(3.62) $$\hat{\nu}(c(s)\cdot\gamma(t))=L_{c(s)*}b(t_0)$$

By Lemma 16 and (3.62),

(3.63) $$<L_{c(s)*}b(t_0),\ \ J_0\eta\ (c(s)\cdot\gamma(t_0))> \ = -\cos\ \alpha \quad \text{for } \forall\ s$$

for any fixed t_0, which shows that $x(M)$ is a circular cylinder in S^3 defined in Definition 4.

Conversely, let $x(M)$ be a circular cylinder in S^3 satisfying

(3.64) $$<\hat{\nu}(p_1),\ \ J_0\eta(p_1)> \ = \ <L_{c(s_1)*}b(t_1),\ \ J_0\eta\ (p_1)>$$

Since γ has torsion –1 and hence its binormal is right invariant,

(3.65) $$b(t_1) \ = \ R_{\left[\gamma(t_0)^{-1}\cdot\gamma(t_1)\right]*}b(t)$$

$J_0\eta$ is also right invariant, so

(3.66) $$J_0\eta(p_1) \ = \ R_{\left[\gamma(t_0)^{-1}\cdot\gamma(t_1)\right]*}J_0\eta(c(s_1)\cdot\gamma(t_0))$$

Hence, using $L_p\circ R_q = R_q\circ L_p$ for $\forall\ p,\ q\in S^3$,

$$(3.67)\qquad <\hat{\nu}(p_1),\ J_0\eta(p_1)> \ = \ <L_{c(s_1)*}b(t_0),\ J_0\eta\ (c(s_1)\cdot\gamma(t_0))> \ = \ -\cos\alpha$$

by (3.63). Hence x is α–slant.

Case (iii)

Let $x(M)=\{c(s)\cdot\gamma(t)\}$ where c and γ are geodesics in S^3 and $c(0)=\gamma(0)$. Since s–curves and t–curves are geodesics intersecting at a constant angle, the immersion $(s, t) \to c(s)\cdot\gamma(t)$ can be extended to a global immersion

$$(3.68)\qquad y\colon \mathbb{R}^2 \to S^3,\quad y(s,\ t) \ = \ c(s)\cdot\gamma(t)$$

Since c and γ are periodic, y induces an immersion of torus

$$(3.69)\qquad \tilde{y}\colon = T^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}) \to S^3$$

c and γ are great circles and hence we can write

$$(3.70)\qquad \begin{cases} c(s) \ = \ A \begin{bmatrix} \cos s \\ \sin s \\ 0 \\ 0 \end{bmatrix} \\ \\ \gamma(t) \ = \ B \begin{bmatrix} \cos t \\ \sin t \\ 0 \\ 0 \end{bmatrix} \end{cases}$$

with suitable $A,\ B \in O(4)$. Let $\tilde{X}$ and $\tilde{Y}$ be the vector field along $y(\mathbb{R}^2)$ determined by s–curves and t–curves respectively. Then, using (1.7) and its right version, we can see that the components of $\tilde{X}$ and $\tilde{Y}$ at $c(s)\cdot\gamma(t)$ with respect to the global frame field $\{\tilde{X}_i\}_{i=1}^3$ are polynomials of $\sin s$, $\cos s$, $\sin t$ and $\cos t$ with coefficients in $\mathbb{R}$. The same holds for $\hat{\nu} = \tilde{X} \times \tilde{Y}$ and $J_0\eta = \psi_*\tilde{X}_1$. Hence if we put

$$(3.71)\qquad F(s,t) \ = \ <\hat{\nu}(c(s)\cdot\gamma(t)),\ \ J_0\eta(c(s)\cdot\gamma(t))>$$

then $F(s,t)$ is a polynomial of $\sin s, \ldots, \cos t$.

