THE SOLUTION OF THE INTEGRAL EQUATION FOR THE PROLATE SPHEROIDAL TRANSMITTING ANTENNA

Thesis for the Degree of Ph. D.

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Patricia James Wells

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This is to certify that the

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Major professor

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By

Patricia James Wells

ABSTRACT

Submitted to the School of Graduate Studies of Michigan State University of Agriculture and Applied Science in partial fulfillment of the requirements for the degree of

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ABSTRACT

This thesis thoroughly studies the problem of the prolate spheroidal transmitting antenna formulated as an integral equation. The Albert-Synge integral equation (Quarterly of Applied Mathematics, Vol. 6, 1948), satisfied by the current on an antenna surface of revolution, is solved for the case when the surface is a prolate spheroid. As a part of this solution, expansions are found in terms of the prolate spheroidal wave functions for the components of the field of an electric dipole. By means of these expansions recursion relations are found between the spheroidal wave functions of orders one and zero. Finally, the Hallén method of successive approximations, which has been used to solve the integral equation for the finite cylindrical antenna, is applied to the Albert-Synge integral equation for the spheroid. The approximate solutions so found are then compared with the known exact solution.

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DEDICATION

To Jacky and Mary Wells

DEDICATION

To Jacky and Mary Wells

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INTRODUCTION

The problem of the perfectly conducting, radiating antenna, when formulated mathematically, becomes a boundary value problem. Consider a mathematical surface S comprised of two parts: 1.) the actual surface of the antenna, and 2.) the gap, or source of radiated energy. Vector functions, $\underline{E}$ and $\underline{H}$, are sought, which, on S, and in the infinite region exterior to S, are regular functions satisfying Maxwell's equations. Certain well-known boundary conditions are prescribed. The tangential component of the electric vector

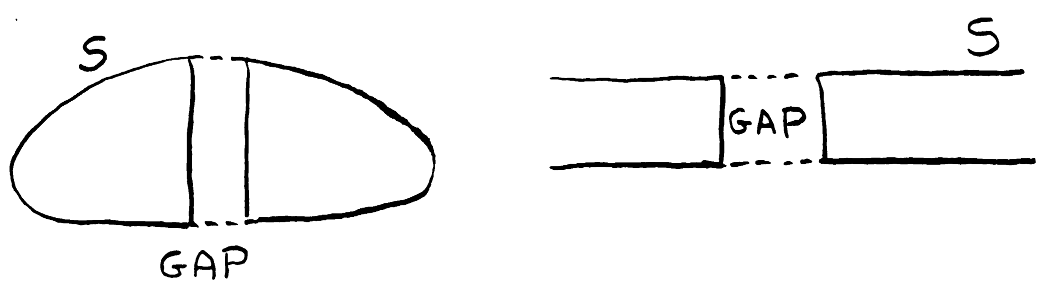

Fig. 1

$\underline{E}$ must vanish everywhere on S except at the gap where this function is given; the behavior of the field ($\underline{E}$,$\underline{H}$) is prescribed at infinity. This condition at infinity, which will be stated more precisely later, is known as the radiation condition. For many problems it is assumed that the field ($\underline{E}$, $\underline{H}$) possesses rotational symmetry, which essentially reduces the vector problem to a scalar one, thereby simplifying the mathematics of the problem.

Despite the fact that an antenna problem may be so simply stated, the number of antenna problems to which exact solutions have been found is few. The difficulty lies in the nature of the surface S. In general, the only antenna problems which may be solved exactly by standard techniques are those for which S coincides with a coordinate surface in a separable coordinate system. One is, therefore, immediately limited to such surfaces as the sphere, the spheroid, the infinite cylinder, etc.. Although some two-part exterior boundary value problems have been solved exactly for LaPlace's equation (Karp, 9), the comparable electromagnetic problems have proven more difficult. There exists no exact solution, for example, for such a simple surface geometrically as the finite cylinder.

Of the eleven coordinate systems in which Maxwell's equations separate, the spherical and spheroidal coordinate systems are the only ones having a family of finite coordinate surfaces; for this reason the mathematical theory of the spherical and spheroidal antennas has been extensively studied. The prolate spheroid, particularly, has been the subject of wide investigation because for eccentricities nearly one it closely approximates the finite cylinder of small width.

The wave functions themselves are basic to the difficulties encountered in prolate (or oblate) spheroidal problems. A suitable representation for the functions is in itself a problem, because at one extreme they must approximate the spherical wave functions and at the other cylindrical functions. The ordinary differential equations, obtained by separation of the wave equation, have series solutions whose coefficients satisfy a three term recursion formula, thereby making computation of the functions difficult. Although a variety of series representations for these functions have been proposed, those defined by Stratton, Morse, Chu, and Hutner (22) are used in this thesis because of their more general acceptance in the literature. Further difficulties arise because relatively little is known concerning the properties of the functions. The many relationships between the functions of different orders and their derivatives, which are known for the hypergeometric functions, have not been made available for the spheroidal functions.

Because other methods have not proven fruitful, the problem of the finite cylindrical antenna has been formulated as an integral equation, (Hallén, 7; Brillouin, 3; Albert and Synge, 23) and the solution sought by various methods of successive approximations. These methods, however, have been subject to considerable criticism and frequent revision. The series solutions found by these approximations are not readily computed, and nearly nothing is known concerning their convergence or divergence.

This thesis studies the problem of the radiating prolate spheroidal antenna formulated as an integral equation. The original motivation lay in the hope that this study might elucidate the cylindrical antenna problem. In Part I the spheroidal functions as defined by Stratton et al (22) are outlined; the problem of the prolate spheroidal antenna is solved by use of a Green's function and Green's formula, which yields an integral equation in the usual way. After Infeld, (8) a step function feed is assumed over a gap of finite width. The solution obtained here becomes that of Stratton and Chu (21) if an infinitesimal gap is assumed instead of a gap of finite width.

The integral equation of Albert and Synge (23) is taken up in Part II, where the main results of this thesis are to be found. This integral equation is an exact formulation of the antenna problem and, being stated for a surface of revolution, it may be readily specialized to either the cylinder or the prolate spheroid. The statement and theory of the Albert-Synge integral equation is summarized; this resume is followed by a solution of the equation for the case of the prolate spheroid. This solution is effected by the expansion of the kernel and the unknown function in series of spheroidal functions of order one. The unknown coefficients in the expansion for the unknown function are determined by means of the orthogonality of the angular spheroidal functions. The solution so obtained, again under the assumption of a gap of finite width, is identical to that found in Part I.

Expansions in terms of the spheroidal wave functions of order one are found for the components of the field of an electric dipole. By means of these expansions, recursion relations are found between the spheroidal wave functions of orders zero and one.

Finally, in Part III, a method of successive approximations, comparable to the method of Hallén, is applied to the Albert-Synge integral equation for the spheroid. The approximate solutions found thereby are then compared with the known exact solution to the integral equation.

