A HISTORY AND DEVELOPMENT OF INDECOMPOSABLE CONTINUA THEORY

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This is to certify that the

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ABSTRACT

A HISTORY AND DEVELOPMENT OF INDECOMPOSABLE CONTINUA THEORY

By

Francis Leon Jones

This thesis is an exposition of the history and development of indecomposable continua theory from its origins in 1910 until the present. It traces the rise of indecomposable continua from the status of pathological examples to that of a general body of knowledge playing a fairly important role in point-set topology.

The theory of ordinary indecomposable continua is explored in great detail. In addition, most of the results arising from the study of various special cases of indecomposability are surveyed. However, no results concerning generalized indecomposable continua are included.

Chapter 2 gives some background material from general topology. The specialized definitions are introduced as they are needed.

Chapter 3 presents some early examples of indecomposable continua in essentially the same terminology as the inventors used. Most of the results of Chapter 4 are structure theorems dating from the 1920's; many are still important today. In Chapter 5, several relationships

between indecomposability and irreducibility are explored. Chapter 6 presents some more examples of indecomposable continua and outlines Knaster's construction of a hereditarily indecomposable continuum.

Chapter 7 deals with some existence questions. In particular, the theorem that every metric space of dimension greater than one contains an indecomposable continuum is proved. A proof is outlined showing that most plane continua are hereditarily indecomposable. Bellamy' non-metrizable indecomposable continuum is also included.

Chapter 8 presents Kuratowski's common boundary theorem for E^2 and several of Knaster's examples. Chapter 9 relates accessibility to indecomposability.

Chapter 10 treats topological groups and inverse limits. Wallace's work on clans constitutes the first part of the chapter, while inverse limits and solenoids make up the last. Chapter 11 examines the results of subjecting indecomposable continua to several usual topological operations.

Chapter 12 surveys the work from Moise's thesis in 1948 to the present. The pseudo-arc and pseudo-circle are discussed, along with theorems for ordinary hereditarily indecomposable continua.

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A HISTORY AND DEVELOPMENT OF INDECOMPOSABLE CONTINUA THEORY

By

Francis Leon Jones

A THESIS

Submitted to
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in partial fulfillment of the requirements
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For Pat and Doug,

Dad and Mom,

Dad and Mom Brumfiel

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TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

This thesis is an exposition of the history and development of indecomposable continua theory from its origin in 1910 until the present. It traces the rise of indecomposable continua from the status of pathological examples to that of a general body of knowledge playing a fairly important role in point-set topology.

We shall explore the theory of ordinary indecomposable continua in some detail. We shall also survey many results arising from the study of various special cases of indecomposable continua. However, we shall not include results from the theory of generalized indecomposable continua, since this vein of research has not yet been as widespread as those of the ordinary and special theories. Much of the work on generalized indecomposable continua has been done by P. M. Swingle and C. E. Burgess.

In Chapter 2, we give some elementary background material from general topology. The specialized definitions we shall use later will be introduced as needed. Chapter 3 presents some early examples of indecomposable continua in essentially the same terminology as the inventors used. Most of the results of Chapter 4 are structure theorems

dating from the 1920's; many are still important today. In Chapter 5, we explore several relationships between indecomposability and irreducibility. Chapter 6 presents some more examples of indecomposable continua and outlines Knaster's construction of a hereditarily indecomposable continuum.

Chapter 7 deals with some existence questions. In particular, we show that every metric space of dimension greater than one contains an indecomposable continuum. We also outline a proof that in the space of all continua of I^2, the set of hereditarily indecomposable continua is a dense G_δ set. Further results of this nature are in Chapter 12. Bellamy's non-metrizable indecomposable continuum is also included in Chapter 7.

Indecomposable continua arose from a study of common boundaries of plane domains, and Chapter 8 continues this investigation. Kuratowski's theorem for E^2 and several of Knaster's examples are included. Chapter 9 relates accessibility to indecomposablity and gives Kuratowski's characterization of the latter in terms of the former.

Chapter 10 treats topological groups and inverse limits. Wallace's work of clans constitutes the first part of the chapter, while inverse limits and solenoids make up the last. Chapter 11 examines the results of subjecting indecomposable continua to several usual topological operations.

Chapter 12 surveys the work from Moise's thesis in 1948 to the present. The pseudo-arc and pseudo-circle are

discussed, along with theorems for ordinary hereditarily indecomposable continua. Because this work is recent and readily available, few proofs are included.

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CHAPTER 2

BACKGROUND DEFINITIONS AND NOTATIONS

This chapter gives the fundamental definitions needed from general topology, beginning with a formal definition of a topological space.

Let X be any set. By a _topology on X_, we mean a collection T of subsets of X which satisfy the following conditions:

(1) $\emptyset$ and X are members of T;

(2) the union of any collection of members of T is a member of T;

(3) the intersection of any finite collection of members of T is a member of T.

The members of T are called _open sets_, and X together with its topology T is a _topological space_. Where no confusion can result, X is used to denote both the underlying set of points, as well as the topological space.

If $x \in X$, a neighborhood for x is any open set in X containing x and will be denoted by U(x). (Some definitions of neighborhood require only that it be a set containing an open set containing x.)

A set may have many distinct topologies on it, with topologies T and T' being distinct if there is an open set

in one that is not in the other. The branch of mathematics known as topologies studies the consequences of imposing a topological structure upon a set. Before giving any more definitions, we make a few remarks about the ones above.

The definition of a topological space did not spring into being as the result of any one person's inspiration, but, like most abstractions, it developed as a result of many persons' work and experiments. Of course, the choice of axioms for a mathematical system is somewhat arbitrary, with the only major restrictions being consistency and completeness. But to be useful, a system must neither be too general nor too restrictive; in either case, very little of value ensues.

Historically, topological spaces had their origin in the process of giving analysis a rigorous foundation [84]. Several concepts from analysis were generalized and abstracted in this process, among them being "limit," "neighborhood," "continuity," and "distance".

In real analysis, given the notion of distance, we may say that "x is near y" if for some real number $r > 0$, $|x-y| < r$, and that all such x for a given positive real number r constitute a neighborhood of y. If this neighborhood concept is abstracted to a more general setting, not necessarily involving distance, then "x is near y" can be given meaning by saying that x is in some neighborhood of y, where neighborhood has been defined, say in the manner described on page 4.

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Next, consider a fixed set $A \subset E^1$ and some point $y \in E^1$. If for any $r > 0$, there exists $x \in A$ such that $|x-y| < r$, then y is a limit point of $A \subset E^1$. Certainly this concept can be defined in terms of neighborhoods, with no reference to distance. On the other hand, given the concept of limit (or cluster) point, neither of the other two notions can be defined in terms of it. This "linear ordering" of the three concepts was known as early as 1914, when Hausdorff noted it in his Grundzuge der Mengenlehre. He used neighborhood axioms to define a topological space, but it was recognized later that the "open set" axioms (p. 4) are simpler.

Once topological spaces have been defined, a precise definition of continuous functions can be given. For example, $f: X \longrightarrow Y$ is continuous at $x_0 \in X$ iff for each neighborhood V in Y of $f(x_0)$, there exists a neighborhood U in X of x_0 such that $f(U) \subset V$. f is continuous on X if it is continuous at x, for each $x \in X$. Alternatively, f is continuous on X iff for each open V in Y, $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X. Continuity can also be specified in terms of "closed sets", which we do, following these definitions.

$A \subset X$ is closed iff $X-A$ is open in X. These sets can be described in other ways. For example, if $A \subset X$, the set of cluster points of A is

$$A' = \{x \in X \mid \forall\ U(x): U(x) \cap (A - x) \neq \emptyset\}.$$

The closure of A is $\bar{A} = A \cup A'$, and A is closed iff $A = \bar{A}$. If A is closed and $A = A'$, then A is perfect.

The open sets can be described in different ways, too. The interior of $A \subset X$ is Int $(A) = X - \overline{X - A}$, and A is open iff $A = \text{Int}(A)$. The boundary of $A \subset X$ is $\text{Fr}(A) = \overline{A} \cap \overline{X-A}$. In terms of the above definitions, continuous functions can be characterized by the property that for each $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$. Furthermore, f is a homeomorphism iff f is one-one, onto, and for each $A \subset X$, $f(\overline{A}) = \overline{f(A)}$; that is, iff f is one-one, onto, and both f and f^{-1} are continuous.

We shall need a few more basic definitions. $A \subset X$ is dense in X if $\overline{A} = X$. A is nowhere dense in X if $A \subset \overline{X - \overline{A}}$. A collection, B, of subsets of X is a basis for a topology T on X if each member of T is the union of members of B. If B is countable, then the space is said to be 2^o countable. Thus, to specify a particular topology T for a set X, we need not specify all the open sets; we can describe a "smaller" collection of open sets and still have the original topology.

In this thesis, we shall be primarily interested in certain special topological spaces, such as Hausdorff or metric spaces.

A metric d on a set X is a function d: $X \times X \to E^1$ satisfying:

(1) $d(x,y) \geq 0$, for all $x,y \in X$;

(2) $d(x,y) = 0$ iff $x = y$;

(3) $d(x,y) = d(y,x)$, for all $x,y \in X$;

(4) $d(x,y) \leq d(x,z) + d(z,y)$, for all $x,y,z \in X$.

$B(x,r) = \{y \in X \mid d(x,y) < r\}$ is called a ball of radius r

at x. A topological space (X,T) is called a metric space if $\{B(x,r) \mid x \in X,\ r > 0\}$ is a basis for T. The distance between nonempty sets A, B in a metric space is

$$d(A,B) = \inf \{d(a,b) \mid a \in A,\ b \in B\},$$

and the diameter of A is

$$\delta(A) = \sup \{d(a,b) \mid a,b \in A\}.$$

The other general class of topological spaces that will be used is the collection of Hausdorff spaces. A topological space is Hausdorff, denoted T_2, if every pair of distinct points of that space have disjoint neighborhoods. A space is regular if there are disjoint open sets containing each closed $A \subset X$, and $x \in X-A$. Clearly every metric space is T_2.

We are now ready to give some definitions from continua theory. A space is connected if it is not the union of two disjoint, nonempty, open subsets. If X is not connected, it is often useful to know its maximal connected subsets. The component of $x \in X$ is the union of all connected subsets of X containing x. If for each $x \in X$, the component $C(x) = x$, then X is totally disconnected. A connected open set is a domain. If S is a closed proper subset of X, then every component of X-S is called a complementary domain of S.

One of the useful properties of connectivity is its invariance under continuous transformations. That is, if $A \subset X$ is connected, and if $f: X \rightarrow Y$ is continuous, then f(A) is connected. We shall also be interested in another

invariant of continuous functions, compactness.

A space X is <u>compact</u> if every open cover has a finite subcover. Thus for any $\{U_\alpha\}_{\alpha \in \mathscr{O}}$ such that each U_α is open and $X = \bigcup_{\alpha \in \mathscr{O}} U_\alpha$, there exists a collection $\{U_{\alpha_i}\}_{i=1}^{n}$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i}$. Compact spaces are "very nice" in several respects. For example, if X is compact, then "all limits" exist in X (in the sense that every maximal filter-base in X converges in X [28, p. 223]). For our purposes, a more useful version of the above is the Bolzano-Weier-strass property, which says that in a compact space every countably infinite subset has at least one cluster point. Also, compact subsets of a T_2 space have many of the same properties as points, the most useful of which to us is that two disjoint compact subsets of a T_2 space have disjoint neighborhoods. Moreover, if X is a metric space, $A \subset X$ closed and $C \subset X$ compact with $A \cap C = \emptyset$, then $d(A,C) \neq 0$ [28, p. 234]. If X is itself a compact metric space, then it is 2^o countable. Finally, if the space is Euclidean, the Heine-Borel theorem states that $A \subset E^n$ is compact iff it is closed and bounded.

Before considering those spaces which are both compact and connected, we describe a very useful example of a compact set, the Cantor set. Let I = [0,1]. Geometrically, the Cantor set can be constructed by removing "open middle thirds" from I. The first step is to remove (1/3, 2/3), leaving two closed intervals, $J_{1,1} = [0,1/3]$, and

$J_{1,2} = [2/3,1]$. Let F_1 be $J_{1,1} \cup J_{1,2}$. At the nth step, $F_n = \bigcup_{k=1}^{2^n} J_{n,k}$, where $J_{n,1}, \ldots, J_{n,2^n}$ are closed intervals, each of length 3^{-n}.

The $(n+1)^{st}$ step is performed by deleting from the middle of each $J_{n,k}$ an open interval of length $3^{-(n+1)}$.

Then $F_{n+1} = \bigcup_{k=1}^{2^{n+1}} J_{n,k}$, and the Cantor set is $F = \bigcap_{1}^{\infty} F_n$.

This set can also be described as the set of all real numbers in I that do not require the use of the digit "1" in their ternary expansion. That is, if $x \in I$, then

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n},$$

$a_n \in \{0,1,2\}$; this gives a ternary (base 3) representation for x. The expansion is unique except when x is of the form $b3^{-m}$, where $0 < b < 3^m$ and b does not divide 3^m. In this case, there are exactly two such expansions for $b \equiv 1 \pmod 3$ and for $b \equiv 2 \pmod 3$. If $b \equiv 2 \pmod 3$, either

$$x = \sum_{n=1}^{m-1} \frac{a_n}{3^n} + \frac{2}{3^m},$$

or

$$x = \sum_{n=1}^{m-1} \frac{a_n}{3^n} + \frac{1}{3^m} + \sum_{n=m+1}^{\infty} \frac{2}{3^n}.$$

If $b \equiv 1 \pmod 3$, then either

$$x = \sum_{n=1}^{m-1} \frac{a_n}{3^n} + \frac{1}{3^m}$$

$$x = \sum_{n=1}^{m-1} \frac{a_n}{3^n} + \frac{0}{3^m} + \sum_{n=m+1}^{\infty} \frac{2}{3^n}$$

We agree to take the first representation in the first case, and the second expansion in the second case, so that we do not force the use of the digit "1". Given this way of expressing numbers between zero and one, we claim that the Cantor set is the set of all such numbers that have a ternary expansion not using "1". Thus in the above expansion for x, $a_n \in \{0,2\}$.

It can be shown by induction, that for each n,

$$F_n = \left\{ x \in I \,\middle|\, \{a_1, \ldots, a_n\} \subset \{0,2\} \right\}$$

where F_n is as described in the geometric construction. This means that the geometric process of deleting (1/3,2/3) from I removes all those numbers which have $a_1 = 1$ in their ternary expansions. Deleting (1/9,2/9) and (7/9,8/9) from F_1 removes all numbers which have $a_2 = 1$, and so on. Therefore, $x \in \bigcap_1^{\infty} F_n$ iff $a_n \in \{0,2\}$, for all n. Note that $x \in I$ has the form $x = b3^{-m}$ iff x is an endpoint of some $J_{n,k}$. (We do not distinguish between the interval on the x-axis from zero to one and the real numbers from zero to one.)

The Cantor set has some interesting topological properties; it is compact, perfect, totally disconnected, and homogeneous. (A set is <u>homogeneous</u> if for each a, b in it, there exists a homeomorphism of the set to itself taking a to b.) For a proof of homogeneity, see Hocking and Young [44, p. 100]. Actually, any metric space possessing the first three of the above properties is homeomorphic to the Cantor set [44, p. 100]. We now establish the other properties mentioned.

$F = \bigcap_{n=1}^{\infty} F_n$, and each F_n closed, imply F is closed.

Since it is also bounded, F is compact by the Heine-Borel theorem. We now show that F is perfect. Let x be any element of F. Since $x \in F_n$, for all n, there exists, for each n, a k_n such that $x \in J_{n,k_n}$. Let $\epsilon > 0$ be given. To establish the existence of elements of F within ϵ of x, choose n large enough so that $3^{-n} < \epsilon$. Then $x \in J_{n,k_n} \subset (x-\epsilon, x+\epsilon)$, so that both endpoints are in the ϵ-neighborhood. Since these endpoints are in F, we have shown that F is perfect.

Since F_n contains no interval of length greater than 3^{-n}, and since $F \subset F_n$, we see that F contains no interval. But the only connected subsets of I of more than one point are intervals [28, p. 107]. Hence, each point is its own component. Therefore F is totally disconnected.

If a topological space is both compact and connected,

it is a continuum. Thus, a continuum is invariant under continuous transformations. A non-degenerate continuum is a continuum which contains more than one point. A semi-continuum is any set S such that for any x,y in S, there is a continuum $C \subset S$ such that $x,y \in C$. A continuum C is a Peano continuum (sometimes called a Jordan continuum) if it is also locally connected; that is, if for each $x \in C$ and each open set U containing x, there exists a connected open set containing x contained in U. We conclude this chapter with a theorem which we shall find very useful.

Theorem 2.1: The monotone intersection of nonempty T_2 continua is a nonempty continuum.

Proof: Let $\mathscr{T}$ be a (well-ordered) index set with α_0 as its first element. Suppose that $\{C_\alpha\}_{\alpha \in \mathscr{T}}$ is a family of nonempty T_2 continua such that $C_{\alpha_0} \supset \ldots \supset C_\alpha \supset C_{\alpha+1} \supset \ldots$.

If $\bigcap_{\alpha \in \mathscr{T}} C_\alpha = \emptyset$, then $\bigcup_\alpha (C_{\alpha_0} - C_\alpha) = C_{\alpha_0}$. Since C_{α_0} is compact, and $C_{\alpha_0} - C_\alpha$ are open, there is a collection $\{C_{\alpha_i}\}_{i=1}^n$ such that $\bigcup_1^n (C_{\alpha_0} - C_{\alpha_i}) = C_{\alpha_0}$. Consequently, $\bigcap_1^n C_{\alpha_i} = \emptyset$, from which it follows that $C_{\alpha_n} = \emptyset$. This contradiction shows that the assumption of a vacuous intersection is false.

Since each C_α is closed, $\bigcap_{\alpha \in \mathscr{T}} C_\alpha$ is closed in the compact set C_{α_0} and is therefore compact. If $\bigcap_\alpha C_\alpha$ is

disconnected, then there exist disjoint, nonempty sets K_1, K_2 that are closed in $\bigcap_\alpha C_\alpha$ and hence are closed in C_{α_0}. Then K_1 and K_2 are compact, and, since C_{α_0} is T_2, there exist disjoint sets O_1, O_2 open in C_{α_0} with $K_1 \subset O_1$ and $K_2 \subset O_2$.

$O_1 \cup O_2 \supset \bigcap_\alpha C_\alpha$ implies that $(C_{\alpha_0} - O_1) \cap (C_{\alpha_0} - O_2)$ is contained in $\bigcup_\alpha (C_{\alpha_0} - C_\alpha)$. Since the latter set is an open covering for the compact set $(C_{\alpha_0} - O_1) \cap (C_{\alpha_0} - O_2)$, we have $(C_{\alpha_0} - O_1) \cap (C_{\alpha_0} - O_2) \subset \bigcup_1^m (C_{\alpha_0} - C_{\alpha_i})$. Therefore, it follows that $O_1 \cup O_2 \supset \bigcap_1^m C_{\alpha_i} = C_{\alpha_m}$.

Now, $C_{\alpha_m} \cap O_i \supset \bigcap_\alpha C_\alpha \cap K_i = K_i \neq \emptyset$, $i = 1, 2$, and $(C_{\alpha_m} \cap O_1) \cap (C_{\alpha_m} \cap O_2) = \emptyset$. Consequently, C_{α_m} is not connected, which is a contradiction, and the theorem holds.

CHAPTER 3

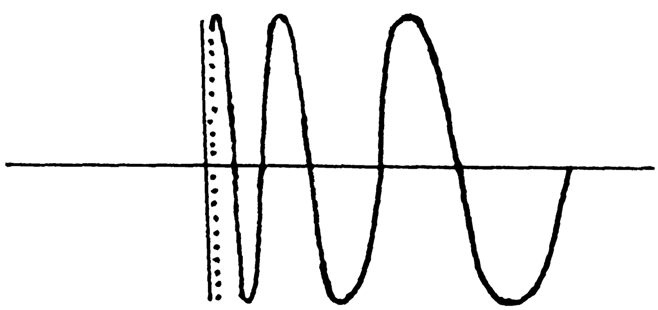
EARLY EXAMPLES

In this chapter we give the major definitions of the thesis, along with some early examples of indecomposable continua.

There are two distinct types of continua: the decomposable and the indecomposable. A continuum is <u>decomposable</u> if it is the union of two proper subcontinua; otherwise, it is <u>indecomposable</u>. The concept of indecomposability is very easy to state; however, it is not so easy to see that such sets actually exist. Most of the usual examples of continua are decomposable. For example, in E^2 the line segment joining two distinct points a and b is "very decomposable": it is the union of the segments ac and cb, where c is any point of the segment except a or b. Even the so-called "topologist's sine curve," the continuum

$$C = \{(x,y) \mid 0 < x \leq 1,\ y = \sin \pi/x\} \cup \{(0,y) \mid -1 \leq y \leq 1\},$$

is not sufficiently pathological to be indecomposable.



However, this continuum does have the related property of being irreducible between (1,0), and any point (0,y) for $|y| \leq 1$. A continuum is irreducible between two of its points if no proper subcontinuum contains both points. These seemingly distinct concepts are not only closely related mathematically, but they also share a common historical origin.

The first indecomposable continuum was constructed by L. E. J. Brouwer in 1910 [15], although he never used the term "unzerlegbaren Kontinuen" in his paper. He used this set to disprove a conjecture made by Schoenflies that if a "closed curve" is the common boundary of two plane domains, then it must be expressible as the union of two proper subsets, each of which is a "curve".

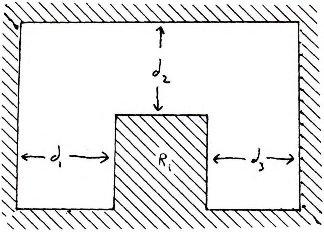
The concept of an irreducible continuum was defined and studied by Zoretti in 1909 [133]. He credited the Schoenflies papers with inspiring his work, although his terminology was different than Schoenflies'. Brouwer was later involved in the developement of irreducible continua, again as a critic. He pointed out several errors in Zoretti's work, saying in particular that his own example of an indecomposable continuum was a counterexample to Zoretti's statement that the "exterior boundary of a domain" can be decomposed into two subcontinua having only two points in common [16]. Zoretti took note of these comments [134] by pointing out that he had already published corrections.

The works of both Schoenflies and Zoretti were at least partially motivated by a desire to give a set-theoretic characterization of "curve". We shall see in later chapters that this same goal inspired the works of several others who contributed to the study of indecomposable continua.

The rest of the chapter will be devoted to describing several of the original examples of indecomposable continua: Brouwer's example, along with certain related examples, and the Lakes of Wada. In view of the opening remarks of this chapter, we would not expect any of these continua to have simple descriptions, and in fact they do not. This may be one reason why indecomposable continua were viewed as being just pathological examples. This opinion seems to have been shared by the discoverers of indecomposable continua and their contemporaries until about 1920, when several theoretical results were published. (We shall discuss these in great detail in the next chapter.) We begin our list of examples with Brouwer's construction (paraphrased slightly in translation) of his first indecomposable continuum [15].

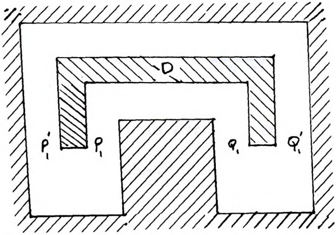
First, form a rectangle R_0 in E^2 of length a and height b. (He called this his "principal rectangle".) The general procedure is to deform the boundary of R_0 by removing a sequence of domains R_n inside R_0 and to simultaneously delete another domain D inside R_0, disjoint from the R_n's.

R_1 is a rectangle based in the middle of R_0's baseline, constructed so that the twice-bent white strip, R_0-R_1, possesses the same width in its three parts: $d_1 = d_2 = d_3$.



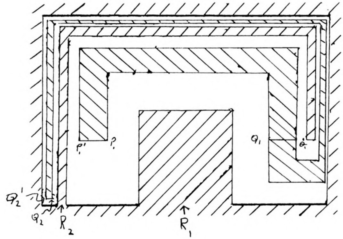
Let the ratio of its baseline to that of R_0 be $\frac{1}{2\alpha+1}$,
where α is any positive real number. Then the white strip has a width of $\left[\frac{\alpha}{2\alpha+1}\right]a$.

Next, draw the shaded strip D between the cross-sections P_1P_1' and Q_1Q_1', as shown. It consists of a strip of width $\left[\frac{1}{2\alpha+1}\right]\left[\frac{\alpha}{2\alpha+1}\right]a$, whose boundary is everywhere parallel to, and at a distance $\left[\frac{\alpha}{(2\alpha+1)^2}\right]a$, from the boundary of the white domain which contains it.



R_2 is now constructed surrounding the already drawn portion of D, beginning on the left hand side of the base-

line of R_o, and ending on the right at the height of Q_1Q_1'. This strip also has a width of $\frac{1}{2\alpha+1}$ that of the white strip in whose center it lies. D is continued from Q_1Q_1' through the middle $\frac{1}{2\alpha+1}$ th part of the new white domain to Q_2Q_2'; these latter points have the same distance from the baseline of R_o as the vertical boundary through Q_2' has from the boundary of R_2.



R_3 is constructed around the existing part of D, beginning on the baseline of R_o and ending at the same height as P_1P_1'. It too removes $\frac{1}{2\alpha+1}$ of the width of the white strip containing it.

Thus, in general, D is extended from both ends; a continuation from Q_nQ_n' to $Q_{n+1}Q_{n+1}'$ is followed by an extension from P_nP_n' to $P_{n+1}P_{n+1}'$. Each R_n and each extension of D is to have a width of $\frac{1}{2\alpha+1}$ that of the white strip which contains it. See Figure 3.1, p. 20.

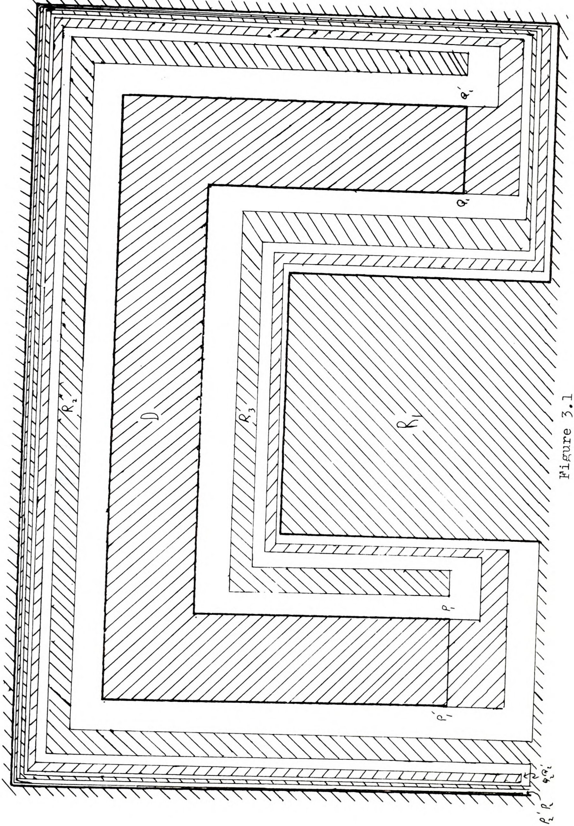


Figure 3.1

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Every point of the boundary of an R_n lies arbitrarily near the domain D by a sufficient continuation of this process. Likewise, every point of the boundary of D finally lies arbitrarily near the R_n's. But D and the R_n's are disjoint, being "fully separated" by K, the complete boundary of D. The complement of K contains only two domains, namely D and the R_n's together with the plane outside R_0 [15, p. 424].

Brouwer gave several other examples by slightly modifying the above construction. He also indicated that his process could be used to construct a continuum that is the common boundary of 3, 4, 5, . . . , or even a countably infinite number of domains.

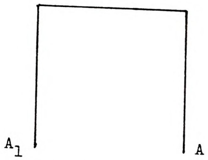
Janiszewski gave a simplification of the above example in his thesis (1911) essentially by taking $\alpha = 1$, $a = b$, and dropping the domain D [48, p. 114, or 49, p. 68]. Thus his example does not have the property of being the common boundary of two domains. Hence his continuum is actually distinct from the quoted one of Brouwer, in spite of the fact that it is based on the latter's work. Note that Janiszewski's technique of describing his continuum is more concise than Brouwer's.

"On the base A_1A of a square, let a sequence of points A_1, A_1', A_2, A_2', . . . be such that $\overline{A_{k+1}A} = (1/3)\overline{A_kA}$, and $\overline{A_k'A} = (2/3)\overline{A_kA}$. Our figure is composed of:

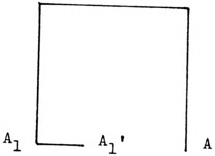
1) the broken line AA_1 formed from three sides of the

1H

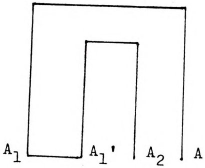
square [as shown];



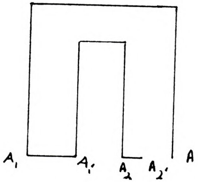
2) the segment A_1A_1';



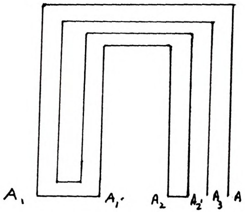
3) the broken line $A_1'A_2$ parallel to A_1A;



4) the segment A_2A_2';



5) the broken line $A_2'A_3$ parallel to the line $A_2A_1' \cup A_1'A_1 \cup A_1A$;



6) and so on [48, p. 114, or 49, p. 68]." He does not say so here, but he intends to include the closure of the imbedded topological ray thus constructed [48, p. 120, or 49, p. 74].

Janiszewski makes no mention of indecomposability in connection with this example. Rather, since his thesis concerned irreducibility, he used this as an example of a continuum irreducible between A and B, where B is in the closure of the imbedded topological ray. Later, [48, p. 120 or 49, p. 74] he notes that if C is any point on a vertical segment whose abscissa is incommensurable with those of A, and B, as well as with any linear combination of them with rational coefficients, then his continuum is irreducible between any two of A, B, C. The fact that a continuum is indecomposable iff it contains three points such that it is irreducible between any two of them (p. 43) was not published (by him and Kuratowski) until 1920, but he seems to

have thought that the property was interesting enough in 1911 to deserve mention.

Knaster developed a simpler means of describing Janiszewski's example. His work appeared in a paper by Kuratowski on irreducible continua [69, pp. 209-210]. The construction is given below, with only a slight change in notation.

Let F denote the Cantor set, and let G_n, for $n \geq 1$, denote the set of points $G_n = \{x \in F | (2/3^n) \leq x \leq (1/3^{n-1})\}$.

Using the point (1/2,0) as center, construct a set of semi-circles above the x-axis having F as its set of endpoints. The points $(5/2)(1/3^n)$ are the centers of the semi-circles below the x-axis whose endpoints are the points of G_n. The set thus formed for all natural numbers n is an indecomposable continuum.

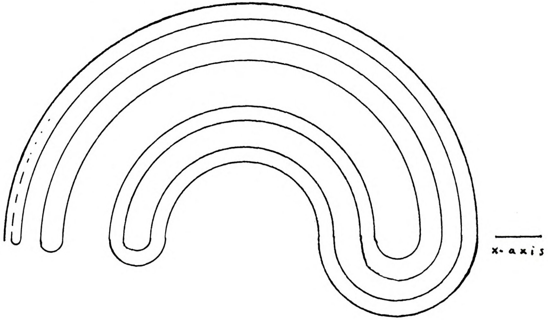


Figure 3.2

Thus, in the span of twelve years, Brouwer's example was modified and condensed from the major point of a paper to a footnote in small print. The verification that this continuum is indeed indecomposable can be more easily given after adequate theory has been developed; see p. 53. There was no proof of indecomposability with the example when it appeared in Kuratowski's paper.

To see that Knaster's first "semi-circle example" is related to Janiszewski's version, we modify the latter's example slightly. Suppose that his square is the unit square and that we delete from it those regions contiguous to its base along $A_n'A_{n+1}$, for n = 1, 2, 3, ... Intersecting this figure with the line y = 1/2 gives the Cantor set on that line. Moreover, if the rectilinear segments are replaced by semi-circles, then we get Knaster's example with left and right reversed.

The Brouwer example is fairly typical of the early work in the study of indecomposable continua. It was viewed as a pathological counterexample, and little else. This may have been at least partially due to the fact that generally the early examples were described by means of rather complicated constructions, as we have seen. A great deal of machinery was required to verify that a given continuum was indecomposable, if indeed explicit verification was given at all. As Paul Urysohn put it, "the reason for this is that the necessary and sufficient conditions for a continuum to be indecomposable are logically simple, but very

few are manageable in practice [111, p. 225].

One of the most famous examples of an indecomposable continuum was presented by the Japanese mathematician Yoneyama in 1917. His extensive English paper entitled "Theory of Continuous Sets of Points" [131] dealt with a theory of "curves" in Euclidean space. But, it was most noted in the literature of that day and this for its presenting the example now known as the Lakes of Wada. This example also occupied the status of being little more than a novelty, so far as the author's major intent is concerned.

Yoneyama used Wada's example to show that in E^2 there exists a continuum C containing three points such that C is irreducible between any two of them, although he did not use this terminology. (See Chapter 4 for his wording.) He was not concerned with indecomposability in this paper. From the direction of his research, it seems doubtful that he was aware that the example has this property. (Again more information is in Chapter 4.)

Essentially, the continuum was described in terms of digging canals in an ocean island containing a fresh water lake. (Motivated perhaps by the geography of his native Japan.) His construction is quoted at length below.

"Suppose that there is a land surrounded by sea and that in this land there is a fresh lake. Also suppose that from the lake and sea canals are built to introduce the waters of them into the land according to the following scheme.

"Let $E_1, E_2, \ldots, E_n, E_{n+1}, \ldots$ be a sequence of positive numbers monotonously [sic] converging to zero; namely let $E_1 > E_2 > \ldots > E_n > E_{n+1} > \ldots$ and $\lim_{n\to\infty} E_n = 0$.

"On the first day a canal is built from the lake such that it does not meet the sea water and such that the shortest distance from any point on the shore of the sea to that of the lake and canal does not exceed E_1. The enp-point of this canal is denoted by L_1.

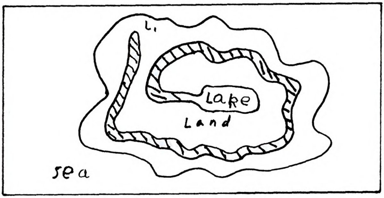


Figure 3.3

"On the second day a canal is built from the sea, never meeting the fresh water of the lake and canal constructed the day before, and the work is continued until the shortest distance from any point on the shore of the lake and canal filled with fresh water to that of the sea and canal filled with salt water does not exceed E_2. The endpoint of this canal is denoted by S_2. [See Fig. 3.4.]

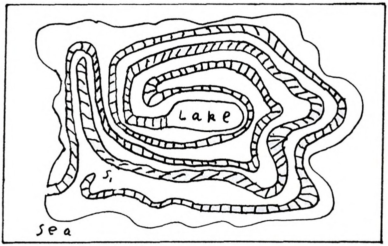


Figure 3.4

"On the third day the work is begun from L_1 never cutting the canals already built, and the work is continued until the shortest distance from any point on the shore of

the sea and canal filled with salt water to that of the lake and canal filled with fresh water does not exceed E_3. The endpoint of this canal is denoted by L_3.

"Now it is clear that we can continue the work day by day in the above way, by adequately narrowing the breadth of the canals, since the land is always semi-continuous [i.e. a semi-continuum] at the end of the work of every day. If we proceed in this way indefinitely, we get at last an everywhere dense set of waters, fresh and salt, which never mingle together at any place.

"Now denote by M_L the shore of the lake and canal filled with fresh water, and by M_S that of the sea and canal filled with salt water, and by M_P the set of limiting points of M_L and M_S not contained in them. Then the sum of M_L, M_S, M_P forms a continuous set [continuum], and any three points, each taken from the above different sets form a system of three points, every two of which form a pair of principal points of the set [i.e. the continuum is irreducible between any two of those three points].

". . . If we suppose that there are many such lakes in the land, we may obtain by the similar method a continuous set having the property [131, pp. 60-62]."

The construction mentioned in the last paragraph is carried out in Hocking and Young's Topology [44, pp. 143-144] for two lakes. Yoneyama supplies no further proof that his set has the desired properties, which is fairly typical of the era prior to 1920.

Parenthetically, it is interesting to note that other new disciplines were studying pathological examples of their own. In the same volume of the Tôhoku Mathematical Journal in which Yoneyama described the Lakes of Wada, Sierpinski gave an example of a non-measurable set which is a slight generalization of today's standard example [110].

A further investigation of the Lakes of Wada was made by Paul Urysohn [116, pp. 231-233] as a tool in his monumental study of Cantor manifolds in a separable metric

space [115], [116], [117]. The goal of his work was to establish the most general possible topological definitions of "line" and "surface". Much of this work was published posthumously under the supervision of Paul Alexandroff, following Urysohn's death in 1924. His untimely death at the age of twenty-six was the result of a swimming accident [1].

Urysohn's contribution to the Lakes of Wada was an outline of a proof of the indecomposability of the continuum, based on a necessary and sufficient condition for indecomposability which he had developed. (See Chapter 5.) He noted that for "a convenient distribution of canals" the continuum is indecomposable, but that he did not know if the "construction always gives an indecomposable continuum for any distribution of canals [116, p. 232]."

He also indicated that the construction can be generalized by:

1) allowing a countable number of lakes, provided that they "converge to a single point";
2) allowing certain lakes or even all lakes to have no canals;
3) allowing other lakes to have several, or even a countable number of canals;
4) allowing certain canals to have only a finite length [116, p. 233].

