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ABSTRACT

STUDIES IN COMPRESSIBLE AND INCOMPRESSIBLE MIXTURES

By

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The work presented herein is focused on identifying qualitative and/or quantitative differences in the behavior of incompressible mixtures with solid constituents being various types of nonlinearly elastic incompressible solids in the context of the Theory of Interacting Continua. In particular, the response of mixtures with Neo-Hookean or Mooney-Rivlin solids with ideal fluid is studied for several cases of simple deformations. The results of these investigations could have a significant impact on material identification studies.Comparison of these results with the experimental results of Treloar are also presented.

Furthermore, attention is also directed towards resolving issues which would permit the solution of boundary value problems involving compressible mixtures. This work also clarifies certain misconceptions pertaining to primitive concepts such as volume additivity and incompressibility of mixtures. In addition, the saturation equations of state for a mixture of a compressible solid and an ideal fluid and an appropriate free energy function for a compressible mixture is derived.

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Finally, a boundary value problem is solved to demonstrate that the presence of constraints such as inserts and inclusions which have differential constitutive characteristics can induce nonhomogeneous swelling characteristics and stress concentrations within the material.

DEDICATED

To

My Grand-Uncle Dr. I. P. Desai for his unique vision and philosophy of life and absolute commitment towards understanding the complex social structure in India.

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My parents and my wife for continuing patience and confidence in me.

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NOMENCLATURE

A	Helmholtz free energy function of the mixture per unit mass of the mixture.
A_1, A_2	Partial derivatives of Helmholtz free energy function with respect to invariants I_1 and I_2, respectively.
A_{S1}, A_{S2}	Helmholtz free energy function of the solid and fluid per unit mass of the solid and fluid, respectively.
A_e, A_m	Helmholtz free energy function of the elastic deformation and mixing per unit mass of the mixture, respectively.
b_i	Components of the interaction body force.
B_{ij}	Components of the Cauchy-Green deformation tensor.
C_1, C_2	Coefficients appearing in the generalised constitutive equations for incompressible solids.
d_{ij}, f_{ij}	Components of the rate of the deformation tensor for the solid and fluid, respectively.
F_{ij}	Components of the deformation gradient tensor.
f_i, g_i	Components of the acceleration vector for the solid and fluid, respectively.
$\underset{\sim}{I}$	Identity tensor.

I_1, I_2, I_3	Invariants of the Cauchy-Green deformation tensor $\underline{B}$.
L_{ij}, M_{ij}	Components of the velocity gradient tensor for the solid and fluid, respectively.
$\underset{\sim}{n}$	Unit outer normal vector.
p	Scalar appearing in the constitutive equations due to the incompressibility constraint.
R, r	Components of the reference and current radial coordinates, respectively.
S_1, S_2	Labels denoting a solid and a fluid particle, respectively.
S_{ij}	Components of the total stress characterizing the state of the mixture in a saturated state (i,j=1,2,3 or r,θ,z).
t_i	Total surface traction vector.
T	Absolute temperature of the mixture continua.
T_{ij}	Components of the total stress tensor for the mixture.
u_i, v_i	Components of the velocity vector for the solid and fluid.
w_i	Components of the mean velocity vector for the mixture.
$\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$	Functions defining the current configuration of the solid and fluid, respectively.
X_i, Y_i	Reference position of the solid and fluid particle, respectively.
x_i, y_i	Current position of a solid and fluid particle, respectively.
X,Z, x,z	Components of the reference and current coordinates in the cartisian coordinate system, respectively.

λ Stretch ratio along the length of the cylindrical mixture.

$\lambda(Z)$ Stretch ratio in the thickness direction of the mixture slab.

λ_r, λ_θ Radial and circumferential stretch ratios.

ν_1 Volume fraction of the solid in the mixture.

ρ Density of the mixture.

ρ_{10}, ρ_{20} Density of the pure solid and fluid, respectively.

ρ_1, ρ_2 Mass per unit volume of the mixture for the solid and fluid, respectively.

Γ_{ij}, Λ_{ij} Components of the vorticity tensor for the solid and fluid, respectively.

γ_i, μ_i Coefficients appearing in the dynamical part of the constitutive equations (i=1,2,3,4).

Θ, θ Components of the reference and current coordinate in the radial coordinate system.

σ_{ij}, π_{ij} Components of the partial stress tensor for the solid and fluid, respectively (i,j=1,2,3 or r,θ,z).

σ_i, π_i Components of the partial surface traction for the solid and fluid, respectively.

χ A constant which depends on the particular combination of the solid and the fluid.

CHAPTER I

INTRODUCTION

Several phenomena involving the presence of two or more constituents are difficult or almost impossible to be mathematically modeled by conventional single constituent theories. Typically examples of such phenomena include the swelling of polymers or rubber-like materials, diffusion phenomena in blood vessels, moisture effects in composite materials. Classical diffusion theories model flow of a fluid through a solid assuming negligible deformation of the solid constituent. However, this assumption is violated in solid-fluid interactions where large deformations are involved. For example,it has been shown by Treloar [1] that a block of rubber can swell into several times its original volume when placed in a bath of an organic fluid. Furthermore the amount of swelling is strongly affected by subjecting the block to uniaxial or bi-axial extension, for example.

The Theory of Interacting Continua, also known as Theory of Mixtures, has been successfully used to model phenomena involving two or more constituents. In this theory, the continua are assumed to be superimposed. Each spatial point in a mixture is assumed to be simultaneously occupied by material particles from each constituent. This essentially amounts to taking into account contributions from each constituent in the neighborhood of a mathematical point and averaging them. The theory accounts for large deformations, dependence of material properties on all constituents and interaction between the constituents. In proof-of-concept studies it has been clearly shown that the Theory of Mixtures has substantially better predictive capabilities than traditional single constituent modeling concept. For example, results obtained by Gandhi [2] using Theory of Interacting Continua had better agreement with experimental results of Paul and Ebra-Lima [3] in modeling pressure-induced diffusion of organic liquids through a rubber layer than results obtained by using Ficks law (Classical Diffusion Theory).

Gandhi, et al.[4] have successfully used Mixture Theory to model behavior of a swollen cylinder under combined extension and torsion and, the results obtained show excellent correlation with the experimental results for global behavior of the swollen cylinder presented by Treloar [5]. They have also been able to predict the distribution of the fluid in the interior of the swollen cylinder and the effect of axial strain on the volume of the swollen cylinder. Currently experimental results pertaining to such detailed field information of these phenomena are not available.

The mathematical basis of the general Theory of Interacting Continua has been well established for a long time. The historical development of the theory and comprehensive surveys of the progress in the field are presented in the review articles by Bowen [6], Atkin and Craine [7], Bedford and Drumheller [8] and Passman, et al.[9]. A critical review of the field makes it clearly evident that the applications of the Theory of Interacting Continua to solve boundary-value problems of physical interest have been very limited. The main difficulty in these problems arises due to the lack of physically obvious ways for specifying the partial tractions, which are an integral part of the Theory of Interacting Continua.

Shi, et al. [10] and Rajagopal, et al. [11] were the first to study equilibrium and steady-state boundary-value problems by employing auxiliary conditions at the boundary of solid-fluid mixtures in an effort to bypass the difficulties associated with specifying partial tractions at the boundary. The use of these auxiliary conditions rendered a whole class of boundary-value problems tractable where the boundary of the mixture could be assumed to be saturated.+ Gandhi, et al. [12,4,13,14] have not only used the above boundary condition to study boundary values problems, such as torsion of a cylindrical mixture, flexure of a mixture cuboid but have also contributed significantly to a better understanding of the saturation boundary condition by providing a strong mathematical basis to the same which was previously derived on an ad-hoc basis.

In previous work reported in the literature, attention has been primarily focused on a mixture of an ideal fluid and a nonlinearly elastic incompressible solid. In particular, for solving the boundary value problems all investigators have assumed the solid constituent to be "Neo-Hookean" which is a nonlinearly elastic incompressible solid. Boundary value problems on mixtures with a nonlinearly elastic compressible solid or compressible mixture have not been addressed in the literature. In this work an ideal fluid is assumed to be one of the constituent for all the mixtures under consideration ,while the other constituent may be an incompressible or a compressible nonlinearly elastic solid . Depending on the type of the solid in the mixture, the mixture will be referred to as an incompressible mixture or a compressible mixture for convenience.

The work presented herein is focused on identifying qualitative and/or quantitative differences in the behavior of incompressible

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+ A saturated state represents an equilibrium state in which material elements of a solid-fluid mixture in a deformed and swollen state are in contact with the fluid with no fluid entering or leaving the mixture.

mixtures with solid constituents being various types of nonlinearly elastic incompressible solids in the context of the Theory of Interacting Continua. In particular, the response of mixtures with Neo-Hookean solids and ideal fluids and mixtures with Mooney-Rivlin solids and ideal fluid is studied for several cases of simple deformations such as uniaxial extension, bi-axial extension and simple shear. The results of these investigations could have a significant impact on material identification studies, for instance solids exhibiting significant swelling characteristics, in their swollen states can be classified as Neo-Hookean-type or Mooney-Rivlin-type mixtures by performing simple experiments such as allowing solid specimens to swell freely and then, subjecting the specimens to uniaxial extension or shear, for example. Comparison of the results from this work with the experimental results of Treloar [1] is also presented. Furthermore, attention is also directed towards resolving issues which would permit the solution of boundary value problems involving compressible mixtures. The qualitative differences between the response characteristics of compressible and incompressible mixtures can be exploited in several practical applications.

This work also clarifies certain misconceptions pertaining to primitive concepts such as volume additivity and incompressibility of mixtures. In addition, the saturation equations of state for a mixture of a compressible solid and an ideal fluid are derived. These equations are instrumental in solving boundary problems where saturation is assumed. A brief rationale for deriving an appropriate free energy function for a compressible mixture is presented herein. Such a function would permit the explicit description of constitutive equations for studying boundary value problems involving compressible mixtures.

Finally, a boundary value problem focused on investigating the swelling characteristics of a composite cylinder is studied in detail. The composite cylinder features an inner core which exhibits material properties which are quite distinct from those of the surrounding concentric material. This work is motivated by the need for modeling the large deformations of polymeric composite materials under various under hygrothermal environments. This work partially addresses this need by characterizing the interaction of ideal fluids and idealized constrained nonlinear elastic solids in the context of Mixture Theory [15]. The deformations of the elastic solid may be restricted due to the presence of extensible/inextensible fibers, rigid inclusions and coatings on reinforcing fibers, for example. Such constraints can quantitatively and qualitatively alter the ability of these materials to undergo dimensional and constitutive changes very significantly. The results of these investigations for large deformations demonstrate that the constraints imposed by the core material induce nonhomogeneous swelling characteristics with significant gradients in the stretch ratios and severe stress concentrations at the fiber/matrix interface. These results could have important implications for a variety of fiber-reinforced composites featuring hygroscopic and temperature-sensitive matrix materials.

The composite cylinder considered herein is an archetypal representative volume element of a unidirectional fiber-reinforced incompressible elastic composite. And study of such a representative volume element when subjected to complicated loads could provide vital information about the neighborhood of fiber.

In chapter II the basic kinematic quantities are defined and basic postulates of the Theory of Interacting Continua are stated. Volume additivity, the incompressibility constraint, and the constitutive equations are stated in chapter III along with the derivation of saturation equations of state for a mixture of compressible solid-ideal fluid. Details of comparative studies between Neo-Hookean type solid-ideal fluid mixtures and Mooney-Rivlin type solid-ideal fluid mixtures are presented in chapter IV. In chapter V the boundary value problem of a cylinder with differential core properties is presented. Finally, concluding remarks and discussion on further application of this work to damage in composite material is presented.

CHAPTER II

PRELIMINARIES: NOTATIONS AND BASIC EQUATION

A brief review of the notations and basic equations of the Theory of Interacting Continua are presented in this section for completeness and continuity. Let Ω and Ω_t denote the reference configuration and the configuration of the body at time t, respectively. For a function defined on $\Omega \times \mathbb{R}$ and $\Omega_t \times \mathbb{R}$, ∇ and grad are used to represent the partial derivative with respect to Ω and Ω_t, respectively. Also $\frac{\partial}{\partial t}$ denotes partial derivative with respect to t. The divergence operator related to grad is denoted by div.

The solid-fluid aggregate will be considered a mixture with S_1 representing the solid and S_2 representing the fluid. At any instant of time t, it is assumed that each place in the space is occupied by particles belonging to both S_1 and S_2. Let $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ denote the reference positions of typical particles of S_1 and S_2. The motion of the solid and the fluid is represented by

$$\underset{\sim}{x} = \underset{\sim}{x}_1 (\underset{\sim}{X}, t), \quad \text{and } \underset{\sim}{y} = \underset{\sim}{x}_2 (\underset{\sim}{Y}, t). \tag{2.1}$$

These motions are assumed to be one-to-one, continuous and invertible. The various kinematical quantities associated with the solid S_1 and the fluid S_2 are

$$\text{Velocity:} \qquad \underset{\sim}{u} = \frac{D^{(1)}\underset{\sim}{x}_1}{Dt}, \qquad \underset{\sim}{v} = \frac{D^{(2)}\underset{\sim}{x}_2}{Dt}, \tag{2.2}$$

Acceleration: $$\underset{\sim}{f} = \frac{D^{(2)}\underset{\sim}{u}}{Dt}, \qquad \underset{\sim}{g} = \frac{D^{(2)}\underset{\sim}{v}}{Dt}, \tag{2.3}$$

Velocity gradient: $$\underset{\sim}{L} = \frac{\partial \underset{\sim}{u}}{\partial \underset{\sim}{x}}, \qquad \underset{\sim}{M} = \frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{y}}, \tag{2.4}$$

Rate of deformation tensor:

$$\underset{\sim}{D} = \frac{1}{2}(\underset{\sim}{L} + \underset{\sim}{L}^T), \ \underset{\sim}{N} = \frac{1}{2}(\underset{\sim}{M} + \underset{\sim}{M}^T) \tag{2.5}$$

where $D^{(1)}/Dt$ denotes differentiation with respect to t, holding $\underset{\sim}{X}$ fixed and $D^{(2)}/Dt$ denotes a similar operation holding $\underset{\sim}{Y}$ fixed. The deformation gradient $\underset{\sim}{F}$ associated with the solid is given by

$$\underset{\sim}{F} = \frac{\partial \underset{\sim}{x}_1}{\partial \underset{\sim}{X}}. \tag{2.6}$$

The total density of the mixture ρ and the mean velocity of the mixture $\underset{\sim}{w}$ are defined by

$$\rho = \rho_1 + \rho_2, \tag{2.7}$$

$$\rho\underset{\sim}{w} = \rho_1\underset{\sim}{u} + \rho_2\underset{\sim}{v}, \tag{2.8}$$

where ρ_1 and ρ_2 are the densities of the solid and the fluid in the mixed state, respectively, defined per unit volume of the mixture at time t.

The basic equations of the Theory of Interacting Continua are presented next.

(1) Conservation of mass

Assuming no interconversion of mass between the two interacting continua, the appropriate forms for the conservation of mass for the solid and the fluid are

$$\rho_1 \,|\det \underset{\sim}{F}| = \rho_{10}, \tag{2.9}$$

and

$$\frac{\partial \rho_2}{\partial t} + \operatorname{div}\,(\rho_2\, \underset{\sim}{v}) = 0, \tag{2.10}$$

where ρ_{10} is the mass density of the solid in the reference state.