I

BACKGROUND

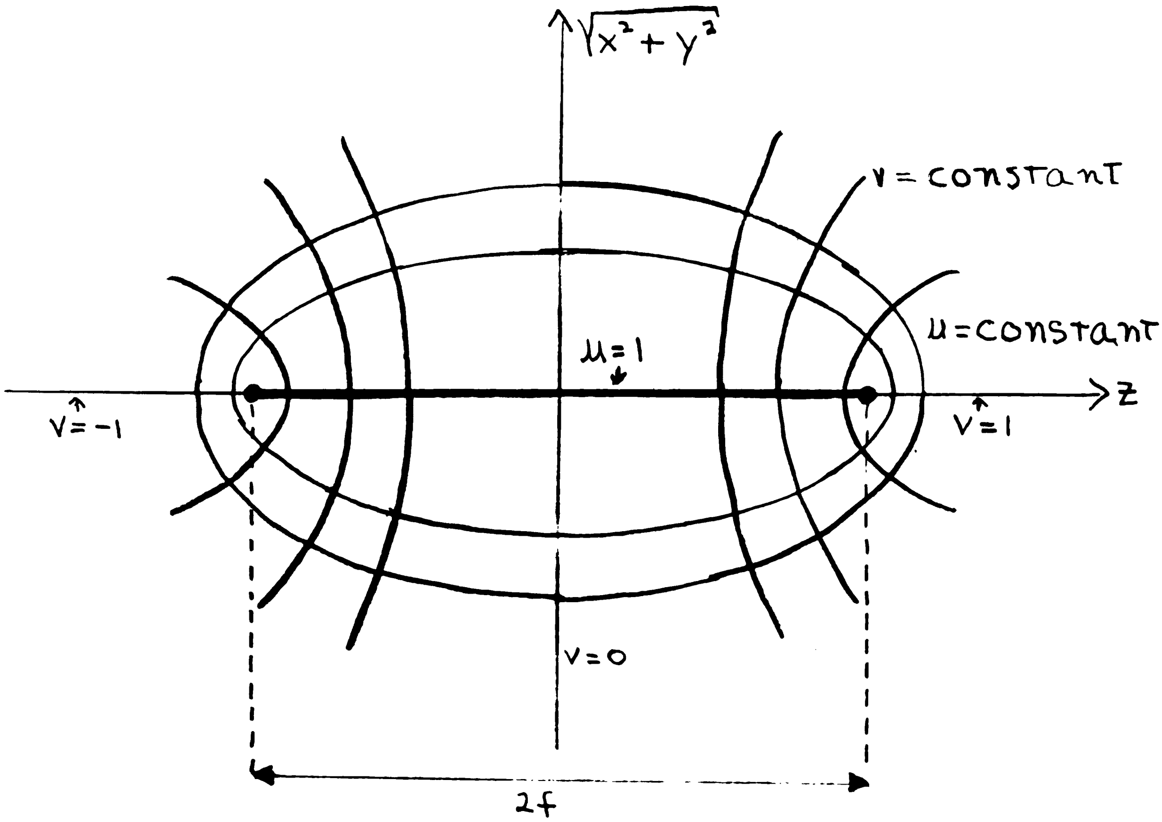

Fig. 2

1.) The Prolate Spheroidal Wave Functions

A cross-section of the prolate spheroidal coordinate system is illustrated above. The surfaces v = constant are confocal hyperboloids of revolution, and the surfaces u = constant are confocal ellipsoids. The third coordinate ϕ is the azimuthal variable about the axis of the system, in this case the z-axis. The surface ϕ = constant is a half plane terminating at the axis. The relations between Cartesian coordinates and the spheroidal coordinates are as follows:

$$x = f(1 - v^2)^{\frac{1}{2}}(u^2 - 1)^{\frac{1}{2}} \cos \phi$$

(1-1) $$y = f(1 - v^2)^{\frac{1}{2}}(u^2 - 1)^{\frac{1}{2}} \sin \phi$$

$$z = fuv$$

where f is the semi-focal length, and $1 \leq u < \infty$, $-1 \leq v \leq 1$, and $0 \leq \phi < 2\pi$. The metric is

(1-2) $$e_v = \frac{f(u^2 - v^2)^{1/2}}{(1 - v^2)^{1/2}}; \quad e_u = \frac{f(u^2 - v^2)^{1/2}}{(u^2 - 1)^{1/2}}; \quad e_\phi = f(1 - v^2)^{\frac{1}{2}}(u^2 - 1)^{\frac{1}{2}}$$

The general properties of the spheroidal wave functions as defined by Stratton et al. (22) are outlined below. The scalar wave equation, $\nabla^2 U + k^2 U = 0$, becomes, in spheroidal coordinates

(1-3) $$\frac{\partial}{\partial v}(1 - v^2)\frac{\partial U}{\partial v} + \frac{\partial}{\partial u}(u^2 - 1)\frac{\partial U}{\partial u} + \frac{(u^2 - v^2)}{(1 - v^2)(u^2 - 1)}\frac{\partial^2 U}{\partial \phi^2} + f^2k^2(u^2 - v^2)\,U = 0$$

Assuming a solution $U = S(v)R(u)Q(\phi)$, (1-3) separates into

$$\frac{d^2Q}{d\phi^2} + m^2Q = 0$$

(1-4) $$\frac{d}{dv}(1 - v^2)\frac{dS_{mn}}{dv} - [m^2/(1 - v^2) - A_{mn} + f^2k^2v^2]S_{mn} = 0$$

$$\frac{d}{du}(u^2 - 1)\frac{dR_{mn}}{du} - [m^2/(u^2 - 1) + A_{mn} - f^2k^2u^2]R_{mn} = 0$$

A_{mn} and m are separation constants. m must be integral to satisfy conditions of periodicity, and the A_{mn} are determined such that the angular functions $S_{mn}(kf, v)$ will be finite at the singular points $v = \pm 1$. The solutions to the problems in this thesis must be finite for v in the range $-1 \leq v \leq 1$, and the angular functions of the second kind will not be discussed. The series expansion for $S_{mn}(kf', v)$ is of the form

(1-5) $$S_{mn}(kf, v) = \sum_{r=0,1}^{\infty}{}' d_r^{mn} P_{m+r}^m(v)$$

where $P_{m+r}^m(v)$ are the associated Legendre functions, and the primed summation indicates that the sum is to be taken over even or odd r according as n is even or odd. The coefficients d_r^{mn} are finally determined by the normalization

(1-6) $$S_{mn}(kf,0) = P_{m+n}^m(0); \quad \left.\frac{dS_{mn}(kf,v)}{dv}\right|_{v=0} = \left.\frac{dP_{m+n}^m(v)}{dv}\right|_{v=0}$$

The radial functions of the first kind, $R_{mn}^{(1)}(kf,u)$, satisfy the same equation as $S_{mn}(kf,v)$, but for a different range of the variable. These functions are finite at $u = \pm 1$, and are normalized to behave like the spherical Bessel functions at ∞.

(1-7) $$R_{mn}^{(1)}(kf,u) \longrightarrow \frac{1}{kfu} \sin\left(kfu - \frac{m+n}{2}\right)$$

The functions $S_{mn}(kf,z)$ and $R_{mn}^{(1)}(kf',z)$ are related by a constant factor λ_{mn}, which depends on m and n.

(1-8) $$S_{mn}(kf',z) = \lambda_{mn} R_{mn}^{(1)}(kf,z)$$

The radial functions of the second kind are characterized by logarithmic singularities at $u = \pm 1$, and have the following asymptotic behavior for large u:

(1-9) $$R^{(2)}_{mn}(kf,u) \longrightarrow -\frac{1}{kfu}\cos\left(kfu - \frac{m+n}{2}\right)$$

Radial functions of the third and fourth kinds are defined:

(1-10) $$R^{(3)}_{mn} = R^{(1)}_{mn} + iR^{(2)}_{mn} \qquad R^{(3)}_{mn} = R^{(1)}_{mn} - iR^{(2)}_{mn}$$

The behavior of $R^{(3)}_{mn}$ and $R^{(4)}_{mn}$ for large u is easily found from (1-7) and (1-9).

(1-11) $$R^{(3)(4)}_{mn} \longrightarrow \frac{1}{kfu}\, e^{\pm i\left(kfu - \frac{m+n}{2}\pi\right)}$$

The Wronskian of the two linearly independent solutions is

(1-12) $$W[R^{(1)}_{mn}, R^{(2)}_{mn}] = \frac{1}{kf(u^2 - 1)}$$

Inasmuch as the parameter kf will remain fixed (except for the computation in Part III), it will be omitted as an argument for both the angular and radial functions.

For the sake of completeness the series for the radial functions is included.

(1-13)
$$R^{(1)(2)}_{mn}(u) = \frac{(u^2 - 1)^{\frac{m}{2}}}{u^m \sum_r' d_r^{mn} \frac{(r+2m)!}{r!}} \sum_{r=0,1}^{\infty}{}' \, i^{n-r} d_r^{mn} \frac{(r+2m)!}{r!} \begin{Bmatrix} j_{m+r}(u) \\ n_{m+r}(u) \end{Bmatrix}$$

Finally, the angular functions form a complete set and are orthogonal on the interval (-1,1).

$$(1\text{-}14) \qquad \int_{-1}^{1} S_{mn}(v)\, S_{mr}(v)\, dv = \begin{matrix} 0 & n \neq r \\ N_{mn} & n = r \end{matrix}$$

2. The Solution to the Boundary Value Problem for the Prolate Spheroidal Antenna

The boundary value problem for the prolate spheroidal antenna, as formulated by Stratton and Chu (21), is outlined in this section. The method of solution differs insofar as a Green's function is used instead of the standard boundary value techniques. Certain modifications are made in the Stratton-Chu theory regarding the applied field at the gap; namely, after Infeld (7), the width of the gap is assumed to be finite.