In closing this chapter, we note one more contribution to the study of the Lakes of Wada. While trying to extend

Schoenflies' results on plane sets of points to higher dimensions, R. L. Wilder showed that the Wada construction does not necessarily yield an indecomposable continuum in E^3 [129, pp. 275-279]. His first result was to use this construction to dig tunnels in a certain solid to get a surface which is a Peano continuum (and hence decomposable), and yet is the common boundary of three (or countably many) domains in E^3. More will be said about this in Chapter 8.

CHAPTER 4

BASIC STRUCTURE THEOREMS

Prior to 1920, there were only two papers on indecomposable continua which could be considered theoretical. The first was by Arnaud Denjoy in 1910 [26], and the second was Yoneyama's in 1917 [131]. However, neither work seems to have been very influential in the study of indecomposable continua.

In his paper, Denjoy announced that he believed "one could construct three domains and even a countable number of domains which all have the same boundary [26, p.28]." In such a case, "the points of such a frontier F situated on an arbitrary straight line must form a perfect everywhere non-dense [nowhere dense] set e if the line contains no continuous [connected] portion of F [26, p. 138]." Since Denjoy's results were stated without proof, and later papers make scant reference to them, we say no more about them.

Yoneyama's paper had only slightly more impact on later theoretical investigations of indecomposable continua. One of his theorems was used by Kuratowski to help establish a theorem on indecomposable continua [69, p. 208].

One reason why Yoneyama's work does not seem to have

been very influential is that his terminology was highly non-standard with respect to the European school of mathematics. For example, he used the word "component" to mean subset, and "continuous set", which corresponds to our word "continuum", to mean a connected perfect set. Furthermore, in place of "irreducible continuum", Yoneyama's concept was stated in the following terms. Let S be a "continuous set" and let a, b be points of S. a and b are principal points of S if no proper "continuous component" of S contains a and b [131, p. 47].

It is interesting to compare this terminology to the European, so we state a theorem of Yoneyama both ways. The original version reads as follows: "When a continuous set has two pairs of principal points, it has always two pairs of them having one point in common [131, p. 48]." On the other hand, Kuratowski, in using the above result, stated it this way: "If e is irreducible between a and b and between c and d then e is irreducible between a and c or a and d [69, p. 208]."

One apparent difference in the above viewpoints seems to be that Yoneyama classifies "continuous sets" according to the number and type of principal points which they possessed. On the other hand, Kuratowski and other Europeans studied the entire continuum, rather than just certain points of it. Their technique seems a little more natural in the sense that irreducibility between two points of a set results from the structure of the set, rather than from

any property inherent in those points alone.

As we proceed into more specialized concepts, we find the differences in terminology and technique growing. Yoneyama's definition of what we would call an indecomposable continuum is given as: "a continuous set having a system of three points, every two of which form a pair of principal points of the set, is called a singular set of points [131, p. 62]."

He proved several theorems concerning some properties of singular sets, but he gave no necessary and sufficient conditions for a continuous set to be singular. Because he was interested primarily in the principal points of a set, rather than in properties of the entire set, it is doubtful that he knew or was interested in the fact that singular sets are indecomposable. For example, his only use of the Lakes of Wada was to show that singular sets exist. Thus, a second reason why his work does not seem to have had a significant influence is that his point of view and direction diverged from those of his western contemporaries.

Beginning in the early 1920's, indecomposable continua were studied more as entities in themselves, rather than just as pathological examples. The first European paper devoted exclusively to studying properties of indecomposable continua — sans examples — was published by the Polish mathematician Stefan Mazurkiewicz in 1920. In it he answered affirmatively the following question posed by Janiszewski, Knaster and Kuratowski [86, p. 35]. Given an

indecomposable continuum C, can one determine two points in C such that C is irreducible between them? In fact, he proved a stronger result. Using Baire category theory, he was able to show that an indecomposable continuum in E^n has three points such that the continuum is irreducible between any two of them. Moreover, it appears that he was the first to use the word "indecomposable" to name these sets, and R. L. Moore credits him with being the originator of the term [100, p. 363]. Instead of giving Mazurkiewicz' results here in more detail, we include them in the next section of the chapter where they can be more naturally presented.

It should be noted here that during this time, the word "continuum" meant a closed connected set rather than a compact connected set. However, Mazurkiewicz restricted his work to bounded, closed, connected sets in E^n. So, thanks to the Heine-Borel theorem, his concept of continuum coincides with ours in E^n. The fact that he worked in E^n was not a restriction as far as his contemporaries were concerned, since they too were working in Euclidean space. Often, papers of this era made no explicit mention of what their underlying space was, perhaps because the geometric nature of the results and examples seemed to be self-evident. Perhaps also, interest in more general spaces had not yet become widespread.

By far the most significant paper published on indecomposable continua theory during the early 1920's was "Sur les Continus Indecomposables", by Janiszewski and Kuratowski.

It appeared in the first volume of the Fundamenta Mathematicae (1920), the same one which contained Mazurkiewicz' above mentioned paper. The importance of the Janiszewski and Kuratowski paper lies in the fact that it gave several necessary and sufficient conditions for a continuum to be indecomposable. The authors also defined the fundamental concept of a "composant" and established some properties of such sets. The significance of this paper is best proved by the many later references to its results.

From the fact that several proofs make explicit use of the metric properties of Euclidean space, it seems likely that Janiszewski and Kuratowski considered an indecomposable continuum as a subset of some E^n. However, their definitions and results can be placed in a more general setting very easily, and we shall do just this.

Before discussing any of the results, we present some definitions. A set A is called a boundary set in X if $A \subset \overline{X-A}$. A subcontinuum K of a continuum C is called a continuum of condensation if $K \subset \overline{C-K}$. Hence, if K is a continuum of condensation of C, it is nowhere dense in C, since it is closed. Equivalently, K is a continuum of condensation of C iff $\overline{C-K} = C$. For if K is a continuum of condensation of C, then $C = K \cup (C-K) = K \cup (\overline{C-K}) = \overline{C-K}$; the converse is trivial. These definitions are still used today, with the only change being that the word "continuum" generally means a compact connected set rather than a closed connected set.

The first three results were not part of the original paper, but we include them here because they simplify some of Janiszewski and Kuratowski's proofs.

Lemma 4.1: A subset Y of a space C is connected iff there do not exist two nonempty subsets A, B of Y such that $Y = A \cup B$ and such that $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$ [44, p. 15].

Proof: If the sets do exist, then $C-\bar{A}$ is an open set containing B and $C-\bar{B}$ is an open set containing A. Therefore

$$Y = [(C-\bar{A}) \cap Y] \cup [(C-\bar{B}) \cap Y],$$

and

$$[(C-\bar{A}) \cap Y] \cap [(C-\bar{B}) \cap Y] = [(C-\bar{A}) \cap (C-\bar{B})] \cap Y =$$

$$= [C-(\bar{A} \cup \bar{B})] \cap Y = \emptyset.$$

Thus Y is disconnected.

If Y is disconnected, then $Y = (O \cap Y) \cup (V \cap Y)$, where O, V are open in C, and $O \cap Y$, $V \cap Y$ are nonempty and disjoint. Set $A = O \cap Y$, $B = V \cap Y$.

Lemma 4.2: Let X be a connected subset of a connected set C. If C-X is disconnected, say $C-X = M \cup N$, then $X \cup M$ and $X \cup N$ are connected. Moreover, if X is closed, then $X \cup M$ and $X \cup N$ are closed [62, pp. 210-211].

Proof: By Lemma 4.1, $C-X = M \cup N$, where $M \neq \emptyset \neq N$, and $(M \cap \bar{N}) \cup (\bar{M} \cap N) = \emptyset$. Suppose $X \cup M = A \cup B$, where $A \neq \emptyset$ and $B \neq \emptyset$, and $(\bar{A} \cap B) \cup (A \cap \bar{B}) = \emptyset$. Since X is connected, we may assume $X \cap A = \emptyset$, whence $A \subset M$. We now disconnect C.

$$C = X \cup M \cup N = A \cup (B \cup N);$$

$$A \neq \emptyset \neq (B \cup N);$$

$$A \cap (\overline{B \cup N}) = (A \cap \bar{B}) \cup (A \cap \bar{N}) \subset (A \cap \bar{B}) \cup (M \cap \bar{N}) = \emptyset;$$

$$\bar{A} \cap (B \cup N) = (\bar{A} \cap B) \cup (\bar{A} \cap N) \subset (\bar{A} \cap B) \cup (\bar{M} \cap N) = \emptyset.$$

This contradiction establishes the first part of the Lemma.

If X is closed, then $\overline{X \cup M} = \overline{X} \cup \overline{M} = X \cup \overline{M} = (X \cup \overline{M}) \cap (X \cup M \cup N) = X \cup M$, since $\overline{M} \cap N = \emptyset$. Therefore, $X \cup M$ is closed. Likewise $X \cup N$ is closed.

Lemma 4.3: Let C be an indecomposable continuum and let K be any proper subcontinuum. Then C-K is connected.

Proof: If C-K is disconnected, then $C-K = M \cup N$, where M, N are nonempty sets such that $(\overline{M} \cap N) \cup (M \cap \overline{N}) = \emptyset$, by Lemma 4.1. By Lemma 4.2, $K \cup M$, $K \cup N$ are continua, and their union is C. Since each is a proper subcontinuum, we have a contradiction.

The next theorem is of major importance. It was included in the Janiszewski and Kuratowski paper, and is due entirely to Janiszewski [50, p. 210].

Theorem 4.4: In order that a T_2 continuum C should be indecomposable, it is necessary and sufficient that each proper subcontinuum of C should be a continuum of condensation [50, p. 212].

Proof: If C is decomposable, then $C = C_1 \cup C_2$, where C_1 and C_2 are proper subcontinua of C. Thus, $C-C_1 \subset C_2$, from which $\overline{C-C_1} \subset C_2 \neq C$. Therefore, C_1 is a proper subcontinuum of C that is not a continuum of condensation of C. (This part of the proof is essentially as Janiszewski gave it.)

Conversely, let C be indecomposable. Suppose that the condition does not hold, so that there exists a proper sub-

continuum K of C such that $\overline{C-K} \neq C$. $C = K \cup (C-K) = K \cup (\overline{C-K})$. By Lemma 4.3, C-K is connected, and hence so is $\overline{C-K}$. Thus, the latter set is a proper subcontinuum of C, and we have contradicted the indecomposability of C.

Corollary 4.5: Let C be a non-degenerate T_2 indecomposable continuum. Then C is not locally connected at any point.

Proof: On the contrary, suppose there is a point $a \in C$ such that C is locally connected at a. There exists $b \in C$, distinct from a, and there exist disjoint open sets U, V containing a, b respectively. By local connectiviety, there exists a connected open set K containing a and contained in U. $\overline{K}$ is a subcontinuum of C, and $\overline{K} \cap V = \emptyset$ implies $\overline{K} \neq C$. Moreover, $K \cap (C-\overline{K}) = \emptyset$ implies $K \cap \overline{C-\overline{K}} = \emptyset$. Thus, $\overline{K}$ is a proper subcontinuum of C which is not a continuum of condensation, contradicting Theorem 4.4.

The converse to the Corollary is false. (Neither this statement nor the Corollary were part of the Janiszewski-Kuratowski paper.) To see that it is false, consider the continuum in E^3 constructed as follows: Construct the Cantor set on the x-axis between (1,0,0) and (-1,0,0) and on the line segment joining (0,0,1) and (0,1,0). Next, construct all the line segments determined by the points of the two Cantor sets. The set so formed is a decomposable continuum that is locally connected at no point.

By definition, an indecomposable continuum is not the union of any two proper subcontinua. Surprisingly, "two" can be replaced by a "countable number", provided C is T_2.

We prove this by using Theorem 4.4 and the following lemmas.

Lemma 4.6: If a topological space is compact and Hausdorff, then it is regular.

Proof: See Dugundji, [28, p.223].

Lemma 4.7: Let X be regular, $x \in X$, and let U be any neighborhood of x in X. Then there is a neighborhood O of x such that $x \in O \subset \overline{O} \subset U$.

Proof: See [28, p. 141].

Lemma 4.8: Let C be a T_2 continuum. Then C is not the union of a countable number of closed nowhere dense subsets.

Proof: Let $\{A_i\}_{i=1}^{\infty}$ be a collection of closed nowhere dense subsets of C, and suppose that $C = \bigcup_{i=1}^{\infty} A_i$. Then $\bigcap_{i=1}^{\infty} (C-A_i) = \emptyset$, which we shall show is false.

Since each A_i is closed and nowhere dense in C, $C-A_i$ is open and dense in C, for each natural number i. We shall show that $\bigcap (C-A_i)$ is dense in C. Suppose U is any nonempty open set in C. Then, for each i, $U \cap (C-A_i) \neq \emptyset$. Hence, $U \cap (C-A_1)$ is nonempty and open in C. If x is any element of this set, then by Lemmas 4.6 and 4.7, there exists an open subset B_1 such that $x \in B_1 \subset \overline{B}_1 \subset U \cap (C-A_1)$. Likewise, there exists an open set B_2 in C such that $\emptyset \neq B_2 \subset \overline{B}_2 \subset B_1 \cap (C-A_2)$.

Inductively, we obtain a sequence $\{B_n\}$ of nonempty open sets such that $\overline{B}_n \subset B_{n-1} \cap (C-A_n)$, for each n. Since Since $\bigcap_{n=1}^{k} \overline{B}_n = \overline{B}_k \neq \emptyset$ and C is compact, then $\bigcap_{n=1}^{\infty} \overline{B}_n \neq \emptyset$.

$\bar{B}_1 \subset U \cap (C-A_1)$ and $\bar{B}_n \subset (C-A_n) \cap B_{n-1}$ imply that $\emptyset \neq \bigcap_{n=1}^{\infty} \bar{B}_n \subset U \cap \bigcap_{n=1}^{\infty} (C-A_n)$. Therefore, $\bigcap_{n=1}^{\infty} (C-A_n)$ is dense and certainly not empty. Therefore, $\bigcup_{n=1}^{\infty} A_n \neq C$.

Theorem 4.9: A Hausdorff indecomposable continuum is not the union of any countable collection of proper subcontinua.

Proof: If $\{K_i\}_{i=1}^{\infty}$ is any family of proper subcontinua of C, then by Theorem 4.4, each K_i is nowhere dense in C. Each K_i is closed, so by Lemma 4.8, $C \neq \bigcup_{i=1}^{\infty} K_i$.

This result was used by Urysohn in a paper [116, p. 243] that we shall consider in the next chapter.

To help establish the rest of their results, Janiszewski and Kuratowski made the following important definition. The notation is theirs. Let C be a continuum, and let $a \in C$. $P(a,C) = \{c \in C \mid a,c$ can be joined by a proper subcontinuum of $C\}$. Hence, $P(a,C) = \bigcup_{\alpha \in \eta} K_\alpha$, where $a \in K_\alpha$ and K_α is a proper subcontinuum of C. Clearly, P(a,C) is a semi-continuum. P(a,C) is called the composant of a in C.

In their paper, Janiszewski and Kuratowski only used the word "composant" when the above sets had the property that for all a, b in C, P(a,C) = P(b,C), or else $P(a,C) \cap P(b,C) = \emptyset$. Current usage is largely as we have given it, although in some cases "subcontinuum" is replaced by "closed connected set". The next theorem was actually proved nearer to the end of the Janiszewski-Kuratowski paper, but we shall make use of it earlier.

Theorem 4.10: If a and b are any two points of an indecom-

posable continuum C, then the composants are either disjoint or coincident [50, pp. 217-218].

Proof: Suppose instead that $P(a,C) \neq P(b,C)$ and that $P(a,C) \cap P(b,C) \neq \emptyset$. Pick $c \in P(a,C)-P(b,C)$, and choose $d \in P(a,C) \cap P(b,C)$. By definition of composant, there exists a proper subcontinuum C_1 of C such that $a,c \in C_1$. Likewise, there exist proper subcontinua C_2 and C_3 of C containing a,d and b,d respectively. C_1, C_2, C_3 being compact imply that $C_1 \cup C_2 \cup C_3$ is compact. $d \in C_2 \cap C_3$ and $a \in C_1 \cap C_2$ imply that $C_1 \cup C_2 \cup C_3$ is connected. Therefore, this union is a continuum, and $b,c \in C_1 \cup C_2 \cup C_3 \subset C$. Since $c \notin P(b,C)$, C is irreducible between b and c. Thus, $C_1 \cup C_2 \cup C_3 = C$. Finally, $C = C_1 \cup C_2$, or else $C = (C_1 \cup C_2) \cup C_3$ show that C is decomposable.

Using the definition of "composant", Janiszewski and Kuratowski restated Mazurkiewicz' theorems as follows:

Theorem 4.11:

(a) If a is any point of a metric indecomposable continuum C, then the set P(a,C) is of first category.

(b) For any point a in a metric indecomposable continuum C, the set P(a,C) is a boundary set in C.

(c) If C is a metric indecomposable continuum, then there exist three points such that C is irreducible between any two of them [50, p. 215].

Proof: (a): (adapted from [44, p. 140]) Let $a \in C$ be arbitrary, and let P(a,C) be as above. C being metric

implies that $C-\{a\}$ is open and has a countable basis $\{O_n\}$ [28, p. 233]. Let $K_n(a)$ be the component of $C-\bar{O}_n$ containing a. Then $\overline{K_n(a)}$ is connected, and, since C is compact, $\overline{K_n(a)}$ is compact. Moreover, this subcontinuum is proper, since $K_n(a) \subset C-\bar{O}_n$ implies that

$$\overline{K_n(a)} \subset \overline{C-\bar{O}_n} \subset C-\bar{O}_n \neq C.$$

Therefore, $\overline{K_n(a)} \subset P(a,C)$, for each n, so that we have $\bigcup_{n=1}^{\infty} \overline{K_n(a)} \subset P(a,C)$.

On the other hand, if $x \in P(a,C)$, then there exists a proper subcontinuum C' of C such that C' contains a and x. Let $p \in C-C'$; $C-C'$ is open in C- a , and $p \neq a$ shows that $p \in C-\{a\}$. Therefore, $p \in O_n$, for some n such that $O_n \subset C-C'$. Since $\bar{O}_n \subset C-C'$, $C-\bar{O}_n \supset C'$. Then by definition of $K_n(a)$, we have $C' \subset K_n(a) \subset C-\bar{O}_n$. Therefore, $x \in P(a,C)$ implies that $x \in K_n(a)$. Hence, $P(a,C) \subset \bigcup_{n=1}^{\infty} \overline{K_n(a)}$. By Theorem 4.4, each $\overline{K_n(a)}$ is nowhere dense in C, and thus P(a,C) is a first category set.

(b): By (a), $P(a,C) = \bigcup_{n=1}^{\infty} \overline{K_n(a)}$, so we have $C-P(a,C) = \bigcap_{n=1}^{\infty} (C-\overline{K_n(a)})$. Each $C-\overline{K_n(a)}$ is open, and, since $K_n(a)$ is nowhere dense, each is dense. By the proof of Lemma 4.8, $\bigcap_{n=1}^{\infty} (C-\overline{K_n(a)})$ is dense. But then $C-P(a,C)$ being dense shows that $P(a,C) \subset C = \overline{C-P(a,C)}$.

(c): By (a), each P(a,C) is the union of a countable number of closed nowhere dense sets. If there were only a

countable number of composants in C, then C would be the union of a countable number of closed nowhere dense subsets, violating Lemma 4.8. Therefore, C has uncountably many composants. By Theorem 4.10, the composants are disjoint, so choose exactly one point from each one. By the definition of "composant", C is irreducible between any two of these points.

The above proof shows that a metric indecomposable continuum has uncountably many composants, and that it is irreducible between each two points of a certain uncountable set. Mazurkiewicz used the same technique in Euclidean space, although he seemed to be satisfied with talking about three points instead of uncountably many. This may have been motivated by the fact that the converse needs only three points. He may have been aware of this, since Janiszewski and Kuratowski established the converse as well as suggesting the original problem to him. Mazurkiewicz later showed [91] that a metric indecomposable continuum has as many composants as there are real numbers.

Using Theorem 4.11, Janiszewski and Kuratowski were able to establish the following necessary and sufficient conditions for a continuum to be indecomposable.

Theorem 4.12: The following are equivalent:

(a) A metric continuum is indecomposable.

(b) For each $a \in C$, there is a point $x \in C$ such that C is irreducible between a and x.

(c) There exists $a \in C$ such that $P(a,C)$ is a boundary

set in C; that is, $P(a,C) \subset \overline{C-P(a,C)}$.

(d) There exist three points of C such that C is irreducible between any two of them [50, p. 215].

Proof: That (a) implies (b), (c),(d) follows from Theorem 4.11 (a), (b), (c) respectively.

Conversely, suppose that C is decomposable; that is assume that $C = C_1 \cup C_2$, where C_1, C_2 are proper subcontinua. To establish that (b) is now false, first choose $a \in C_1 \cap C_2$. Then $C_1 \subset P(a,C)$ and $C_2 \subset P(a,C)$. Therefore, we have that $C = C_1 \cup C_2 \subset P(a,C)$, so that C is not irreducible between a and any other point of C.

Statement (c) is also false now. Let $a \in C$ be such that (c) holds, and without loss of generality, suppose $a \in C_1$. Since $C_1 \subset P(a,C)$, then $C-P(a,C) \subset C-C_1 \subset C_2$. C_2 being closed shows that $\overline{C-P(a,C)} \subset C_2$. If (c) were true, then $P(a,C) \subset \overline{C-P(a,C)}$ would imply that $C_1 \subset P(a,C) \subset \overline{C-P(a,C)} \subset C_2$. Thus, we would have $C_1 \subset C_2$, which would imply that $C = C_1 \cup C_2 = C_2$. This contradicts $C \neq C_2$. Therefore, $P(a,C) \not\subset \overline{C-P(a,C)}$, for any $a \in C$. Thus, (c) is false.

Finally, let a, b, c be any three points of C. Without loss of generality, $a, b \in C_1$. Therefore, C is not irreducible between a, b. This shows (d) is false.

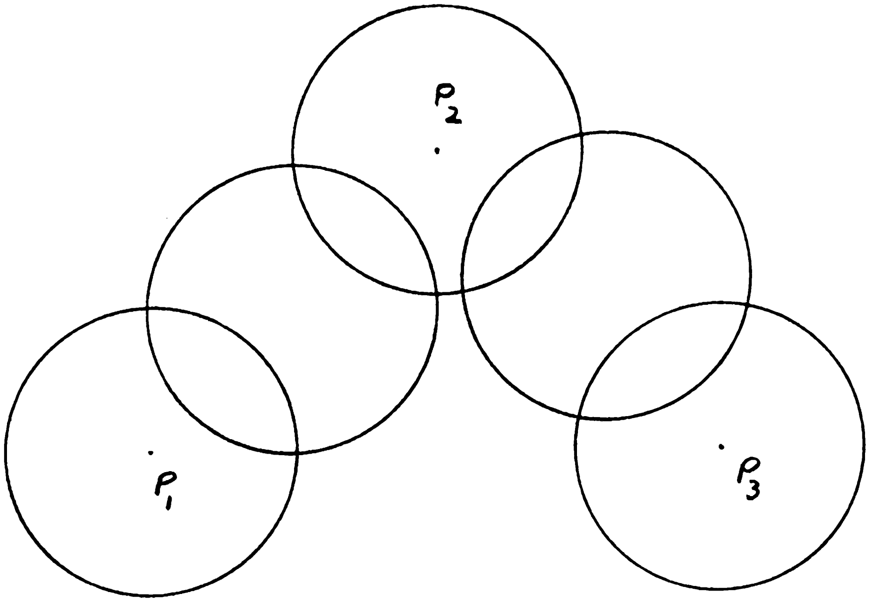
Corollary 4.13: Let C be a metric continuum. C is indecomposable iff it is irreducible between some point $p \in C$ and each $d \in D$, where D is a dense subset of C.

Proof: If C is indecomposable, then by Theorem 4.11 (b), $C = \overline{C-P(a,C)}$. Conversely, if D is any dense subset of C, and if C is irreducible between some point p and all points of D, then $C-P(p,C) \supset D$. Thus, $\overline{C-P(p,C)} \supset \overline{D} = C$, so that $P(p,C) \subset \overline{C-P(p,C)}$. By Theorem 4.12 (c), C is indecomposable.

This result was used often in the literature, but it was never explicitly stated nor proved, possibly because the proof is not difficult.

As an interesting application, Theorem 4.12 (d) can be used to construct an indecomposable continuum in the manner discussed in Hocking and Young's Topology, [44, p. 142]. This example was not part of the Janiszewski-Kuratowski paper.

Let p_1, p_2, p_3 be any three distinct points of E^2. Construct C_1, a finite simple chain of connected open sets from p_1 to p_3, containing p_2, as shown:



Inside C_1, construct another finite chain of open connected sets, C_2, from p_2 to p_3 containing p_1. Inside C_2, construct another such chain C_3 from p_1 to p_2 containing p_3, as shown in Figure 4.1.

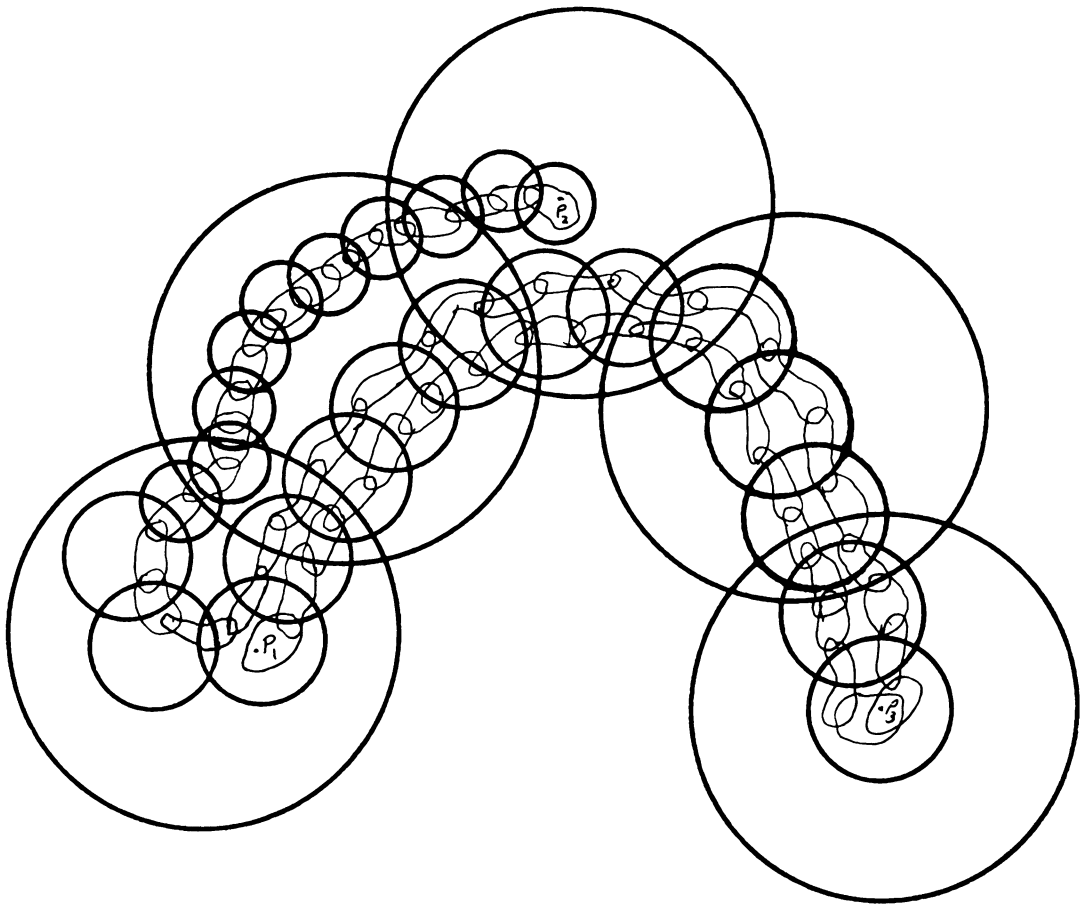


Figure 4.1

In general, C_{3n+1} is a chain from p_1 to p_3 containing p_2, C_{3n+2} is a chain from p_2 to p_3 containing p_1, C_{3n+3} is a chain from p_1 to p_2 containing p_3, and for all k, we have $C_k \subset C_{k+1}$.

Note that $C = \bigcap_{n=0}^{\infty} C_{3n+1} = \bigcap_{n=0}^{\infty} C_{3n+2} = \bigcap_{n=0}^{\infty} C_{3n+3}$. Moreover $\bigcap_{n=0}^{\infty} C_{3n+1}$ is irreducible between p_1 and p_3; $\bigcap_{n=0}^{\infty} C_{3n+2}$ is irreducible between p_2 and p_3; $\bigcap_{n=0}^{\infty} C_{3n+3}$ is irreducible between p_1 and p_2. Therefore, C is indecomposable by Theorem 4.12.

Although we have explicitly given only three points, p_1, p_2, p_3, such that C is irreducible between any two of them, it follows from the proof of Theorem 4.11 (c) that C actually has uncountably many such points. (In more current terminology, the continuum C is said to be cellular, since it is the monotone intersection of a countable number of 2-cells.)

Janiszewski and Kuratowski established a further characterization of indecomposability in terms of composants:

Theorem 4.14: In order that a metric continuum C should be indecomposable, it is necessary and sufficient that it contains two disjoint composants [50, p. 219].

Proof: By Theorems 4.10 and 4.11, a metric indecomposable continuum has uncountably many disjoint composants, and hence certainly has two.

Conversely, suppose P(a,C) and P(b,C) are two disjoint composants of C. Assume C is decomposable: $C = C_1 \cup C_2$, where C_1 and C_2 are proper subcontinua of C. Either $C_1 \subset P(a,C)$, or else $C_2 \subset P(a,C)$. But, in any case, we have $C_1 \cap C_2 \subset P(a,C)$. Likewise, $C_1 \cap C_2 \subset P(b,C)$. Therefore, $\emptyset \neq C_1 \cap C_2 \subset P(a,C) \cap P(b,C)$, contradicting the

hypothesis of disjointness. Hence, C is indecomposable, and the theorem is established.

In the last three pages of their monumental paper, Janiszewski and Kuratowski considered bounded closed connected sets, while the rest of their results held (at least in E^2) regardless of boundedness. The principal theorem established in this section may be stated in our terminology as follows: "Each composant of a T_2 continuum is dense." Their proof was done via metric properties, but we give a more general argument.

Definition: Let X be a topological space. Define a relation "$\sim$" on X by $x \sim y$ iff there is no decomposition of X into two nonempty, disjoint, open subsets, one of which contains x and the other of which contains y.

It can easily be seen that "$\sim$" is an equivalence relation. The equivalence classes are called the quasi-components of X, and we denote the quasi-component containing $x \in X$ by Q(x). Moreover, Q(x) is the intersection of all closed open subsets of X containing x [76, p. 148]. Furthermore, the component of x, C(x), is contained in Q(x). For if A is any closed open set containing x, then $C(x) \subset A$, and hence $C(x) \subset Q(x)$. To see that $C(x) \subset A$ holds, suppose $C(x) \not\subset A$; then $C(x) = A \cup [C(x)-A]$, contradicting the connectedness of C(x).

Lemma 4.15: In a compact T_2 space, the components coincide with the quasi-components.

Proof: By the above remarks, we have $C(x) \subset Q(x)$, for

each $x \in X$. By the maximality of $C(x)$, it suffices to show that $Q(x)$ is connected in order to establish the opposite inclusion.

Suppose $Q(x) = A \cup B$, where A, B are nonempty, disjoint, closed subsets of $Q(x)$. Then A, B are closed in X, since $Q(x)$ is closed in X. Therefore, A, B are compact, since X is compact. X being T_2 implies there exist disjoint open sets in X, U, V, such that $A \subset U$, $B \subset V$ [28, p. 225].

Let $M = X-(U \cup V)$; M is closed in X. Let $\{F_\alpha\}_{\alpha \in \mathcal{A}}$ be all the closed open sets in X containing x, so that $Q(x) = \bigcap_{\alpha \in \mathcal{A}} F_\alpha$. Now, $\bigcap_\alpha (M \cap F_\alpha) = M \cap \bigcap_\alpha F_\alpha = M \cap (A \cup B) = \emptyset$. By applying De Morgan's laws to the definition of compactness, it follows that there exists $\{\alpha_i\}_{i=1}^n$ such that $\bigcap_{i=1}^n (M \cap F_{\alpha_i}) = \emptyset$. Therefore, $\bigcap_{i=1}^n F_{\alpha_i} \subset U \cup V$.

<u>Claim</u>: $(\bigcap_{i=1}^n F_{\alpha_i}) \cap U$ is closed open in X.

$\bigcap_{i=1}^n F_{\alpha_i}$ is certainly closed open and $(\bigcap_{i=1}^n F_{\alpha_i}) \cap U$ is clearly open. Moreover, $(\bigcap_{i=1}^n F_{\alpha_i}) \cap \overline{U}$ is closed, and $(\bigcap_{i=1}^n F_{\alpha_i}) \cap U = \bigcap_1^n (F_{\alpha_i} \cap \overline{U})$, since $\bigcap_{i=1}^n F_{\alpha_i} \subset U \cup V$ and $\overline{U} \cap V = \emptyset$. Therefore, the claim holds.

There exists $w \in Q(x)$ such that $w \in \bigcap_{i=1}^n F_{\alpha_i} \cap U$, since $\emptyset \neq A \subset (\bigcap_{i=1}^n F_{\alpha_i}) \cap U$. $\emptyset \neq B$ implies that there exists a $z \in Q(x)$ such that $z \in X-\bigcap_{i=1}^n F_{\alpha_i} \cap U$. But then $\bigcap_{i=1}^n F_{\alpha_i} \cap U$ and its complement are nonempty, disjoint, open subsets of X, one of which contains w and the other of which contains z, contradicting the fact that $Q(x)$ is a quasi-component.

Therefore, $Q(x)$ is connected, and hence $Q(x) \subset C(x)$.

We use these results to establish the following important lemma. Janiszewski and Kuratowski also established it, although they did not use the "quasi-component technique".

<u>Lemma 4.16</u>: If K is a proper subcontinuum of a T_2 continuum C, then there exists a subcontinuum L such that $K \subsetneq L \subsetneq C$ [50, p. 220].

<u>Proof</u>: From C being compact and T_2, it follows that C is regular. Then $x \in C-K$ and K closed imply there exists an open V such that $x \in V$ and $\overline{V} \cap K = \emptyset$. Therefore, $K \subset C-\overline{V}$, and $\overline{C-\overline{V}} \neq C$. Let L be the component of $\overline{C-\overline{V}}$ containing K. L is connected , closed in the closed set $\overline{C-\overline{V}}$ and hence also closed in the compact set C. Thus, L is a proper subcontinuum of C. It remains to show that $L \neq K$.

On the one hand, $K \cap \overline{C-\overline{C-\overline{V}}} = \emptyset$, because $\emptyset = K \cap (C-C-\overline{V}) = K \cap \overline{C-C-\overline{V}} \supset K \cap \overline{C-\overline{C-\overline{V}}}$. On the other hand, suppose that $L \cap \overline{C-\overline{C-\overline{V}}} = \emptyset$. Since C is compact and T_2, $\overline{C-\overline{V}}$ is compact and T_2. Therefore, L is a quasi-component and hence is the intersection of all closed open sets in $\overline{C-\overline{V}}$ containing a given point $y \in K$, by Lemma 4.15. Thus, $L = \bigcap_{\beta \in B} G_\beta$, G_β closed open in $\overline{C-\overline{V}}$, and $y \in G_\beta$. By assumption, we have $(\bigcap_\beta G_\beta) \cap \overline{C-\overline{C-\overline{V}}} = \emptyset$, whence $\bigcap_\beta (G_\beta \cap \overline{C-\overline{C-\overline{V}}}) = \emptyset$. As before, there exists a set $\{\beta_i\}_{i=1}^{m}$ such that $\bigcap_1^m G_{\beta_i} \cap \overline{C-\overline{C-\overline{V}}} = \emptyset$. In such a case, $C = \bigcap_1^m G_{\beta_i} \cup [(\overline{C-\overline{V}} - \bigcap_1^m G_{\beta_i}) \cup \overline{C-\overline{C-\overline{V}}}]$,

which contradicts the connectivity of C. Therefore, we have $L \cap C-\overline{C-\overline{V}}$, and hence $L \neq K$.

The above proof is patterned after several in volume two of Kuratowski's Topology [76].

Theorem 4.17: In a T_2 continuum, each composant is dense [50, p. 221].

Proof: $\overline{P(a,C)}$ is a continuum, since it is connected and closed in the compact set C. If $\overline{P(a,C)} \neq C$, then it is a proper subcontinuum of C containing a. Therefore, $\overline{P(a,C)} \subset P(a,C)$, whence P(a,C) is closed and is therefore a continuum. But, since P(a,C) is then a proper subcontinuum of C, there exists a proper subcontinuum K of C properly containing P(a,C), by Lemma 4.16. However, by definition of P(a,C), $K \subset P(a,C)$. Therefore, P(a,C) is dense.

Corollary 4.18: In a T_2 continuum, a composant is not a proper subcontinuum.

Proof: This was shown in the proof of Theorem 4.17.

Corollary 4.19: If C is a metric indecomposable continuum, then for any $a \in C$, $\overline{P(a,C)} = C$; for any $a \in C$, $P(a,C) \subset \overline{C-P(a,C)} = C$.

