This is to certify that the

dissertation entitled

COHEN-MACAULAY UNIONS OF LINES IN $P^n_k$

AND THE COHEN-MACAULAY TYPE

presented by

Frank Judson Curtis III

has been accepted towards fulfillment
of the requirements for

Ph.D. degree in Mathematics

Dr. William C. Brown
Major professor

Date February 12, 1990

# COHEN–MACAULAY UNIONS OF LINES IN $\mathbf{P}_k^n$ AND THE COHEN–MACAULAY TYPE

by

Frank Judson Curtis III

A THESIS

Submitted to
Michigan State University
in partial fulfillment of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

Department of Mathematics

1990

# ABSTRACT

## COHEN–MACAULAY UNIONS OF LINES IN $P_k^n$ AND THE COHEN–MACAULAY TYPE

by

Frank Judson Curtis III

Let V be a union of projective lines in projective n–space. We consdier the questions of when $k[V]_m$, the coordinate ring localized at the irrelevant maximal ideal, is Cohen–Macaulay, and what the Cohen–Macaulay type is when $k[V]_m$ is Cohen–Macaulay.

Chapter 1 reviews the definitions and some of the history of the problem. Chapter 2 shows how it can be interpreted as a problem involving the graded ring k[V] and modules over k[V]. This approach yields a linear algebra algorithm, similar to an algorithm by M. Baruch and W. C. Brown, which can answer both questions (Chapter 3). The method also provides a graph theoretical solution in the case where V is a graph on n+1 linearly independent points (Chapter 4).

Chapters 5 and 6 are applications of a method, developed by A. V. Geramita and C. Weibel, which uses pullback rings. In Chapter 5, the class of simply connected unions V is defined and both questions are answered for this class. A Hilbert function condition on k[V], which is necessary for Cohen–Macaulay V. (due to A. V. Geramita, P. Maroscia, and L. Roberts), is reviewed in chapter 6, and an example constructed to show the condition is not sufficient.

The last chapter expands the class of known Cohen–Macaulay unions of linear varieties by showing that any unmixed union of linear varieties W is contained in a union of linear varieties V of the same dimension which is Cohen–Macaulay. The type of V is also established.

## ACKNOWLEDGMENTS

I would like to thank my advisor, Professor William C. Grown, for his guidance and encouragement, and particularly for the careful criticism and many suggestions which he provided at every stage in the preparation of this thesis.

I would also like to thank my typist, Loretta Ferguson, for a job well done.

## TABLE OF CONTENTS

## LIST OF FIGURES

# CHAPTER 1
# INTRODUCTION

The study of commutative algebra arose historically as an adjunct to the study of algebraic geometry, but it has not been limited to this role in the development of its methods and concepts. In fact, while commutative algebra has been able to provide a more rigorous foundation for algebraic geometry, it has at the same time produced new concepts whose significance in algebraic geometry remains largely unexplored.

In this paper I will be concerned with two such properties, the Cohen–Macaulay property for rings and the Cohen–Macaulay type of a Cohen–Macaulay ring. The geometric objects I will consider are the localized coordinate rings of reducible projective varieties consisting of unions of line in $\mathbf{P}^n_k$, where k is an arbitrary algebraically closed field. The two basic questions are:

(1) When are such rings Cohen–Macaulay?

(2) What is the Cohen–Macaulay type in those cases where the coordinate ring is Cohen–Macaulay?

Given the motivation for considering these questions, it is to be understood that any answers to them will be of greater interest to the extent to which the criteria invoked are clearly geometric, and that algebraic computational solutions are not as informative.

Unfortunately, it appears that a satisfactory general solution to even the first question is very difficult to obtain. In fact, one of the basic problems in studying these questions is the scarcity of examples of unions of lines whose coordinate rings are known to be Cohen–Macaulay. For this reason, much of this paper will be devoted to the task of identifying classes of examples whose coordinate rings are Cohen–Macaulay.

Before reviewing the history of the subject, I will pause to review some definitions and introduce notation to be used throughout the paper.

Let $k$ be an algebraically closed field. Let $\mathcal{P}$ be a homogeneous prime ideal of height $n-1$ in $k[X_0,\dots,X_n]$ with $\mathcal{P}=(f_1,\dots,f_{n-1})$, where $\deg f_i=1$ for $i=1,\dots,n-1$. $\mathcal{P}$ is a prime ideal and $L=\mathcal{V}(\mathcal{P}) \subseteq \mathbf{P}^n_k$ is a line in $\mathbf{P}^n_k$. Thus $L$ is a linear variety of projective dimension 1. Linear varieties of projective dimension 0 are points in $\mathbf{P}^n_k$. $V$ will denote a union of $s$ lines in $\mathbf{P}^n_k$, $V=\bigcup_{i=1}^{s} L_i$.

Let $I=\mathcal{I}(V)=\bigcap_{i=1}^{s} \mathcal{P}_i$. $A$ will denote the coordinate ring,

$$A=k[V]=k[X_0,\dots,X_n]/I.$$

We wish to consider the local ring $A_m$, where $m=(x_0,\dots,x_n)$ is the homogeneous maximal ideal of $A$. A 2–generated $mA_m$–primary ideal $q$ is a parameter ideal, and any set of 2 generators for such an ideal is a system of parameters. The multiplicity of a parameter ideal $q$ is the leading coefficient of its Hilbert polynomial.

**Definition/Theorem 1.1.** [ZS, p. 400; HK] The following are equivalent for a Noetherian local ring $(R,m)$ with $\dim R=d$:

1) $R$ is a Cohen–Macaulay ring
2) $m$ contains a regular sequence of length $d$.
3) $e(q)=l(R/q)$ for some parameter ideal $q$, where $e(q)$ is the multiplicity and $l(R/q)$ is the length of $R/q$.
4) $e(q)=l(R/q)$ for every parameter ideal $q$.
5) One system of parameters is a regular sequence.
6) Every system of parameters is a regular sequence.

7) $\operatorname{Ext}^i_R(R/m,R)=0$ for $i=0,\ldots,d-1$.

Any reduced Noetherian local ring of dimension 1 contains regular elements, and is thus Cohen–Macaulay, by (2). In particular, the localized coordinate ring of a union of points in $P^n_k$ is always Cohen–Macaulay.

**Definition.** The Cohen–Macaulay type $r$ of a Cohen–Macaulay ring $(R,m)$ is a $r=\dim_{R/m}(\mathscr{S}(R/q))$ for any parameter ideal $q$ [K, p. 189]. $\mathscr{S}(R/q)$ is the socle, $\mathscr{S}(R/q)=\{y\in R/q \mid ym=0\} \cong [q:m]/q$. $R$ is Gorenstein if $r=1$.

**Theorem 1.2** $r$ is also [K, p.202, HK, p.4]

1) The number of irreducible components in an irredundant decomposition of $q$ into irreducible ideals.
2) $\dim_{R/m}(\operatorname{Ext}^d_R(R/m, R))$.

For the rings $A_m$ which are Cohen–Macaulay, $r$ is also the last nonzero betti number in a minimal free resolution of $A_m$ over $k[X_0,\ldots,X_n]_{(X_0,\ldots,X_n)}$.

It will sometimes be convenient to call $V$ Cohen–Macaulay whenever $A_m$ is and refer to the Cohen–Macaulay type of $A_m$ as the Cohen–Macaulay type of $V$.

The first partial answer to the first question (when is $A_m$ Cohen–Macaulay) is due to Hartshorne.

**Theorem 1.3** (by [H1, prop. 2.1]). Let $A$ be the homogeneous coordinate ring of a union of irreducible curves in $P^n_k$. If $A$ is Cohen–Macaulay, then the union of curves is connected.

This says that $V$ must be connected in order to be Cohen–Macaulay. So, for example, if $V$ consists of two skew lines in $P^3_k$, then $V$ is not Cohen–Macaulay.

Unfortunately the converse is false, as the following theorem illustrates. We recall that the nonsingular quadratic surface, $\mathcal{V}(X_0X_2-X_1X_3)$, is isomorphic to $P_k^2 \times P_k^2$; i.e., it is a ruled surface with two rulings [M, p. 27].

**Theorem 1.4**. Geramita and Weibel ([GW, 5.1]). Let V be a union of lines on a quadric surface, with m lines from one ruling and n lines from the other ruling. Then $A_m$ is Cohen–Macaulay if and only if $|m-n| \leq 1$.

In the same paper it is shown unions of lines through a common point (pencils) are always Cohen–Macaulay. Geramita and Weibel have also proven the following theorem. Here, a <u>vertex</u> of V is a point of intersection of two or more lines. For a reduced Noetherian ring R, with total quotient ring Q(R), R is said to be <u>seminormal</u> if the following property holds: whenever $a \in Q(R)$; $a^2, a^3 \in R$, then $a \in R$ [S].

**Theorem 1.5** [GW, 5.9]. If V is connected, and the lines through each vertex are linearly independent, then V is Cohen–Macaulay if and only if V is seminormal.

As a consequence, the examples of seminormal unions V given in [DR] are all Cohen–Macaulay.

The Cohen–Macaulay type of V has, so far a I know, been previously studied only in some special cases which will be discussed in Chapter 4.

The Cohen–Macaulay type of unions of points has been studied more extensively (see bibliography).

# CHAPTER 2

# GRADED RINGS

The ring $A$, as the coordinate ring of a projective variety, is naturally a graded ring. In this chapter, we will relate the local properties of $A_m$ (being Cohen–Macaulay of a certain type) to the global properties of the graded ring $A$.

Let $f_1, f_2 \in k[X_0,\dots,X_n]$, with $\deg f_1 = \deg f_2 = 1$. Denote by $\bar{f}_i$ the image of $f_i$ in $A_m$ and in $A$. As $\dim A_m = 2$,

$\bar{q} = (\bar{f}_1, \bar{f}_2) \subseteq A_m$ is a parameter ideal (i.e., $\{\bar{f}_1, \bar{f}_2\}$ is a system of parameters)

$\Leftrightarrow$ $(\bar{f}_1, \bar{f}_2)$ is $mA_m$–primary in $A_m$

$\Leftrightarrow$ $(\bar{f}_1, \bar{f}_2)$ is $m$–primary in $A$

$\Leftrightarrow$ $\sqrt{(f_1, f_2)+I} = (X_0,\dots,X_n)$

$\Leftrightarrow$ $\mathcal{V}(f_1) \cap \mathcal{V}(f_2) \cap V = \emptyset$ in $P^n_k$

Because $k$ is infinite, we can choose elements $f_1$ and $f_2$ of degree 1 so that $f_1 \notin \mathcal{P}_i$, for $i=1,\dots,s$, and $f_2 \notin (f_1, \mathcal{P}_i)$, for $i=1,\dots,s$. Then $\sqrt{(f_1, f_2)+I} = (X_0,\dots,X_n)$ and $\{\bar{f}_1, \bar{f}_2\}$ will be a system of parameters.

In the following proposition, $e(A)$ denotes the multiplicity of the graded ring $A$, defined as follows. If we let $P_A(n)=\dim_k A_n$ be the Hilbert function of $A$, and $\overline{P}_A(n)$ the corresponding Hilbert polynomial, then $e(A)$ is the leading coefficient of $\overline{P}_A(n)$. By [K, VI, prop. 2],

$$gr_{mA_m}(A_m) \cong k[X_0,\dots,X_n]/(\{L(F) \mid F \in I\})$$

$$= k[X_0,\dots,X_n]/I = A.$$

As $P_{m_m}(n) = l(m_m^n/m_m^{n+1})$, it follows that the Hilbert functions of $m_m$ and the graded ring $A$ are the same. So $e(A)=e(m_m)$.

**Proposition 2.1.** If $l(A/q)=e(A)$, then $A_m$ is Cohen–Macaulay.

**Proof.** As $q$ is m–primary, $A/q$ is local with maximal ideal $m/q$. So $A/q \cong (A/q)_{m/q} \cong A_m/q_m$, so the lengths are the same. $l(A_m/q_m) \geq e(q_m)$, as this is true for any parameter ideal, by [ZS II, p. 296].

Let $Pq_m(n)$ and $\overline{P}q_m(n)$ denote the Hilbert function and polynomial respectively for $q_m$, then $Pq_m(n) \geq Pm_m(n)$ for all n. Thus $\overline{P}q_m(n) \geq \overline{P}m_m(n)$ for all $n>>0$. As these polynomials both have degree $2=\dim A_m$, we can compare leading coefficients to obtain $e(q_m) \geq e(m_m)$. As noted in the discussion preceding the proposition, $e(m_m)=e(A)$, so $e(q_m) \geq e(A)$.

We have now shown that the following chain of inequalities holds:

$$l(A/q)=l(A_m/q_m) \geq e(q_m) \geq e(A).$$

So if $l(A/q)=e(A)$, then $l(A_m/q_m)=e(q_m)$, and $A_m$ is Cohen–Macaulay(1.1(3)). □

The converse of 2.1 is also true.

**Proposition 2.2.** If $A_m$ is Cohen–Macaulay, then $l(A/q)=e(A)$.

**Proof.** If $A_m$ is Cohen–Macaulay, then $\{\overline{f}_1, \overline{f}_2\}$ is a regular sequence in $A_m$, so $\{f_1, f_2\}$ is a regular sequence in A. Consider the Poincaré series:

$$Q_A(z) = \sum_{i=0}^{\infty} (\dim_k A_i) z^i$$

$$Q_{A/q}(z) = \sum_{i=0}^{\infty} (\dim_k (A/q)_i) z^i,$$

where $A_i$ denotes the ith graded part of A. We then have an exact sequence

$$0 \to A(-1) \xrightarrow{\mu_{f_1}} A \to A/f_1A \to 0$$

is exact. So $Q_{A/f_1A}(z)=Q_A(z)-Q_{A(-1)}(z)$

$$= Q_A(z)-zQ_A(z).$$

Repeating the argument with $f_2$ we obtain $Q_A(z)=(\sum_{i=0}^{\infty} z^i)^2 Q_{A/q}(z)$. So if $m^{j+1} \subseteq q$, $\sum_{i=0}^{\infty}(\dim_k A_i)z^i = \sum_{i=0}^{\infty}(i+1)z^i \sum_{i=0}^{j} \dim_k(A/q)_i z^i$. Equating coefficients of $z^i$ for $i \geq j$, $\dim_k A_i = \sum_{k=0}^{j}(i+1-k)\dim_k(A/q)_k = \sum_{k=0}^{j}(i+1)\dim_k(A/q)_k - \sum_{k=0}^{j} k \dim_k(A/q)_k = (i+1)l(A/q) - \sum_{k=0}^{j} k \dim_k(A/q)_k$. So $e(A)=l(A/q)$. □

$e(A)$ is the leading coefficient of the Hilbert polynomial of $A$. As projdim $(A)=1$, the degree of $V$ is $e(A)\cdot 1!=e(A)$. We now apply [H, prop.I.7.6]. Each line has degree 1; all lines have projective dimension 1; and any two intersect in a variety of projective dimension 0 or $-1$. So the degree of $V$ is the sum of the degrees of the $L_i$, i.e. deg $(V)=s$. Thus $e(A)=s$.

We have now proven:

**Proposition 2.3.** $A_m$ is Cohen–Macaulay if and only if $l(A/q)=s$.

As $l(A/q)$ can be determined by choosing as a composition series of $A/q$ a refinement of:

$$A/q \supseteq m/q \supseteq m^2q/q \supseteq \ldots \supseteq 0,$$

it follows that $A_m$ is Cohen–Macaulary precisely when $\dim_k A/q=s$.

The Cohen–Macaulay type can also be determined using $q$.

**Proposition 2.4.** $r=\dim_k(q{:}m/q)$.

**Proof.** By definition, $r=\dim_{A_m/m_m}(q_m{:}m_m/q_m)$. As $A_m/m_m \cong A/m \cong k$, it suffices to note that $\dim_k(q_m{:}m_m/q_m)=\dim_k(q{:}m/q)$. But this is clear, for if $\{x_1,\ldots,x_n\}$ is a basis for $(q{:}m/q)$, then $\{\frac{x_1}{1},\ldots,\frac{x_n}{1}\}$ is a linearly independent set of elements of $(q{:}m/q)_m=(q_m{:}m_m/q_m)$. And, conversely, given a basis for $q_m{:}m_m/q_m$, we can choose representatives of the basis elements and clear fractions to obtain a linearly independent subset of $q:m/q$. □

The significance of these propositions is that $A$, $A/q$, and $q:m/q$ are all graded, and all vector spaces over $k$. Thus $l(A/q)=\sum_{i=0}^{j} \dim_k(A/q)_i$ for some $j>>0$ (where the subscript $i$ denotes the ith graded piece of $A/q$), and $\dim_k(q:m/q)=\sum_{i=0}^{j} \dim_k(q:m/q)_i$.

In the next two chapters we will use these formulas to reduce determination of the Cohen–Macaulay property and computation of type to elementary problems in linear algebra, and to obtain a formula for the type for one class of unions $V$ which are Cohen–Macaulay.