Since $x\colon M \to S^3 \subset \mathbb{C}^2$ is α-slant, we can choose an open domain U in $\mathbb{R}^2$ such that

(3.72) $$\hat{y} = y_{|U} : U \to S^3 \subset \mathbb{C}^2$$

is proper slant. Then, by Lemma 16,

(3.73) $$F(s,t) = -\cos\alpha = const \ on \ U$$

and hence,

(3.74) $$F(s,t) = -\cos\alpha \ \ on \ \mathbb{R}^2$$

which shows y is proper slant globally on $\mathbb{R}^2$ and hence $y\colon T^2 \to S^3 \subset \mathbb{C}^2$ is a proper slant immersion, contradicting Proposition 4.

Therefore, x can be proper slant only in cases (i), (ii) and (ii′) and hence, Proposition 6 is proved.

Q.E.D.

CHAPTER 5

SLANT SURFACES IN $\mathbb{C}^2$ WITH RANK $\nu<2$

Let $x : M \to \mathbb{C}^2=(\mathrm{E}^4, J_0)$ be a slant immersion of an oriented surface. In this chapter, we consider slant surfaces with the rank of the Gauss map less than 2.

(1) $$\text{rank } \nu < 2.$$

We note that proper slant surfaces among Eg 1–Eg 7 have this property (Ch2 §2). Eg 4 has rank $\nu = 2$ and proper slant with respect to J_0 but totally real with respect to J_1. We also note that rank $\nu < 2$ means rank $\nu_+ < 2$ and rank $\nu_- < 2$, hence, by Lemma 3,

(2) $$G \equiv G^D \equiv 0$$

and $x(M)$ is a flat slant surface in $\mathbb{C}^2$.

What we are going to do here is a version of the classification of flat surfaces in E^3. As was pointed out by Spivak ([S1] vol.4 Ch 4, [S2]), the classical "classification" of flat surfaces was not complete. Likewise, if we try here to classify flat slant surfaces with rank $\nu < 2$ completely, we cannot avoid some messy argument. But it is not our main concern, so we will just consider typical surfaces and will not go into the problem of gluing pieces of these surfaces together. The result is stated in Proposition 7.

If we assume some additional conditions, the shapes of these surfaces become more rigid. For example if $x(M)$ is slant and contained in some 3–plane in E^4, then rank $\nu<2$ and we get Proposition 8.

We first prove the following lemma.

Lemma 19

If x is slant and rank $\nu < 2$, then $x(M)$ is a flat ruled surface in E^4.

(Proof)

Let $\tilde{\nabla}$, ∇ and $\tilde{R}$, R be the connections and curvature tensors of E^4, M with respect to the induced metric of M. By $G^D \equiv 0$, (h^r_{ij})'s are simultaneously diagonalized, so we can choose an adapted from $\{e_A\}$ such that

(3) $$(h^3_{ij}) = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}, \quad (h^4_{ij}) = \begin{bmatrix} d & 0 \\ 0 & e \end{bmatrix}.$$

By Gauss' equation Ch1 (1.21),

(4) $$bc + de = 0$$

Put

(5) $$M_1 = \{p \in M \mid H(p) \neq 0\} \quad open \subset M$$

(6) $$M_0 = Interior\ of\ (M - M_1)$$

If $M_0 \neq \emptyset$, then by (4) and $b + c = d + e = 0$, $b = c = d = e = 0$ *on* M_0, i.e., $x(M_0)$ is totally geodesic in E^4 and hence

(7) $$x(M_0) = a\ portion\ of\ 2\text{-}plane\ in\ E^4.$$

In the following assume $M = M_1$ (i.e., we don't think about ∂M_1). Let $e_3 = H/\|H\|$. Then, since rank $\nu < 2$,

(8) $$bc = 0, \quad d = e = 0$$

by a direct computation using Ch1 (3.7) and Ch 1 (1.19). Without loss generality $c = 0$, $b \neq 0$, i.e.,

(9) $$(h^3_{ij}) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix}, \quad (h^4_{ij}) = 0$$