Let the surface, S, described in the introduction, be, in prolate spheroidal coordinates, the spheroid $u = u'$, which coincides with the antenna and covers the gap from which energy is radiated. Harmonic time dependence is assumed. That is, all components of the vectors E and H vary with time as e^{-iwt}, where $w = 2\pi\nu = kc = \frac{2\pi c}{\lambda}$, with ν the frequency, λ the wave length, and c the velocity of light. For mathematical simplicity, it is also assumed that $\underline{E}$ and $\underline{H}$ are independent of ϕ; this is equivalent, physically, to assuming that the voltage at the gap is applied symmetrically. Under this hypothesis, Maxwell's equations, curl $\underline{E} = ik\underline{H}$, and curl $\underline{H} = -ik\underline{E}$, consist of two independent sets. The one which is of interest here involves only E_u, E_v, and H_ϕ:

$$- ikf(u^2-v^2)^{\frac{1}{2}} E_v = \frac{\partial}{\partial u} (u^2 - 1)^{\frac{1}{2}} H_\phi$$

$$ikf(u^2-v^2)^{\frac{1}{2}} E_u = \frac{\partial}{\partial v} (1 - v^2)^{\frac{1}{2}} H_\phi$$

(1-15)

$$(1 - v^2)^{\frac{1}{2}} \frac{\partial}{\partial v}(u^2 - v^2)^{\frac{1}{2}} E_u - (u^2 - 1)^{\frac{1}{2}} \frac{\partial}{\partial u}(u^2 - v^2)^{\frac{1}{2}} E_v = ikf\ (u^2 - v^2) H_\phi$$

If E_v and E_u are eliminated from (1-15) the following equation for H_ϕ is obtained.

$$L[H_\phi] = [\frac{\partial}{\partial u}(u^2 - 1)\frac{\partial}{\partial u} + \frac{\partial}{\partial v}(1 - v^2)\frac{\partial}{\partial v} - \frac{(u^2 - v^2)}{(1 - v^2)(u^2 - 1)} + f^2k^2(u^2 - v^2)]\ H_\phi = 0$$

(1-16)

Upon separation of variables and comparison with (1-4), $S_{1n}(v)$ and $R_{1n}(u)$ are found to be the eigenfunctions of equation (1-16). There are two boundary conditions, as mentioned in the introduction. First, the total tangential component of the electric field must vanish of the surface:

(1-17)
$$E_v + E_A \Big|_{u=u'} = 0$$

where E_v and E_A are the tangential components of the induced and applied fields respectively. E_A is given. The second boundary condition concerns the behavior of the field at infinity:

$$\lim_{R \to \infty} R\,[(\underline{n} \cdot \nabla)\underline{E} - ik\underline{E}] = 0 \tag{1-18}$$

where $\underline{n}$ is the exterior unit normal to S, and R the radius of a large sphere with center at the center of the spheroid. This limit must hold uniformly in all directions. A more general statement of this condition by Synge and Albert is given later.

For the applied field E_A a step function is assumed: on the gap E_A is equal to a constant voltage divided by the length of arc across the gap, while elsewhere on the surface E_A is zero. For a gap extending from $-v_1$ to v_1, we have

$$(1\text{-}19)\qquad \begin{aligned} E_A &= -\frac{V}{2v_1 e_v} = -\frac{V(1-v^2)^{1/2}}{2v_1 f(u^2-v^2)^{1/2}} && |v| \leq v_1 \\ &= 0 && |v| > v_1 \end{aligned}$$

The function E_A may be expanded in a series of the orthogonal angular functions.

$$(u^2 - v^2)^{1/2} E_A = -\frac{V}{f} \sum_{n=0}^{\infty} B_n \frac{S_{1n}(v)}{N_{1n}} \tag{1-20}$$

$$B_n = \int_{-v_1}^{v_1} \frac{(1-v^2)^{1/2}}{2v_1} S_{1n}(v)\, dv \tag{1-21}$$

Stratton and Chu (21) assume instead an infinitesimal gap. Under this assumption the coefficients in (1-20) become

$B_n = S_{1n}(0)$. However, as is pointed out by Infeld (8), the solution obtained using (1-20) becomes logarithmically infinite as the gap width goes to zero. Infeld shows that the assumption of a finite gap width, long compared to the width of the antenna but short in comparison with its length, leads to a solution without this singularity. For this reason we assume a finite gap width.

We now apply Green's formula, which, for $P = P(u,v)$ and $P_o = P_o(u_o,v_o)$, reads:

$$\int_S \left[G(P,P_o) \frac{\partial}{\partial n_o} H_\phi(P_o) - H_\phi(P_o) \frac{\partial}{\partial n_o} G(P,P_o) \right] dS_o \tag{1-22}$$

$$= H_\phi(P) \qquad P \text{ outside } S$$

$$= 0 \qquad P \text{ inside } S$$

where $\frac{\partial}{\partial n_o}$ is the normal derivative to S, and $G(P, P_o)$ is the Green's function satisfying the following conditions:

1. G satisfies $L[G] = 0$ except at $P = P_o$.
2. $G(P,P_o) = G(P_o,P)$.
3. $\lim_{\rho \to 0} L[G]\, d\sigma = -1$, where ρ is the radius of a small sphere σ with center at P_o.
4. $\frac{\partial}{\partial u}(u^2 - 1)^{1/2}\, G(P,P_o) = 0$ when $u = u'$.

The particular Green's function which satisfies these conditions was found by Leitner and Hatcher (13).

$$(1\text{-}23)\qquad G(P,P_o) = -\frac{k}{2\pi}\sum_{n=o}^{\infty}\frac{S_{1n}(v)\;S_{1n}(v_o)}{w_{1n}^{(3)\prime}(u')\;N_{1n}}\begin{cases}R_{1n}^{(I)}(u)R_{1n}^{(3)}(u_o) & u\leq u_o\\ R_{1n}^{(I)}(u_o)R_{1n}^{(3)}(u) & u_o\leq u\end{cases}$$

where the prime indicates differentiation, $w_{1n}^{(\,)}(u) = (u^2-1)^{1/2}R_{1n}^{(\,)}(u)$, and $R_{1n}^{(I)}(u)$ is the following radial function:

$$(1\text{-}24)\qquad R_{1n}^{(I)}(u) = w_{1n}^{(2)\prime}(u')\;R_{1n}^{(1)}(u) - w_{1n}^{(1)\prime}(u')\;R_{1n}^{(2)}(u)$$

In the Green's function (1-23), the angular functions of the first kind only are chosen to satisfy regularity conditions, and the radial functions of the third kind are chosen to satisfy the radiation condition (1-18).

On the surface S: $u_o = u' \leq u$, we have

$$dS_o = 2\pi f^2(u'^2-1)^{1/2}(u'^2-v_o^2)^{1/2}dv_o$$

$$(1\text{-}25)\qquad \frac{\partial}{\partial n_o}G(P,P_o) = \frac{(u'^2-1)^{1/2}}{f(u'^2-v_o^2)^{1/2}}\frac{\partial}{\partial u'}\;G(P,P_o)$$

$$= -\frac{u'}{f(u'^2-1)^{1/2}(u'^2-v_o^2)^{1/2}}\;G(P,P_o)$$

$$\frac{\partial}{\partial n_o}H_\phi(u',v_o) = ikE_A(v_o) - \frac{u'}{f(u'^2-1)^{1/2}(u'^2-v_o^2)^{1/2}}\;H_\phi(u',v_o)$$

Substitution of these expressions into (1-22) yields:

$$(1\text{-}26)\quad H_\phi(u,v,u') = 2\pi i k f^2 \int_{-1}^{1} (u'^2 - 1)^{1/2}(u'^2 - v_o^2)^{1/2} G(u,v,u',v_o) E_A(v_o)\,dv_o$$

The substitution of (1-20) and (1-23) into (1-26), followed by integration making use of the orthogonality of the $S_{1n}(v_o)$, gives

$$(1\text{-}27)\quad H_\phi(u,v,u') = ikV \sum_{n=0}^{\infty} B_n \frac{S_{1n}(v)\, R_{1n}^{(3)}(u)}{N_{1n} \frac{d}{du'}[(u'^2-1)^{1/2} R_{1n}^{(3)}(u')]}$$

Infeld shows that the convergence of this series is assured if v_1 is chosen so that $f(u'^2-1)^{1/2}\log v_1$ is finite.