# CHAPTER 3

## MATRIX METHODS

In this chapter, we will use the results of Chapter 2 to show that we can tell whether $A_m$ is Cohen–Macaulay, and compute the type if it is, by doing elementary matrix computations. These results are more of theoretical than practical interest, as the matrices whose ranks must be computed will be very large, given $V$ with a large number $s$ of lines.

The reader should note that the method used for computing the type is generalization to lines of Baruch and Brown's method for computing the type in the case of points [BB, B].

Each line in $V$ can be described as the linear span of two projective points, $L_i=\text{span}((b_{io},\ldots,b_{in}),(c_{io},\ldots,c_{in}))$. Let $S=k[X_o,\ldots,X_n]$. Let $\nu$ be the map defined by

$$S \xrightarrow{\nu} k[T_1^{(1)}, T_2^{(1)}] \oplus\cdots\oplus k[T_1^{(s)}, T_2^{(s)}],$$
$$\nu(X_i)=(b_{1i}T_1^{(1)} + c_{1i}T_2^{(1)},\ldots,b_{si}T_1^{(s)}+c_{si}T_2^{(s)}),$$
$$\text{for } i=0,\ldots,n.$$

Let $\Pi_j$ be projection into the jth coordinate,

$$\Pi_j{:}k[T_1^{(1)}, T_2^{(1)}] \oplus\cdots\oplus k[T_1^{(s)}, T_2^{(s)}],$$
$$\to k[T_1^{(j)}, T_2^{(j)}].$$

Then $\pi_j\circ\nu$: $S \to k[T_1^{(j)}, T_2^{(j)}]$ is the map defined by $\pi_j\circ\nu(X_i)=b_{ji}T_1^{(j)} + c_{ji}T_2^{(j)}$. So $\ker\ \nu=\bigcap_{j=1}^{s} \ker\ (\pi_j\circ\ \nu)$

$$=\bigcap_{j=1}^{s} \{f\in S\,|\,f(b_{jo}T_1^{(j)} + c_{jo}T_2^{(j)},\ldots,$$
$$b_{jn}\ T_1^{(j)} + c_{jn}T_2^{(j)}) \equiv 0\}$$
$$=\bigcap_{j=1}^{s} \{f\in S\,|\,f \in \mathcal{I}(L_j)\} = \bigcap_{j=1}^{s} \mathcal{P}_i=I$$

So $A=S/I \cong \operatorname{im} \nu$.

We now use $\nu$ to obtain a matrix representation for each graded piece $A_d$ of the graded ring $A$.

Lexicographically order the monic monomials in $S_d$ via $\prod_{i=0}^{n} X_i^{e_i} > \prod_{i=0}^{n} X_i^{f_i}$ if $e_i=f_i$ for $i=0,\dots t$, and $e_{t+1}>f_{t+1}$, as $g_1^{(d)},\dots,g_{\eta(d)}^{(d)}$, where $\eta(d)=\begin{bmatrix} n+d \\ n \end{bmatrix}$. Let $g_j^{(d)}=X_o^{\alpha_o}\cdot\dots\cdot X_n^{\alpha_n}$ $(\alpha_o+\dots+\alpha_n=d)$. Then

$$\nu(g_j^{(d)})=\nu(X_o)^{\alpha_o}\cdot\dots\cdot\nu(X_n)^{\alpha_n}$$
$$=<\{b_{1o}T_1^{(1)}+c_{1o}T_2^{(1)}\}^{\alpha_o}\cdots\{b_{1n}T_1^{(1)}+c_{1n}T_2^{(1)}\}^{\alpha_n},$$
$$\dots,\ b_{so}T_1^{(s)}+c_{so}T_2^{(s)}\}^{\alpha_o}\cdots\{b_{sn}T_1^{(s)}+c_{sn}T_2^{(s)}\}^{\alpha_n}>$$
$$<\sum_{\alpha=0}^{d} c_{1\alpha}^{(j)}\left[T_1^{(1)}\right]^{d-\alpha}\left[T_2^{(1)}\right]^{\alpha},\dots,$$
$$\sum_{\alpha=0}^{d} c_{s\alpha}^{(j)}\left[T_1^{(s)}\right]^{d-\alpha}\left[T_2^{(s)}\right]^{\alpha}>,$$

for some constants $c_{r\alpha}^{(j)}$; $j=1,\dots,\nu(d)$; $r=1,\dots,s j$ $\alpha=0,\dots,d$. Define $\left[g_j^{(d)}\right]_r=\left[c_{ro}^{(j)}, c_{r1}^{(j)},\dots,c_{rd}^{(j)}\right]$, and let $\Gamma_d(A)$ be the $\eta(d)\times s(d+1)$ matrix defined by

$$\mathrm{Row}_j\{\Gamma_d(A)\}=\left[\left[g_j^{(d)}\right]_r\middle|\cdots\middle|\left[g_j^{(d)}\right]_s\right];\ \text{i.e.,}$$

$$\begin{bmatrix} c_{1o}^{(1)},\dots\dots,c_{1d}^{(1)}, & c_{20}^{(1)},\dots\dots,c_{so}^{(1)},\dots\dots,c_{sd}^{(1)} \\ \vdots & \\ c_{10}^{(\eta(d))},\dots,c_{1d}^{(\eta(d))}, & c_{20}^{(\eta(d))},\dots,c_{so}^{(\eta(d))},\dots c_{sd}^{(\eta(d))} \end{bmatrix}$$

Thus $\Gamma_d(A)$ is a matrix representation of a spanning set for $A_d$, as embedded in $\bigoplus_{r=1}^{s} k[T_1^{(r)}, T_2^{(r)}]_d$. In particular, $\dim_k A_d=\operatorname{rk}\Gamma_d(A)$.

Next, choose an m–primary ideal $q$ generated by two linear forms as in the last chapter. We can assume coordinates on $V$ have been chosen so that $q=(x_0,x_1)$. Such a change of coordinates takes $k[V]$ to an isomorphic graded ring, so none of the numbers we are computing will change. Then $\dim_k(A/q)_d$ and $\dim_k(q{:}\ m/q)_d$ can be determined by routine matrix computations.

Because $(q)_d$ is generated by the images in $A$ of all $g_j^{(d)}=X_0^{\alpha_0}\cdots X_n^{\alpha_n}$ such that $\alpha_0\neq 0$ or $\alpha_1\neq 0$, $(q)_d$ can be represented in $\Gamma_d(A)$ as the span of the first $\begin{bmatrix} n+d \\ n \end{bmatrix} - \begin{bmatrix} n-2+d \\ d \end{bmatrix}$ rows. If we partition $\Gamma_d(A)$ as $\left[\begin{array}{c} \Gamma_d^1 \\ \hline \Gamma_d^2 \end{array}\right]$, where $\Gamma_d^1$ is $\left[\begin{bmatrix} n+d \\ n \end{bmatrix} - \begin{bmatrix} n-2+d \\ d \end{bmatrix}\right] \times s(d+1)$ and $\Gamma_d^2$ is $\begin{bmatrix} n-2+d \\ d \end{bmatrix} \times s(d+1)$, then $\dim_k(q)_d=\text{rk}\ \Gamma_d^1$ and $\dim_k(A/q)_d=\text{rk}\ \Gamma_d(A)-\text{rk}\ \Gamma_d^1$. As $l(A/q)=\sum_{d=0}^{\infty}\dim_k(A/q)_d$, it follows (by proposition 2.3) that we can determine whether $A_m$ is Cohen–Macaulay by checking the equality:

$$s = \sum_{d=0}^{\infty}(\text{rk}\ \Gamma_d(A) - \text{rk}\ \Gamma_d^1).$$

As $\dim_k(A/q)_d=0$ implies that $\dim_k(A/q)_{d+1}=0$, we only need to compute ranks for values of $d$ up to (at most) $d=s$.

We recall from proposition 2.4 that, if $A_m$ is Cohen–Macaulay, then the Cohen–Macaulay type is given by $r=\dim_k(q{:}m/q). = \sum_{d=0}^{\infty}\dim_k(q{:}\ m/q)_d$. If $A_m$ is Cohen–Macaulay, then $\dim_k(A/q)_d\neq 0$ is possible only for $d=0,1,\ldots,s-1$. Thus $(q{:}m/q)_d\neq 0$ is possible only for $d<s$. So we need only compute $\dim_k(q{:}m/q)_d$ for $d < s$.

In the case $d=0$, $\dim_k(q{:}m/q)_0=0$, unless $s=1$, in which case $q=m$ and $\dim_k(q{:}m/q)_0=1$.

To compute $\dim_k(q:m/q)_d$ for $d \geq 1$, we can use the following procedure.

We first locate a spanning set for $(q: x_i)_d$ for each $i=2,\ldots,n$.

Let $M_i$ be the matrix whose rows correspond to the $\nu\left[g_j^{(d)} X_i\right]$, where $g_j^{(d)}$ runs through the last $\begin{bmatrix} n-2+d \\ d \end{bmatrix}$ monic monomials of degree $d$. Note that the $g_j^{(d)}$ are precisely the monomials in $X_2,\ldots,X_n$. Note also that $M_i$ is a submatrix of $\Gamma_{d+1}(A)$, consisting of all rows corresponding to monomials in $X_2,\ldots,X_n$ where the exponent on $X_i$ is nonzero.

Let $M=\left[\frac{\Gamma^1_{d+1}}{M_i}\right]$, and suppose $M$ acts on the right on row vectors. Let $N=NS(M)=(N_1 | N_2)$, where $N_2$ has $\begin{bmatrix} n-2+d \\ d \end{bmatrix}$ columns. Let $Q_i=\left[\frac{\Gamma^1_d}{N_2\Gamma^2_d}\right]$.

Claim. The rows of $Q_i$ represent a spanning set for $(q: X_i)_d$. Consequently, the row space of $Q_i$ satisfies $RS(Q_i) \cong (q:X_i)_d$, and thus

$$\operatorname{rank}\left[\frac{\Gamma^1_d}{M_2\Gamma^2_d}\right] = \dim_k(q:X_i)_d.$$

Proof. To show that each row represents an element of $(q:X_i)_d$, we need only consider the rows of $N_s\Gamma^2_d$. Each row represents a linear combination $\Sigma\, a_j g_j^{(d)}$ where the coefficients $a_j$ are given by a row of $N_2$. Since $N_1\Gamma^1_{d+1} + N_2 M_i = 0$ $\nu(\Sigma a_j g_j^{(d)} X_i) = \Sigma a_j\, \nu(g_j^{(d)} X_i) \in RS(N_2 M_i) = RS(N_1 \Gamma^1_{d+1}) \subseteq RS(\Gamma^1_{d+1})$. So $\Sigma\, a_j\, g_j^{(d)}$ represents an element of $(q: X_i)_d$.

Conversely, let $g = \sum_{j=1}^{\eta(d)} a_j\, g_j^{(d)} \in (q: x_i)_d$. In order to show that $\nu(g)$ is in the span of the rows of $Q_i$, we need only consider $g' = \Sigma\, a_j\, g_j^{(d)}$ where the $g_j^{(d)}$ are the last $\begin{bmatrix} n-2+d \\ d \end{bmatrix}$ monic monomials of degree $d$. We have the relation in $A$:

$$0 = \Sigma\, \bar{a}_k\, \bar{g}_k^{(d+1)} + \Sigma\, \bar{a}_j\, \bar{g}_j^{(d)} x_i,$$

where $k=1,\ldots,\begin{bmatrix} n-d+1 \\ d+1 \end{bmatrix} - \begin{bmatrix} n-2+d+1 \\ d+1 \end{bmatrix}$, and $j=\begin{bmatrix} n+d \\ d \end{bmatrix}-\begin{bmatrix} n-2+d \\ d \end{bmatrix}+1,\ldots,\begin{bmatrix} n+d \\ d \end{bmatrix}$.

Then, by definition of $N$, $(a_1,\ldots,a_k,\ldots,a_j,\ldots)\in RS(N)$. In particular, $(a_1,\ldots,a_k,\ldots,a_j,\ldots)\in RS(N_2)$. So $\nu(g')= \Sigma\, a_j\, \nu(g_j^{(d)})$ is represented by some element in $RS(N_2\Gamma_2^d)$, i.e., some element in $RS(Q_i)$.

From $RS(Q_i)\cong(q_i:x_i)_d$, it follows that $(q:m)_d \cong \bigcap_{i=2}^{n} RS(Q_i) = \Sigma\, (RS(Q_i)^\perp)^\perp$, which can be computed using the following procedure.

Let $B_i$ be the row reduced echelon form of $Q_i$, with the zero rows deleted, so that $rk(B_i)$ is the number of nonzero rows. Let $C_i$ be a permutation matrix such that $B_iC_i=(I|D_i)$, where $I$ is an identity matrix of size $rk(B_i)$. Then $(I|D_i)\begin{bmatrix} -D_i \\ \hline I \end{bmatrix}=0$, so $(RS(I|D_i))^\perp = CS\begin{bmatrix} -D_i \\ \hline I \end{bmatrix}$, as the column space has the correct dimension. Thus $(RS(I|D_i))^\perp = RS(-D_i^T|I)$, and $(RS(Q_i))^\perp = (RS(B_i))^\perp = (RS((I|D_i)C_i^{-1}))^\perp = RS((-D_i^T|I)C_i^{-1})$. Let $E_i=(-D_i^T|I)C_i^{-1}$, for $i=2,\ldots,n$.

Let $E$ be the matrix $E=(E_2^T|E_3^T|\cdots|E_n^T)^T$. Then $RS(E)=\sum_{i=2}^{n}(RS(Q_i))^\perp$, and $(RS(E))^\perp$ can be computed by the same method used to compute $(RS(Q_i))^\perp$.

So we can compute both $\dim_k(q)_d = rk\, \Gamma_d^1$, and $\dim_k(q:m/q)_d = \dim_k(q:m)_d - \dim_k(q)_d$. As previously noted, this gives the Cohen–Macaulay type as

$$r=\sum_{d=0}^{\infty} \dim_k(q:m/q)_d = \sum_{d=0}^{s-1} \dim_k(q:m/q)_d.$$

Thus, in principle, we can determine whether $A_m$ is Cohen–Macaulay, and compute the Cohen–Macaulay type if it is, by doing matrix computations.

For large $s$, the matrices involved become very large, and accurately computing ranks would be difficult. But for small $s$, the method is practical.

# CHAPTER 4

# GRAPHS

We now consider examples of A which are also Stanley–Reisner Rings. [Ho, Re], i.e., rings which correspond to simplicial complexes. In order for a Stanley–Reisner ring to be k[V] for some union of lines V, it is necessary and sufficient for it to be the coordinate ring of a graph over a field.

Let G be a graph on $\{0,\ldots,n\}$, $n > 1$, that is, $V(G) = \{0,\ldots,n\}$ is the <u>vertex</u> <u>set</u> of G. The <u>edge</u> <u>set</u> of G, E(G) is a subset of the set of unordered pairs of distinct vertices. Let $s = |E(G)|$. We assume that G has no isolated vertices, so, in particular $s \geq n$. For each edge $e = ij \in E(G)$, Let $\mathscr{P}_e$ be the ideal in $k[X_0,\ldots,X_n]$ generated by $\{X_0,\ldots,X_n\}\backslash\{X_i, X_j\}$, and $I = \bigcap_{e \in E(G)} \mathscr{P}_e$. As each $\mathscr{P}_e = \mathscr{V}(L_e)$ for some $L_e$ in $\mathbb{P}^n_k$ (in fact, a coordinate axis), $A = k[X_0,\ldots,X_n]/I$ is k[V] for some union of lines V. A is usually denoted k[G] to indicate its construction from the graph G.

I is generated by monomials, for if $f \in \bigcap_{e \in E(G)} \mathscr{P}_e$, then each monomial term of f is in each $\mathscr{P}_e$, as these are generated by monomials. We first locate a generating set for I.

Suppose I contains an element of degree 1, say $X_i$. Then

$$X_i \in \mathscr{P}_e,\ \forall\ e \in E(G) \Rightarrow ij \notin E(G) \text{ for any } j$$
$$\Rightarrow i \text{ is an isolated vertex of } G.$$

As we assume that G has no isolated vertices, I contains no elements of degree 1.

As I is reduced, it contains no elements $X_i^2$. As for other elements of degree 2,

$$X_iX_j \in I \Leftrightarrow X_i \in \mathcal{P}_e \text{ or } X_j \in \mathcal{P}_e, \ \forall e \in E(G)$$
$$\Leftrightarrow ij \notin E(G).$$

$I$ will not contain any elements of the form: (1) $X_i^k$, (2) $X_i^k X_j^l$, where $ij \in E(G)$.

Finally, if $i$, $j$, and $k$ are distinct, each $P_e$ will contain one of $X_i$, $X_j$ or $X_k$, so all other monomials are in I. Thus, I is generated by

$$\{X_iX_j | ij \notin E(G)\} \cup \{X_i\ X_j\ X_k | \neq i, j, k\},$$

and A is a Stanley–Reisner ring [Ho, §1].

Note that for each graded part $A_d$ of A, a basis is given by the set of all monomials of degree d which do not occur in $I_d$. For example, $A_2$ has a basis $\{x_i^2\} \cup \{x_ix_j | ij \in E(G)\}$.