(2) Conservation of linear momentum

Let $\underset{\sim}{\sigma}$ and $\underset{\sim}{\pi}$ denote the partial stress tensors associated with the solid S_1 and the fluid S_2, respectively. Then, assuming that there are no external body forces, the balance of linear momentum equations for the solid and fluid are given by

$$\operatorname{div} \underset{\sim}{\sigma} - \underset{\sim}{b} = \rho_1 \underset{\sim}{f}, \tag{2.11}$$

$$\operatorname{div} \underset{\sim}{\pi} + \underset{\sim}{b} = \rho_2 \underset{\sim}{g}. \tag{2.12}$$

In equations (2.11) and (2.12), $\underset{\sim}{b}$ denotes the interaction body force vector, which accounts for the mechanical interaction between the solid and the fluid. By defining the total stress as

$$\underset{\sim}{T} = \underset{\sim}{\sigma} + \underset{\sim}{\pi}, \tag{2.13}$$

the equilibrium equations for the mixture may be written as

$$\operatorname{div} \underset{\sim}{T} = \rho_1 \underset{\sim}{f} + \rho_2 \underset{\sim}{g}. \tag{2.14}$$

It may be pointed out that it is sufficient to satisfy any two of equations (2.11), (2.12) and (2.14) to satisfy the balance of linear momentum.

(3) Conservation of angular momentum

This condition states that

$$\underset{\sim}{\sigma} + \underset{\sim}{\pi} = \underset{\sim}{\sigma}^T + \underset{\sim}{\pi}^T. \tag{2.15}$$

However, the partial stresses $\underset{\sim}{\sigma}$ and $\underset{\sim}{\pi}$ need not be symmetric.

(4) Surface conditions

Let $\underset{\sim}{t}$ and $\underset{\sim}{q}$ denote the surface traction vectors taken by S_1 and S_2, respectively, and let $\underset{\sim}{n}$ denote the unit outer normal vector at a point on the surface of the mixture region. Then the partial surface tractions are related to the partial stress tensors by

$$\underset{\sim}{t} = \underset{\sim}{\sigma}^T\underset{\sim}{n}, \text{ and } \underset{\sim}{q} = \underset{\sim}{\pi}^T\underset{\sim}{n}. \tag{2.16}$$

(5) Thermodynamical considerations

The laws of conservation of energy and the entropy production inequality are not explicitly mentioned here for brevity. However, the relevant results are quoted. A complete discussion of these issues is presented in [14].

Let the Helmholtz free energy per unit mass of S_1 and S_2 be denoted by A_1 and A_2, respectively. The Helmholtz free energy per unit mass of the mixture is defined by

$$\rho A = \rho_1 A_1 + \rho_2 A_2. \tag{2.17}$$

Note that by setting

$$\underset{\sim}{b} = - \text{grad } \phi_1 + \bar{\underset{\sim}{b}} = \text{grad } \phi_2 + \bar{\underset{\sim}{b}}, \tag{2.18}$$

$$\underset{\sim}{\sigma} = \phi_1 \underset{\sim}{I} + \underset{\sim}{\bar{\sigma}}, \tag{2.19}$$

$$\underset{\sim}{\pi} = \phi_2 \underset{\sim}{I} + \underset{\sim}{\bar{\pi}}, \tag{2.20}$$

where,

$$\phi_1 = \rho_1(A_1-A), \ \phi_2 = \rho_2(A_2-A), \ \phi_1 + \phi_2 = 0, \tag{2.21}$$

equations (2.11), (2.12), (2.14) and (2.15) become

$$\text{div}\ \underset{\sim}{\bar{\sigma}} - \underset{\sim}{\bar{b}} = \rho_1 \underset{\sim}{f}, \tag{2.22}$$

$$\text{div}\ \underset{\sim}{\bar{\pi}} + \underset{\sim}{\bar{b}} = \rho_2 \underset{\sim}{g}, \tag{2.23}$$

$$\text{div}\ \underset{\sim}{\bar{T}} = \rho_1 \underset{\sim}{f} + \rho_2 \underset{\sim}{g}, \tag{2.24}$$

$$\underset{\sim}{\bar{\sigma}} + \underset{\sim}{\bar{\pi}} = \underset{\sim}{\bar{\sigma}}^T + \underset{\sim}{\bar{\pi}}^T. \tag{2.25}$$

The terms in $\underset{\sim}{\sigma}$, $\underset{\sim}{\pi}$ and $\underset{\sim}{b}$ which depend on ϕ_1 and ϕ_2 do not contribute to the equations of motion or the total stress.

CHAPTER III

VOLUME ADDITIVITY, INCOMPRESSIBILITY CONSTRAINT AND CONSTITUTIVE EQUATIONS

a) Volume additivity

Volume additivity is an intrinsic property of mixtures which simply says, the total volume of the mixture at any state (time or deformation) is equal to the sum of the respective volumes of the superimposed continua at that state. In the present notation it can be stated as

$$V_m = V_{S_1} + V_{S_2} \tag{3.1}$$

where V_{S_1} and V_{S_2} are the current volume of the respective constituents and V_m is the current volume of the mixture. (3.1) can also be written as

$$1 = \nu_1 + \nu_2 \ . \tag{3.2}$$

where ν_1, ν_2 are volume fractions of the respective constituents. Prior to this work volume additivity was misinterpreted as a constraint and was defined as volume of the mixture is the sum of the volumes of the respective continua in their reference state, which is true only if both the constituents of the mixture are incompressible. When two continua are superimposed to form a mixture, assuming no interconversion of mass, and both the constituents being chemically inert to each other the sum of the volumes of the constituents at any state (time or deformation) is equal to the total volume of the mixture at that state even though the individual constituents might undergo volume changes as in a case of compressible constituents, for example. Let the mixture under consideration constitute of a nonlinearly elastic solid S_1, and an ideal fluid S_2. Consider

$$\frac{\rho_1}{\rho_{10}} = \frac{M_{S_1}}{V_m}\,\frac{V_{RS_1}}{M_{S_1}} = \frac{V_{RS_1}}{V_m} = \frac{V_{RS_1}}{V_{S_1}}\,\frac{V_{S_1}}{V_m} = \frac{V_{RS_1}}{V_{S_1}}\,\nu_1 \tag{3.3}$$

where V_{RS_1} is the reference volume of the solid and V_{S_1} is the current volume of the solid. Similarly

$$\frac{\rho_2}{\rho_{10}} = \frac{V_{RS_2}}{V_m} = \frac{V_{S_2}}{V_m} = \nu_2 \tag{3.4}$$

where V_{RS_2} is the reference volume of the fluid and V_{S_2} is the current volume of the fluid. As the fluid is ideal in nature $V_{RS_2} = V_2$. Substituting (3.3)-(3.4) in (3.2) results into

$$\frac{V_{S_1}}{V_{RS_1}}\,\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1 \tag{3.5}$$

This relationship is valid for a mixture of any solid and an ideal fluid.

b) Incompressibility constraint

Now if the nonlinearly elastic solid is incompressible that is $V_{S_1} = V_{RS_1}$ then (3.5) yields

$$\frac{\rho_1}{\rho_{10}} + \frac{\rho_2}{\rho_{20}} = 1 \tag{3.6}$$

which can be considered as an incompressibility constraint relationship. This relationship was referred to as volume additivity constraint prior to this work [16].

c) CONSTITUTIVE EQUATIONS

i) Mixture of incompressible solid and an ideal fluid

A mixture of an elastic solid and a fluid is considered. The solid is assumed to be nonlinearly elastic and incompressible, the fluid is assumed to be ideal. Thus all the constitutive functions are required to depend on the following variables:

$$\underset{\sim}{F},\ \nabla\underset{\sim}{F},\ \rho_2,\ \text{grad}\ \rho_2,\ T,\ \text{grad}\ T,\ \underset{\sim}{u}\ \text{and}\ \underset{\sim}{v},$$

where T denotes the common absolute temperature of the solid and the fluid.

A lengthy but standard argument, based on the balance of energy, entropy production inequality, restrictions due to material frame indifference and the assumption that the solid is isotropic in its reference state, leads to the following results [10].

The constitutive equations are written in terms of the Helmholtz free energy function A per unit mass of the mixture, and the form of this function is given by

$$A = \hat{A}\ (I_1,\ I_2,\ I_3,\ \rho_2,\ T), \tag{3.7}$$

where I_1, I_2, I_3 are the principal invariants of $\underset{\sim}{B} = \underset{\sim}{F}\underset{\sim}{F}^T$ defined through

$$I_1 = \mathrm{tr}\ \underset{\sim}{B} \tag{3.8}$$

$$I_2 = \frac{1}{2}\ [(\mathrm{tr}\ \underset{\sim}{B})^2 - \mathrm{tr}\ \underset{\sim}{B}^2], \tag{3.9}$$

$$I_3 = \det\ \underset{\sim}{B} = (\det\ \underset{\sim}{F})^2. \tag{3.10}$$

Using (2.9), (2.26) and (3.4), I_3 can be expressed in terms of ρ_2 by the relation

$$I_3^{\ 1/2} = \det\ \underset{\sim}{F} = (1 - \rho_2/\rho_{20})^{-1}. \tag{3.11}$$

Furthermore, on restricting attention to isothermal conditions equation (3.1) reduces to

$$A = A\ (I_1,\ I_2,\ \rho_2). \tag{3.12}$$

The components of the partial stresses in the solid and fluid, and the interaction body force for isothermal conditions are given by

$$\bar{\sigma}_{ki} = -\ p\ \frac{\rho_1}{\rho_{10}}\ \delta_{ki} + 2\rho\ \left(\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\ I_1\right)\ B_{ki} - \frac{\partial A}{\partial I_2}\ B_{km}\ B_{mi}\right), \tag{3.13}$$

$$\bar{\pi}_{ki} = -\ p\ \frac{\rho_2}{\rho_{20}}\ \delta_{ki} - \rho\rho_2\ \frac{\partial A}{\partial \rho_2}\delta_{ki}, \tag{3.14}$$

$$\bar{b}_k = -\ \frac{p}{\rho_{10}}\ \frac{\partial \rho_1}{\partial x_k} + \rho_1\ \frac{\partial A}{\partial \rho_2}\ \frac{\partial \rho_2}{\partial x_k} - \rho_2\ \left(\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2}\ I_1\right)\delta_{i\ell}\right.$$

$$\left. - \frac{\partial A}{\partial I_2}\ B_{i\ell}\right)\ B_{i\ell,k} + \alpha\ \frac{\rho_1}{\rho_{10}}\ \frac{\rho_2}{\rho_{20}}\ (u_k - v_k). \tag{3.15}$$

In equations (3.7) - (3.9), p is a scalar which arises due to the volume additivity and incompressibility constraint. The constitutive parameter α accounts for a contribution to the interaction body force due to relative motion between the solid and the fluid. The interaction between the solid and the fluid is evident in these equations, where the partial stress of each constituent is affected by the deformed state of both the constituents. It is also useful to record the representation for the total stress

$$T_{ki} = \bar{\sigma}_{ki} + \bar{\pi}_{ki} = -p\delta_{ki} - \rho\rho_2 \frac{\partial A}{\partial \rho_2} \delta_{ki} + 2\rho \left[\left(\frac{\partial A}{\partial I_1} + \frac{\partial A}{\partial I_2} I_1 \right) B_{ki} - \frac{\partial A}{\partial I_2} B_{km} B_{mi} \right]. \tag{3.16}$$

In the remainder of this thesis, only $\bar{\underset{\sim}{\sigma}}$, and $\bar{\underset{\sim}{\pi}}$ and $\bar{\underset{\sim}{b}}$, will be used. Hence, for notational convenience, the superposed bars are dropped.

ii) <u>Mixture of compressible solid and an ideal fluid</u>

Green and Steel [17], Crochet and Naghdi [18] have presented the constitutive equations for a mixture of a nonlinearly elastic solid and a viscous fluid couple of decades ago and are not restated here for brevity. Rigorous verification of the constitutive equations given by the above researchers is not addressed here but at equilibrium of the solid and the fluid (at saturation) the constitutive equations derived by the above mentioned authors is in agreement with the constitutive equations obtained below.

Consider a block of pure solid compressible material which a unit cube in its unstrained unswollen state. The unit cube is placed in a bath of an ideal fluid and is subjected to a triaxial extension due to tractions on its surfaces and the absorption of the fluid. Eventually, the block, which is now a mixture of the solid and the liquid, reaches an equilibrium strained swollen state while in contact with the bath. This is what is meant by a saturation state. The mixture in the block has mass M_m and dimensions $\lambda_1, \lambda_2, \lambda_3$ in the direction of triaxial extension. In triaxial extension the triaxial directions are the principal directions. Hence, the derivation which follows would yield equations of state for principal directions only. Later, the equations of state will be tensorially transformed to general directions.

The condition for the equilibrium of the block with the surrounding liquid and thus of saturation is

$$\delta\,(M_m A) = \delta W \tag{3.17}$$

where, $A = A(I_1, I_2, I_3, \rho_2)$ is the Helmholtz free energy function per unit mass of the mixture. Equation (3.17) can be written as

$$M_m \delta A + A\,\delta M_m = \delta W \tag{3.18}$$

where

$$\delta A = \frac{\partial A}{\partial I_1} \delta I_1 + \frac{\partial A}{\partial I_2} \delta I_2 + \frac{\partial A}{\partial I_3} \delta I_3 + \frac{\partial A}{\partial \rho_2} \delta \rho_2 \tag{3.19}$$

Also, for triaxial extension the invarients of Cauchy-Green tensor,

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{3.20}$$

$$I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 \tag{3.21}$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{3.22}$$

The mass balance equation for the solid yields

$$\rho_1 = \frac{\rho_{10}}{\lambda_1\lambda_2\lambda_3} \tag{3.23}$$

The variations of I_1, I_2 and I_3 from equations (3.20)-(3.23) are given by

$$\delta I_1 = 2\lambda_1\delta\lambda_1 + 2\lambda_2\delta\lambda_2 + 2\lambda_3\delta\lambda_3 \tag{3.24}$$

$$\delta I_2 = 2(\lambda_2^2 + \lambda_3^2)\ \lambda_1\delta\lambda_1 + 2(\lambda_1^2 + \lambda_2^2)\ \lambda_2\delta\lambda_2 + 2(\lambda_2^2 + \lambda_1^2)\ \lambda_3\delta\lambda_3 \tag{3.25}$$

$$\delta I_3 = 2\lambda_1\lambda_2^2\lambda_3^2\delta\lambda_1 + 2\lambda_2\lambda_1^2\lambda_3^2\delta\lambda_2 + 2\lambda_3\lambda_2^2\lambda_1^2\delta\lambda_3 \tag{3.26}$$

The variation of ρ_1 from equation (3.23)

$$\delta\rho_1 = \rho_{10}\left(\frac{-1}{\lambda_1^2\lambda_2^2\lambda_3^2}\right)(\lambda_2\lambda_3\delta\lambda_1 + \lambda_1\lambda_3\delta\lambda_2 + \lambda_1\lambda_2\delta\lambda_3) \tag{3.27}$$

The mass of the mixture is related to the density of the mixture by

$$\begin{aligned} M_m &= \rho(\lambda_1\lambda_2\lambda_3) \\ &= \rho_1(\lambda_1\lambda_2\lambda_3) + \rho_2(\lambda_1\lambda_2\lambda_3) \end{aligned} \tag{3.28}$$

Hence, using (3.27)

$$\delta M_m = \delta\rho\ (\lambda_1\lambda_2\lambda_3) + \rho\ (\lambda_2\lambda_3\delta\lambda_1 + \lambda_1\lambda_3\delta\lambda_2 + \lambda_1\lambda_2\delta\lambda_3) \tag{3.29}$$

The tractions on the surfaces of the block are considered to be the total tractions. The expression for the virtual work done is given by

$$\delta W = T_{11}\lambda_2\lambda_3\delta\lambda_1 + T_{22}\lambda_1\lambda_3\delta\lambda_2 + T_{33}\lambda_1\lambda_2\delta\lambda_3 \tag{3.30}$$

Equation (3.19), (3.24)-(3.30) when substituted in equation (3.18) results

$$
\begin{aligned}
&[\rho(\lambda_1\lambda_2\lambda_3](A_1\delta I_1 + A_2\delta I_2 + A_3\delta I_3 + A\rho_2\delta\rho_2) + A(\lambda_1\lambda_2\lambda_3\delta\rho_2 \\
&+ \delta_2(\lambda_2\lambda_3\delta\lambda_1 + \lambda_1\lambda_3\delta\lambda_2 + \lambda_1\lambda_2\delta\lambda_3)) = T_{11}\lambda_2\lambda_3\delta\lambda_1 \\
&+ T_{22}\lambda_1\lambda_3\delta\lambda_2 + T_{33}\lambda_1\lambda_2\delta\lambda_3
\end{aligned}
$$

where $A_1 = \frac{\partial A}{\partial I_1}$, $A_2 = \frac{\partial A}{\partial I_2}$ and, $A_3 = \frac{\partial A}{\partial I_3}$

On further simplification the resulting equation must be satisfied for all arbitrary variations of λ_1, λ_2, λ_3, and ρ_2 the following saturation equations are obtained

$$T_{11} = A,\rho_2 + 2\rho\ [A_1\lambda_1^2 + A_2\lambda_1^2(\lambda_2^2 + \lambda_3^2) + A_3\lambda_1^2\lambda_2^2\lambda_3^2] \tag{3.31}$$

$$T_{22} = A,\rho_2 + 2\rho\ [A_1\lambda_2^2 + A_2\lambda_2^2(\lambda_1^2 + \lambda_3^2) + A_3\lambda_1^2\lambda_2^2\lambda_3^2] \tag{3.32}$$

$$T_{33} = A,\rho_2 + 2\rho\ [A_1\lambda_3^2 + A_2\lambda_3^2(\lambda_2^2 + \lambda_1^2) + A_3\lambda_1^2\lambda_2^2\lambda_3^2] \tag{3.33}$$

and

$$\rho(\lambda_1\lambda_2\lambda_3)\ A,_{\rho 2}\delta\rho_2 + A(\lambda_1\lambda_2\lambda_3)\delta\rho_2 = 0$$

or

$$\rho A,_{\rho 2} + A = 0 \tag{3.34}$$

where $A,_{\rho_2} = \frac{\partial A}{\partial \rho_2}$.