Equation (1-27) becomes identical to the Stratton-Chu solution (21) if the coefficients B_n are taken equal to $S_{1n}(0)$. This summary of the work which has been done pertaining to the exact solution of the spheroidal antenna problem (with the modifications in method and gap assumptions mentioned above) is included as general background, in order that it may be compared later with the solution to the Albert-Synge integral equation

II

THE INTEGRAL EQUATION OF ALBERT AND SYNGE

The work of Albert and Synge (23) leading to an integral equation which exactly expresses the antenna problem for a surface of revolution is summarized below in section 1. Also in section 1, the integral equation is specialized to the case where the surface is a prolate spheroid. This equation is then solved and the solution so found is identical to (1-27). In section 2 the main results of this thesis are found. The Albert-Synge integral equation is used to find the expansions for the components of the field of an electric dipole. By means of these expansions recursion relations are found between the spheroidal functions of orders one and zero.

1. The Albert-Synge Integral Equation and its Solution for the Prolate Spheroid

The integral equation of Albert and Synge (23) is a statement of the boundary value problem for the perfectly conduction, radiating antenna, described in the introduction. Its derivation does not make use of the scalar or vector potentials, which are used in the derivation of the Hallén equation (7) for the cylinder. Furthermore, the Albert-Synge equation is a statement of the complete vector problem.

Basic to the work of Albert and Synge is the following theorem: (23)

> Let (E,H) and (E', H') be two electromagnetic fields, each satisfying Maxwell's equations in an infinite region R, bounded internally by one or more surfaces S; let neither field have any singularities in R or on S, and let each field satisfy certain conditions for outward radiation at infinity; then

$$\int_S \underline{n}\cdot(\underline{E} \times \underline{H}' - \underline{E}' \times \underline{H})\, dS = 0 \tag{2-1}$$

> where $\underline{n}$ is the exterior unit vector normal to S.

This theorem may be expected to hold in any region R for which Green's theorem is true, (Kellogg-10). Albert and Synge impose a more general condition on the field at infinity than (1-18); this condition, referred to in the statement of the theorem, is:

> A constant B exists such that at all sufficiently great distances ρ from some fixed point 0 we have
>
> $|\underline{E}| < B/\rho \qquad |\underline{H}| < B/\rho$
>
> $|\underline{E} + \underline{n} \times \underline{H}| < B/\rho^2 \qquad |\underline{H} - \underline{n} \times \underline{E}| < B/\rho^2$
>
> where $\underline{n}$ is the unit vector drawn from 0 towards the point at which the field is considered.

Synge and Albert now let the field ($\underline{E}'$, $\underline{H}'$) be the field of a dipole of vector strength $\underline{A}$ at the point P_o .

$$\underline{E}' = \nabla(\underline{A} \cdot \nabla \Psi) + k^2 \Psi \underline{A}; \quad \underline{H}' = ik\underline{A} \times \Psi$$

(2-2) $$\text{where } \Psi = \frac{e^{ik\overline{PP}_o}}{PP_o}$$

This field satisfied both the above condition at infinity and Maxwell's equations everywhere except at the point P_o where it has a pole (of third order for the electric vector and of second order for the magnetic vector). If P_o lies inside S, then the dipole field is regular in R, and (2-1) holds; if, on the other hand, P_o lies in R, then

(2-3) $$\int_S \underline{n} \cdot (\underline{E} \times \underline{H}' - \underline{E}' \times \underline{H}) dS = 4\pi ik\underline{E} \cdot \underline{A}$$

where $\underline{E}$ on the right hand side is evaluated at P_o. The proofs of (2-1) and (2-3) together may be viewed mathematically as a proof that Green's formula for the vector wave equation, $\nabla \times \nabla \times \underline{E} - k^2 \underline{E} = 0$, is valid in the infinite region R, when the fields in question satisfy the given condition at infinity. The field of the dipole plays the role of the Green's function for the vector wave equation. (Morse and Feshbach, 16) Equation (2-3) gives the field ($\underline{E}$,$\underline{H}$) (for appropriate choices of $\underline{A}$) in terms of its tangential components on the boundary S of the region R.

Albert and Synge apply equation (2-1) to the special case of an antenna, whose surface, including again the ficticious surface over the gap, has rotational symmetry about the z-axis. Symmetric excitation of the antenna is assumed, as it will be throughout the rest of this thesis, so that $\underline{E}$ and $\underline{H}$ are independent of the azimuthal angle. Cylindrical coordinates (z,R,ϕ) are introduced, and the dipole A is taken to be a unit vector in the positive z direction. It is located on the z-axis inside the surface S of the antenna.

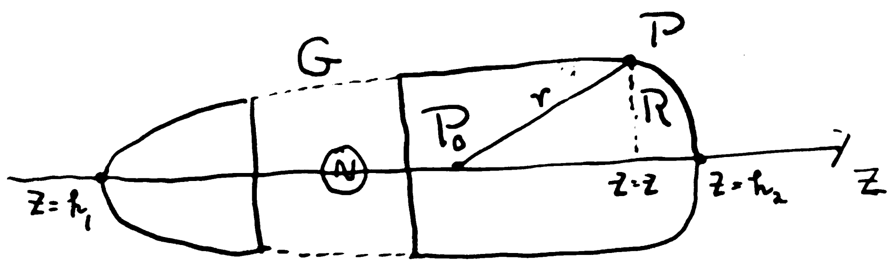

Fig. 3

For the sake of completeness, the final integral equation of Albert and Synge, satisfied by the current on the surface of an antenna of revolution is included:

$$(2\text{-}4)\quad \int_{h_1}^{h_2} \left(k^2\psi - \frac{\partial^2\psi}{\partial z\,\partial z_0}\right) I(z)\,dz = 2\pi ikc \int_{gap} E(z)R^2[1+R'^2]^{\frac{1}{2}}\frac{1}{r}\frac{d\psi}{dr}dz$$

where the antenna extends from $z = \ell_1$ to $z = \ell_2$; $R = R(z)$ is the equation in cylindrical coordinates of S; E(z) is the tangential component of the electric vector at the gap, and E(z) = 0 outside the gap; I(z) is the current across a section z = constant. $\Psi = e^{ikr}/r$, where $r^2 = [R(z)]^2 + (z - z_0)^2$.

Having now summarized the theory of Albert and Synge, we apply it to the case of the prolate spheroid. Although equation (2-4) may be transformed directly to spheroidal coordinates, it is easier as well as enlightening to consider equation (2-1). $\underline{A}$ remains a unit vector in the z direction and may be chosen to lie on the degenerate surface u = 1.

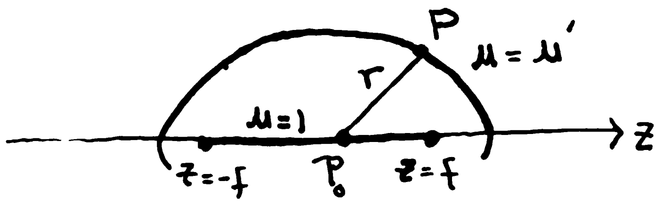

Fig. 4

The components of $\underline{A}$ are

$$(2\text{-}5) \quad A_v = \frac{u(1 - v^2)^{\frac{1}{2}}}{(u^2 - v^2)^{\frac{1}{2}}} \qquad A_u = \frac{v(u^2 - 1)^{\frac{1}{2}}}{(u^2 - v^2)^{\frac{1}{2}}}$$

Again $\Psi = e^{ikr}/r$, where $r^2 = f^2(u^2 + v^2 + v_0^2 - 1 - 2uvv_0)$. It is easily verified that the following identity holds.