<u>Proposition 4.1</u> (Reisner, [Ho, p. 180]) $A_m$ is Cohen–Maculay if and only if G is connected.

<u>Proof</u>. We give a different proof than the the one in [Ho], by computing $l(A/q)$ for an m–primary ideal q. We note first that the proposition holds in one direction by Hartshorne's result (Theorem 1.3).

Assume G is connected.

let $q = (f_1, f_2)$, where

$$f_1 = \sum_{i=0}^{n} x_i,\ f_2 = \sum_{i=0}^{n} a_ix_i, \text{ where:}$$
$$0 \neq a_i \in k, \text{ and } a_i \neq a_j \text{ for } i \neq j.$$

We have immediately,

$$q_o = \langle 0 \rangle, \text{ so } \dim_k q_o = 0$$
$$q_1 = \langle f_1, f_2 \rangle, \text{ so } \dim_k q_o = 0.$$
$$q_2 = \langle \{x_jf_i | j=0,\ldots,n;\ i = 1, 2\} \rangle.$$

Suppose $\sum_{i=0}^{n} c_i x_i f_1, + \sum_{i=0}^{n} d_i x_i f_2 = 0$. Then $(c_i + d_i a_i)x_i^2 = 0$ for each i, by linear independence of basis elements of $A_2$, so $c_i = -d_i a_i$, and we have

$$(*) \qquad \sum_{i=0}^{n} -d_i a_i x_i f_1 + \sum_{i=0}^{n} d_i x_i f_2 = 0.$$

<u>Claim</u>. $d_0 = d_1 = .... = d_n$

<u>Proof of Claim</u>. If $ij \in E(G)$, then $x_i x_j \neq 0$. So the coefficient of $x_i x_j$ in (*) is zero, i.e., $-d_i a_i - d_j a_j + d_i a_j + d_j a_i = 0$,

$$(d_i - d_j)(a_j - a_i) = 0 \text{ and } a_i \neq a_j,$$

so $d_i = d_j$. The claim now follows because any two vertices of G are connected by a path in G. So $c_i = -d_0 a_i$, and

$$\sum_{i=0}^{n} c_i x_i f_1 + \sum_{i=0}^{n} d_i x_i f_2$$

$$= -d_0 \Big( \sum_{i=0}^{n} a_i x_i f_1 - \sum_{i=0}^{n} x_i f_2 \Big).$$

So every linear relation on the generators of $q_2$ is a multiple of the relation

$$\sum_{i=0}^{n} a_i x_i f_1 - \sum_{i=0}^{n} x_i f_2 = 0.$$

So $\dim_k(q)_2 = 2(n+1) - 1 = 2n + 1$.

Next, we not that for i, j, $i \neq j$,

$$x_i x_j f_1 = x_i^2 x_j + x_i x_j^2, \text{ as } x_i x_j x_k = 0,$$
$$\text{for } \neq i, j, k,$$

$$x_i x_j f_2 = a_i x_i^2 x_j + a_j x_i x_j^2, \text{ and } a_i \neq a_j,$$

so $x_i^2 x_j \in q$ for any $i \neq j$.

Then q contains

$$x_i^2 f_1 = x_i^3 + \sum_{j \neq i} x_i^2 x_j \equiv x_i^3 \pmod{q}.$$

So $q_3 = m_3 = m^3$. So $q$ is an m–primary ideal. Moreover,

$$\dim_k \ A_0/q_0 = 1 - 0 = 1.$$

$$\dim_k \ A_1/q_1 = n + 1 - 2 = n - 1.$$

$$\dim_k \ A_2 = n + 1 + s,$$

as $I_2$ contains every element of degree 2 except $x_i^2$ for each $i$, and $x_i x_j$ for the $s$ edges $ij \in E(G)$. So $\dim_k \ A_2/q_2 = n + 1 + s - (2n+1) = s - n$. Thus $l(A/q) = s$, and $A_m$ is Cohen–Macaulay (prop. 2.3). ■

In order to compute the type of $A_m$, we will need some definition from elementary graph theory. Consider a sequence of vertices $v_1,...,v_i$ such that $v_j v_{j+1} \in E(G)$ for $j=1,...,i-1$. If $v_j \neq v_k$ for $j \neq k$, the sequence is a <u>path</u> , and if $v_1 = v_i$, but $v_j \neq v_k$ for $j \neq k$ otherwise, the sequence is a <u>cycle</u>. G–v denotes the graph obtained from G by deleting the vertex v: $V(G-v) = V(G) - \{v\}$, and $E(G-v) = E(G) - \{uv \mid uv \in E(G)\}$. A vertex v is a cut vertex if G–v is disconnected (i.e., not path–connected). G is said to be a <u>block</u> if G has no cut vertices, and the <u>blocks</u> of G are the subgraphs which are maximal with respect to the property of being a block. For example, a graph consisting of two vertices and one edge is a block, and a graph consisting of a single cycle is a block. the graph depicted in Figure 4.1 is not a block, but it has as blocks the graphs in Figure 4.2.

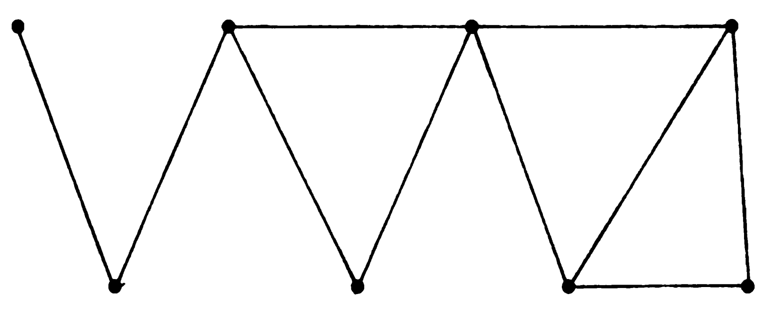

Figure 4.1

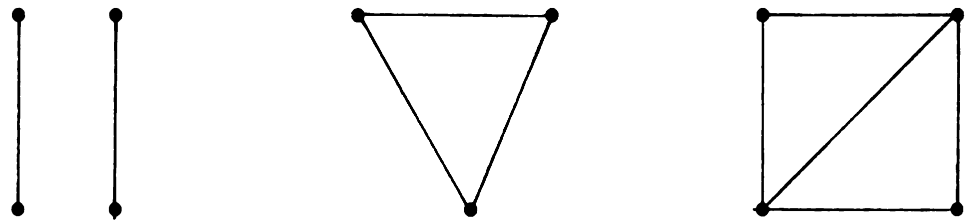

Figure 4.2

We will make use of the following fact: If $|V(G)| \geq 3$, then G is a block if and only if any two vertices lie on a common cycle (for this and other elementary results, see [CL]).

We now return to consideration of the ring $A = k[G]$.

If $g \in A_2$, we can use the basis for $A_2$ given previously to write g uniquely as

$$g = \sum_{i \in V(G)} g_{ii} x_i^2 + \sum_{ij \in E(G)} g(i, j) x_i x_j$$

where in the second sum, each edge ij is counted only once. To simplify notation, we can refer to $g(i, j)$ as $g_{ij}$ or $g_{ji}$, either one denoting the unique coefficient of $x_i x_j$ in g. So

$$g = \sum_{i \in V(G)} g_{ii} x_i^2 + \sum_{ij \in E(G)} g_{ij} x_i x_j.$$

Proposition 4.2. Suppose 0, 1, ..., k, 0 is a cycle in G. Then there is no element

$$g = \sum_{j \in V(G)} g_{jj} x_j^2 + \sum_{ij \in E(G)} g_{ij} x_i x_j \in q_2 \text{ such that.}$$

1) $g_{jj} = 0.$ for $j = 0, ...., n.$

2) $g_{01} \neq 0$, $g_{ok} = 0$, $g_{jj+1} = 0$ for $j = 1, ..., k - 1.$

Proof. k = 1. The proposition is trivial.

k > 1. Suppose there is such a g, and let $g = \sum_{j=0}^{n} c_jx_jf_1 - d_jx_jf_2$. Then $g_{jj} = 0$ implies $c_j = d_ja_j$ (as $x_j^2 \neq 0$). So $g = \sum_{j=0}^{n} d_jx_j(a_jf_1 - f_2)$, and

$$\begin{aligned} g_{ij} &= d_ia_i - d_ia_j + d_ja_j - d_ja_i \\ &= (d_i - d_j)(a_i - a_j), \quad \text{for each } i\,j \in E(G). \end{aligned}$$

Then $a_i - a_j \neq 0$ for $i \neq j$, but, for $j = 1, \ldots, k-1$ $g_{jj+1} = 0$. So $d_j = d_{j+1}$, and similarly $d_k = d_0$, so $d_0 = d_1$. But $g_{01} \neq 0$, a contradiction. ∎

Proposition 4.3. Let $h = \sum_{i=0}^{n} b_ix_i \in (q\colon m)_1$. If $b_{i_o} = b_{j_o} = 0$ for $i_oj_o \in E(G)$, then $b_l = 0$ for all vertices $l$ in the block $B$ containing the edge $i_oj_o$.

Proof. If $V(B) = \{i_o, j_o\}$, there is nothing to show. So assume $|V(B)| > 2$, and let $l \in V(B)$ be any third vertex. Suppose $b_l \neq 0$.

Claim 1. There is path $v, v_0, v_1$ in $B$ such that $b_v = 0$, $b_{v_0} = 0$, $b_{v_1} \neq 0$.

Proof. Because $B$ is a block, we can choose path $u_1, \ldots, u_k$ from $j_0 = u_1$ to $l = u_k$, which does not contain $i_o$. Let $P$ be the path $i_0, u_1, \ldots, u_k$. Let $v_1$ be the first vertex along $P$ such that $b_{v_1} \neq 0$, and let $v$ and $v_0$ be the two preceding vertices.

There is a path $P_2$ in $B$ from $v_1$ to $v$ which does not contain $v_0$, say $P_2$ is the path $v_1, v_2, \ldots, k_{k-1}, v_k = v$. Then $v_0, \ldots, v_k, v_0$ is a cycle in $B$. Relabel $G$, if necessary, so that $v_j = j$ for

$j = 0, \ldots, k$.

Claim 2. $g = x_0 h$ satisfies the conditions of proposition 4.2.

Proof. Let $g = x_0 h = \sum_{ij \in E(G)} g_{ij} x_i x_j$. Here $x_0 h = \sum_{i=0}^{n} b_i x_o x_i$. If $i \neq 0 \neq j$, then $g_{ij} = 0$, so $g_{jj+1} = 0$ for $j = 1, \ldots, k-1$. Moreover, $b_k = 0$, so $g_{0k} = 0$, and $b_1 \neq 0$ so $g_{01} \neq 0$. So condition (2) of 4.2 is satisfied. If $j \neq 0$, then $g_{jj} = 0$. As $b_o = 0$ implies $g_{oo} = 0$ also, condition (1) is also satisfied.

Then 4.2 implies $x_o h \notin q_2$, contrary to $h \in (q: m)$. therefore $b_l = 0$ for all $l \in V(B)$. ∎

Proposition 4.4. Suppose $i$ is a cut vertex of $G$, and $V_o$ the set of vertices in some connected component of $G - i$. Let $g = \sum_{j \in V_o} (a_i - a_j) x_j$. then $g \in (q: m)_1$.

Proof. We show $x_k g \in q$, for $k = 0, \ldots, n$.

Case 1. $k \notin V_o$, $k \neq i$. Here $kj \notin E(G)$ for $j \in V_o$, so $x_k x_j = 0$ and $x_k g = 0 \in q$.

Case 2. $k \in V_o$. $x_k g = x_k \sum_{j \in V_o} (a_i - a_j) x_j = \sum_{j \in V_o} (a_i - a_j) x_k x_j$. If $p \notin V_o \cup \{i\}$ then $x_k x_p = 0$, so if $p \notin V_o$, $(a_i - a_p) x_k x_p = 0$. Thus

$$x_k g = \sum_{j \in V(G)} (a_i - a_j) x_k x_j = a_i x_k f_1 - x_k f_2 \in q.$$

Case 3. $k = i$.

$$x_i g = f_1 g - \sum_{j \neq i} x_g g \in q. \qquad ∎$$

Proposition 4.5. If $G$ is a block, then $(q: m)_1 = q_1$.

Proof. Let $h = \sum_{i=0}^{n} b_i x_i \in (q: M)$. Relabel so that $01 \in G(G)$. Define

$$h' = h - b_o f_1 - \left(\frac{b_1 - b_o}{a_o - a_1}\right)(a_o f_1 - f_2) = \sum_{i=0}^{n} c_i x_i.$$

Then $c_o = c_1 = 0$, and $h' \in (q: m)$, so $h' = 0$ by prop. 4.3. So $h \in q$. ∎

Let $C = \{i \in \{0, ..., n\} \mid i$ is a cut vertex$\}$. $C$ may be empty. For each $i \in C$, let $V_{i1},...,V_{ip(i)}$ denote the connected components of $G - i$. Let $W \subseteq A_1$, be the vector space generated by

$$\{f_1, f_2\} \cup \{\sum_{j \in V_{ip}} (a_i - a_j)x_j \mid i \in C, \ p = 1, ..., p(i)\}.$$

Proposition 4.6. $(q: m)_1 = W$.

Proof. By prop. 4.4., it is enough to show $(q: m)_1 \subseteq W$.

If $G$ is a block, this inclusion follows immediately from prop. 4.5. So assume $G$ is not a block and relabel so that 0 is a cut vertex. For any $h = \sum_{i=0}^{n} b_i x_i \in A_1$, let $\mathcal{U}(h) = \{i \in V(G) \mid b_i = 0\}$. Let $\langle \mathcal{U}(h) \rangle$ be the subgraph generated by $\langle \mathcal{U}(h) \rangle$; i.e., $V(\langle \mathcal{U}(h) \rangle) = V(G) \cap \mathcal{U}(h)$, and $E(\langle \mathcal{U}(h) \rangle) = \{e = ij \in E(G) \mid i,j \in \mathcal{U}(h)\}$.

Let $\mathcal{U}_o(h)$ denote the set of all vertices in the connected component of $\langle \mathcal{U}(h) \rangle$ which contains 0 (thus $\mathcal{U}(h) = \phi$ if $b_o \neq 0$).

Suppose the proposition is false. Let $h \in (q: m)_1 \setminus W$, $h$ as above. We can replace $h$ mod $W$ by $h' = h - b_o f_1$ and so assume that

$b_0 = 0$. So the following set is nonempty

$$\mathscr{S} = \{h \in (q: m)_1 \setminus W \mid b_0 = 0\}.$$

Choose $h \in \mathscr{S}$ so that $\mathscr{U}_0(h)$ is a maximal (w.r.t. inclusion) element of $\mathscr{U}_0(l) \mid l \in \mathscr{S}\}$.

If $\mathscr{U}_0(h) = V(G)$, then $h = 0 \in W$, so we can choose $v \in V(G) \setminus \mathscr{U}_0(h)$ and let $u_0 = 0$, $u_1, \ldots, u_m = v$ be a path from 0 to v. If necessary, replace v by the first $u_i$ on the path such that $u_i \notin \mathscr{U}_0(h)$. Relabel G so that $u_i = i$, for $i = 1, \ldots, m$.

If $b_m = 0$, then $u_m = v \in \mathscr{U}(h)$. Since there is a path in $\mathscr{U}(h)$ from 0 to v, $v \in \mathscr{U}_0(h)$, a contradiction. So $b_m \neq 0$.

<u>Claim</u>. $m - 1$ is a cut vertex.

<u>Proof</u>. <u>Case 1</u>. $m = 1$.

There is nothing to show.

<u>Case 2</u>. $m > 1$.

If $m - 1$ is not a cut vertex, then $G - (m-1)$ is connected, so there is a path from $m - 2$ to $m$ which does not contain $m - 1$. But then $m - 2$, $m - 1$, and $m$ lie on a common cycle and thus in the same block. Now $b_{m-2} = b_{m-1} = 0$. So $b_m = 0$ by prop. 4.3. This is a contradiction. Therefore $m - 1$ is a cut vertex of G.

Let $V_0$ be the set of all vertices in the component of $G - (m - 1)$ which contains m.

<u>Claim</u>. $\mathscr{U}_0(h) \cap V_0 = \emptyset$

<u>Proof</u>. Suppose not. Then $\langle \mathscr{U}_0(h) \rangle \cup \langle V_0 \rangle$ is connected. So there is a path P in $\langle \mathscr{U}_0(h) \rangle \cup \langle V_0 \rangle$ from $m - 1 \in \mathscr{U}_0(h)$ to $m \in V_0$, say P is $v_0 = m - 1, v_1, \ldots, v_t = m$. Then $v_0, \ldots, v_t, v_0$ is a cycle in G, so $m - 1$, $v_1$, and m are in the same block of G. As

$m-1 \notin V_0$, $(m-1)v_1 \in E(\langle \mathcal{U}_0(h) \rangle)$ and $v_1 \in \mathcal{U}_0(h)$. So $b_{m-1} = b_{v_1} = 0$ forces $b_{m-1} = b_{v_1} = 0$ forces $b_m = 0$, by prop. 4.3, a contradiction.