Proof: Theorem 4.17 establishes the first statement, and the second follows from Theorem 4.11 (b).

Thus, an indecomposable continuum in a metric space is "very irreducible" in the sense that given any point $a \in C$, there is a point $x \in C$, arbitrarily near a, such that C is irreducible between a and x. For a given $a \in C$, the

set of all such $x \in C$ is dense in C. On the other hand, given any $a \in C$, C is not irreducible between a and all points of a dense subset.

Furthermore, an indecomposable continuum is "very connected" in the sense that any proper subcontinuum may be removed without disconnecting it (Lemma 4.3). Knaster and Kuratowski proved [64, p. 37] a similar result which showed that any point could be removed from a (non-degenerate) indecomposable continuum without disconnecting it. R. L. Moore established an even stronger result for Hausdorff spaces along these lines.

Theorem 4.20: Let C be a T_2 indecomposable continuum, and let K be any proper subcontinuum. If L is any subset of K, then C-L is connected [100, p. 361].

Proof: If C-L is not connected, then by Lemma 4.1, $C-L = A \cup B$, where A, B are nonempty and $(\overline{A} \cap B) \cup (A \cap \overline{B}) = \emptyset$. By Lemma 4.3, C-K is connected, so $C-K \subset A$ and $B \subset K$. By Theorem 4.4, $C = \overline{C-K} \subset \overline{A}$. Therefore, $K \subset \overline{A}$, and hence $B \subset \overline{A}$, which is a contradiction. Thus, C-L is connected.

For a related result, see p. 76.

Janiszewski and Kuratowski also established two other results for indecomposable continua. We shall present them, but since they play no role in our later work, we do not prove them.

Theorem 4.21: Let C be a metric indecomposable continuum. Each subcontinuum situated in a composant is a boundary set with respect to that composant [50, p. 221].

The following is due to Mazurkiewicz.

Definition: The relative distance between $x,y \in S$ is $d_r(x,y) = \inf \delta(E)$, where $\delta(E)$ is the diameter of E, and the infimum is taken over all connected sets $E \subset S$ containing x,y. The relative diameter of $A \subset S$ is $\delta_r(A) = \sup d_r(x,y)$, for $x,y \in A$. The oscillation of S at $p \in S$ is $\omega(p) = \inf \delta_r(A)$, where A runs over all subsets of S such that $p \in \text{Int}(A)$ [87, p. 170].

Theorem 4.22: For any point of an indecomposable continuum C in a metric space, the oscillation of C at the point is a constant and equal to the diameter of C [50, p. 217].

As our final result of the chapter, we prove Knaster's first semi-circle example (see p. 24) really is an indecomposable continuum (cf p. 161). Let D_0 be the set of semi-circles with non-negative ordinates, centered at (1/2,0) and having as endpoints the points of the Cantor set, F. For $n \geq 1$, let G_n be as before, and let D_n be the set of semi-circles with non-positive ordinates, centered at $(5/[2 \cdot 3^n],0)$ and having as endpoints the points of G_n. Then the set $B = \bigcup_0^\infty D_n$ will be shown to be an indecomposable continuum.

Let $S = \bigcup_1^\infty S_n$, where S_n is an infinite sequence of semi-circles used in constructing B, satisfying:

(a) (0,0) and (1,0) are in S_1;

(b) for $n \geq 1$, $S_n \cap S_{n+1}$ is the point of F common to both.

Let K represent the points of the Cantor set, F, which are

endpoints of some $J_{n,k}$ (see p. 9 for the notation). It can be shown by induction that $K \subset S$; that is, for each "endpoint" in F, there is a semi-circle S_m having it as an endpoint. The proof that F is perfect shows (p. 12) that K is dense in F. Therefore, $K \subset S$ and $\overline{K} = F$ imply that $\overline{S} = B$. S is clearly connected, and hence so is B. $\overline{S}$ is compact by the Heine-Borel theorem, so we have shown that B is a continuum.

We shall next show that $\overline{F-K} = F$. Given $x \in K$, and $\epsilon > 0$, we must find $y \in F-K$ such that $|x-y| < \epsilon$. Since $x \in K$, then

$$x = \sum_{n=1}^{\infty} \frac{b_n}{3^n}, \text{ where } b_n \in \{0,2\},$$ and there exists N such that

for all $n \geq N$, $b_n = 0$, or else $b_n = 2$. Choose $N_1 \geq N$ so large that $1/(3^{N_1 -1}) < \epsilon$. Any element y in F-K has a ternary

expansion of the form $y = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$. For the desired $y \in F-K$,

set

$$a_n = \begin{cases} b_n & \text{if } n < N_1 \\ 0 & \text{if } n \text{ is even and } \geq N_1 \\ 2 & \text{if } n \text{ is odd and } \geq N_1. \end{cases}$$

Then $$|x-y| = \left| \sum_{n=1}^{\infty} \frac{a_n}{3^n} - \sum_{n=1}^{\infty} \frac{b_n}{3^n} \right| \leq \sum_{n=N_1}^{\infty} \frac{|a_n - b_n|}{3^n} < \sum_{n=N_1}^{\infty} \frac{2}{3^n} =$$

$1/(3^{N_1 -1}) < \epsilon$.

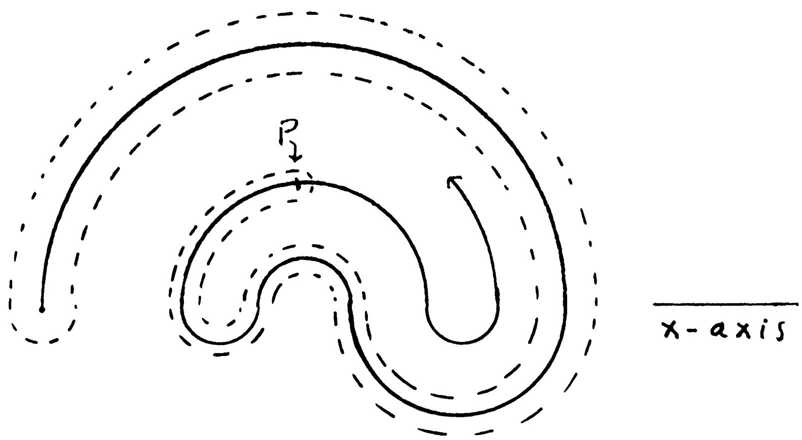
Since $\overline{F-K} = F$, then $\overline{B-S} = B$. Therefore, B-S is dense in B. By Corollary 4.13, it suffices to show that B is irreducible between (0,0) and each point of B-S. To do this, we first prove the following

<u>Lemma</u>: If L is a proper subcontinuum of B such that $L \cap S \neq \emptyset$, then there exists n such that $L \subset \bigcup_{1}^{n} S_i$.

<u>Proof</u>: Let $q \in L \cap S$. Since S is dense in B, there is a point $p \in S$ in the nonempty open set B-L. In fact, p can be chosen in such a way that if A denotes the arc in S between (0,0) and p, then $q \in A$. This can be done by simply adding the arc from (0,0) to q to the continuum L.

We shall next show $L \subset A$. If not, then there is a point $r \in L-A$. Let n_0 be a natural number such that the distance from p to L exceeds 3^{-n_0} (and hence the distance from r to A exceeds 3^{-n_0}) and such that $A \subset \bigcup_{1}^{2^{n_0}-1} S_i$.

Consider the band P formed by all the circles of radius $4/(3^{n_0+2})$ centered on A. The boundary of P is composed of two lines parallel to A and two semi-circles centered at (0,0) and p respectively.



The first three of these lines are disjoint from B, for their points of intersection with the x-axis are in the intervals that were deleted in constructing F (since $A \subset \bigcup_{1}^{2^{n_o-1}} S_i$). The fourth is disjoint from L, since the distance from p to L is greater than $1/(3^{no}) > 4/(3^{no+2})$.

Therefore, the entire boundary of P is disjoint from L. Since L is a continuum, we either have $L \subset P$ or $L \cap P = \emptyset$, contradicting $r \in L-P$ and $q \in L \cap P$. Therefore $L \subset A \subset \bigcup_{1}^{2^{n_o-1}} S_i$, which establishes the lemma.

To verify the irreducibility of B between (0,0) and any point of B-S, assume that there exists a proper subcontinuum L of B such that (0,0) L, and $y \in L$, for some $y \in B-S$. Then $L \cap S \neq \emptyset$, and $L \cap (B-S) \neq \emptyset$, so that $L \not\subset S$. This contradicts the lemma. Therefore, we conclude that no such L can exist. Thus, B is irreducible between (0,0) and each point of B-S, which proves the indecomposability of B.

The above proof is slightly modified from the one that appeared in a paper by Knaster and Kuratowski [64]. They were dealing with closed, connected, non-bounded sets in E^n. This bounded example and proof were included because they wanted to invert B-(1/2,1/2) with respect to a unit circle centered at (1/2,1/2) to obtain a closed, connected, non-bounded indecomposable set in E^2.

Thus, the proof of the indecomposability of Knaster's first semi-circle example appeared two years after the example itself.

We conclude this chapter with a few historical observations. Zygmund Janiszewski made several great contributions to mathematics in general and to continua theory in particular [49]. His thesis established many results on irreducible continua that continue to be of use today. Of course, the above paper with Kuratowski developed many of the fundamental properties of indecomposable continua. Janiszewski was also very instrumental in establishing both the Polish school of mathematics and the journal, Fundamenta Mathematicae. Sadly, the first volume carried his obituary. He died January 3, 1920 at the age of 32, as a result of a long illness.

The second remark concerns the Fundamenta Mathematicae itself. It was founded by Janiszewski, Mazurkiewicz, and Sierpinski to be a journal dealing with set-theoretic problems written in French, English, German, or Italian. This restriction of topic did not put the journal out of print for lack of papers, as some mathematicians of that day had feared. It even survived Nazi occupation in World War II, although many of its contributors did not [78].

CHAPTER 5

INDECOMPOSABLE SUBCONTINUA OF IRREDUCIBLE CONTINUA

In this chapter we shall consider indecomposability as a special case of irreducible continua theory. In particular, we shall exhibit some conditions that are both necessary and sufficient for an irreducible continuum to be indecomposable. This will give a partial answer to the question: "How much stronger is the condition of indecomposability than that of irreducibility?" Moreover, the interrelations between the two concepts will be more clearly exposed.

Historically, the papers cited here date from 1922 to 1927, and all but one of them were written, at least in part, by Kuratowski. Some of the results obtained in those papers were valid only for non-bounded sets in Euclidean spaces. These are omitted, not only because such sets are not continua by our definition of a continuum, but also because they do not contribute to our later developments.

The first result to be considered here was proved by Paul Urysohn [116, p. 226] in 1926. His work was done in a general metric space under the same definitions that we use today. His definition of "compact" is actually our term "countably compact", but there is no distinction between

these concepts in a metric space [28, p. 230]. It is interesting to note that this is the first paper we have discussed in which the definitions agree with current usage.

<u>Theorem 5.1</u>: In order that a metric continuum C, irreducible between a and b, should be indecomposable, it is necessary and sufficient that it contain a semi-continuum S such that:

(a) either a or b is in S;

(b) $\overline{S} = C$;

(c) $\overline{C-S} = C$.

<u>Proof</u>: If C is indecomposable, then by Theorem 4.12, C is irreducible between a and some $x \in C$. Set $S = P(a,C)$. By Theorem 4.11, $\overline{C-S} = C$, and by Theorem 4.17, $\overline{S} = C$.

On the other hand, suppose the conditions of the theorem are satisfied, and without loss of generality, assume $a \in S$. We claim that $P(b,C) \cap S = \emptyset$. If not, choose x in the intersection. Since $x \in S$, there exists a continuum $K \subset S$, with $a, x \in K$. $x \in P(b,C)$ implies that there is a continuum $K' \subset P(b,C)$, with $x, b \in K'$. C being irreducible between a and b implies $C = K \cup K'$. Therefore, $C-S \subset C-K \subset K'$, from which it follows that $C = \overline{C-S} \subset K' \subset P(b,C)$. But then $a \in P(b,C)$, which contradicts the irreducibility of C between a and b. Thus, the claim is established.

Since $P(b,C) \cap S = \emptyset$, $S \subset C-P(b,C)$. Therefore, we have $\overline{C-P(b,C)} \supset \overline{S} = C$. By Theorem 4.12, C is indecomposable. (This proof is essentially as Urysohn gave it [116, pp. 226-227].)

Urysohn notes that as a necessary condition, the theo-

rem is not very interesting. However, it does provide precisely sufficient conditions, which he shows by removing each condition one at a time and constructing counter-examples.

As mentioned earlier (p. 29), he used this theorem to outline a proof of the indecomposability of the Lakes of Wada. Essentially, he lets M_S play the role of S, and he states that the irreducibility of $C = M_S \cup M_L \cup M_P$ follows from a convenient distribution of the canals [116, p. 232].

We next consider several results which Kuratowski established in [69]. This work was the major portion of his thesis, written under the direction of Mazurkiewicz and Sierpinski in 1920 [69, p. 201]. It also contained the previously discussed Knaster's "semi-circle example" (pp. 24, 53), and is seemingly the only paper to use results of Yoneyama.

Using Kuratowski's notation, let C be a T_2 continuum irreducible between a and b, and define R(a,C) to be the empty set together with the set of all subcontinua L of C containing a such that $L = \overline{C-\overline{C-L}}$. The equation simply requires that $L = \overline{\text{Int}\ (L)}$, a condition sometimes referred to by saying that L is a regular set. This is not to be confused with the separation axiom of the same name.

Before we can prove any of the major theorems, we must establish some background results.

Lemma 5.2: Let C be irreducible between a and b, and let K be a closed connected subset. Then C-K is either connected

or else it is the union of two connected sets, one of which contains a and the other of which contains b. If $a \in K$, then C-K is connected [69, pp. 202-203].

Proof: Suppose C-K is not connected. Then $C-K = P \cup Q$, where P, Q are nonempty, disjoint, open subsets of the open set C-K. By Lemma 4.2, $K \cup P$, $K \cup Q$ are closed connected subsets of C, and hence are subcontinua. But then we have $C = (K \cup P) \cup (K \cup Q)$. If $a \in K$, then either $K \cup P$ or $K \cup Q$ is a proper subcontinuum of C containing a, b. This violates the irreducibility of C. Therefore, if $a \in K$, then C-K is connected.

Since a, b are not both in either $K \cup P$ or $K \cup Q$, we may assume $a \in P$, and $b \in Q$. Hence, $C-(K \cup P) = Q$ is connected by the first part of this lemma, and likewise P is connected.

Lemma 5.3: If A, B are two closed connected subsets of C, with C irreducible between a, b, where $a \in A$, $b \in B$, then $C-(A \cup B)$ is connected [76, p. 193].

Proof: We may assume $A \cap B = \emptyset$, for if not, then $A \cup B$ is a subcontinuum of C containing a, b. Thus, $C = A \cup B$ by the irreducibility of C, and consequently $C-(A \cup B) = \emptyset$.

C-A is connected by Lemma 5.2. Suppose that the set $(C-A)-B = C-(A \cup B)$ is disconnected. Then it is the union of two nonempty sets U, V such that $(U \cap \overline{V}) \cup (\overline{U} \cap V) = \emptyset$. By Lemma 4.2, $B \cup U$ and $B \cup V$ are connected. Hence, their closures, $B \cup \overline{U}$, $B \cup \overline{V}$ are connected.

Since $C = (A \cup B) \cup \overline{(C-A)-B}$, and $A \cap B = \emptyset$, then $A \cap \overline{(C-A)-B} \neq \emptyset$. For if not, then A and $B \cup \overline{(C-A)-B}$ show C

is disconnected. Moreover, $\emptyset \neq A \cap \overline{(C-A)-B} = A \cap \overline{U} \cup \overline{V}$ implies that $A \cap \overline{U} \neq \emptyset$, or $A \cap \overline{V} \neq \emptyset$. Without loss of generality, suppose $A \cap \overline{U} \neq \emptyset$. Therefore, $A \cup \overline{U} \cup B$ is connected and contains a, b. By the irreducibility of C, $C = A \cup \overline{U} \cup B$. Therefore, $V \subset (C-A)-B \subset \overline{U}$; however, $V \cap \overline{U} = \emptyset$, so we must conclude that $V = \emptyset$. This contradicts the fact that $V \neq \emptyset$. Hence, $C-(A \cup B)$ is connected.

Lemma 5.4: Let C be a continuum irreducible between a,b, and let K be a subcontinuum. Then Int (K) is connected [76, p. 194].

Proof: If K = C, then the result is clear. So suppose $K \neq C$. By the irreducibility of C, not both a, b are in K; assume $a \in C-K$.

If C-K is connected, then so is $\overline{C-K}$. In this case, Lemma 4.2 shows that $C-\overline{C-K} = \text{Int}(K)$ is connected. If C-K is not connected, then by Lemma 5.2, it is the union of two connected sets, P, Q with $a \in P$, and $b \in Q$. Let $A = \overline{P}$ and $B = \overline{Q}$ in Lemma 5.3, and it follows that $C-\overline{C-K} = C-\overline{P \cup Q} = C-(\overline{P} \cup \overline{Q})$ is connected.

We need two more sequences of lemmas to enable us to establish the major results, Theorems 5.14, 5.16, 5.17.

Lemma 5.5: Let S be a topological space with subsets A, B. Then $\overline{A}-\overline{B} \subset \overline{A-B}$ [68, p. 183].

Proof: Let $x \in \overline{A}-\overline{B}$, and let O be any neighborhood of x. We must show that there is a $y \in O$ such that $y \in A$ and $y \notin B$. But, there is an open set U, $x \in U$, such that $U \cap B = \emptyset$. Since $x \in O \cap U$, this intersection is nonempty and open. So

choose any $y \in O \cap U$.

It is easily seen that the following formulas hold.

Lemma 5.6: For any sets A, B, C:

(a) $(A-C)-(B-C) = (A-B)-C$;

(b) $(A-C) \cup (B-C) = (A \cup B)-C$;

(c) $(A-C) \cap (B-C) = (A \cap B)-C$;

(d) $(A-B)-(A-C) = A \cap (C-B)$;

(e) $(A-B) \cup (A-C) = A-(B \cap C)$;

(f) $(A-B) \cap (A-C) = A-(B \cup C)$.

Finally, we prove a sequence of lemmas which will show $\overline{\text{Int}\,\overline{X \cup Y}} = \overline{\text{Int}\,\overline{X}} \cup \overline{\text{Int}\,\overline{Y}}$.

Lemma 5.7: Let S be a topological space. If X, Y are nowhere dense in S, then so is $X \cup Y$.

Proof: Since $\overline{S-\overline{X}} = S = \overline{S-\overline{Y}}$, then $S-\overline{Y} = \overline{S-\overline{X}}-\overline{Y} \subset \overline{(S-\overline{X})-Y}$, by Lemma 5.5. Thus, $S = \overline{S-\overline{Y}} \subset \overline{S-(\overline{X \cup Y})} \subset S$, from which the Lemma follows at once.

Lemma 5.8: If X is nowhere dense in a topological space S, and if O is open in S, then $X \cap O$ is nowhere dense in S.

Proof: $X \cap O \subset X$ implies $\overline{S-\overline{X \cap O}} \supset \overline{S-\overline{X}} \supset X \supset X \cap O$.

Lemma 5.9: X is nowhere dense in a topological space S iff $\text{Int}\,\overline{X} = \emptyset$; X is a boundary set in S iff $\text{Int}\,X = \emptyset$.

Proof: The result is obvious from the definitions.

Definition: X is locally nowhere dense (respectively a boundary set) at $p \in S$ if there is a neighborhood O of p such that $O \cap X$ is nowhere dense (respectively, a boundary set).

Lemma 5.10: X is nowhere dense at p iff $\overline{X}$ is a boundary set

at p.

Proof: If X is not nowhere dense at p, and if G is any open set containing p, then $G \cap X$ is not nowhere dense. By Lemma 5.9, there is an open set H such that $\emptyset \neq H \subset \overline{G \cap X}$. Therefore, $H = H \cap \overline{G \cap X} \subset \overline{H \cap G \cap X}$, since H is open. Hence, $H \cap G \neq \emptyset$. Then $\emptyset \neq H \cap G \subset G \cap \overline{G \cap X} \subset G \cap \overline{X}$ and Lemma 5.9 show $G \cap \overline{X}$ is not a boundary set. Hence, $\overline{X}$ is not a boundary set at p.

Conversely, if $\overline{X}$ is not a boundary set at p, and if G is any neighborhood of p, then there exists an open set H such that $\emptyset \neq H \subset G \cap \overline{X}$. Since G is open, $\emptyset \neq H \subset G \cap \overline{X} \subset \overline{G \cap X}$. Therefore, X is not nowhere dense at p.

Lemma 5.11: $\overline{\text{Int } X}$ is the set of points of S where X is not locally a boundary set.

Proof: $p \in \overline{\text{Int } X}$ and G any neighborhood of p imply that $G \cap \text{Int } X \neq \emptyset$. Then $\emptyset \neq G \cap \text{Int } X \subset G \cap X$ shows that X is not a boundary set at p.

On the other hand, $p \in S - \overline{\text{Int } X}$ implies $(S - \text{Int } X) \cap X$ is a boundary set, since $\text{Int } [(S - \overline{\text{Int } X}) \cap X] = \text{Int } (S - \text{Int } X) \cap \text{Int } X$ $= (S - \overline{\text{Int } X}) \cap \text{Int } X = \emptyset$.

Lemma 5.12: $\overline{\text{Int } \overline{X}}$ is the set of points of S where X is not locally nowhere dense.

Proof: The result is immediate from Lemmas 5.10, 5.11.

Lemma 5.13: $\overline{\text{Int } \overline{X \cup Y}} = \overline{\text{Int } \overline{X}} \cup \overline{\text{Int } \overline{Y}}$.

Proof: One inclusion is easy and requires none of the above machinery.

$$\overline{S - \overline{S - \overline{X}}} \cup \overline{S - \overline{S - \overline{Y}}} = \overline{(S - \overline{S - \overline{X}}) \cup (S - \overline{S - \overline{Y}})} = \overline{S - (\overline{S - \overline{X}} \cap \overline{S - \overline{Y}})} \subset$$

$\overline{S-\overline{[(S-\overline{X}) \cap (S-\overline{Y})]}} = \overline{S-\overline{S-(\overline{X} \cup \overline{Y})}}$.

Conversely, if $p \notin \overline{\text{Int } \overline{X}}$, then by Lemma 5.12, X is locally nowhere dense at p. Therefore, there is a neighborhood O of p such that $O \cap X$ is nowhere dense in S. Likewise, $p \notin \overline{\text{Int } \overline{Y}}$ implies there is a neighborhood U of p such that $U \cap Y$ is nowhere dense.

Then $X \cap O \cap U$ and $Y \cap O \cap U$ are nowhere dense in S by Lemma 5.8. By Lemma 5.7, we have $(X \cap O \cap U) \cup (Y \cap O \cap U) = (X \cup Y) \cap (O \cap U)$ is nowhere dense. Therefore, $(X \cup Y)$ is nowhere dense at p, and hence by Lemma 5.12, $p \notin \overline{\text{Int } \overline{X \cup Y}}$. The desired inclusion follows by taking complements.

This last sequence of lemmas has been adapted from material in Volume I of Kuratowski's <u>Topology</u> [75]. We may now return to indecomposable continua theory.

<u>Definition</u>: Two members K_1, K_2 of a family of sets $\mathcal{K}$ form a jump [saut] if for each $K \in \mathcal{K}$ such that $K_1 \subset K \subset K_2$, then either $K = K_1$, or else $K = K_2$.

<u>Theorem 5.14</u>: If K is a nonempty indecomposable continuum contained in a T_2 continuum C which is irreducible between a and b, then K is either a continuum of condensation or else $K = \overline{C-\overline{C-K}}$. In the latter case, there is a member R_o of R(a,C) such that R_o and $R_o \cup K$ form a jump [69, pp. 210-212].

<u>Proof</u>: $C-K \subset \overline{C-K}$ implies $C-\overline{C-K} \subset K$. Then, since K is closed, $\overline{C-\overline{C-K}} \subset K$. By Lemma 5.4, $C-\overline{C-K}$ is connected, and hence so is $\overline{C-\overline{C-K}}$. Therefore, $\overline{C-\overline{C-K}}$ is a continuum contained in K. If it is a proper sub-continuum of K, then,

since K is indecomposable, Theorem 4.4 requires that $\overline{C-\overline{C-K}}$ be a continuum of condensation of K and hence of C. If it is not proper, then $K = \overline{C-\overline{C-K}}$.

We now establish the second part of the theorem. If $K = C$, then $R_0 = \emptyset$ will do. For let R be any member of $R(a,C)$, $\emptyset \subset R \subset \emptyset \cup C$. If $R = C$, there is nothing to prove. If $R \neq C$, then by Theorem 4.4, $\overline{C-R} = C$. But, $R \in R(a,C)$ implies $\overline{C-\overline{C-R}} = R$. Therefore, $R = \overline{C-C} = \emptyset$.

We now assume that $K \neq C$, and that $a \in C-K$. By Lemma 5.2, $\overline{C-K} = \overline{P} \cup \overline{Q}$, where $P \cap Q = \emptyset$, and P, Q are open in the open set C-K. Furthermore, $a \in P$, and either $b \in Q$ or else $Q = \emptyset$. We shall show that in each case, $R_0 = \overline{P}$ will do.

We must first show that $\overline{P} \in R(a,C)$. Clearly $\overline{P}$ is a subcontinuum of C which contains a. It only remains to show $\overline{P} = \overline{C-\overline{C-\overline{P}}}$. As in the first part of the proof of this theorem, $\overline{C-\overline{C-\overline{P}}} \subset \overline{P}$. $P \subset \overline{P}$ and P open in C imply $C-P = \overline{C-P} \supset \overline{C-\overline{P}}$. Therefore, $P \subset C-\overline{C-\overline{P}}$, from which the desired result follows by taking closures.

The next step is to show $\overline{P} \cup K \in R(a,C)$, and the first result needed for this is that $\overline{P} \cup K$ be a continuum containing a. To establish this, it suffices to show that $\overline{P} \cap K \neq \emptyset$. Since $C-K = P \cup Q$, with $a \in C-K$ and either $Q = \emptyset$ or $b \in Q$, then $\overline{C-K}$ is a continuum containing a and b or else a and not b. In the first case, the irreducibility of C shows $C = \overline{C-K}$. Therefore, $\overline{P} = \overline{P} \cup \overline{Q} = C \supset K$, and hence $\overline{P} \cap K = K \neq \emptyset$. In the second case, $\overline{P}-P \neq \emptyset$, for if not,

then P is an open and closed nonempty subset of C which is proper since $b \in Q$. This contradicts the connectivity of C. Suppose that $\overline{P} \cap K = \emptyset$. Then $\overline{P} \subset C-K$. Let $x \in \overline{P}-P$; then $x \in Q$ or $x \in K$. But, $x \in K$ implies $x \notin C-K$, whence $\overline{P} \not\subset C-K$. On the other hand, if $x \in Q$, then $x \notin \overline{P}$, since $P \cap Q = \emptyset$. Therefore, $\overline{P} \cap K \neq \emptyset$.

As before, $\overline{P} \cup K \supset \overline{C-\overline{C-(\overline{P} \cup K)}}$. For the opposite inclusion, $K = \overline{C-\overline{C-K}}$ and $\overline{P} = \overline{C-\overline{C-\overline{P}}}$ imply that

$$\overline{P} \cup K = \overline{(C-\overline{C-K}) \cup (C-\overline{C-\overline{P}})}$$

$$= \overline{C-(\overline{C-K} \cap \overline{C-\overline{P}})}$$

$$\subset \overline{C-\overline{[(C-K) \cap (C-\overline{P})]}}$$

$$= \overline{C-\overline{C-(K \cup \overline{P})}}.$$

We shall next show that $\overline{P}$ and $\overline{P} \cup K$ form a jump. Let $S \in R(a,C)$ be such that $\overline{P} \subset S \subset \overline{P} \cup K$. We must show that $S = \overline{P}$ or else $S = \overline{P} \cup K$. Since $S \in R(a,C)$, $a \in S$. We wish to show that $\overline{S-\overline{P}}$ is connected, and this can be done by applying Lemma 5.2, provided S is irreducible between a and some other point. Thus, we first show that S is irreducible between a and all points of $Fr(S)$. Note that $Fr(S) = \emptyset$ iff S is closed open in C iff C is disconnected, provided $\emptyset \neq S \neq C$. Therefore, $Fr(S) \neq \emptyset$.

To prove the irreducibility of S, let F be any subcontinuum of S such that $a \in F$, and $F \cap Fr(S) \neq \emptyset$. We must show $S \subset F$. $F \cap Fr(S) \neq \emptyset$ implies $F \cap (\overline{C-S} \cap \overline{S}) \neq \emptyset$. Hence, in particular, $F \cap \overline{C-S} \neq \emptyset$. Moreover, by Lemma 5.2, $a \in S$

implies $\overline{C-S}$ is connected. By the irreducibility of C, $b \in \overline{C-S}$. Therefore, $F \cup \overline{C-S}$ is a continuum containing a, b, and thus must be C. Consequently, $C-\overline{C-S} \subset F$, from which it follows that $S = \overline{C-\overline{C-S}} \subset F$. Thus, S is irreducible between a and all points of the nonempty set Fr(S).

By Lemma 5.2, $\overline{S-\overline{P}}$ is connected, and hence is a subcontinuum of K, since $S \subset \overline{P} \cup K$. By Theorem 4.4, $\overline{S-\overline{P}}$ is either not proper, or else it is a continuum of condensation. That is, either $\overline{S-\overline{P}} = K$, or else $\overline{K-\overline{S-\overline{P}}} = K$. In the first case, $\overline{P} \cup K = \overline{P} \cup \overline{S-\overline{P}} \subset \overline{P} \cup S = S$, so that $S = \overline{P} \cup K$.

The other case is slightly more complicated. We shall investigate it with the help of the following lemma, which will also be useful in proving the next theorem.

<u>Lemma 5.15</u>: Let T be any topological space, let A be a subset such that $A = \overline{T-\overline{T-A}}$, and let B be closed in T.

(a) Then $\overline{A-B} = \overline{A-\overline{A-\overline{A-B}}}$.

(b) Moreover, if $D \subset A$ is such that $D = \overline{A-\overline{A-D}}$, then $D = \overline{T-\overline{T-D}}$ [68, p. 184].

<u>Proof</u>: (a) Since $A-\overline{A-B} \subset \overline{A-\overline{A-B}}$, then it follows that $\overline{A-B} = A-(A-\overline{A-B}) \supset A-\overline{A-\overline{A-B}}$. Therefore, $\overline{A-B} \supset \overline{A-\overline{A-\overline{A-B}}}$. On the other hand, $A-\overline{A-B} \subset \overline{A-(A-B)}$ by Lemma 5.5. Consequently, $A-\overline{A-B} \subset \overline{B} = B$, whence $\overline{A-\overline{A-\overline{A-B}}} \supset \overline{A-B}$.

(b) We need only prove that $\overline{T-\overline{T-D}} = \overline{A-\overline{A-D}}$. Let x be an element of $\overline{T-\overline{T-D}}$, and let O be any neighborhood of X in

T. We must show that $O \cap (A-\overline{A-D}) \neq \emptyset$. That is, we must prove that there is a $z \in O$ such that $z \in A$ and $z \notin \overline{A-D}$. But, since $O \cap (T-\overline{T-D}) \neq \emptyset$, there exists a $w \in O$ such that $w \in T$ and $w \notin \overline{T-D}$. (Hence $w \notin \overline{A-D}$.) $w \notin \overline{T-D}$ implies that $w \in T-\overline{T-D} = \text{Int } D \subset D$. Therefore, $w \in D \subset A$, and $w \notin \overline{A-D}$, so $z = w$ will do.

On the other hand,

$$\begin{aligned}
\overline{T-\overline{T-A}} - \overline{T-\overline{T-D}} &\subset \overline{(T-\overline{T-A}) - (T-\overline{T-D})} \\
&= \overline{T \cap [(\overline{T-D}) - (\overline{T-A})]} \\
&= \overline{\overline{T-D} - \overline{T-A}} \\
&= \overline{\overline{(T-D) - (T-A)}} \\
&= \overline{A-D}.
\end{aligned}$$

Since $A = \overline{T-\overline{T-A}}$, we have that $A-\overline{T-\overline{T-D}} \subset \overline{A-D}$, by the above equations. Therefore, $\overline{T-\overline{T-D}} \supset A-\overline{A-D}$, whence $\overline{T-\overline{T-D}} \supset \overline{A-\overline{A-D}}$. This concludes the proof of the Lemma.

Applying the Lemma with $T = C$, $A = S$, $B = \overline{P}$, we conclude that $\overline{S-\overline{P}} = \overline{S-\overline{S-\overline{S-\overline{P}}}}$. Then in (b) of the Lemma, taking $D = \overline{S-\overline{P}}$, we get that $\overline{C-\overline{C-\overline{S-\overline{P}}}} = \overline{S-\overline{P}}$. Furthermore, since $\overline{S-\overline{P}} \subset K$, and $K = \overline{C-\overline{C-K}}$, we can apply the proof of (b) with $T = C$, $D = \overline{S-\overline{P}}$, and $A = K$, to conclude $\overline{C-\overline{C-\overline{S-\overline{P}}}} = \overline{K-\overline{K-\overline{S-\overline{P}}}}$. Therefore, $\overline{S-\overline{P}} = \overline{K-\overline{K-\overline{S-\overline{P}}}}$. Since $K = \overline{K-\overline{S-\overline{P}}}$, the equation of the last sentence shows that $\overline{S-\overline{P}} = \emptyset$, whence $S \subset \overline{P}$. Thus, the second case reduces to $S = \overline{P}$. Therefore, $\overline{P}$ and $\overline{P} \cup K$

form a jump, and the theorem is established.

The above proof is not the original one, which is even more complicated. This proof is from Kuratowski's <u>Topology</u> vol. II, with the (many) details supplied.

As a converse to the above theorem, we have

<u>Theorem 5.16</u>: If the elements R_o and R_1 of $R(a,C)$ form a jump, then $\overline{R_1-R_o}$ is either empty or an indecomposable continuum [69, p. 211].

<u>Proof</u>: Assume that $R_o \subset R_1$, since these elements form a jump, and consequently one such inclusion must hold. Moreover, we may assume $R_o \neq R_1$, for if not then $\overline{R_1-R_o} = \emptyset$.

Since $a \in R_1$, $\overline{R_1-R_o}$ is connected by the same argument that showed $\overline{S-\overline{P}}$ is connected in Theorem 5.14. Thus, $\overline{R_1-R_o}$ is a continuum. Suppose that $R_1-R_o = A \cup B$, where A, B are subcontinua of $\overline{R_1-R_o}$. We must show that one of them is $\overline{R_1-R_o}$.

Let $A^* = \overline{C-\overline{C-A}}$, and $B^* = \overline{C-\overline{C-B}}$. By Lemma 5.4, A^*, B^* are connected. By Lemma 5.15, $\overline{C-\overline{C-(A \cup B)}} = A \cup B$. By Lemma 5.13, $A^* \cup B^* = A \cup B$, whence $A^* \cup B^* = \overline{R_1-R_o}$. There are two possibilities; $R_o \neq \emptyset$, or $R_o = \emptyset$.

If $R_o \neq \emptyset$, then we shall show $R_o \cap A^* \neq \emptyset$, or else $R_o \cap B^* \neq \emptyset$. Suppose though that both those sets are empty. Then $R_o \cap (A^* \cup B^*) = \emptyset$. Now, $R_o \subset R_1$, and $R_1 \supset R_1-R_o$, so that $R_1 \supset \overline{R_1-R_o} = A^* \cup B^*$. Therefore, $\overline{R_1-R_o} = A^* \cup B^* \subset R_1-R_o$, whence R_1-R_o is closed in C and in the closed set R_1. But, since R_1 is closed, R_1-R_o is open in R_1. This, together with the fact that $\emptyset \neq R_1-R_o \neq R_1$, violates the connectivity

of R_1. Therefore, one of the two above sets, say $R_0 \cap A^*$, is nonempty. Therefore, $R_0 \cup A^*$ is a continuum containing a, and $R_0 \cup A^* = \overline{C-\overline{C-(R_0 \cup A^*)}}$, by the same proof used in Theorem 5.14 to show $\overline{P} \cup K = \overline{C-\overline{C-(\overline{P} \cup K)}}$. Consequently, $R_0 \cup A^* \in R(a,C)$. If $R_0 = \emptyset$, then let A^* be the one of A^*, B^* containing a. In this case too, $R_0 \cup A^* \in R(a,C)$.

By definition of "jump", either $R_0 \cup A^* = R_0$, or $R_0 \cup A^* = R_1$. In the first case,

$$R_1 = R_0 \cup \overline{R_1-R_0} = R_0 \cup A^* \cup B^* = R_0 \cup B^*.$$

Therefore, $R_1-R_0 \subset B^* \subset B$, whence $\overline{R_1-R_0} \subset B$. But since $B \subset A^* \cup B^* = \overline{R_1-R_0}$, so that $\overline{R_1-R_0} = B$.

If $R_0 \cup A^* = R_1$, then $R_1-R_0 \subset A^* \subset A$, and $\overline{R_1-R_0} = a$, as above.

This proof is also from Kuratowski [76], again with the details supplied.

<u>Theorem 5.17</u>: Let C be a T_2 continuum irreducible between a and b. Then C is indecomposable iff $R(a,C) = \{\emptyset, C\}$.