(2-6) $$\frac{\partial\Psi}{\partial v_o} = -\frac{v(u^2 - 1)}{(u^2 - v^2)}\frac{\partial\Psi}{\partial u} - \frac{u(1 - v^2)}{(u^2 - v^2)}\frac{\partial\Psi}{\partial v}$$

Using (2-5) and (2-6) the existing components of the field ($\underline{E}'$ $\underline{H}'$) are found from (2-2).

$$H'_\phi = \frac{ik}{f}\frac{(1 - v^2)^{1/2}(u^2 - 1)^{1/2}}{(u^2 - v^2)}\left[u\frac{\partial\Psi}{\partial u} - v\frac{\partial\Psi}{\partial v}\right]$$

$$= \frac{ik}{f}(u^2 - 1)^{1/2}(1 - v^2)^{1/2}\frac{1}{r}\frac{d\Psi}{dr}$$

(2-7)

$$E'_v = \frac{1}{f^2}\frac{(1 - v^2)^{1/2}}{(u^2 - v^2)^{1/2}}\left[f^2k^2 u\Psi - \frac{\partial^2\Psi}{\partial v\partial v_o}\right]$$

$$E'_u = \frac{1}{f^2}\frac{(u^2 - 1)^{1/2}}{(u^2 - v^2)^{1/2}}\left[f^2k^2 v\Psi - \frac{\partial^2\Psi}{\partial u\partial v_o}\right]$$

Since $\underline{n} = \underline{i}_u$ and $dS = 2\pi f^2(u'^2 - 1)^{1/2}(u'^2 - v^2)dv$, with v ranging from -1 to 1, the integral equation (2-1) becomes:

(2-8) $$\int_{-1}^{1}(u'^2 - v^2)^{1/2}\,(E'_v H_\phi - E_v H'_\phi)dv = 0$$

The current I(fu'v) across any cross-section z = fu'v, may be found from H(fu'v) by

$$I(fu'v) = 2\pi f(u'^2 - 1)^{1/2}\,(1 - v^2)^{1/2}\,H(fu'v)$$

Since a symmetrically excited field is assumed H_ϕ satisfies $L[H_\phi] = 0$ defined in (1 - 16). Therefore a series expansion for H_ϕ of the following form may be formally assumed.

(2-9) $$H_\phi = \sum_{n=0}^{\infty} \alpha_n \frac{S_{1n}(v)\, R^{(3)}_{1n}(\mu)}{N_{1n}}$$

Where the α_n are unknown coefficients independent of u and v. The $S_{1n}(v)$ are chosen since the solution must be regular at ± 1, and the radial functions of the third kind are chosen to satisfy the radiation condition. For $E_v = E_A$, the expansion (1 - 20) may be used.

It remains to find expansions for H'_ϕ and E'_v . An expansion for e^{ikr}/r is known, (Meixner-15), where r is the distance from the point $P(v,u,\phi)$ to the point $P_o(v_o, u_o, \phi_o)$

(2-10) $$\frac{e^{ikr}}{r} = ik \sum_{m,n=0}^{\infty} e_m \cos m(\phi - \phi_o) \frac{S_{mn}(v) S_{mn}(v_o)}{N_{mn}}$$

$$\times \begin{cases} R^{(1)}_{mn}(u)\; R^{(3)}_{mn}(u_o) & u \le u_o \\ R^{(1)}_{mn}(u_o)\; R^{(3)}_{mn}(u) & \mu_o \le u \end{cases}$$

$$e_m = 2 \quad \text{for } m = 0$$
$$ = 1 \quad \phantom{\text{for }} m \neq 0$$

Because $R^{(1)}_{mn}(1) = 0$, $m \neq 0$, (2 -10) becomes for $u_o = 1$

(2-11) $$\frac{e^{ikr}}{R} = 2ik \sum_{n=0}^{\infty} \frac{S_{on}(v)\; S_{on}(v_o)}{N_{on}} R^{(1)}_{on}(1) R^{(3)}_{on}(u);\; u > 1$$

Using (2-11) and the expressions (2-7) for the components of the dipole field, series expansions for H'_ϕ and E'_v may be found in terms of the spheroidal functions of order zero and their derivatives. Substitution of these series, as well as the series (2-9) and (1 - 20) into the integral equation (2-8) leads to an infinite system of equations in infinitely many unknowns, because only a partial orthogonality relation exists between the spheroidal angular functions of orders one and zero. Since no recursion relations were known which would transform the functions of order zero into those of order one, or conversely, another approach was taken. The question of recursion relations for the spheroidal functions will be taken up later.

The field ($\underline{E}'$, $\underline{H}'$) of the dipole satisfies Maxwell's equations, and is independent of ϕ. Consequently H'_ϕ will satisfy $L[H'_\phi] = 0$. By the same reasoning that was used to form the series (2-9), we have:

$$\text{(2-12)} \qquad H'_\phi = \sum_{n=0}^{\infty} C_n \frac{S_{1n}(v)}{N_{1n}} R_{1n}^{(3)}(u)$$

where the coefficients C_n are independent of u and v. The distance r is symmetric in v and v_0, so that $\frac{1}{r}\frac{d\Psi}{dr}$ must also possess this symmetry. Similarly, $(1 - v_0^2)^{1/2} H'_\phi$ is symmetric in v and v_0, so that

$$\text{(2 - 13)} \qquad H'_\phi = \frac{1}{(1 - v_0^2)^{1/2}} \sum_{n=0}^{\infty} D_n \frac{S_{1n}(v)S_{1n}(v_0)}{N_{1n}} R_{1n}^{(3)}(u)$$

where the coefficients D_n are now independent of u, v, and v_0.

Proceeding formally, we have from Maxwell's equations:

$$(2\text{-}14) \qquad -ikf\,(u^2 - v^2)^{\frac{1}{2}}(1 - v_o^2)^{\frac{1}{2}} E_v' = \sum_{n=o}^{\infty} D_n \frac{S_{1n}(v)S_{1n}(v_o)}{N_{1n}} \frac{d}{du}[(u^2 - 1)^{1/2} R_{1n}^{(3)}(u)]$$

Substitution of (2-13), (2-14), (2-9), and (1-20) into the integral equation (2-8), followed by integration yields:

$$(2\text{-}15) \qquad \sum_{n=o}^{\infty} D_n \alpha_n \frac{S_{1n}(v_o)}{N_{1n}} R_{1n}^{(3)}(u') \frac{d}{du'}[(u'^2 - 1)^{1/2} R_{1n}^{(3)}(u')] = ikV \sum_{n=o}^{\infty} B_n D_n \frac{S_{1n}(v_o)}{N_{1n}} R_{1n}^{(3)}(u')$$

Multiplication of equation (2-15) by $S_{1r}(v_o)$, followed by integration from -1 to 1 with respect to v_o, gives us the unknown coefficients:

$$(2\text{-}16) \qquad \alpha_r = ikV \frac{B_r}{\frac{d}{du'}[u'^2 - 1)^{1/2} R_{1r}^{(3)}(u')]}$$

These coefficients are precisely those in the Stratton-Chu expansion. When they are substituted into equation (2-9) we have

$$H_\phi = ikV \sum_{n=o}^{\infty} B_n \frac{S_{1n}(v)}{N_{1n}} \frac{R_{1n}^{(3)}(u)}{\frac{d}{du'}[(u'^2 - 1)^{1/2} R_{1n}^{(3)}(u')]}$$

The fact that this solution is identical to (1-27) may be regarded as a verification of the exactness of the Albert-Synge integral equation.

2. Recursion Relations

If the coefficients D_n in the expansion (2-13) for H'_ϕ could be found, then two series representations for the function H'_ϕ would be known. The first is the expansion (2-13) in terms of the spheroidal functions of order one, while the other is obtained from (2-11) and is in terms of the spheroidal functions of order zero. Equating these series would yield a relationship between these functions.