Put

(10) $$\omega_2^1 = \lambda\omega^1 + \mu\omega^2$$

(11) $$\omega_4^3 = l\omega^1 + m\omega^2$$

Then,

(12) $$\tilde{R}(e_1, e_2)e_1 = R(e_1, e_2)e_1 - (e_2 b + \lambda b)e_3 + bme_4$$

(13) $$\tilde{R}(e_1, e_2)e_2 = R(e_1, e_2)e_1 + b\,\mu e_3$$

Since $\tilde{R} = 0$, $R = 0$ and $b \neq 0$

(14) $$e_2 b + \lambda b = 0$$

(15) $$m = \mu = 0$$

(16) $$\tilde{\nabla}_{e_2} e_2 = \mu e_1 = 0$$

which means the integral curves of e_2 in $x(M)$ is geodesics in E^4, i.e. straight lines.

Therefore, $x(M)$ is a flat ruled surface.

Q.E.D.

A flat ruled surface in E^4 is, "in general", a cylinder, a cone or a tangent developable ([S1] vol. 4 p 127). Let us consider slant surfaces of each type.

(Case A) $M = a$-slant cylinder

In this case

(17) $$x(M) = c \times l$$

where $c(s)$ and $l(t)$ are integral curves of e_1 and e_2 through $p_o = c(o) = l(o) \in x(M)$ and l is a straight line. If $x(M)$ is a-slant,

(18)
$$\begin{aligned} \cos a &= -\Omega(e_1, e_2) \\ &= \langle e_1, -J e_2 \rangle \\ &= \langle c'(s), -J e_2 \rangle \end{aligned}$$

If we denote by W the orthogonal complement of l in E^4, then $c(s)$ is a curve in W and $-Je_2$ is a fixed vector in W, hence $c(s)$ is a generalized helix in W whose tangents have a constant angle a between $-Je_2$.

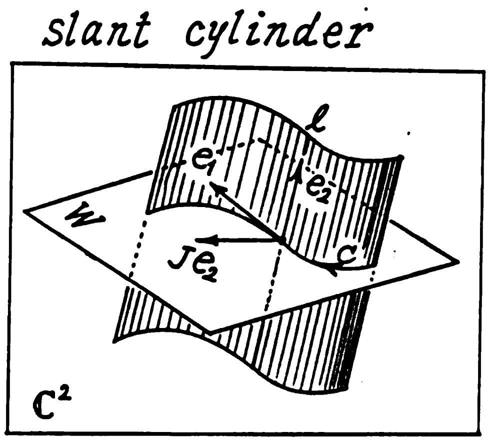


FIGURE 12

(<u>Case B</u>) M = a-slant cone.

Let $c(s)$, $l(t)$ be the integral curves of e_1 and e_2 through a point $p_o = c(o) = l(o)$. Without loss of generality, $\|p_o\|=1$ and the vertex of the cone is the origin of E^4. Since

$$(19) \qquad c(s)/|c(s)| = e_2(c(s)) \qquad \text{for } \forall s$$

i.e.

$$(20) \qquad \langle c(s), c'(s) \rangle = 0 \qquad \text{for } \forall s$$

hence

$$(21) \qquad \|c(s)\| = const = 1$$

which shows $c(s)$ is a curve on $S^3 = S^3_1(0)$.

If $x(M)$ is a-slant, then as before,

$$(22) \qquad \cos a = \langle c'(s), -J\, e_2(c(s)) \rangle \quad \text{for } \forall s$$

But

(23) $$J\, e_2(c(s)) = J\, \eta(c(s))$$

where η is the exterior normal of S^3 and hence by the argument of Ch 4 §2, $(\psi \circ c)(s)$ is a generalized helix in S^3 with the axis vector field $\tilde{X}_1$.