It is essential to note here that now four equations are available instead of three, as obtained incase of the mixture constituting only incompressible constituents [2].

iii) <u>Free energy function for a mixture of a nonlinearly elastic compressible solid and an ideal fluid.</u>

The free energy function for the mixture of a solid and an ideal fluid is obtained by adding the change in free energy due to mixing of the solid and the fluid and the elastic free energy of the solid. The basis for adding up the two quantities is not clear but does seem reasonable. The elastic free energy function, that of the solid can be written as

$$A_s = A_s(I_1, I_2, I_3) \tag{3.35}$$

where I_1, I_2, I_3 are these invarients of the Cauchy Green tensor. The free energy function reduces to zero when the solid is stress free.

The free energy of mixing A_m of the solid and the fluid is complicated as compared to that of the solid and is obtained by considering the change in Gibbs free energy when 'n_1' moles of fluid are mixed with the solid as given by Flory-Huggins [19] equations stated below.

$$\frac{\partial \Delta G_m}{\partial n_1} = RT\ \{\ln\ (1 - \nu_1) + \nu_1 + \chi \nu_1^2\} \tag{3.36}$$

where

R = universal gas constant

T = temperature

ΔG_m = change in the Gibbs free energy

χ = interaction parameter that depends on the combination of the specific solid and the fluid.

If V_1 is the molar volume of the fluid in the mixture, n_1 number of moles present at any state and V_S volume of the solid at that state, then volume additivity can be written as

$$V_m = V_S + n_1V_1 \tag{3.37}$$

From (3.4)

$$\frac{\rho_2}{\rho_{20}} = \frac{n_1V_1}{V_m} \tag{3.38}$$

Using (2.9), (3.38) can be stated as

$$n_1 = \frac{\rho_2 I_3^{\frac{1}{2}} V_{RS}}{V_1 \rho_{20}} \tag{3.39}$$

Differentiating (3.39) with respect to ρ_2 yields

$$\frac{\partial n_1}{\partial \rho_2} = \frac{I_3^{\frac{1}{2}}}{V_1 \rho_{20}} \left[1 - \frac{\rho_2}{2I_3} \frac{\partial I_3}{\partial \rho_2}\right] \tag{3.40}$$

Since the fluid is ideal and hence incompressible (3.5) can be written as

$$1 = \nu_1 + \frac{\rho_2}{\rho_{20}} \tag{3.41}$$

Differentiating (3.41) with respect to ρ_2 we have

$$\frac{\partial \nu_1}{\partial \rho_2} = \frac{-1}{\rho_{20}} \tag{3.42}$$

Now (3.30) can be written as

$$\frac{\partial \Delta G_m}{\partial n_1} = \frac{\partial \Delta G_m}{\partial \nu_1} \frac{\partial \nu_1}{\partial \rho_2} \frac{\partial \rho_2}{\partial \nu_1} \tag{3.43}$$

Using (3.41) and (3.42), (3.43) yields

$$\frac{\partial \Delta G_m}{\partial \nu_1} = \frac{-RT}{V_1}\left[\ln(1-\nu_1) + \nu_1 + \chi\nu_1^2\right](I_3^{½})\left[1 - \frac{\rho_2}{2I_3\rho_{20}}\frac{\partial I_3}{\partial \nu_1}\right] \quad (3.44)$$

Considering mass balance (2.9) and rewriting yields,

$$I_3^{½} = \frac{V_S}{V_{RS}\nu_1} \quad (3.45)$$

Differentiating with respect to ν_1, substituting in (3.44) yields

$$\frac{\partial \Delta Gm}{\partial \nu_1} = -\frac{RT}{V_1}\left(\frac{\ln(1-\nu_1)}{\nu_1} + 1 + \chi\nu_1\right)\left(\frac{V_S}{\nu_1} - (1-\nu_1)\frac{\partial V_S}{\partial \nu_1}\right) \quad (3.46)$$

Integrating with respect to ν_1, we get

$$\Delta Gm = \frac{-RT}{V_1}\int(\ln(1-\nu_1) + \nu_1 + \chi\nu_1^2)\left[I_3^{½} + \frac{(1-\nu_1)}{2I_3^{½}}\frac{\partial I_3}{\partial \nu_1}\right]\partial\nu_1 + C \quad (3.47)$$

Where C is the constant of integration which is evaluated considering no change in Gibbs free energy in absence of solid constituent results in

$C = 0$.

The thermodynamic system includes the mixture and the infinite bath surrounding of the fluid. Thus the total volume change of the thermodynamic system can be assumed to be zero and hence

$$\Delta G_m = \Delta A$$

The free energy of mixing per unit mass of the mixture becomes,

$$A_m = \Delta A = \frac{-RT\ \nu_1}{V_1\ \rho\ V_s M_m} \int \left(\frac{\ln(1-\nu_1)}{\nu_1} + 1 + \chi\nu_1 \right) \left(\frac{V_s}{\nu_1} - (1-\nu_1)\frac{\partial V_s}{\partial \nu_1} \right) \partial \nu_1 \quad (3.48)$$

The total free energy per unit mass of the mixture is thus given as follows

$$A = A_s + A_m , \quad (3.49)$$

where, A_s and A_m are defined per unit mass of the mixture

Now consider a mixture of an incompressible nonlinearly elastic solid such as a Neo-Hookean solid and an ideal fluid, the elastic free energy per unit mass is given by

$$A_e = \frac{\nu_1 \rho_{10}\ RT}{2\ \rho} (I_1 - 3) \quad (3.50)$$

The free energy of mixing is obtained by letting the volume of the solid V_S = constant, i. e. $V_S = V_{RS}$, equation (3.48) becomes

$$A_m = \frac{\nu_1\ RT}{\rho\ \ V_1} \left(\frac{(1-\nu_1)\ln(1-\nu_1)}{\nu_1} + (1 - \nu_1)\chi \right) \quad (3.51)$$

It should be noted that the total free energy per unit mass of the mixture obtained by substituting (3.50) and (3.51) in to (3.49) is the same as derived and employed by Gandhi et al [4] for an incompressible solid-fluid mixture.

It is clear that the free energy function in case of a mixture of a compressible nonlinearly elastic solid and an ideal fluid can only be obtained if the variation of the volume of the solid is known when a known amount of fluid has entered the mixture. Hence specific boundary value problem with compressible constituents cannot be solved unless experimental results are available.

Physical interpretation of the free energy function of mixing.

The free energy function of mixing can be obtained in the closed form if and only if the variation in volume of the mixture is known when a known quantity of fluid enters the mixture. In other words the change in the volume of the solid with change in the fluid content of the system. This relationship needs to be experimentally determined. Currently such experimental results are not available.

CHAPTER IV

MATERIAL CHARACTERIZATION USING SIMPLE DEFORMATIONS OF A CUBOID MIXTURE

The problem of swelling of an incompressible solid cube under simple deformations is reported in this chapter. A unit cube of nonlinearly elastic material is placed in an infinite bath of an ideal fluid and then is allowed to swell freely untill it attains saturation. The swollen cuboid is then subjected to uniaxial extension, equibiaxial extension and shear. The results obtained have a significant impact on material characterization or identification of incompressible solids. Let (X_1, X_2, X_3) denote the position of a typical particle in reference configuration. The particle denoted by (X_1, X_2, X_3) in the reference configuration may be represented in deformed configuration by the coordinates (x_1, x_2, x_3). The problem is formulated in a generalized form and then reduced to the specific case of uniaxial, biaxial and shear deformation as follows,

The general form of the deformation field is assumed to be

$$
\begin{aligned}
x_1 &= \lambda_1 (X_1 + CX_2) \\
x_2 &= \lambda_2 X_2 \\
x_3 &= \lambda_3 X_3
\end{aligned}
\tag{4.1}
$$

The deformation gradient $\underset{\sim}{F}$ associated with the mixture is given by

$$\underset{\sim}{F} = \begin{bmatrix} \lambda_1 & \lambda_1 C & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \tag{4.2}$$

The Cauchy-Green strain tensor $\underset{\sim}{B}$ which is defined as

$$\underset{\sim}{B} = \underset{\sim}{F}\,\underset{\sim}{F}^T \tag{4.3}$$

takes the following form for the above deformation:

$$\underset{\sim}{B} = \begin{bmatrix} \lambda_1^2(1+C^2) & C\lambda_1\lambda_2 & 0 \\ c\lambda_1\lambda_2 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} \tag{4.4}$$

The principal invariants of $\underset{\sim}{B}$ are given as

$$I_1 = \lambda_1^2\,(1 + C^2) + \lambda_2^2 + \lambda_3^2 \tag{4.5}$$

$$I_2 = \lambda_1^2\lambda_2^2\,(1 + C^2) + \lambda_2^2\lambda_2^2\,(1 + C^2) \tag{4.6}$$

$$I_3 = \lambda_1^2\lambda_2^2\lambda_3^2 \tag{4.7}$$

The balance of mass equation for the solid constituent (2.9) may be expressed in terms of the stretch ratios as

$$\frac{\rho_1}{\rho_{10}} = \frac{1}{\lambda_1\lambda_2\lambda_3} = \nu_1 \tag{4.8}$$

where ν_1 represents the volume fraction of the solid. The equilibrium equations are expressed in terms of the coordinates in the reference configuration for computational convenience. Assuming no external body forces, the equations of equilibrium for the mixture and solid take the form

$$\frac{\partial \sigma_{ij}}{\partial x_p} \quad F_{pj}^{-1} = 0 \tag{4.9}$$

$$\frac{\partial \pi_{ij}}{\partial x_p} \quad F_{pj}^{-1} = 0 \tag{4.10}$$

$$\frac{\partial T_{ij}}{\partial_{xp}} \quad F_{pj}^{-1} = 0 \tag{4.11}$$

The tensor $\underset{\sim}{F}^{-1}$ that appears in these equations has the form given by

$$\underset{\sim}{F}^{-1} = \begin{bmatrix} 1/\lambda_1 & -C/\lambda_2 & 0 \\ 0 & 1/\lambda_2 & 0 \\ 0 & 0 & 1/\lambda_3 \end{bmatrix} \tag{4.12}$$

For the deformation field under consideration, (3.13)-(3.15) become,

$$\sigma_{11} = - p \frac{\rho_1}{\rho_{10}} + 2\rho \Big((A_1 + A_2 I_1) \lambda_1^2 (1+C^2) - A_2[\lambda_1{}^4 (1 + C^2)^2 + C^2 \lambda_1^2 \lambda_2^2] \Big) \tag{4.13}$$

$$\sigma_{22} = -p \frac{\rho_1}{\rho_{10}} + 2\rho \Big((A_1 + A_2 I_1)\lambda_2^2 - A_2 \; (C^2 \lambda_1^2 \lambda_2^2 + \lambda_2^4) \Big) \tag{4.14}$$

$$\sigma_{33} = -p \frac{\rho_1}{\rho_{10}} + 2\rho \Big((A_1 + A_2 I_1)\lambda_3^2 - A_2 \lambda_3^4 \Big) \tag{4.15}$$

$$\sigma_{12} = 2\rho \Big((A_1 + A_2 I_1) C \lambda_1 \lambda_2 - A_2 [C \lambda_1^3 \lambda_2 (1+C^3) + C \lambda_1 \lambda_2^3] \Big) \tag{4.16}$$

$$\sigma_{13} = \sigma_{23} = 0, \tag{4.17}$$

$$\pi_{11} = \pi_{22} = \pi_{33} = -p\,\frac{\rho_2}{\rho_{20}} - \rho\rho_2\,\frac{\partial A}{\partial \rho_2} \tag{4.18}$$

$$b_k = 0 \qquad k = 1,2,3 \tag{4.19}$$

where,

$$A_1 = \frac{\partial A}{\partial I_1} \text{ and } A_2 = \frac{\partial A}{\partial I_2}.$$

It is sufficient to satisfy any two of the three equilibrium equations (4.9)-(4.11) for the deformation field under consideration, the third equilibrium equation is automatically satisfied. By virtue of equation (3.16) total stresses are given as follows:

$$T_{11} = -p - \rho_{\rho_2}\frac{\partial A}{\partial \rho_2} + 2\rho\Big[(A_1 + A_2I_1)\lambda_1^2(1 + C^2) - A_2(\lambda_1^4(1 + C^2)^2 + C^2\lambda_1^2\lambda_2^2)\Big] \tag{4.20}$$

$$T_{22} = -\,p - \rho_{\rho_2}\frac{\partial A}{\partial \rho_2} + 2\rho\Big[(A_1 + A_2I_2)\lambda_2^2 - A_2(C^2\lambda_1^2\lambda_2^2 + \lambda_2^4)\Big] \tag{4.21}$$

$$T_{33} = -\,p - \rho_{\rho_2}\frac{\partial A}{\partial \rho} + 2\rho\Big[(A_1 + A_2I_1)\lambda_3^3 - A_2\lambda_3^4\Big] \tag{4.22}$$

$$T_{12} = 2\rho\Big[(A_1 + A_2I_1)C\lambda_1\lambda_2 - A_2(C\lambda_1^3\lambda_2(1 + C^2) + C\lambda_1\lambda_2^3)\Big] \tag{4.23}$$

$$T_{13} = T_{23} = 0 \tag{4.24}$$

In the subsequent part of this section special cases of free swelling, uniaxial extension, biaxial extension and shear are considered.