Consider equation (2-3) where the dipole is taken outside the surface in question. For the case where S is a prolate spheroid, $\underline{A} = \underline{i}_z'$, and the field $(\underline{E},\underline{H})$ is independent of the azimuthal variable, the only 'unknowns' appearing in the integral equation are the coefficients in the expansions for E'_v and H'_ϕ. By use of the orthogonality of the angular functions equation (2-3) may be solved for these coefficients.

It will be noticed, however, that the expansion for H'_ϕ will be somewhat different when P_o is taken outside the surface u = u'.

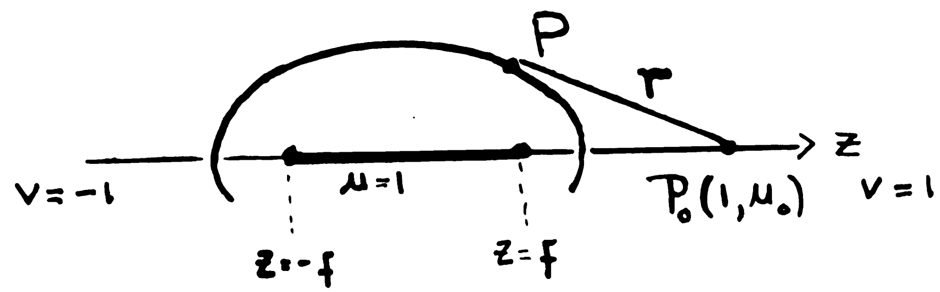

Fig. 5

In spheroidal coordinates, the z-axis is divided into three parts: $z \leq f$, $-f \leq z \leq f$, and $z \geq f$, which are respectively the surfaces v = -1, u = 1, and v = 1. In the expansion (2-13) only the case where P_o lay on the surface u = 1 was considered. Now let $P_o = P_o(\pm 1, u_o)$. The expression for r becomes $r^2 = f^2(u^2 + u_o^2 + v^2 - 1 - 2uu_ov)$ which is symmetric is u and u_o. It follows that $(u_o^2 - 1)^{1/2}H''_\phi$ is also symmetric in these variables. (The double prime notation is used to indicate that the point of singularity is $P_o(\pm 1, u_o)$ instead of $P_o(v_o', 1)$.) An expansion for H''_ϕ is therefore

$$H''_\phi = \frac{1}{(u_o^2 - 1)^{1/2}} \sum_{n=o}^{\infty} F_n \frac{S_{1n}(v)}{N_{1n}} \begin{cases} R_{1n}^{(1)}(u) R_{1n}^{(3)}(u_o) & u \leq u_o \\ R_{1n}^{(1)}(u_o) R_{1n}^{(3)}(u) & u_o \leq u \end{cases}$$

(2-17)

where the coefficients F_n are independent of u, u_o, and v. For P held fixed away from the z-axis, the field of the dipole is a regular function of P_o. As a consequence, the expansion (2-13) for H'_ϕ eveluated at $v_o = \pm 1$ must be identical to the expansion (2-17) for H''_ϕ evaluated at $u_o = 1$. Thus we have

$$\sum_{n=o}^{\infty} D_n \frac{S_{1n}(v)}{N_{1n}} \left[\frac{S_{1n}(v)}{(1 - v^2)^{1/2}}\right]_{v=\pm 1} R^{(3)}_{1n}(u) = \sum_{n=o}^{\infty} F_n \frac{S_{1n}(v)}{N_{1n}} \left[\frac{R^{(3)}_{1n}(u)}{(u^2 - 1)^{1/2}}\right]_{u=1} R^{(3)}_{1n}(u)$$

Again multiplying by $S_{1r}(v)$ and integrating over the interval of orthogonality, yields

$$(2\text{-}18) \qquad F_n \left[\frac{R^{(1)}_{1n}(u)}{(u^2 - 1)^{1/2}}\right]_{u=1} = D_n \left[\frac{S_{1n}(v)}{(1 - v^2)^{1/2}}\right]_{v=1}$$

That these expressions are well-defined follows from (1-5) and (1-13) which define the angular and radial functions.

The integral equation (2-3) may now be used to find the coefficients F_n, and, by (2-18), the coefficients D_n. For $P_o = P_o(1, u_o)$, we write (2-3) in spheroidal coordinates.

(2-19)

$$4\pi ik E_v(1,u_o) = 2\pi f^2 \int_{-1}^{1} (u'^2 - v^2)^{1/2} (u'^2 - 1)^{1/2} [E_v H''_\phi - E''_v H_\phi]\, dv$$

Using Maxwell's equations and (1-27), the left hand side of (2-19) becomes:

(2-20)
$$4\pi ikE_v(1,u_o) = -\frac{4\pi ikV}{f(u_o^2-1)^{1/2}}\sum_{n=o}^{\infty} B_n \frac{R_{1n}^{(3)}(u_o)\,\frac{d}{dv}[(1-v^2)^{1/2}S_{1n}(v)]_{v=1}}{\frac{d}{du'}[(u'^2-1)^{1/2}R_{1n}^{(3)}(u')]\,N_{1n}}$$

If the appropriate series are substituted into the right hand side of (2-19), we have, after integration with respect to v.

(2-21)
$$2\pi Vf\frac{(u'^2-1)}{(u_o^2-1)^{1/2}}\sum_{n=o}^{\infty} B_n\,F_n\,\frac{R_{1n}^{(3)}(u_o)\,W[R_{1n}^{(1)}(u')R_{1n}^{(3)}(u')]}{N_{1n}\,\frac{d}{du'}[(u'^2-1)^{1/2}R_{1n}^{(3)}(u')]}$$

where $W[R_{1n}^{(1)}\,R_{1n}^{(3)}]$ is the Wronskian of the radial functions of the first and third kinds and may be found from (1-12). Setting (2-20) equal to (2-21), and equating coefficients of the $R_{1n}^{(3)}(u_o)$, yields:

$$-\frac{4\pi ikV}{f}\,\frac{d}{dv}[(1-v^2)^{1/2}S_{1n}(v)]\Big|_{v=1} = 2\pi VfF_n\,\frac{i\,(u'^2-1)}{kf(u'^2-1)}$$

The unknown coefficients F_n and D_n are therefore found to be

(2-22)
$$F_n = 2k^2/f\left[\frac{S_{1n}(v)}{(1-v^2)^{1/2}}\right]_{v=1} \qquad D_n = 2k^2/f\left[\frac{R_{1n}^{(1)}(u)}{(u^2-1)^{1/2}}\right]_{u=1}$$

Dropping now the double prime notation, we have, finally, for the expansion for the ϕ component of the magnetic vector of the dipole field:

(2-23)

$$H'_{\phi} = \frac{ik}{f} \frac{(u^2-1)^{1/2}(1-v^2)^{1/2}}{(u^2-v^2)} \left[u \frac{\partial \psi}{\partial u} - v \frac{\partial \psi}{\partial v} \right]$$

$$= 2k^2/f \sum_{n=o}^{\infty} \frac{S_{1n}(v)\, S_{1n}(v_o)}{(1-v_o^2)^{1/2}(u_o^2-1)^{1/2}} \frac{1}{N_{1n}} \begin{cases} R_{1n}^{(1)}(u) R_{1n}^{(3)}(u_o) & u \leq u_o \\ R_{1n}^{(1)}(u_o) R_{1n}^{(3)}(u) & u_o \leq u \end{cases}$$

where P_o is understood to lie on the z-axis. That is, either $u_o = 1$, or $v_o = 1$, or both.

The function H'_{ϕ} when multiplied by either $(u_o^2 - 1)^{1/2}$ or by $(1 - v_o^2)^{1/2}$, depending upon the location of P_o, satisfies all the requirements for a Green's function for the infinite domain except that, at the point $P = P_o$ it possesses a singularity of higher order than that specified for the Green's function. The series (2-23) of eigenfunctions may nonetheless be expected to converge uniformly in any closed and bounded domain which does not contain the singular point P_o.

We now equate the series (2-23) for H'_{ϕ} with $u_o = 1$ to the series for H'_{ϕ} obtained from (2-11).