Let $g = \sum_{j \in V_0} (a_{m-1} - a_j)x_j$, and $h' = h - \frac{b_m}{a_{m-1} - a_m} g$. Then $g \in W$, so $h' \equiv h \mod W$, and $h' \in \mathcal{A}$. But by the claim, $\mathcal{U}_0(h) \subseteq \mathcal{U}_0(h')$, and the coefficient of $x_m$ is zero in $h'$, so $\mathcal{U}_0(h) \subsetneq \mathcal{U}_0(h')$, contrary to choice of $h$. So 4.6 holds. ■

In the following lemma, $b(G)$ will denote the number of blocks of $G$.

Lemma 4.7. Let $G$ be a graph with cut vertices $v_1, \ldots, v_s$, $s \geq 1$. Then there exists $V_1, \ldots, V_t$ such that

1) Each $V_i$ is the vertex set of some connected component of $G - v_j$ for some $j$.

2) For each $j = 1, \ldots, s$, $G - v_j$ has all of its connected components in $\{V_1, \ldots, V_t\}$ except for one component.

3) $V_i \not\subseteq \bigcup_{j<i} V_j$ for $i = 1, \ldots, t$

4) $|V(G) - \bigcup_{i=1}^{t} V_i| > 1$.

5) $t = b(G) - 1$.

Proof. We prove 4.7 by induction on the number of blocks of $G$. As $s \geq 1$, the smallest case is when $G$ has two blocks. Letting $V_1$ be the vertex set of either component of $G - v_1$ satisfies the conditions.

Suppose $G$ has $m$ blocks and the lemma holds for all $G$ with $m-1$ blocks. Choose a block $B$ which contains only one cut vertex $v$ of $G$. Let $G'$ be the subgraph of $G$ obtained by deleting all edges and vertices of $B$, except $v$. Choose $V_1', \ldots, V_t'$ satisfying 1 – 5 for $G'$.

Let $V_1 = V(B - v)$, and for $i = 2, ..., t + 1$ let

$$V_i = \begin{cases} V'_{i-1}, & \text{if } v \in V'_{i-1} \\ V'_{i-1} \cup V_1 & \text{if } v \in V'_{i-1} \end{cases}.$$

It follows immediately that 1–4 hold for $V_1, ..., V_{t+1}$, and $t + 1 = b(G') - 1 + 1 = b(G') = b(G) - 1$, so 5 holds also. ■

For the next proposition, it should be noted for each $i = 1,..., t$, there is a unique cut vertex $v_j$ such that $V_i$ is a component of $G - v_j$. The proof is as follows.

Let $v_1$ and $v_2$ be cut vertices with $v_1 \neq v_2$. Let $C$ be a connected component of $G - v_1$, $D$ a connected component of $G - v_2$, and suppose $C = D$. Choose $w \in V(C)$ so that $wv_1 \in E(G)$. If $w \neq v_2$, then $wv_1 \in E(D)$. But $wv_1 \notin E(C)$, so $C \neq D$, a contradiction. Suppose now that $w = v_2$. then $v_2 \in V(C) - V(D)$, again contrary to $C = D$. Therefore, $C = D$ is impossible.

Proposition 4.8. $\dim_k((q:m)_1/q_1) = b(G) - 1$.

Proof. If $b(G) = 1$, this is prop. 4.5. So assume $b(G) > 1$. We use prop. 4.6 and show that $\dim_k W/q_1 = b(G) - 1$.

Let $V_1, ...., V_t$ be as in lemma 4.7. Let $v_{j(i)}$ be the cut vertex corresponding to $V_i$, and define $g_i = \sum_{k \in V_i} (a_{v_{j(i)}} - a_k)x_k$. Then $g_1,....,g_t \in W$. Fix i. The connected components of $G - v_i$ are among $V_1$, ...., $V_t$, except for one, say $V_o$. We can assume $G - v_i = V_o \vee V_1 \vee \cdots \vee V_m$ by relabeling $V_1,...,V_t$ if needed ($m \leq t$). Let $h = \sum_{j \in V_o} (a_{v_i} - a_j)x_j$.

Suppose the vertex sets of the other components are $V_1,\dots,V_m$. Then

$$a_{v_i}f_1 - f_2 - (\sum_{j=1}^{m} g_j) = a_{v_i}\sum_{k=0}^{n} x_k - \sum_{k=0}^{n} a_k x_k - \sum_{j=1}^{m}\sum_{k\in V_j}(a_{v_i} - a_k)x_k =$$

$$\sum_{k=0}^{n}(a_{v_i} - a_k)x_k - \sum_{k\in V(G)-V_0}(a_{v_i} - a_k)x_k = \sum_{k\in V_0}(a_{v_i} - a_k)x_k = h.$$ So $h$ is in the span of $\{g_1,\dots,g_t, f_1, f_2\}$. But $h$ was any generator of $W$ not in this set, so $\{g_1,\dots,g_t, f_1, f_2\}$ spans $W$. So $\dim_k W/q_1 \le t = b(G) - 1$.

By 4.7 (3), $g_1,\dots,g_t$ are linearly independent, and by 4.7 (4) (and choice of $f_1, f_2$), $\langle g_1,\dots,g_t\rangle \cap \langle f_1, f_2\rangle = 0$, so $\bar{g}_1,\dots,\bar{g}_t$ are linearly independent over $W/q_1$. So $\dim_k W/q_1 = b(G) - 1$. ■

Using prop. 4.8 it is now easy to determine the type of $A_m$.

Theorem 4.9. Let $A = k[G]$, for some connected graph $G$, and let $m$ be the homogeneous maximal ideal. Let $|V(G)| = n+1$, and assume $n \ge 2$. If $|E(G)| = s$, then the Cohen–Macaulay type of $A_m$ is $r = s - (n + 1) + b(G)$.

For example, the graph in Figure 4.1 has 10 edges, 8 vertices, and 4 blocks, so the type is 6.

Proof of 4.9. As $|E(G)| > 1$ $q \neq m$, so $\dim_k(q\colon m/q)_0 = 0$. By prop. 4.8, $\dim_k(q\colon m/q)_1 = b(G) - 1$. Finally, by the proof of 4.1, $q_i = m_i$ for $i \ge 3$, so $(q\colon m)_2 = A_2$, and thus $\dim_k(q\colon m/q)_2 = \dim_k A_2/q_2 = s - n$ (again from the proof of 4.1). Moreover, $\dim_k(q\colon m/q)_i = 0$ for $i \ge 3$. So

$$r = \sum_{i=0}^{\infty} l((q\colon m/q)_i) = \sum_{i=0}^{2} \dim_k(q\colon m/q)_i$$

$$= 0 + b(G) - 1 + s - n = s - (n + 1) + b(G). \;■$$

We conclude this chapter with some applications of Theorem 4.9.

Example 4.10. A tree is a connected graph which contains no cycles. If $G$ is a tree on $n + 1$ vertices then $G$ has $n$ edges, and every edge is a block, so $r = n - 1$. In particular, if $G$ is a path on $n + 1$ vertices $(n \geq 2)$, then $r = n - 1$. ■

Example 4.11. A graph is planar if it can be embedded in $\mathbb{R}^2$ (This does not imply that the corresponding union of lines embeds in $\mathbb{P}^2$). By Euler's formula, $|V(G)| - |E(G)| + |R(G)| = 2$, where $R(G)$ is the set of regions determined by any planar embedding of $G$. So if $G$ is a planar graph, then $r = |R(G)| + b(G) - 2$. In particular, if $G$ is a cycle on $n + 1$ points, $r = 1$. ■

Example 4.12. A complete graph on $n+1$ points is the graph which contains all possible $\begin{bmatrix} n+1 \\ 2 \end{bmatrix}$ edges. If $G$ is a complete graph, then $r = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} - (n + 1) + 1 = \begin{bmatrix} n+1 \\ 2 \end{bmatrix} - \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ 2 \end{bmatrix}$. ■

Example 4.13. Let $G$ be the graph obtained by adding $i$ edges to a cycle on $n + 1$ points, $0 \leq i \leq \begin{bmatrix} n \\ 2 \end{bmatrix} - 1$. Then $|V(G)| = n + 1$, $|E(G)| = n + 1 + i$, and $b(G) = 1$, so $r = 1 + i$. So a graph on $n + 1$ points can have any type $r$, $1 \leq r \leq \begin{bmatrix} n \\ 2 \end{bmatrix}$. ■

If $G = G_1 \cup G_2$, where $V(G_1) \cap V(G_2)$ consists of a single vertex, and $|V(G_1)| > 2$, $|V(G_2)| > 2$, then Theorem 4.9 implies $r = r_1 + r_2 + 1$, where $r_i$ is the type of $G_i$. This makes it easy to prove the following.

Proposition 4.14. If $G$ is a graph on $n + 1 > 2$ vertices, then $r \leq \begin{bmatrix} n \\ 2 \end{bmatrix}$. If $n + 1 > 3$, then equality holds only when $G$ is a complete graph on $n + 1$ vertices.

<u>Proof</u>. We induct on the number of blocks of G.

If $b(G) = 1$, then $|E(G)| \le \left[\begin{smallmatrix} n+1 \\ 2 \end{smallmatrix}\right]$ so $r \le \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]$, by 4.9.

Suppose $b(G) = m > 1$ and that the proposition is true for $b(G) < m$. Then $G = G_1 \cup G_2$, where $G_1$ is a block with $t$ vertices, $G_2$ is a graph on $n - t + 2$ vertices, $b(G_2) = m - 1$, and $V(G_1) \cap V(G_2)$ consists of a single vertex.

Suppose $t > 2$ and $n - t + 2 > 2$, i.e., $2 < t < n$. Then, by inductive assumption, $r_1 \le \left[\begin{smallmatrix} t-1 \\ 2 \end{smallmatrix}\right]$, $r_2 \le \left[\begin{smallmatrix} n-t+1 \\ 2 \end{smallmatrix}\right]$, so $r = r_1 + r_2 + 1 \le \left[\begin{smallmatrix} t-1 \\ 2 \end{smallmatrix}\right] + \left[\begin{smallmatrix} n-t+1 \\ 2 \end{smallmatrix}\right] + 1 < \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]$.

Suppose $t = 2$ (or similarly $t = n$). Then

$$|E(G)| = 1 + |E(G_2)|,$$
$$|V(G)| = 1 + |V(G_2)|,$$
$$b(G) = 1 + b(G_2)$$

If $G_2$ also consists of a single edge, then $r = 1$ and the proposition holds. Otherwise, $r = 1 + r_2 \le 1 + \left[\begin{smallmatrix} n-1 \\ 2 \end{smallmatrix}\right] < \left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]$.

Note that if G is a block, equality holds only if G contains all $\left[\begin{smallmatrix} n \\ 2 \end{smallmatrix}\right]$ possible edges. And if G is not a block, equality can hold only if $n = 3$. This is a genuine exception as a path on 3 points has type 1. ∎

The last result in this chapter was first proven by Hochster [Ho, p. 199], using different methods.

<u>Proposition 4.15 (Hochster)</u>. $r = 1$ if and only if G is a cycle or consists of one or two edges.

<u>Proof</u>. the result is trivial if $|E(G)| = 1$, so assume $|E(G)| > 1$, thus $|V(G)| > 2$. Any connected graph satisfies $|E(G)| \ge |V(G)| - 1$, with equality precisely when G is a tree. So if $r = 1$, $-1 + b(G) \le 1$,

$b(G) \leq 2$. If $b(G) = 2$, $|E(G)| = |V(G)| - 1$, and G is a tree with two blocks, i.e. G has two edges. If $b(G) = 1$, then $|E(G)| = |V(G)|$. Removing one edge yields a tree, which must be a path, as $b(G) = 1$. So G is a cycle. ∎

It should be noted that Hochster [Ho, p. 194] has given formulas for the betti numbers for all Stanley–Reisner rings. The aim of this chapter has been to show that in the case of graphs, the type (i.e., the last betti number) is given by a simple formula which can be established using more elementary methods than those used in [Ho].

# CHAPTER 5

## SIMPLY CONNECTED UNIONS OF LINES

Pencils of projective lines are the simplest non–trivial examples of Cohen–Macaulay unions of lines [GW, 4.1].

In this chapter we review these examples and apply some results of [GW], to extend them to a larger class of examples. We say that union $V$ of $s$ lines is <u>simply connected</u> if it is connected and there is no sequence of distinct lines in $V$, $L_1,\dots,L_t$, with $t \geq 3$, such that $L_i \cap L_{i+1} \neq \phi$, $L_1 \cap L_t \neq \phi$, and $L_i \cap L_j = \phi$, for $1 < |i - j| < t - 1$. That is, a simply connected union is one which contains no cycles. The goal of this chapter is to characterize those simply connected $V$ which are Cohen–Macaulay, and to obtain some results regarding the type.

If $V$ consists of lines through a single projective point (i.e., $V$ is a pencil), then $V$ is simply connected, and $V$ is always Cohen–Macaulay. A proof of the first part of the following proposition was given in [GW].

<u>Proposition 5.1</u>. ([GW, 4.1]). Let $V = \bigcup_{i=1}^{s} L_i$ be a union of lines through a point $v$ in $P^n$. Then $V$ is Cohen–Macaulay, and the type of $V$ is the same as the type of a union of points in $P^{n-1}$.

<u>Proof</u>. By a linear change of coordinates, we may assume $v = (1: 0:\dots:0)$. Then each line is determined by $v$ and one other point $v_i = (0: a_{i1}:\dots:a_{in})$. As before, let $\mathcal{P}_i = \mathcal{I}(L_i)$. Then $\mathcal{P}_i = (f_{i1},\dots,f_{in-1})$, with $\deg f_{ij} = 1$, $f_{ij} \in k[X_1,\dots,X_n]$, as no other linear polynomial vanishes at $v$.

Let $S = k[X_0,\dots,X_n]$, $T = k[X_1,\dots,X_n]$ $\bar{\mathcal{P}}_i = \mathcal{P}_i \cap T$, $I = \bigcap_{i=1}^{s} \mathcal{P}_i$ and $\bar{I} = \bigcap_{i=1}^{s} \bar{\mathcal{P}}_i$. As before, $A = k[V]$, so that $A = S/I$.

Claim. $A = T/\bar{I}[x_0]$.

Proof. Let $\nu$: $T \to S/I$ be the canonical map. Then $\ker \nu = I \cap T = (\cap \mathcal{P}_i) \cap T = \cap(\mathcal{P}_i \cap T) = \cap \bar{\mathcal{P}}_i = \bar{I}$. So $\bar{\nu}$: $T/\bar{I} \to S/I$ is a monomorphism. For each $i$, $f_{i1},\dots,f_{in-1}$ are linearly independent in $S$ over $k$, thus algebraically independent over $k$, and thus algebraically independent in $T$ over $k$. So $\operatorname{ht} \bar{\mathcal{P}}_i = n - 1$ for each $i$, and $\dim T/\bar{I} = 1$. But $\dim S/I = 2$, and the canonical map $T/\bar{I}\,[x_0] \to S/I$ is subjective, so $x_0$ is transcendental over $T/\bar{I}$, by consideration of dimensions, and so this map is also injective. This proves the claim.

Each $\bar{\mathcal{P}}_i$ has height $n - 1$ and is generated by linear forms, so it is the ideal of a point in $\mathbf{P}^{n-1}$. As $v_i' = (a_{i1}:\dots a_{in}) \in \mathcal{V}(\bar{\mathcal{P}}_i)$, this implies $T/\bar{I} = k[\cup v_i']$ is the coordinate ring of a union of points, and thus contains a regular element $g$. As $x_0$ is regular by choice of coordinates, $\{x_0, g\}$ is a regular sequence in $A$, so $A$ is Cohen–Macaulay. By [HK, Satz 1.22], the type of $A_m$ is the same as the type of $(A/(x_0))_{\bar{m}}$, where $\bar{m}$ is the image of $m$ in $A/(x_0)$. That is, the type is the same as that of $(T/\bar{I})_{\bar{m}}$. ∎

In order to study simply connected unions of lines, we use the notion of Cohen–Macaulification of the graded ring $A$.

Proposition 5.2. ([GW, 1.7]). Let $C = \cup M^{-n} \subseteq \bar{A}$, where $M$ is the homogeneous maximal ideal of $A$. Then $C$ is a Cohen–Macaulay ring, finitely generated as an A–module, and $C = M^{-n}$ for $n >> 0$. Moreover, every Cohen–Macaulay ring containing A and finite over A contains C.

Remark. C is called the Cohen–Macaulification of A, as it is the natural generalization to the graded ring A of the notion of Cohen–Macaulification of a local ring, as introduced by Grothendreck.

Clearly A is Cohen–Macaulay if and only if $A = C$. The main goal of this chapter is to determine when $A = C$ in the case of a simply connected V. We will use the following construction of C given by Geramita and Weibel.