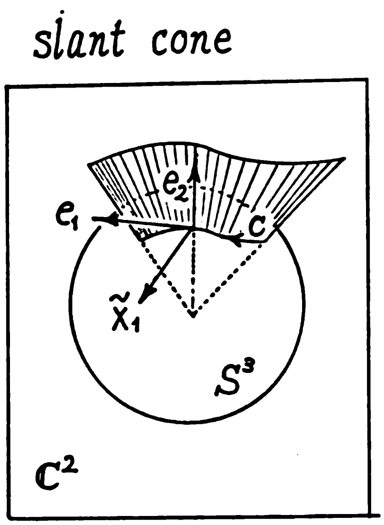


FIGURE 13

(<u>Case C</u>) M = a-slant tangent developable. Put

(24) $$x(s) = c(s) + (t-s)c'(s)$$

where $c(s)$ is a curve in $\mathbf{E}^4$ parameterized by arc length.

Put

(25) $$\begin{cases} v_1(s) = c'(s) \\ \kappa_1(s) = \|v_1(s)\| \\ v_2(s) = (1/\kappa_1(s))v_1'(s) \end{cases}$$

then

(26) $$\langle v_1, v_2 \rangle = 0, \quad \|v_1\| = \|v_2\| = 1.$$

Put

(27) $$\begin{cases} e_1\,(s,t) = v_2(s) \\ e_2\,(s,t) = v_1(s) \end{cases}$$

then, $\{e_1, e_2\}$ is a positive orthonormal frame on M.

Assume x is a-slant, then

$$(28) \qquad \cos a = -\Omega_J(e_1 \wedge e_2) = \langle v_2(s), -J\, v_1(s) \rangle = \langle v'(s)/\|v_1'(s)\|, -J\, v_1(s) \rangle$$

We consider $v_1(s)$ a curve in S^3. Then (28) means

$$(29) \qquad \cos a = \langle t(s), -J\, \eta(v_1(s)) \rangle$$

where t is the tangent of the curve $v(s)$. Hence, by Ch 4§1, $(\psi \circ v(s))$ is a generalized helix in S^3 with the axis vector field $\tilde{X}_1$.

slant tangent developable

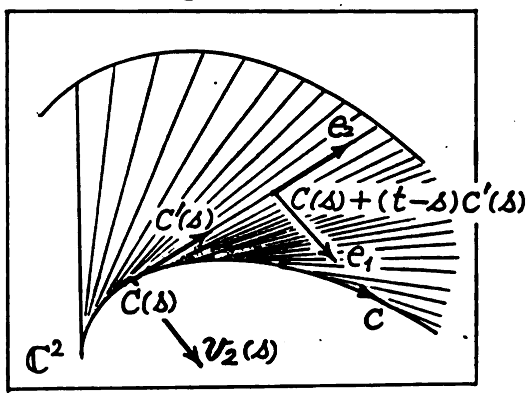


FIGURE 14

In cases B and C, a generalized helix in S^3 with axis vector field $\tilde{X}_1$ is defined, analogous to Euclidean case, by Definition 4 with (3.4) replaced by

$$(30) \qquad \begin{cases} f_1(s) = b \\ (f_2(s))^2 + (f_3(s))^2 = 1-b^2. \end{cases}$$

In each case A–C, the converse is also true. Summing, up, we have the following proposition.

Proposition 7

If $x(M)$ *is a slant surface with rank* $\nu<2$, *then* $x(M)$ *is a flat ruled surface in* $\mathbb{C}^2$. *Furthermore,*

(*i*) *A cylinder in* $\mathbb{C}^2$ *is a slant surface if and only if it is of the form* $c(s) \times l(t)$ *where,* l *is a straight line generated by a unit vector, say* e, *and* c *is a generalized helix with axis* Je *in a 3–plane perpendicular to* l.