(2-24) $$2k^2/f \sum_{n=o}^{\infty} \left[\frac{R_{1n}^{(1)}(u)}{(u^2-1)^{1/2}}\right]_{u=1} \frac{S_{1n}(v) S_{1n}(v_o)}{N_{1n}\,(1-v_o^2)^{1/2}} R_{1n}^{(3)}(u)$$

$$= 2k^2/f \frac{(u^2-1)^{1/2}(1-v^2)^{1/2}}{(u^2-v^2)} \sum_{n=o}^{\infty} \frac{S_{on}(v)\, R_{on}^{(1)}(1)}{N_{on}}$$

$$\times \quad [u\, S_{on}(v) R_{on}^{(3)\prime}(u) - v\, S'_{on}(v) R_{1n}^{(3)}(u)]$$

Multiplication of (2-24) by $S_{or}(v_o)$, followed by integration with respect to v_o, removes the summation sign from the right hand side. Further multiplication by $S_{1m}(v)$ with integration with respect to v removes the summation from the left hand side and yields the following recursion relation.

$$\text{(2-25)} \qquad + R^{(3)}_{1m}(u)\left[\frac{R^{(1)}_{1m}(u)}{(u^2-1)^{1/2}}\right]_{u=1} \int_{-1}^{1}(1 - v_o^2)^{-1/2} S_{1m}(v) S_{or}(v)\,dv$$

$$= (u^2 - 1)^{1/2} R^{(1)}_{or}(1)\ [uR^{(3)\prime}_{or}(u) \int_{-1}^{1} \frac{(1 - v^2)^{1/2}}{(u^2 - v^2)} S_{or}(v) S_{1m}(v)\,dv$$

$$- R^{(3)}_{or}(u) \int_{-1}^{1} \frac{v(1 - v^2)^{1/2}}{(u^2 - v^2)} S'_{or}(v) S_{1m}(v)\,dv$$

where the primes denote differentiation and (m - r) is even. The integrals in the coefficients vanish for (m - r) odd. Equation (2-25) gives the functions $R^{(3)}_{1m}(u)$ in terms of $R^{(3)}_{or}(u)$ and its derivative. Because $R^{(3)}_{mn}(u) = R^{(1)}_{mn}(u) + iR^{(2)}_{mn}(u)$, it follows that (2-25) holds for the radial functions of all four kinds. Equation (2-25) may be somewhat modified. Substitution of the asymptotic forms (1-11) for the radial functions into the terms of (2-25) which are dominant as $u \to \infty$, yields:

(2-26)

$$\pm kf \int_{-1}^{1}(1 - v^2)^{1/2} S_{1m}(v) S_{or}(v)\,dv$$

$$= \left[\frac{R^{(1)}_{1m}(u)}{(u^2 - 1)^{1/2}}\right]_{u=1} \frac{1}{R^{(1)}_{or}(1)} \int_{-1}^{1} \frac{S_{1m}(v)\ S_{or}(v)}{(1 - v^2)^{1/2}}\,dv$$

The plus or minus sign is chosen according as $\frac{m - r}{2}$ is odd or even. Equation (2-25) may now be written:

$$(2\text{-}27) \quad \pm kfR^{(3)}_{1m}(u) \int_{-1}^{1} (1 - v^2)^{1/2} S_{1m}(v) S_{or}(v)\, dv$$
$$= (u^2 - 1)^{1/2} [uR^{(3)\prime}_{or}(u) \int_{-1}^{1} \frac{(1 - v^2)^{1/2}}{(u^2 - v^2)} S_{1m}(v) S_{or}(v)\, dv$$
$$- R^{(3)}_{or}(u) \int_{-1}^{1} \frac{v(1 - v^2)^{1/2}}{(u^2 - v^2)} S^{\prime}_{or}(v)\; S_{1m}(v)\, dv]$$

Another relationship between the spheroidal functions, although this time only involving functions of order zero, may be found by using the identity (2-6). If the expansion (2-11) is substituted into this identity we obtain, after multiplication by $S_{om}(v)\; S_{or}(v_o)$ and integration with respect to both v and v_o:

$$(2\text{-}28) \quad - R^{(1)}_{om}(1)\; R^{(3)}_{om}(u) \int_{-1}^{1} S_{or}(v) S^{\prime}_{om}(v)\, dv$$
$$= R^{(1)}_{or}(1)\; [R^{(3)\prime}_{or}(u) \int_{-1}^{1} \frac{v(u^2 - 1)}{(u^2 - v^2)} S_{or}(v)\; S_{om}(v)\, dv$$
$$+ R^{(3)}_{or}(u) \int_{-1}^{1} \frac{u(1 - v^2)}{(u^2 - v^2)} S^{\prime}_{or}(v)\; S_{om}(v)\, dv]$$

where (m - r) is odd. By use of the asymptotic forms (1-11) we obtain an identity similar to (2-26):

$$(2\text{-}29) \quad \frac{R^{(1)}_{om}(1)}{R^{(1)}_{or}(1)} \int_{-1}^{1} S_{or}(v) S^{\prime}_{om}(v)\, dv \; = \pm kf \int_{-1}^{1} v\; S_{or}(v)\; S_{om}(v)\, dv$$

In (2-29) the upper or lower sign is chosen as $\frac{m - r - 1}{2}$ is even or odd. The recursion relation (2-28) may now be rewritten:

$$(2\text{-}30)\qquad \mp kf\, R_{om}^{(3)}(u)\int_{-1}^{1} vS_{or}(v)S_{om}(v)dv = R_{or}^{(3)\prime}(u)\int_{-1}^{1}\frac{v(u^2-1)}{(u^2-v^2)}S_{or}(v)S_{om}(v)dv + R_{0r}^{(3)}(u)\int_{-1}^{1}\frac{u(1-v^2)}{(u^2-v^2)}S'_{or}(v)S_{om}(v)dv$$

Recursion relation of the same form as (2-27) and (2-30), but with certain coefficients unknown, were shown by I. Marx (14) to exist for all orders of the functions $R_{mn}^{(1)}(u)$. Later, after the above equations had been derived independently by this author, it was discovered that complete recursion relations had been published in a report by the University of Michigan Willow Run Research Center (20) for the radial functions of all kinds and orders. It is shown in this report how these relations may be modified to hold for the angular functions or both the first and second kinds. When the Willow Run recursion relations are specialized to the present case, they become identical to (2-27) for m = r, and to (2-30) for m = r+1. The derivation of the recursion relations which is employed by Marx (15) and (20) is based on a method used by E. T. Whittaker for Mathieu functions. It is altogether different from the derivation employed in this thesis.

The comment is made in the Willow Run report (20) that the coefficients in the recursion relations "are at least as difficult to compute as the functions whose computation they are supposed to simplify." So much is certainly true. On the other hand, the problem treated here, that of solving the Albert-Synge integral equation for the prolate spheroid, is witness to the fact that problems do exist to which the recursion relations may be directly applied. The application of the recursion relations to the infinite system of equations in infinitely many unknowns, mentioned on page 23, leads directly to a linear equation from which the unknown coefficients α_n, and hence the solution, may be obtained.

Other relationships may be derived in similar fashion if the method is modified by changing the position and orientation of the dipole.

III

SUCCESSIVE APPROXIMATIONS

An integral equation satisfied by the current on the surface of a finite cylindrical radiating antenna was derived by Hallén (7) in 1938, and a solution based on a method of successive approximations was advanced by him at this time. This method has been the subject of much revision and some criticism. Among the many authors on this subject other than Hallén are Bouwkamp (2), Middleton (12), King (11) and (12), Harrison (11), Gray (6), Brillouin (3), Schelkunoff (18), and Dike (5).