As before, we let the vertices of V be the points of the form $L_i \cap L_j \neq \phi$, for $i \neq j$. Label these points $P_1,\ldots,P_t$, and let $\mathcal{Q}_j = \mathcal{I}(P_j)$ for $i = 1,\ldots t$. Let $J_j = \cap\{\mathcal{P}_i | \mathcal{P}_i \subset \mathcal{Q}_j\}$, so that $S/J_j$ (where $S = k[X_0,\ldots,X_n]$) is the coordinate ring of the lines of V which pass through $P_j$. The set of ideals $\{J_j\} \cap \{\mathcal{P}_i\} \cup \{\mathcal{Q}_j\}$ is partially ordered by inclusion and corresponds to a directed diagram $\Gamma$ of the rings $\{S/J_j\} \cup \{S/\mathcal{P}_i\} \cup \{S/\mathcal{Q}_j\}$ as in Figure 5.1.

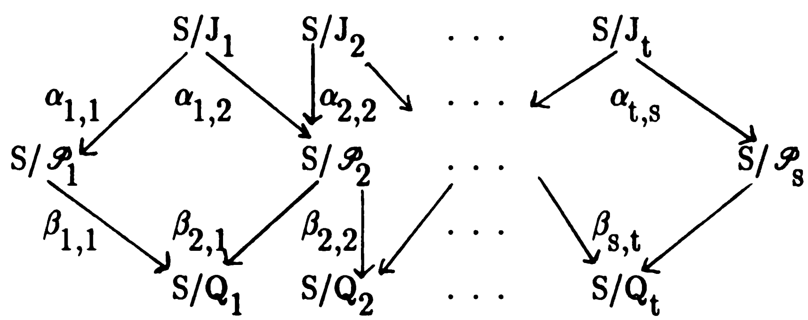


Figure 5.1. The diagram $\Gamma$

The maps of $\Gamma$ are defined as follows. For $\mathcal{P}_i \subseteq \mathcal{Q}_j$, $\alpha_{j,i}$ is the canonical map $\alpha_{j,i}$: $S/J_j \to S/\mathcal{P}_i$, and $\beta_{j,i}$ is the canonical map $\beta_{j,i}$: $S/\mathcal{P}_i \to \mathcal{Q}_j$.

We will assume throughout the rest of the chapter that V is connected, and consists of more than one line. Thus every line passes through at least one vertex, and the initial rings of the diagram $\Gamma$ will be the rings $S/J_j$.

The <u>pullback</u> (or inverse limit) of the diagram $\Gamma$ is the ring:

$$\{(f_1,\dots,f_t) \in \prod_{j=1}^{t} S/J_j \mid A(f_j) = B(f_l)$$

whenever the maps A and B are defined and have the same image}.

<u>Theorem 5.3</u>. ([GW, 8.4]) The pullback ring of the diagram $\Gamma$ is the Cohen–Macaulification of A.

We first prove as a corollary a slightly stronger version of the theorem.

<u>Proposition 5.4</u>. Let $\Gamma'$ be the directed diagram of the rings $\{S/J_j\} \cup \{S/\mathcal{P}_i\}$ (i.e., $\Gamma'$ is a subdiagram of $\Gamma$). Then the pullback ring of $\Gamma'$ is the Cohen–Macaulfication of A.

<u>Proof</u>. Let $C'$ be the pullback ring of $\Gamma'$. Clearly $C \subseteq C'$ as $\Gamma'$ is a subdiagram of $\Gamma$ with the same initial rings. So it suffices to shown $C' \subseteq C$.

Let $(\bar{f}_1,\dots,\bar{f}_t) \in C' \subseteq \prod_{1 \le j \le t} S/J_j$, and suppose $(\bar{f}_1,\dots,\bar{f}_t) \not\subseteq C$. If $j \neq k$, there is at most one i such that $\mathcal{P}_i \subseteq Q_j \cap Q_k$, so $\beta_{j,i}\ \alpha_{j,i}$: $S/J_j \to S/\mathcal{Q}_j$ is the only path in $\Gamma$ from $S/J_j$ to $S/\mathcal{Q}_k$. If $j = k$, and

$\mathcal{P}_i \subseteq \mathcal{Q}_j$, then $\beta_{i,j}\ \alpha_{j,i}$: $S/J_j \to S/\mathcal{Q}_j$ is just the canonical map, and so does not depend on i. So without ambiguity we can let $\gamma_{j,k}$: $S/J_j \to S/\mathcal{Q}_k$ be the map defined by any path in $\Gamma$ from $S/J_j$ to $S/\mathcal{Q}_k$.

Because $(\bar{f}_1,...,\bar{f}_t) \in C'$, $\alpha_{j,i}(\bar{f}_j) = \alpha_{k,i}(\bar{f}_k)$ whenever both maps are defined. If $(\bar{f}_1,...,\bar{f}_r) \notin C$, then $\gamma_{j,k}(\bar{f}_j) \neq \gamma_{l,k}(\bar{f}_l)$ for some j, k, l. Let $\gamma_{j,k} = \beta_{i(1),k}\alpha_{j,i(1)}$ and $\gamma_{l,k} = \beta_{i(2),k}\alpha_{l,i(2)}$ for some i(1), i(2). Then $P_{i(1)} \subset \mathcal{Q}_k$ and $P_{i(2)} \subset \mathcal{Q}_k$. We can write $\gamma_{k,k} = \beta_{i(1),k}\alpha_{k,i(1)} = \beta_{i(2),k}\alpha_{k,i(2)}$. Then $\gamma_{k,k}(\bar{f}_k) = \beta_{i(1),k}\alpha_{k,i(1)}(\bar{f}_k)$, and $\gamma_{j,k}(\bar{f}_j) = \beta_{i(1),k}\alpha_{j,i(1)}(\bar{f}_j)$. But $\alpha_{k,i(1)}(\bar{f}_k) = \alpha_{j,i(1)}(\bar{f}_j)$ because $(\bar{f}_1,...,\bar{f}_r) \in C'$. So $\gamma_{k,k}(\bar{f}_k) = \gamma_{j,k}(\bar{f}_j)$, and $\gamma_{k,k}(\bar{f}_k) = \gamma_{l,k}(\bar{f}_l)$, similarly. But then $\gamma_{j,k}(\bar{f}_j) = \gamma_{l,k}(\bar{f}_l)$, a contradiction. Therefore $C' \subseteq C$. ■

Proposition 5.5. Let $\Gamma^2$ be the directed diagram contained in $\Gamma'$, and consisting of $\{S/J_j\} \cup \{S/\mathcal{P}_i \mid L_i$ contains at least two vertices$\}$, where $\mathcal{P}_i = \mathcal{J}(L_i)$. Then $\Gamma'$ and $\Gamma^2$ have the same pullback.

Proof. The initial rings of the two diagrams are the same. $\Gamma^2$ is obtained from $\Gamma'$ by eliminating only those terminal rings $S/P_i$ which are the image of precisely one initial ring. So the pullback is unchanged. ■

Lemma 5.6. Let V be a simply connected union of lines. Then there is a vertex $P_j$ such that among $L_1^{(j)},...,L_r^{(j)}$, the lines through $P_j$, there is at most one $L_i^{(j)}$ which contains some other vertex of V.

Proof. Suppose V has no such vertex. Then V has more than one vertex. Choose a sequence of distinct lines in V, $L_1,...,L_m$ such that $L_i \cap L_{i+1} \neq \phi$, for $i=1,...,m-1$, and $L_i \cap L_j \neq \phi$, if $|i - j| > 1$, with m as large as possible. As $V$ is simply connected, $m \geq 3$. Let $P_i = L_i \cap L_{i+1}$ for $1 \leq i \leq m-1$. Then $L_{m-1}$ contains $P_{m-1}$ and $P_{m-2}$, and, by assumption, there is at least one other line $L_o$ which contains $P_{m-1}$ and some other vertex. $L_o = L_i$ for $i < m - 1$, otherwise $V$ would contain a loop. So wlog we may assume $L_o \neq L_m$. Thus there is a vertex $P_m \neq P_{m-1}$ on $L_m$, and $P_m = L_m \cap L_{m+1}$ for some $L_{m+1}$. This contradicts the maximality of $m$. ■

The following proposition characterizes those simply connected $V$ which are Cohen–Macaulay.

<u>Proposition 5.7</u>. Let $V$ be a simply connected union of lines. Then $A = k[V]$ is Cohen–Macaulay if and only if the degree 1 graded part of $A$ is the pullback of the degree 1 graded parts of the rings of $\Gamma^2$.

<u>Proof</u>. (G). Suppose $A$ is Cohen–Macaulay. Then $A = C$, which is the pullback of $\Gamma^2$, by propositions 5.4 and 5.5. In particular, $A$ is a pullback in degree 1.

(F). We induct on the number of vertices $t$. Suppose $V$ contains $t$ vertices and $A$ is a pullback in degree 1. If $t = 1$, then $A$ is Cohen–Macaulay, by prop. 5.1. Suppose $t > 1$ and the proposition is true when $V$ contains at most $t - 1$ vertices.

By lemma 5.6, choose $P_1$ satisfying the condition of 5.6. Let $L_1$ be the unique line containing $P_1$ as well as some other vertex $P_2$ ($L_1$ exists as $t > 1$). Let $\Delta_1$ be the directed diagram $\{S/J_j | \ j \neq 1\} \cup \{S/P_i | L_i$ contains at least two vertices$\}$, Let $\Delta_2$ be the directed diagram $\{S/J_1\} \cup \{S/P_1\}$. Thus $\Gamma^2 = \Delta_1 \cup \Delta_2$.

Let $V_1$ be the union of those lines of $V$ which contain at least one vertex not equal to $P_1$, and let $V_2$ be the union of all lines of $V$ which contain $P_1$. As $k[V_2]$ is Cohen–Macaulay, $k[V_2]$ is the pullback of $\Delta_2$. We show next that $k[V_1]$ is the pullback of $\Delta_1$.

Let $\deg f_i = 1$, $i = 2,\ldots,t$, and suppose that $(f_2,\ldots,f_t) \in \prod_{i \neq 1} S/J_i$ is in the pullback of $\Delta_1$. As $V_1 \cap V_2$ consists of a single line $(f_2, f_2 \ldots, f_t) \in \Pi\, S/J_i$ is in the pullback of $\Gamma^2$. So, by the assumption that A is a pullback in degree 1, there is some $f \in A_1$ such that $f \equiv f_2$ mod $J_1$ and $f \equiv f_i$ mod $J_i$ for $i = 2,\ldots,t$. If we replace f by the image of f in $k[V_1]$, then these congruences still hold. Thus $k[V_1]_1$ is the pullback of the degree one parts of the diagram $\Delta_1$. As $V_1$ contains $t - 1$ vertices, and is simply connected, it follows by inductive assumption that $k[V_1]$ is Cohen–Macaulay, and thus that $k[V_1]$ is the pullback of $\Delta_1$.

Let $L_1 = V_1 \cap V_2$. In order to show that A is the pullback of the diagram $\Gamma^2 = \Delta_1 \cup \Delta_2$, it now suffices to show that A is the pullback of the diagram in Figure 5.2.

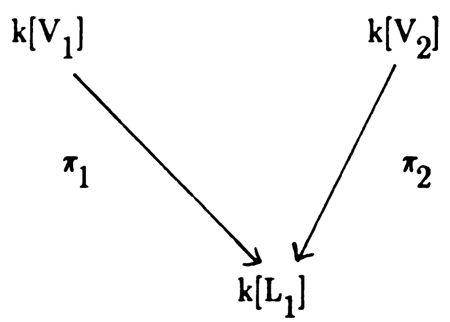


Figure 5.2

As A is the pullback in degree 1, and the maps $\pi_1$ and $\pi_2$ are subjective, we have:

(*) $$\text{edim } k[V] = \text{edim } k[V_1] + \text{edim } k[V_2] - \text{edim } k[L_1]$$

Let $m_i = \text{edim } k[V_i]$, for $i = 1, 2$. For any projective variety W, $\text{edim } k[W] = \dim_k \text{Span }(W)$, where Span (W) denotes the linear span in $A_k^{n+1}$ of the cone of W (by [K, p. 165]). So it follows from (*) that

$$\dim_k \text{Span }(V) = \dim_k \text{Span }(V_1 \cup V_2)$$
$$= \dim_k \text{Span }(V_1) + \dim_k \text{Span }(V_2) - \dim_k(\text{Span }(V_1) \cap \text{Span }(V_2))$$
$$= \dim_k \text{Span }(V_1) + \dim_k \text{Span }(V_2) - \dim_k(\text{Span }(L_1).$$

Thus, $\dim_k \text{Span }((V_1) \cap \text{Span }(V_2)) = \dim_k \text{Span }(L_1)$, and so $\text{Span }(V_1) \cap \text{Span }(V_2) = \text{Span }(L_1)$.

It follows that we can change coordinates on V as follows. First, we may assume by an initial change of coordinates that V is minimally embedded; i.e. edim (V) = n + 1. [To do this, change coordinates so that the affine subspace Span (V) consists of all points with non-zero coordinates only in the first n + 1 coordinates. Then (if $V \subseteq P_k^m$) $k[V]/(X_{n+1},\dots,X_m) \cong k[V]$.] Next, change coordinates so that

$$\mathcal{J}(L_1) = (X_2,\dots,X_n),$$
$$\mathcal{J}(\text{Span }(V_1)) = (X_{m_1},\dots,X_n\ )$$
$$\mathcal{J}(\text{Span }(V_2)) = (X_2,\dots, X_{m_1-1}).$$

Thus $\mathcal{J}(V_1) \supseteq \mathcal{J}(X_{m_1},\dots,X_n\ )$, $\mathcal{J}(V_2) \supseteq (X_2,\dots, X_{m_1-1})$, and then

$$k[L_1] = k[X_o, X_1],$$
$$k[V_1] = k[X_o, X_1,\dots,X_{m_1-1}],$$
$$k[V_2] = k[X_o, X_1, X_{m_1},\dots,X_n].$$

[Note that there are algebraic relations among the generators of the last two rings.]

As $\mathcal{I}(V_i) \subseteq \mathcal{I}(V)$, there are canonical surjective maps $p_i\colon k[V] \to k[V_i]$. Let $(f, g) \in k[V_1] \times k[V_2]$ be in the pullback of the diagram in Figure 5.2. Choose $F, G$ so that $F \in k[X_0,\dots,X_{m_1-1}] \subseteq k[V]$, $G \in k[X_0, X_1, X_{m_1},\dots,X_n] \subseteq k[V]$, and such that $p_1(F) = f$, $p_2(G) = g$. Let $F = F_1 + H_1$ and $G = G_1 + H_2$, where $H_1 = F(x_0, x_1, 0,\dots,0)$, $H_2 = G(x_0, x_1, 0,\dots,0)$. Then $\pi_1(p_1(H_1)) = \pi_1(f) = \pi_2(g) = \pi_2(p_2(H_2))$. As $p_i \circ \pi_i\colon k[V] \to k[L_1]$ is the identity when restricted to $k[x_0,x_1]$, it follows that $H_1 = H_2$. Moreover, $F_1 + H_1 + G_1 \in A = k[V]$, and $p_2(F_1) = 0$, $p_1(G_1) = 0$, so

$$(p_1(F_1 + H_1 + G_1), p_2(F_1 + H_1 + G_1)) = (f, g).$$

So $A$ is the pullback of the diagram in figure 5.2., and therefore also the pullback of $\Gamma^2$. So $A$ is Cohen–Macaulay. ∎

The preceding proof allows an alternative characterization of which simply connected unions of lines are Cohen–Macaulay. For $i = 1,\dots,t$, let $V^i = \mathcal{V}(J_i)$. Thus $V^i$ consists of all lines through the vertex $P_i$.

Theorem 5.8. Let $V$ be a simply connected union of lines. Then $\operatorname{edim} k[V] \le \Sigma \operatorname{edim} k[V^i] - 2(t-1)$, where $t$ is the number of vertices of $V$. Moreover, equality holds if and only if $A = k[V]$ is Cohen–Macaulay.

Proof. We induct on $t$. For $t = 1$, $V = V^1$, and there is nothing to show. Let $V$ have $t > 1$ vertices and assume the corollary is proven when $V$ has less than $t$ vertices. Let $P_1$ be as in the proof of prop.