(*ii*) *A cone in* $\mathbb{C}^2$ *is a slant surface if and only if it is of the form (modulo translations)*

$$tc(s)$$

where $(\psi \circ c)(s)$ *is a generalized helix in* S^3 *with axis vector field* $\tilde{X}_1$.

(*iii*) *A tangent developable*

$$x(s,t) = c(s) + (t-s)c'(s)$$

is a slant surface if and only if $\psi \circ c'$ *is a generalized helix in* S^3 *with the axis vector field* $\tilde{X}_1$.

Next, we consider slant surfaces in $\mathbb{C}^2$ contained in a hyperplane in $\mathbf{E}^4$. We note first the following lemma.

Lemma 20

Let $x : M \to \mathbb{C}^2 = (\mathbf{E}^4, J_0)$ *be a slant immersion of an oriented surface. If* $x(M)$ *is contained in some* $W \in G(3,4)$, *then* rank $\nu < 2$

and x is doubly slant with the same slant angle.

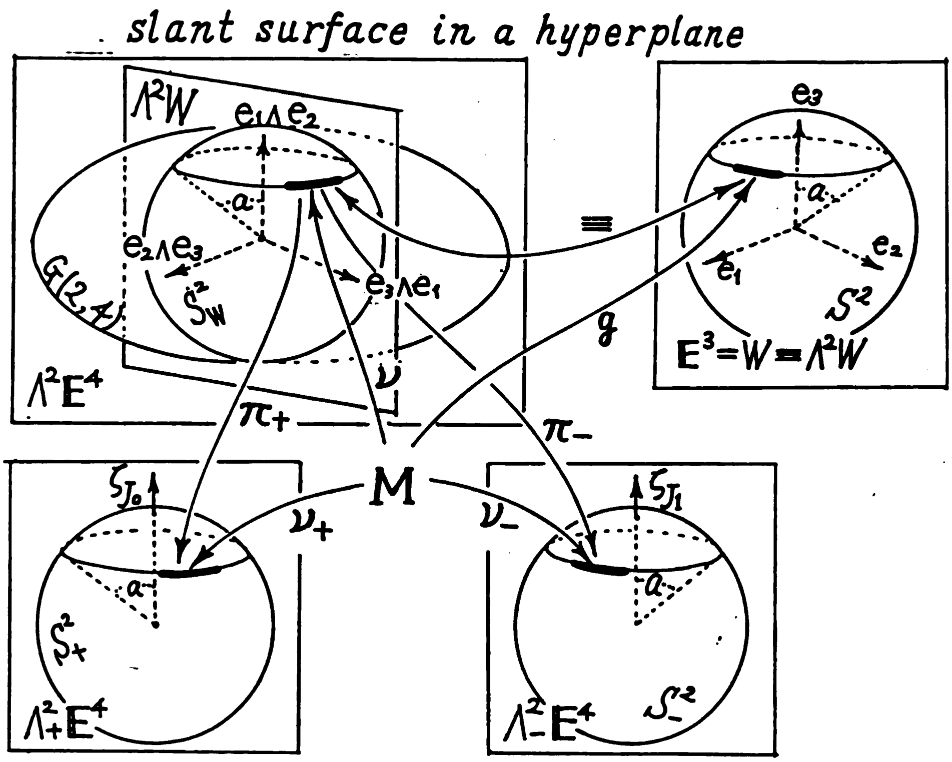


FIGURE 15

(Proof)

As in the proof of Lemma 6, we choose a positive orthonormal basis $\{e_A\}$ of $\mathbf{E}^4$, such that

(31) $$e_1,\ e_2 = J_0 e_1 \in W \cap JW,$$

(32) $$e_4 = Je_3 = \eta,$$

where η is the positive unit normal of W in $\mathbf{E}^4$. Put

(33) $$\Lambda^2 W = \{2\text{-}vectors\ of\ \ W\},$$

(34) $$G_W = G(2,4) \cap \Lambda^2 W$$

$\Lambda^2 W$ is a 3-dimensional linear subspace of $\Lambda^2\mathbf{E}^4$ spanned by $\{e_1\wedge e_2,$