a. Free Swelling of a Unit Cube

In the case of free swelling the cube of unit length is allowed to swell freely untill it achieves saturation and, at saturation the total stress on its boundary vanishes, hence

$$C = 0, \tag{4.25}$$

$T_{ij} = 0$, on boundary of swollen cuboid yields

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda \tag{4.26}$$

$$p = -\rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_1)\lambda^2 - A_2\lambda^4 \right] \tag{4.27}$$

and

$$T_{12} = T_{13} = T_{23} = 0. \tag{4.28}$$

b. Uniaxial Extension

The swollen cube is stretched in x_1 direction which implies that the stretches in the other two directions are equal and hence,

$$\lambda_2 = \lambda_3 = \lambda, \tag{4.29}$$

$$C = 0, \tag{4.30}$$

and total stresses are given by

$$T_{11} = -p - \rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_1)\lambda_1^2 - A_2\lambda_1^4 \right] \tag{4.31}$$

$$T_{22} = T_{33} = 0$$

$$p = -\rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_1)\lambda^2 - A_2\lambda^4 \right] \tag{4.32}$$

and

$$T_{12} = T_{13} = T_{23} = 0 \tag{4.33}$$

c. Equibiaxial Extension

The case of equibiaxial extension is given by

$$\lambda_1 = \lambda_2 = \lambda, \tag{4.34}$$

$$C = 0 \tag{4.35}$$

and total stresses are given by

$$T_{11} = T_{22} = -p \; -\rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_1)\lambda^2 - A_2\lambda^4 \right] \tag{4.36}$$

$$T_{33} = 0$$

$$p = -\rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_2)\lambda_3^2 - A_2{\lambda_3}^4 \right] \text{ and,} \tag{4.37}$$

$$T_{12} = T_{13} = T_{23} = 0 \tag{4.38}$$

d. Simple Shear

In the case of simple shear the unit cube is allowed to swell freely untill it achieves saturation and then is sheared by an angle γ, i. e. shear is superimposed on a freely swollen cuboid. The stretch ratios ratios take form as follows,

$$\lambda_1 = \lambda_2 = \lambda_3 = \lambda, \tag{4.39}$$

$$C = \tan(\gamma) \tag{4.40}$$

and total stresses given by

$$T_{11} = -p - \rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_1)\, \lambda^2 (1 + \tan^2\gamma\,) - A_2\lambda^4\, (1 + 3\tan^2\gamma \; + \tan^4\gamma\,) \right] \tag{4.41}$$

$$T_{22} = -p - \rho\rho_2 \frac{\partial A}{\partial \rho_2} + 2\rho \left[(A_1 + A_2 I_1)\lambda^2 - A_2\lambda^4 (1 + \tan^2\gamma) \right] \tag{4.42}$$

$$T_{33} = 0$$

$$p = -\rho\rho_2 \frac{\partial A}{\partial \rho_2}\; 2\rho \left[(A_1 + A_2 I_1)\lambda^2 - A_2\lambda^4 \right] \tag{4.43}$$

$$T_{12} = 2\rho \left[\tan\gamma(A_1 + A_2 I_1)\lambda^2 - A_2(\tan\gamma(1 + \tan\gamma^2)\lambda^4 + \tan\gamma\lambda^4) \right] \tag{4.44}$$

and

$$T_{13} = T_{23} = 0. \tag{4.45}$$

Cases (a-c) cannot be solved for stretch ratios due to the presence of indeterminate scalar p. To eliminate the indeterminate scalar p saturation boundary condition is assumed where p becomes determinate [2].

The equations of state at saturation for a mixture of an incompressible solid and an ideal fluid were obtained by Gandhi et al are stated here for completeness

$$T_{ij} = [\rho_{20}A + \rho A_{\rho_2}(\rho_{20} - \rho_2)]\,\delta_{ij} + 2\rho\,[(A_1 + A_2 I_1)B_{ij} - A_2 B_{im}B_{mj}] \qquad (4.46)$$

Comparing (4.46) with (4.20-4.22) , it is observed that at saturation the indeterminate scalar p becomes determinate and is given by

$$p = -\rho\rho_{20}A_{\rho_2} - \rho_{20}A \qquad (4.47)$$

Now assuming that saturation is attained by the swollen cuboid when subjected to uniaxial extension, and equibiaxial extension. And then substituting equation (4.47) for p in equations (4.27), (4.32) and, (4.37) and the variation of stretch ratios, stresses and volume fractions is obtained. The numerical example and discussion on these results is presented next.

NUMERICAL EXAMPLE AND DISCUSSION

The explicit forms of equations (4.7), (4.8), (4.13) and (4.14) may be obtained for a specific choice of the Helmholtz free energy function A. The free energy function per unit mass of the mixture is assumed to be given [6] by

$$A = \frac{\nu_1}{\rho}\left[C_1\,(I_1 - 3) + C_2(I_2 - 3) + \frac{RT}{V_1}\left[\frac{1-\nu_1}{\nu_1}\,\ell n(1-\nu_1) + \chi(1-\nu_1)\right]\right], \qquad (4.48)$$

where,

V_1 is the molar volume of the fluid,

χ is the constant which depends on the particular combination of the solid and the fluid,

R is the universal gas constant,

T is the absolute temperature,

M_c is the molecular weight of the solid.

The free energy function given by equation (4.48) is for a "Mooney-Rivlin type" nonlinear solid fluid mixture. When $C_2 = 0$ the free energy function represents a "Neo-Hookean type" nonlinear solid mixture.

For numerical calculations the following material properties as given by Treloar [1] were used:

Density of natural rubber in the reference state	ρ_{10}	$= .9016$	gm/cc
Density of solvent (benzene) in the reference state	ρ_{20}	$= .862$	gm/cc
Molar volume of the solvent	V_1	$= 106$	cc/mole
The molecular weight of rubber between cross links	M_c	$= 9151$	gm/mole
Natural rubber-benzene interaction constant	χ	$= .425$	

The numerical value of the universal gas constant R is given by 8.317×10^7 Dyne-cm/mole - °K, and the absolute temperature T was assumed to be 303.16°K. The constitutive co-efficients C_1 and C_2 are given as follows:

$$C_1 = \frac{RT\rho_{10}}{M_c} \tag{4.49}$$

$$C_2 = \alpha C_1 \tag{4.50}$$

Numerical results presented in figures (1-6) for a cuboid mixture under simple deformations are for both Mooney-Rivlin and Neo-Hookean type mixture.

The variation of the stretch ratios for uniaxial extension of a cuboid mixture is presented in figure 1. Corresponding experimental results obtained by Treloar [1] are also presented on the same graph for reference. The experimental results are in excellent agreement for tension and not quantitatively in agreement for the compression case. It is observed that qualitatively there is no difference between Neo-Hookean type and Mooney-Rivlin type mixtures. Figure 1 clearly shows that Neo-Hookean mixtures admit more states as compared to Mooney-Rivlin type mixtures. The variation of the volume fraction of the solid in the mixture with the stretch ratio is presented in figure 2. Figure 2 conveys that Neo-Hookean solids absorb more fluid to achieve saturation as compared to Mooney-Rivlin solids. These results have a substantial bearing on material characterization or material identification and, can be exploited as follows. Consider a block of rubber of unknown properties and known dimensions, place it in a fluid bath for a sufficient length of time so that the block is fully saturated. Then stretch the block in x_1 direction by some amount. Measure the new dimensions of the swollen block and obtain the stretch ratios in the other two directions. Place the value of the proper stretch ratios in figure 1 and relate its position to the existing variation for Neo-Hookean and various Mooney-Rivlin mixtures and hence the material can be identified. Also figure 2 can be used for the same purpose if the amount of fluid required for complete saturation is measured. A parametric study can be done to incorporate the values of χ which depends on the particular combination of the solid and the fluid. The variation of χ for all known combinations is very small.

Similar results are obtained for Equi-Biaxial extension and the variation of the stretch ratios is presented in figure 3. The variation of the stresses for the Neo-Hookean and various Mooney-Rivlin mixtures subjected to simple shear with shear angle is presented in figures 4, 5 and 6. These results clearly satisfy Universal relationship for simple shear. Linear behavior of stresses in Neo-Hookean mixtures is observed as anticipated whereas nonlinear nature of variation of stresses is seen in case of Mooney-Rivlin mixtures.

CHAPTER V

INTERACTION OF CONSTRAINED NONLINEAR ELASTIC SOLIDS AND IDEAL FLUIDS

The problem

The behavior of a constrained nonlinear elastic solid in the presence of an ideal fluid is investigated in this section. The constraints under consideration are those which result due to the presence of two or more elastic materials, rigid inclusions or inextensible/extensible fibers, for example. The representative problem is presented in order to demonstrate the role of constraints in modifying the swelling characteristics of reinforced elastic solids.

a. Swelling of a Composite Cylindrical Mixture with differential core properties.

Consider a solid cylinder composed of two different materials M_1 and M_2, with the material M_1 occupying the region $R \epsilon [0,R_i]$ and the material M_2 occupying the region $R \epsilon [R_i,R_o]$ in the reference configuration, such that, $R_o > R_i > 0$. Both cylinders are assumed to have a length L_0. A schematic of the reference configuration and the current configuration is presented in figure 7. It is assumed that both materials are perfectly bonded to each other at the interface. The co-ordinates of a typical material particle in the reference configuration will be denoted by cylindrical co-ordinates (R,θ,Z). In the deformed swollen state the co-ordinates of the same particle are assumed to be described by

$$r = r(R),\ \theta = \Theta,\ \text{and } z = \lambda Z, \tag{5.1}$$

where λ is a constant axial stretch ratio assumed to be unity. The deformation gradient associated with the mixture is,

$$F = \begin{bmatrix} \lambda_r & 0 & 0 \\ 0 & \lambda_\theta & 0 \\ 0 & 0 & \lambda \end{bmatrix} \tag{5.2}$$

The Cauchy Green tensor $\underset{\sim}{B}$ which is defined as

$$B = FF^T \tag{5.3}$$

takes the following form for the above deformation:

$$\underset{\sim}{B} = \begin{bmatrix} \lambda_r^2 & 0 & 0 \\ 0 & \lambda_\theta^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} \tag{5.4}$$

where $\lambda_r = dr/dR$ and $\lambda_\theta = r/R$ denote the stretch ratios in the r and θ directions. The principal invariants of B are then given as

$$I_1 = \lambda_r^2 + \lambda_\theta^2 + \lambda^2, \tag{5.5}$$

$$I_2 = \lambda^2(\lambda_r^2 + \lambda_\theta^2) + \lambda_\theta^2\lambda_r^2 \quad \text{and ,} \tag{5.6}$$

$$I_3 = \lambda_r^2\lambda_\theta^2\lambda^2 \ . \tag{5.7}$$

The balance of mass equation for the solid constituent (2.9) may be expressed in terms of the stretch ratios or the volume fraction of the solid ν_1 as

$$\frac{\rho_1}{\rho_{10}} = \frac{1}{\lambda_r\lambda_\theta\lambda} = \nu_1 \ . \tag{5.8}$$

The equations of equilibrium which are appropriate for the deformation being considered are documented next. Since the assumed form of deformation implies that the stresses depend only on the radial coordinate r , the equations of equilibrium for the solid constituent , namely (2.11), reduce to

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} - b_r = 0 \tag{5.9}$$

where σ_{rr} and $\sigma_{\theta\theta}$ denote appropriate components of σ , and b_r denotes the component of the interaction body force b in the radial direction. The equilibrium equations for the fluid constituent, namely (2.12), reduce to

$$\frac{d\pi_{rr}}{dr} + \frac{\pi_{rr} - \pi_{\theta\theta}}{r} + b_r = 0 \tag{5.10}$$

where π_{rr} and $\pi_{\theta\theta}$ denote the components of π. Equations (5.9), (5.10) and (2.14) yield

$$\frac{dT_{rr}}{dr} + \frac{T_{rr} - T_{\theta\theta}}{r} = 0 \tag{5.11}$$

which is the equation of equilibrium for the mixture.

For the deformation under consideration, it follows from (5.3) and equations (3.13) - (3.15) that the non-zero components of the partial stress tensors for the solid and fluid constituents are given by

$$\sigma_{rr} = -P\frac{\rho_1}{\rho_{10}} + 2\rho\Big[(A_1 + A_2 I_1)\lambda_r^2 - A_2\lambda_r^4\Big] , \tag{5.12}$$

$$\sigma_{\theta\theta} = -P\frac{\rho_1}{\rho_{10}} + 2\rho\Big[(A_1 + A_2 I_1)\lambda_\theta^2 - A_2\lambda_\theta^4\Big] , \tag{5.13}$$

$$\sigma_{zz} = -P \frac{\rho_1}{\rho_{10}} + 2\rho\left[(A_1 + A_2 I_1)\lambda^2 - A_2\lambda^4\right] \text{ and,} \tag{5.14}$$

$$\pi_{rr} = \pi_{\theta\theta} = \pi_{zz} = - P \frac{\rho_2}{\rho_{20}} - \rho\rho_2 A_{\rho_2}. \tag{5.15}$$

The only non-zero component of interaction body force is given by

$$b_r = - \frac{P}{\rho_{10}\lambda_r} \frac{d\rho_1}{dr} + \rho_1 \alpha_{\rho_2} \frac{d\rho_2}{dr} \rho_2 \left[(A_1 + A_2 I_1) \frac{d\,(\lambda_r^2 + \lambda_\theta^2)}{\lambda_r dr} - \right.$$

$$\left. 2A_2(\lambda_r^2 \frac{d\lambda_r}{dr} + \frac{\lambda_\theta^3}{\lambda_r} \frac{d\lambda_\theta}{dr}) \right] \tag{5.16}$$

where, $A_1 = \frac{\partial A}{\partial I_1}$, $A_2 = \frac{\partial A}{\partial I_2}$ and, $A_{\rho_2} = \frac{\partial A}{\partial \rho_2}$.