<u>Proof</u>: If C is indecomposable, then $C = \overline{C-\overline{C-C}}$. So by Theorem 5.14, $R_0 = \emptyset$, and $C \cup \emptyset = C$ form a jump. If K is any element of $R(a,C)$, then $\emptyset \subset K \subset C$ implies that $K = \emptyset$, or $K = C$. Therefore, $R(a,C) = \{\emptyset, C\}$.

If $R(a,C) = \{\emptyset, C\}$, then $\emptyset$ and C certainly form a jump. Then Theorem 5.16 shows $C = \overline{C-\emptyset}$ is an indecomposable continuum.

The fact that Theorems 5.14, 5.16, and 5.17 are not used a great deal in the later literature leads one to

believe that they are not as useful as those given in Chapter 4. It is interesting to note, however, that all of the theorems of this chapter hold in an arbitrary T_2 space. In Chapter 4, all the major theorems except the first required "metric".

We conclude this chapter by presenting a result (Theorem 5.20) from Kuratowski's "Théorie des continus irréductibles entre deux points II" [71] dealing with the set C-P(a,C). Before presenting the theorem, we give a brief description of the problem which Kuratowski was studying when he proved the theorem. This digression seems appropriate because it also involves Zoretti and Brouwer, both of whom we have met before.

Zoretti, considering that an irreducible continuum is a generalization of a simple are, conjectured that any irreducible continuum could be given a linear ordering. Moreover, he published a theorem which would provide the basis for this ordering [133]. When it was pointed out to him that his method failed for an irreducible continuum that is also indecomposable, he published a new method based on a weaker theorem [135, p. 2o2]. Brouwer also observed that this theorem was false for an indecomposable continuum. The most that could be done in this case is to order the points of each composant [17, pp. 144-145]. Thus, Brouwer continued (see p. 16) to play the role of critic in the development of indecomposable continua theory.

In 1927, however, Kuratowski provided a surprising cor-

ollary to the linear ordering question. Given a continuum C irreducible between points a and b, Kuratowski proved that C has a decomposition $\mathcal{D}$ which is "linearly ordered"[71, pp. 225-228].

A decomposition of C is semi-continuous and linear (terminology is due to R. L. Moore [99]) if C is decomposed into a single element or else into a disjoint collection of sets T_x, $0 \leq x \leq 1$, such that $\lim_{n\to\infty} x_n = x_0$ implies that the $\limsup_{n\to\infty} T_{x_n} \subset T_{x_0}$. (For a definition of "lim sup", see [44, p. 100].)

Kuratowski showed that $\mathcal{D}$ has the following properties:

(1) $\mathcal{D}$ is semi-continuous and linear, having continua for the T_x;

(2) if $\mathcal{D}^*$ is any decomposition satisfying (1), then each T_x of $\mathcal{D}^*$ is either in $\mathcal{D}$ or else is the union of members of $\mathcal{D}$.

A complete discussion of Kuratowski's paper would carry us too far afield. However, we do require one of the theorems from it. The following lemma will be used in the proof.

Lemma 5.18: (a) Let K be a component of a compact T_2 space S, and let U be any open set containing K. Then U contains a closed open set V containing K.

(b) Let C be a T_2 continuum, and let O be an open proper subset of C. Let K be a component of O. Then we have $(O-\bar{O}) \cap \bar{K} \neq \emptyset$ [44, p. 47].

Proof: (a) By Lemma 4.15, each component is a quasi-component. Therefore, $K = \bigcap_{\alpha \in \eta} F_\alpha$, where each F_α is

closed open in S and contains K. Let $K \subset U$. If $K = U$, then K is open and closed, so let $K = V$. If $K \neq U$, there is an α_0 such that $\bigcap_{\alpha \geq \alpha_0} F_\alpha \subset U$. Therefore, $\bigcup_{\alpha \geq \alpha_0} (S-F_\alpha) \supset S-U$, and since S-U is compact, there is a collection $\{S-F_{\alpha_i}\}_{i=1}^{n}$ such that $\bigcup_{1}^{n} (S-F_{\alpha_i}) \subset S-U$. Hence, $K \subset \bigcap_{1}^{n} F_{\alpha_i} \subset U$, so let $V = \bigcap_{1}^{n} F_{\alpha_i}$. V is both open and closed, so (a) holds.

(b) $\emptyset \neq O \neq C$ and C connected imply that $\overline{O}-O \neq \emptyset$. Suppose $(\overline{O}-O) \cap \overline{K} = \emptyset$. Then $K \subset \overline{K} \subset O$, so by the maximality of K, $\overline{K} \subset K$. Therefore, K is closed and hence compact. $\overline{O}-O$ is closed in C and hence is compact. Therefore, there exist two disjoint sets U_1, U_2 open in $\overline{O}$ such that $U_1 \supset \overline{O}-O$, and $U_2 \supset K$. Thus, $U_2 \subset O$, and K is a component of $\overline{U}_2$. By (a), there is a set $U_3 \subset U_2$ that is both closed and open in $\overline{U}_2$. U_3 is closed in C, and since $U_3 \subset U_2$ and the latter is open, then U_3 is open in C. But, $\emptyset \neq U_3 \neq C$, which contradicts the connectivity of C. Therefore, $(\overline{O}-O) \cap \overline{K} \neq \emptyset$.

<u>Theorem 5.19</u>: Let C be a T_2 continuum irreducible between a and b. Then C-P(a,C) is connected.

<u>Proof</u>: We note first that it is no loss of generality to consider only P(a,C), rather than P(d,C), where d is a point such that C is not irreducible between d and any other point of C. For in this case, P(d,C) = C.

Suppose $C-P(a,C) = A \cup B$, where $(A \cap \overline{B}) \cup (\overline{A} \cap B) = \emptyset$. $b \in C-P(a,C)$ by the irreducibility of C, so without loss of generality, we may assume $b \in B$. Let $O = C-\overline{A}$; O is open, and $B \cap \overline{A} = \emptyset$, so $B \subset O$. $A \cap \overline{O} \subset \overline{A \cap O} = \overline{A \cap (C-\overline{A})} = \emptyset$. Therefore,

$A \cap \overline{O} = \emptyset$.

By Lemma 5.6, $(\overline{O}-O) \cap (A \cup B) = [(A \cup B)-O] - [(A \cup B)-\overline{O}] = [(A-B) \cup (B-O)] - [(A-\overline{O}) \cup (B-\overline{O})] = (A \cup \emptyset) - (A \cup \emptyset) = \emptyset$.

Let K be a component of b in O. If $A \neq \emptyset$, then $O \neq C$, since $O = C$ implies $C = C-\overline{A}$, whence $\overline{A} = \emptyset$. Consequently, O is a proper open subset. Therefore, $\overline{K}-O \neq \emptyset$, for by Lemma 5.18, $(\overline{O}-O) \cap \overline{K} \neq \emptyset$. Let $p \in \overline{K}-O$. $K \subset O$ implies $p \in \overline{O}-O$. Now, $\overline{O}-O \subset C-(A \cup B) = P(a,C)$, so $p \in P(a,C)$. Therefore, there exists a proper subcontinuum L containing a, p. Since b,p are in $\overline{K}$, $\overline{K} \cup L$ is a continuum joining a, b. By the irreducibility of C, $C = \overline{K} \cup L$. Thus, $A \subset \overline{K} \cup L$. But, $K \subset O$ implies $A \cap \overline{K} \subset A \cap \overline{O} = \emptyset$. $L \subset P(a,C)$ implies $A \cap L \subset A \cap P(a,C)$ and the latter set is empty. Therefore, $A \cap (\overline{K} \cup L) = \emptyset$, whence $A = \emptyset$. [76, p. 210].

Theorem 5.20: Let C be a T_2 continuum irreducible between a and b. Then C-P(a,C) is either a continuum of condensation or else a non-closed boundary set whose closure is an indecomposable continuum such that $\overline{C-P(a,C)} = \overline{C-\overline{C-\overline{C-P(a,C)}}}$ [68, p. 239].

Proof: Theorem 5.19 implies C-P(a,C) is connected. If it is closed in compact C, then it is a continuum. Moreover $\overline{C-[C-P(a,C)]} = \overline{P(a,C)} = C$, by Theorem 4.17. Thus, C-P(a,C) is a continuum of condensation.

If C-P(a,C) is not closed, then the above equation shows it is a non-closed boundary set. $\overline{C-\overline{C-P(a,C)}} = \overline{\text{Int } P(a,C)}$. This last set is a proper subset of $\overline{P(a,C)} = C$, since C-P(a,C) is not closed. Let $Q = \overline{C-\overline{C-P(a,C)}}$; $Q \neq C$.

$\overline{C-P(a,C)}$ is a continuum by Theorem 5.19, and contains b. Therefore, $C-\overline{C-P(a,C)}$ is connected by Lemma 5.2, and hence $Q = \overline{C-\overline{C-P(a,C)}}$ is a continuum.

We claim that $Q \subset P(a,C)$. If $Q = \emptyset$, the claim is clearly true. $Q \neq \emptyset$ implies $C \neq \overline{C-P(a,C)}$. $a \in C-\overline{C-P(a,C)}$ by the irreducibility of C. Then $a \in Q$, $Q \neq C$ is a continuum so that $Q \subset P(a,C)$. Therefore, $C-P(a,C) \subset C-Q$, so we have $\overline{C-P(a,C)} \subset \overline{C-Q} = \overline{C-\overline{C-\overline{C-P(a,C)}}} \subset \overline{C-P(a,C)}$. Thus, $\overline{C-Q} = \overline{C-P(a,C)}$, and so $\overline{C-P(a,C)} = \overline{C-\overline{C-\overline{C-P(a,C)}}}$.

It only remains to show the indecomposability. Suppose $\overline{C-P(a,C)} = M \cup N$, where M, N are proper subcontinua of $\overline{C-P(a,C)}$. We shall show that $M-P(a,C) \neq \emptyset \neq N-P(a,C)$. If, say $N-P(a,C) = \emptyset$, then $C-P(a,C) \subset C-N$. But this implies $C-P(a,C) \subset \overline{C-P(a,C)} - N \subset M$. Therefore, $\overline{C-P(a,C)} \subset M$, which is a contradiction to $M \neq \overline{C-P(a,C)}$. The assertion holds.

If $Q = \emptyset$, then $M \cup N = \overline{C-P(a,C)} = C$. $a \in C$, so without loss of generality, $a \in M$. M is a continuum, and $M-P(a,C) \neq \emptyset$, so $M = C$. This can not hold, for $M \neq \overline{C-P(a,C)} \subset C$.

If $Q \neq \emptyset$, then $a \in Q$, as above. $C = \overline{C-\overline{C-P(a,C)}} \cup \overline{C-P(a,C)} = Q \cup (M \cup N)$. Since C is connected, $Q \cap (M \cup N) \neq \emptyset$.

Therefore, $Q \cup M = C$, so $C-Q \subset M$. Since M is closed, $\overline{C-Q} \subset M$. Thus, $\overline{C-P(a,C)} \subset M$; since $M \subset \overline{C-P(a,C)}$, we conclude that $\overline{C-P(a,C)} = M$. This contradicts M being a proper subcontinuum.

The above proof is adapted from Kuratowski [76, p.211].

In Chapter 4, we saw that any subset of any proper

subcontinuum can be removed from a T_2 indecomposable continuum without disconnecting it. We can say more for a metric indecomposable continuum. We know by Theorem 4.12 that such a set is irreducible between any point and some other point. It then follows from Theorem 5.19 that if the composant of any point is removed, the resulting nonempty set is still connected.

CHAPTER 6

KNASTER'S THESIS

In this chapter we briefly consider more examples of indecomposable continua that appeared in the early 1920's. In particular, we shall present two examples constructed by Knaster, as well as a simplification of one of them. The second example has a property not shared with any previously discussed example. Not only is it indecomposable, but also each of its subcontinua is indecomposable. In today's terminology, such a set is called a hereditarily indecomposable continuum, although no special name was given to it originally.

It is surprising enough that indecomposable continua exist, but it seems truly remarkable that there are hereditarily indecomposable continua. Even more remarkable is the fact that an example was discovered comparatively early in the study of indecomposable continua. However, we shall defer a detailed study of such continua until Chapter 12, since most of the investigations of hereditarily indecomposable continua have been made recently. For the moment, we present the first construction of a hereditarily indecomposable continuum, not only for historical completeness, but also because we shall need the existence of such continua

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Knaster and Kuratowski had asked [62] if such a set could exist in E^2. In 1922, the answer was shown to be affirmative. Bronislaw Knaster described the continuum in his thesis, which he wrote under the direction of Mazurkiewicz and Sierpinski [59, p. 248]. We will not give his construction in detail, since it constitutes the major portion of his forty page paper.

He called his construction technique the "method of bands", and he credits Sierpinski with originating the concept in 1918 [59, p. 247]. Essentially, the method of bands provides a way of constructing a nested sequence of continua in the plane in which the "nesting" is done in a special manner. By varying this manner slightly, Knaster first constructed a previously unknown example of an ordinary indecomposable continuum. Then by placing more restrictions on the nesting, he constructed the first hereditarily indecomposable continuum. Since each continuum in the nested sequence resembles a band, it is not hard to see where the name of the method probably originated.

We now show how Knaster used the method of bands to construct an ordinary indecomposable continuum. Partition the unit square into twenty-five equal squares. Our first continuum, or band, Q_0, is the union of a certain number of those small squares, as shown in Figure 6.1 a. The band is not allowed to intersect itself, hence the rows of buffer squares. We construct the continuum Q_1 as a subset of Q_0

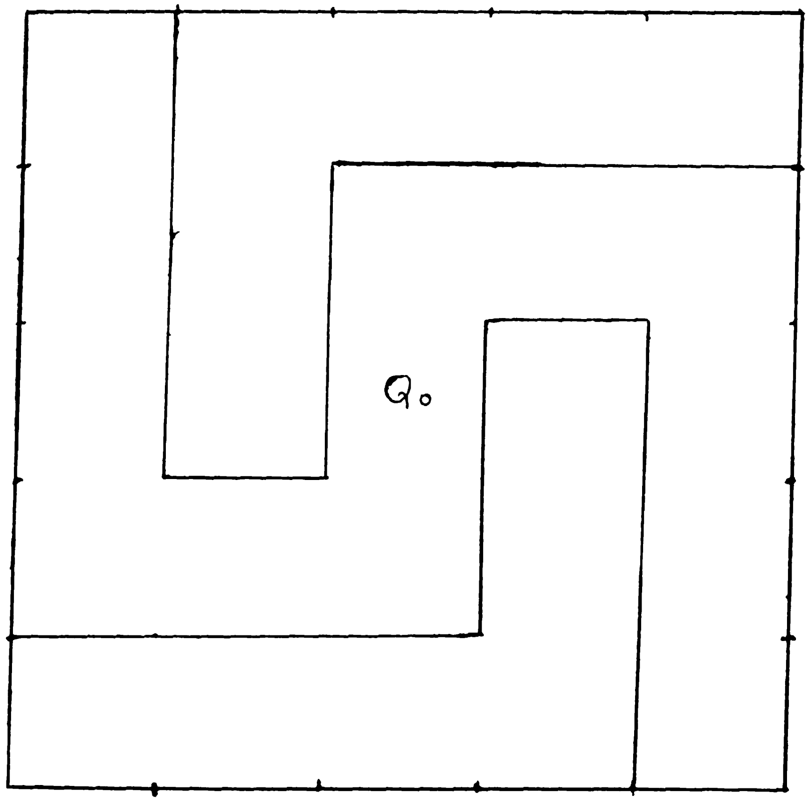
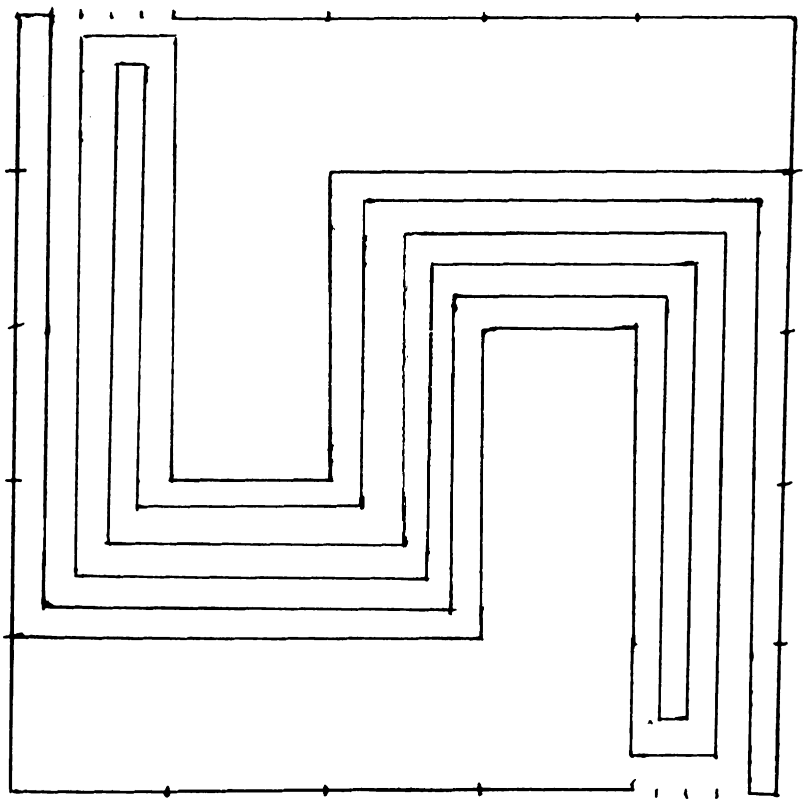


Figure 6.1 a/b



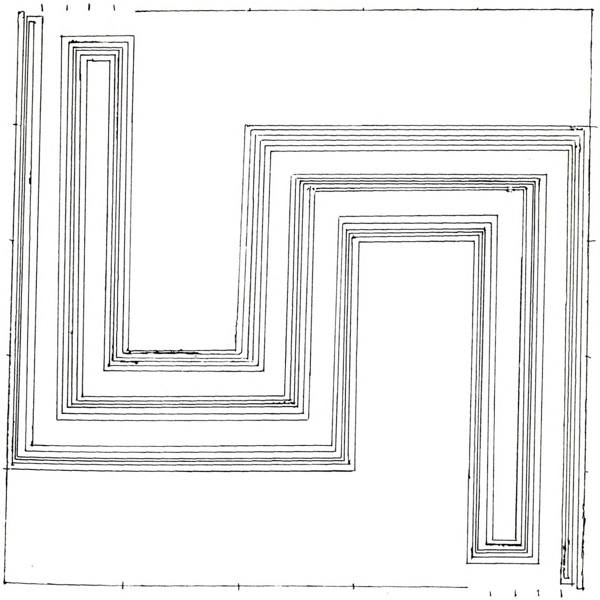


Figure 6.2

by partitioning I^2 into 5^4 equal squares and selecting squares as shown in Figure 6.1 b. Likewise, each continuum Q_{n+1} is formed by partitioning I^2 into $5^{2(n+1)}$ equal squares and selecting a band in Q_n.

The desired continuum is $Q = \bigcap_{0}^{\infty} Q_n$. Note that it is a continuum by Theorem 2.1. Knaster actually proved the

indecomposability of this continuum using Theorem 4.4. However, it took ten pages of machinery to give a sufficiently precise description of the Q_n's and Q to allow the theorem to be used.

Knaster later gave a much simpler description of his indecomposable continuum in Kuratowski's paper "Théorie des continus irréductibles entre deux points I", the same paper which presented Knaster's simplification of Brouwer's example [69, p. 216]. We now give the new construction, which we will call Knaster's second semi-circle example.

Let E be the set of numbers of the segment [0,1] which can be written in base five without the use of the digits 1 and 3. Let E_n $(n \geq 0)$ be the set of points e of E such that $2/(5^{n+1}) \leq e \leq 1/(5^n)$. Let F_n be the set of points e such that that 1-e belongs to E_n.

For a given n, draw semi-circles below the x-axis centered at $(7/10)5^{-n}$ to each point of the set E_n. Likewise, draw semi-circles above the x-axis centered at the points $1 - (7/10)5^{-n}$ to each point of the set F_n. The set formed by the union of these semi-circles for all non-negative n is the desired indecomposable continuum [69, p. 216]. See Figure 6.3, p. 83.

Knaster does not give a proof of the indecomposability of this continuum, but it could be obtained by modifying the proof given for Knaster's first semi-circle example (p. 53).

There is a major difference between these two semi-

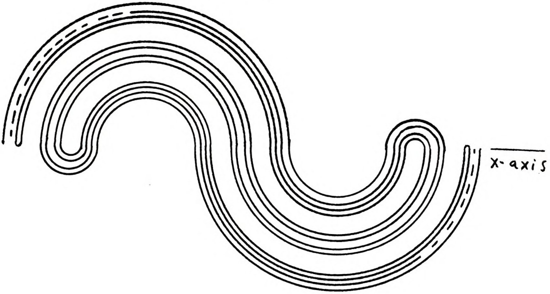


Figure 6.3

circle examples. In order to describe it, we need another definition. A point p is <u>accessible</u> from the set A if there exists a continuum C such that $p \in C \subset A \cup \{p\}$, and $C \neq \{p\}$. The first example has only one composant (the one containing (0,0))containing points accessible from the complement of the set, while the second has two such composants (the one containing (0,0), and the one containing (1,0)).

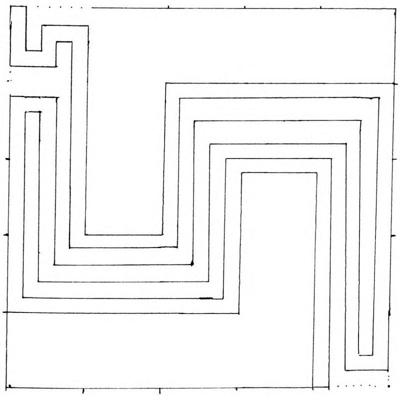
Knaster notes [59, p. 271] that Vietoris had independently constructed the example which Knaster had described by the method of bands. Vietoris' example appeared near the end of his thesis (Vienna) in 1920. He used it as an example of a continuum irreducible between points a and b which contains no connected subset irreducible between a and b [120].

In his thesis, Knaster also constructed a type of indecomposable continuum having the property that each of

its subcontinua contains an indecomposable continuum. Since the construction of this continuum is very similar to that of his hereditary example, we do not include it.

We turn now to his hereditarily indecomposable continuum. The basic technique of construction is the same as that discussed on pp. 79-82. The essential difference between the two constructions lies in how the adjacent squares making up each Q_n are determined from those formed by partitioning Q_{n-1}.

Partition the unit square into twenty-five equal squares, and choose Q_o as before. Partition I^2 into 5^4 equal squares, and select a band $Q_1 \subset Q_o$, as shown:



$Q_2 \subset Q_1$ is shown in Figure 6.4, p. 85. The desired continuum is $Q = \cap Q_n$. Since this construction is a special

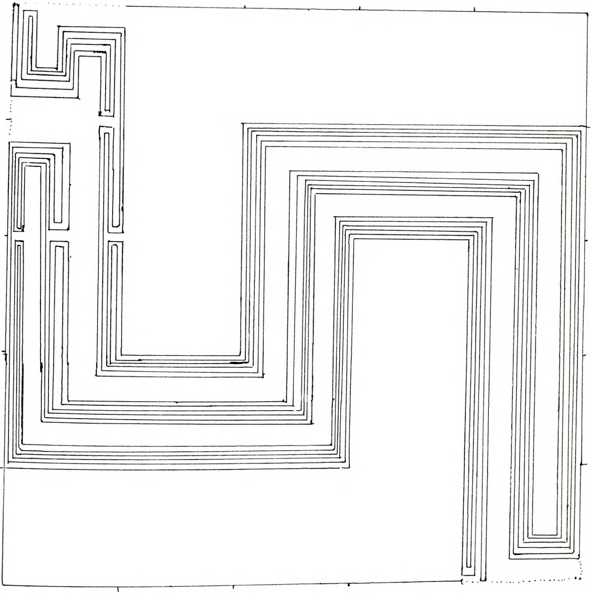


Figure 6.4

case of the construction of the first example in his thesis, Knaster did not have to prove indecomposability again. He established the hereditariness of this continuum by applying Theorem 4.4 to each subcontinuum. In this case, the major difficulty also lies in giving a sufficiently precise description of the set to apply the theorem. It took him twelve pages to set up the notation to describe the Q_n's and the Q's of his examples.

We have seen that Knaster was able to describe two ordinary indecomposable continua in terms of Cantor sets and semi-circles. There is no hope that such a simplification can be given for the hereditarily indecomposable continuum, since the arcs of the semi-circles are decomposable subcontinua.

We shall discuss hereditarily indecomposable continua in great detail in Chapter 12. At that time we shall also give a precise description of an example of a hereditarily indecomposable continuum which is homeomorphic to Knaster's. The more recent description and construction techniques are not so cumbersome as Knaster's.

CHAPTER 7

EXISTENCE OF INDECOMPOSABLE CONTINUA

In previous chapters, we have presented several examples of indecomposable continua in E^2 and many theorems dealing with the properties of metric and non-metric indecomposable continua. This chapter is devoted to showing several existence theorems about these continua.

First, we shall prove Mazurkiewicz' theorem which says that every compact metric space of dimension greater than one contains an indecomposable continuum. In Chapter 12, we shall discuss Bing's result that there exist hereditarily indecomposable continua of all dimensions and the related result of J. L. Kelley. Second, we shall show that non-metric indecomposable continua exist. This will be done by an example, rather than by a general existence theorem. The last part of the chapter will be used to discuss he question of how frequently indecomposable continua ccur in I^2.

Before establishing Mazurkiewicz' result, we summarize he needed definitions and theorems from general topology.

<u>efinition</u>: Let X, Y be topological spaces, and let f, g be ny two continuous functions from X to Y. The functions re <u>homotopic</u> if there exists a continuous $\Phi : X \times I \longrightarrow Y$ such

that $\Phi(x,0) = f(x)$, and $\Phi(x,1) = g(x)$, for all $x \in X$. f is nullhomotopic if it is homotopic to a constant map.

Definition: Let f, g, X, Y be as above, and let $A \subset X$. Then f, g are homotopic relative to A if there exists a continuous $\Phi : X \times I \to Y$ such that $\Phi(x,0) = f(x)$, $\Phi(x,1) = g(x)$, and $\Phi(a,t) = f(a) = g(a)$, for all $x \in X$, $a \in A$, $t \in I$.

Definition: Let $f: R \to W^n$ be a continuous surjection, where W^n denotes a homeomorph of I^n. If every continuous mapping $g: R \to W^n$ which is homotopic to f relative to $f^{-1}[Fr(W^n)]$ satisfies $g(R) = W^n$, then f is called essential. If f is not essential, then it is inessential.

The above terminology follows Alexandroff's "Dimensiontheorie" [3] and Nagata's Modern Dimension Theory [103].

Lemma 7.1: A continuous surjection $f: R \to B^n$, where $B^n = \{x \in E^n / |x| \leq 1\}$ is essential iff every continuous mapping $g: R \to B^n$ which coincides with f on $f^{-1}(S^{n-1})$ satisfies $g(R) = B^n$.

Proof: If f is essential, the conclusion follows at once. Conversely, if the condition holds, we only need to prove that f is homotopic to g relative to $f^{-1}(S^{n-1})$. The homotopy given by $\Phi(x,t) = t \cdot g(x) + (1-t) \cdot f(x)$ establishes this.

Lemma 7.2: Let X be normal and $A \subset X$ closed, with $f: A \to S^n$ continuous. Then there exists a neighborhood $U \supset A$ over which f can be extended relative to S^n.

Proof: See [28, p. 151] for a proof of this corollary to Tietze's extension theorem.

emma 7.3: (Borsuk) Let X be a compact metric space, and let $\subset X$ be closed. Let f, g: $D \rightarrow S^n$ be homotopic. If f has n extension F: $X \rightarrow S^n$, then so does g, G: $X \rightarrow S^n$, and G an be chosen so that F and G are homotopic.

Proof: Let φ be the homotopy of f and g. Define the ap $\Phi : X\times\{0\} \cup D\times I \rightarrow S^n$ by $\Phi(x,0) = F(x)$, $\Phi(d,t) = \varphi(d,t)$. e extend Φ to all of $X\times I$. By Lemma 7.2, Φ has an exten- ion $\hat{\Phi}$ over some neighborhood $U \supset (X\times\{0\} \cup D\times I)$. Since I s compact, there exists a neighborhood $V \supset B$ such that $\times I \subset U$ [28, p. 228]. Since B and X-V are disjoint closed ets, there exists a continuous function $f: X \rightarrow I$, say

$$(x) = \frac{d(x, X-V)}{d(x, X-V) + d(x, B)}$$, such that $f(B) = 1$ and $f(X-V)$ 0. Then $\Psi(x,t) = \hat{\Phi}(x, t \cdot f(x))$ is the required homotopy. tting $G(x) = \Psi(x,1)$ completes the proof.

There are also less stringent conditions on X under ich the lemma holds; for our purposes, we only need it as ated. We now present the first of Mazurkiewicz' lemmas.

mma 7.4: If f is an essential transformation from a com- ct metric space A onto B^2, then f: $A_1 = f^{-1}(S^1) \rightarrow S^1$ is t nullhomotopic [95, p. 327].

Proof: Suppose $f|_{A_1}: A_1 \rightarrow S^1$ is nullhomotopic. Then by mma 7.3, $f|_{A_1}$ can be extended to a continuous $F: A \rightarrow S^1$ ich is also nullhomotopic). Then $F(A) \subset S^1 \neq B^2$, so by mma 7.1, f is inessential.

mma 7.5: Let X be any space, and let f: $X \rightarrow S^1$ be con- uous. If $f(X) \neq S^1$, then f is nullhomotopic.

Proof: Choose $s_0 \in S^1 - f(X)$, and define $g: X \to S^1$ by $g(x) = -s_0$.

$\Phi(x,t) = \frac{t \cdot g(x) + (1-t) \cdot f(x)}{|t \cdot g(x) + (1-t) \cdot f(x)|}$ is the required homotopy.

Corollary 7.6: If f is an essential transformation from a compact metric space A onto B^2, then $A_1 = f^{-1}(S^1)$ contains a component K such that f is a nullhomotopic transformation of K into S^1, and consequently, $f(K) = S^1$.

Proof: Suppose that for every component $K \subset A_1$, $f_{|K}$ is nullhomotopic. (We now follow a proof of Eilenberg [29, p. 164].) There exists a continuous $\Phi : K \times I \to S^1$ such that $\Phi(k,0) = f_{|K}(k)$ and $\Phi(k,1) = s_0 \in S^1$.

Let $B = (A_1 \times \{0\}) \cup (K \times I) \cup (A_1 \times \{1\}) \subset A_1 \times I$. Set $\hat{\Phi}(x,0) = f(x)$, $\hat{\Phi}(x,1) = s_0$, for all $x \in A_1$, and $\hat{\Phi}(x,t) = \Phi(x,t)$, for all $x \in K$, and $t \in I$. There exists an open set $U \supset B$ such that $\hat{\Phi}$ can be extended to U. As before, there is a neighborhood $V \supset K$, such that $V \times I \subset U$. Since K is a component of a compact metric space, it is a quasi-component (see p. 48). Consequently, there is a closed open set V_1' in A_1 such that $K \subset V_1' \subset V$, from which $V_1' \times I \subset U$. Let $\bar{\Phi}_1 : V_1' \times I \to S^1$ be the extension of $\hat{\Phi}$ on U restricted to $V_1' \times I$. $\bar{\Phi}_1$ establishes a nullhomotopy of $f_{|V_1'} : V_1' \to S^1$.

Carry out this process for all components K to get an open covering $\{V_\alpha'\}$ of the compact space A_1. Then there is an open subcover $\{V_{\alpha_i}'\}_{i=1}^{n}$ of A_1. Moreover, we may choose these sets to be pairwise disjoint, since each of the finitely many sets is both open and closed. Since $f|_{V'_{\alpha_i}}$: $V'_{\alpha_i} \to S^1$ is nullhomotopic by $\bar{\bar{\Phi}}_{\alpha_i}$, we define $\Psi : A_1 x I \to S^1$ by $\Psi(x,t) = \bar{\bar{\Phi}}_{\alpha_i}(x,T)$, for the unique α_i such that $x \in V'_{\alpha_i}$. Ψ is clearly continuous, $\Psi(x,0) = f(x)$, for all $x \in A_1$, and $\Psi(x,1) \in \{s_{\alpha_i}\}_{i=1}^{n}$. $\Psi(x,1)$ is not a surjection, so by Lemma 7.5, it is nullhomotopic, say by Δ.

Therefore, $f|_A : A_1 \to S^1$ is nullhomotopic by

$$\hat{\Psi}(x,t) = \begin{cases} \Psi(x,2t) & 0 \leq t \leq 1/2 \\ \Delta(x, 2t-1) & 1/2 \leq t \leq 1. \end{cases}$$

But, $f|_{A_1} : A_1 \to S^1$ being nullhomotopic implies, by Lemma 7.4, that $f: A \to B^2$ is inessential. Thus, the result holds. Finally, $f|_{A_1}$ being not nullhomotopic implies $f(K) = S^1$, by Lemma 7.5.

<u>Lemma 7.7</u>: Let f be an essential transformation from A onto B^2. Let $J \subset B^2$ be a simple closed curve, with H denoting the one of the two domains determined by J in the plane that lies in B^2. Then, f is an essential transformation from $f^{-1}(\bar{H})$ onto $\bar{H} = H \cup J$ [95, p. 328].

<u>Proof</u>: $\bar{H}$ is homeomorphic to B^2. For convenience, let D denote $f^{-1}(\bar{H})$. Suppose that $f|_D$ is an inessential transformation from D onto $\bar{H}$. Then there is a homotopy $\Phi : DxI \to \bar{H}$ such that $\Phi(x,0) = f(x)$, and $\Phi(x,1)$ is, say $g(x)$, where $g(x) \neq a$, for all $x \in D$ and some $a \in H$, and such that Φ fixes $f^{-1}(J)$. We extend $f|_D$ to a function F on all A by:

$$F(x) = \begin{cases} f(x) & \text{if } x \in \overline{A-D} \\ g(x) & \text{if } x \in D. \end{cases}$$

If $x \in D \cap \overline{A-D}$, then $x \in f^{-1}(J)$: $D \cap \overline{A-D} = f^{-1}(\overline{H}) \cap \overline{A-f^{-1}(\overline{H})}$ implies $f(D \cap \overline{A-D}) = \overline{H} \cap f(\overline{A-f^{-1}(\overline{H})}) \subset \overline{H} \cap \overline{f(A-f^{-1}(\overline{H}))}$ by continuity of f. The last set is contained in the set $\overline{H} \cap \overline{B^2-\overline{H}}$, which is J. Therefore, $f(x) = g(x)$ for all $x \in D \cap \overline{A-D}$, and it follows that F is continuous.

F is homotopic to f relative to $f^{-1}(S^1)$ by

$$\Psi(x,t) = \begin{cases} \Phi(x,t) & \text{if } x \in D \\ f(x) & \text{if } x \in \overline{A-D}. \end{cases}$$

But, since $g(x) \neq a$, for all $x \in D$ by construction, and since $f|_{\overline{A-D}}(x) \neq a$, for all $x \in \overline{A-D}$ by definition of inverse image, we have that $F(X) \neq B^2$. Therefore, f is inessential.

<u>Corollary 7.8</u>: If f is an essential transformation from A onto B^2, with $J \subset B^2$ a simple closed curve, then A contains a continuum K such that $f(K) = J$.

<u>Proof</u>: By Lemma 7.7, $f|_D : D \rightarrow \overline{H}$ is essential. Let $\varphi: \overline{H} \rightarrow B^2$ be a homeomorphism. We shall show that $\varphi \circ f|_D$ is an essential transformation of D onto B^2.

Let g be any continuous function which is homotopic, say by Φ, to $\varphi \circ f|_D$, relative to $(\varphi \circ f|_D)^{-1}(S^1) = f^{-1}(J)$. It only remains to show that $g(D) = B^2$. But, $\varphi^{-1}\Phi : D \times I \rightarrow \overline{H}$ shows that $f|_D$ is homotopic to $\varphi^{-1} \circ g$ relative to $f^{-1}(J)$, and since $f|_D$ is essential, $\varphi^{-1} \circ g(D) = \overline{H}$. Therefore, $g(D) = \varphi(\overline{H}) = B^2$.

By Corollary 7.6, there exists a component (which is, of course, a continuum) $K \subset (\varphi \circ f|_D)^{-1}(S^1)$ such that $\varphi \circ f|_D(K) = S^1$. Therefore, $f(K) = \varphi^{-1}(S^1) = J$.

Before we can establish Mazurkiewicz' first major

result (Theorem 7.9), we need some results from set-theoretic topology.

<u>Definition</u>: Let $\{X_n\}$ be a sequence of subsets of a topological space S. The limit inferior of $\{X_n\}$ is defined to be the set $\liminf_{n\to\infty} X_n = \{x \in S \mid$ for all nbds $N(x)$, $N \cap X_n \neq \emptyset$, for all but a finite number of X_n's$\}$. $\limsup_{n\to\infty} X_n = \{x \in S \mid$ for all nbds $N(x)$, $N \cap X_n \neq \emptyset$, for an infinite number of $X_n\}$. If these two sets are equal, then this set is denoted by $\lim_{n\to\infty} X_n$, and the sequence of sets <u>converges</u>.

References to this concept may be found in [44, p. 100] or [75, p. 335]. It is clear that $\liminf_{n\to\infty} X_n \subset \limsup_{n\to\infty} X_n$. For the remainder of the discussion, S denotes a compact metric space.