$e_2 \wedge e_3$, $e_3 \wedge e_1\}$. Any $\xi \in \Lambda^2 W$ satisfies $\xi \wedge \xi = 0$, i.e., ξ is decomposable, and hence G_W is the unit sphere in $\Lambda^2 W$. For $a \in [0, \pi]$, put

(35) $$G_{W,a} = \{V \in G_W \mid V \textit{ is a-slant with respect to } J_0\}$$

Then, by Ch 2(1.26), $G_{W,a}$ is a circle on $G_W = S^2$ expressed by

(36) $$G_{W,a} = \{V \in G_W \mid \langle V, e_1 \wedge e_2 \rangle = \cos a\}$$

Put as before

(37) $$J_1 = J_{e_1 \wedge e_2}$$

by Ch 2(1.29).

Then, using notations of Ch2 §1,

(38) $$\pi_+(G_{W,a}) = S^1_{J_0,a} \subset S^2_+,$$

(39) $$\pi_-(G_{W,a}) = S^1_{J_0,a} \subset S^2_-$$

If x is a-slant with respect to J_0 and $x(M) \subset W$, then

(40) $$\nu(M) \subset G_{W,a}$$

which implies

(41) $$\text{rank } \nu < 2,$$

and by (39) and Proposition 2, x is also a-slant with respect to J_1.

Q.E.D.

Remark

If we identify $\Lambda^2 W$ with Euclidean 3-space $\mathbf{E}^3$ spanned by $\{e_1, e_2, e_3\}$ through the isometry $X \wedge Y \to X \times Y$, where $\times$ is the usual vector product, then $\nu : M \to G_W \subset \Lambda^2 W$ is nothing but the classical Gauss map $g : M \to S^2 \subset \mathbf{E}^3$. So, $x(M)$ is a slant surface if and only if $g(M)$ is contained in a circle S^1_a on $S^2 \cup W = \mathbf{E}^3$ perpendicular to $e_1 \times e_2 = e_3 = -J\eta$:

(42) $$S^1_a = \{Z \in W \mid \langle Z, -J\eta \rangle = \cos a\}$$

By Lemma 20, we consider slant surfaces contained in a hyperplane according to the three cases of Proposition 7. We choose a local frame field $\{e_A\}$ used in the proof of [BYC5] Theorem 2 as follows. Let P, F be as in Ch 1 (4.5) and t, f be as in [BYC5], namely,

(43) $$J_0 Y = tY + f\,Y \qquad for \quad Y \in T^{\perp}M$$

where $tY \in TM$ and $f\,Y \in T^{\perp}M$. Pick a local unit vector field e on M which takes value in $T^{\perp}_pM \cap W$ at $p \in M$. Let a be the slant angle of $x(M)$ and assume $x(M)$ is proper slant. Then, we can put

(44) $$\begin{cases} e_1 = te/|te| \\ e_2 = (sec\ a)\ Pe_1 \\ e_3 = (cosec\ a)\ Fe_1 \\ e_4 = (cosec\ a)\ Fe_2 \end{cases}$$

then $\{e_A\}$ is an adapted frame field on M and satisfies

(45) $$e_3(p) \in T^{\perp}_pM \cap W$$

(46) $$\begin{cases} te_3 = -(sin\ a)\ e_1 \quad , \quad te_4 = -(sin\ a)\ e_2 \\ fe_3 = -(cos\ a)\ e_4 \quad , \quad fe_4 = (cos\ a)\ e_3. \end{cases}$$

Since $\tilde{\nabla}\ e_4 = 0$,

(47) $$(h^4_{ij}) = 0.$$

By [BYC5] Proposition 2 and Lemma 3, we can use $A_{FX}Y = A_{FY}X$, and hence

(48) $$(h^4_{ij}) = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}$$

(47) and (48) means our frame $\{e_A\}$ coincide with the one we chose in the proof of Lemma 19 and all equations there are also valid here. As before, we consider the case $a \neq 0$. By $\tilde{\nabla}_X(Je_4) = 0$ and (46),