It is sufficient to satisfy any two of the three equilibrium equations (5.9) - (5.11). Equations (5.12) - (5.15) are substituted into the equilibrium equations for the solid and the mixture (5.9) and (5.11), respectively, to get the following functional functional form of the equilibrium equations, which are stated in terms of the co-ordinates in the reference configuration for computational convenience:

$$- \frac{dp}{dR} \frac{\rho_1}{\rho_{10}} + g_1\,(A_1, A_2, A_{\rho_2}, \lambda_r, \lambda_\theta, R, \lambda_r', \lambda_\theta', \lambda) = 0 \tag{5.17}$$

The mixture is assumed to be of a "New-Hookean-type", that is, A is a linear function of I_1. Following this assumption the explicit forms of the equilibrium equations for the mixture and the solid for the deformation under consideration are given by

$$- \frac{dp}{dR} - \frac{d}{dR}\left(\rho\rho_2 \frac{\partial A}{\partial \rho_2}\right) + 2\rho\, A_1\lambda_r \left[\frac{d\lambda_r}{dR} - \frac{\lambda_r}{\lambda_\theta R}\left(\lambda_r - \lambda_\theta\right)\right.$$

$$\left. - \frac{1}{\lambda_\theta R}\left(\lambda_\theta^2 - \lambda_r^2\right)\right] = 0 \qquad (5.18)$$

and

$$- \nu_1 \frac{dp}{dR} + 2\rho\, A_1\lambda_r \left[\frac{d\lambda_r}{dR} - \frac{\lambda_r}{\lambda_\theta R}\left(\lambda_r - \lambda_\theta\right) - \frac{1}{\lambda_\theta R}\left(\lambda_\theta^2 - \lambda_r^2\right)\right]$$

$$- \nu_1\, \rho_1\rho_{20} \frac{\partial A}{\partial \rho_2}\left\{\frac{1}{\lambda_\theta}\frac{\lambda_r - \lambda_\theta}{R} + \frac{1}{\lambda_r}\frac{d\lambda_r}{dR}\right\} + \rho_2 A_1 \frac{d}{dR}\left\{\lambda_r^2 - \lambda_\theta^2\right\} = 0,$$

respectively (5.19)

In equations (5.18) and (5.19) the radial and the tangential stretch ratio λ_r and λ_θ, respectively, are related through the compatibility condition given by

$$\frac{d\lambda_\theta}{dR} = \frac{\lambda_r - \lambda_\theta}{R} . \qquad (5.20)$$

Equations (5.18) and (5.19) may be solved for p, λ_r and λ_θ once the specific form of the Helmholtz free energy function for the mixture is known, and the appropriate boundary conditions are specified. The Helmholtz free energy function per unit mass of the mixture is assumed to be given by

$$A = \frac{\nu_1}{\rho}\left[\frac{RT\rho_{10}}{2M_c}(I_1 - 3) + \frac{RT}{V_1}\left[\frac{1-\nu_1}{\nu_1}\ell n(1-\nu_1) + \chi(1-\nu_1)\right]\right] \qquad (5.21)$$

where, R, T, V_1, M_c, and χ are constants [1]. The appropriate boundary conditions for solving the set of ordinary differential equations (5.18) and (5.19) in regions $[0,R_o]$ are given by

$$\lambda_r(0) = \lambda_\theta(0) \ , \tag{5.22}$$

$$T_{rr}^{(1)}(R_i) = T_{rr}^{(2)}(R_i) \ , \tag{5.23}$$

$$\lambda_\theta^{(1)}(R_i) = \lambda_\theta^{(2)}(R_i) \ , \text{ and} \tag{5.24}$$

$$T_{rr}(R_o) = 0 \ . \tag{5.25}$$

The boundary condition given by equation (5.22) arises due to the compatibility requirement between the radial and tangential stretch ratios at the axis of the cylinder. The boundary condition (5.23) on radial stress tensor at $R = R_i$ is due to the requirement that the radial stress be continuous at the interface. The boundary condition (5.24) arises due to the assumption of perfect bonding at the interface. The boundary condition on total traction vector represented by (5.25) is a consequence of the requirement that the outer surface of the composite cylinder be traction-free. Since a boundary condition for the partial traction vectors is not physically obvious, following the arguments presented in [10, 11, 12] it is assumed that the outer surface of the cylinder is in a saturated state. This assumption results in the boundary condition represented by

$$S_{rr}(R_o) = 0 \ , \tag{5.26}$$

where, S_{rr} represents the radial stress component for a saturated state, and is given by [15]

$$S_{rr} = \rho \ (\rho_{20} - \rho_2) \frac{\partial A}{\partial \rho_2} + \rho_{20} A + 2 \ \rho \ A_1 \lambda_r^2 \ . \tag{5.27}$$

For computational convenience, equations (5.18) and (5.19) may be combined to eliminate p, and for the Helmholtz free energy function given by (5.21) the resulting equation is given by

$$\frac{R\lambda_\theta}{\lambda_r}\frac{d\lambda_r}{dR} = - \frac{(\lambda_r - \lambda_\theta)\left[K\left(2\chi - \frac{1}{1-\nu_1}\right)\nu_1 - \lambda_r\lambda_\theta\right]}{K\left(2\chi - \frac{1}{1-\nu_1}\right)\nu_1 - \lambda_r^2} \qquad (5.28)$$

where,

$$K = \frac{M_c}{\rho_{10}V_1} .$$

The set of ordinary differential equations given by (5.20) and (5.28) subjected to boundary conditions given by (5.22) - (5.25) were solved numerically. For the computational work the following properties were employed [1,6a]:

	M_1	M_2	
Density of rubber in the reference state	ρ_{10} = .9016	.9016	mg/cc
Density of solvent in the reference state	ρ_{20} = .862	.862	gm/cc
Molar volume of the solvent	v_1 = 106.0	106.0	cc/mole
The molecular weight of rubber between cross-links	M_c = 8891.0	4000	gm/mole
Rubber-solvent interaction constant	χ = .400	.400	

The computational results are presented in Figures 8, 9, 10, 11 and 12 where it is clearly evident that the presence of two different elastic materials results in nonhomogeneous swelling characteristics, nonlinear distribution of the radial stress, discontinuity in the circumferential stress and the volume fraction of the solid and finally a nonlinear volume change in the outer circumferential cylinder due to swelling. Figure 8 shows that the composite cylinder swells nonhomogeneously with the radial and tangential stretch ratios λ_r and λ_θ, varying nonlinearly with R in the interval $[R_i, R_o]$. In addition, there is a discontinuity in the radial stretch ratio λ_r at the interface, where the material characteristics change abruptly. This is in sharp contrast to a cylinder composed of a single material, which would swell homogeneously with λ_r and λ_θ equal and constant throughout the domain $R \in [0, R_o]$. Figure 9 shows the nonlinear distribution of the radial stress in the region $R \in [R_i, R_o]$, which is induced by the presence of the material M_1 occupying the region $R \in [0, R_i]$. The nonlinear variation of the circumferential stress in the region $R \in [R_i, R_o]$ with the radial coordinate appears in figure 10. A discontinuity of the circumferential stress at the interface of the two different properties is observed. Figure 11 shows a discontinous but almost constant volume fraction of the solid along the radial coordinate. In figure 12 the percentage volume change of the cylinder with the radial coordinate is presented. This variation is particularly useful for experimentalist.

b. <u>Swelling of a Nonlinearly Elastic Cylinder with a Rigid Core.</u>

A special case of the problem considered in section (5a) is when the inner material M_1 is assumed to be rigid. The governing equations for this problem are again given by equations (5.18) and (5.19). The appropriate boundary conditions are given by

$$\lambda_\theta(R_i) = 1, \text{ and}$$

$$S_{rr}(R_o) = 0 \ .$$

The computational results are presented in Figures 13 and 14. It is clearly evident from these results that the presence of the rigid core induces nonhomogeneous swelling characteristics. Furthermore, it has been demonstrated that the rigid core initiates significant gradients in the stretch ratios and severe stress concentrations at the bond interface. A complete detail of this special case appears in [20].

CONCLUDING REMARKS AND DISCUSSION ON APPLICATIONS OF THIS WORK TO DAMAGE IN COMPOSITE MATERIALS

The principal contributions of the work presented are

a. Qualitative and/or quantitative differences in the behavior of incompressible mixtures with solid constituents being various types of nonlinearly elastic incompressible solids were identified. In particular, the response of the mixtures with Neo-Hookean solids and ideal fluids and mixtures with Mooney-Rivlin solids and ideal fluid is studied and compared with experimental results by Treloar for several cases of simple deformations such as uniaxial extension, bi-axial extension and simple shear. The results of these investigations could have a significant impact on material identification studies.

b. This work presents sufficient basis to motivate experimentalist to conduct certain experiments with compressible solids to yield change in volume of the solid when a known amount of fluid enters in the solid. These results would provide the missing link to solve boundary value problems involving mixtures with compressible constituents.

c. Certain misconceptions pertaining primitive concepts like volume additivity and incompressibility which apparently existed in previous literature were clarified.

d. Finally, a boundary value problem focused on investigating the swelling characteristics of a composite cylinder is presented. The composite cylinder features an inner core which exhibits material properties which are quite distinct from those of the surrounding concentric material. This work is motivated by the need for modeling the large deformations of polymeric composite materials under various under hygrothermal environments. This work partially addresses this need by characterizing the interaction of ideal fluids and idealized constrained nonlinear elastic solids in the context of Mixture Theory. The deformations of the elastic solid may be restricted due to the presence of extensible/inextensible fibers, rigid inclusions and coatings on reinforcing fibers, for example. Such constraints can quantitatively and qualitatively alter the ability of these materials to undergo dimensional and constitutive changes very significantly. The results of these investigations for large deformations demonstrate that the constraints imposed by the core material induce nonhomogeneous swelling characteristics with significant gradients in the stretch ratios and severe stress concentrations at the fiber/matrix interface. These results could have important implications for a variety of fiber-reinforced composites featuring hygroscopic and temperature-sensitive matrix materials. This problem also has significant implications in ability to predict local (in the neighborhood of the fiber/matrix interface in composites, for example) events occuring due to global (on the whole composite structure) effects. This is very critical in composites, particularly in damage studies for instance, from the boundary value problem it can be speculated that large gradients of stretch ratios and stresses in the neighborhoods of the fiber matrix interface due to smaller global effects occur. Damage in composite materials seemingly macroscopic phenomena is truely a microscopic phenomena and hence representative volume element considered in the above presented boundary value problem should be studied in damaged and undamaged configuration in detail and a continuum relationship should be developed to predict macroscopic response. This is taken up by the author as a part of Doctoral studies.

FIGURES

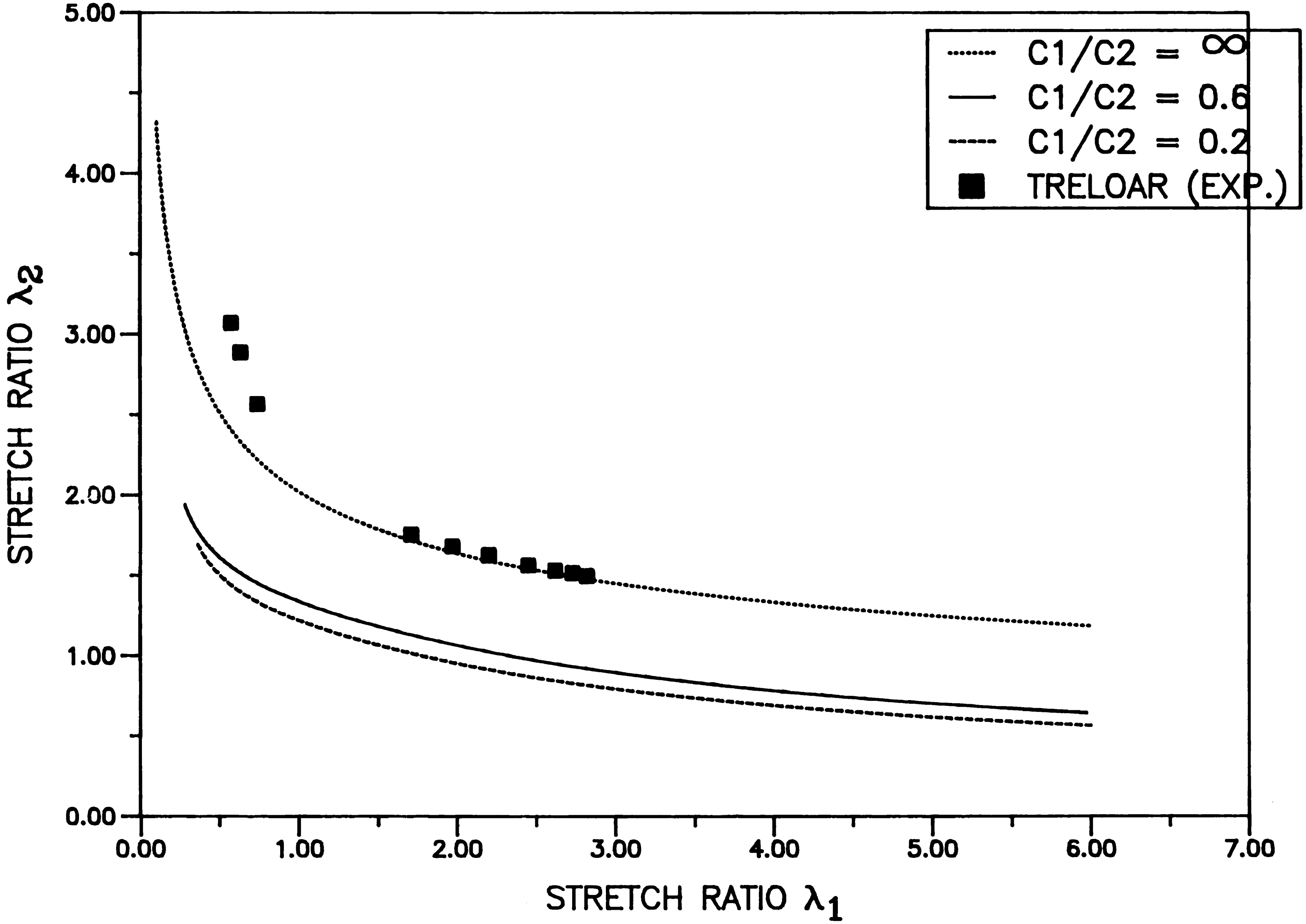


FIG.1: Variation of stretch ratios for uniaxial extension of a cuboid mixture.

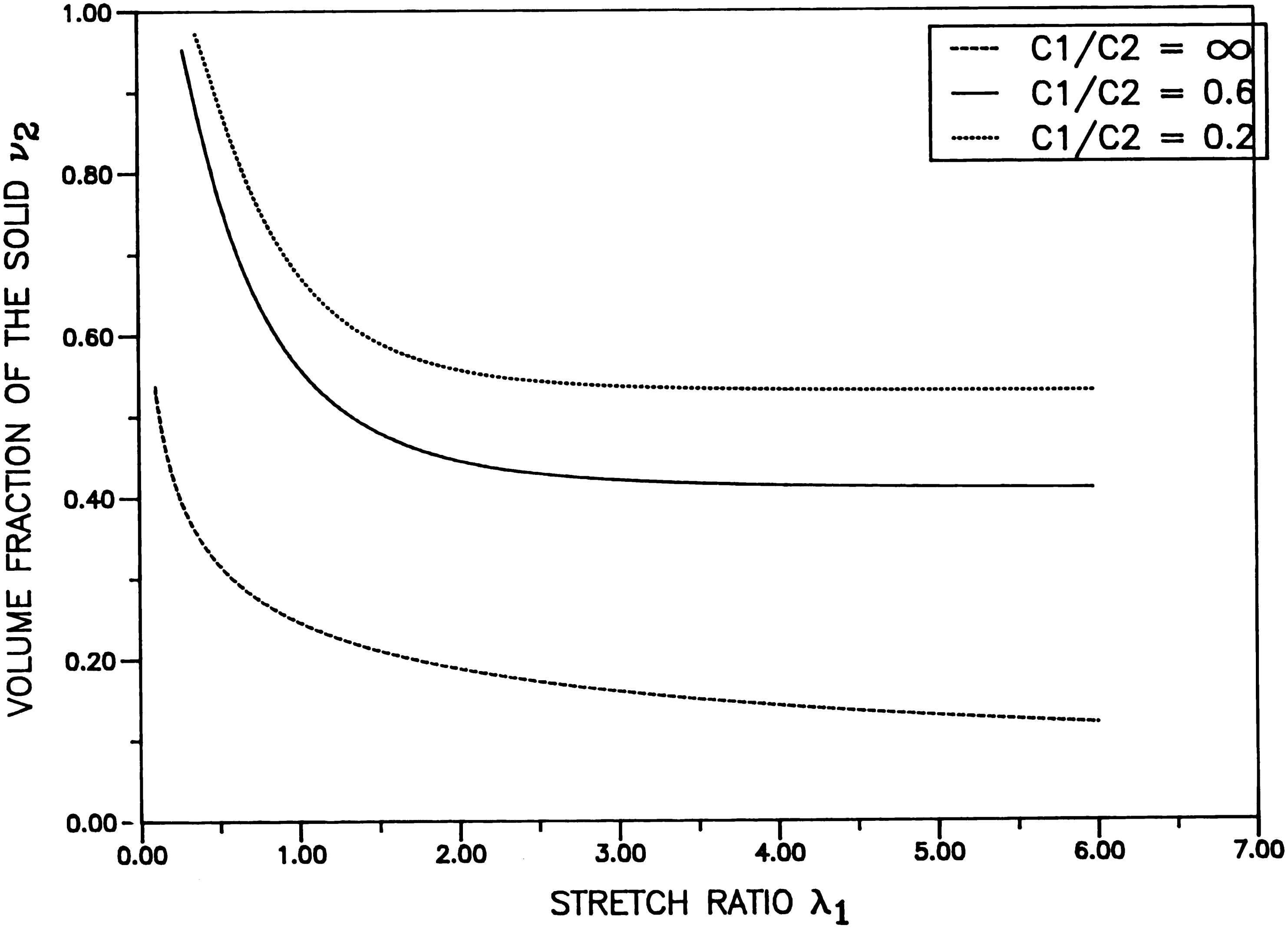


FIG.2: Variation of the volume fraction of the solid with the stretch ratio for uniaxial extension of a cuboid mixture.