5.7. Let $V_2 = V^1$, and $V_1 = \cup_{i>1} V^i$, and $L_1 = V_1 \cap V_2$. then, as in the proof of 5.7, A is Cohen–Macaulay

FG A is the pullback of the diagram in Figure 5.2

We show first that the inequality holds. We always have

$$\begin{aligned} \text{edim } k[V] &\le \text{edim } k[V_1] + \text{edim } k[V_2] - \text{edim } k[L_1] \\ &= \text{edim } k[V_1] + \text{edim } k[V_2] - 2. \end{aligned}$$

Then, by inductive assumption, there follows

$$\begin{aligned} \text{edim } k[V] \le \sum_{i>1} \text{edim } k[V^i] - 2(t - 2) \\ + \text{edim } k[V_2] - 2. \end{aligned}$$

As $V_2 = V^1$, we conclude

$$\text{edim } k[V] \le \sum_{i>1} \text{edim } k[V^i] - 2(t - 1).$$

Suppose A is Cohen–Macaulay. By prop. 5.7, $A_1$ is the pullback of the degree 1 parts of the rings of $\Gamma^2$. So by the proof of 5.7, $k[V_1]_1$ is the pullback of the degree 1 parts of the rings of $\Delta_1$, and thus is Cohen–Macaulay, by 5.7. So, by inductive hypothesis,

$$\text{edim } k[V_1] \le \sum_{i>1} \text{edim } k[V^i] - 2(t - 2)$$

But also,

$$\begin{aligned} \text{edim } k[V] &= \text{edim } k[V_1] + \text{edim } k[V_2] - \text{edim } k[L_1] \\ &= \text{edim } k[V_1] + \text{edim } k[V_2] - 2 \\ &= \sum_{i>1} \text{edim } k[V^i] + \text{edim } k[V^1] - 2(t - 2) - 2 \\ &= \sum_{i\geq 1} \text{edim } k[V^i] - 2(t - 1) \end{aligned}$$

For the converse, suppose that $\text{edim } k[V] = \sum_{i\geq 1} \text{edim } k[V^i] - 2(t - 1)$.

Then

$$\text{edim } k[V] = \sum_{i>1} \text{edim } k[V^i] - 2(t - 2) + \text{edim } k[V^1] - \text{edim } k[L_1]$$

Also

$$\text{edim } k[V] \leq \text{edim } k[V_1] + \text{edim } k[V_2] - \text{edim } k[L_1].$$

If equality does not hold, then

$$\begin{aligned} &\text{edim } k[V_1] + \text{edim } k[V_2] - \text{edim } k[L_1] \\ &\quad > \sum_{i>1} \text{edim } k[V^i] - 2(t-2) + \text{edim } k[V^1] - \text{edim } k[L_1] \end{aligned}$$

But, as $V_2 = V^1$, this implies $\text{edim } k[V_1] > \sum_{i>1} \text{edim } k[V^i] - 2(t-2)$ which is impossible. So

$$(1) \qquad \text{edim } k[V] = \text{edim } k[V_1] + \text{edim } k[V_2] - \text{edim } k[L_1],$$

and also

$$(2) \qquad \text{edim } k[V_1] = \sum_{i>1} \text{edim } k[V^i] - 2(t-2).$$

By inductive hypothesis, (2) implies that $k[V_1]$ is Cohen–Macaulay. In particular, $k[V_1]_1$ is a pullback of $\Delta_1$ in degree 1. It then follows from (1), that $k[V]_1$ is a pullback of diagram 5.1 in degree 1, and thus a pullback of $\Gamma^2$ in degree 1. So, by proposition 5.7, A is Cohen–Macaulay. ∎

<u>Remark</u>. These results are false if V is not simply connected. Let V be a 2 × 2 configuration of lines on a nonsingular quadratic surface. Then k[V] is Cohen–Macaulay (theorem 1.4). Also edim k[V] = 4. However, $\Sigma \text{ edim } k[V_i] - 2(t-1) = 4(3) - 2(3) = 6 \neq 4$.

We conclude the chapter by showing how to compute the Cohen–Macaulay type of $A_m$ in the case where V is simply connected and A is Cohen–Macaulay. As noted in Chapter 2, we can compute the type by computing the graded analogue of the socle.

For each $i = 1,\ldots,t$, let $r_i$ be the Cohen–Macaulay type of the ring $k[V^i]$, and r the Cohen–Macaulay type of k[V].

Proposition 5.9. If $V$ is simply connected and Cohen–Macaulay, then

$$r = \sum_{i=1}^{t} r_i .$$

Proof. We induct on T. For $t = 1$, there is nothing to show. As in the proof of prop. 5.7, we reduce to consideration of the diagram in figure 5.2. By inductive assumption $k[V_1]$ has type $\sum_{2 \le i \le t} r_i$, and it suffices to show that the type of $k[V]$ is the sum of the types of $k[V_i]$, $i = 1, 2$. For ease of notation, let $r_i$ be the type of $k[V_i]$.

Assume coordinates have been changed as in the proof of prop. 5.7. Choose a system of parameters $\{f_1, f_2\}$ for $A$, with deg $f_i = 1$, $i = 1, 2$. Let $F_i$ be the preimage of $f_i$ in $k[X_0,\ldots,X_n]$. Then $\sqrt{(F_1, F_2) + \cap P_i} = M$ and $\mathcal{J}(L_1) \supseteq \cap P_i$ so $\sqrt{(F_1, F_2) + \mathcal{J}(L_1)} = M$. As $(F_1, F_2)$ and $\mathcal{J}(L_1)$ are generated by degree 1 forms, $(F_1, F_2) + \mathcal{J}(L_1)$ is the ideal of a linear variety, and thus prime. So $(F_1, F_2) + \mathcal{J}(L_1) = M$; i.e., $(F_1, F_2) + (X_2,\ldots,X_n) = M$. So $X_0 = a_1F_1 + b_1F_2 - G_1$, $X_1 = a_2F_1 + b_2F_2 - G_2$, for some $a_i, b_i \in k$, $G_i \in (X_2,\ldots,X_n)$ . Replace $F_i$ by $F_i' = a_iF_1 + b_iF_2$, so as to assume wlog that

$$F_1 = X_0 + G_1, \quad F_2 = X_1 + G_2, \quad \text{with}$$

$G_i \in (X_2,\ldots,X_n)$ . [Note that $\{F_1', F_2'\}$ is still a system of parameters because

$$X_0 + G_1 \notin (X_1 + G_2)$$

$$\mathcal{G} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$$

$$\mathcal{G}\ F_1, F_2 \in (F_1', F_2')$$

$$) \sqrt{(F_1', F_2') + \cap P_i} = M.]$$

Let $g_1,\ldots,g_{r_1} \in k[V_1]$ such that $g_1 + \bar{q},\ldots,g_{r_1} + \bar{q}$ is a basis for the socle of $k[V_1]/\bar{q}$, and $h_1,\ldots,h_{r_2} \in k[V_2]$ such that $h_1 + \bar{q},\ldots,h_{r_2} + \bar{q}$ is a basis for the socle of $k[V_2]/\bar{q}$. Suppose $g_i = F_i(x_0, x_1) + G_i(x_0,\ldots,x_n)$, where each monomial term of $G_i$ contains one of $x_2,\ldots,x_n$ as factor. then replace $g_i$ by $g_i - F_i(f_1, f_2) \equiv g_i \bmod (f_1, f_2)$, so that we may assume wlog that each monomial term of $g_i$ contains one of $x_2,\ldots,x_n$ as a factor. Similarly, we may assume that each monomial term of each $h_i$ contains one of $x_2,\ldots,x_n$ as a factor. Then $\pi_1(g_i) = \pi_2(h_j) = 0 \in k\{L_1]$ for all i, j, so $(g_i, 0), (0, h_j) \in k[V]$ for all i, j.

Claim 1. $(g_i, 0)$ is in the socle of $k[V]/q$ for each i.

Proof. It suffices to show that $(x_k, x_k)(g_i, 0) = (0, 0) = 0 \in k[V]/q$, for $k = 0,\ldots,n$. Thus it suffices to show $x_k g_i = 0$ in $k[V_1]/\bar{q}$. But this true because $g_1$ is in the scole of $k[V_1]/\bar{q}$.

Claim 2. $(0, h_j)$ is in the socle of $k[V]/q$ for each j.

Proof. Similar.

Clearly the elements $\{(g_i, 0)\} \cup \{0, h_j)\}$ are linearly independent over k. In order to show $r = r_1 + r_2$, it now suffices to show that this set generates the socle of $k[V]/\bar{q}$.

Let $k_1,\ldots,k_r \in k[V]$ such that $(k_1, k_1),\ldots,(k_r, k_r)$ generate the socle of $k[V]/\bar{q}$. As in the previous argument, let $k_l = F_l(x_0, x_1) + G_l(x_0,\ldots,x_n)$, where each monomial term of $G_l$ contains one of $x_2,\ldots,x_n$ as a factor. Then replace $k_l$ by $k_l - F_l(f_1, f_2) \equiv k_l \bmod (f_1, f_2)$ so that we may assume wlog that each monomial term of $k_l$ contains on of $x_2,\ldots,x_n$ as a factor. As before, it follows that $(k_l, 0), (0, k_l) \in k[V]$ for each i.

For each $j = 1,\ldots,n$, $(k_l x_j, k_l x_j) = (0, 0) = 0 \in k[V]/q$. Consequently, $k_l x_j = 0$ in $k[V_1]/\overline{q}$ and in $k[V_2]\overline{q}$. Thus $k_l$ is in the socles of both $k[V_1]/\overline{q}$ and $k[V_2]/\overline{q}$. Let

$$k_l = \sum_{i=1}^{r_1} a_i g_i \in k[V_1]/\overline{q}$$

$$k_l = \sum_{j=1}^{r_2} b_j h_j \in k[V_2]/\overline{q}.$$

Then $(k_l, k_l) = \; = \sum_{i=1}^{r_1} a_i(g_i, 0) + \sum_{j=1}^{r_2} b_j(0, h_j)$. So $\{(g_i, 0)\} \cup \{(0, h_j)\}$ generates the socle of $k[V]/\overline{q}$ as claimed. ∎

Note that by proposition 5.1, the Cohen–Macaulay type for a union of lines through a single vertex can be reduced to the computation of the type for a union of points obtained by intersecting with a hyperplane.

Thus prop. 5.9 reduces the problem of computing the type of a simply connected union of lines to computations of the types of unions of points.

As any union of points in $P^{n-1}$ can be coned to a union of lines through a single vertex in $P^n$, prop. 5.9 is in some sense the strongest result we can expect for an arbitrary simply connected Cohen–Macaulay union of lines.

Finally, note that prop. 5.9 is false if $V$ is not simply connected.

<u>Example 5.10</u>. Let $V$ consist of 3 lines in $P^2$ with 3 vertices. Then $k[V^i]$ has type 1 for each $i = 1, 2, 3$, as $V^i$ is a hypersurface. But $V$ is also a hypersurface, so $k[V]$ has type $1 \neq 3$.

# CHAPTER 6

## HILBERT FUNCTIONS

One necessary condition for $A = k[V]$ to be Cohen–Macaulay is that its Hilbert function be twice differentiable. In this chapter, this condition will be discussed, and an example constructed which shows that the condition is not sufficient.

Let $P_A(i)$ be the Hilbert function of $A$, $P_A(i) = \begin{cases} \dim_k A_i, & i \geq 0 \\ 0, & i < 0 \end{cases}$.
Let $a_i = P_A(i)$. Then the first sequence of differences is $\{b_i\}_{i\in\mathbb{Z}}$, where $b_i = a_i - a_{i-1}$, and the second sequence of differences is $\{c_i\}_{i\in\mathbb{Z}}$, where $c_i = b_i - b_{i-1}$. If $c_i \geq 0$ for all i, we say that $P_A(i)$ is twice differentiable.

As in Chapter 2, let $Q_A(z)$ denote the Poincaré series of $A$, $Q_A(z) = \sum_{i=0}^{\infty} P_A(i)z^i$. As noted in the proof of proposition 2.2, if $A$ is Cohen–Macaulay, then we can choose $f_1, f_2 \in A_1$ such that the set of images $\{\bar{f}_1, \bar{f}_2\}$ is a system of parameters for $A_m$, in which case $Q_{A/f_1A}(z) = Q_A(z) - Q_{A(-1)}(z)$. So $\sum_{i=0}^{\infty} b_i z^i = Q_{A/f_1A}(z)$. Similarly, with $q = (f_1, f_2)$, we have $Q_{A/q}(z) = Q_{A/f_1A}(z) - Q_{(A/f_1A)(-1)}(z)$, so $\sum_{i=0}^{\infty} c_i z^i = Q_{A/q}(z) = \sum_{i=0}^{\infty} P_{A/q}(i)z^i$. Thus, for $i \geq 0$, $c_i = P_{A/q}(i) \geq 0$, as Hilbert functions are nonnegative. As $c_i = 0$ for $i < 0$, it follows that $c_i \geq 0$ for all $i$ and $P_A(i)$ is twice differentiable. We conclude the following:

Proposition 6.1. ([GMR]) Let $A = k[V]$ be the coordinate ring of a union $V$ of projective lines in $\mathbf{P}_k^n$. If $A$ is Cohen–Macaulay, then the Hilbert function $P_A(i)$ is twice differentiable.

Proposition 6.1 gives a useful criterion for showing that certain unions of lines are not Cohen–Macaulay. Here is a simple application.

Example 6.2. Let $L_1 = \mathcal{V}(X_0, X_1)$, $L_2 = \mathcal{V}(X_2, X_3)$, $V = L_1 \cup L_2 \subseteq \mathbf{P}^3_k$. then $\mathcal{I}(V) = (X_0, X_1) \cap (X_2, X_3) = (X_0X_2, X_0X_3, X_1X_2, X_1X_3)$. $P_A(i)$ is then easily computed,

$$P_A(i) = \begin{cases} 1, & i = 0 \\ 2(i+1), & i < 0 \end{cases}$$

Then $b_1 = 3$, $b_2 = 2$, and $c_2 = -1$, so $V$ is not Cohen–Macaulay.

Unfortunately, the condition of proposition 6.1 is not sufficient.

<u>Example 6.3</u>. Let $A = k[V]$, where $V = \bigcup_{i=1}^{6} L_i$ is the union of the following six lines:

$$L_1 = \mathcal{V}(X_2, X_3)$$
$$L_2 = \mathcal{V}(X_1, X_3)$$
$$L_3 = \mathcal{V}(X_3, X_1 - X_2)$$
$$L_4 = \mathcal{V}(X_3, X_1 + X_2)$$
$$L_5 = \mathcal{V}(X_0, X_2)$$
$$L_6 = \mathcal{V}(X_1, X_0 - X_3)$$

Then $A$ is not Cohen–Macaulay, but $P_A(i)$ is twice differentiable.

<u>Proof</u>. We show first that $A$ is not Cohen–Macaulay. Figure 6.1 illustrates the intersection relations among the $L_i$:

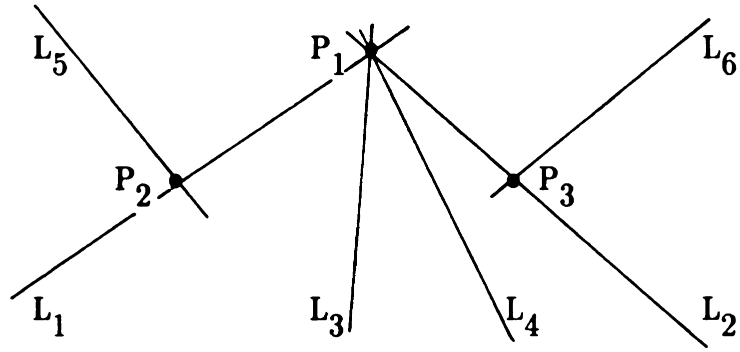


Figure 6.1

Here $P_1 = \mathcal{V}(X_1, X_2, X_3)$, $P_2 = \mathcal{V}(X_0, X_2, X_3)$, and $P_3 = \mathcal{V}(X_0, X_1, X_3)$.

Let $V^1 = \bigcup_{i=1}^{4} L_i$, $V^2 = L_1 \cup L_5$, and $V^3 = L_2 \cup L_6$. Then each $V^i$ is contained in a hyperplane $H^i$, where $H^1 = \mathcal{V}(X_3)$, $H^2 = \mathcal{V}(X_2)$ and $H^3 = \mathcal{V}(X_1)$. So edim $k[V^i] = 3$ for $i = 1, 2, 3$, and thus $\sum_{i=1}^{3}$ edim $k[V^i] - 2(3 - 1) = 5$, while edim $k[V] = 4 < 5$. As $V$ is simply connected, $V$ is not Cohen–Macaulay, by Theorem 5.8.

We now show that the Hilbert function is twice differentiable. We begin by computing the Hilbert function for $C$, the Cohen–Macaulfication of $A$, as this provides an upper bound for the Hilbert function of $A$.

By prop. 5.5., $C$ is the pullback ring of the diagram in Figure 6.2

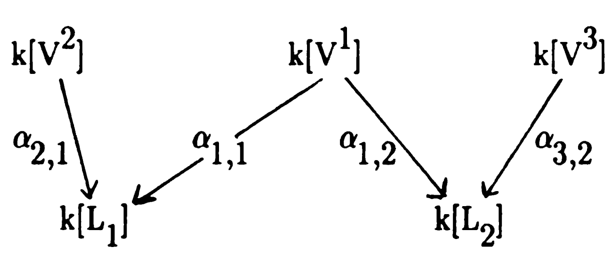


Figure 6.2

Fix the degree = i for all the graded rings in the diagram of Figure 6.2. Let $\gamma_i$ be the map

$$\gamma_i \colon k[V^2]_i \oplus k[V^1]_i \oplus k[V^3]_i \to k[L_1]_i \oplus k[L_2]_i,$$

defined by

$\gamma_i(f, g, h) = (\alpha_{2,1}(f) - \alpha_{1,1}(g),\ \alpha_{3,2}(h) - \alpha_{1,2}(g))$. Let $(\bar{f}, \bar{h}) \in k[L_1]_i \oplus k[L_2]_i$, and choose preimages $f \in k[V^2]_i$ and $h \in k[V^3]_i$ (using the fact that $\alpha_{2,1}$ and $\alpha_{3,2}$ are surjective). Then $\gamma_i(f, 0, h) = (\bar{f}, \bar{h})$, so $\gamma_i$ is also surjective.