<u>Lemma A</u>: $p \in \limsup_{n\to\infty} X_n$ iff there exists a sequence of points $\{p_{n_k}\}$ such that $n_k < n_{k+1}$, for $k = 1, 2, \ldots$, $p = \lim_{n_k\to\infty} p_{n_k}$, and $p_{n_k} \in A_{n_k}$.

<u>Proof</u>: Suppose that such a subsequence exists. Let U be a neighborhood of p. Then there exists $N \in Z^+$ such that $n_k \geq N$ implies $p_{n_k} \in U$. Consequently, $U \cap X_{n_k} \neq \emptyset$, for an infinite number of X_n's. Therefore, $p \in \limsup_{n\to\infty} X_n$.

Conversely, if $p \in \limsup_{n\to\infty} X_n$, then $B(p, 1/m) \cap X_n \neq \emptyset$ for an infinite number of X_n's, say X_{n_k}. Choose a p_{n_k} in each intersection. Then $d(p, p_{n_k}) < 1/m$ implies $\{p_{n_k}\} \to p$.

Lemma B: Let f be continuous on S. If $\{X_n\}$ converges, then $f(\lim_{n\to\infty} X_n) = \lim_{n\to\infty} f(X_n)$.

Proof: We need only show

$$f(\liminf_{n\to\infty} X_n) \subset \liminf_{n\to\infty} f(X_n) \subset \limsup_{n\to\infty} f(X_n) \subset f(\limsup_{n\to\infty} X_n).$$

To establish the first inclusion, note that $y \in f(\liminf_{n\to\infty} X_n)$ implies that there exists an $x \in \liminf_{n\to\infty} X_n$ such that $f(x) = y$. Let V be any neighborhood of y. $f^{-1}(V)$ is a neighborhood of x, and therefore, $f^{-1}(V) \cap X_n \neq \emptyset$ for all but a finite number of X_n's. Consequently, $V \cap f(X_n) \neq \emptyset$ for all but a finite number of X_n's.

The second inclusion is trivial. To prove the third inclusion, note by Lemma A that $p \in \limsup_{n\to\infty} f(X_n)$ implies there exists a sequence $\{p_{n_k}\} \to p$, with $p_{n_k} \in f(X_{n_k})$. Thus, there exist $q_{n_k} \in X_{n_k}$ such that $f(q_{n_k}) = p_{n_k}$. Since S is a compact metric space, $\{q_{n_k}\}$ has a convergent subsequence, $\{q_{n_{k_i}}\} \to q \in S$. Therefore, $q \in \limsup_{n\to\infty} X_n$, and since f is continuous, we have that $f(q_{n_{k_i}}) \to f(q)$, which must be p, since $f(q_{n_k}) \to p$. Hence $p \in f(\limsup_{n\to\infty} X_n)$.

Lemma C: Every sequence of sets in S has a convergent subsequence of sets.

Proof: See Hocking and Young [44, p. 102].

Lemma D: If $\lim_{n\to\infty} X_n$ exists, then $\lim_{n_k\to\infty} X_{n_k} = \lim_{n\to\infty} X_n$.

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Proof: See Kuratowski [75, p. 339].

Theorem 7.9: Let f be an essential transformation from A onto B^2, and let $C \subset B^2$ be a continuum. Then there exists a continuum $L \subset A$ such that $f(L) = C$ [95, p. 328].

Proof: We first indicate a proof that there exists a sequence of simple closed curves $J_n \subset B^2$ such that $\lim J_n = C$. For each $n \geqslant 1$, cover C by $\{B(c,1/n)\}_{c \in C}$. By compactness of C, there is a subcover $\{B(c_{(i,n)},1/n)\}_{i=1}^{m_n}$. For each n, construct a simple closed curve M_n passing through the points $c_{(i,n)}$. M_n has a convergent subsequence $\{M_{n_k}\}$ by Lemma C. We claim that $\lim_{n\to\infty} M_{n_k} = C$. It suffices to show that $\liminf_{n\to\infty} M_{n_k} \subset C \subset \limsup_{n\to\infty} M_{n_k}$. From $x \in \liminf_{n\to\infty} M_{n_k}$, it follows that x is a limit point of C, and hence is in C. If $y \in C$, then for all n_k there exist $c_{(i,n_k)} \in C$ such that $d(y,c_{(i,n_k)}) < 1/n_k$. $B(y,1/n_k) \cap X_{n_k} \neq \emptyset$, and it follows that every neighborhood of y meets infinitely many of the X_n's.

We now prove the theorem. By Corollary 7.8, there is a continuum $K_n \subset A$ such that $f(K_n) = J_n$, for $n = 1, 2, \ldots$. $\{K_n\}$ has a convergent subsequence $\{K_{n_i}\}$, so let $\lim_{n\to\infty} K_{n_i} = L$. $L \subset A$ and L is easily seen to be closed, and hence compact. L is nonempty and by a theorem in Hocking and Young [44, p. 102], L is connected. Thus, $f(L) = \lim_{n\to\infty} f(K_{n_i}) = \lim_{n\to\infty} J_{n_i} = \lim_{n\to\infty} J_n = C$.

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We are now ready to establish the principal result of Mazurkiewicz' paper.

Theorem 7.10: Every compact metric space of dimension greater than one contains an indecomposable continuum [95, p. 328].

Proof: For a definition and discussion of dimension, see Hurewicz and Wallman [45] or Nagata [103]. Alexandroff established the following result [3, p. 170], the proof of which can also be found in [103, p. 59]. "A metric space A has dimension less than or equal to one iff every continuous mapping of A into B^2 is inessential." Therefore, dim $A > 1$ implies there exists a continuous function $f: A \rightarrow B^2$ that is essential.

Let C_o be an indecomposable continuum contained in B^2, say Knaster's first semi-circle example, shrunk sufficiently to be contained in B^2. By Theorem 7.9, A contains a continuum L_o such that $f(L_o) = C_o$. Knaster and Mazurkiewicz showed [65, p. 87] that L_o contains as indecomposable continuum L_1 such that $f(L_1) = f(L_o) = C_o$. The proof is as follows. Using Zorn's lemma, it is easy to show that there is a subcontinuum K of L_o, irreducible with respect to the property that $f(K) = f(L_o)$. If $K = A \cup B$, where A, B are proper subcontinua, then we have $L_o = f(K) = f(A) \cup f(B)$. f(A), f(B) are subcontinua, but by the minimality of K, neither is all of L_o, whence L_o is decomposable. This is a contradiction, so the result holds, with $L_1 = K$.

This concludes Mazurkiewicz' two page paper. We shall consider related questions for hereditarily indecomposable continua in Chapter 12.

The existence question for non-metric indecomposable continua seems to be more difficult. This is not too surprising in view of the fact that the only major theorems we have presented thus far that deal with this case are 4.4 and 5.1; the rest deal with the metric case. However, in 1968, Bellamy constructed an example of a non-metric indecomposable continuum in his thesis [6]. We again need some preliminary definitions.

A topological space X is <u>completely</u> <u>regular</u> if for each $p \in X$ and closed set A not containing p, there exists a continuous function $\varphi: X \longrightarrow I$ such that $\varphi(p) = 1$, and $\varphi(a) = 0$ for all $a \in A$ [28, p. 153]. Let I^X denote the set of all continuous functions $f: X \longrightarrow I$, and let $\{I_f | f \in I^X\}$ be a family of unit intervals indexed by I^X. Let $P^X = \pi\{I_f | f \in I^X\}$; its points are denoted $\{t_f\}$ [28, p. 155].

<u>Lemma 7.11</u>: If X is a completely regular T_2 space, then it can be embedded in P^X. That is, $\varphi : X \longrightarrow P^X$ given by $\varphi(x) = \{f(x)_f\}$ is a homeomorphism of X and $\varphi(X) \subset P^X$.

<u>Proof</u>: See [28, p. 155].

A <u>compactification</u> of a space X is a pair (X,h), where X is a compact T_2 space, and h is a homeomorphism of X onto a dense subset of X. The <u>Stone-Čech</u> <u>compactification</u> of X is $(\beta(X), \varphi)$, where $\beta(X) = \overline{\varphi(X)}$.

<u>Lemma 7.12</u>: For each compact space Y and each continuous

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f: $X \to Y$, there is a unique continuous extension $F: \beta(X) \to Y$ ($f = F \circ \varphi$). Moreover, any other compactification of X having this property is homeomorphic to $\beta(X)$.

<u>Proof</u>: See [28, p.243].

We now follow Bellamy, with only slight changes in notation. Let $A = [1, \infty)$, and $A^* = \beta(A) - A$. Actually, $A^* = \beta(A) - \varphi(A)$, but we identify A with its image in $\beta(A)$.

<u>Lemma 7.13</u>: Let U be an open set which meets A^*. Then $U \cup A$ is unbounded [6, p.30].

<u>Proof</u>: Suppose $U \cup A$ is not bounded. Then $U \cup A \subset [1,x]$, for some $x \in A$. Thus, $U \cap (\beta(A)-[1,x])$ is a nonempty (since $\emptyset \neq U \cap (\beta(A)-A) \subset U \cap (\beta(A)-[1,x])$ open subset of $\beta(A)$ which misses A. This is impossible, since A is dense in $\beta(A)$.

<u>Lemma 7.14</u>: A^* is a T_2 continuum.

<u>Proof</u>: For all $n \geq 1$, let $A_n = [n, \infty)$, and set $P_n = A_n \cup A^*$. Then $A^* = \bigcap_1^\infty P_n$. But, $P_n = \overline{A_n}$ (closure in $\beta(A)$), so that each P_n is a T_2 continuum. The intersection is monotone, so by Theorem 2.1, A^* is a T_2 continuum.

Note that the above two lemmas hold for any compactification of A, as Bellamy showed.

<u>Theorem 7.15</u>: A^* is a non-metrizable indecomposable continuum.

<u>Proof</u>: By Lemma 7.14, A^* is a continuum. Suppose that $A^* = X \cup Y$, where X, Y are proper subcontinua of A^*. We shall show that X is not connected.

Let $x \in X-Y$, $y \in Y-X$. Let U, V be open sets in $\beta(A)$ such that: 1) $x \in U$, $y \in V$, and 2) $\overline{U} \cap \overline{V} = \overline{U} \cap Y = \overline{V} \cap X = \emptyset$. Choose sequences $\{p_i\}_{i=1}^{\infty}$, $\{q_i\}_{i=1}^{\infty}$, and $\{r_i\}_{i=1}^{\infty}$ from A as follows: Let $p_1 \in U \cap A$. Choose $q_1 > p_1$ such that $q_1 \in V$; this is possible since by Lemma 7.13, $V \cap A$ is unbounded. Next, choose $r_1 > q_1$ such that $(q_1, r_1) \subset V$; this can be done since V is open and hence q_1 lies in some open interval in V.

Suppose p_k, q_k, r_k have been choosen for $k < n$ such that for each k:

1) $p_k \in U$,

2) the interval $(q_k, r_k) \subset V$,

3) $p_k < q_k < r_k$, and if $k < n-1$, then $r_k < p_{k+1}$.

Then, since $U \cap A$ is unbounded, there exists a $q_n > p_n$ such that $q_n \in V$. Since V is open, r_n may be chosen greater than q_n such that $(q_n, r_n) \subset V$.

The sequences $\{p_n\}_1^{\infty}$, $\{q_n\}_1^{\infty}$, $\{r_n\}_1^{\infty}$ are all unbounded. For if not, they would have a common supremum which would have to belong to $\overline{U} \cap \overline{V}$, a contradiction.

Define $f: A \to I$ as follows:

$$f(t) = 0 \quad \text{if } t < p_1$$

$$f(p_i) = \begin{cases} 0 & \text{if } i \text{ is odd} \\ 1 & \text{if } i \text{ is even} \end{cases}$$

$$f(q_i) = \begin{cases} 1/3 & \text{if } i \text{ is odd} \\ 2/3 & \text{if } i \text{ is even} \end{cases}$$

$$f(r_i) = \begin{cases} 1/3 & \text{if } i \text{ is even} \\ 2/3 & \text{if } i \text{ is odd} \end{cases}$$

Now extend f linearly to each of the intervals $[p_i, q_i]$, $[q_i, r_i]$, and $[r_i, p_{i+1}]$. Then f is a continuous function from A to I. By Lemma 7.12, f has a continuous extension $F: \beta(A) \to I$. $F^{-1}(0)$ is a closed subset of $\beta(A)$ containing $\{p_{2k+1}\}_1^{\infty}$, and hence containing all limit points of the sequence in $\beta(A)$. Therefore, $F^{-1}(0) \cap A^* \neq \emptyset$. But, since the sequence lies in U, any limit point of it is an element of $\overline{U}$, and hence does not lie in Y. Therefore, $F^{-1}(0) \cap X \neq \emptyset$, and thus $0 \in F(X)$. Likewise, $1 \in F(X)$.

But, let $a \in F^{-1}(1/3, 2/3)$. a is a limit point of $f^{-1}(1/3, 2/3) = \bigcup_1^{\infty} (q_k, r_k) \subset V$. Therefore, $a \in \overline{V}$, and hence $a \notin X$. Consequently, $F(X) \cap (1/3, 2/3) = \emptyset$. Then, since F is continuous and takes on the values 0 and 1, its domain can not be connected, since its range is not.

The proof that A^* is non-metrizable follows from a corollary [6, p. 40] which says that A^* has 2^c points. Thus A^* can't be embedded in the Hilbert cube, as it could be if it were metrizable. The proof that A^* has 2^c points is rather long and will not be presented.

We conclude the chapter by briefly mentioning some results dealing with a different type of existence question. Namely, how frequently do indecomposable continua occur in the space of all continua of a given space? This appears to be a rather difficult question to answer. But the surprising result is that "most" continua in I^2 are not only indecomposable, but are also hereditarily indecomposable. In this context, "most" means that the set of hereditarily

indecomposable continua in I^2 constitute a dense G_δ set in the space of all continua of I^2, when the latter set is given the Hausdorff (see [76, p. 47]) metric [94, pp. 151-159]. In Chapter 12, we shall see that Bing established a similar result for an even more singular type of continuum (the pseudo-arc) in any Euclidean or Hilbert space. Kuratowski also notes [76, p. 202] that in any compact metric space, the set of indecomposable or hereditarily indecomposable continua are a G_δ set. There does not appear to be much known about the frequency of occurrence of indecomposable continua in spaces other than Euclidean or Hilbert.

We shall content ourselves with an outline of Mazurkiewicz' proof that the hereditarily indecomposable continua in I^2 are a dense G_δ set. The nine pages of details are not difficult and the interested reader may consult the original paper for them.

He defined a sequence of sets of continua in I^2, $\{\Gamma_{1/n}\}_1^\infty$ as follows: $\Gamma_{1/n}$ is the set of all continua C in I^2 such that $C \supset K$, where K is a subcontinuum such that $K = K_1 \cup K_2$, and K_1, K_2 are continua with the property that $\sup_{p \in K_2} \{d(K_1, p)\} \geq 1/n$ and $\sup_{q \in K_1} \{d(K_2, q)\} \leq 1/n$. The major portion of his paper was devoted to showing that the $\Gamma_{1/n}$'s are closed nowhere dense sets. Letting Γ_K denote the set of hereditarily indecomposable continua in I^2, Γ the set of all continua in I^2, and Γ_0 the one point continua, it follows that $\Gamma_K = \Gamma - (\Gamma_0 \cup \bigcup_1^\infty \Gamma_{1/n})$. Thus, the result holds.

CHAPTER 8

THE COMMON BOUNDARY QUESTION

We have considered various examples and properties of indecomposable continua in the preceding chapters, but we have seen no application of them, except in their original role of being pathological examples. In this chapter and in Chapter 10, we shall present some other situations in which indecomposable continua arise.

The topic we are going to discuss in this chapter is the structure of sets in E^2 and E^3 which are common boundaries to three or more domains, which we recall is the problem that Brouwer was considering when he discovered indecomposable continua. In the plane, such common boundaries must be indecomposable or else the union of two indecomposable continua. It seems remarkable that by shifting our setting to E^3, nothing of the sort is true. In fact, there is a set in E^3 which is the common boundary of three domains and is not only decomposable but is also an absolute neighborhood retract.

Kuratowski and Knaster did the above mentioned work on the planar case. Most of the chapter will be devoted to their results, but we shall also mention Eilenberg's work on the common boundary question for S^2, various papers on

prime end theory, and Burgess' thesis, which makes all of the above planar results into special cases of a more general theorem.

In 1924, Kuratowski wrote a paper on irreducible cuts of the plane (to be defined on p. 104) in which he showed that if a compact set cuts E^2 into three or more domains and if it is the boundary of each of them, then the set is either an indecomposable continuum or the union of two indecomposable continua [70, p. 138]. In 1928, he was able to establish the same conclusion while only requiring the set in question to be the boundary of at least three domains. There are no restrictions on its relationships to any other domains it may determine in E^2 [72, p. 36].

In 1925, Knaster gave examples B_n, C_n which cut E^2 irreducibly into n domains such that each B_n is indecomposable and each C_n is the union of two indecomposable continua. Examples of the first type were already available from the work of Brouwer and Wada. But, the existence of the second class of examples was not previously known. Thus, the "either-or" conclusion to Kuratowski's theorems can not be improved, since there are common boundaries of both types.

Starting in the early 1920's, many papers dealt with the idea of cutting the plane. In particular, several papers we will present in this chapter were written using this terminology. More recently, the idea of a set separating another set has become quite widely used. We shall show that for closed subsets of E^n, the concepts agree. We

first define the necessary terminology.

Definition: Let X be a topological space. The subset $A \subsetneq X$ cuts X if X-A is not a semi-continuum. A separates X if X-A is not connected.

If A cuts X, we may express X-A as a disjoint union of semi-continua which do not meet A. (The term used in the older literature for these semi-continua is "regions composants.") If A separates X, X-A is the usual disjoint union of components, and if A is closed, recall that these sets are called the complementary domains of A (see p. 8).

Definition: A cuts (respectively separates) X between p, q if p, q lie in different semi-continua (respectively components) of X-A.

It is clear that if A separates X, then it cuts X. To see that the converse is not necessarily true unless more is assumed about X, A, consider the space

$$X = \{(x,y) \mid y = \sin \pi/x,\ 0 < x \leq 1\} \cup \{(0,y) \mid |y| \leq 1\}$$

and take $A = \{(0,0)\}$. Then A cuts X between (0,1) and (0,-1), but A certainly does not separate X. However, we do have the following

Lemma 8.1: If $A \subset E^n$ is closed, then A separates E^n iff A cuts E^n.

Proof: If A separates, then it cuts. If A does not separate E^n, then E^n-A is open and connected. By [28, p. 116], E^n-A is path connected, which in our terminology means there is a continuum disjoint from A joining any pair of points in E^n-A. Therefore, A does not cut E^n.

Thus, we may use the phrases "A cuts B" and "A separates B" interchangeably, whenever A is closed and B is a connected open subset of E^n.

Lemma 8.2: Let A be a closed set in E^n. Then:

(a) each complementary domain of A is a path connected open set;

(b) the boundary of each complimentary domain of A is contained in A;

(c) if A separates E^n, but no proper closed subset does, then the boundary of each complementary domain of A is exactly A;

(d) if A is compact, then A has exactly one unbounded complementary domain.

Proof: See [28, p. 356], and note that Dugundji's hypothesis of compactness is not required for a - c.

Because the separators in (c) are quite important, they are given a special name below.

Definition: If a closed set A cuts E^n between a, b and if no proper closed subset does, then A is said to cut E^n irreducibly between a, b. If A cuts E^n irreducibly between all points that it cuts between, then A is a completely irreducible cut of E^n.

Using a Zorn's lemma argument, it is easy to see that any closed set which cuts E^2 between p, q contains a closed subset which cuts E^2 irreducibly between p, q. However, if such a subset is to be completely irreducible, then the original cut must determine only finitely many domains [70,

p. 135]. Moreover, a cut is itself completely irreducible iff it is the common boundary to all its complementary domains (see the Corollary to Theorem 8.6). The following is due to Mazurkiewicz.

<u>Lemma 8.3</u>: Let R be a domain and let S be a complementary domain of $E^2-\overline{R}$. Then Fr(S) is an irreducible cut of E^2 between all points a, b, where $a \in R$, $b \in S$ [90, p. 193].

<u>Proof</u>: It is clear that the boundary cuts E^2 between such points a, b, for otherwise we could disconnect any continuum joining a,b.

$Fr(S) \subset Fr(R)$ holds, for by Lemma 8.2 (c), $Fr(S) \subset \overline{R}$, and always $Fr(S) \subset \overline{S} \subset \overline{E^2-\overline{R}}$. Then $Fr(S) \subset \overline{R} \cap \overline{E^2-R} = Fr(R)$. Let A be any closed proper subset of Fr(S), and choose $p \in Fr(S)-A \subset Fr(R)$. There must be points $x \in R$, $y \in S$ in the neighborhood $B(p,[d(p,A)]/2)$. By Lemma 8.2(a), we can join a to x by a path P_1 lying in S (and hence disjoint from A), and we can join b to y by a path P_2 lying in R. The line segment S joining a and y lies in the ball, so it too is disjoint from A. $P_1 \cup S \cup P_2$ is therefore a continuum joining a and b which does not meet A. Thus, A does not separate.

We shall also have need of the following result, first proved by L. E. J. Brouwer in 1910.

<u>Theorem 8.4</u>: (Phragmen-Brouwer property) If the boundary of a complementary domain of a continuum is compact, then it is a continuum.

<u>Proof</u>: See [102, p. 176], or [127, p. 106].

For the more general case in which the boundary is not assumed to be compact, see Wilder [130, p. 48].

Theorem 8.5: If the compact set C cuts E^2 irreducibly between a and b, then it is connected [70, p. 134].

Proof: Let R be the complementary domain of C containing a. Let S be the complementary domain of $\overline{R} \subset R \cup C$ (by Lemma 8.2 (b)) which contains b.

By Lemma 8.3, $Fr(S) \subset Fr(R)$, and by Lemma 8.2(b), the latter is in C. By Lemma 8.3, Fr(S) is an irreducible cut, as is C. Therefore, C = Fr(S). By Theorem 8.4, Fr(S) is connected; hence C is connected.

Theorem 8.6: In order that a closed set C should be the common boundary of two domains D_1, D_2 in E^2, it is necessary and sufficient that C be an irreducible cut of E^2 between a, b for all $a \in D_1$, $b \in D_2$ [70, p. 133].

Proof: If C is an irreducible cut, then by Lemma 8.2 (c), it is the common boundary of D_1, D_2.

Suppose $C = Fr(D_1) = Fr(D_2)$. C cuts E^2 between all points of D_1 and D_2, since any connected set containing such a pair of points must meet the boundary, C. We shall now show its irreducibility.

Let D_3 be the complementary domain of $E^2-\overline{D}_1$ containing b. Consequently, $b \in D_3 \cap D_2$, and we shall show $D_3 = D_2$. From Lemma 8.3, $Fr(D_3) \subset Fr(D_1) = Fr(D_2)$. Since all points of D_2 can be joined to b by a continuum (path) not meeting $Fr(D_2)$, the same can be done while missing $Fr(D_3)$. Since D_2 is connected and $D_2 \cap D_3 \neq \emptyset$, we have $D_2 \subset D_3$.

On the other hand, since D_3 is a complementary domain of $E^2-\bar{D}_1$, $D_3 \cap \bar{D}_1 = \emptyset$. Thus, $D_3 \cap Fr(D_1) = \emptyset$, and hence $D_3 \cap Fr(D_2) = \emptyset$. Therefore, all points of D_3 can be joined to b by a continuum disjoint from $Fr(D_2)$ (Lemma 8.2(a)). Hence $D_3 \subset D_2$.

$D_2 = D_3$ implies $Fr(D_2) = Fr(D_3) = C$, so by Lemma 8.3, C cuts E^2 irreducibly between a and b.

As a corollary, note that C is a completely irreducible cut of E^2 iff it is the common boundary of all its complementary domains. We shall also have need of

Theorem 8.7: (Janiszewski)

(a) If A and B are continua such that $A \cap B$ is disconnected, then $A \cup B$ separates the plane.

(b) If A and B are compact sets, neither of which cuts E^2 between p, q and if $A \cap B$ is a continuum (perhaps empty), then $A \cup B$ does not cut E^2 between p, q.

Proof: See [49, p. 192], or [102, pp 175 and 173].

We note in passing that Kuratowski has shown that (a) and (b) are equivalent in any locally connected continuum [73, p. 311].

Lemma 8.8: Let K be a subcontinuum of a compact set C which cuts E^2 irreducibly between p, q. Then C-K is connected [70, p. 136].

Proof: The result is clear if $K = \emptyset$ or $K = C$. So suppose $\emptyset \neq K \neq C$. If C-K is disconnected, then by Theorems 8.5 and 4.2, $C-K = M \cup N$, where M, N are nonempty, disjoint, closed subsets of C-K. Moreover, $K \cup M$ and $K \cup N$ are proper subcontinua of C. By the irreducibility of C, neither cuts

E^2 between p, q. So by Theorem 8.7 (b), $(K \cup M) \cup (K \cup N) = C$ does not cut either. This is a contradiction. Therefore, $C-K$ is connected.

<u>Lemma 8.9</u>: If a decomposable continuum C is the common boundary of two domains in E^2, then $C = A \cup B$, where A, B are proper subcontinua such that $\overline{C-A} = B$, and $\overline{C-B} = A$.

<u>Proof</u>: By Theorem 4.4, C contains a proper subcontinuum K that is not nowhere dense: $\emptyset \neq \overline{C-K} \neq C$. Let $A = \overline{C-K}$, $B = \overline{C-A}$. Then

$$\overline{C-B} = \overline{C-\overline{C-A}} = \overline{C-\overline{C-\overline{C-K}}},$$

which is $\overline{C-K} = A$ by Lemma 5.15.

By Theorem 8.6, C is an irreducible cut of E^2 between all pairs of points a, b, for a and b in different complementary domains of C. Therefore, by Lemma 8.8, A is a continuum and hence so is B. It follows from the choice of K that A is a proper subcontinuum of C. $A \neq \emptyset$ implies $B \neq C$, and $A \neq C$ implies $B \neq \emptyset$.

Finally, $C = A \cup B$, for

$$A \cup B = \overline{C-K} \cup \overline{C-A} = \overline{(C-A) \cup (C-\overline{C-K})} = \overline{C-Fr_C(K)} = C.$$

We have now established the foundation needed to prove both of Kuratowski's common boundary theorems. To continue with the proof of the first such theorem, we prove

<u>Lemma 8.10</u>: If C is a compact set which is a completely irreducible cut of E^2, and if K is a non-degenerate proper subcontinuum of C that is not nowhere dense in C, then $K \cap \overline{C-K}$ is disconnected and $\overline{C-K}$ is a continuum irreducible between all pairs of points belonging to different com-

ponents of $K \cap \overline{C-K}$ [70, p. 136].

Proof: By Lemma 8.8, $\overline{C-K}$ is connected. Since $\overline{C-K} \neq C$, $C \neq K$, and $K \cup \overline{C-K} = C$, it follows from Theorem 8.7 (b) that $K \cap \overline{C-K}$ is disconnected.

Let a, b be in different components of this set, and suppose that L is a continuum containing a, b and contained in $\overline{C-K}$. If $L \cap K$ is connected, then $\{a,b\} \subset K \cap L \subset K \cap \overline{C-K}$ implies that a, b are in the same component. Thus, $L \cap K$ must be disconnected, whence Theorem 8.7 (a) implies $K \cup L$ cuts E^2. By the irreducibility of C, $C = K \cup L$. Then $\overline{C-K} \subset L$ so that $L = \overline{C-K}$. Therefore, $\overline{C-K}$ is an irreducible continuum.

Lemma 8.11: (Straszewicz) If A, B are compact sets which do not cut E^2 and if $A \cup B$ cuts E^2 into more than two domains, then $A \cap B$ contains at least three components.

Proof: See [112] or [76, p. 551]. For the case where $A \cup B$ cuts E^2 into more than n domains, see [113, pp. 159-187]. Straszewicz showed in the latter paper that if the union of two continua, neither of which cuts the plane, and which have $n \geq 1$ components in their intersection, then the union cuts the plane into n domains. He also showed that "n" can be replaced by "countably infinite" [113, 174].

Theorem 8.12: (Kuratowski's first common boundary theorem) A compact set C which is a completely irreducible cut of E^2 and which determines at least three domains is either an indecomposable continuum or else the union of two indecomposable continua [70, p.138].

Proof: By Theorem 8.5, C is a continuum. Suppose it is decomposable. By Lemma 8.9, $C = L \cup M$, where L, M are proper subcontinua of C such that $\overline{C-L} = M$ and $\overline{C-M} = L$. L is not nowhere dense in C. For if it were, then $C = \overline{C-L} = M$ implies $\emptyset = \overline{C-M} = L$, contradicting Lemma 8.9.

$L \cap \overline{C-L} = L \cap M = M \cap \overline{C-M}$. We apply Lemma 8.10 to both L and M to conclude that $L \cap M$ is not connected and that L and M are continua irreducible between all pairs of points belonging to different components of $L \cap M$. By the irreducibility of C, L and M do not cut E^2, so by Lemma 8.11, $L \cap M$ must contain at least three components. By the first part of this proof, L and M are each irreducible continua between all pairs of points in different components of $L \cap M$. Therefore, by Theorem 4.11, L and M are indecomposable continua.

Eilenberg established a similar result for the sphere S^2 in [30, p.82].

We now prove Kuratowski's second common boundary theorem. The distinction between Theorems 8.12 and 8.13 is that in the former, C is required to be the common boundary of <u>all</u> its complementary domains, while in the latter, C is only assumed to be the common boundary of <u>some</u> (3) of them.

<u>Theorem 8.13</u>: If the plane continuum C is the boundary of at least three domains, then it is either indecomposable or the union of two indecomposable continua [72, p.36].

<u>Proof</u>: We follow the original proof, with only slight modifications. Let D_1, D_2, D_3 be the domains of the hypothesis. $C = Fr(D_k)$, $k = 1, 2, 3$. If C is decomposable, then by Lemma 8.9, there are proper subcontinua K_1 and L_1

in C such that $\overline{C-K_1} = L$ and $\overline{C-L} = K_1$.

Suppose further that C is not the union of two indecomposable continua. Then without loss of generality, L is decomposable: $L = K_2 \cup K_3$, where K_2, K_3 are proper subcontinua of L. Now $K_1 \cup K_2 \neq C$, since if equality held, then $C-K_1 \subset K_2$. This would imply that $L = \overline{C-K_1} \subset K_2$, contradicting the decomposition of L. Likewise, $K_1 \cup K_3 \neq C$. Thus $C = K_1 \cup K_2 \cup K_3$, where the union of any two of the K_i's is a proper subcontinuum of C.

Consequently, there are points $x_i \in K_i$, i = 1, 2, 3 that are not in the other K_j's. Let ϵ_i denote one half the distance from x_i to the union of the other two K_j's. Then $B_i = B(x_i, \epsilon_i)$ are pairwise disjoint sets such that $B_i \cap K_i \neq \emptyset$, and $B_i \cap K_j = \emptyset$, for $j \neq i$.

Since $\bigcup_1^3 K_i = C = Fr(D_k)$, k = 1, 2, 3, we have $B_i \cap D_k \neq \emptyset$, for i = 1, 2, 3, and k = 1, 2, 3. Because connected open sets in E^2 are polygonally arc-wise connected [44, p.108], there are polygonal lines P_1, P_2 such that $P_k \subset D_k$, k = 1, 2 and $P_k \cap B_i \neq \emptyset$, for k = 1, 2, and i = 1, 2, 3. The sets $P_k \cap \overline{B}_i$, k = 1, 2, i = 1, 2, 3 are compact. So there exist points $y_i \in P_1 \cap \overline{B}_i$, and $z_i \in P_2 \cap \overline{B}_i$, i = 1, 2, 3 such that $d(y_i, z_i)$ is the minimum with respect to the distances between all pairs of points, one from $P_1 \cap \overline{B}_i$ and one from $P_2 \cap \overline{B}_i$.

Let T_i be a triangle in B_i having y_i and z_i for the vertices of two of its acute angles. $P_1 \cap T_i = y_i$, and

$P_2 \cap T_i = z_i$, for i = 1, 2, 3. Since $y_i \in D_1$, $z_i \in D_2$, there are points of C in the interior of each triangle T_i, and consequently, there are points of D_3 in the interior of each T_i.

Thus, we can construct a polygonal line $Z \subset D_3$ such that $Z \cap T_i \neq \emptyset$ for i = 1, 2, 3, since D_3 is polygonally arc-wise connected. Let $W \subset Z$ be a polygonal line minimal with respect to meeting each T_i. Therefore, W must meet two of the T_i's, say T_1 and T_3 , only at their endpoints r_1, r_2: $W \cap T_1 = r_1$, $W \cap T_2 \neq \emptyset$, $W \cap T_3 = r_3$.

We agree that $(y_i z_i)$ shall denote either the segment joining those points, or else the line formed from the other two sides of T_i, depending on whether the segment contains points of W. Thus, $W \cap (y_i z_i) = r_i$, i = 1, 3 and $W \cap (y_2 z_2) \neq \emptyset$.

1) $P_1 \cap (y_i z_i) = y_i$, $P_2 \cap (y_i z_i) = z_i$, i = 1, 2, 3.

Since $(y_i z_i) \subset T_i \subset B_i$, it follows from the construction of the B_i's that

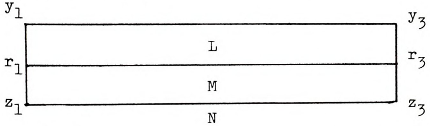
2) $(y_i z_i) \cap (y_{i+1} z_{i+1}) = \emptyset = (y_i z_i) \cap (K_{i+1} \cup K_{i+2})$, where the indices i+1, i+2 are reduced mod 3. Therefore,

3) $(y_i z_i) \cap K_i \neq \emptyset$.

Consider $S = (y_1 z_1) \cup (z_1 z_3) \cup (z_3 y_3) \cup (y_3 y_1)$, where $(z_1 z_3)$ and $(y_3 y_1)$ denote polygonal lines extracted from P_2, P_1 respectively.

S is a simple closed polygonal path by equations 1 and 2. Moreover, since W was a polygonal line having only its end-

points on S, $S \cup W$ cuts the plane into three domains, L, M, N such that $Fr(L) = (y_1r_1) \cup W \cup (r_3y_3) \cup (y_3y_1)$, $Fr(M) = (z_1r_1) \cup W \cup (r_3z_3) \cup (z_3z_1)$, and $Fr(N) = S$.



By 2), $(S \cup W) \cap K_2 = \emptyset$. Since K_2 is connected, it must be in exactly one of the three domains. It can not be in N, because $K_2 \cup [(y_2z_2)-\{y_2, z_2\}]$ is connected by 3), it has points in common with W (since $W \cap (y_iz_i) \neq \emptyset$, $i = 1, 2, 3$), and it is disjoint from S, which is the boundary of N, while $N \cap W = \emptyset$. Without loss of generality, $K_2 \subset L$.

Starting from y_2, let t be the last point of (y_2z_2) which belongs to K_2. Then (tz_2) has no points in common with C except t, and since $W \subset D_3$, $z_2 \notin D_2$, we conclude that $W \cap (tz_2) = \emptyset$. Therefore, $P_2 \cup (tz_2)$ is disjoint from $Fr(L)$. Consequently, $P_2 \cup (tz_2) \subset L$, since $t \in K_2 \subset L$, and $P_2 \cup (tz_2)$ is connected. This is impossible, since $(z_1z_3) \subset P_2$ and $(z_1z_3) \subset S$ imply that $S \cap P_2 \neq \emptyset$, while $S \cap L = \emptyset$. Thus, P_2 is not in L. Therefore, the assumption that C was not the union of two indecomposable continua leads to a contradiction.

We next present some of Knaster's examples of common boundaries of plane domains. Let A be the numbers in I that can be written in base five without using the digits "1" and

"3". We now describe a continuum B_n that is both indecomposable and the common boundary of n domains. Fix $n \geq 2$, and let $B_n = \bigcup_0^n E_k \cup \bigcup_1^{n-1} F_k$, where E_0, E_k $(0 < k < n)$, and E_n are composed of semi-circular arcs satisfying respectively the following conditions:

$$(x-5/2)^2 + y^2 = r^2, \qquad \text{if } x \leq 1/2;$$
$$(x-2k-1/2)^2 + y^2 = r^2, \qquad \text{if } 2k-1/2 \leq x \leq 2k+3/2;$$
$$(x-2n+3/2)^2 + y^2 = r^2, \qquad \text{if } 2n-1/2 \leq x,$$

where $y \geq 0$ and $(r-1/2) \in A$.

For each k, F_k is composed of the following semi-circles:

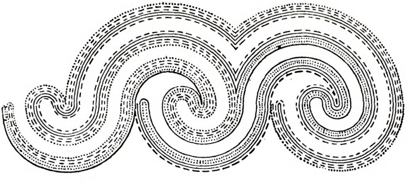
$$(x-2k-1)^2 + y^2 = r^2, \qquad y \leq 0,\ r \in A;$$
$$(x-2k-[7/2]5^{-m})^2 + y^2 = r^2,\ y > 0,\ (5^m r-1/2) \in A,\ m > 0.$$

See Figure 8.1 (a). The solid, dashed, dotted lines represent respectively the sets D_o, D_1, D_2 to be defined on p. 117.