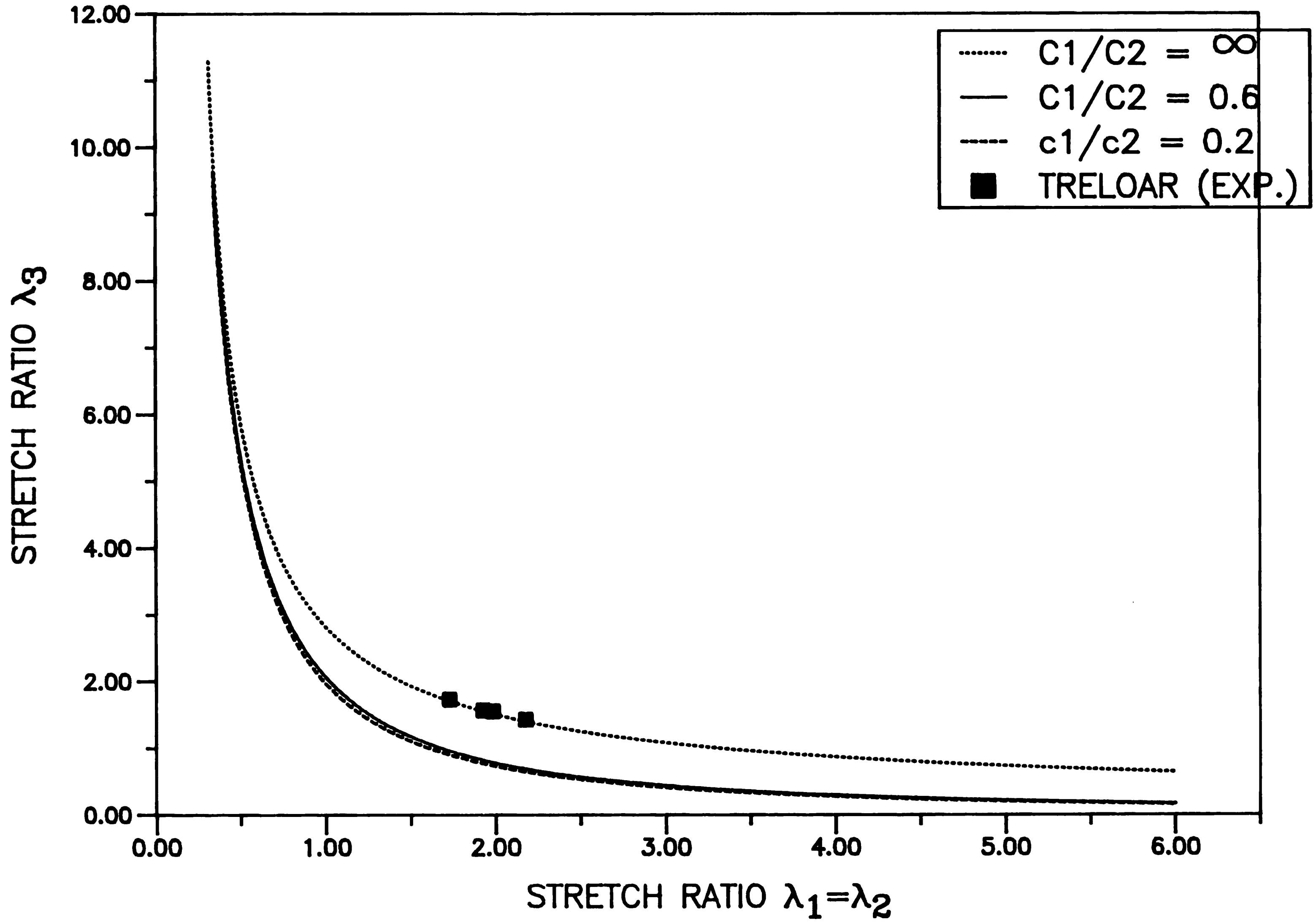


FIG.3: Variation of stretch ratios for bi-axial extension for a cuboid mixture

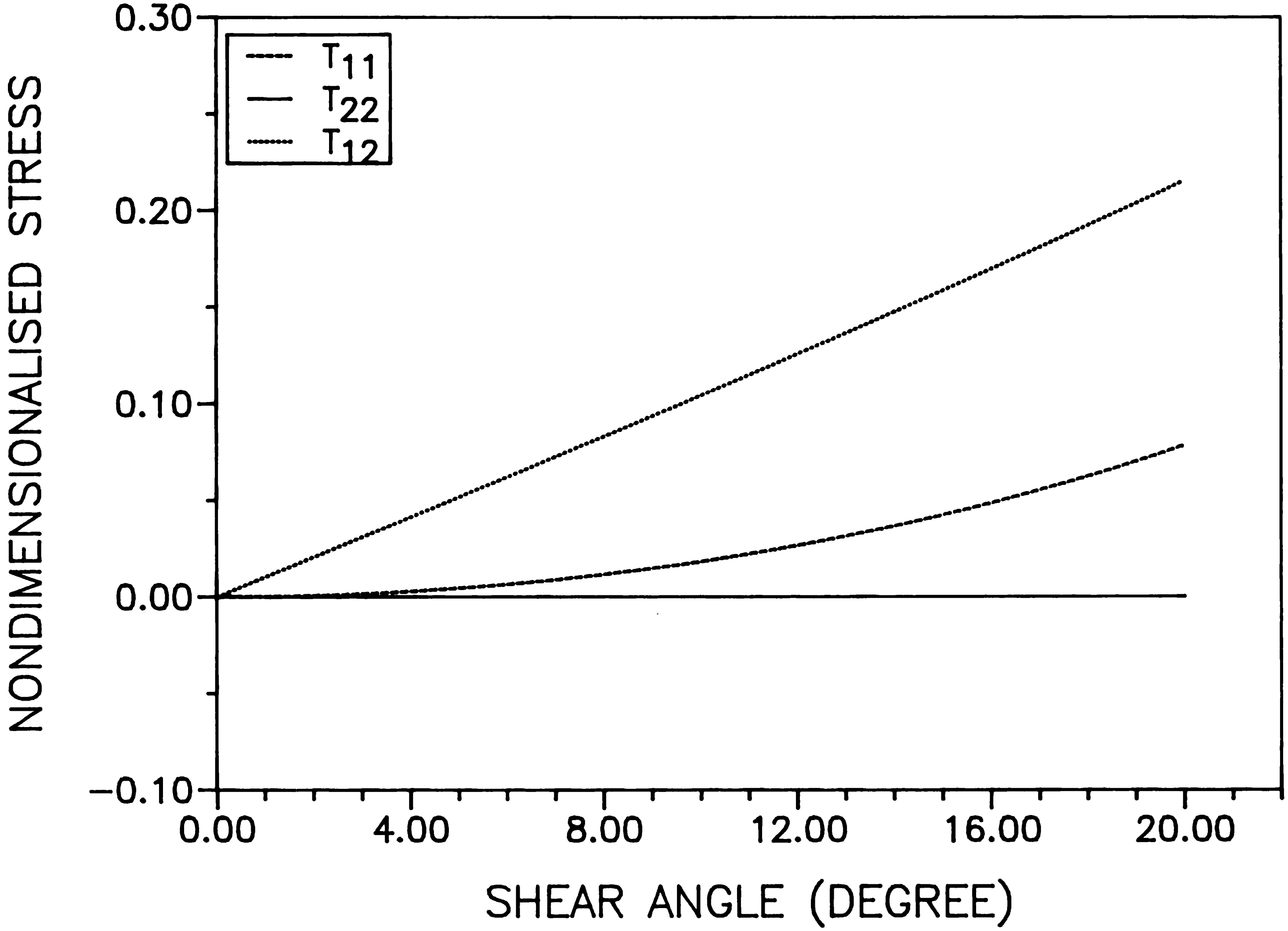


FIG. 4: Variation of nondimensional stress with shear angle of a cuboid Neo−Hookean type mixture ($C1/C2=\infty$) when subjected to simple shear.

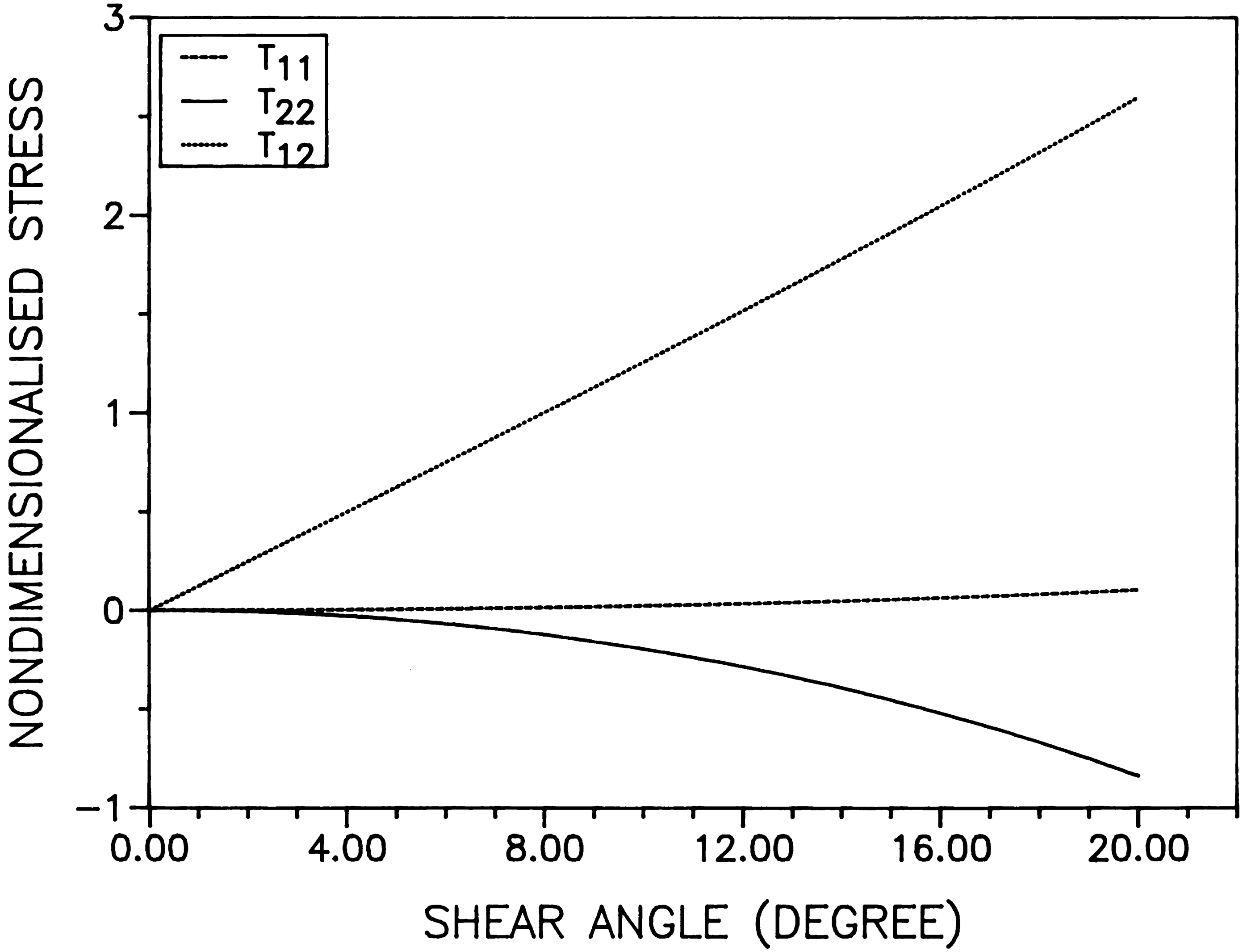


FIG. 5: Variation of nondimensional stress with shear angle of a cuboid Mooney–Rivlin type mixture (C1/C2=0.2) subjected to simple shear.