For each i, $C_i = \ker \gamma_i$, so

$$(*) \qquad P_C(i) = \sum_{j=1}^{3} P_{k[V^j]}(i) - \sum_{k=1}^{2} P_{k[L_k]}(i).$$

$\mathcal{J}(V^1) = (X_1 X_2(X_1 - X_2)(X_1 + X_2), X_3)$. and $k[V^1] = k[X_0, X_1, X_2, X_3]/\mathcal{J}(V^1) \cong k[X_0, X_1, X_2]/(X_1 X_2(X_1 - X_2)(X_1 + X_2))$, which is the coordinate ring of a hypersurface in $P_k^2$ of degree 4. So we have the following exact sequence, where $S = k[X_0, X_1, X_2]$,

$$0 \to S(-4) \to S \to k[V^1] \to 0.$$

So $P_{k[V^1]}(i) = P_S(i) - P_{S(-4)}(i)$. As $P_S(i) = \begin{bmatrix} i+2 \\ 2 \end{bmatrix}$ and $P_{S(-4)}(i) = \begin{bmatrix} i-2 \\ 2 \end{bmatrix}$, we conclude that

$$P_{k[V^1]}(i) = \begin{cases} \begin{bmatrix} i+2 \\ 2 \end{bmatrix}, & \text{if } 0 \le i \le 3 \\ 4i - 2, & \text{if } i \ge 4 \end{cases}.$$

Similarly, $k[V^2]$ is isomorphic to the coordinate ring of a hypersurface of degree 2 in $P_k^2$, so $P_{k[V^2]}(i) = P_S(i) - P_{S(-2)}(i)$

$$= \begin{cases} \begin{bmatrix} i+2 \\ 2 \end{bmatrix}, & \text{if } 0 \le i \le 1 \\ 2i + 1, & \text{if } i \ge 2 \end{cases} .$$

$V^3$ also consists of two lines in a plane, so $P_{k[V^3]}(i) = P_{k[V^2]}(i)$. Finally $P_{k[L_k]}(i) = i + 1$, as $L_k$ is isomorphic to $P^1_k$.

It now follows from equation (*) that

$$P_C(0) = 3\begin{bmatrix} 2 \\ 2 \end{bmatrix} - 2(1) = 1,$$

$$P_C(1) = 3\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 2(2) = 5,$$

$$P_C(2) = \begin{bmatrix} 4 \\ 2 \end{bmatrix} + 2(5) - 2(3) = 10$$

$$P_C(3) = \begin{bmatrix} 5 \\ 2 \end{bmatrix} + 2(7) - 2(4) = 16$$

and, for $i \ge 4$,

$$\begin{aligned} P_C(i) &= (4i - 2) + 2(2i + 1) - 2(i + 1) \\ &= 6i - 2. \end{aligned}$$

We have established:

<u>Claim 1</u>. $P_C(0) = 1$, $P_C(1) = 5$, and $P_C(i) = 6i - 2$, for $i \ge 2$.

We next show:

<u>Claim 2</u>. $C$ is generated by elements of degree 1.

<u>Proof of Claim 2</u>. Choose lines $L_1', \ldots, L_6'$ in $P^4_k$ as follows: For $i = 1, 2, \ldots, 5$,

$$L_i' = \mathcal{V}(\mathcal{J}(L_i)k[X_0, \ldots, X_4], X_4),$$

$$L_6' = \mathcal{V}(X_1, X_0 - X_3, X_0 - X_4)$$

The same intersection relations hold among $L_i'$ as hold among the $L_i$. Let $V^{1'} = \bigcup_{i=1}^{4} L_i'$, $V^{2'} = L_1' \cup L_5'$, and $V^{3'} = L_2' \cup L_6'$. Then, for $i = 1, 2$, $k[V^{i'}] \cong k[V^i]$ via the map induced by $\pi_1(X_i) = X_i$, for $i = 0, 1, 2, 3$, and $\pi_1(X_4) = 0$. $k[V^{3'}] \cong k[V^3]$ under a different map. Note first that

$$\begin{aligned}
\mathcal{J}(V^{3'}) &= (X_1, X_3, X_4) \cap (X_1, X_0 - X_3, X_0 - X_4) \\
&= (X_1, X_4 - X_3, X_3) \cap (X_1, X_4 - X_3, X_0 - X_3) \\
&= (X_1, X_4 - X_3, X_3(X_0 - X_3)). \\
\mathcal{J}(V^3) &= (X_1, X_3) \cap (X_1, X_0 - X_3) \\
&= X_3(X_0 - X_3).
\end{aligned}$$

Let $\pi_2$: $k[X_0,\dots,X_4] \to k[X_0,\dots,X_3]$ be defined by $\pi_2(X_i) = X_i$, for $i = 1, 2, 3$, and $\pi_2(X_4) = X_3$. Then $\pi_2(\mathcal{J}(V^{3'})) = \mathcal{J}(V^3)$, and $\pi_2$ induces the desired isomorphism. Note finally that $\pi_1$ induces isomorphisms $k[L_i'] \cong k[L_i]$, for $i = 1, 2$.

We have now constructed the following commutative diagram.

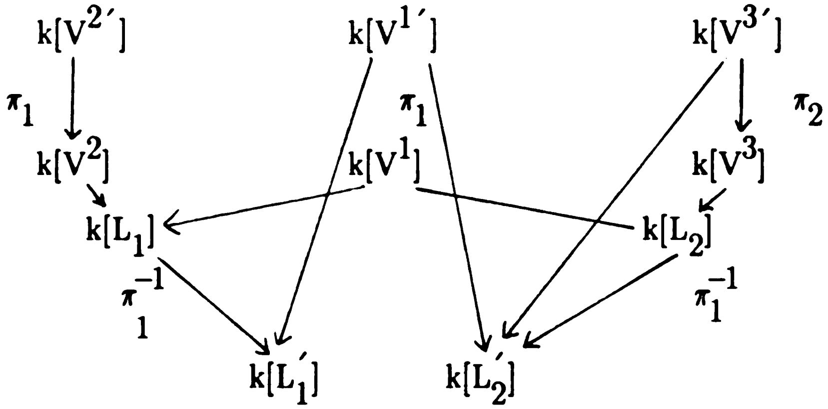


Figure 6.3

Commutativity is easily checked, as most of the maps are projections. As the maps induced by $\pi_1$ and $\pi_2$ are isomorphisms, it follows easily that $k[V]$ and $k[V']$ have isomorphic Cohen–Maculifications. As edim $k[V^{i'}]$ = edim $k[V^i]$, for $i = 1, 2, 3$, and edim $k[V'] = 5$ with $V'$ simply connected, it follows from Theorem 6.8 that $k[V']$ is

Cohen–Macaulay. Thus $k[V']\cong C$, and $C$ is generated by elements of degree 1, as claimed.

Claim 3. $P_{k[V]}(0) = 1$, $P_{k[V]}(1) = 4$, $P_{k[V]}(2) = 10$.

Proof of Claim 3. Only the last equality is non–trivial, and $P_{k[V]}(2) \leq 10$ is obvious. Suppose $P_{k[V]}(2) < 10$. then $V$ lies in a hypersurface $W$ of degree 2. Suppose $W \supseteq H^1$. Then $W \cap H^1 \supseteq V^1$ which is a reducible curve of degree 4, contrary to Bézout's theorem. Thus $W \supseteq H^1$. As deg $W = 2$, we must have $W = H^1 \cup H^4$ for some hyperplane $H^4$. But then $H^4 \supseteq L_5 \cup L_6$, which is impossible, as $L_5 \cup L_6 = \phi$. So there is no such $W$ and $P_{k[V]}(2) = 10$.

Claim 4. $P_{k[V]}(3) = P_s(3) - P_{\mathcal{J}(V)}(3) = 20 - P_{\mathcal{J}(V)}(3)$, so it suffices to show that $P_{\mathcal{J}(V)}(3) = 4$. Let $f \in \mathcal{J}(V)_3$, and $W = \mathcal{V}(f)$. As in the previous claim, it follows by Bézout's theorem that "$W \supseteq H^1$. So $W \supseteq H^1 \cup W_1$ for some hypersurface $W_1$ of degree 2. As before $W_1 \supseteq L_5 \cup L_6$,. But $L_5 \cup L_6$, is a union of two skew lines in $P^3_k$, and as such, is isomorphic by a change of coordinates to the union of two skew lines given in example 6.2. In that example, we found $P_A(2) = 6$. Thus there are $10 - 6 = 4$ linearly independent elements of degree 2 vanishing on two skew lines. So there are only four linearly independent elements of degree 3 vanishing on $V$.

Therefore $P_{k[V]}(3) = 16$.

Claim 5. $k[V]_i = C_i$, for $i \geq 2$.

Proof of Claim 5. By claims 1, 3, and 4, $P_{k[V]}(i) = P_c(i)$ for $i = 2, 3$. As $k[V] \subset C$, $k[V]_i = C_i$ for $i = 2, 3$. If $f \in C_i$ for $i \geq 4$, then, by claim 2, $f = \Sigma\, g_j$, where each $g_j$ is a product of degree 1 elements of $C$. Thus, each $g_j$ is a product of elements of $C_2$ and $C_3$. As each of these factors is in $k[V]$, so is $g_j$, and therefore, $f \in k[V]_i$.

Claim 6. $P_{k[V]}(i) = \begin{cases} 1 & \text{if } i = 0 \\ 6i-2, & \text{if } i \geq 1 \end{cases}.$

Proof of Claim 6. For $i = 0$ and $1$, this follows by claim 3. For $i \geq 2$, this follows by claims 1 and 5.

Now, letting $a_i = P_{k[V]}(i)$, we have a first sequence of differences $\{b_i\}_{i=1}^{\infty}$ with $b_0 = 1$, $b_1 = 3$, $b_i = 6$, for $i \geq 2$. The second sequence of differences is $c_1 = 1$, $c_2 = 2$, $c_3 = 3$, $c_i = 0$, for $i \geq 4$. So $P_{k[V]}(i)$ is twice differentiable. ∎

Example 6.3 allows us to answer a related question posed by Geramita, Maroscia and Roberts, relating to differentiable $\mathcal{O}$-sequences. $\mathcal{O}$-sequences can be defined as follows [St, p. 60]. If h and i are positive integers, then h can be written uniquely in the form

$h = \begin{bmatrix} n_i \\ i \end{bmatrix} + \begin{bmatrix} n_i - 1 \\ i - 1 \end{bmatrix} + \dots + \begin{bmatrix} n_j \\ j \end{bmatrix}$, where $n_i > n_{i-1} > \dots > n_j \geq j \geq 1$. Define $h^{<i>} = \begin{bmatrix} n_i + 1 \\ i + 1 \end{bmatrix} + \begin{bmatrix} n_{i-1} + 1 \\ i \end{bmatrix} + \begin{bmatrix} n_j + 1 \\ j + 1 \end{bmatrix}$. An $\mathcal{O}$-sequence is a sequence $\{a_i\}_{i=1}^{\infty}$ such that $a_0 = 1$ and $a_{i+1} \leq a_i^{<i>}$ for $i \geq 1$. It can be shown that the $\mathcal{O}$-sequences are precisely the sequences of values of Hilbert functions of Standard G–algebras R with $k_0 = k$[St, Theorem 2.2] Thus $P_{k[V]}(i)$ defines an $\mathcal{O}$-sequences. An $\mathcal{O}$-sequence is a 2–differentiable $\mathcal{O}$-sequence if its first and second sequences of differences are also $\mathcal{O}$-sequence.

Geramita, Maroscia and Roberts have asked [GMR, remark 7.13] whether a union of lines in $P_k^3$ whose Hilbert function is a 2–differentiable $\mathcal{O}$-sequence must be Cohen–Macaulay. It is easy to check that the first and second sequences of differences in example 6.3 are $\mathcal{O}$-sequences, so the answer is negative.

# CHAPTER 7

# EXISTENCE THEOREMS

In order to be Cohen–Macaulay, a union of lines must be connected, and the Hilbert function of its coordinate ring must be a twice differentiable $\mathcal{O}$-sequence. These global necessary conditions suggest that "most" unions of lines are not Cohen–Macaulay.

In this chapter we will show that any union of lines $V$ is contained in a union of lines $W$ which is Cohen–Macaulay. That is, it is always possible to add lines to $V$ so as to make some larger union $W$ Cohen–Macaulay. This suggests: (1) that the class of Cohen–Macaulay unions of lines may contain much more than just the classes of examples discussed in previous chapters, and (2), that there may be no simple local necessary condition (i.e., condition on the vertices) for a union of lines to be Cohen–Macaulay. The second observation, if true, would contrast the Cohen–Macaulay property sharply with seminormality, as a union of lines cannot be seminormal if the lines through each vertex are not linearly independent (by [DR, 2, 3.5]).

The results in this chapter hold not only for unions of lines, but also for all unions of linear varieties of unmixed projdim $\geq 1$, so we will prove the results in this form.

We will need the following proposition about minimal generating sets of ideals of definition of certain unions of points in $\mathbf{P}^n_k$.

<u>Proposition 7.1</u>. Let $f_1,\ldots,f_s \in k[X_0,\ldots,X_n]$ be polynomials of degree 1, with $s > n$, with the following property. For every choice of $1 \leq i_1 < \cdots < i_{n+1} \leq s$, $f_{i_1},\ldots,f_{i_{n+1}}$ are linearly independent over $k$. For each such

choice of $i_1,\dots,i_{n+1}$ define

$$g_{i_1,\dots,i_{n-1}} = \prod_{j \notin \{i_1,\dots,i_{n-1}\}} f_j.$$

Then

(a) $V = \mathcal{V}(\bigcap_{1 \le i_1 < \dots < i_n \le s} (f_{i_1},\dots,f_{i_n})) \le P_k^n$ is a set of $\begin{bmatrix} s \\ n \end{bmatrix}$ points.

(b) V is not contained in any hypersurface of degree $< s - n + 1$.

(c) The $g_{i_1,\dots,i_{n-1}}$ are linearly independent over k. (d) If $I = (\{g_{i_1,\dots,i_{n-1}}\})$, then $I = \mathcal{J}(v)$.

<u>Proof</u>. (a) it follows immediately from the linear independence condition that each of the $\begin{bmatrix} s \\ n \end{bmatrix}$ ideals $(f_{i_1},\dots,f_{i_n})$ is the ideal of a point in $P_k^n$ and that any two of these ideals are distinct.

(b) Let $H = \mathcal{V}(h)$ with $\deg h \le s - n$, and suppose $H \supset V$. Then, for any choice of $1 \le i_1 < \dots < i_{n-1} \le s$, H contains each of the $s - n + 1$ points $\mathcal{V}(f_{i_1},\dots,f_{i_{n-1}}, f_j)$, where $j \notin \{i_1,\dots,i_{n-1}\}$, so by Bézout's theorem, H contains the line $\mathcal{V}(f_{i_1},\dots,f_{i_{n-1}})$. So $h \in \bigcap_{1 \le i_1 < \dots < i_{n-1} \le s} (f_{i_1},\dots,f_{i_{n-1}})$. Repeat the argument. For any choice of $1 \le i_1 < \dots < i_{n-2} \le s$, H contains each of the $s - n + 2$ lines $\mathcal{V}(f_{i_1},\dots,f_{i_{n-2}},f_j)$ for $j \notin \{i_1,\dots,i_{n-2}\}$. So by Bézout's theorem, H contains the plane $\mathcal{V}(f_{i_1},\dots,f_{i_{n-2}})$, and $h \in \bigcap_{1 \le i_1 < \dots < i_{n-1} \le s} (f_{i_1},\dots,f_{i_{n-2}})$. In a finite number of steps we find that $\bigcap_{i=1}^{s} (f_i) = f_1 \dots f_s)$. So $f_1 \dots f_s | h$, contrary to $\deg h \le s - n$. Therefore $V \not\subseteq H$.

(c) Suppose $\sum_{1 \le i_1 < \dots < i_{n-1} \le s} a_{i_1,\dots,i_{n-1}} g_{i_1,\dots,i_{n-1}} = 0$, for some $a_{i_1,\dots,i_{n-1}} \in k$. Suppose one of these coefficients is nonzero, w log $a_{1,\dots,n-1} \neq 0$. Let $(b_0:\dots:b_n)$ be any point on the line $\mathcal{V}(f_1,\dots,f_{n-1})$ which is distinct from the points $\mathcal{V}(f_1,\dots,f_{n-1}, f_j)$, $j = n,\dots, s$. Then $g_{1,\dots,i_{n-1}}$ does not vanish at $(b_0:\dots:b_n)$, but all the other $g_{i,\dots,i_{n-1}}$ do, so if we put $X_i = b_i$ in the equation above, we find $a_{1,\dots,n-1} = 0$, a contradiction. So there is no nontrivial linear relation among the $g_{i_1,\dots,i_{n-1}}$.