(49) $$-(sin\ a)\ \nabla_X e_2 - (cos\ a)\ A_3 X = 0$$

hence

(50) $$\omega_2^1(e_1) = -b \cot a$$

and by (10) and (14)

(51) $$e_2 b = b^2 \cot a.$$

Case (A′): a slant cylinder in W

(52) $$x(M) = c \times l \subset W$$

where l is a straight line generated by some unit vector $\boldsymbol{v}$ and c is a curve in a 2–plane W' in W perpendicular to $\boldsymbol{v}$. Then, $x(M)$ is totally real with respect to $J^{\pm}_{W'}$. So $x(M)$ is either a portion of a 2–plane (the case of Ch 2 §2(h)) or a non–minimal totally real cylinder (the case of Ch 2 §2 (i)) and hence $J_0 = J^{+}_{W'}$, or $J^{-}_{W'}$.

Case (B′): a slant cone in W.

(53) $$x(M) = \{t\, c(s)\} \subset W$$

where $c(s)$ is a curve in

(54) $$S^2 = S^3 \cap W.$$

Assume $x(M)$ is proper slant and define $\{e_A\}$ by (44). Let $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$, κ *and* τ be the tangent, normal, binormal, curvature and torsion of $c(s)$ in $W = \mathbf{E}^3$. Then,

(55) $$e_1(s, t) = c'(s) = (1/t)\partial/\partial s$$

(56) $$e_2(s, t) = c(s) = \partial/\partial t$$

(57) $$e_3(s, t) = e_1(s, t) \times e_2(s, t)$$

where x is the vector product in $W = \mathbf{E}^3$. Hence

(58)
$$b = <\tilde{\nabla}_{e_1} e_1, e_3 >$$
$$= (1/t) <\tilde{\nabla}_{\partial/\partial s} e_1, e_3 >$$
$$= (1/t) < t', t \times c >$$
$$= -(\kappa/t) < n, t \times c >$$
$$= -(\kappa/t) < n \times t, c >$$
$$= -(\kappa/t) < b, c >.$$

Differentiating $< c, c > = 1$ by s twice,

(59)
$$< n, c > = -1/\kappa$$

note that $\kappa \neq 0$ since c is spherical. Differentiating again,

(60)
$$< b, c > = -(1/\tau)(1/\kappa)'.$$

By (58) and (60)

(61)
$$b = (-\kappa'/\kappa\tau)(1/t).$$

Hence by (51), (61) and (56)

(62)
$$\kappa' = \kappa\tau \tan \theta.$$

By (59), (60) and $< t, c> = 0$

(63)
$$c = -(1/\kappa)n - (1/\tau)(1/\kappa)'\mathrm{b}$$

Since $|c| = 1$,

(64)
$$\tau^2\kappa^4 = \kappa^2\tau^2 + (\kappa')^2$$

By (62) and $\kappa \neq 0$

(65)
$$\tau^2(\kappa^2 - 1 - \tan^2 a) = 0$$

(65) and (62) imply

(66)
$$\tau = 0$$

which means $c(s)$ is a circle on S^2 and hence $x(M)$ is a circular cone. By the remark after Lemma 20, we can see that the axis of this cone is $-J_0\eta$.

Case (C′): a slant tangent developable in W.