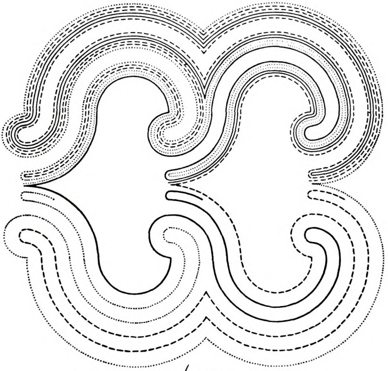
Knaster also discovered the continua $\{C_n\}$, each being the common boundary of n domains and the union of two indecomposable continua [60, p. 274]. $C_n = B_n' \cup B_n''$, where B_n' is obtained from B_n by replacing each F_k by F_k':

$$(x-2k-1)^2 + y^2 = r^2, \qquad 2k+1 \leq x,\quad y \leq 0,\ r \in A;$$
$$(x-2k-1)^2 + (y+1-[7/2]5^{-m})^2 = r^2,\ x \leq 2k+1,$$

$(5^m r-1/2) \in A$, $m > 0$. B_n'' is obtained from B_n' by reflecting it through the line $y = -1$. See Figure 8.1 (b). The solid, dashed, and dotted lines represent respectively D_0', D_1', D_2', to be defined on page 117.



a : B_3 / b : C_3



b : C_3 / c : upper C_∞

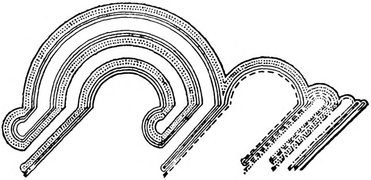


Figure 8.1

The verification that the B_n's are indecomposable can be found in Knaster's paper [60, pp. 274-281]. We will not present it due to its length. However, the essential idea is to show that every proper subcontinuum is nowhere dense, and then apply Theorem 4.4. Knaster accomplished this for B_n by showing that every proper subcontinuum must be contained in the union of an infinite number of successive simple arcs (compare Knaster's first semi-circle example, p. 24). The first arc in the first union is the semi-circle in F_0 starting at (0,0) and terminating at (2,0). The next arc in this union is the semi-circle in E_1 starting at (2,0) and terminating at (3,0), and so on. This union is denoted D_0, and it is represented by the solid line in Figure 8.1 (a). D_0' is the corresponding set in B_n', with its first arc starting at (1,-1).

The set D_k, for $0 < k < n$, is constructed in the same way as D_0. However, we begin with the arc in F_k joining 2k to 2k+2. D_1 is the dashed line and D_2 the dotted line in Figure 8.1 (a). An analogous description holds for the lines in Figure 8.1 (b).

Knaster was able to prove that B_n cuts the plane into exactly n nonempty domains and that B_n is the common boundary of each of these domains [60, pp. 278-279]. Moreover, he showed none of the B_n', B_n'' cut the plane. But, since $B' \cap B_n'' = \{(2k+1,0) \mid 0 \leq k < n\}$, the intersection has exactly n components. Then by the generalized Straszewicz Theorem (8.11), C_n cuts the plane into exactly n domains.

Knaster showed C_n is the common boundary of all those domains [60, p. 281]. Hence, by Theorem 8.12, C_n is either indecomposable or the union of two indecomposable continua. But C_n is clearly decomposable, so B_n', B_n'' must be indecomposable. Note that if Knaster would have had Theorem 8.13, he would only have had to show that C_n is the boundary of three of its complimentary domains.

We now have an abundant supply of indecomposable continua that do not cut the plane. Although some earlier examples have this property, it was not discussed by the original authors. For instance, Knaster's second semicircle example is homeomorphic to B_2' [60, p. 281], so it is not a "common boundary".

Brouwer, Wada, and Urysohn knew it was possible to have plane continua being the common boundary of a countably infinite collection of domains. However, Knaster seems to have been the first to actually publish a specific description of such sets, which he denoted B_∞. He was certainly the pioneer in constructing a continuum C_∞ which is the union of two indecomposable continua, and which is the common boundary of infinitely many domains. See Figure 8.1 (c) for the upper half of C_∞.

Knaster's construction is as follows: $B_\infty = (1,0) \cup \bigcup_0^\infty L_n \cup \bigcup_0^\infty M_n$, where each L_n is composed of circular arcs satisfying:

(1) $y \geq 0$ for all cases;

(2) for $n = 0$, $(x-3/10)^2 + y^2 = r^2$, $0 \leq x \leq 29/50$,

$(5r-1/2) \in A$;

(3) for n odd, $[x-1+(3/2)5^{-(1+n)/2}]^2 + y^2 = r^2$,

$1-(21/2)5^{(1-n)/2} \leq x \leq 1-(9/2)5^{(1-n)/2}$,

$[(5^{(n+3)/2})r-5/2] \in A$;

(4) for n even, $[x-1+(7/2)5^{-(n+2)/2}]^2 + y^2 = r^2$,

$1-(9/2)5^{-(n+2)/2} \leq x \leq 1-(21/2)5^{-(n+4)/2}$,

$[(5^{(n+2)/2})r-1/2] \in A$.

Each M_n is the union of semi-circles satisfying:

(1) $y \leq 0$ for the first type:

(a) for n = 0, $x^2 + y^2 = r^2$, $5r \in A$;

(b) for n odd, $[x-1+7 \cdot 5^{-(n+3)/2}]^2 + y^2 = r^2$,

$5^{(n+1)/2} \in A$;

(c) for n even, $[x-1+3 \cdot 5^{-(n+2)/2}]^2 + y^2 = r^2$,

$[(5^{(n+4)/2})r-2] \in A$;

(2) $y > 0$ and $[5^{m+1}r-1/2] \in A$ for the second type:

(a) for n = 0, $[x+(1/15)-(7/2)5^{-(m+1)}]^2 + y^2 = r^2$,

$m > 0$;

(b) for n odd, $[x-1+4 \cdot 5^{-(1+n)/2}-(7/2)5^{-(1+m)}]^2 +$

$y^2 = r^2$, where $m > (n-1)/2$;

(c) for n even, $[x-1+2 \cdot 5^{-(n/2)}-(7/2)5^{-(m+1)}]^2 +$

$y^2 = r^2$, where $m > n/2$.

On the other hand, $C_\infty = B'_\infty \cup B''_\infty$, where B''_∞ is the symmetric image of B'_∞ with respect to $y = -1$, and B'_∞ is obtained from B_∞ by replacing each M_n by M'_n, where M'_n is

composed of semi-circles and straight line segments satisfying:

(1) for the semi-circles, $x < 0$, $[(5^{m+1})r-1/2] \in A$, and:

(a) for $n = 0$, $x^2 + [y-(1/5)+(7/2)5^{-(m+1)}]^2 = r^2$, $m > 0$;

(b) for n odd, $[x-(4/5)+2 \cdot 5^{-(n+1)/2}]^2 + [y+(1/5)-(7/2)5^{-(m+1)}]^2 = r^2$, $m > (n-1)/2$;

(c) for n even, $[x-(4/5)+4 \cdot 5^{-(n/2)}]^2 + [y+(1/5)-(7/2)5^{-(m+1)}]^2 = r^2$, $m > n/2$.

(2) The straight line segments are to have slope +1 and are to join to the x-axis not only the extremities of the semi-circles in M_n', but also the latter's points of accumulation.

Knaster explains the relationship between B_n and C_n on one hand, and B_∞ and C_∞ on the other this way:

"It was easy to see that in the case of the continua B_n and C_n, each domain R_k $(0 < k < n-1)$ wormed its way into the next ones by means of an infinite number of narrower and narrower blind alleys terminating in dead ends. They were directed forward and could only reach a neighborhood of an analogous blind alley of a preceding domain after having wound through those of all the following [domains]. This is impossible in the case of an infinite number of more and more distant domains.

Now one succeeds in restoring the dense disposition of these blind alleys [in the infinite case] (in order that they should have a common boundary) by occassionally directing them directly backward toward those of the preceding domains. This is precisely the case of the continua B_∞ and C_∞ where the first blind alley of each odd region is directed, without winding through the following ones, toward its point on the y-axis

$$[(7/2)5^{-(n+1)/2} - 1],$$

where "n" designates the number of that domain"[60, p.284].

The fact that there exist indecomposable continua that are the common boundaries of several plane domains allows us to give examples where some familiar double integral formulas fail, as Hocking and Young show [44, pp. 143-145]. Suppose we modify the Lakes of Wada construction by taking the island to be I^2. We further modify it by digging sufficiently long and narrow canals from the ocean and the two internal lakes so that the three resulting domains, D_1, D_2, D_3 each have measure 1/10. Then, if $f \equiv 1$, we have that $\int_{I^2}\int f = 1$, while $\sum_1^3 \int_{D_i}\int f = 3/10$. Moreover, since $\bigcup_1^3 D_i$ is dense and since $C = Fr(D_i)$, $i = 1, 2, 3$, we have

$$\sum_1^3 \int_{\overline{D}_i}\int f = 3[(1/10)+(7/10)] \neq 1.$$

Before presenting some examples which show Kuratowski's common boundary theorem fails in E^3, we shall mention some of the work done on the "prime end theory" and Burgess' generalization of both it and Kuratowski's result.

Caratheodory introduced the term "prime end" in his 1912 paper on conformal mappings [22]. A prime end of the boundary F of a domain D is the set of limit points of a sequence of nested domains determined in D by a chain of transversals tending to zero in length. A transversal is a simple arc contained in $D \cup F$ joining two points of F. Transversals form a chain if they are pairwise disjoint and the subdomain D_j, determined in D by the transversal T_j, contains the domain D_k determined by T_k, for all T_k following

T_j. This version of the definition was given by Marie Charpentier [23, p. 303].

In 1939, Charpentier investigated irreducible cuts of E^2 that were sufficiently complicated so that the entire cut was one prime end. She showed [23, p. 306] that if the continuum C cuts E^2 into two domains and if C has a prime end identical to itself, then C must be either an indecomposable continuum or else the union of two indecomposable continua K, L such that $K = \overline{C-L}$ and $L = \overline{C-K}$.

In 1935, Rutt considered the question of when "the set of prime ends of a plane bounded simply connected domain includes one which contains all the boundary points of the domain." [109, p. 265] He established the following results:

(1) "In order that the collection of prime ends of the plane bounded simply connected domain D with boundary F should include one containing F, it is sufficient that F be indecomposable." [109 p.268]

(2) "In order that a plane bounded simply connected domain D with boundary F should have a prime end containing F, it is necessary that F should be either indecomposable or the union of two indecomposable continua." [109, p. 278]

In his thesis, C. E. Burgess investigated continua and their complementary domains in E^2, using some results of P. M. Swingle on generalized indecomposable continua. He showed (among other things) that the following theorem holds:

"Suppose H is a closed set and M is a continuum in $\overline{E^2-H}$ and intersecting E^2-H such that if R_1, R_2, R_3 are three domains intersecting M, there exist three complementary domains of $M \cup H$ each intersecting each of the domains R_1, R_2, R_3. Then either M is indecomposable or there is only one pair of indecomposable proper subcontinua of M whose union is M." [19, p. 907]

Kuratowski's theorem (8.13), Charpentier's theorem, and the second result of Rutt are now special cases of Burgess' theorem [19, p. 908].

To conclude the chapter, we shall consider the situation in E^3. Kuratowski's common boundary theorem fails there, a fact he knew when he published his works on E^2 [70, p. 132], [72, p. 36]. In fact, he gave the following example. Let C be a plane continuum that is the boundary of three plane domains. Join each point of C to a point above the plane and to a point below the plane. The resulting continuum is the common boundary of three domains in E^3, but it is certainly neither indecomposable nor the union of two indecomposable continua.

R. L. Wilder showed in 1933 that there exists in E^3 a peano continuum which is the boundary of three domains [129, pp. 275-278]. He constructed this set by generalizing the Lakes of Wada technique to three dimensions. That is, the island becomes a solid ball, the lakes become two (tangent) balls removed from inside the first ball, and the canals become tunnels. His method can easily be generalized to

give a peano continuum that is the common boundary of n domains, or even a countably infinite number, although in the latter case, the diameters of the inside balls must tend to zero. In connection with this, P. M. Swingle has shown in 1961 that if Wilder's tunnels of circular cross section are replaced by tunnels of annular cross section, then the resulting closed connected set in E^n is indecomposable [114].

There is a familiar name in the background of Wilder's work: Schoenflies. It is Wilder's belief that Schoenflies' methods of investigating the topology of plane domains and their boundaries could be extended to higher dimensions, even though some topologists felt otherwise [129, pp. 273-274]. His above paper was a step in that direction.

Perhaps the most frequently cited example showing that the three dimensional case differs from the two dimensional one is Lubanski's ANR which is the boundary of three (or more) domains and which can be decomposed into a finite union of AR's whose diameters are arbitrarily small [81]. (See [28, pp. 151-152] for a definition of ANR, AR.) Thus, we can have a "nice" continuum being a common boundary of three or more complementary domains in E^3.

Lubanski notes that a mathematician named Gruba constructed the first such example. It is not known if it had Lubanski's decomposition property, since the paper was lost in WW II and never published [81, p. 29].

CHAPTER 9

ACCESSIBILITY OF PLANE INDECOMPOSABLE CONTINUA

In this chapter we shall present another characterization of indecomposable continua in the plane. This characterization was given by Kuratowski in 1929, based on work done by Mazurkiewicz in the same year.

Kuratowski asked whether every plane indecomposable continuum contains a composant which contains no accessible point. Mazurkiewicz' surprising answer was that "almost all" composants have no accessible points, in the sense that the union of all composants containing accessible points is of first category with respect to the given continuum [92, p. 107]. In a later paper [93], he showed that in a plane indecomposable continuum, the collection of composants which contain more than one accessible point is either finite or countably infinite.

Using the first of these results, Kuratowski showed [74] that a plane continuum is indecomposable iff it is nowhere dense and contains a point which is contained in no proper accessible subcontinuum.

Before proving this theorem, we need to mention the ways in which the term "accessible" is being used here. During the late 1920's, a point p contained in a set $S \subset E^2$

was said to be accessible from E^2-S if there were a simple arc A having p as an endpoint and such that $A \cap S = p$. Mazurkiewicz used this definition in his above-mentioned papers. Note that this definition differs slightly from the one given on p. 83. However, Kuratowski's definition was different than both of the above. $X \subset C$ is accessible iff there is a continuum L such that $L \cap C = X$, and $L - C \neq \emptyset$.

Instead of establishing Mazurkiewicz' first result, we shall prove Kuratowski's generalization of it.

Theorem 9.1: If C is an indecomposable continuum in E^2, the union of its proper accessible subcontinua is of first category with respect to C [74, p. 116].

Proof: Let a be any point of C, and let P(a,C) be its composant. Let $G_{n,k}$ be any complementary domain of the set $C \cup \overline{B(a, 1/n)}$ in E^2, and let $F_{n,k}$ be its boundary. Let K(y) be the component of $y \in F_{n,k}$ contained in C-B(a,1/n), and set

$$Q_n = \bigcup_k \bigcup_{y \in F_{n,k}} K(y), \qquad Q = \bigcup_n Q_n.$$

Suppose D is an accessible subcontinuum of C. By Kuratowski's definition of an accessible subset, there exists a continuum L such that $L \cap C = D$, and $L - C \neq \emptyset$. Either $D \cap P(a,C) \neq \emptyset$, or else $D \cap P(a,C) = \emptyset$. In the former case, $D \subset P(a,C)$. In the latter case, $L \cap P(a,C) = L \cap (C \cap P(a,C)) = D \cap P(a,C) = \emptyset$. In particular, $a \notin L$. Consequently, there is an n sufficiently large that $L \cap \overline{B(a,1/n)} = \emptyset$, say $1/n = [d(a,L)]/2$. Therefore, $L-(C \cup \overline{B(a,1/n)}) = (L-C) \cap (L-\overline{B(a,1/n)}) = (L-C) \cap L = L-C \neq \emptyset$. Thus, there is a

k such that $L \cap G_{n,k} \neq \emptyset$. Moreover, $\emptyset \neq L \cap C \subset L - G_{n,k}$. Therefore, $L \cap F_{n,k} \neq \emptyset$. $F_{n,k} \subset C \cup \overline{B(a,1/n)}$, and, since $L \cap \overline{B(a,1/n)} = \emptyset$, we have $\emptyset \neq L \cap F_{n,k} \subset L \cap C = D$. Hence, $D \cap F_{n,k} \neq \emptyset$. Thus, $L \cap C \cap \overline{B(a,1/n)} = \emptyset$, and so $D \cap B(a,1/n) = \emptyset$. Therefore, $D \subset C - B(a,1/n)$, and consequently, $D \subset Q_n$.

Thus, the union of all proper accessible subcontinua of C is contained in $Q \cup P(a,C)$. Theorem 4.11 (a) shows that $P(a,C)$ is first category with respect to C. Lemma 9.2 will show that Q is also first category with respect to C. Therefore, D is first category.

Lemma 9.2: Under the hypotheses and notations of Theorem 9.1, Q is first category with respect to C [92, pp. 112-115].

Outline of Proof: We shall show that each Q_n is nowhere dense in C. Suppose that the indices n, k are fixed, and that $c \in C - \overline{B(a,1/n)}$; $d(c, \overline{B(a,1/n)}\ ) > \epsilon > 0$. Let P_1 be some composant of C, not $P(a,C)$. By Theorem 4.17, $\overline{P}_1 = C$, so $P_1 \cap B(a,1/n) \neq \emptyset$. Let x be in this intersection, and let $L(x)$ be the component of x in $C - B(c, \epsilon/2)$. It is a nonempty proper subcontinuum of C, so by Theorem 4.4, it is a continuum of condensation of C. That is, it is nowhere dense in C. It follows that $(P_1 \cap B(a,1/n)) - L(x) \neq \emptyset$. Let y be in this set. Since x,y are in P_1, there is a continuum $J \subset P_1 \subset C$, irreducible between x, y. (The irreducibility follows from the fact that any continuum containing a pair of points contains a continuum irreducible between them.)

Set $J_1 = J - B(a,1/n)$, $J_2 = J \cap \overline{B(a,1/n)}$, and let c_1 be

in the nonempty set $J \cap B(c, \epsilon/2)$. J_1 and J_2 are closed and satisfy $x, y \in J_2$, $c_1 \in J_1 - J_2$, and $J_1 \cup J_2 = J$. Thus, the hypotheses of the following lemma are satisfied.

Lemma: Let C be a continuum irreducible between p, q of a compact metric space. Let F_1 and F_2 be two closed subsets such that $p, q \in F_2$, $F_1 - F_2 \neq \emptyset$, and $F_1 \cup F_2 = C$. If $z \in F_1 - F_2$, and if $\epsilon > 0$, then there exists a continuum $K \subset F_1$ such that $K \cap F_2 \neq \emptyset$ and non-connected, and $d(z,K) < \epsilon$ [92, pp. 107-109].

Therefore, there exists a continuum M such that $M \subset J_1 \subset (C - B(a,1/n)) \cap P_1$, $d(c_1, M) < \epsilon/2$, (hence $d(c,M) < \epsilon$), and, since $M \subset J - B(a,1/n)$, $M \cap \overline{B(a,1/n)} = M \cap (J \cap B(a,1/n))$. The right hand side of this equation is $M \cap J_2$, so by the lemma, it is a nonempty non-connected set. Then, by Theorem 8.7, $M \cup \overline{B(a,1/n)}$ cuts the plane. Consequently, M cuts $E^2 - \overline{B(a,1/n)}$, and hence also cuts $E^2 - B(a,1/n)$ [92, p. 110].

Let $V(P_1)$ denote the component of $C - B(a,1/n)$ containing M. Clearly, $V(P_1) \subset P_1$ and $d(c_1, V(P_1)) < \epsilon$. By Theorem 4.4, $V(P_1)$ is nowhere dense in C, and hence in $E^2 - B(a,1/n)$. Then since M cuts $E^2 - B(a,1/n)$, it follows that $V(P_1)$ cuts this set [92, p. 110].

By the proof of Theorem 4.11 (c), the composants are uncountable. Thus, the collection of components $V(P_1)$ is uncountable. Distinct composants are disjoint by Theorem 4.10, so $V(P_1') \cap V(P_1'') \subset P_1' \cap P_1'' = \emptyset$, for $P_1' \neq P_1''$. We can now apply the following lemma of Mazurkiewicz.

Lemma: Let A be a continuum in E^2 which is locally connected. Let Φ be an uncountable collection of disjoint continua, each of which cuts A. Then Φ contains three continua such that one cuts A between the other two. That is, two continua are in different complementary domains of the one continuum [92, pp. 109-110].

Thus, there are three composants P_1', P_1'', P_1''', such that $V(P_1)$ cuts $E^2-B(a,1/n)$ between $V(P_1'')$ and $V(P_1''')$. We claim that either $V(P_1'') \cap F_{n,k} = \emptyset$, or $V(P_1''') \cap F_{n,k} = \emptyset$. If not, then choose $v_1 \in V(P_1'') \cap F_{n,k}$ and $v_2 \in V(P_1''') \cap F_{n,k}$. $V(P_1')$ cuts $E^2-B(a,1/n)$ between v_1 and v_2 [92, p. 109]. Since v_1, $v_2 \in F_{n,k}$, all neighborhoods of v_1, v_2 must contain points of $G_{n,k}$. Hence, $V(P_1')$ cuts $E^2-B(a,1/n)$ between some pair of points v_1', v_2' of $G_{n,k}$. However, since $G_{n,k}$ is a domain, there is a continuum $N \subset G_{n,k}$ containing those points. Then $N \subset G_{n,k} \subset E^2-(C \cup \overline{B(a,1/n)}) \subset (E^2-B(a,1/n))-V(P_1')$. This contradicts the fact that $V(P_1')$ cuts $E^2-B(a,1/n)$ between v_1 and v_2. Therefore, take V to be the one of $V(P_1')$, $V(P_1'')$ which is disjoint from $F_{n,k}$. Consequently, $V \cap \bigcup_{y \in F_{n,k}} K(y) = \emptyset$.

But, $d(c,V) < \epsilon$, since $d(c,V(P_1')) < \epsilon$, so that we have $d(c,C-\bigcup_{y \in F_{n,k}} K(y)) < \epsilon$. Since c is an arbitrary point of the set $C-\overline{B(a,1/n)}$ and $\epsilon > 0$ is arbitrarily small, we have that $\overline{C-\bigcup_{y \in F_{n,k}} K(y)} \supset \overline{C-\overline{B(a,1/n)}}$, and $\overline{C-\bigcup_{y \in F_{n,k}} K(y)} \supset \overline{C \cap B(a,1/n)}$.

Moreover, $C = \overline{C-\overline{B(a,1/n)}} \cup \overline{C \cap B(a,1/n)}$. If not, then C would contain either a point of dimension zero, or else an arc of Fr[B(a,1/n)] which would not be a continuum of condensation of C. This violates Theorem 4.4.

Therefore, $\overline{C-\bigcup_{y\in F_{n,k}} K(y)} = C$. Consequently, for any n, k the closed set $\bigcup_{y\in F_{n,k}} K(y)$ is nowhere dense with respect to C. Thus Q is first category.

Theorem 9.3: A plane continuum C is indecomposable iff it is nowhere dense and contains a point which is contained in no proper accessible subcontinuum [74, p. 116].

Proof: Suppose C is decomposable, say $C = A \cup B$, where A, B are proper subcontinua of C. If C is not nowhere dense, then there is nothing to prove. If C is nowhere dense, then C = Fr(C). Since the set of points accessible from E^2-C is dense on Fr(C), both A and B must contain points that are accessible from E^2-C. (To prove the statement about density, let $p \in Fr(C)$, and let $\epsilon > 0$ be given. There is a point q in E^2-C within ϵ of p. Starting from q, let r be the first point of the segment qp in Fr(C). Then qr lies in E^2-C, except for r, and hence r is accessible from E^2-C and is within ϵ of p.)

Let p, q be accessible points of C contained in A, B respectively. Thus, there exist simple arcs L, K having p, q as respective extremities, and such that $L \cap C = p$, $K \cap C = q$. Then $L \cup A$, $K \cup B$ show that A, B are accessible in the sense of Kuratowski. Since the union of these proper subcontinua contain all points of C, we have contradicted

the condition given in the theorem.

If C is indecomposable, then C is nowhere dense in E^2 by an easy corollary of Theorem 4.4. The accessibility condition follows by Theorems 9.1 and 4.8, the latter of which says that no T_2 indecomposable continuum is first category with respect to itself.

CHAPTER 10

TOPOLOGICAL GROUPS AND INVERSE LIMITS

A. D. Wallace nicely expressed the reaction of many mathematicians to the concept of indecomposable continua when he said, "We commonly think of indecomposable spaces as being monstrous things created by set-theoretic topologists for some evil (but purely mathematical) purpose." [123, p. 96] We hope to dispel any such feelings about indecomposable continua by showing that they can play a role in areas other than point set topology.

In particular, we are going to explore two roles of indecomposable continua in topological groups. First, we shall present Wallace's proof of the fact that if we have a continuous multiplication with a two sided identity defined on a continuum, (such a structure is called a <u>clan</u>) then the clan is a topological group, provided the continuum is indecomposable. The second part of the chapter will be devoted to the famous class of examples called solenoids, which are both indecomposable continua and topological groups.

We begin by defining some of the above terminology. A <u>group</u> is a set G together with an operation, *, such that:

(a) $a, b \in G$ implies $a*b \in G$;

(b) * is associative;

(c) there is an identity element e, such that $x*e = x = e*x$, for all $x \in G$;

(d) for each $x \in G$, there is an inverse x^{-1}, such that $xx^{-1} = e = x^{-1}x$.

If only a and b hold, we have a semi-group, while if only a, b, and c hold, we have a monoid. If all of the axioms hold and the operation is also commutative, the group is abelian.

Definition: A topological group is a set G which has both a group structure, (G,*) and a topological structure (G,T) such that:

(a) $f: G\times G \longrightarrow G$ given by $f((x,y)) = x*y$ is continuous;

(b) $g: G \longrightarrow G$ given by $g(x) = x^{-1}$ is continuous.

For example, the real numbers under the usual addition and topology form a topological group. Later in this chapter we shall need the fact that $\{z \mid z \text{ is complex and } |z| = 1\}$ is a topological group under complex multiplication. Its underlying topological space is S^1.

A mob is a T_2 space (S,T) together with a continuous associative multiplication, m. If (S,T) is also a continuum and if (S,m) has a two sided identity, then the mob is a clan [123, p. 96]. Let $\emptyset \neq L \subset S$, where S is a mob. L is a left ideal if $SL \subset L$. Similarly, one defines a right ideal and a two sided ideal.

The next theorem shows that the algebraic property of existence of inverses in a clan is implied by the topological property of indecomposability. Wallace first proved the result in 1953 under the additional restriction that the space be metrizable [122]. He later simplified it [66], but the version we give appeared in [123, p. 103].

Theorem 10.1: An indecomposable clan is a topological group.

Proof: Let S be an indecomposable clan. Let H denote the maximal subgroup of S containing the identity. Let K denote the minimal closed ideal of S. To prove its existence, let $\{I_\alpha\}_{\alpha \in \mathfrak{a}}$ denote the set of all closed ideals of S. The collection is nonempty since S is in it, as well as the closure of any ideal. If L and R are respectively left and right ideals, then $L \cap R \neq \emptyset$, for $x \in L$ and $y \in R$ show $yx \in L \cap R$. Thus, $\{I_\alpha\}_{\alpha \in \mathfrak{a}}$ is a collection of closed subsets satisfying the finite intersection property. Since S is compact, $K = \bigcap_{\alpha \in \mathfrak{a}} I_\alpha \neq \emptyset$. K is the minimal closed two sided ideal of S.

If S = H, then S is a group. So suppose $S \neq H$. X = S-H is a right ideal in S. For, let $x \in X$, and let y be any element of S. If $xy \in H$, then there exists $(xy)^{-1} \in H$. Then $x[y(xy)^{-1}] = e$ implies $x \in H$, a contradiction. Hence, $K \subset S-H$, so $K \cap H = \emptyset$. Let U be a neighborhood of e such that $\overline{U} \cap K = \emptyset$ (recall that compact spaces are regular). Let J be the union of all ideals of S contained in $S-\overline{U}$. J is nonempty since it contains all the elements of K. We shall show that J is open.

Let $x \in J \subset S-\overline{U}$. $S\{x\}S$ is an ideal contained in $J \subset S-\overline{U}$. Define $\tilde{m}$: $S \times \{x\} \times S \rightarrow S$ by $\tilde{m}(s,x,t) = m(m(s,x),t) = sxt$.

Since m is continuous, $\tilde{m}$ is continuous. $S\times\{x\}\times S \subset \tilde{m}^{-1}(S-\overline{U})$. By repeated applications of Corollary 2.6 of [28, p. 228], there exist open sets V_1, V_2, V_3 in S such that $S\times\{x\}\times S \subset V_1\times V_2\times V_3 \subset \tilde{m}^{-1}(S-\overline{U})$. $V_1 = V_3 = S$, so $S\times\{x\}\times S \subset S\times V_2\times S \subset \tilde{m}^{-1}(S-\overline{U})$. Therefore, $SV_2S \subset S-\overline{U}$, and hence it is contained in J. Since $e \in S$, we have $x \in V_2 \subset SV_2S \subset J$, and thus J is open.

Moreover, J is connected. For let $a \in J$; $\{a\}S$ is a right ideal and is connected since it is the continuous image of the connected set S under the map m_a, defined by $m_a(x) = m(a,x)$. On the other hand, each $x \in J$ satisfies $x \in S\{x\} \subset J$, so that J is the union of connected left ideals. As earlier in this proof, each of the $S\{x\}$'s meets $\{a\}S$, whence the union is connected.

Therefore, $\overline{J}$ is a continuum. Moreover, it is a proper subcontinuum of S: $\overline{J} \subset \overline{S-\overline{U}} \subset \overline{S-U} = S-U \neq S$. There are two cases: $S-\overline{J}$ connected or not connected.

If $S-\overline{J}$ is connected, then we have $S = \overline{J} \cup \overline{S-\overline{J}}$. $\overline{J}$ is a proper subcontinuum of S, and $\overline{S-\overline{J}}$ is a continuum. It is also proper: $S-\overline{S-\overline{J}} \supset S-\overline{S-J} = J \neq \emptyset$, since J is a nonempty open set. Thus, S is decomposable, which is a contradiction.

If $S-\overline{J} = A \cup B$, where A, B are nonempty, disjoint, open subsets of $S-\overline{J}$ (hence open in S), then $S = (A \cup \overline{J}) \cup (B \cup \overline{J})$ is a decomposition of S (see Lemma 4.2).

Therefore, we must have H = S, so that S is a group. To show that it is a topological group, we must show that

the map $g: S \rightarrow S$ given by $g(x) = x^{-1}$ is continuous. To do this, we introduce the concept of a net.

The following definitions are from Kelley [58], chapter II. A binary relation $\geqslant$ on a set A is called a *preorder* if it satisfies: (a) for all $a \in A$, $a \geqslant a$, and (b) $a \geqslant b$ and $b \geqslant c$ imply $a \geqslant c$. A *directed set* D is a preordered set such that for all a, $b \in D$, there is a $c \in D$ such that $a \leqslant c$, and $b \leqslant c$. A *net* in a space X is a function $\varphi: D \rightarrow X$ where D is a directed set. We will use the notation $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ to represent the range of φ. A net $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ *converges* to $x \in X$ if for all neighborhoods U(x), there is an $a \in D$ such that for all $b \geqslant a$, $x_b \in U$. A net $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ *accumulates* at $x \in X$ if for all neighborhoods U(x) and for all $a \in D$, there is a $b \in D$, $b \geqslant a$, such that $x_b \in U$. $\psi: F \rightarrow X$ is a *subnet* of $\varphi: D \rightarrow X$ iff there exists a function $f: F \rightarrow D$ such that $\psi = \varphi \circ f$ and for each $m \in D$, there is n in F such that if $p \geqslant n$, then $f(p) \geqslant m$.

Lemma 10.2: Consider (G,m,T), where (G,m) is a group and (G,T) is a compact T_2 space. If m is continuous, then so is g, where $g(x) = x^{-1}$. That is, (G,m,T) is a topological group.

Proof: Let $x \in G$ be arbitrary, and let $\{x_\alpha\}_{\alpha \in \mathcal{D}}$ be any net converging to x. To show that g is continuous, we must show $\{g(x_\alpha)\} \rightarrow g(x)$ [58, Theorem 3.1 (f), p. 86]. $\{g(x_\alpha)\}$ is a net and since S is compact, there is a subnet $\{g(x_{\alpha_\beta})\}$ converging to some unique $y \in S$ [58, p. 136].

$\{x_{\alpha_\beta}\}$ converges to x since $\{x_\alpha\}$ does. $\{(x_{\alpha_\beta}, g(x_{\alpha_\beta}))\}$ is a net in GxG and it converges to (x,y). Since m is continuous, $\{m(x_{\alpha_\beta}, g(x_{\alpha_\beta}))\} = \{e\}$ converges to $m(x,y) = xy$. Therefore, xy = e, which implies $y = x^{-1} = g(x)$. Hence, $\{g(x_{\alpha_\beta})\} \to g(x)$.

To see that $\{g(x_\alpha)\}$ converges to g(x), we note that if $\{g(x_\alpha)\}$ has any other accumulation point z, then the above argument can be applied again to get $z = x^{-1}$, so that z = y. Hence, g(x) is the unique accumulation point of $\{g(x_\alpha)\}$. Since S is compact, $\{g(x_\alpha)\}$ must in fact converge to g(x). For otherwise, there exists a neighborhood U[g(x)] such that for all $a \in D$, there exists $b \in D$, $b \geq a$, such that $g(x_b) \notin U$. Then choosing $a_0 \in D$, there is $a_1 \geq a_0$ such that $g(x_{a_1}) \notin U$. In general, given a_γ such that $g(x_{a_\gamma}) \notin U$, there exists $a_\beta \geq a_\gamma$ such that $g(x_{a_\beta}) \notin U$. Then $\{g(x_{a_\alpha})\}$ is a net none of whose terms is in U. Hence, it can not accumulate at g(x). Since S is compact, it accumulates elsewhere, contradicting the uniqueness of accumulation point.

There is a great deal of information in the literature on the subject of when an algebraic structure and a topological structure are sufficiently compatible to yield a topological group. See, for example, the references at the end of Wallace's paper [123; 110-112]. Also, the book by Husain [46] does a great deal in this direction.

It is interesting to note that the conclusion of Theorem 10.1 holds if the hypothesis of being indecomposable is replaced by that of being a manifold [122, p.2]. This is especially curious since, in Wallace's words,"certainly manifolds and indecomposable continua are antipodal points on the sphere of topology." [122, p. 2]

The question naturally arises as to whether there are any indecomposable continua that are also topological groups. We would certainly hope so, for otherwise, the preceding theorem would be of very little interest. In fact we shall show that all solenoids have the desired properties. Since one of the most useful ways of describing these spaces involves inverse limits, the next few paragraphs are devoted to stating some properties of inverse limits.

Paul Alexandroff introduced the concept in 1929, although in a slightly different form and context than ours [2]. The following material follows the treatment in Eilenberg and Steenrod [31], Chapter 8, except where otherwise noted.

Let M be a directed set. A subset $M' \subset M$ is <u>cofinal</u> in M if for each $m \in M$, there is an $m' \in M'$ such that $m \leq m'$. Let $X = \{X_\alpha\}_{\alpha \in M}$ be a collection of sets and $F = \{f_{\alpha\beta}\}$ a collection of functions such that whenever $\alpha \leq \beta$, then $f_{\alpha\beta}(X_\beta) \subset X_\alpha$, for all $\alpha \in M$, $f_{\alpha\alpha}(X_\alpha) = X_\alpha$, and $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$, $\alpha \leq \beta \leq \gamma$. In such a case, $\{X_\alpha, f_{\alpha\beta}\}_M$ is called an <u>inverse system of sets</u>. If $M = Z^+$,

we say it is an <u>inverse sequence</u>. The sets X_α are called the <u>factor spaces</u> of the system, and the functions F are called the <u>binding maps</u>. If each X_α is a topological space and each binding map is continuous, then we have an inverse system of topological spaces. If each X_α is a topological group and each map is a continuous homomorphism, then we have an inverse system of topological groups.

The <u>inverse limit</u> of an inverse system of topological spaces (or groups) $\{X_\alpha, f_{\alpha\beta}\}$ is $\underleftarrow{\text{Lim}}\,(X\quad) =$

$$X_\infty = \{x = \{x_\alpha\} \in \prod X_\alpha \mid \alpha \leq \beta \Rightarrow f_{\alpha\beta} \circ p_\beta(x) = p_\alpha(x)\}$$

where $\prod X_\alpha$ is the product of the spaces X_α (see [28, pp. 21, 98]) and $p_\gamma : \prod X_\alpha \longrightarrow X_\gamma$ is given by $p_\gamma(\{x_\alpha\}) = x_\gamma$. X_∞ is given the topology it inherits as a subspace of $\prod X_\alpha$. If each X_α is a topological group, then X_∞ is a topological group.

<u>Lemma 10.3</u>: If $\{X_\alpha, f_{\alpha\beta}\}$ is an inverse system of topological spaces (groups), then each p_α is a continuous function (homomorphism).

<u>Proof</u>: See [31, p. 216].

<u>Lemma 10.4</u>: Let $\{X_\alpha, f_{\alpha\beta}\}$ be an inverse system of topological spaces. (a) If for each $\alpha \leq \beta$, $f_{\alpha\beta}$ is one-to-one, then p_α is one-to-one; if $f_{\alpha\beta}$ is one-to-one and onto, then so is p_α. (b) If the index set M is countable or if each X_α is a compact T_2 space, and if each $f_{\alpha\beta}$ is onto, then p_α is onto.

<u>Proof</u>: See [31, pp. 216, 218].