(d) Let $S = k[X_0,\dots,X_n]$ and $a = S/\mathcal{J}(V)$. By (b), $\mathcal{J}(V) \cup S_i = (0)$ for $i = 0,\dots,s - n$, so for these $i$, the Hilbert function has value $P_A(i) = \binom{i+n}{n}$. $P_A(i)$ is the Hilbert function of a reduced variety, so it is differentiable, and thus nondecreasing. And $V$ consists of $\binom{s}{n}$ points in $P^n_k$, so the Hilbert polynomial is $\bar{P}_A(i) \equiv \binom{s}{n}$. This forces $P_A(i) = \binom{s}{n}$, for all $i \ge s - n$. In particular, if $d_0 = s - n + 1$, then $P_A(d_0) = \binom{s}{n}$. so $\dim_k (\mathcal{J}(v) \cap S_{d_0}) = \binom{d_0+n}{n} - \binom{s}{n}$. As $d_0$ is the least degree of a polynomial vanishing on $V$, and also the least integer such that $\binom{d_0+n}{n} > \binom{s}{n}$, it follows that the points of $V$ are in generic $\binom{s}{n}$ – position [GO1, prop. 3]. Moreover, $\mathcal{J}(V) \cap S_{d_0} = I \cap S_{d_0}$, as $I \subseteq \mathcal{J}(V)$ and $\dim_k(I \cap S_{d_0})$ is, by (c), $\binom{s}{n-1} = \binom{s+1}{n} - \binom{s}{n} = \binom{d_0+n}{n} - \binom{s}{n} = \dim_k(\mathcal{J}(V) \cap S_{d_0})$.

It now follows by [GO2, prop. 4]. that $\mathcal{J}(V) = I$. ∎

We now construct Cohen–Macaulay unions of linear varieties which are higher dimensional analogues of the unions of points in prop. 7.1.

Prop. 7.2. Let $f_1,\dots,f_s \in k[X_0,\dots,X_n]$ be homogeneous of degree 1. Let $t < s$, $2 \le t \le n$, and suppose that for every choice of $1 \le i_1 < \cdots < i_{t+1} \le s$, $f_{i_1},\dots,f_{i_{t+1}}$ are linearly independent over $k$. For each choice of $1 \le i_1 < \cdots < i_{t+1} \le s$,, define $g_{i_1,\dots,i_{t-1}} = \prod_{j \notin \{i_1,\dots i_{t-1}\}} f_j$ . Then:

(a) $V = \mathcal{V}(\bigcap_{1 \le i_1 < \dots < i_t \le s} (f_{i_1},\dots,f_{i_t})) \subseteq \mathbf{P}^n_k$ is a union of $\begin{bmatrix} s \\ t \end{bmatrix}$ distinct linear varieties of dimension $n - t$.

(b) The $g_{i_1,\dots,i_{t-1}}$ are linearly independent over $k$.

(c) if $I = (\{g_{i_1,\dots,i_{t-1}}\})$, then $I = \mathcal{I}(V)$. (d) If $A = k[V]$, then $A$ is Cohen–Macaulay.

Proof. (a). The proof is the same as the proof of prop. 7.1. (a).

(b): The proof is similar to the proof of prop. 7.1

(c): If $(b_0:\dots:b_n)$ is a point of $\mathcal{V}(f_1,\dots,f_{t-1})$ which is not contained in any of the proper subvarieties $\mathcal{V}(f_1,\dots,f_{t-1},f_j)$, $j = t,\dots,s$, then $g_{1,\dots,t-1}$ does not vanish at this point, but all the other $g_{i_1,\dots,i_{t-1}}$ do. The rest of the argument is the same. (c) and (d). We proceed by induction on $\dim V = n - t$. For $n = t$, (c) is just 7.1 (d), and (d) is also true as unions of points are always cohen–Macaulay. So suppose (c) and (d) are true if $\dim V < d$ and suppose $\dim V = d$.

As $t < n$, each $(f_{i_1},\dots,f_{i_{t+1}}) \subseteq M$, where $M$ is the homogeneous maximal ideal of $S$. So, by a vector space argument applied to $S_1$, we can choose an $x \in M - \bigcup_{1 \le i_1 < \dots < i_{t+1} \le s} (f_{i_1},\dots,f_{i_{t+1}})$ of degree 1. By changing coordinates we can assume wlog that $x = X_n$ and define $f'_i = f_i(X_0,\dots,X_{n-1}, 0) \in S'_i = k[X_0,\dots,X_n]$. Also, define

$$g_{i_1,\dots,i_{t-1}} = \prod_{j \notin \{i_1,\dots i_{t-1}\}} f'_j.$$

Note that for any choice of $1 \le i_1 < \dots < i_{t+1} \le s$, if we we had a nontrivial relation $\sum_{j=1}^{t+1} c_j f'_{i_j} = 0$, $c_j \in k$, then either $\sum_{j=1}^{t+1} c_j f_{i_j} = 0$ or $X_n \in (f_{i_1},\dots,f_{i_{t+1}})$, both being contradictions. So $f'_{i_1},\dots,f'_{i_{t+1}}$ are linearly independent over $k$. Finally, let $V' \subseteq \mathbf{P}_k^{n-1}$, where $V' = \mathcal{V}(\bigcap_{1 \le i_1 < \dots < i_t \le s} (f'_{i_1},\dots,f'_{i_t}))$, and $A' = k[V']$.

If we identify $S/(X_n) \cong S'$, then $(X_n) + I/(X_n) = (X_n) + (\{g_{i_1,\dots,i_{t-1}}\})/(X_n) = (X_n) + (\{g'_{i_1,\dots,i_{t-1}}\})/(X_n) \cong (\{g'_{i_1,\dots,i_{t-1}}\}) = \mathcal{J}(V')$, by the inductive assumption, as $n - 1 - t = d - 1 = \dim(V')$. As the preimage in $S$ is $(X_n) + I$, this must also be a radical ideal, so $(X_n) + I = (X_n) + \sqrt{I}$. But

$$\prod_{1 \le i_1 < \dots < i_t \le s} (f_{i_1},\dots,f_{i_t}) \subseteq I \subseteq \mathcal{J}(V),$$

so $\sqrt{I} = \mathcal{J}(V)$, and $(X_n) + I = (X_n) + \mathcal{J}(V)$. By choice of coordinates,

$$X_n \notin \bigcup_{1 \le i_1 < \dots < i_t \le s} (f_{i_1},\dots,f_{i_t})$$

so $X_n$ is regular in $A$. Suppose $h \in \mathcal{J}(V)$ and $\deg h < s - t + 1$. If $h = X_n^m h'$, then the image in $A$ is $\bar{h} = X_n^m \bar{h}' = 0$, so $\bar{h}' = 0$, i.e., $h' \in \mathcal{J}(V)$. So we can replace $h$ by $h'$ and assume wlog that $X_n$ th. but then $0 \ne h(X_0,\dots,X_{n-1}, 0) \in \mathcal{J}(V) = (\{g'_{i_1,\dots,i_{t-1}}\})$, which is a contradiction, as $\deg g_{i_1,\dots,i_{t-1}} = s - t + 1$. Thus $\mathcal{J}(V)$ contains no elements of degree $< s - t + 1$.

As $X_n$ is regular in $S/\mathcal{J}(V)$, it follows by [G, lemma 1.1] that $\nu(\mathcal{J}(V)) = \nu((X_n, \mathcal{J}(V))/(X_n))$. As $(X_n, \mathcal{J}(V))/(X_n) = (X_n, I)/(X_n) \cong \mathcal{J}(V')$, we have $\nu(\mathcal{J}(V)) = \nu(\mathcal{J}(V') = \begin{bmatrix} s \\ t-1 \end{bmatrix}$, by (b) and the inductive assumption on $\mathcal{J}(V')$. Then by (b), we have $\mathcal{J}(V) = (\{g_{i_1,\ldots,i_{t-1}}\})$. So the induction step holds for (c).

Also, $A/(X_n) \cong (S/I)/((I, X_n)/I) \cong S/(I, X_n) \cong S'/\mathcal{J}(V') = k[V']$, and $X_n$ is regular in A. So by the inductive assumption and (d), $k[V'] = A/(x_n)$ is Cohen–Macaulay, and it follows that A is Cohen–Macaulay. So the inductive step holds for (d).

Proposition 7.3. Let V be as in prop. 7.2. Then the Cohen–Macaulay type of V is $\begin{bmatrix} s-1 \\ t-1 \end{bmatrix}$.

Proof. As in the previous proof, we induct on dim $V = n - t$. For $n = t$, V consists of $\begin{bmatrix} s \\ n \end{bmatrix}$ points in generic position in $P^n_k$. by [GO2, prop. 16], the Cohen–Macaulay type is $\begin{bmatrix} s-1 \\ n-1 \end{bmatrix} = \begin{bmatrix} s-1 \\ t-1 \end{bmatrix}$.

Suppose the proposition is true for dim $V < d$ and that dim $V = d$. By the proof of the inductive step in prop. 7.2 (c) and (d), there is a regular element x in A (corresponding to $x_n$ under change of coordinates), such that $A/(x) \cong k[V']$. Then k[V] and k[V'] have the same Cohen–Macaulay type, which is, by inductive assumption, $\begin{bmatrix} s-1 \\ t-1 \end{bmatrix}$. ∎

We come now to the main result of the chapter.

Proposition 7.4. Let $V = \bigcup_{i=1}^{s} V_i \subseteq P^n_k$ be a union of linear varieties $V_i$, each of dimension $n - t \geq 0$, with $t \geq 1$. Then $V \subseteq W \subseteq P^n_k$, where $W$ is a union of linear varieties of dimension $n - t$ and $k[W]$ is Cohen–Macaulay.

Proof. if $t = 1$, then $V$ is a hypersurface, thus Gorenstein, so we can let $W = V$. If $n - t = 0$, then $V$ is a union of points. Again, we let $W = V$. So assume that $2 \leq t \leq n - 1$.

We use the fact that any vector space over an infinite field is not the union of a finite number of proper subspaces. Let $\mathscr{P}_i = \mathscr{I}(V_i)$ and $W_i = \mathscr{P}_i \cap S_1$, for $i = 1,\ldots,s$. Choose $f_1,\ldots,f_{st}$ as follows. Each $W_1 \cap W_j \nsubseteq W_1$, if $j \neq 1$, so choose $f_1 \in W_1 - \bigcup_{j=2}^{s} W_j$. Suppose we have chosen $f_1,\ldots,f_m$ so that: (1) If $t(i-1) < j \leq ti$, then $f_j \in W_i - \bigcup_{k \neq i} W_k$, and (2) If $1 \leq i_1 \leq \ldots \leq i_t < j$, then $f_j \notin (f_{i_1},\ldots,f_{i_t})$. Let $t(i - 1) < m + 1 \leq ti$. If $1 \leq i_1 \leq \ldots \leq i_t \leq m$ and $W_i \subseteq (f_{i_1},\ldots,f_{i_t})$, then $W_i \subseteq (f_{i_1},\ldots,f_{i_t}) \cap S_1$, by dimensions. But then $i_1 \leq ti - t = t(i-1)$, so for some $j < i$, $f_{i_1} \in W_j - W_i$, by inductive assumption (1), contrary to $f_{i_1} \in W_i$. Thus $W_i \cap (f_{i_1},\ldots,f_{i_t})$ is a proper subspace of $W_i$. So we can choose some element

$$f_{m+1} \in W_i - [(\bigcup_{j \neq i} W_j) \cup (\bigcup_{1 \leq i_1 \leq \ldots \leq i_t \leq m} (f_{i_1},\ldots,f_{i_t}))],$$

and then properties (1) and (2) hold for $f_1,\ldots,f_{m+1}$. So we can choose $f_1,\ldots,f_{st}$ also that properties (1) and (2) hold. It follows immediately from property (2) that any $t + 1$ of the $f_i$ are linearly independent. If $s = 1$, then $W = V$ is already Cohen–Macaulay, so there is nothing to prove. So assume $s > 1$. Let $W = \mathscr{V}(\bigcap_{1 \leq i_1 < \ldots < i_t \leq s} (f_{i_1},\ldots,f_{i_t}))$. By prop. 7.2., W is a union of $\begin{bmatrix} st \\ t \end{bmatrix}$ linear varieties of dimension $n-t$ in $P_k^n$, and $k[W]$ is Cohen–Macaulay. As $f_{t(i+1)+1}, \ldots, f_{ti} \in W_i$ are linearly independent, $\mathcal{P}_i = (f_{t(i+1)+1}, \ldots, f_{ti})$ and so $V \subseteq W$. ■

We close with a simple example to illustrate the construction involved in obtaining W from V.

Example 7.5. Recall example 6.2, $L_1 = \mathcal{V}(X_0, X_1)$, $L_2 = \mathcal{V}(X_2, X_3)$. Letting $f_i = X_{i-1}$ for $i = 1, \ldots, 4$, we satisfy conditions (1) and (2) in the proof of prop. 7.4 (here $s = t = 2$). Then $W = \mathcal{V}\left( \bigcap_{0 \le i_1 < i_2 \le 3} (X_{i_1}, X_{i_2}) \right)$ consists of $\begin{bmatrix} 4 \\ 2 \end{bmatrix} = 6$ lines in $P_k^3$. By prop. 7.3, $k[W]$ has type $\begin{bmatrix} 4-1 \\ 2-1 \end{bmatrix} = 3$.

In general, the number of linear varieties in W is much larger than the number of linear varieties in V, and may be larger than the smallest number required to obtain a Cohen–Macaulay variety containing V. In the example above, we can add the line $L_3 = \mathcal{V}(X_1, X_2)$, to $L_1 \cup L_2$ to obtain a simply connected Cohen–Macaulay union of lines of type 2. There does not appear to be any general procedure for finding the minimum number of new lines required.

## BIBLIOGRAPHY


[B] William C. Brown, "A Note on the Cohen–Macaulay Type of Lines in Uniform Position in $A^{n+1}$," Proceedings of the American Mathematical Society, v. 87, no. 4, 591–595 (April 1983).

[BB] Marjory Baruch and William C. Brown, "A Matrix Computation for the Cohen–Macaulay Type of s–Lines in Algebra 85, 1–13 (1985).

[CL] Gary Chartrand and Linda Lesniak, Graphs and Digraphs, second edition, Wadsworth, 1986.

[DR] Barry H. Dayton and Leslie G. Roberts, "Seminormality of Unions of Planes," Lecture Notes in Mathematics, Vol. 854, 93–126, Springer–Verlag.

[G] Anthony V. Geramita, "Remarks on the Number of Generators of some Homogeneous Ideals," Bull. Sc. Math., $2^e$ Série, 107, 1983, p. 193–207.

[GMR] A. V. Geramita, P. Maroscia, L. Roberts, "The Hilbert Function of a Reduced K–Algebra, "The Curves Seminar at Queen's, Volume II, Queen's Papers in Pure and Applied Mathematics, no. 61, Queen's University, 1982.

[GO1] A. V. Geramita and F. Orecchia, "On the Cohen–Macaulay Type of s–Lines in $A^{n+1}$," Journal of Algebra 70, 116–140 (1981).

[GO2] A. V. Geramita and F. Orecchia, "Minimally Generating Ideals Defining Certain Tangent Cones," Journal of Algebra 78, 36–57 (1982).

[GW] A. V. Geramita and C. A. Weibel, "On the Cohen–Macaulay and Buchsbaum Property for Unions of Planes in Affine Space, "Journal of Algebra 92, 413–445.

[H] Robin Hartshorne, Algebraic Geometry, corrected third printing, Springer–Verlag, 1983.

[H1] Robin Harshorne, "Complete Intersections and Connectedness," Amer. J. Math. 84 (1962), pp. 497–508.

[HK] Jurgen Herzog and Ernest Kunz, Der kanonische Modul eines Cohen–Macaulay–Rings, Lecture Notes in Mathematics no. 238, Springer–Verlag, 1971.

[HO] Melvin Hochester, "Cohen–Macaulay Rings, Combinatorics, and simplicial complexes," Lecture Notes in Pure and Applied Mathematics," No. 26, pp. 171–221, Dekker, 1977.

[K] Ernest Kunz, Introduction to Commutative Algebra and Algebraic Geometry, translated by Michael Ackerman, Birkauser, 1985.

[M] David Mumford, Algebraic Geometry I Complex Projective Varieties, corrected second printing, Springer–Verlag, 1976.

[Re] Gerald Allen Reisner, "Cohen–Macaulay Quotients of Polynomial Rings," Advances in Mathematics 21, 30–49 (1976).

[S] Richard G. Swan, "On Seminorality," Journal of Algebra 67, 210–229 (1980).

[St] Richard P. Stanley, "Hilbert Functions of Graded Algebras," Advances in Mathematics 28, 57–83 (1978).

[ZS] Oscar Zariski and Pierre Samuel, Commutative Algebra, volumes I and II, second printing, Springer–Verlag, 1979.