(67) $$x(s,t) = c(s) + (t-s)c'(s)$$

Assume x be proper slant and $\{e_A\}$, $t = v_1$, $n = v_2$, b, $\kappa = \kappa_1$, τ be as in case (B′) and case (C). Then,

(68) $$e_1(s, t) = n(s) = (1/t-s)\ \partial/\partial s$$

(69) $$e_2(s, t) = t(s) = \partial/\partial t$$

(70) $$e_3(s, t) = e_1(s, t) \times e_2(s, t) = -b(s).$$

Hence

(71) $$\tilde{\nabla}_{e_1} e_1 = (1/(t-s)\kappa)\mathbf{n}'(s)$$
$$= -1/(t-s)t + \tau/(t-s)\kappa\ b$$
$$= -1/(t-s)e_2 - \tau/(t-s)\kappa e_3$$

so by (9)

(72) $$b = -\ \tau/(t-s)\kappa$$

By (72) and (69)

(73) $$e_2 b = \tau/\kappa(t-s)^2$$

By (51) and (72)

(74) $$\tau/\kappa = \tan a = const.$$

Hence, $c(s)$ is a generalized helix in W and $x(s, t)$ is a helicoid. The axis of this helix is $-J_0\eta$.

In each case A–C, the converse holds. For any circular cone in W, we can choose a complex structure J on $\mathbf{E}^4$ such that $-J\eta$ is the axis of this cone, and the same for a helicoid.

Thus we have proved the following Proposition.

Proposition 8

Let $x : M \to \mathbf{C}^2 = (\mathbf{E}^4, J_0)$ *be a proper slant immersion of an oriented surface* M. *If* $x(M)$ *is contained in a 3–plane* W, *then* $x(M)$ *is a flat ruled surface in* W. *And,*

(A) *A cylinder in* W *is a proper slant surface with respect to a complex structure* J *on* $\mathbf{E}^4$ *if and only if it is a portion of 2–plane.*

(B) *A cone in* W *is a proper slant surface with respect to a complex structure* J *on* $\mathbf{E}^4$ *if and only if it is a circular cone.*

(C) *A tangent developable in* W *is a proper slant surface with respect to a complex structure* J *on* $\mathbf{E}^4$ *if and only if it is a helicoid.*

SUMMARY

We have proved the following:

(1) The set of 2–planes in $\mathbb{C}^2=(E^4, J_0)$ with constant Wirtinger angle α is described as $S^1_{J_0,\alpha} \times S^2_-$ where $G(2,4)=S^2_+ \times S^2_-$ is the decomposition of the Grassmannian into the product of two 2–spheres of radius $1/\sqrt{2}$ in the eigenspaces of the star operator of $\Lambda^2 E^4$ and $S^1_{J_0,\alpha}$ is a circle in S^2_+ determined by J_0 and α.

(2) An immersion x of an oriented surface M into $\mathbb{C}^2$ is a slant immersion with the Wirtinger angle α if and only if $\nu_+(M) \subset S^1_{J_0,\alpha}$ where $\nu_+ = \pi_+ \circ \nu$, ν is the Gauss map and π_+ is the projection of $G(2,4)$ onto S^2_+.

(3) Any surface without complex tangent points in a 4–dimensional almost Hermitian manifold is a proper slant surface with given Wirtinger angle with respect to a suitable almost complex structure.

(4) No compact proper slant submanifolds exist in complex spaces $\mathbb{C}^m$.

(5) For a surface contained in S^3 of E^4, we can define another Gauss map by means of left–invariant vector fields. A surface in S^3 is a slant surface with respect to a complex structure if and only if the image of this Gauss map is contained in a circle.

(6) A surface in S^3 is a slant surface if and only if it is a "helical cylinder" where "helixes" and "cylinders" in S^3 are defined by analogies of those of E^3 replacing parallel translations with left translations.

(7) If a slant surface in $\mathbb{C}^2$ has the rank of the Gauss map less than 2, then it is a flat slant ruled surface in $\mathbb{C}^2$ and we can apply the

classification of flat ruled surfaces in E^4. In particular a proper slant surface contained in a 3–plane in $\mathbb{C}^2$ is, in general, a portion of a 2–plane, a circular cone or a helicoid.

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