<u>Lemma 10.5</u>: Let $\{X_\alpha, f_{\alpha\beta}\}$ be an inverse system of topo-

logical spaces. (a) If each X_α is T_2, then X_∞ is a closed subspace of $\prod X_\alpha$. (b) If each X_α is compact and T_2, then X_∞ is compact and T_2. If also each $X_\alpha \neq \emptyset$, then $X_\infty \neq \emptyset$.

Proof: See [31, pp. 216-217].

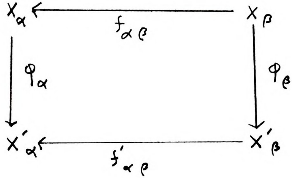
Lemma 10.6: If each X_α in the inverse system is a nonempty T_2 continuum, then X_∞ is a nonempty continuum.

Proof: See Engelking [32, p. 244].

Lemma 10.7: If $\{X_\alpha, f_{\alpha\beta}\}$ is an inverse system of spaces, then $p_\alpha^{-1}(U)$, where U runs over all open sets of X_α and α over M, is a basis for the topology of X_∞.

Proof: See [31, p. 218].

If we have two inverse systems $\{X_\alpha, f_{\alpha\beta}\}$, $\{X'_\alpha, f'_{\alpha\beta}\}$ over M, we can sometimes define a collection of functions $\{\varphi_\alpha\}: \{X_\alpha, f_{\alpha\beta}\} \longrightarrow \{X'_\alpha, f_{\alpha\beta}\}$, $\varphi_\alpha: X_\alpha \longrightarrow X'_\alpha$ such that if $\alpha \leq \beta$ then the following diagram commutes:



In such a case, $\{\varphi_\alpha\}$ is called a map from $\{X_\alpha, f_{\alpha\beta}\}$ to $\{X'_\alpha, f'_{\alpha\beta}\}$.

Lemma 10.8: Let $\{X_\alpha, f_{\alpha\beta}\}$, $\{X'_\alpha, f'_{\alpha\beta}\}$ be inverse systems of topological spaces (groups) over M with a map $\{\varphi_\alpha\}$, where each of the φ_α's is a continuous function (homo-

morphism). Then $\{\varphi_\alpha\}$ induces a continuous function (homomorphism) $\varphi_\infty : X_\infty \longrightarrow X'_\infty$ given by $\varphi_\infty(\{x_\alpha\}) = \{\varphi_\alpha(x_\alpha)\}$.

Proof: See [31, p. 218].

Lemma 10.9: Let $\{\varphi_\alpha\} : \{X_\alpha, f_{\alpha\beta}\} \longrightarrow \{X'_\alpha, f'_{\alpha\beta}\}$ be a map of the systems. If each φ_α is one-to-one, then so is the induced map φ_∞. If each $\varphi_\alpha \circ p_\alpha$ is onto, then $\varphi_\infty(X_\infty)$ is dense in X'_∞. Hence, if each X_α is compact, φ_∞ is onto.

Proof: See [28, p. 430].

Lemma 10.10: Let M be a directed set, $M' \subset M$ cofinal. Let $\{X_\alpha, f_{\alpha\beta}\}$ be an inverse system over M and $\{X_{\alpha'}, f_{\alpha'\beta'}\}$ the inverse system extracted from the first by choosing each α', β' to be in M'. Then $\varprojlim X_\alpha$ is homeomorphic to $\varprojlim X_{\alpha'}$.

Proof: See [28, p. 431].

As our final result of this section, we present a recent (1971) theorem of Kuykendall giving necessary and sufficient conditions for an inverse limit of an inverse sequence of metric continua to be indecomposable. We prove only the sufficiency, since that is the only part that we will be using in the chapter.

Theorem 10.11: Let $\{X_n, f_{n,m}\}$ be an inverse sequence such that for each n, X_n is a non-degenerate metric continuum with metric d_n, and surjective binding maps. The following are equivalent:

(a) X_∞ is indecomposable.

(b) If n is a positive integer and $\epsilon > 0$, there is a positive integer $m > n$ and three points of X_m such that if K is a subcontinuum of X_m containing two

of them, then $d_n(x, f_{n,m}[K]) < \epsilon$ for each $x \in X_n$.

Proof: For (a) implies (b), see Kuykendall's thesis [77]. We only note that this proof makes use of Theorem 4.11, as we would expect.

(b) implies (a): Suppose that (a) does not hold. Since M is non-degenerate, there are proper subcontinua H and K of M such that $M = H \cup K$. There is an n such that $p_n(H)$ and $p_n(K)$ are proper subcontinua of X_n: Since H is proper, there is an $x = \{x_i\} \in X_\infty - H$. For each $y = \{y_i\} \in H$, there exists i_y such that $x_{i_y} \neq y_{i_y}$. Further, if $n \geq i_y$, then $x_n \neq y_n$. Otherwise, we would have $x_{i_y} = p_{i_y}(x) = f_{i_y,n} \circ p_n(x) = f_{i_y,n} \circ p_n(y) = p_{i_y}(y) = y_{i_y}$, a contradiction.

Thus, for each $y \in H$, there exists an open set (see Lemma 10.7) $U_{i_y} = p_{i_y}^{-1}(X_{i_y} - \{x_{i_y}\})$ such that $p_{i_y}(x) \cap p_{i_y}(U_{i_y}) = \emptyset$. Since H is compact, there exists a finite subcover of H, $\left\{U_{i_{y_j}}\right\}_{j=1}^{k}$. Then, for $n_1 \geq \max\left\{i_{y_j}\right\}$, $x_{n_1} \neq y_{n_1}$, for all $y = \{y_i\} \in H$. Hence $p_{n_1}(x) \notin p_{n_1}(H)$, and we have that $p_{n_1}(H)$ is a proper subcontinuum of X_{n_1}. Likewise, there exists n_2 such that $p_{n_2}(K)$ is a proper subcontinuum of X_{n_2}. Setting $n = \max\{n_1, n_2\}$ gives the desired result.

Thus, there is a point $q_1 \in X_n - p_n(H)$ and an $\epsilon_1 > 0$ such that $d_n(q_1, p_n(H)) \geq \epsilon_1$. There is a point $q_2 \in X_n - p_n(K)$ and

an $\epsilon_2 > 0$ such that $d_n(q_2, p_n(K)) \geq \epsilon_2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. If $m \geq n$, and a, b, c are three points of X_m, then two of a, b, c are in one of $p_m(H)$, $p_m(K)$. Since $f_{n,m} \circ p_m(H) = p_n(H)$, $f_{n,m} \circ p_m(K) = p_n(K)$, $d_n(q_1, p_n(H)) \geq \epsilon$, and $d_n(q_2, p_n(K)) \geq \epsilon$, it follows that (b) does not hold.

We can now define and study the solenoids. Let $\{a_j\}$ be a sequence of real numbers with $|a_j| > 1$. Let $S^1 = \{z \mid z$ is complex and $|z| = 1\}$. Let $f_{n,n+1}: S^1 \longrightarrow S^1_n$ be given by $f_{n,n+1}(z) = z^{a_n}$, n = 1, 2, ... The inverse limit of this sequence is called a <u>solenoid</u>, Σ. If the sequence $\{a_j\}$ is replaced by a sequence $P = \{p_1, p_2, \ldots\}$, where the p_j are prime (1 is not considered to be a prime), then the resulting inverse limit is the <u>P-adic solenoid</u>, ΣP. If all the p_j's are equal to say p, then we have the <u>p-adic solenoid</u>, Σ_p. However, if we took p = 1, then the circle results, and we say that the solenoid is degenerate. Historically, Σ_2 was the first solenoid to be discovered, although its original formulation will be given later. We make one last preliminary remark about ΣP. Note that the set of all P-adic solenoids contains the set of all solenoids whose binding maps are z^n, where n is any integer and may vary with the binding map. For we can replace this binding map by a finite sequence of factor spaces and binding maps which arise from the prime factorization of n.

We now discuss some properties of Σ. Note that since each factor space is a non-degenerate T_2 continuum,

Σ is a non-degenerate T_2 continuum. Moreover, since each factor space is a topological group and each binding map is a continuous homomorphism, Σ is a topological group. The indecomposability of Σ follows at once from Theorem 10.11. First, any subcontinuum K of S^1_m is an arc of the form $\{e^{i\theta} \mid \alpha \leq \theta \leq \beta,\ 0 < \beta - \alpha < 2\pi\}$. Then, given $n \geq 1$, $\epsilon > 0$, choose the three points to be e^{i0}, $e^{\frac{\pi i}{2}}$, $e^{\frac{3\pi i}{2}}$ in S^1_m, where $m > n$ is large enough that $|a_{m-1} \cdot \ldots \cdot a_n| \geq 4$, which is possible since $|a_j| > 1$. Then, if K is any subcontinuum of S^1_m containing any two of the above points, K must contain an arc of length $\pi/2$ containing them. The image of this arc under $f_{n,m}$ has length $|a_{m-1} \cdot \ldots \cdot a_n|(\pi/2) \geq 2\pi$. Hence, $f_{n,m}(K) = S^1_n$, so that for any $x \in S^1_n$, $d(x, f_{n,m}]K]) = 0 < \epsilon$.

The indecomposability of Σp can also be shown directly. Suppose $\Sigma p = A \cup B$, where A, B are proper subcontinua. There exists n such that $p_n(A) \neq S^1_n \neq p_n(B)$, (see p. 142). Since p_n is continuous and A is a continuum, $p_n(A)$ is a continuum. Hence, $S^1_n - p_n(A)$ must be an arc sans endpoints, say $K = \{e^{i\theta} \mid \alpha < \theta < \beta,\ \beta - \alpha \leq 2\pi\}$. Then we have

$$L = f^{-1}_{n,n+1}(K) = \bigcup_{1}^{q_n} \left\{e^{i\psi} \;\middle|\; \frac{\alpha}{q_n} + \frac{m2\pi}{q_n} < \psi < \frac{\beta}{q_n} + \frac{m2\pi}{q_n}\right\}$$

$= \cup\ L_m$. Moreover, $p_{n+1}(A) \subset f^{-1}_{n,n+1}[p_n(A)] \subset S^1_{n+1} - L$. But L is the disjoint union of the q_n arcs $L_1, L_2, \ldots, L_{q_n}$, and since $p_{n+1}(A)$ is connected, we must have $p_{n+1}(A) \subset J$, where J is one of the components of S^1_{n+1}-L. Thus, the arc length of $p_{n+1}(A)$ is strictly less than π. Likewise, the arc length of $p_{n+1}(B)$ is strictly less than π. Therefore, the arc length of $p_{n+1}(\sum p) = p_{n+1}(A \cup B) = p_{n+1}(A) \cup p_{n+1}(B) < 2\pi$, contradicting the fact that the projections are surjective (Lemma 10.4 (b)). This is essentially the way the indecomposability of $\sum$ was proved by A. van Heemert in 1938 [119, p. 323], although he did not supply the details of the arc length argument.

D. van Dantzig described each $\sum_n$ as an intersection of solid tori in 1930 [118, pp. 102-125], but he did not mention indecomposability. We shall give this description later and show its equivalence to the inverse limit definition. H. Freudenthal announced the indecomposability of $\sum_n$, described in terms of inverse limits, but he gave no proof [40, pp. 232-233]. Finally, Vietoris, the discoverer of the first solenoid, $\sum_2$, mentioned it was indecomposable, but again he gave no proof [121, pp. 459-460]. This is not surprising in view of his formulation of $\sum_2$, which we now present without further comment.

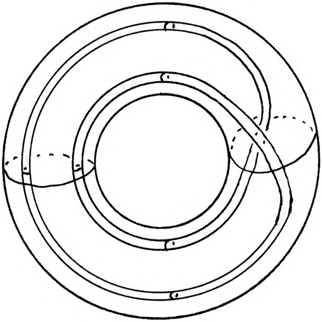
Let F denote the Cantor set in the unit interval, I, and consider FxI. For each $x \in F$, identify (x,0) with (f(x),1), where f is defined on F as on p. 146. For convenience, we use the triadic expansion for the numbers used in defining f.

Let $x \in F$:

$$f(x) = \begin{cases} x + 0.2 & \text{if} \quad 0 \leq x \leq 0.1 \\ x + 0.12 - 1 & \text{if} \quad 0.2 \leq x \leq 0.21 \\ x + 0.012 - 1 & \text{if} \quad 0.22 \leq x \leq 0.221 \\ x + 0.0012 - 1 & \text{if} \quad 0.222 \leq x \leq 0.2221 \\ \vdots & \vdots \quad \vdots \\ 0 & \text{if} \quad x = 1 \end{cases}$$

See Figure 10.1, p. 147. For a discussion of why this is homeomorphic to the toroidal description, see [118, pp.106-108].

The solenoids Σ_p are often described as the intersection of solid tori. The basic procedure is to put one torus inside another in a special fashion. Namely, the torus T_{n+1} must be wrapped longitudally p_n times around the inside of T_n. Thus, for $p_1 = 2$, T_2 is embedded in T_1 as shown:



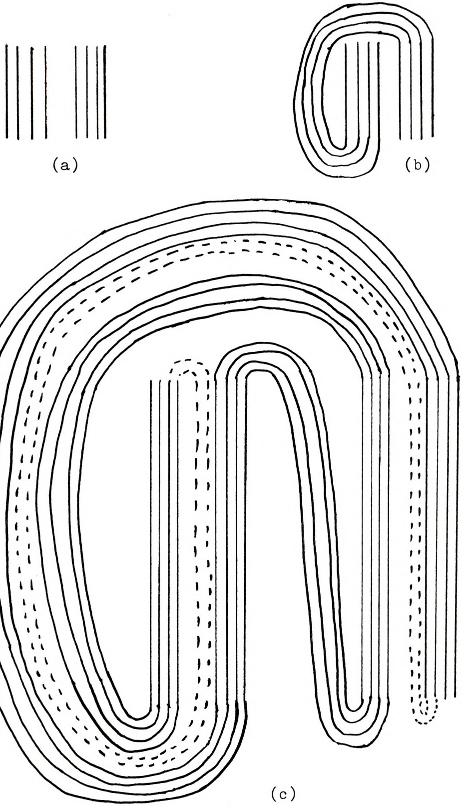
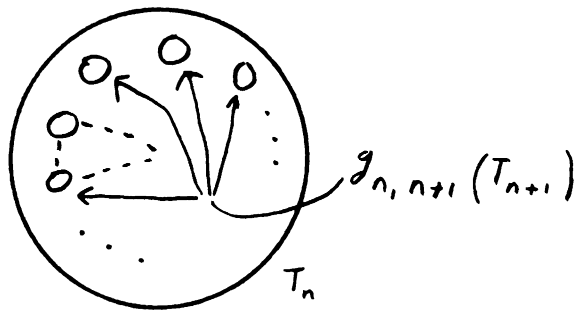


Figure 10.1

More precisely, let T denote the solid torus, $T = S^1 x D^2 = \{(s,z) \mid |s| = 1, |z| \leq 1, s, z \text{ complex}\}$. Define $g_{n,n+1}$ from T_{n+1} to T_n by $g_{n,n+1}((s,z)) = (s^{p_n}, [z/c_n] + [s/2])$, where $0 < c_n < (1/2) \sin \pi / p_n$. Then, $g_{1,2}(T_2)$ is a torus embedded in T_1, running p_1 times around T_1. The choice of c_n assures us that for each cross section of $g_{n,n+1}(T_{n+1})$, the p_n disks are disjoint:



Consider the inverse sequence $\{T_n, g_{n,m}\}$, where each $T_n = T$, and $g_{n,n+1}$ is as above. Let T_∞ denote the inverse limit. We want a homeomorphism from T_∞ to Σ_P, so we look at the map φ_∞ induced by $\{\varphi_n\}: \{T_n, g_{n,m}\} \longrightarrow \{S^1, f_{n,m}\}$, where the function $\varphi_n : T_n \longrightarrow S^1_n$ is given by $\varphi_n((s,z)) = s$:

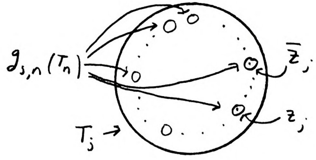
$$
\begin{array}{ccccccccccccc}
T_1 & \xleftarrow{g_{1,2}} & T_2 & \xleftarrow{g_{2,3}} & T_3 & \leftarrow \cdots \leftarrow & T_n & \xleftarrow{g_{n,n+1}} & T_{n+1} & \leftarrow & \cdots & & T_\infty \\
\downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & \cdots & \downarrow \varphi_n & & \downarrow \varphi_{n+1} & & & & \downarrow \varphi_\infty \\
S^1_1 & \xleftarrow{f_{1,2}} & S^1_2 & \xleftarrow{f_{2,3}} & S^1_3 & \leftarrow \cdots \leftarrow & S^1_n & \xleftarrow{f_{n,n+1}} & S^1_{n+1} & \leftarrow & \cdots & & \Sigma_P
\end{array}
$$

Everything in sight commutes, so there is a function $\varphi_\infty : T_\infty \longrightarrow \Sigma_P$ induced by $\{\varphi_n\}$. By Lemma 10.8, φ_∞ is continuous, since each φ_n is. Since each φ_n is onto, so is

φ_∞, by Lemma 10.9. If we can show φ_∞ is one-to-one, then it will be a homeomorphism, since T_∞ is compact and $\sum p$ is T_2 [28, p. 226].

So suppose $w = \{(s_i, z_i)\}$, $x = \{(\overline{s}_i, \overline{z}_i)\}$ are such that $w \neq x$. Then there exists j such that $(s_j, z_j) \neq (\overline{s}_j, \overline{z}_j)$. If $s_j \neq \overline{s}_j$, then $\varphi_\infty(w) = \{s_i\} \neq \{\overline{s}_i\} = \varphi_\infty(x)$, so that φ_∞ is one-to-one in this case.

If $z_j \neq \overline{z}_j$, then let the distance between them be ϵ. Choose $n > j$ large enough that $[2/(c_{n-1}) \cdot \ldots \cdot c_j] < \epsilon$. Note that $g_{j,n}(T_n)$ is a torus running $[p_{n-1} \cdot \ldots \cdot p_j]$ times around inside T_j. When we take the s_j-cross section of T_j, we also get $[p_{n-1} \cdot \ldots \cdot p_j]$ cross sections of $g_{j,n}(T_n)$, corresponding to the $[p_{n-1} \cdot \ldots \cdot p_j]^{\text{th}}$ roots of s_j. Each of the latter cross sections is a disk of diameter $2/(c_{n-1} \cdot \ldots \cdot c_j)$. Since z_j, $\overline{z}_j$ are coordinates of points in the inverse limit, they must be in $g_{j,n}(T_n)$. But, since $2/(c_{n-1} \cdot \ldots \cdot c_j) < \epsilon$, they must be in different disks; that is, in disks corresponding to different $(p_{n-1} \cdot \ldots \cdot p_j)^{\text{th}}$ roots of s_j:



Hence, at the n^{th} coordinates of w and x, we must have $(s_n, z_n) \neq (\bar{s}_n, \bar{z}_n)$ by virtue of $s_n \neq \bar{s}_n$, since s_n is the $(p_{n-1} \cdot \ldots \cdot p_j)^{th}$ roots of s_j in whose disk z_j lay, and similarly for $\bar{s}_n$ with respect to $\bar{s}_j$, $\bar{z}_j$. Thus, φ_∞ is one-to-one, and is therefore a homeomorphism.

The solid tori description of Σp can be given as $\bigcap_1^\infty g_{1,n}(T_n)$. To see that this is homeomorphic to the "unit circle" definition, we construct another inverse sequence. Let $X_n = g_{1,n}(T_n)$, n = 1, 2, . . . , let $h_{n,n+1} = i$, the inclusion map, and consider the inverse sequence $\{X_n, h_{n,n+1}\}$. We first show that $\varprojlim \{X_n, h_{n,n+1}\}$ is homeomorphic to $\bigcap_1^\infty g_{1,n}(T_n)$.

Choose $x = \{x_i\} \in \varprojlim \{X_n, h_{n,n+1}\}$. Since the binding maps are all inclusion maps, $i(x_n) = x_{n-1} = x_n$, for all n. Thus, all the coordinates of x must be the same, say x_o. Hence, x_o is in each factor space, and therefore, we have $x_o \in \bigcap_1^\infty g_{1,n}(T_n)$. Define $\varphi: \varprojlim \{g_{1,n}(T_n), h_{n,n+1}\} \longrightarrow \bigcap_1^\infty g_{1,n}(T_n)$ by $\varphi(\{x_o\}) = x_o$. φ is clearly one-to-one and onto. Since the domain is compact and the range is T_2, we need only show that φ is continuous. But, the map $\varphi: \varprojlim \{g_{1,n}(T_n), h_{n,n+1}\} \longrightarrow T_1$ is just p_1, the first projection map, which is known to be continuous (although it is not onto). Therefore, $\varphi : \varprojlim \{g_{1,n}(T_n), h_{n,n+1}\} \longrightarrow$

$\varphi(\underleftarrow{\text{Lim}}\{g_{1,n}(T_n), h_{n,n+1}\}) = \bigcap\limits_{1}^{\infty} g_{1,n}(T_n)$ is continuous by restricting the range [28, p. 79]. Finally, we show that $\underleftarrow{\text{Lim}}\{T_n, g_{n,n+1}\}$ is homeomorphic to $\underleftarrow{\text{Lim}}\{g_{1,n}(T_n), h_{n,n+1}\}$.

Consider the following diagram:

$$\begin{array}{ccccccccccc}
T_1 & \xleftarrow{g_{1,2}} & T_2 & \xleftarrow{g_{2,3}} & T_3 & \leftarrow \cdots \leftarrow & T_n & \xleftarrow{g_{n,n+1}} & T_{n+1} & \leftarrow \cdots & T_\infty \\
\downarrow 1 & & \downarrow g_{1,2} & & \downarrow g_{1,3} & & \downarrow g_{1,n} & & \downarrow g_{1,n+1} & & \downarrow G \\
T_1 & \xleftarrow{i} & g_{1,2}(T_2) & \xleftarrow{i} & g_{1,3}(T_3) & \leftarrow \cdots \leftarrow & g_{1,n}(T_n) & \xleftarrow{i} & g_{1,n+1}(T) & \leftarrow \cdots & X_\infty
\end{array}$$

Since all the "blocks" in the diagram commute, there is an induced map $G\colon T_\infty \longrightarrow X_\infty$. Moreover, since each $g_{1,n}$ is a homeomorphism, so is G.

Thus, Σ_P, as first defined, is homeomorphic to $\bigcap\limits_{1}^{\infty} g_{1,n}(T_n)$ (by $\varphi \circ G \circ \varphi_\infty^{-1}$).

An obvious question to ask is this: are the solenoids Σ_P homeomorphic? The answer is no, as the following theorem shows. First, some terminology. Let P, Q be sequences of prime numbers. P is <u>equivalent</u> to Q if a finite number of terms can be deleted so that every prime number occurs the same number of times (possibly infinitely often) in the deleted sequences.

<u>Theorem 10.12</u>: Σ_P is homeomorphic to Σ_Q iff P is equivalent to Q. Thus, in the case of Σ_n and Σ_m, Σ_n is homeomorphic to Σ_m iff m and n have the same prime factors.

<u>Proof</u>: Van Dantzig was the first to prove this for Σ_n [118, p. 122]. Since his proof was based on the torus

description of a solenoid, it was fairly involved. The first proof based on inverse limits was given by McCord in 1965 and relies heavily on Čech cohomology theory [83,p. 198]. (Cook has also generalized van Dantzig's results, but in a different direction [25].)

For the sake of completeness, we include McCord's proof, but lack of space precludes introducing all the terminology needed to make it self-contained.

"From the continuity theorem for Čech cohomology [31, p. 261], one sees that $H^1(\Sigma_P, Z)$ is isomorphic to the group F_P of P-adic rationals (all rationals of the form $k/(p_1 \cdot \cdot \cdot \cdot p_n)$, where k is an integer and n is a positive integer.) Also it can be seen that Σ_P, as a topological group, is topologically isomorphic to the character group of F_P. By number-theoretic considerations one can see that F_P is isomorphic to F_Q iff P is equivalent to Q." See his thesis [82, pp. 22-26] for more details.

As our next major result (10.17), we shall give van Heemert's answer to the question of which metrizable compact connected abelian topological groups are indecomposable.

Theorem 10.13: Let G be a metrizable compact aonnected abelian topological group. Then G is topologically isomorphic to the inverse limit of a sequence of Lie groups. In particular, these factor spaces are metrizable compact connected locally connected abelian topological groups, and the binding maps are continuous surjective homomorphisms.

Proof: The first statement is proved in Husain [46,

p. 154], although its original discoverer seems to be H. Freudenthal [39, p. 69]. In the proof, the $n^{\underline{th}}$ factor space was shown to be G/N_n, where $\{N_n\}$ is a decreasing sequence of closed normal subgroups. Hence, each factor space is abelian. Since the binding maps are continuous and onto and the index set is countable, the projections are continuous surjections. Thus, each factor space is compact and connected. Since each factor space is a Lie group, it is also locally connected. The metrizability follows from the construction of the factor spaces.

Theorem 10.14: Let H be a metrizable compact connected locally connected abelian topological group. Then H is topologically isomorphic to a torus group $T^k = \prod_1^k S^1$, where k is a positive integer or $\aleph_0$.

Proof: See Pontrjagin [104, p. 380].

Thus, the G of Theorem 10.13 is topologically isomorphic to $\varprojlim \{T^{k_n}, f_{n,m}\}$.

It will be useful to express a torus T^m as E^m/Z^m, where Z denotes the integers. If $h: E \rightarrow T$ is given by $h(x) = e^{ix}$, then E^m/Z^m is homeomorphic to T^m under the map $H_m = \prod_1^m h$. Moreover, if we have $f: T^m \rightarrow T^n$ which is a continuous surjective homomorphism, then there exists a continuous surjective homomorphism $F: E^m \rightarrow E^n$ such that $f \circ H_m = H_n \circ F$ [14, p. 82]. Such an F is, of course, a linear transformation. Moreover, we see that if we have an inverse sequence of torii, T^{k_n}, with surjective binding maps, the dimension, k_n, must be a non-decreasing function of n.

Lemma 10.15: Let f: $E^n \longrightarrow E^m$, $n \geq m \geq 1$, be a continuous surjective homomorphism. Then f is monotonic. That is, if $C \subset E^m$ is connected, then $f^{-1}(C)$ is connected.

Proof: Let ker (f) = $\{x \in E^n \mid f(x) = (0,0, \ldots, 0)\}$. It is well known that ker (f) is a linear subspace, and since f is onto, it has dimension k = n-m. Therefore, ker (f) is topologically isomorphic to E^k.

Moreover, $E^n \cong E^m x E^k \cong [E^n/E^k] x E^k$; let Φ denote this composite homeomorphism. Since f is onto, it is easy to see that f is an open map. Hence, by [28, p. 130], the following diagram commutes:

$$\begin{array}{ccccc} [E^n/E^k] x E^k & \xrightarrow{\Phi^{-1}} & E^n & \xrightarrow{f} & E^m \\ & \searrow^{p} & & \swarrow_{g} & \\ & & E^n/E^k & & \end{array}$$

where p is the natural projection, and g is the natural homeomorphism. Therefore, $f^{-1}(C) = \Phi^{-1} \circ p^{-1}[g(C)] = \Phi^{-1} \circ [g(C) x E^k]$. Since g(C) and E^k are connected and Φ^{-1} is a homeomorphism, $f^{-1}(C)$ is connected. Thus, $f^{-1}(C)$ is homeomorphic to CxKer (f), where C is the obvious homeomorphic image of C in E^n.

Theorem 10.16: Let $\{T^{k_n}, f_{n,m}\}$ be an inverse sequence of k_n-dimensional torii with binding maps that are continuous surjective homomorphisms. If there is an n such that $k_n > 1$, then $\varprojlim \{T^{k_n}, f_{n,m}\}$ is decomposable [119, p. 322].

Proof: Let m be the first integer such that $k_m > 1$. By Lemma 10.10, $\varprojlim \{T^{k_n}, f_{n,n+1}\}_{n \geq 1}$ is homeomorphic to $\varprojlim \{T^{k_n}, f_{n,m}\}_{n \geq m}$. There exist proper subcontinua A_m, B_m

whose union is T^{k_m} and such that $H_{k_m}^{-1}(A_m)$, $H_{k_m}^{-1}(B_m)$ are connected in E^{k_m}:

$$T^{k_m} = \left\{(e^{i\theta_1}, \ldots, e^{i\theta_{k_m}}) \mid 0 \le \theta_i \le 2\pi,\ i = 1, \ldots, k_m\right\};$$ let

$$A_m = \left\{(e^{i\theta_1}, \ldots, e^{i\theta_{k_m}}) \mid 0 \le \theta_i \le 2\pi,\ i = 1, \ldots, k_m - 1,\right.$$
$$\left. 0 \le \theta_{k_m} \le \pi\right\} \cup$$
$$\left\{(e^{i\theta_1}, \ldots, e^{i\theta_{k_m}}) \mid 0 = \theta_i,\ i = 1, \ldots, k_m - 1,\right.$$
$$\left. 0 \le \theta_{k_m} \le 2\pi\right\},$$ and

$$B_m = \left\{(e^{i\theta_1}, \ldots, e^{i\theta_{k_m}}) \mid 0 \le \theta_i \le 2\pi,\ i = 1, \ldots, k_m - 1,\right.$$
$$\pi \le \theta_{k_m}$$
$$\left\{(e^{i\theta_1}, \ldots, e^{i\theta_{k_m}}) \mid 0 = \theta_i,\ i = 1, \ldots, k_m - 1,\right.$$
$$0 \le \theta_{k_m}$$

We next construct an inverse sequence of proper subcontinua A_{m+n}, B_{m+n} of $T^{k_{m+n}}$. Let $A_{m+1} = f_{m,m+1}^{-1}(A_m)$, $B_{m+1} = f_{m,m+1}^{-1}(B_m)$. These sets are closed in the compact space $T^{k_{m+1}}$ and hence are compact. Moreover, we have seen that the following diagram commutes:

$$\begin{array}{ccc} E^{k_m} & \xleftarrow{\ F_{m,m+1}\ } & E^{k_{m+1}} \\ {\scriptstyle H_{k_m}}\downarrow & & \downarrow{\scriptstyle H_{k_{m+1}}} \\ T^{k_m} & \xleftarrow{\ f_{m,m+1}\ } & T^{k_{m+1}} \end{array}$$

Therefore, $f_{m,m+1}^{-1}(A_m) = H_{k_{m+1}} \circ F_{m,m+1}^{-1} \circ H_{k_m}^{-1}(A_m)$. By choice of A_m and Lemma 10.15, A_{m+1} is connected. Similarly, B_{m+1} is connected, and it is clear that these are two proper subcontinua whose union is $T^{k_{m+1}}$.

Proceding inductively, we define $X_{m+j+1} = f^{-1}_{m+j,m+j+1}(X_{m+j})$, where X is A or B. As above, the sets are easily seen to be compact and proper. To show that they are connected and cover, consider the following diagram:

$$\begin{array}{ccccc}
E^{k_{m+j-1}} & \xleftarrow{F_{m+j-1,m+j}} & E^{k_{m+j}} & \xleftarrow{F_{m+j,m+j+1}} & E^{k_{m+j+1}} \\
\downarrow H_{k_{m+j-1}} & & \downarrow H_{k_{m+j}} & & \downarrow H_{k_{m+j+1}} \\
T^{k_{m+j-1}} & \xleftarrow{f_{m+j-1,m+j}} & T^{k_{m+j}} & \xleftarrow{f_{m+j,m+j+1}} & T^{k_{m+j+1}}
\end{array}$$

Since the two small blocks commute, so does the large one. We can establish the desired properties as before, since $H^{-1}_{k_{m+j}}(X_{m+j}) = F^{-1}_{m+j-1,m+j} \circ H^{-1}_{k_{m+j-1}}(X_{m+j-1})$.

Let $X = \underleftarrow{\mathrm{Lim}}\{X_{m+n}, f_{m+n,m+n+i}\}$, X = A, B. We claim that A and B form a decomposition of $Y = \underleftarrow{\mathrm{Lim}}\{T^{k_{m+n}}, f_{m+n,m+n+i}\}$. A and B are subcontinua of Y by Lemma 10.6. $A \cup B = Y$ holds, since given $y = \{y_i\} \in Y$, we have $y_m \in T^{k_m}$, and hence either in A_m or B_m. Without loss of generality, suppose $y_m \in A_m$. Then $y_{m+1} \in f^{-1}_{m,m+1}(y_m) \subset A_{m+1}$, since $y \in Y$. Inductively, if $y_{m+j} \in A_{m+j}$, then $y_{m+j+1} \in f^{-1}_{m+j,m+j+1}(y_{m+j}) \subset A_{m+j+1}$. Therefore, $y \in A$.

Finally, A, B are proper subcontinua. Choose $y = \{y_{m+i}\} \in A_m - B_m$, which exists since the projections are surjective. Then $y_{m+1} \in f^{-1}_{m,m+1}(y_m) \subset A_{m+1}$. Since $B_{m+1} = f^{-1}_{m,m+1}(B_m)$,

then $y_{m+1} \notin B_{m+1}$. Again induction shows that $y_{m+i} \in A_{m+i} - B_{m+i}$, for all $i \in Z^+$, so that $y \notin A-B$. Likewise, $B-A \neq \emptyset$.

Theorem 10.17: Let G be a compact connected metrizable abelian topological group. G is indecomposable iff G is a non-degenerate solenoid Σ.

Proof: We have only to examine the binding maps from Theorem 10.13. By [14, pp. 9, 82], the binding maps must have the form $f_{j,j+1}(\theta) = e^{a_j i\theta}$, $a_j \in E'$, if they are merely assumed to be into. For them to be onto, we need $|a_j| \geq 1$: given any $e^{i\psi} \in S^1$, it has $e^{i\psi/a_j}$ as a pre-image, provided $\psi/a_j < 2\pi$. That is, provided $\psi < 2\pi |a_j|$. Taking ψ arbitrarily close to 2π forces $|a_j| \geq 1$. However, if all but a finite number of the a_j's have absolute value 1, the solenoid is degenerate and hence decomposable.

We know from their toroidal descriptions, that Σ_P can be considered as subspaces of E^3. It is natural to ask if these solenoids can actually be embedded in the plane. The answer is no, as the following stronger result shows.

Theorem 10.18: A solenoid Σ_P is not a continuous image of any plane continuum.

Proof: Fort first proved this result for the dyadic solenoid $[P = (p_i),\ p_i = 2]$ in 1959 [38, p. 512]. In his Yale thesis (1963), McCord established the theorem as stated for a general class of spaces which include Σ_P [82, p. 80]. The embeddability question seems to have been answered earlier, at least in the folklore of the subject.

CHAPTER 11

OPERATIONS ON INDECOMPOSABLE CONTINUA

It is natural to ask whether indecomposability is preserved by any of the usual set-theoretic operations. It is clear that if A, B are two indecomposable continua, neither contained in the other, then, even if the union is connected, it is decomposable. However, $A \cap B$ may be disconnected, even if it is nonempty, so this case does not appear to be too interesting either. Of course, if we assume the continua are hereditarily indecomposable and the intersection is connected, then $A \cap B$ is hereditarily indecomposable. There is no hope for products at all:

Theorem 11.1: If A and B are non-degenerate T_2 continua, then AxB is decomposable [119, p. 319].

Proof: Let $A = C \cup D$, where C, D are proper compact subsets of A. ($x, y \in A$ and A being T_2 imply there exist U, V open disjoint sets containing x and y respectively. $x \notin \overline{V}$, $y \notin \overline{U}$, since $U \cap V = \emptyset$. Take $C = \overline{U}$, $D = \overline{A-C}$.) Let $b \in B$ be arbitrary. We claim that the following is a decomposition:

$$AxB = [(CxB) \cup (Ax\{b\})] \cup [(DxB) \cup (Ax\{b\})] \equiv C_1 \cup C_2.$$

If C is connected, then C_1 is clearly connected. If C is disconnected, and if $\{F_\alpha\}$ are its components, then $C = \bigcup_\alpha F_\alpha$, and $C_1 = [(\bigcup_\alpha F_\alpha)xB] \cup [Ax\{b\}] = [\bigcup_\alpha(F_\alpha xB] \cup [Ax\{b\}] =$

$= \bigcup_{\alpha} [(F_{\alpha} \times B) \cup (A \times \{b\})]$. Since $F_{\alpha} \times B$, $A \times \{b\}$ are connected, and since $F_{\alpha} \subset C \subset A$, $b \in B$, then $(F_{\alpha} \times B) \cup (A \times \{b\})$ is connected. Since each set contains the connected set $A \times \{b\}$, the union, C_1, is connected. C_1 is closed in compact $A \times B$, so C_1 is a continuum.

Likewise, C_2 is a continuum. Both are proper subcontinua of $A \times B$. For $C \neq A$ implies there is a $d_1 \in D-C$, and B non-degenerate implies there is a $b_1 \neq b$ in B. $(d_1, b_1) \notin C_1$. Likewise, C_2 is proper.

However, the situation changes for inverse limits. J. H. Reed proved the following in 1967.

Theorem 11.2: (a) Let $\{X_{\alpha}, f_{\alpha\beta}, \mathcal{D}\}$ be an inverse system of T_2 indecomposable continua over a directed set $\mathcal{D}$, where the binding maps are continuous surjections. Then the inverse limit, X_{∞}, is an indecomposable continuum.

(b) If each of the above X_{α} are also assumed to be hereditarily indecomposable while the binding maps are only assumed to be continuous, then X_{∞} is hereditarily indecomposable [105, pp. 597-599].