Hallén's integral equation is one of the first kind:

$$\int_0^{\ell} K(z,z_0)\; I(z)\; dz = f(z_0) \tag{3-1}$$

where $I(z)$ is the unknown function and the kernel and $f(z_0)$ are known. The general scheme of Hallén's method of approximations, as outlined by King and Middleton (12), depends upon two functions defined by

$$g(z,z_0) = \frac{I(z)}{I(z_0)} \tag{3-2}$$

and

$$W(z_0) = \int_0^{\ell} K(z,z_0)\; g(z,z_0)\; dz$$

Using this notation, equation (3-1) may be written:

$$(3\text{-}3)\qquad f(z_0) = I(z_0)\int_0^h K(z,z_0)\, g(z,z_0)\, dz + \int_0^h K(z,z_0)\,[I(z) - I(z_0)g(z,z_0)]\, dz$$
$$= I(z_0)W(z_0) + \int_0^h K(z,z_0)[I(z) - I(z_0)g(z,z_0)]dz.$$

This formulation of equation (3-1) is an attempt to write the equation as an integral equation of the second kind so that an iterative process of approximations may be applied. It is now argued that for a good choice of $g(z,z_0)$, the difference integral will be small, and if equation (3-3) is solved for $I(z_0)$

$$(3\text{-}4)\qquad I(z_0) = \frac{f(z_0)}{W(z_0)} - \frac{1}{W(z_0)}\int_0^h K(z,z_0)[I(z) - I(z_0)g(z,z_0)]dz$$

then the quantity $f(z_0)/W(z_0)$ may be taken as a good first approximation, $I_1(z_0)$. The second approximation, $I_2(z_0)$ is found by substitution of $I_1(z_0)$ into the difference integral, and similarly for each successive approximation. In this manner, $I(z_0)$ is 'expanded' in a series of powers of $1/W(z_0)$.

It is a matter of some controversy whether or not the choice of the expansion parameter $W(z_0)$, which of course depends upon the choice of $g(z,z_0)$, is important. However, it seems reasonable, as is contended by King and Middleton, that at least the first few approximations will be better if $g(z,z_0)$ is carefully chosen. On this basis, they use $\frac{\sin k\,(h - |z|)}{\sin k\,(h - |z_0|)}$, instead of e^{ikr} used by Hallén, because $\sin k(h - |z|)$ is known to be a close approximation to the current. Other authors have tried various other functions in arriving at an expansion parameter.

A number of difficulties with Hallén's method of solution arise. First, even the first few approximations are difficult to compute owing to the complexity of the functions involved. Furthermore, the computed solutions do not agree as closely with experimental values as might be expected. (Dike: 5) These discrepancies may be due to the fact that so few terms of the series can be practically computed, or to the unknown behavior of the series solution with respect to convergence.

In the hope that some light be shed upon the dependability of the method, a comparable series of approximations is sought for the integral equation for the prolate spheroid. In this case the exact solution is known, and the approximated series solution may be compared with it. To this end equation (2-1) is put in the form of (3-1).

(3-5) $$\int_{-1}^{1} K(v,v_o)\, H(v)\,dv = ikV\, M(v_o)$$

where $$M(v) = \sum_{n=o}^{\infty} B_n \frac{S_{1n}(v)}{N_{1n}} \left[\frac{R^{(3)}_{1n}(u)}{(u^2-1)^{1/2}}\right]_{u=1} R^{(3)}_{1n}(u)$$

$$K(v,v_o) = \sum_{n=0}^{\infty} \frac{S_{1n}(v)S_{1n}(v_o)}{N_{1n}} \left[\frac{R^{(1)}_{1n}(u)}{(u^2-1)^{1/2}}\right]_{u=1} \frac{d}{du}[(u^2-1)^{1/2} R^{(3)}_{1n}(u)]$$

as can be verified using the results of Part II. If it is considered that the dominant term of the series (1-27) is a reasonably good approximation for H, (the subscript ϕ is dropped now inasmuch as numerical subscripts will be needed) then

$$(3\text{-}6) \qquad g(v,v_o) = \frac{S_{1n}(v)}{S_{1n}(v_o)}$$

with the choice of n increasing with kf, n = 0, 2, 4, ...
For values of kf which are not too large, the function $S_{1o}(v)$ may be used. Because $S_{1o}(v)$ is an even function ranging from 0 to one as v goes from 1 to 0, it is seen to behave similarly to $\sin k(h - |z|) = \sin kf\,(1 - |v|)$. For this choice of $g(v,v_o)$ we have:

$$(3\text{-}7) \qquad W(u) = \left[\frac{R_{10}^{(1)}(u)}{(u^2-1)^{1/2}}\right]_{u=1} \frac{d}{du}\left[(u^2-1)^{1/2} R_{10}^{(3)}(u)\right]$$

which is independent of v_o and constant for any given spheroid. If equation (3-5) is written in the form of (3-4) we have

$$(3\text{-}8) \quad \frac{1}{ikV}H(v_o) = \frac{1}{W}M(v_o) - \frac{1}{W}\int_{-1}^{1} K(v,v_o)\,[H(v) - H(v_o)\,g(v,v_o)]dv$$

The first few approximations are given:

$$\frac{1}{ikV}H_1(v_o) = M(v_o)/W = \frac{1}{W}\sum_{n=o}^{\infty} B_n \frac{S_{1n}(v_o)}{N_{1n}}\left[\frac{R_{1n}^{(1)}(u)}{(u^2-1)^{1/2}}\right]_{u=1} R_{1n}^{(3)}(u)$$

$$H_2(v_o) = 2H_1(v_o) - \frac{ikV}{W}J_2(v_o)$$

$$H_3(v_o) = H_1(v_o) + H_2(v_o) - \frac{2ikV}{W^2}J_2(v_o) + \frac{1ikV}{W^3}J_3(v_o)$$

where

$$J_n(v_o) = \sum_{r=o}^{\infty} B_r \left[\frac{R_{1r}^{(1)}(u)}{(u^2-1)^{1/2}}\right]_{u=1}^{n} \frac{S_{1r}(v_o)}{N_{1r}} R_{1r}^{(3)}(u)\left[\frac{d}{du}[(u^2-1)\, R_{1r}^{(3)}(u)]\right]^{n-1}$$

In general $H_n(v_o)$ contains a term $\frac{1}{W^n} J_n(v_o)$.

The approximations H_1, H_2, and H_3 were computed for kf = 2, various values of u, and for appropriate gap widths. The value given for v in each case was chosen so that the series involved would converge faster. These results are given below. The numerical computation is not very satisfactory, because, although the series for H_1 and H converge, and converge relatively rapidly, the convergence of the series for J_1 and J_2 is open to question. The sums were taken to four terms: n = 0, 2, 4, 6. The odd terms vanish because for a symmetrically located gap the B_r for odd r are zero.

	$H = ikV(.911 - 1.682i)$
$u = 1.005$	$H_1 = ikV(1.730 - 2.470i)$
$v_1 = .25$	$H_2 = ikV(1.453 - 5.295i)$
$v = .25$	$H_3 = ikV(-10.616 - 12.769i)$

	$H = ikV(.318 - 1.105i)$
$u = 1.020$	$H_1 = ikV(.624 - 1.486i)$
$v_1 = .3$	$H_2 = ikV(.348 - 2.220i)$
$v = .25$	$H_3 = ikV(-1.272 - 2.701i)$

$u = 1.044$ | $H = ikV(.127 - .843i)$
$v_1 = .35$ | $H_1 = ikV(.287 - 1.017i)$
$v = .25$ | $H_2 = ikV(.052 - 1.288i)$
$H_3 = ikV(-.284 - 1.156i)$

$u = 1.001$ | $H = ikV(.053 + .064i)$
$v_1 = .2$ | $H_1 = ikV(.068 + .067i)$
$v = .2$ | $H_2 = ikV(.017 + .765i)$
$H_3 = ikV(2.242 + .668i)$

No positive conclusions, of course, can be drawn on the basis of these figures. They are, nonetheless, suggestive. The first approximation is in most cases also the best approximation. It is not too surprizing that the H_1 be good approximations since the first terms of the series for H and H_1 are identical. In both cases the first terms of the series is also the dominant term. This is an immediate consequence of the choice of $g(v,v_o)$. If one assumes that the series of successive approximations converges, it is to be expected that the second and third approximations be better than they are. However, the terms of the approximations should be computed for other values of kf, before more can be said about the possible divergence of the approximation series.

The basic difficulty with respect to this method seems to lie in the fact that the integral equation is one of the first kind. The initial choice of $g(v,v_o)$ appears in every term of the series, and is never affected by the iteration. Difficulties are in general to be expected with integral equations of the first kind. (Courant and Hilbert : 4) If the problem could be formulated as an integral equation of the second kind, the extensive theory dealing with such equations would be available; at least the usual methods of successive approximations could be applied.

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