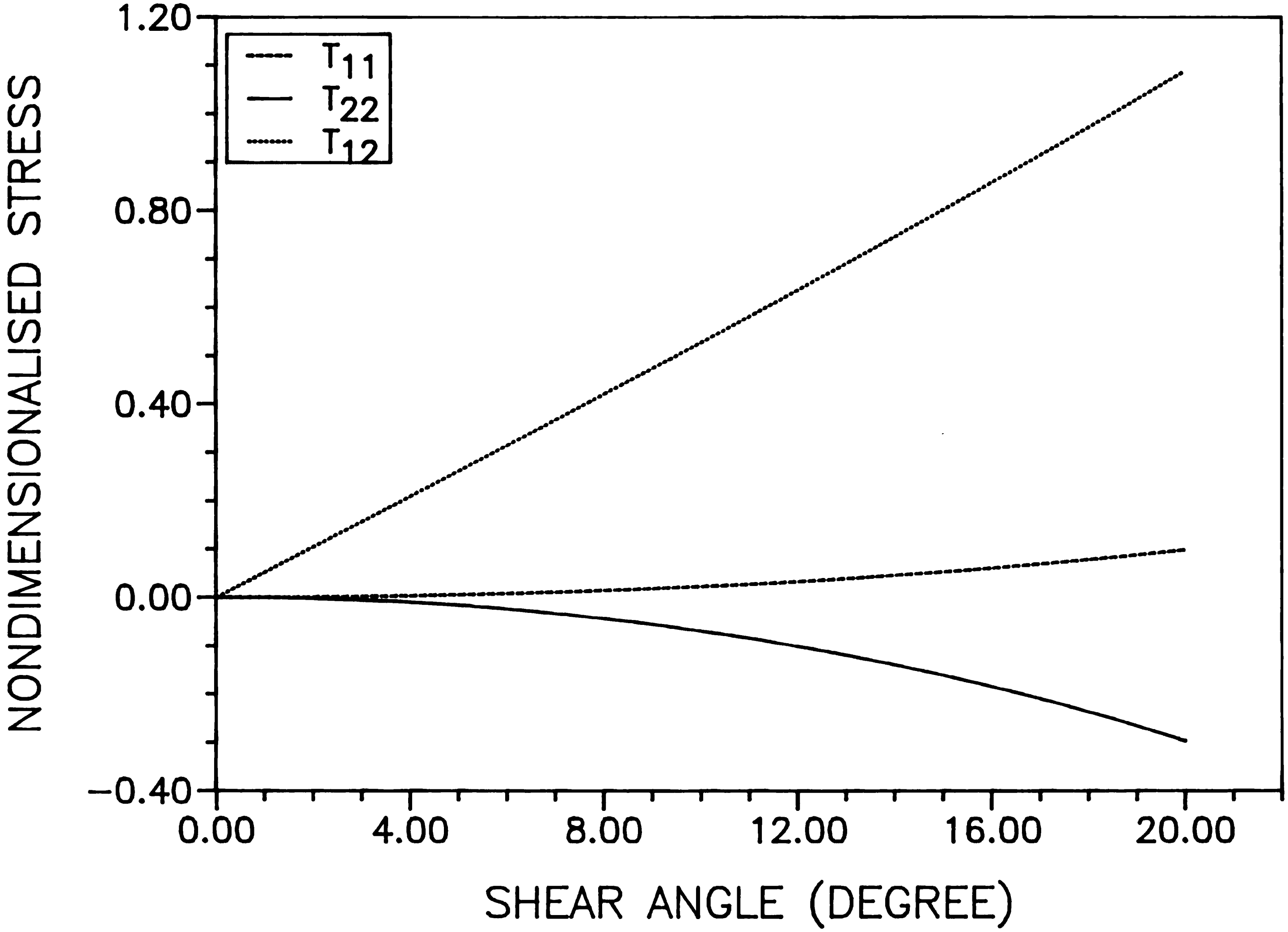
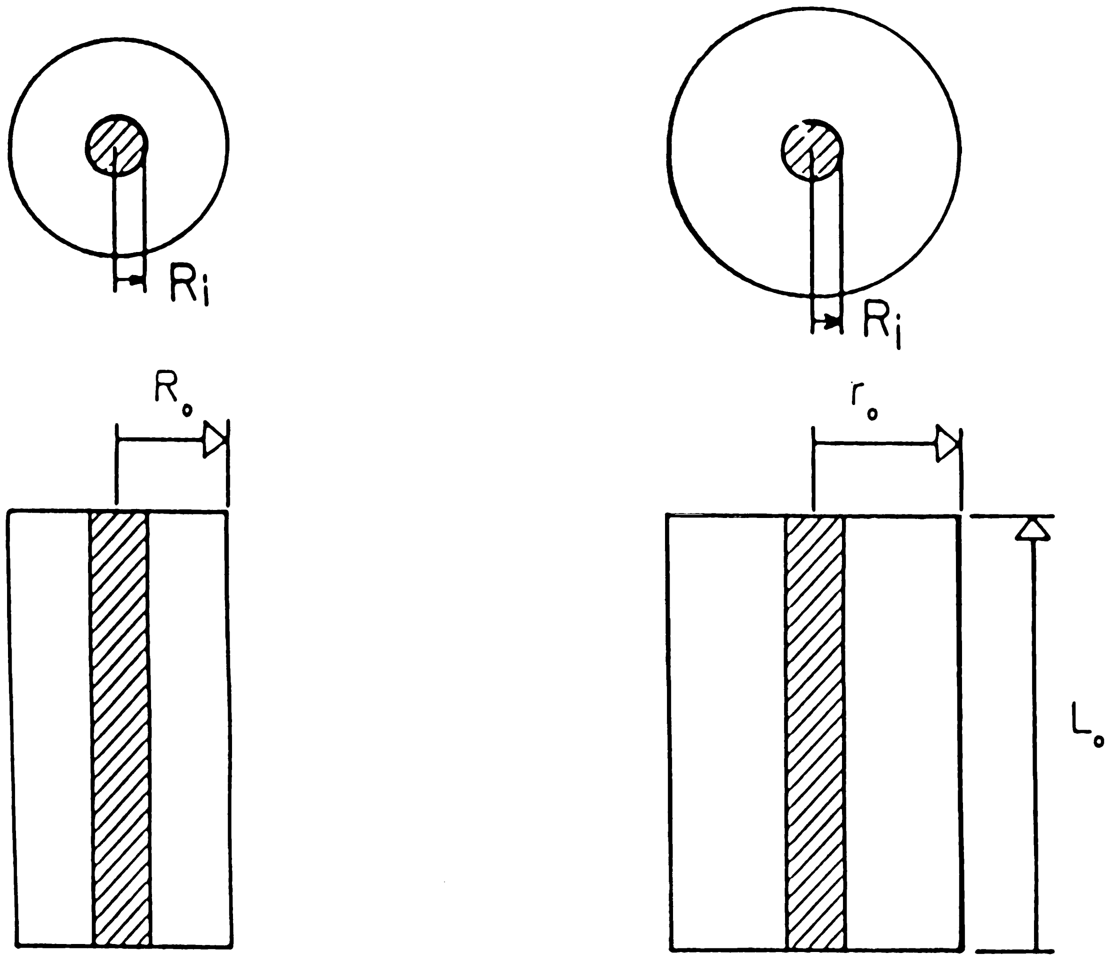


FIG. 6: Variation of nondimensional stress with shear angle of a cuboid Mooney–Rivlin type mixture (C1/C2=0.6) when subjected to simple shear.



REFERENCE CONFIGURATION   DEFORMED CONFIGURATION

FIG. 7: Finite swelling of elastic cyclinder with core of differential properties.

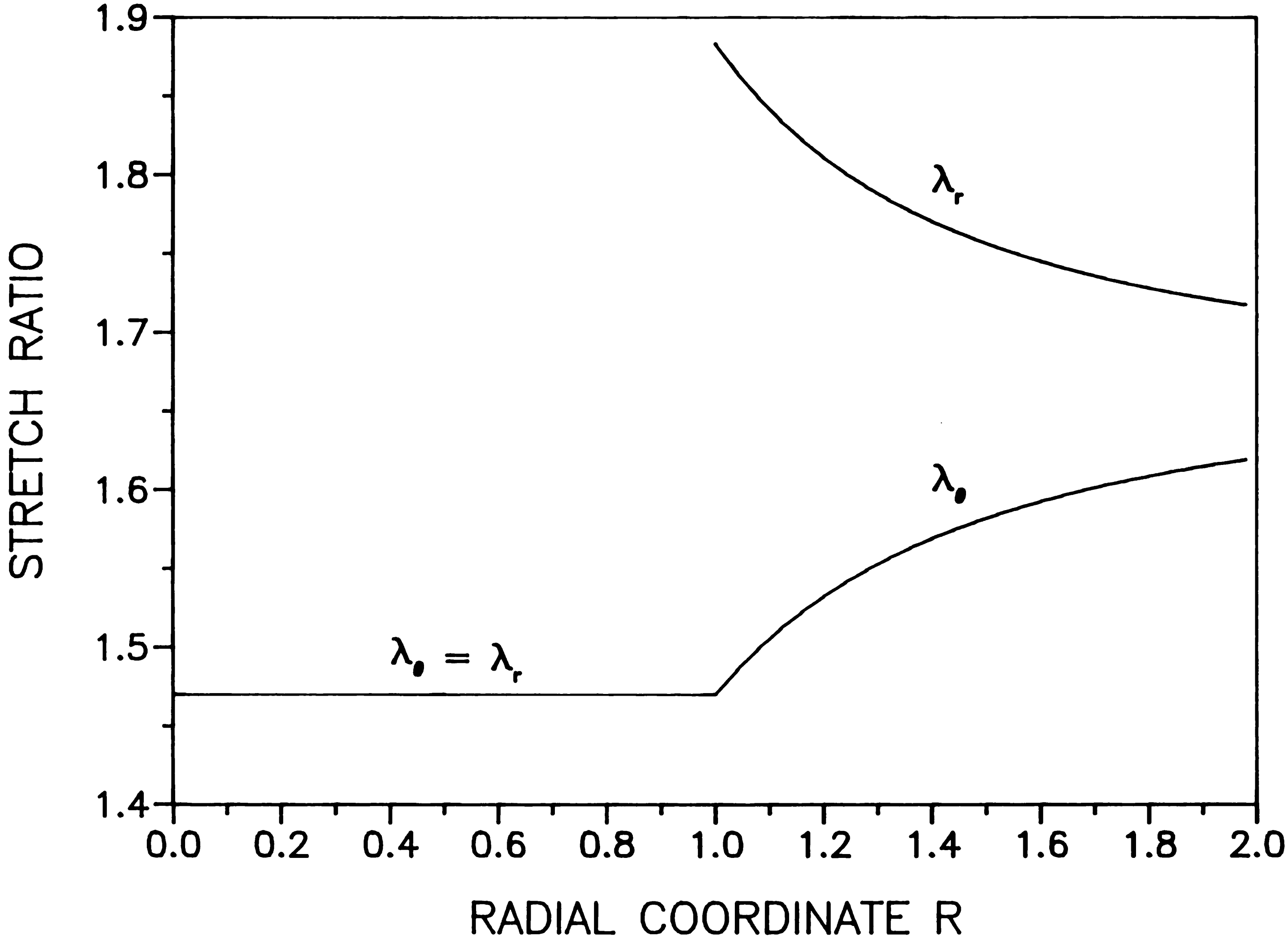


FIG. 8: Variation of the stretch ratios with the reference radial coordinate R

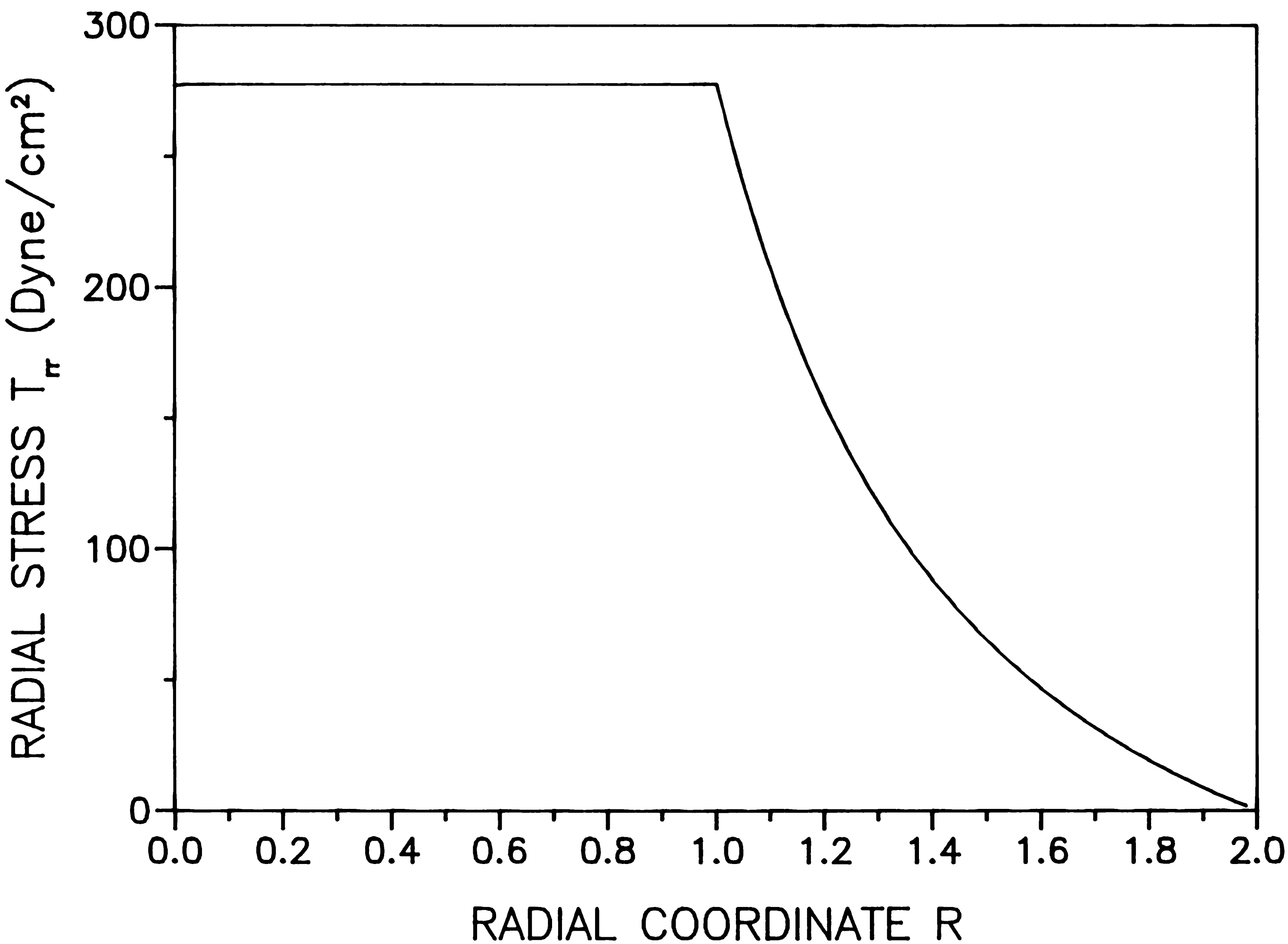


FIG. 9: Variation of the radial stress with the reference radial coordinate R

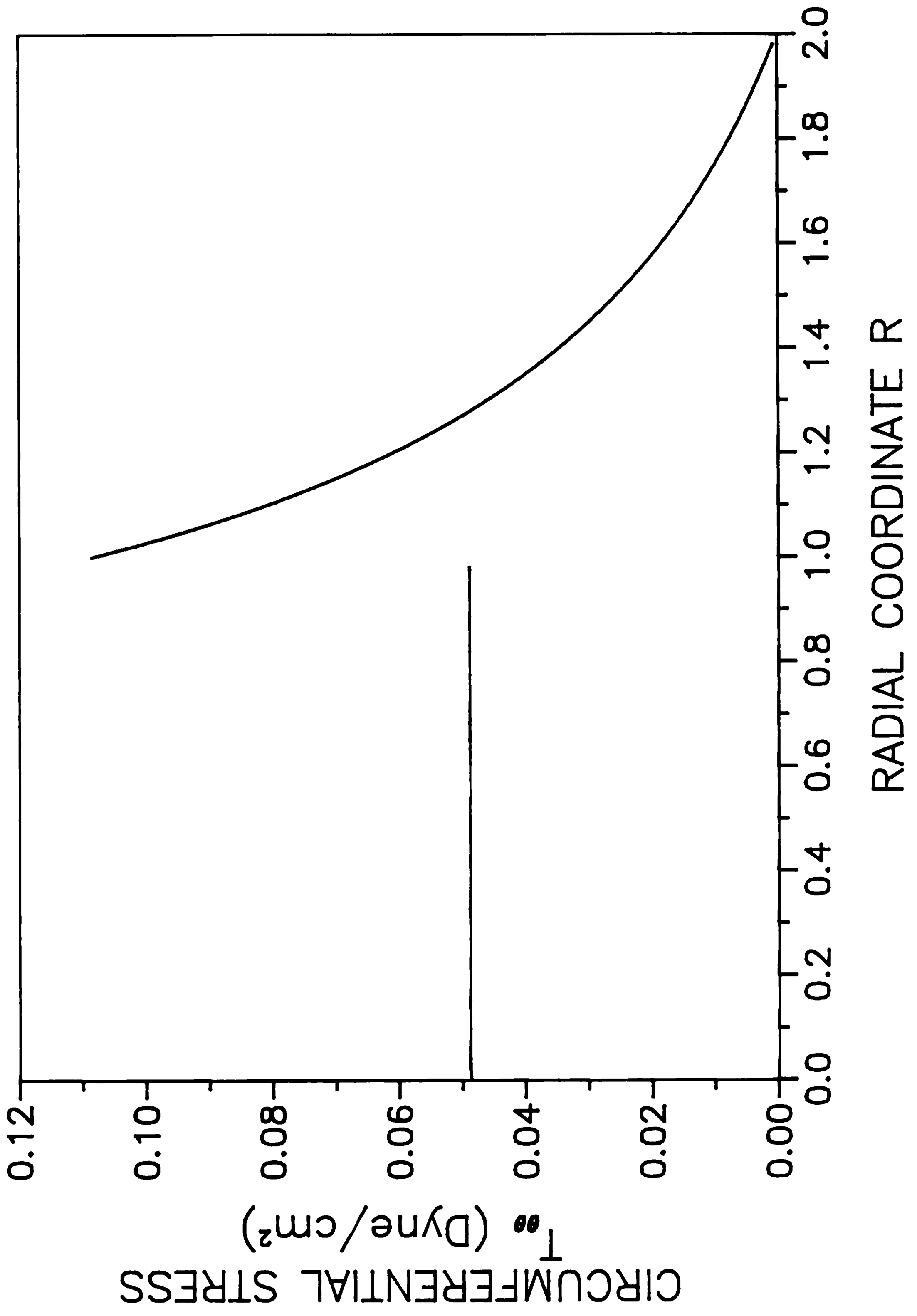


FIG. 10: Variation of the circumferential stress with radial coordinate R.