Proof: (a) We have seen (Lemma 10.6) that X_{∞} is a continuum. Suppose $X_{\infty} = A \cup B$, where A, B are proper subcontinua of X_{∞}. As in the proof of Theorem 10.11, there is an α such that $p_{\alpha}(A)$, $p_{\alpha}(B)$ are proper subcontinua of X_{α}. Since p_{α} is surjective, we have $X_{\alpha} = p_{\alpha}(A) \cup p_{\alpha}(B)$, contradicting the indecomposability of X_{α}.

(b) Let K be any subcontinuum of X_{∞}, let $K_{\alpha} = p_{\alpha}(K)$, and let $g_{\alpha\beta} = f_{\alpha\beta}|K_{\alpha}$. Each K_{α} is an indecomposable continuum. $\{K_\alpha, g_{\alpha\beta}, \gamma\}$ is an inverse system with continuous surjective binding maps, and K_∞ is homeomorphic to K [21, p. 235, #2.8]. By (a), K is indecomposable.

We shall see in the next chapter that, under suitable modifications, a similar theorem holds for pseudo-arcs.

We now consider mappings of indecomposable continua. A continuous image of an indecomposable continuum need not be indecomposable, as the projection of Knaster's first semi-circle example onto the unit interval shows. On the other hand, homeomorphisms clearly preserve indecomposability. Is there any type of mapping satisfying intermediate conditions that preserves indecomposability? The answer is yes, as the following theorem shows.

Theorem 11.3: Let X be an indecomposable continuum and let f be continuous and monotone. Then f(X) is an indecomposable continuum.

Proof: Since f is continuous, f(X) is a continuum. If $f(X) = A \cup B$, where A, B are proper subcontinua, then we have $X = f^{-1}(A) \cup f^{-1}(B)$. Since f is continuous, $f^{-1}(A)$, $f^{-1}(B)$ are closed and therefore compact. Since f is monotone, they are connected. $A-B \neq \emptyset \neq B-A$ implies $\emptyset \neq f^{-1}(A-B) = f^{-1}(A)-f^{-1}(B)$ and $\emptyset \neq f^{-1}(B-A) = f^{-1}(B)-f^{-1}(A)$. Therefore, $f^{-1}(A)$, and $f^{-1}(B)$ are proper subcontinua , contradicting the indecomposability of X.

We conclude this chapter with the following rather startling result of J. W. Rogers Jr.: There is a plane indecomposable continuum that is a continuous image of every

indecomposable continuum. Consider the inverse sequence $\{I_n, g_{n,m}\}$ where, for each natural number n, I_n is the unit interval and

$$g_n(x) = \begin{cases} 2x & \text{if } 0 \le x \le 1/2 \\ 2-2x & \text{if } 1/2 \le x \le 1 \end{cases} .$$

Let D denote the inverse limit of this sequence. By Kuykendall's theorem (10.11), this continuum is indecomposable (take the three points to be 0, 1/2, 1 and choose m = n+1).

Theorem 11.4: Let M be any metric indecomposable continuum. Then there exists a continuous function f such that f(M) = D.

Proof: See [108, p. 452].

It is a "folk theorem" of the subject that D is actually homeomorphic to the first Knaster semi-circle example [108, p. 450].

CHAPTER 12

HEREDITARILY INDECOMPOSABLE CONTINUA

In this chapter, we shall survey some of the more important results of the last twenty-five years in the study of indecomposable continua. We shall be dealing with hereditarily indecomposable continua in general and with such special cases as the pseudo-arc and the pseudo-circle. The change in emphasis of this chapter from ordinary indecomposability to the more restrictive hereditarily indecomposable continua reflects the changing areas of major interest in the investigation of indecomposable continua. We shall also see that certain examples of hereditarily indecomposable continua have been studied intensively because of their relationships to long-standing problems in plane topology.

Not only has the subject changed directions since the late 1940's, but it has also undergone a "change of personnel." That is, most of the work done on indecomposability prior to then was done by Europeans, primarily from the Polish school of mathematics. However, since 1948 most of the work seems to have been done by Americans, primarily by first, second or third generation R. L. Moore students. (For an interesting account of Moore's famous teaching method, see the paper by Lucille Whyburn [128, pp. 35-39].)

We shall also see that some problems originating in the Polish school were either partially or fully solved in the last quarter century by Moore descendants.

We recall from Chapter 6 that Knaster discovered the first example of a hereditarily indecomposable continuum in 1922. His motivation was simply to prove that there exists a continuum each of whose subcontinua is indecomposable. Many of the examples of hereditarily indecomposable continua presented in this chapter were constructed in order to have an example of a continuum satisfying property P, where P was something other than being hereditarily indecomposable.

After Knaster's thesis in 1922, there were only a few theoretical results concerning hereditarily indecomposable continua, and no really significant theorems or examples, until 1948. In that year, E. E. Moise, in a thesis written under the supervision of R. L. Moore, found a homogeneous (see p. 184) plane hereditarily indecomposable continuum, the pseudo-arc, with the property that it is homeomorphic to each of its non-degenerate subcontinua. This answered negatively the question posed by Mazurkiewicz in 1921 [88] as to whether every plane continuum homeomorphic to each of its non-degenerate subcontinua is an an arc (that is, a homeomorph of I). However, Henderson showed in his thesis (1959) that in any metric space, any decomposable continuum that is homeomorphic to each of its non-degenerate subcontinua is an arc [42]. Hence, the conjecture of Mazurkiewicz was partially correct.

Bing, Moise, and others have extensively studied the pseudo-arc, and we shall present their findings later. For now, we give one method of constructing this continuum.

A <u>chain</u> is a finite collection C of open (though not necessarily connected) sets $(c_1, \ldots, c_n)$ called <u>links</u> such that $c_i \cap c_j \neq \emptyset$ iff $|i-j| \leq 1$. If each link has a diameter less than ϵ, C is called an <u>ϵ-chain</u>. If $p \in c_1$, and $q \in c_n$, then we have a chain from p to q. Chain D <u>refines</u> chain C if each link of D is a subset of a link of C. If D refines C in such a way that for each link c of C, the set of all links of D that lie in c is a subchain of D, then D is <u>straight</u> with respect to C.

The basic terminology we need to describe the construction of the pseudo-arc is that of "crooked chain." Consider a chain $C = (c_1, \ldots, c_n)$ from p to q. If $n \leq 4$, then a chain D from p to q is <u>very crooked</u> with respect to C if D is straight with respect to C. If $n \geq 5$, then a chain D from p to q is very crooked with respect to C if D is a refinement of C, and D is the union of:

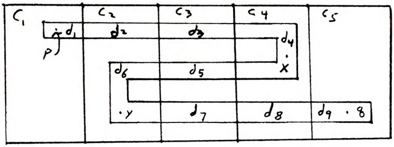
(a) a chain from p to $x \in c_{n-1}$;

(b) a chain from x to $y \in c_2$;

(c) a chain from y to q,

such that these chains are very crooked with respect to $C-\{c_n\}$, $C-\{c_1, c_n\}$, and $C-\{c_1\}$ respectively, and such that no two of them have in common any link that is not an end link of both of them. (If a chain C goes from p to q, then the

<u>end</u> <u>links</u> of C are the links containing p and q.)



Note the similarity to Knaster's "method of bands" (Chapter 6). The above definitions are essentially as Moise gave them [97, pp. 581-583]. We can now define the pseudo-arc.

<u>Definition</u>: Let C_1, C_2, . . . be a sequence of chains from p to q such that:

(a) $\overline{C_1^*} = \overline{\bigcup_j c_{1,j}}$ is a compact metric space;

(b) for each i, C_{i+1} is very crooked with respect to C_i and $\overline{C_{i+1}^*}$ is contained in the interior of $\overline{C_i^*}$;

(c) C_1 contains five links;

(d) if c is a link of C_i and X is a subchain of C_{i+1} which is maximal with respect to the property of being a subchain of C_{i+1} and a refinement of the chain whose only link is c, then X consists of five links;

(e) for each i, each link of C_i has diameter less than 1/i.

Let $M = \bigcap \overline{C_i^*}$; M is called a <u>pseudo-arc</u> [97, p. 583].

<u>Theorem 12.1</u>: If M, N are any sets satisfying the definition of a pseudo-arc, then they are homeomorphic.

Proof: See [97, p. 585].

Theorem 12.2: Every pseudo-arc is hereditarily indecomposable [97, p. 585].

Outline of proof: M is clearly compact. Since we did not assume the chains have connected links, we must show that M is connected. Suppose it is the union of disjoint nonempty open sets H, K. Let i be such that $3/i < d(H,K)$. It follows that C_i is not a chain, which is a contradiction.

Suppose N is any subcontinuum of M. We will use Janiszewski's theorem (4.4) to show it is indecomposable. Let C_i' be the subchain of C_i consisting of all links of C_i that intersect N. Let K be any proper subcontinuum of N, and let C_i'' be the subchain of C_i' consisting of all links of C_i' which intersect K. It can be shown that for all but a finite number of integers, $C_i' - C_i''$ contains two adjacent links of C_i'. It follows from this, that for such i, the set of all links of C_{i+1}' which lie in links of C_i'' contains two chains which "lie close together" such that one has C_{i+1}'' for a refinement. It follows from this that $\overline{N-K} \subset N$. Thus, by Theorem 4.4, N is indecomposable.

Moise comments [97, p.586] that his proof of M's being hereditarily indecomposable is quite similar to the corresponding proof in Knaster's thesis [59, p. 279]. We have also mentioned that Knaster and Moise used similar methods of construction. In fact, Moise suspected that their continua might be homeomorphic [97, p. 581]. Thus,

the following theorem of Bing should come as no surprise. It was published in 1951, three years after the appearance of the psuedo-arc. First, another definition. A metric continuum is <u>chainable</u> or <u>snake-like</u> (a term Bing credits to Choquet [8, p. 653]) if it can be covered by an ϵ-chain for each $\epsilon > 0$.

<u>Theorem 12.3</u>: If M, M' are non-degenerate compact metric continua that are hereditarily indecomposable and chainable, then they are homeomorphic. Moreover, if p, q are in different composants of M, and p', q' are in different composants of M', then there is a homeomorphism of M to M' sending p to p' and q to q'.

<u>Proof</u>: See [10, pp. 44-45].

However, not all hereditarily indecomposable continua are homeomorphic to the pseudo-arc. In fact, Bing proved

<u>Theorem 12.4</u>: There are as many non-homeomorphic plane hereditarily indecomposable continua as there are real numbers.

<u>Proof</u>: See [10, p. 50].

Bing also extended Mazurkiewicz' results (Chapter 7) on the frequency of occurrence of indecomposable continua. In his monumental 1951 paper, Bing proved

<u>Theorem 12.5</u>: Let S be E^n ($n \geq 2$) or a Hilbert space. Then most continua are pseudo-arcs in the sense that if the set of all continua in S is given the Hausdorff metric, then the set of pseudo-arcs is of second category and in fact is a dense G_δ set.

Proof: See [10, p. 46].

He extended this result in 1964. Recall that the links of a chain in E^2 need not be connected, let alone open disks. However, Bing showed that the above theorem is true for pseudo-arcs constructed from chains with open disks for links:

Theorem 12.6: Most (in the sense of Theorem 12.5) plane continua are pseudo-arcs which for each $\epsilon > 0$ can be covered with a linear chain whose links are open disks of diameter less than ϵ.

Proof: See [12, p. 122].

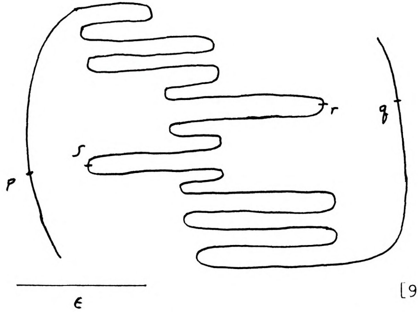
It might be conjectured by now that all indecomposable continua are at most one dimensional. In his thesis, directed by G. T. Whyburn, J. L. Kelley proved (1940) that if there is a hereditarily indecomposable continuum of dimension greater than one, then there is one of infinite dimension [57, pp. 22-35]. However, the major result in this direction was proved by Bing in 1951:

Theorem 12.7: There are infinite dimensional hereditarily indecomposable continua in a Hilbert cube and n-dimensional hereditarily indecomposable continua in E^{n+1}. More generally, each (n+1)-dimensional continuum contains an n-dimensional hereditarily indecomposable continuum.

Proof: See [9, p. 270].

This is not merely an existence theorem: Bing's proof actually gives a way of constructing higher dimensional hereditarily indecomposable continua. We shall give that

construction in E^3 after these definitions. An arc is <u>ϵ-crooked</u> if for each pair of points p, q there exist points r, s between p, q on the arc such that r lies between p and s, and $d(p,s) < \epsilon$, $d(r,q) < \epsilon$:



[9, p. 267]

Let a, b be two distinct points in E^3. The desired continuum is to be the intersection of a decreasing sequence of bounded domains D_i, where $D_i \supset \overline{D_{i+1}}$; D_i separates a and b; $E^3 - D_i$ has only two components, and no point of D_i is more than 1/i from either of them; and finally, each arc in D_i is 1/i crooked.

$D_1 = \{x \in E^3 \mid \min[1/4, d(a,b)/3] < d(a,x) <$ $\min[1/2, 2d(a,b)/3]\}$. Bing proves a general theorem which allows him to construct D_2, D_3, . . . satisfying the above conditions. Since $D_i \supset \overline{D_{i+1}}$, then $C = \bigcap D_i = \bigcap \overline{D}_i$; hence C is a continuum by Theorem 2.1. C separates a from b, and it separates E^3 irreducibly into two complementary domains of C, by the third and fourth properties of the D_i. Thus, by [45, pp. 98-99], the dimension of C is two. The last condition on the D_i gives the hereditarily indecomposability of C: Let K be any subcontinuum of C, and suppose $K = A \cup B$, where A, B are proper subcontinua of K. Then there is an $n > 0$, $p \in A$, $q \in B$ such that $d(p,B) > 1/n$, $d(q,A) > 1/n$. Let U, V be connected open sets of D_n containing A, B respectively and such that $d(p,V) > 1/n$, $d(q,U) > 1/n$. Let $x \in U \cap V$, and consider an arc pxq in $U \cup V$. Since $d(p,xq) > 1/n$, and $d(px,q) > 1/n$, then nxq is not 1/n crooked. This is a contradiction [9, p. 268].

Although Bing and Moise proved that the pseudo-arc is homogeneous (we shall say much more about this at the end of the chapter), no such result is true for higher dimensional hereditarily indecomposable continua. In fact, Bing proved the following

Theorem 12.8: If n is an integer greater than 1, then no n-dimensional hereditarily indecomposable continuum is homogeneous.

Proof: See [9, p. 272].

There are very few results in the literature characterizing hereditarily indecomposable continua in terms of other properties such as there are for ordinary indecomposable continua. (c.f. Theorems 4.4, 4.11) However, we do have two theorems along this line. In 1929, Roberts and Dorroh answered a question of G. T. Whyburn [125] by proving

Theorem 12.9: A necessary and sufficient condition that a metric continuum C be hereditarily indecomposable is that no subcontinuum of C contain an irreducible separator of itself.

Proof: See [106, p. 61].

A much more recent (1966) result due to Zame is

Theorem 12.10: A T_2 continuum C is hereditarily indecomposable iff for each pair of subcontinua, M, N, M-N is connected.

Outline of proof: Suppose there exists a pair of subcontinua M, N such that M-N is disconnected: $M-N = A \cup B$, where A, B are open in M-N, disjoint, and nonempty. Then by Lemma 4.2, $N \cup A$, $N \cup B$ are continua. Hence $N \cup A \cup B$ is a decomposable subcontinuum of C.

Conversely, suppose M is a decomposable subcontinuum of C, say $M = H \cup K$. If $H \cap K$ is connected, then it is a subcontinuum of C. $(H \cup K)-(H \cap K) = [H-(H \cap K)] \cup [K-(H \cap K)]$ is disconnected.

The case of $H \cap K$ disconnected is somewhat longer and will not be presented here. See [132, pp. 709-710].

Recall that a corresponding theorem for ordinary indecomposable continua (4.20) says that if C is a T_2 indecomposable continuum and if K is any proper subcontinuum, then C-L is connected, where L is any subset of K.

Just as for ordinary indecomposable continua, hereditarily indecomposable continua can sometimes be effectively represented by inverse limits. There always exists such a representation for metric continua of any sort, which is a

result first proved by Freudenthal [40]. See also [85, p. 149], and [47, pp. 75-76].

Conversely, Isbell proved in 1959 that if we are given an inverse sequence of compact subspaces of E^n, then the inverse limit is a subspace of E^{2n}. Moreover, for $n = 1$, the hypothesis of "compact" may be dropped [47, p. 78]. McCord proved a related theorem in his thesis: The inverse limit of a sequence of compact metric spaces of dimension n may be embedded in I^{2n+1}. We would like to start with such an inverse sequence and know when the inverse limit is a hereditarily indecomposable continuum. In 1960, M. Brown established a criterion for this to happen.

Let $f: X \longrightarrow Y$, where X, Y are metric spaces, and let $\epsilon > 0$. Let $L(\epsilon, f) = \sup\{\delta \mid x, y \in X,\ d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \epsilon\}$. Suppose now that X is only assumed to be a topological space. f is <u>ϵ-crooked</u> if for every path $g: I \longrightarrow X$, there exists t_1, t_2, $0 \leq t_1 \leq t_2 \leq 1$ such that $|fg(0) - fg(t_2)| < \epsilon$, and $|fg(t_1) - fg(1)| < \epsilon$.

<u>Theorem 12.11</u>: Let $\{X_i, f_{i,j}\}$ be an inverse sequence of locally connected metric continua with diameter d_i. Suppose for all n that $f_{n,n+1}$ is ϵ_n-crooked, where

$$\epsilon_n < \min_{i < n-1} \{L(2^{-n} d_i, f_{i,n-1})\}.$$

Then X_∞ is hereditarily indecomposable.

<u>Proof</u>: See [18, p. 130].

At this point, we seem to have exhausted the supply of structure theorems for general hereditarily indecomposable

continua. It would be interesting to know if such continua can be characterized in terms of irreducibility or some other property. Intuitively, a hereditarily indecomposable continuum ought to be "more irreducible" than an ordinary indecomposable continuum. However, there are some results of this nature for the pseudo-arc.

In 1951, Bing gave the following characterization of the pseudo-arc (Theorem 12.14). Let p be a point of a metric continuum C such that for each $\epsilon > 0$, there exists an ϵ -chain covering C such that only the first link of each chain contains p. Then p is an <u>endpoint</u> of C. (Under this definition, the only endpoint of Knaster's first semi-circle example is the origin [8, p. 662].)

<u>Lemma 12.12</u>: Let C be as above; the following are equivalent:

(a) Each non-degenerate subcontinuum of C containing p is irreducible between p and some other point of C.

(b) If each of two subcontinua of C contain p, then one contains the other.

<u>Proof</u>: See [8, p. 661].

<u>Lemma 12.13</u>: Let C be a metric snake-like continuum. A point $p \in C$ is an endpoint of C iff it satisfies either a or b above.

<u>Proof</u>: See [8, p. 661].

<u>Theorem 12.14</u>: A non-degenerate snake-like continuum is a pseudo-arc iff each point of it is an endpoint.

<u>Proof</u>: See [8, p. 662].

We can restate the above results as follows. Let C be

a non-degenerate snake-like continuum. C is a pseudo-arc iff for each $p \in C$ and for each non-degenerate subcontinuum K containing p, K is irreducible between p and some other point. Compare this with the case of an ordinary snake-like indecomposable continuum, such as Knaster's first semi-circle example: Let p be any point except the origin and let $K = \bigcup_{1}^{3} C_i$, where C_2 is the semi-circle containing p; C_3 is a semi-circle having an endpoint in common with C_2. C_1 is the other such semi-circle, provided C_2 is not the semi-circle having (0,0) and (1,0) as endpoints; it is the empty set in this case. Then K is not irreducible between p and anything else.

In contrast to the higher dimensional hereditarily indecomposable continua that Bing constructed, we have <u>Theorem 12.15</u>: The pseudo-arc does not separate the plane. Moreover, there exist plane hereditarily indecomposable continua which are not homeomorphic to the pseudo-arc that do not separate the plane. However, there exists a hereditarily indecomposable continuum which does separate the plane.

<u>Proof</u>: Moise showed that the pseudo-arc is planar and homeomorphic to each of its non-degenerate subcontinua [97, p. 581]. However, in 1930, G. T. Whyburn had shown that no such continuum could separate the plane [126, pp. 319-320].

The second statement is due to R. D. Anderson [5, p. 185]. In fact, he announced that there exist in E^2 uncountably many hereditarily indecomposable continua not sepa-

rating E^2 and not homeomorphic to the pseudo-arc, including one containing no pseudo-arc and another one, all of whose proper non-degenerate subcontinua are pseudo-arcs.

Finally, Bing showed that there exists an example of the third type by constructing the example later known as the pseudo-circle [10, p. 48]. We shall say more about this example later.

The pseudo-arc has other interesting properties with respect to the plane.

Theorem 12.16: (a) There is a continuous collection of pseudo-arcs filling the plane.

(b) There exists in the plane an uncountable set of disjoint continua, no one of which contains an arc.

Proof: For (a), see R. D. Anderson [4, p. 350].

(b) This was first proved by R. L. Moore in 1928 [101, p. 86]. Of course, this was during the time when the only known example of a hereditarily indecomposable continuum was that of Knaster, which is homeomorphic to the pseudo-arc. So let C be a pseudo-arc. By the proof of Theorem 4.11, C has uncountably many disjoint composants. From each composant, select a subcontinuum of C contained in that composant. This set of continua is uncountable, each two elements are disjoint, and none can be an arc. (C can be any hereditarily indecomposable continuum, of course.)

We now consider some of the mapping properties of the pseudo-arc. Shortly after Moise announced the discovery of the pseudo-arc, F. B. Jones asked O. H. Hamilton whether

the pseudo-arc has the fixed point property with respect to continuous functions. That is, does the pseudo-arc C satisfy the condition that for every continuous function f taking C to itself there exists $x \in C$ such that $f(x) = x$? Hamilton showed that the answer is yes for not only the pseudo-arc, but for arbitrary snake-like continua.

Theorem 12.17: Let Y_1, Y_2, . . . be a sequence of chains such that:

(a) $\overline{Y}_1$ is a nonempty compact metric space, where $\overline{Y}_i$ is the closure of the set of points lying in the links of Y_i;

(b) for each i, $Y_i \supset \overline{Y_{i+1}}$;

(c) $\lim_{i \to \infty} \text{diam}(Y_i) = 0$, where $\text{diam}(Y_i)$ is the maximum diameter of the links of the chain Y_i.

Let M denote the continuum which is the intersection of the sets $\overline{Y}_i$. If T is any continuous transformation of M into a subset of itself, then there exists a $p \in M$ such that $T(p) = p$.

Proof: See [41].

At the Summer Institute on Set-Theoretic Topology, 1955, R. H. Bing raised the question of what characterizes the continuous images of the pseudo-arc. In other words, is there an analog for the pseudo-arc of the Hahn-Mazurkiewicz theorem for the arc. (This theorem says that a metric continuum C is a continuous image of an arc iff C is locally connected [44, p. 129].) One result in this direction is

Theorem 12.18: Every snake-like continuum is a continuous image of a pseudo-arc.

Proof: J. Mioduszewski proved this in 1962 [96] using inverse limits. He also remarked that it seemed to follow from one of Bing's theorems [8, Theorem 5] and one of Lehner's [79, Theorem 1]. (G. Lehner was a thesis student of Bing.) L. Fearnley also proved this theorem in 1964 [33, p. 389].

The first characterization seems to have been given by A. Lelek in 1962, using the following terms. A weak chain in a metric space is a finite sequence of sets $X_1, \ldots, X_m$ such that $X_i \cap X_j \neq \emptyset$ if $|i-j| \leq 1$. Note that the X_i are not assumed to be either open or connected. Moreover, $X_i \cap X_j \neq \emptyset$ does not imply $|i-j| \leq 1$. A weak chain $\{X_i\}_1^m$ is a refinement of a weak chain $\{Y_j\}_1^n$ provided that each X_i is contained in some Y_{j_i} such that $|j_i - j_k| \leq 1$ if $|i-k| \leq 1$. Finally, a continuum C is weakly chainable provided there exists an infinite sequence $\{G_i\} = \left\{\{O_{i,j}\}_{j=1}^{n_i}\right\}$ of finite open covers of C such that each G_i is a weak chain, each link of G_i has diameter less than $1/i$, and G_{n+1} is a refinement of G_n.

Theorem 12.19: A metric continuum is a continuous image of a pseudo-arc iff it is weakly chainable.

Proof: See [80, p. 274].

Note that Theorem 12.18 follows directly from 12.19, since if C is chainable, it is weakly chainable.

L. Fearnley also established some characterizations, using the following terminology. A p-chain is a finite sequence of sets such that each, except the last, intersects its successor. (c.f. "weak chain") If $P = (p_1, \ldots, p_n)$ and $Q = (q_1, \ldots, q_m)$ are p-chains and each link p_i of P is a subset of a link q_{x_i} of Q, then the sequence of ordered pairs $\{(i, x_i)\}$ is a <u>pattern</u> of P in Q. If $|x_i - x_j| \leq 1$ whenever $|i-j| \leq 1$, $1 \leq i, j \leq n$, then the pattern is an <u>r-pattern</u> of P in Q. If a p-chain $P = (p_1, \ldots, p_n)$ has an r-pattern of the form $(1, x_1 = 1), (2, x_2), \ldots, (n, x_n = m)$ in a p-chain $Q = (q_1, \ldots, q_m)$ then P is a <u>normal refinement</u> of Q. Finally, let H be a closed connected separable metric space. H is <u>p-chainable</u> if there is a sequence of p-chains $P_1, \ldots$ such that for each i:

(a) the union of the elements of P_i is H;

(b) P_{i+1} is a normal refinement of P_i;

(c) the diameter of each link of P_i is less than $1/i$;

(d) the closure of each link of P_{i+1} is a subset of the link of P_i to which it corresponds under the r-pattern of P_{i+1} in P_i.

<u>Theorem 12.20</u>: (a) In order that H (as defined above) be a continuous image of the pseudo-arc, it is necessary and sufficient that H be p-chainable.

(b) A metric continuum C is a continuous image of the pseudo-arc iff C is p-chainable with p-chains whose links are open sets.

<u>Proof</u>: For (a), see [33, p. 387], and for (b), see [33, p. 388].

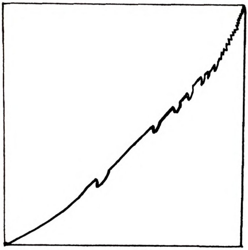
<u>Theorem 12.21</u>: (a) Let K be a chainable separable closed connected metric space. Then K is a continuous image of the pseudo-arc.

(b) The class of continuous images of the pseudo-arc and the class of continuous images of all such sets as K are identical.

<u>Proof</u>: For (a), see [33, p. 389]. Also compare (a) with Theorem 12.17. (b) is a corollary of (a).

These results still do not fully answer the question of whether there is an analog of the Hahn-Mazurkiewicz theorem for the pseudo-arc. However, Fearnley went on to show that there is no characterization of the continuous images of the pseudo-arc in terms of local properties by constructing locally homeomorphic metric continua H and K such that H is a continuous image of the pseudo-arc and K is not [33, pp. 391-395].

It might be conjectured that using inverse limits to describe the pseudo-arc would be easier than using chain conditions. However, in most cases there are infinitely many different binding maps, so that the situation is not greatly improved. In 1964, Henderson was able to construct the pseudo-arc as an inverse limit of a sequence of arcs and one binding map. Roughly speaking, the map was obtained by taking $f(x) = x^2$ on I and "notching its graph with an infinite set of non-intersecting ∿'s which accumulate at (1,1)." [43, p. 421] See the figure on the next page. The proof of this function's existence may be found in the



paper cited above. (See also the Math Reviews 29-4039 for some comments about errors.)

The pseudo-arc is preserved by inverse limits and monotone maps. Explicitly, Bing has shown

Theorem 12.22: Let M denote the pseudo-arc and let N be any non-degenerate monotone continuous image of M. Then M and N are homeomorphic.

Proof: See[10, p. 47].

Theorem 12.23: Let $\{X_i, f_{i,j}\}$ be an inverse sequence of pseudo-arcs, and let X_∞ be the inverse limit. If X_∞ is non-degenerate, then it is a pseudo-arc [105, p. 599].

Proof: By Theorem 11.2, X_∞ is a hereditarily indecomposable continuum. Reed has shown that X_∞ is snake-like [105, p. 598]. Therefore, by Theorem 12.3, X_∞ is a pseudo-arc.

We now discuss one last example of a hereditarily indecomposable continuum. In 1951, R. H. Bing described a plane non-snake-like circularly chainable hereditarily indecomposable continuum, which has since become known as a pseudo-circle. (A metric continuum is circularly chainable if it can be covered for each $\epsilon > 0$ by an ϵ-chain whose first and last links intersect each other.)

Bing described his example this way [10, p. 48]. Let $D_1, D_2, \ldots$ be a sequence of circular chains in E^2 such that:

(a) each link of D_i is an open circular disk of diameter less than $1/i$;

(b) the closure of each link of D_{i+1} is contained in a link of D_i;

(c) the union, A_i, of the links of D_i is homeomorphic to the interior of an annulus;

(d) each complementary domain of A_{i+1} contains a complementary domain of A_i;

(e) if E_i is a proper subchain of D_i and E_{i+1} is a proper subchain of D_{i+1} contained in E_i, then E_{i+1} is very crooked in E_i.

$M = \bigcap_1^{\infty} A_i$ is called a pseudo-circle. Bing proved that such sets exist and that they separate the plane. Fearnley pointed out that every proper non-degenerate subcontinuum of it is a pseudo-arc [34, p. 491]. After defining it, Bing asked if all such continua are homogeneous and whether they are homeomorphic. Fearnley has recently (1969) answered these questions, as well as some others.

Theorem 12.24: (a) The pseudo-circle is unique in the sense

that any two continua satisfying the above definition are homeomorphic.

(b) The pseudo-circle is not homogeneous.

Proof: See [35, pp. 398-401] or [37] for a proof of (a). Fearnley's proof of (b) may be found in [36]. J. T. Rogers Jr. also proved (b) [107] in a thesis supervised by F. B. Jones.

Fearnley also investigated some mapping properties of the pseudo-circle. We need the following definition in order to state his result. A circular p-chain is a p-chain in which the first and last links intersect.

Theorem 12.25: In order that a continuum C be a continuous image of a pseudo-circle, it is necessary and sufficient that C be circularly p-chainable.

Proof: See [34, p. 507].

Theorem 12.26: (a) Every plane circularly chainable continuum is a continuous image of the pseudo-circle.

(b) Every snake-like continuum is a continuous image of the pseudo-circle.

Proof: See [34, p. 510] for (a) and [34, p. 512] for (b).

Thus, the pseudo-arc is a continuous image of the pseudo-circle. It is not known if the converse is true. However, Fearnley has indicated that he has a paper in progress which answers this question as well as whether every solenoid is a continuous image of the pseudo-circle.

We have mentioned that the pseudo-arc is homogeneous,

while the pseudo-circle fails to have this property. The homogeneity of the pseudo-arc has an interesting history which is closely related to that of finding all homogeneous plane continua. Consequently, we shall discuss this problem in some detail. We begin with some developments in the early Polish school of mathematics.

Sierpinski formulated the definition of homogeneity in a paper which appeared in the first volume of the Fundamenta Mathematicae [111, pp. 15-16]. In the same issue, Knaster and Kuratowski posed the question of whether every non-degenerate homogeneous plane continuum is a simple closed curve (that is, a homeomorph of S^1) [61]. Mazurkiewicz proved in 1924 that the answer is yes if the continuum is also assumed to be locally connected [89].

During the years between 1924 and 1948, two false solutions were published. Of course, it was not known that these solutions were false until Bing and Moise showed that the pseudo-arc is homogeneous. Waraszkiewicz announced in 1937 that all non-degenerate homogeneous plane continua are simple closed curves [124]. Choquet's paper [24] went even further in 1944. In it, he asserted that every compact homogeneous plane set is either:

(a) a finite set of points;

(b) a totally disconnected perfect set;

(c) homeomorphic to the union of a collection of concentric circles such that the intersection of this union and a line through the center of the circles

is either a finite set or a totally disconnected perfect set.

In 1949, F. B. Jones proved that under slightly stronger hypotheses, Waraszkiewicz' theorem is correct. We need the following definition in order to give a precise statement of Jones' result. A continuum C is aposyndetic at x if for each point $y \in C-\{x\}$, there exists a subcontinuum K of C and an open set U of C such that $C-\{y\} \supset K \supset U \supset \{x\}$. Jones gives the following explaination of his term "aposyndetic": In Greek, "apo" means "away from", "syn" means "together", while "deo" signifies "to bind". Thus, the word "aposyndetic" means "bound together away from" [51, p. 546].

Theorem 12.27: Let C be a non-degenerate homogeneous plane continuum. If C is either aposyndetic at all points or if no point of C is a cut point, then C is a simple closed curve.

Proof: See [52].

Jones also suggested that Waraszkiewicz' error may have been to confuse the idea of a cut point of a continuum with that of a separating point [54, p. 66].

Shortly after seeing Moise's pseudo-arc, R. H. Bing proved that it is homogeneous. Moise gave his own proof shortly thereafter.

Theorem 12.28: The pseudo-arc is homogeneous.

Proof: See [7] for Bing's (1948) proof, and [98] for Moise's proof (1949).

In view of the results of Warszkiewicz and Choquet, it

is not surprising that some people questioned the homogeneity of the pseudo-arc. Isaac Kapuano presented a paper in 1953 in which he claimed that the pseudo-arc is not homogeneous [55]. He noted that it would be interesting to know exactly what part of Bing's paper is contradicted by his work. However, an error was discovered in his own work, so later in 1953. he published an attempt to correct it [56]. Moreover, neither paper received much criticism in the Mathematical Reviews [Math Reviews 15: 146, 335]. However, mathematicians seem more inclined to accept the results of Bing and Moise than those of Kapuano. Thus, what might have developed into a "lengthy debate" just faded away.

Theorem 12.29: Each non-degenerate homogeneous snake-like continuum is a pseudo-arc. Thus, a non-degenerate snake-like continuum is a pseudo-arc iff it is either hereditarily indecomposable or homogeneous.

Proof: For the first statement, see [11]. The second statement is a summary of Theorems 12.2, 12.3, 12.28, and the first statement of 12.29.

Knaster and Kuratowski's question can now be expanded to the problem of finding all homogeneous plane continua. F. B. Jones gave the following classification of possible homogeneous plane continua [54, p. 67]:

(a) those which do not separate E^2;

(b) those which are decomposable;

(c) those which separate E^2 and are indecomposable.

This is a reasonable approach, in view of one of Jones'

earlier theorems:

Theorem 12.30: If C is a homogeneous plane continuum which does not separate the plane, then C is indecomposable.

Proof: See [53, p. 859].

At the time Jones gave his classification, a point and the pseudo-arc were the only known non-homeomorphic examples of type (a). A simple closed curve and an example discovered simultaneously by Bing and Jones, called a circle of pseudo-arcs [13] were the only known examples of type (b).

Many people conjectured that the pseudo-circle was an example of type (c), but, in view of Fearnley's result, this conjecture was false. There are no known examples of type (c). C. E. Burgess summarized the state of the art in 1969 when he proved

Theorem 12.31: A non-degenerate circularly chainable plane continuum is homogeneous iff it is either a simple closed curve, a pseudo-arc, or a circle of pseudo-arcs.

Proof: See [20].

There have been other partial solutions to the homogeneity question. In his thesis [67], H. V. Kronk showed that if C is a homogeneous, continuously near-homogeneous plane continuum, then C is a simple closed curve [67, p. 18]. (See [67] for the terminology.) He also showed that if the hypothesis of homogeneity is dropped, the conclusion becomes: "C is a simple closed curve or is indecomposable [67, p. 25]." There is a corresponding result for continuously invertible spaces. P. H. Doyle and J. G. Hocking have shown [27] that the only decomposable continuously invertible plane continua are the simple closed curves [27, p. 503]. It is an open question whether there are any indecomposable continuously invertible plane continua. However, there do exist indecomposable continuously near homogeneous plane continua [67, pp. 16-18].

In concluding, we mention a few more open questions and make some conjectures as to where future research in indecomposable continua theory may lead. With only a few exceptions, most of the recent work done in the field of indecomposability has been done in special cases, such as hereditarily indecomposable continua or special cases of this. (In spite of this, there are no known structural theorems for hereditarily indecomposable continua paralleling Theorem 4.12, for example.) It seems likely that future work will also be concentrated in certain special cases of indecomposability theory. Perhaps further work on the homogeneity problem will result in some new developments in indecomposability theory. It also seems likely that there will be more effort devoted to solving certain mapping questions. For example, J. W. Rogers Jr. has indicated (1970) that the following questions are of interest:

(a) Is the pseudo-arc a continuous image of every nondegenerate hereditarily indecomposable continuum?

(b) Is every continuum a continuous image of some (hereditarily) indecomposable continuum? [108,

p. 449].

Some results are known for various special cases of (b). In his thesis [6] (1968), Bellamy proved that every metric continuum is a continuous image of A^* (Chapter 7), his non-metrizable indecomposable continuum [6, p. 39]. More recently, he has shown that every metric continuum is a continuous image of a metrizable indecomposable continuum [To appear, Proc. Am. Math. Soc.]. In an unpublished result, Gordh has shown that this theorem holds when metric and metrizable are replaced by T_2.

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