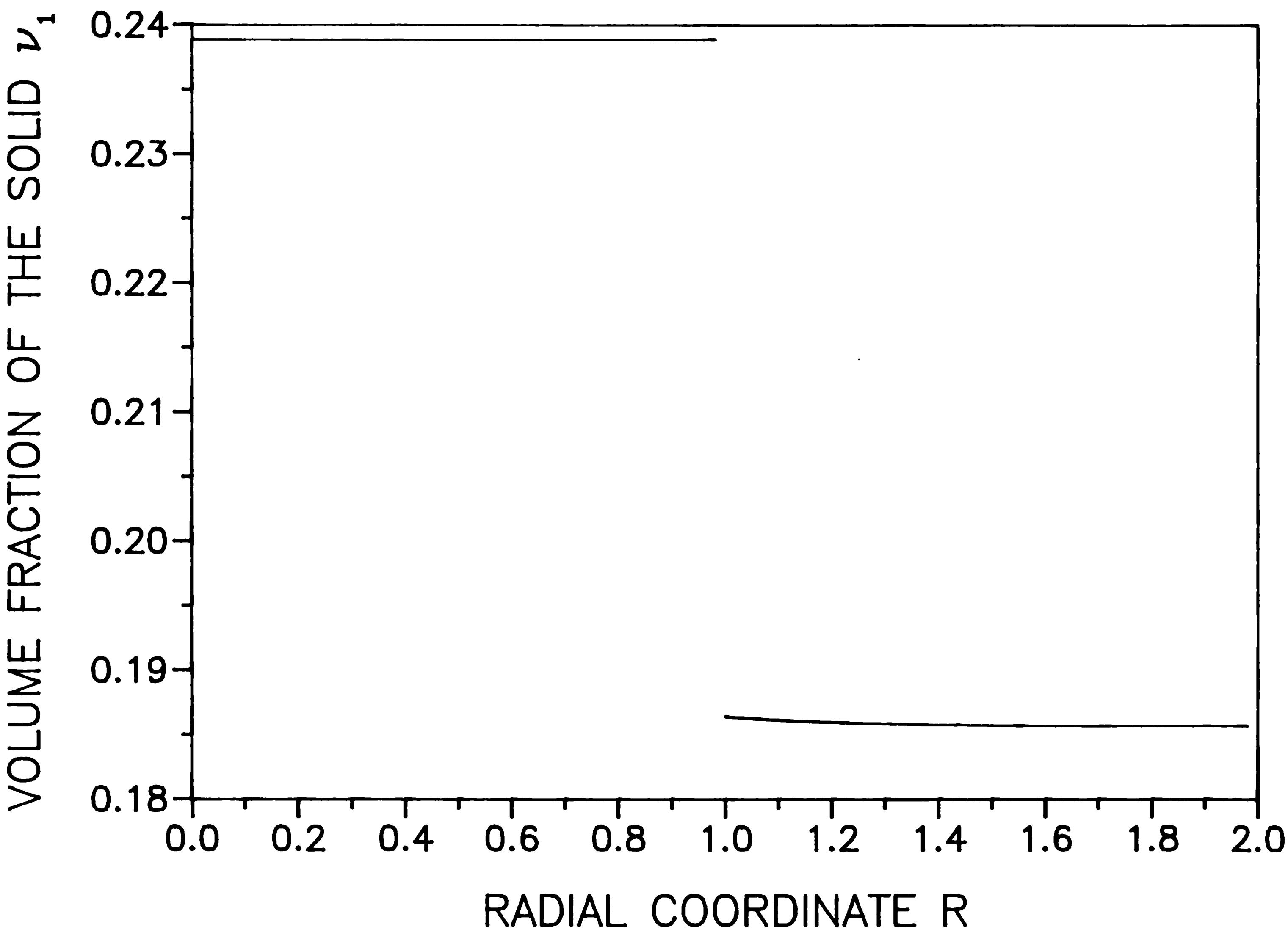


FIG. 11: Variation of the volume fraction of the solid with the reference radial coordinate R.

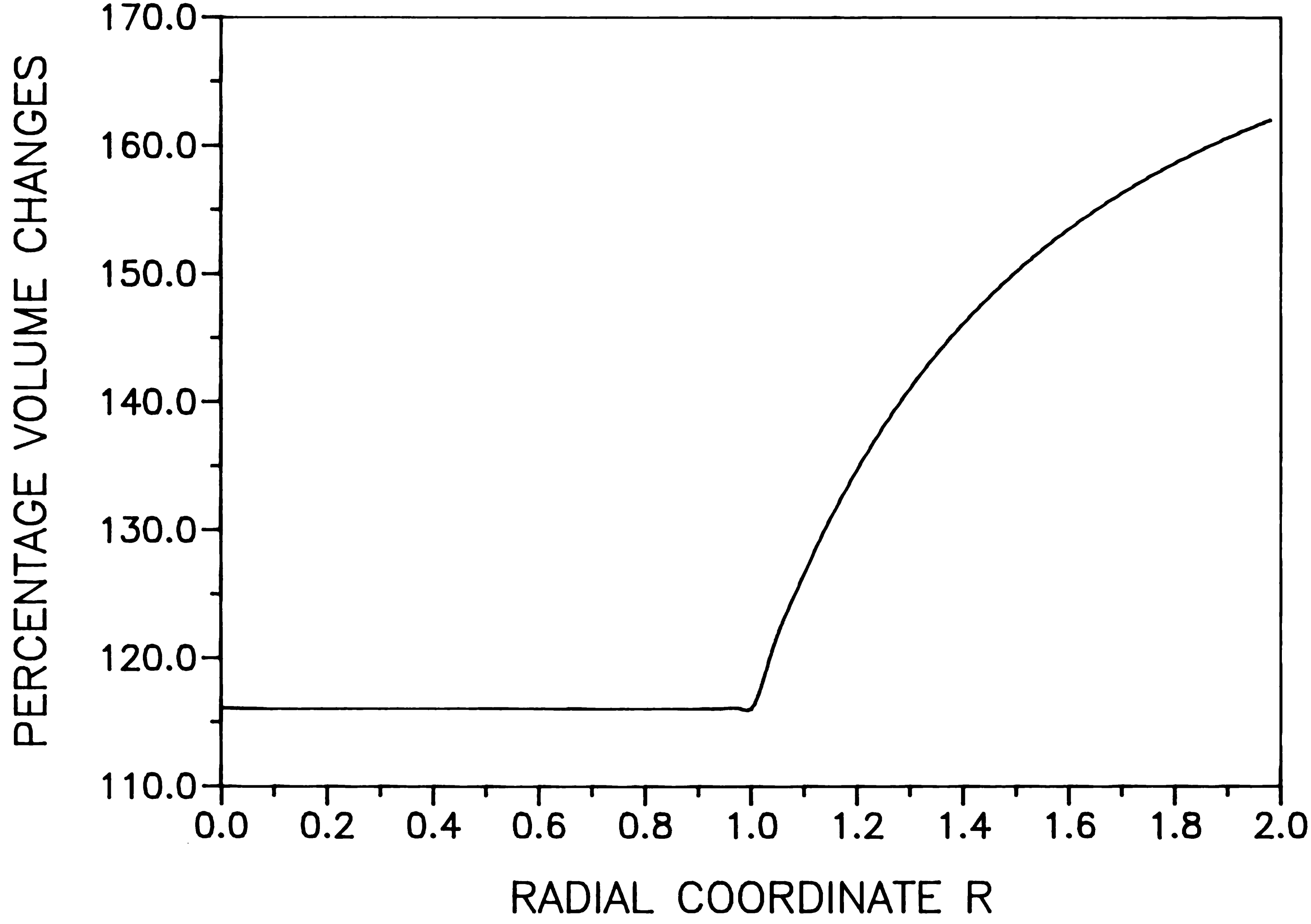


FIG. 12: Percentage volume changes of the swollen cylinder with reference radial coordinate R.

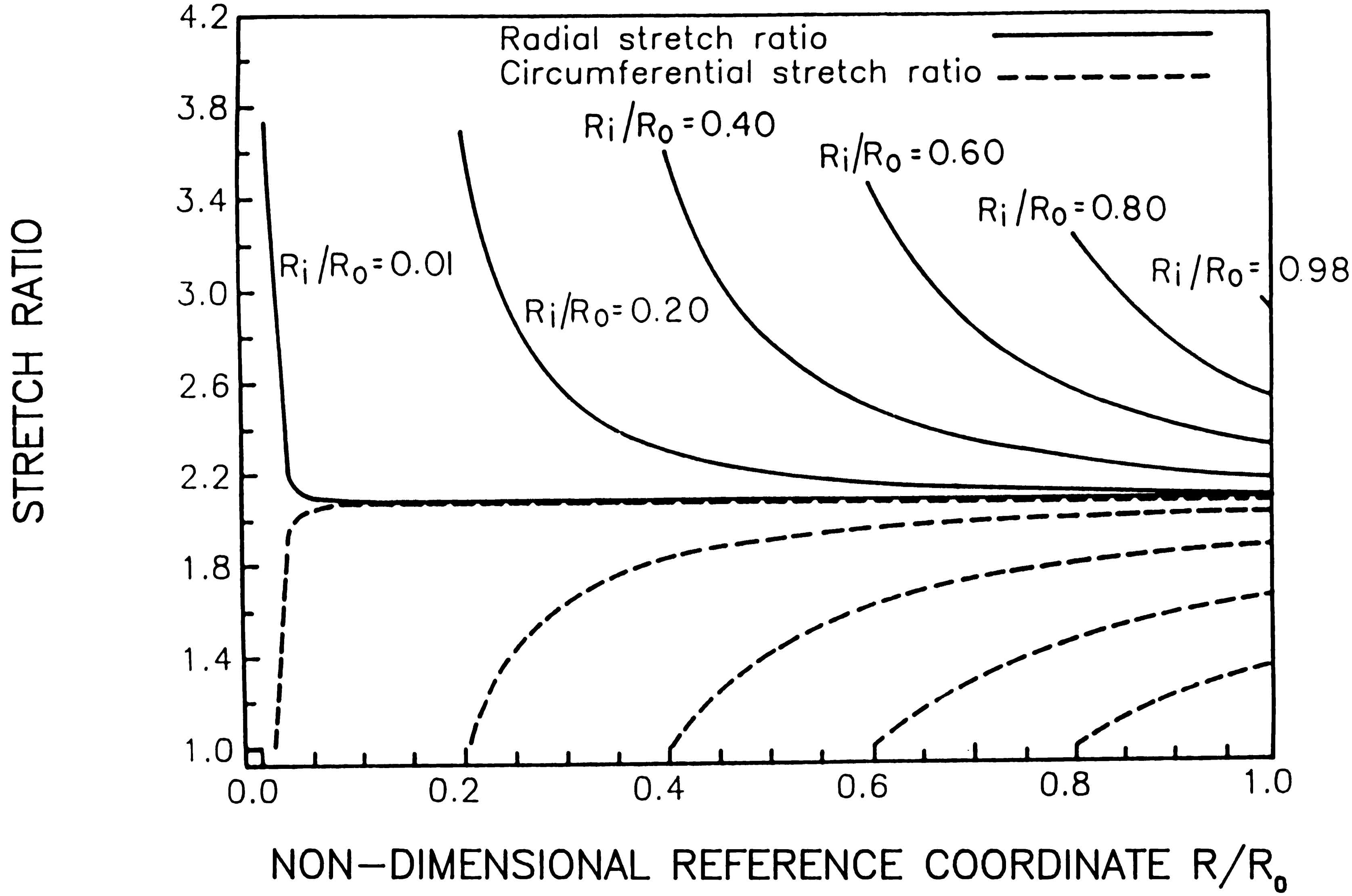


FIG. 13: Variation of radial and circumferential stretch ratios with the reference radial coordinate R for rigid core.

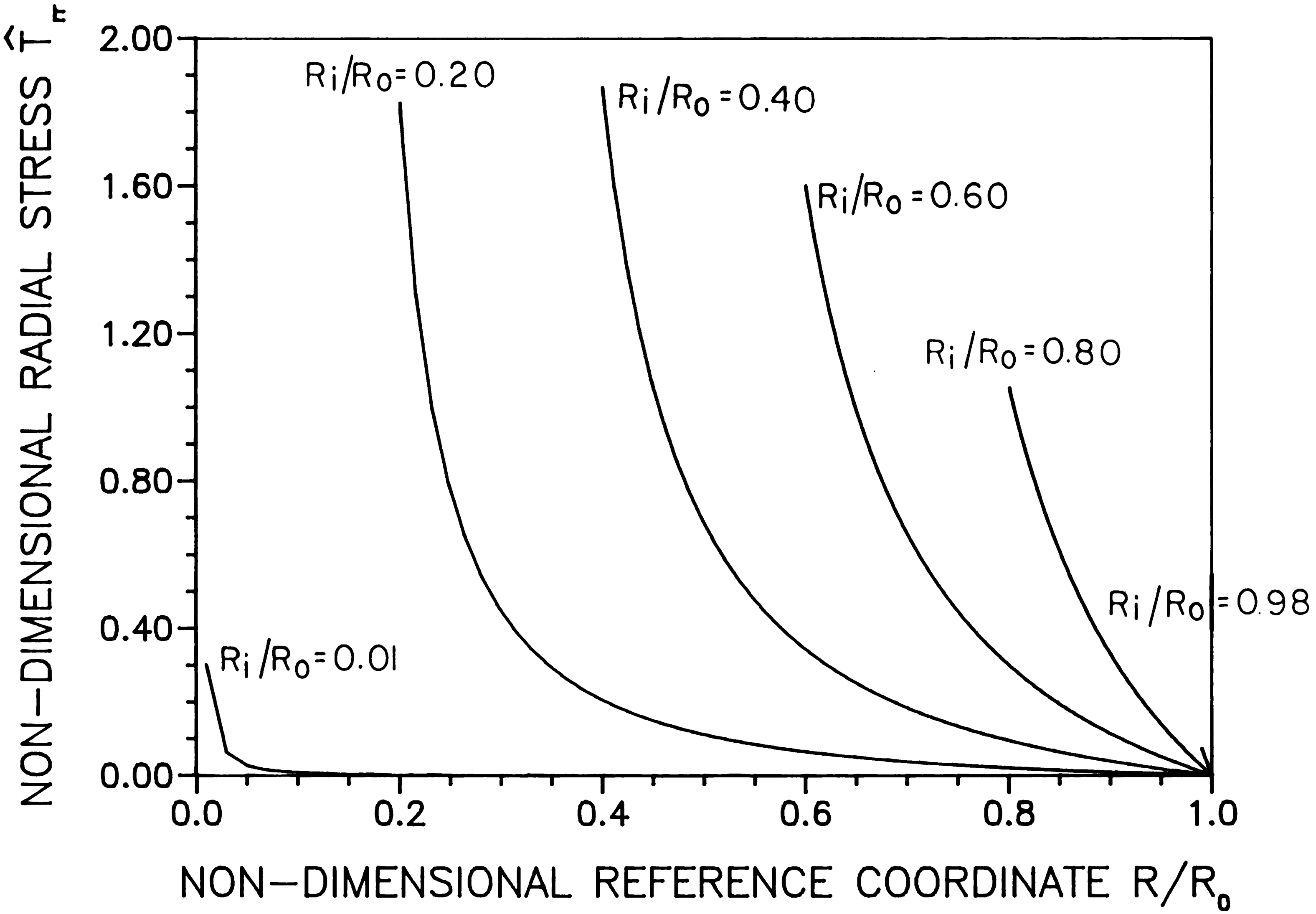


FIG. 14: Variation of the radial stress with the reference reference radial coordinate R for rigid core.

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BIBLIOGRAPHY

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